Solution :-

Given $M = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

General case of M being an $n \times n$ tridiagonal matrix by replacing 2 with a and 1 with b.

$$T = \begin{bmatrix} a & b & 0 & \cdots & 0 & 0 & 0 \\ b & a & b & \cdots & 0 & 0 & 0 \\ 0 & b & a & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b & 0 \\ 0 & 0 & 0 & \cdots & b & a & b \\ 0 & 0 & 0 & \cdots & 0 & b & a \end{bmatrix}_{n * n}$$

Here, every diagonal entry in principal diagonal is 'a' and the entries just below and above the principal diagonal elements are 'b', and every other position in matrix has 0. Hence, this is a symmetric tridiagonal matrix.

For Eigenvalues ($\lambda$) such that there exists a vector $x$ with

$Tx = \lambda x$ , if $x = (x_1, x_2, \ldots, x_n)^T$ is a vector

$$\begin{bmatrix} a & b & 0 & \cdots & 0 & 0 & 0 \\ b & a & b & \cdots & 0 & 0 & 0 \\ 0 & b & a & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b & 0 \\ 0 & 0 & 0 & \cdots & b & a & b \\ 0 & 0 & 0 & \cdots & 0 & b & a \end{bmatrix}_{n \times n} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{bmatrix}_{n \times 1} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \\ \vdots \\ \vdots \\ \vdots \\ \lambda x_n \end{bmatrix}$$

$\therefore$ from above eqn we get

$a x_1 + b x_2 = \lambda x_1$ — (i) $\rightarrow$ from $1^{st}$ row multiplied by vector $x$

Similarly, $b x_1 + a x_2 + b x_3 = \lambda x_2$

$b x_2 + a x_3 = \lambda x_3$

Last row can be represented as:

$$bx_{n-1} + ax_n = \lambda x_n \quad \text{— (ii)}$$

Any middle row can be represented as

$$bx_{j-1} + ax_j + bx_{j+1} = \lambda x_j \quad \text{where } j = 2, 3, \ldots n-1$$

Hence,

each entry of $x$ is linked by its neighbours by this recurrence..

$$bx_{j-1} + ax_j + bx_{j+1} - \lambda x_j = 0$$

$$\therefore \quad bx_{j-1} + (a-\lambda)x_j + bx_{j+1} = 0 \quad \text{— (iii)}$$

So, matrix condition is now a recurrence relation, where every entry is tied to the previous & the next.

At ends, we can pretend there are 2 extra entries:

$x_0 = 0$, before first row & $x_{n+1} = 0$ after last row

This way, the recurrence applies uniformly from $j = 1$ to $n$

So, matrix problem is reduced to,

$$bx_{j-1} + (a-\lambda)x_j + bx_{j+1} = 0 \quad \text{— (iv)} \quad \text{where } j = 1, 2, \ldots n \text{ \& } x_0 = 0 \text{ \& } x_{n+1} = 0$$

For such recurrences, a standard trick is to guess that the entries look like powers of some number '$r$':

Let $x_j = r^j$

Substituting this to above equation (iv)

$$br^{j-1} + (a-\lambda)r^j + br^{j+1} = 0 \quad \text{— (v)}$$

Divide by $r^{j-1}$

$$\Rightarrow \frac{br^{j-1}}{r^{j-1}} + \frac{(a-\lambda)r^j}{r^{j-1}} + \frac{br^{j+1}}{r^{j-1}} = 0$$

$$\Rightarrow b + (a-\lambda)r + br^2 = 0 \quad \text{— (vi)}$$

This is a quadratic eqn in '$r$'.

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Solving the equation $br^2 + (a-\lambda)r + b = 0$

product of roots $= \frac{b}{b} = \underline{\underline{1}}$

$\left[\because ax^2+bx+c=0 \quad \text{Product of roots} = \left(\frac{c}{a}\right) \quad \text{Sum of roots} = \left(-\frac{b}{a}\right)\right]$

& Sum of roots $= \left(-\frac{(a-\lambda)}{b}\right) = r + \frac{1}{r}$ — (vii)

$\because$ If one of the solution of above equation is $r$, then other will be $1/r$.

$\because$ Roots are complex conjugate pair that are also reciprocals, they must be located on the unit circle in complex plane.

$\therefore$ Any complex number with a modulous of 1 can be written in the form $r = e^{i\theta}$ using Euler's formula.

$\therefore \frac{1}{r} = e^{-i\theta}$

$\because$ Sum of roots $= r + \frac{1}{r} = e^{i\theta} + e^{-i\theta}$

Now,

$$r + \frac{1}{r} = (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta) \quad \text{[Using Euler's formula]}$$

$$r + \frac{1}{r} = 2\cos\theta$$

$$-\frac{(a-\lambda)}{b} = 2\cos\theta \quad \left(\because r + \frac{1}{r} = \left(-\frac{(a-\lambda)}{b}\right)\right)$$

$$\therefore \boxed{\lambda = a + 2b\cos\theta} \quad \text{— (viii)}$$

This is the formula for Eigen Values

Now, we need to find which values of $\theta$ are allowed in above eq$^n$.

We set $x_0 = 0$ & $x_{n+1} = 0$. Therefore General Solution is

$$x_j = Ae^{ij\theta} + Be^{-ij\theta} \quad \text{— (ix)}$$

Substitute $e^{ij\theta} = \cos(j\theta) + i\sin(j\theta)$ & $e^{-ij\theta} = \cos(j\theta) - i\sin(j\theta)$ in above eq$^n$ (ix)

we get $x_j = (A+B)\cos(j\theta) + i(A-B)\sin(j\theta)$

Let $C = i(A-B)$ & $D = (A+B)$

$$\therefore x_j = D\cos(j\theta) + C\sin(j\theta) \quad \text{— (x)}$$

Applying Boundary Conditions:

(i) At $x_0 = 0$ in eq (x)

$$x_0 = D\cos(0) + C\sin(0) \qquad [\because \sin(0) = 0 \ \& \ \cos(0) = 1]$$

$$x_0 = D(1) + C(0) = D$$

$\because x_0 = 0 \ \therefore D = 0$

plugging $D = 0$ in eq (x)

$$\boxed{x_j = C\sin(j\theta)} \quad \text{— (xi)}$$

(ii) At $x_{n+1} = 0$ in eq (x)

$$x_{n+1} = D\cos((n+1)\theta) + C\sin((n+1)\theta)$$

$[\because x_{n+1} = 0 \ \& \ D = 0]$

$$C\sin((n+1)\theta) = 0$$

$$\sin((n+1)\theta) = 0$$

$$(n+1)\theta = \sin^{-1}(0)$$

[$\because$ C is scaling factor & we don't want C=0, as it will make whole vector as zero, which then will not be eigen vector.]

$$\theta_m = \frac{m\pi}{(n+1)} \quad \text{— (xii)}, \quad \text{where } [m = 1, 2, \ldots n]$$

Substituting value of $\theta_m$ to eigenvalue formula, we get,

$$\boxed{\lambda_m = a + 2b\cos\left(\frac{m\pi}{n+1}\right)} \quad \text{— (xiii)}$$

where $[m = 1, 2, \ldots n]$