Cut Elimination in Cyclic Proofs for Temporal Logic

Johannes Kloibhofer
(j.w.i.p.w. Bahareh Afshari)

April 28, 2023

Institute for Logic, Language and Computation
University of Amsterdam, Netherlands

Outline

- Introduce modal logic with eventually
- Show problems with usual cut elimination
- Present our method

Modal logic with eventually operator

Formulas of *modal logic with the eventually operator* MLe are defined by

$$\varphi := p \mid \varphi \wedge \varphi \mid \neg\varphi \mid \Diamond\varphi \mid \mathsf{F}\varphi$$

$\mathsf{F}\varphi$ is interpreted as "eventually φ":

- there is a reachable world, where φ holds,
- in μ-calculus: $\mathsf{F}\varphi \equiv \mu x.(\varphi \vee \Diamond x)$.

Proof rules

Ax: $\dfrac{}{\varphi \Rightarrow \varphi}$

$\Diamond$: $\dfrac{\varphi \Rightarrow \Gamma}{\Diamond\varphi \Rightarrow \Diamond\Gamma}$

$\wedge_L$: $\dfrac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}$

$\wedge_R$: $\dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$

$\neg_L$: $\dfrac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta}$

$\neg_R$: $\dfrac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$

w_L: $\dfrac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$

w_R: $\dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}$

F_R: $\dfrac{\Gamma \Rightarrow \Delta, \varphi, \Diamond\mathrm{F}\varphi}{\Gamma \Rightarrow \Delta, \mathrm{F}\varphi}$

F_L: $\dfrac{\Diamond\mathrm{F}\varphi, \Gamma \Rightarrow \Delta \qquad \varphi, \Gamma \Rightarrow \Delta}{\mathrm{F}\varphi, \Gamma \Rightarrow \Delta}$

cut: $\dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$

Annotated sequents

Trace condition
A trace τ is *successful* if it is principal in an F_L rule, where it goes to the left.

Annotated sequents

Trace condition
A trace τ is *successful* if it is principal in an F_L rule, where it goes to the left.

We work with annotated sequents:

- Only formulas in antecedent are annotated
- Formulas in focus are of form $\mathsf{F}\varphi$ or $\Diamond\mathsf{F}\varphi$
- On every sequent there is at most one formula *in focus*

$$\text{Ax: } \frac{}{\varphi^{u} \Rightarrow \varphi}$$

$$\Diamond\text{: } \frac{\varphi^{a} \Rightarrow \Gamma}{\Diamond\varphi^{a} \Rightarrow \Diamond\Gamma}$$

$$\wedge_L\text{: } \frac{\varphi^{u}, \psi^{u}, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi^{u}, \Gamma \Rightarrow \Delta}$$

$$\wedge_R\text{: } \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$$\neg_L\text{: } \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi^{u}, \Gamma \Rightarrow \Delta}$$

$$\neg_R\text{: } \frac{\varphi^{u}, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$$

$$\mathrm{w}_L\text{: } \frac{\Gamma \Rightarrow \Delta}{\varphi^{u}, \Gamma \Rightarrow \Delta}$$

$$\mathrm{w}_R\text{: } \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}$$

$$\mathrm{F}_R\text{: } \frac{\Gamma \Rightarrow \Delta, \varphi, \Diamond \mathrm{F}\varphi}{\Gamma \Rightarrow \Delta, \mathrm{F}\varphi}$$

$$\mathrm{F}_L\text{: } \frac{\Diamond \mathrm{F}\varphi^{a}, \Gamma \Rightarrow \Delta \qquad \varphi^{u}, \Gamma \Rightarrow \Delta}{\mathrm{F}\varphi^{a}, \Gamma \Rightarrow \Delta}$$

$$\text{cut: } \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi^{u}, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$\text{f: } \frac{\varphi^{u}, \Gamma \Rightarrow \Delta}{\varphi^{f}, \Gamma \Rightarrow \Delta}$$

$$\text{u: } \frac{\varphi^{f}, \Gamma \Rightarrow \Delta}{\varphi^{u}, \Gamma \Rightarrow \Delta}$$

Successful path

Definition
A $\mathsf{GKe}^{\mathsf{a}}$ *pre-proof* is a derivation defined by the above rules.

Definition
A path α in a $\mathsf{GKe}^{\mathsf{a}}$ pre-proof is *successful* if it holds that

1. Every sequent on α has a formula in focus and
2. α passes through an application of F_L, where the principal formula is in focus.

GKe^a **proof system**

We call a leaf v in a GKe^a pre-proof π a *discharged leaf* if there is a proper ancestor $c(v)$ such that

1. v and $c(v)$ are labelled by the same annotated sequent and
2. the path from $c(v)$ to v is successful.

Definition
A GKe^a *proof* is a finite GKe^a pre-proof, where every leaf is either an axiom or a discharged leaf.

Standard cut elimination

Usual procedure:

1. Take infinite unfolding of cyclic proof
2. Push cuts upwards
3. Obtain limit proof
4. Check soundness condition

Critical case

$$\dfrac{\dfrac{\overset{\pi_0}{\Gamma \Rightarrow \Delta, \varphi, \Diamond \mathsf{F}\varphi}}{\Gamma \Rightarrow \Delta, \mathsf{F}\varphi}\ \mathsf{F}_R \qquad \dfrac{\overset{\pi_1}{\Diamond \mathsf{F}\varphi^f, \Gamma \Rightarrow \Delta} \qquad \overset{\pi_2}{\varphi^u, \Gamma \Rightarrow \Delta}}{\mathsf{F}\varphi^f, \Gamma \Rightarrow \Delta}\ \mathsf{F}_L}{\Gamma \Rightarrow \Delta}\ \mathsf{cut}$$

Critical case

$$\dfrac{\dfrac{\overset{\pi_0}{\Gamma \Rightarrow \Delta, \varphi, \Diamond \mathsf{F}\varphi}}{\Gamma \Rightarrow \Delta, \mathsf{F}\varphi}\,\mathsf{F}_R \qquad \dfrac{\overset{\pi_1}{\Diamond \mathsf{F}\varphi^f, \Gamma \Rightarrow \Delta} \qquad \overset{\pi_2}{\varphi^u, \Gamma \Rightarrow \Delta}}{\mathsf{F}\varphi^f, \Gamma \Rightarrow \Delta}\,\mathsf{F}_L}{\Gamma \Rightarrow \Delta}\,\mathsf{cut}$$

will be transformed to

$$\dfrac{\dfrac{\overset{\pi_0}{\Gamma \Rightarrow \Delta, \varphi, \Diamond \mathsf{F}\varphi} \qquad \overset{\pi_1}{\Diamond \mathsf{F}\varphi^f, \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta, \varphi}\,\mathsf{cut} \qquad \overset{\pi_2}{\varphi^u, \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}\,\mathsf{cut}$$

F_L-rule is removed:

- Might produce unsuccessful paths!

Idea

We call a cut unimportant, if no descendant of the cut-formula is in focus and important otherwise.

Treat them separately:

Idea

We call a cut unimportant, if no descendant of the cut-formula is in focus and important otherwise.

Treat them separately:

Important cuts can be brought in to the following form, where c is a companion node:

$$\dfrac{\Gamma \Rightarrow \Delta, \mathsf{F}\varphi \qquad \mathsf{c}\colon\ \dfrac{\mathsf{F}\varphi^{f}, \Gamma \Rightarrow \Delta}{\mathsf{F}\varphi^{u}, \Gamma \Rightarrow \Delta}\ \mathsf{u}}{\Gamma \Rightarrow \Delta}\ \mathsf{cut}$$

Important cuts

Consider the following important cut:

$$
\dfrac{
\begin{array}{c}
\text{v: } \dfrac{\overset{\pi_v}{\Gamma_v \Rightarrow \Delta_v, \varphi, \Diamond \mathsf{F}\varphi}}{\Gamma_v \Rightarrow \Delta_v, \mathsf{F}\varphi}\ \mathsf{F}_R \\
\vdots \\
\pi_l \\
\vdots \\
\Gamma \Rightarrow \Delta, \mathsf{F}\varphi
\end{array}
\qquad
\begin{array}{c}
\text{w: } \dfrac{\overset{\vdots}{\Diamond \mathsf{F}\varphi^f, \Gamma_w \Rightarrow \Delta_w} \qquad \overset{\pi_w}{\varphi^u, \Gamma_w \Rightarrow \Delta_w}}{\mathsf{F}\varphi^f, \Gamma_w \Rightarrow \Delta_w}\ \mathsf{F}_L \\
\vdots \\
\pi_r \\
\vdots \\
\dfrac{\mathsf{F}\varphi^f, \Gamma \Rightarrow \Delta}{\mathsf{F}\varphi^u, \Gamma \Rightarrow \Delta}\ \mathsf{u}
\end{array}
}{\Gamma \Rightarrow \Delta}\ \text{cut}
$$

Important cuts

$$\dfrac{\begin{array}{c}\pi_v\\ \text{v:}\ \dfrac{\Gamma_v \Rightarrow \Delta_v, \varphi, \Diamond \mathsf{F}\varphi}{\Gamma_v \Rightarrow \Delta_v, \mathsf{F}\varphi}\ \mathsf{F}_R \\ \vdots\\ \pi_l\\ \vdots\\ \Gamma \Rightarrow \Delta, \mathsf{F}\varphi\end{array} \qquad \begin{array}{c} \text{w:}\ \dfrac{\begin{array}{c}\vdots\\ \Diamond \mathsf{F}\varphi^f, \Gamma_w \Rightarrow \Delta_w\end{array} \quad \begin{array}{c}\pi_w\\ \varphi^u,\Gamma_w\Rightarrow\Delta_w\end{array}}{\mathsf{F}\varphi^f,\Gamma_w\Rightarrow\Delta_w}\ \mathsf{F}_L\\ \vdots\\ \pi_r\\ \vdots\\ \dfrac{\mathsf{F}\varphi^f,\Gamma\Rightarrow\Delta}{\mathsf{F}\varphi^u,\Gamma\Rightarrow\Delta}\ \text{u}\end{array}}{\Gamma\Rightarrow\Delta}\ \text{cut}$$

Delete occurrences of $\mathsf{F}\varphi$ and $\Diamond\mathsf{F}\varphi$ in left subproof:

$$\begin{array}{c}\pi_v\\ v:\ \dfrac{\Gamma_v \Rightarrow \Delta_v, \varphi, \cancel{\Diamond \mathsf{F}\varphi}}{\Gamma_v \Rightarrow \Delta_v, \cancel{\mathsf{F}\varphi}}\ \mathsf{F}_R \\ \vdots\\ \pi_l\\ \vdots\\ \Gamma \Rightarrow \Delta, \cancel{\mathsf{F}\varphi}\end{array}$$

Important cuts

$$\dfrac{\begin{array}{c} \text{v: } \dfrac{\overset{\pi_v}{\Gamma_v \Rightarrow \Delta_v, \varphi, \Diamond \mathsf{F}\varphi}}{\Gamma_v \Rightarrow \Delta_v, \mathsf{F}\varphi}\ \mathsf{F}_R \\ \vdots \\ \pi_l \\ \vdots \\ \Gamma \Rightarrow \Delta, \mathsf{F}\varphi \end{array} \qquad \begin{array}{c} \text{w: } \dfrac{\overset{\vdots}{\Diamond \mathsf{F}\varphi^f, \Gamma_w \Rightarrow \Delta_w} \quad \overset{\pi_w}{\varphi^u, \Gamma_w \Rightarrow \Delta_w}}{\mathsf{F}\varphi^f, \Gamma_w \Rightarrow \Delta_w}\ \mathsf{F}_L \\ \vdots \\ \pi_r \\ \vdots \\ \dfrac{\mathsf{F}\varphi^f, \Gamma \Rightarrow \Delta}{\mathsf{F}\varphi^u, \Gamma \Rightarrow \Delta}\ \text{u} \end{array}}{\Gamma \Rightarrow \Delta}\ \text{cut}$$

Combine it with cut of smaller complexity:

$$\begin{array}{c} \dfrac{\overset{\pi_v}{\Gamma_v \Rightarrow \Delta_v, \varphi} \qquad \overset{\pi_w}{\varphi^u, \Gamma_w \Rightarrow \Delta_w}}{\Gamma_v, \Gamma_w \Rightarrow \Delta_v, \Delta_w}\ \text{cut} \\ \vdots \\ \pi_l, \pi_r \\ \vdots \\ \Gamma \Rightarrow \Delta \end{array}$$

Unimportant cuts

Let C be an unimportant cut on a cycle.

Strategy:

- push C upwards in unfolding of cycle
- C unimportant → does not affect formulas in focus
- reach repeat below cut
- same formulas in focus → successful repeat

Conclusion

Summary of method:

- Working with *annotated proof system*
- Treat critical case *separately*
- Usual argumentation for rest
- Produces *cyclic proof*

Further work

- Complexity
- Which properties of proof system do we need?
- Can this be extended to other logics? Candidates:
  - PDL
  - Alternation-free μ-calculus
  - …

Thank you !