Lecture 4

$$A_{m\times n} = \begin{pmatrix} a_{11} & & a_{1n} \\ \vdots & & \\ a_{m1} & & \end{pmatrix}$$

Basis $\left\{ \begin{pmatrix} 1 & \cdots & 0 \\ 0 & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & & \cdots & 0 \\ 0 & & & 0 \end{pmatrix}, \cdots \begin{pmatrix} 0 & & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{pmatrix} \right\}$

Standard basis & dim = $m \times n$

(ex) $A_{2\times 2}$. $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \& \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

$\left\{ \begin{pmatrix} 1 & 7 \\ 6 & 9 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 1 & 8 \end{pmatrix}, \qquad \right\}$

② $\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}$ $\mathbb{R}^2$

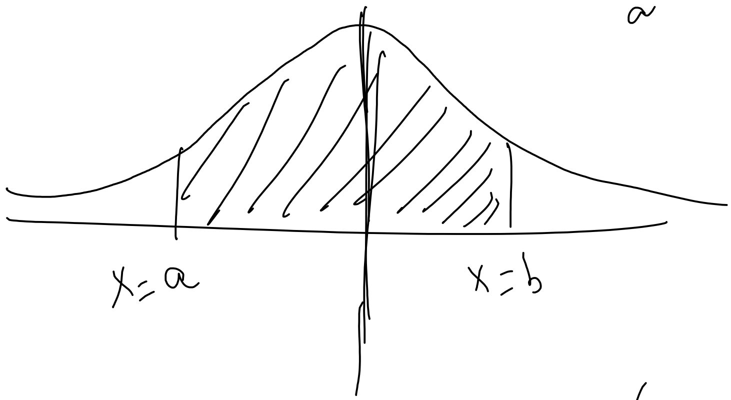
Q $A = ② \begin{pmatrix} 2 & 1 & 2 \\ 3 & 1 & 2 \end{pmatrix}$ $\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ $\mathbb{R}^2$

Basis = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \}$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \qquad (\underline{pdf})$$

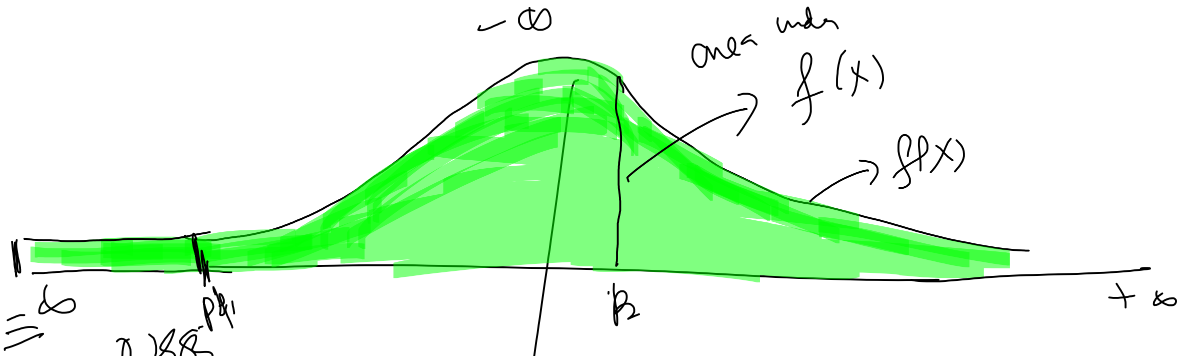
$$\int_a^b f(x)\,dx$$



drand48(); rand2; rand (ocline),

Unif dis in (0,1),

Cdf $\int_{-\infty}^{x} f(x)\,dx$



$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

, rand();

0.2983

0.6024

$$\underline{BA} = \underline{I} = AC$$

$$\underline{B} = BI = B\,AC = (\underline{BA})\,C = I.C = \underline{C}$$

$$\boxed{\begin{aligned} BA &= I \\ A^T B^T &= I^T = I \end{aligned}}$$

A is the inverse of B iff $AB = BA = I$

$AB = I$ $\quad B = A^{-1}$ $\quad 5x = 2$, $x = 2/5$

Null space of $\underline{A}$ & Nullspace of A^TA are the same $\underline{\underline{U}} = \{x \mid Ax = 0\}$ $\quad \underline{\underline{V}} = \{y \mid A^TA y = 0\}$

Let $x \in U \Rightarrow Ax = 0$, multiply both sides by A^T

$$\underline{A^T.Ax} = A^T.0 = 0 \Rightarrow A^TAx = 0$$

$$\Rightarrow x \in V \Rightarrow \underline{U} \subseteq \underline{V}.$$

Let $y \in V \Rightarrow A^TAy = 0$

$$y^TA^TAy = y^T.0 = 0.$$

$$(Ay)^T(Ay) = 0 \Rightarrow \|Ay\|^2 = 0$$

$$\Rightarrow Ay = 0$$

$(Ay)^T(Ay) = 0 \Rightarrow$

$\Rightarrow Ay = 0$

$\Rightarrow y \in \mathcal{U}$

$V \subseteq \mathcal{U}$

$\mathcal{U} \subseteq V \ \& \ V \subseteq \mathcal{U} \Rightarrow \mathcal{U} = V$

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 9 \\ 3 & 0 & 2 \end{bmatrix}$$

$\boxed{R_2 \to R_2 - 2R_1}$

$\boxed{R_3 \to R_3 - 3R_1}$

$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & -2 & -1 \\ 3 & 0 & 2 \end{bmatrix}$$

$$E = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_1} [A]$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}}_{E_2} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

How many elementary matrices are req. to convert a given matrix $A_{n \times n}$ to an up triangular form?

$$1 + 2 + \cdots + n-1 = \frac{(n-1)\,n}{2}$$

$$\underbrace{E_m \cdots E_3 E_2 E_1}_{\text{Lower Tr.}} A = \mathcal{U} \text{ (upper Tr. matrix)}$$

$E_m \ldots E_1 = L^{-1}$

Lower Δr.

Sym $E_m E_{m-1} \ldots E_2 E_1 = L^{-1}$

$$L^{-1}A = U \Rightarrow A = LU$$

$$\left[A^TA \mid A^T \right]$$

$\boxed{\text{If } A^TA = LU}$ $\quad \left[LU \mid A^T \right]$ or $\left[U \mid L^{-1}A^T \right]$

Designate. $Q^T = L^{-1}A^T$

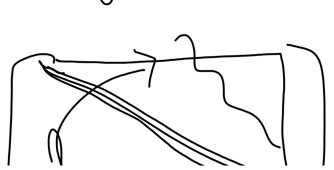
$$Q^TQ = (L^{-1}A^T)(L^{-1}A^T)^T$$
$$= L^{-1}A^T \cdot (A^T)^T \cdot (L^{-1})^T$$
$$= L^{-1} \underline{A^T \cdot A} \, (L^{-1})^T$$
$$= \cancel{L^{-1}} \cancel{L} U \, (L^{-1})^T$$
$$= U (L^{-1})^T.$$

If L is L.Triangular, $\Rightarrow$ L^{-1} is also L Triangular

$(L^{-1})^T$ is upper Δr $\quad U \cdot (L^{-1})^T$ is also upper Δr

Q^TQ is upper Δr.

Is Q^TQ Symmetric? $\underline{(Q^TQ)^T} = Q^T(Q^T)^T$
$= \underline{Q^T \cdot Q}$



Q^TQ is diagonal

Q^TQ is diagonal

$$\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} & 0 \\ & \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & 0 & & 0 \\ a_{21} & a_{22} & \cdot & 0 \\ & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & 0 & & 0 \\ b_{21} & b_{22} & 0 \cdots & 0 \\ & & & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

LT

$$C = \begin{bmatrix} - & 0 & 0 & \cdots \\ & & & \\ & & & \end{bmatrix}$$

Lecture 4

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \det(A - \lambda I) = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix}$$

$$= (a_{11}-\lambda)\left[(a_{22}-\lambda)(a_{33}-\lambda) - a_{23}a_{32}\right]$$

$$\left[a_{21}(a_{33}-\lambda) - a_{23}\,a_{31}\right]$$

$$-a_{12}\left(a_{21}(a_{33}-\lambda) - a_{23}a_{31}\right)\Big]$$

$$+\,a_{13}\left(a_{21}a_{32} - (a_{22}-\lambda)a_{31}\right)$$

$$= (a_{11}-\lambda)\left[a_{22}a_{33} - a_{22}\lambda - a_{33}\lambda + \lambda^2 - a_{23}a_{32}\right]$$

$$- a_{12}\left(a_{21}a_{33} - a_{21}\lambda - a_{23}a_{31}\right)$$

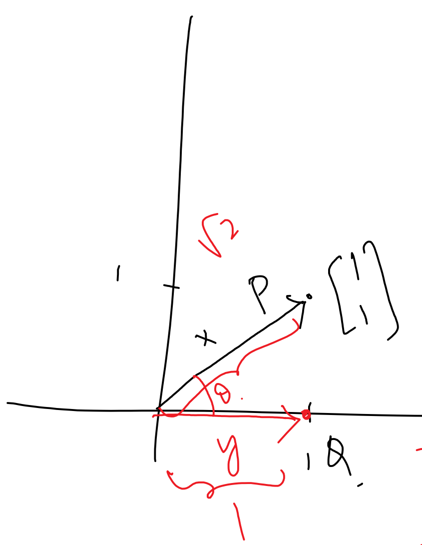
$$+ a_{13}\left(a_{21}a_{32} - a_{22}a_{31} + a_{31}\lambda\right)\Big)$$

$$= (-1)^3\lambda^3 + \lambda^2(a_{11}+a_{22}+a_{33}) + \lambda(\ \dots\ )$$

$$1x^3 - 7x^2 + 6x + 4 = 0$$

$$x_1, x_2, x_3$$

$$x_1+x_2+x_3 = -\frac{(-7)}{1} = 7$$



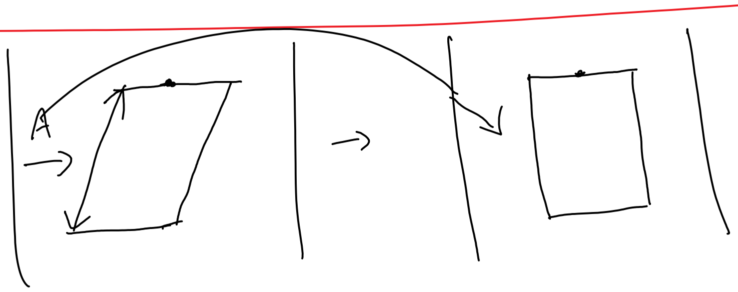
$$A = \begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}$$

$$Ax = \begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}$$

Reduction in length

rotation

rotation

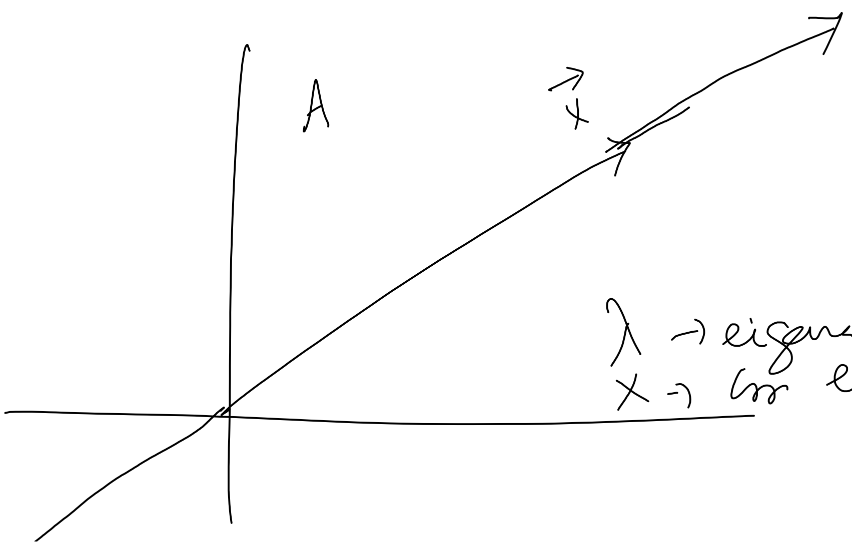


x is a vector & A is a matrix

(Ax → rotation + scaling.)

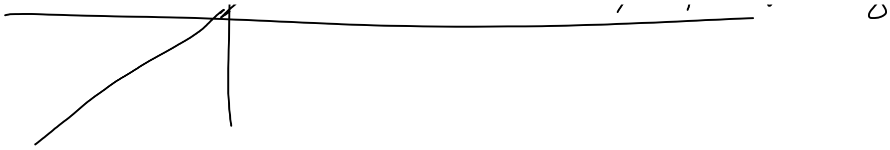
Only rotation : A is orthogonal matrix
→ if does not change lengths

only Scaling . $A = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$, $\alpha < 1$ lessen, $\alpha > 1$ increase.



$A\vec{x} = \lambda \vec{x}$

λ → eigenvalue
x → is eigenvector



(eg) $A = \begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix}$

$$Ax = \lambda I x$$

$$(A - \lambda I)x = 0$$

x is non-zero

$(A - \lambda I)$ is invertible

$(A - \lambda I)^{-1}$ exists

$(A - \lambda I)^{-1}(A - \lambda I)x = 0$

$\Rightarrow x = 0$

$A - \lambda I$ is not invertible

$(A - \lambda I)x$ has at least one sol.

$\Rightarrow \det(A - \lambda I) = 0$

$$A = \begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix}, \quad \det(A - \lambda I) = \det \begin{bmatrix} 5-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}$$

$$= (5-\lambda)(3-\lambda) - 2 = 0$$

$$\Rightarrow \lambda^2 - 8\lambda + 15 - 2 = 0 \Rightarrow \lambda^2 - 8\lambda + 13 = 0$$

$$\lambda = \frac{8 \pm \sqrt{64 - 52}}{2} = \frac{8 \pm \sqrt{12}}{2}$$

$$= 4 \pm \sqrt{3}$$

$\lambda_1 = 4 + \sqrt{3}$, $\lambda_2 = 4 - \sqrt{3}$

$\downarrow$ $\quad$ $\downarrow$

x_1 $\quad$ x_2