

## 1 | nth-derivatives

Find all of the first, second, and third partial derivatives of this function (including the mixed ones):

$$f(x, y) = 4x^2y^5 + 3x^3y^2 \quad (1)$$

$$f_x = 8xy^5 + 9x^2y^2 \quad (2)$$

$$f_y = 20x^2y^4 + 6x^3y \quad (3)$$

$$f_{xx} = 8y^5 + 18xy^2 \quad (4)$$

$$f_{yy} = 80x^2y^3 + 6x^3 \quad (5)$$

$$f_{xy} = 40xy^4 + 18x^2y \quad (6)$$

$$f_{xxx} = 18y^2 \quad (7)$$

$$f_{yyy} = 240x^2y^2 \quad (8)$$

$$f_{xxy} = 40y^4 + 36xy \quad (9)$$

$$f_{yyx} = 160xy^3 + 18x^2 \quad (10)$$

## 2 | Jacobian Derivative Matrix

$$2.1 \mid f(x, y) = \frac{xy}{x^2+y^2}$$

$$\frac{\partial f}{\partial x} = \frac{(x^2 + y^2)y - xy(2x)}{(x^2 + y^2)^2} \quad (11)$$

$$\frac{\partial f}{\partial y} = \frac{(x^2 + y^2)x - xy(2y)}{(x^2 + y^2)^2} \quad (12)$$

$$f'(x, y) = \begin{bmatrix} \frac{(x^2 + y^2)y - xy(2x)}{(x^2 + y^2)^2} \\ \frac{(x^2 + y^2)x - xy(2y)}{(x^2 + y^2)^2} \end{bmatrix} \quad (13)$$

$$2.2 \mid f(x, y, z) = [xy + 2yz, 2x^2y^2]$$

$$\frac{\partial f^0}{\partial x} = y \quad (14)$$

$$\frac{\partial f^1}{\partial x} = 4xy^2 \quad (15)$$

$$\frac{\partial f^0}{\partial y} = x + 2z \quad (16)$$

$$\frac{\partial f^1}{\partial y} = 4yx^2 \quad (17)$$

$$f'(x, y, z) = \begin{bmatrix} y & x + 2z \\ 4xy^2 & 4yx^2 \end{bmatrix} \quad (18)$$

### 3 | Total First Derivatives

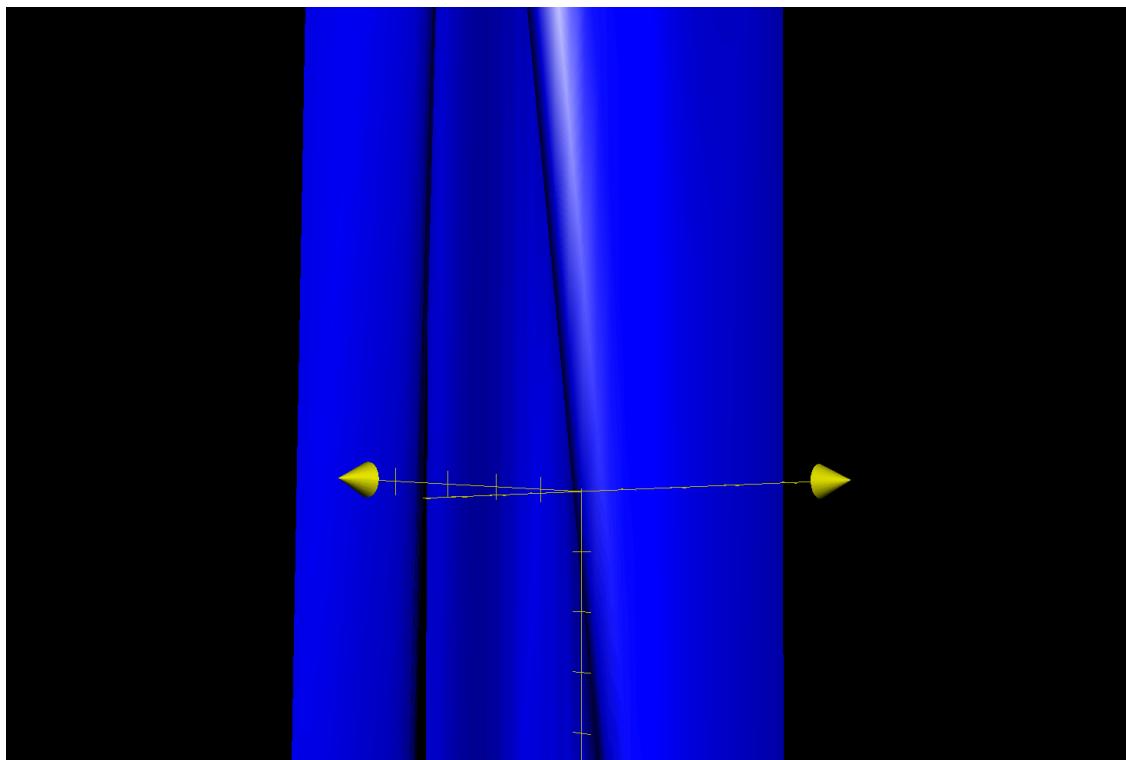
Suppose you have a function  $\mathbb{R}^2 \rightarrow \mathbb{R}^1$ , how many total first partial derivatives does it have?  
 What about second partial derivatives? ... What about k'th partial derivatives.

For a function  $\mathbb{R}^2 \rightarrow \mathbb{R}^1$ , it has 2 first partial derivatives, 3 second partial derivatives, 4 third partial derivatives, and  $k + 1$  k-th partial derivatives.

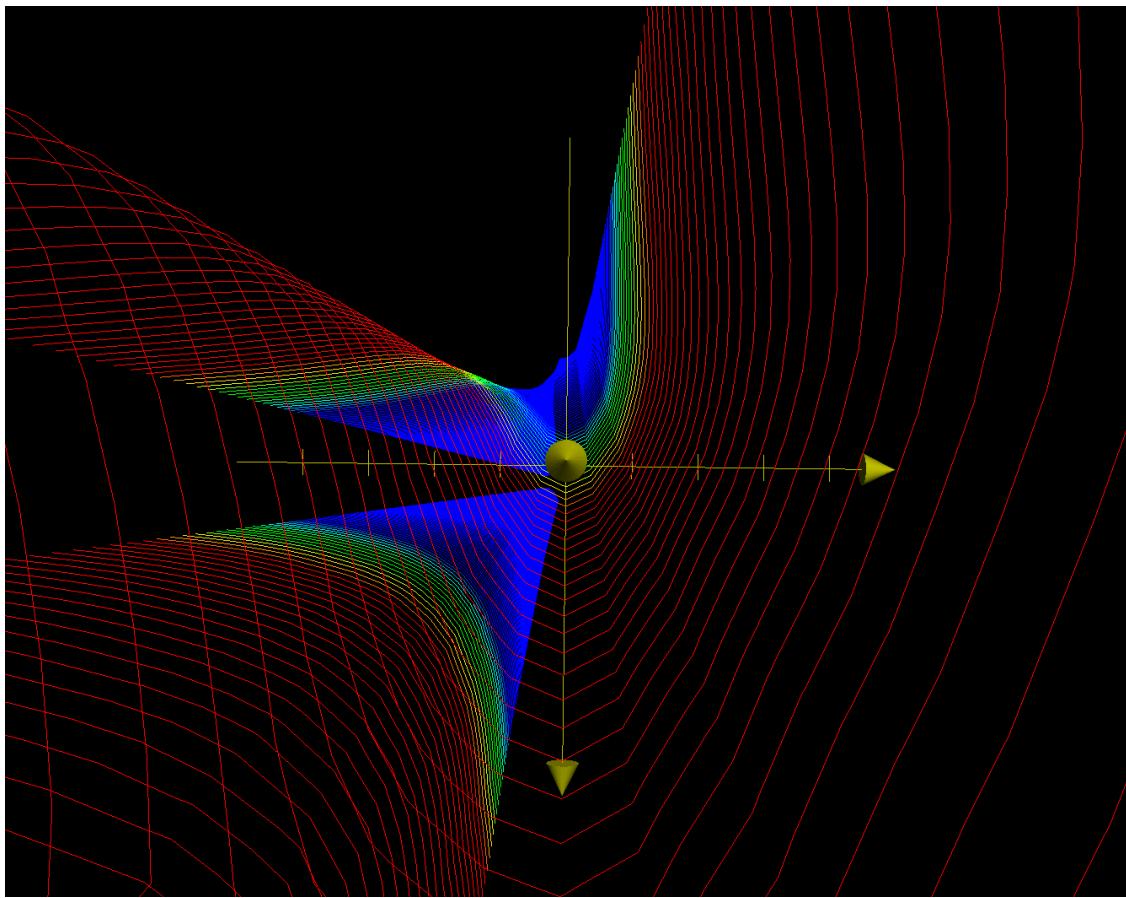
### 4 | Multi-Variable Slopes

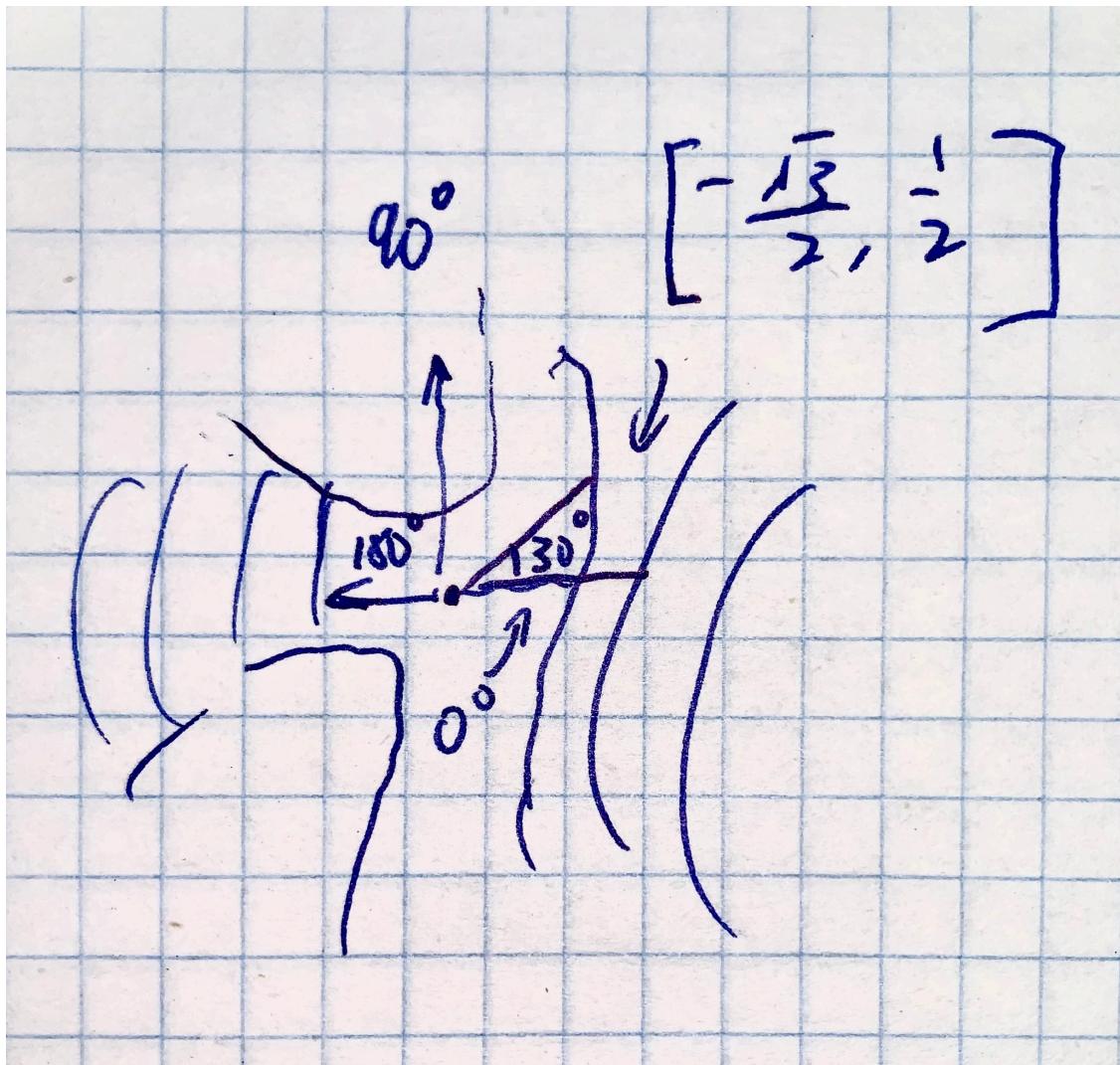
$$f(x, y) = 7x + 2x^2y^3 + 10y^2 \quad (19)$$

Plot of the manifold:



Plot of the contours:





$$f_x = 7 + 4xy^3 \quad (20)$$

$$f_y = 6x^2y^2 + 20y \quad (21)$$

#### 4.1 | Steepness of function at $[-\frac{\sqrt{3}}{2}, \frac{1}{2}]$

The gradient at the  $(1, 2)$ , as a vector, is  $[39, 64]$

##### 4.1.1 | Steepness of the Function at Point

Projecting that onto the direction of  $[-\frac{\sqrt{3}}{2}, \frac{1}{2}]$  would result in:

$$\begin{bmatrix} 39 \\ 64 \end{bmatrix} \cdot \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{64 - 39\sqrt{3}}{2} \quad (22)$$

Hence,  $\frac{64-39\sqrt{3}}{2}$  is the "slope" of the function at  $(1, 2)$  facing  $[-\frac{\sqrt{3}}{2}, \frac{1}{2}]$ . The angle of which the slope represents, therefore, would be  $\arctan(\frac{64-39\sqrt{3}}{2}) \approx -60.60^\circ$ .

## 4.2 | Optimizing for steepness

We want to figure an angle  $\theta$  such that the gradient would be most steep, that is, a  $\theta$  such that...

$$39\cos\theta + 64\sin\theta \quad (23)$$

is maximized. We know that there exists a  $\theta_{max}$  such that this expression would be maximized, meaning that if there exists a critical point for this expression we would arrive at such an optima.

The derivative w.r.t.  $\theta$  of the expression above is

$$-39\sin\theta + 64\cos\theta \quad (24)$$

Solving for a value of that expression at 0 to arrive at a critical point,

$$64\cos\theta = 0 \quad (25)$$

$$\Rightarrow 64\cos\theta = 39\sin\theta \quad (26)$$

$$\Rightarrow \frac{64}{39} = \frac{\sin\theta}{\cos\theta} \quad (27)$$

$$\Rightarrow \frac{64}{39} = \tan\theta \quad (28)$$

$$\Rightarrow \theta = \arctan(\frac{64}{39}) \approx 58.64^\circ \quad (29)$$

Hence, the angle at which the steepness would be optimized would be  $58.64^\circ$ .

## 4.3 | Optimizing for flatness

Similar to before, we want to figure an angle  $\theta$  such that the value of the slope at that point would be 0, that is:

$$39\cos\theta + 64\sin\theta = 0 \quad (30)$$

Solving for the value of  $\theta$  in this expression:

$$39\cos\theta + 64\sin\theta = 0 \quad (31)$$

$$\Rightarrow 64\sin\theta = -39\cos\theta \quad (32)$$

$$\Rightarrow \tan\theta = \frac{-39}{64} \quad (33)$$

$$\Rightarrow \theta = \arctan(\frac{-39}{64}) \approx -31.36^\circ \quad (34)$$

Hence, the angle at which the steepness would be minimized would be  $-31.36^\circ$ .