Birla Institute of Technology and Science, Pilani

Work Integrated Learning Programmes Division

Cluster Programme - M.Tech. in AIML.

I Semester 2023-24

Course Number	AIMLCZC416
Course Name	Mathematical Foundation for Machine Learning
Nature of Exam	Closed Book
Weightage for grading	30%
Duration	120 minutes
Date of Exam	21/01/2024 (AN)

# Pages	4
# Questions	4

Instructions

1. All questions are compulsory
2. Questions are to be answered in the order in which they appear in this paper and in the page numbers mentioned before each of them.

Q1 Answer the following questions with justifications.

1) Consider the system of equations $\boldsymbol{AX} = \boldsymbol{b}$ where $\boldsymbol{A} = [\boldsymbol{A}_1, \cdots, \boldsymbol{A}_n]$ with $\boldsymbol{A}_i$s are columns of $\boldsymbol{A}$. Prove or disprove that if matrix $\boldsymbol{C}$ is obtained by interchanging $\boldsymbol{A}_i$ with $\boldsymbol{A}_j$ where $i < j \leq n$, the consistency of the new system of equations $\boldsymbol{CX} = \boldsymbol{b}$ is same as that of $\boldsymbol{AX} = \boldsymbol{b}$. (1.5 marks)

2) Define $f : \mathbb{R}^{3\times 2} \to \mathbb{R}^{3\times 3}$ as $f(\boldsymbol{B}) = \boldsymbol{B}\boldsymbol{B}^T$. Compute $\dfrac{\partial f}{\partial \boldsymbol{B}}$. (2.5 marks)

3) Let $\boldsymbol{A}, \boldsymbol{B}$ be two square matrices of order n and $f(x_1, x_2) = x_2 \cos(x_1)$
   i. If rank of $\boldsymbol{A}$ is n and $\boldsymbol{B}$ is symmetric positive definite matrix then prove $\boldsymbol{A}^T\boldsymbol{B}\boldsymbol{A}$ is symmetric positive definite. (1 mark)

   ii. If $\nabla f(x_1, x_2) =$ grad $f(x_1, x_2)$ then define $\boldsymbol{A} = \nabla^2 f(x_1, x_2)$ as the Hessian of f. Then find the condition on $x_1, x_2 \in \mathbb{R}$ such that $\langle \boldsymbol{x}, \boldsymbol{y}\rangle_C := \boldsymbol{x}^T\boldsymbol{C}\boldsymbol{y}$ is an inner product where $\boldsymbol{C} = \boldsymbol{A}^T\boldsymbol{B}\boldsymbol{A}$ and $\boldsymbol{B}$ is symmetric positive definite matrix of order 2. (2 marks)

   iii. Find Taylor's 2nd degree polynomial approximation of f about $[\frac{\pi}{2}, 1]^T$ (1 mark)

Q2 Answer the following questions with justifications.

1) Let $\mathbf{x} \in \mathbb{R}^n$. We define a function $f(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{Z}^+ \cup \{0\}$ as $f(\mathbf{x}) =$ *total number of non-zero entries in* $\mathbf{x}$. Assuming that $f(\mathbf{x})$ satisfies the triangle inequality property of norms, prove or disprove whether $f(\mathbf{x})$ satisfies the other two properties (i.e. absolutely homogeneous, positive definiteess) of norms. (1+1 marks)

2) Let $\beta \in \mathbb{R}$ be an unknown constant and let $i = \sqrt{-1}$, a symbol commonly used in definition of complex numbers. Now consider a square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$. It is given to you that $\mathbf{A}$ can be factorized as $\mathbf{A} = \mathbf{B^T B}$ where $\mathbf{B} \in \mathbb{R}^{n \times n}$. Let $\mathbf{x} \in \mathbb{R}^n$ be an eigenvector of $\mathbf{A}$ corresponding to an eigenvalue λ i.e. $\mathbf{Ax} = \lambda \mathbf{x}$. It is also known that this eigenvalue λ is of the form $\lambda = 7 + i\beta$.
   i) Is it possible to find the exact value of λ by deriving the constant β based on the given information? If yes, derive β. If no, provide proper reason(s) as to why it is not possible to finf β with the given information. (1 mark)
   ii) Is the matrix $\mathbf{A}$ a positive semi-definite matrix ? If yes prove it. If no, give proper reason why it cannot be a positive semi-definite matrix. (1 mark)

3) Consider a square matrix $\mathbf{B} \in \mathbb{R}^{\mathbf{2 \times 2}}$ defined as follows:

$$\mathbf{B} = \begin{bmatrix} 3 & 0 \\ 4 & 0 \end{bmatrix}$$

   Assume that $\mathbf{B} = \mathbf{U \Sigma V^T}$ is the singular value decomposition of $\mathbf{B}$.
   i) Derive the largest singular value σ_1 of this matrix $\mathbf{B}$. (1 mark)
   ii) Derive left singular vector $\mathbf{u_1}$ of this matrix $\mathbf{B}$ corresponding to largest singular value σ_1. (1 mark)
   iii) Derive right singular vector $\mathbf{v_1}$ of this matrix $\mathbf{B}$ corresponding to largest singular value σ_1. (1 mark)
   iv) Derive the rank-1 approximation matrix $\mathbf{B_1} = \boldsymbol{\sigma_1}(\mathbf{u_1 v_1^T})$. (1 mark)

Q3 Answer the following questions with justifications.

(1) Following matrix $\boldsymbol{Q}$, where α, β, γ and ω are real numbers and solutions of the polynomial equation $\sum_{i=0}^{2024} c_i x^{2024-i} = 0$, has 3 linearly independent eigenvectors (as columns) in $\boldsymbol{P}$. Find the sum of all the elements of $\mathbf{P^{-1}QP}$. (1 mark)

$$\mathbf{Q} = \begin{bmatrix} 3 & \gamma & \alpha+\beta \\ \gamma & 0 & \omega \\ \alpha+\beta & \omega & 3 \end{bmatrix}$$

(2) Compute $\boldsymbol{A}^{\mathbf{9}}$ without using the conventional matrix multiplication. Show all the computations. (2 marks)

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix}$$

(3) Determine which of the following are subspaces of $\mathbb{R}^3$? Justify your answer. If it is a subspace find its basis and dimension. (2 marks)

(a) All vectors of the form (a, b, c) where $b = a + c$

(b) All vectors of the form (a, b, c) where $c = a + b + 1$

(4) Find dimension of a vector space spanned by the following vectors $\{(1, 1, -2, 0, 1), (1, 2, 0, -4, 1), (0, 1, 3, -3, 2), (2, 3, 0, -2, 0)\}$. (1 mark)

(5) For the matrix below construct an elementary matrix for every elementary row operation that is performed on $\mathbf{A}$ and write the decomposition of $\mathbf{A}$ as $\mathbf{LU}$, where $\mathbf{L}$ is a lower triangular matrix and $\mathbf{U}$ is an upper triangular matrix (2 Marks)

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

Q4 Answer the following questions.

a) You are building an application to scan documents using a mobile phone camera. When a user captures a document, the edges of the document are not aligned with the horizontal and vertical axis of the phone screen. Let us assume that user takes precaution to ensure that the bottom edge of the document is parallel to the x-axis when photo is taken (as shown in the figure **P0** below, A,B,C and D are the corners of the document in the captured image)

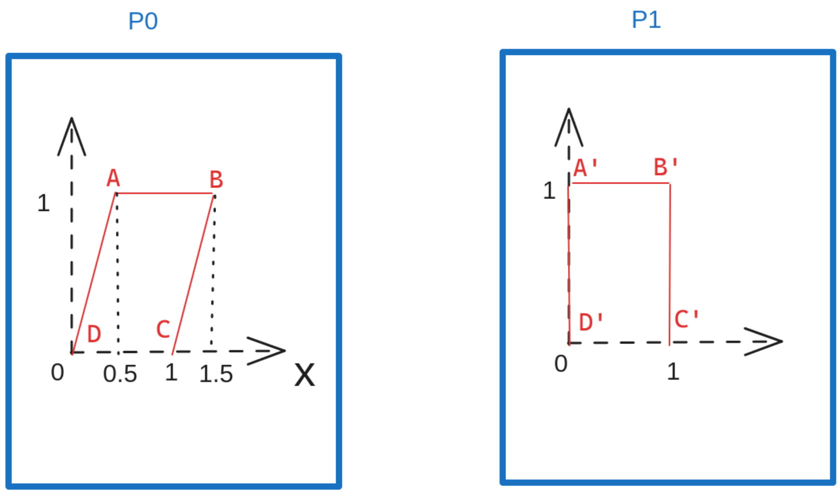

FIGURE 1. Document Scanner

You need to construct a matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$ such that, when $\mathbf{A}$ is applied on the vectors representing the document corners in diagram **P0**, the result is as shown in diagram **P1**. (2 marks)

b) Find the eigenvalues and eigenvectors of $\mathbf{A}^{-1}$. (2 marks)

c) Find the algebraic and geometric multiplicities for eigenvalues found in part (b) of this question? (2 marks)