1 | **nth-derivatives**

Find all of the first, second, and third partial derivatives of this function (including the mixed ones):

$$f(x, y) = 4x^2y^5 + 3x^3y^2 \tag{1}$$

$$\begin{aligned}
f_x =& 8xy^5 + 9x^2y^2 && (2)\\
f_y =& 20x^2y^4 + 6x^3y && (3)\\
f_{xx} =& 8y^5 + 18xy^2 && (4)\\
f_{yy} =& 80x^2y^3 + 6x^3 && (5)\\
f_{xy} =& 40xy^4 + 18x^2y && (6)\\
f_{xxx} =& 18y^2 && (7)\\
f_{yyy} =& 240x^2y^2 && (8)\\
f_{xxy} =& 40y^4 + 36xy && (9)\\
f_{yyx} =& 160xy^3 + 18x^2 && (10)
\end{aligned}$$

2 | **Jacobian Derivative Matrix**

2.1 | $f(x, y) = \frac{xy}{x^2+y^2}$

$$\frac{\partial f}{\partial x} = \frac{(x^2 + y^2)y - xy(2x)}{(x^2 + y^2)^2} \tag{11}$$

$$\frac{\partial f}{\partial y} = \frac{(x^2 + y^2)x - xy(2y)}{(x^2 + y^2)^2} \tag{12}$$

$$f'(x, y) = \begin{bmatrix} \frac{(x^2+y^2)y-xy(2x)}{(x^2+y^2)^2} \\ \frac{(x^2+y^2)x-xy(2y)}{(x^2+y^2)^2} \end{bmatrix} \tag{13}$$

2.2 | $f(x, y, z) = [xy + 2yz, 2x^2y^2]$

$$\frac{\partial f^0}{\partial x} = y \tag{14}$$

$$\frac{\partial f^1}{\partial x} = 4xy^2 \tag{15}$$

$$\frac{\partial f^0}{\partial y} = x + 2z \tag{16}$$

$$\frac{\partial f^1}{\partial y} = 4yx^2 \tag{17}$$

$$f'(x,y,z) = \begin{bmatrix} y & x+2z \\ 4xy^2 & 4yx^2 \end{bmatrix} \tag{18}$$

3 | Total First Derivatives

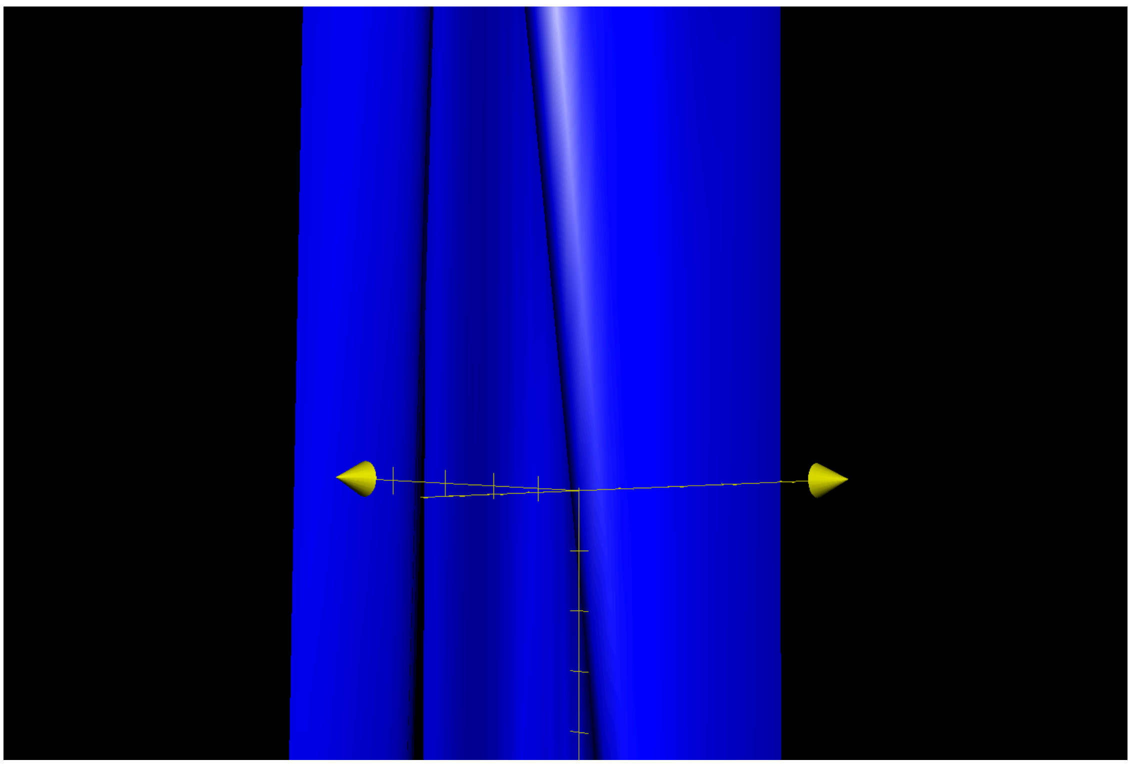
> Suppose you have a function $\mathbb{R}^2 \to \mathbb{R}^1$, how many total first partial derivatives does it have? What about second partial derivatives? … What about k'th partial derivatives.

For a function $\mathbb{R}^2 \to \mathbb{R}^1$, it has 2 first partial derivatives, 3 second partial derivatives, 4 third partial derivatives, and $k+1$ k-th partial derivatives.

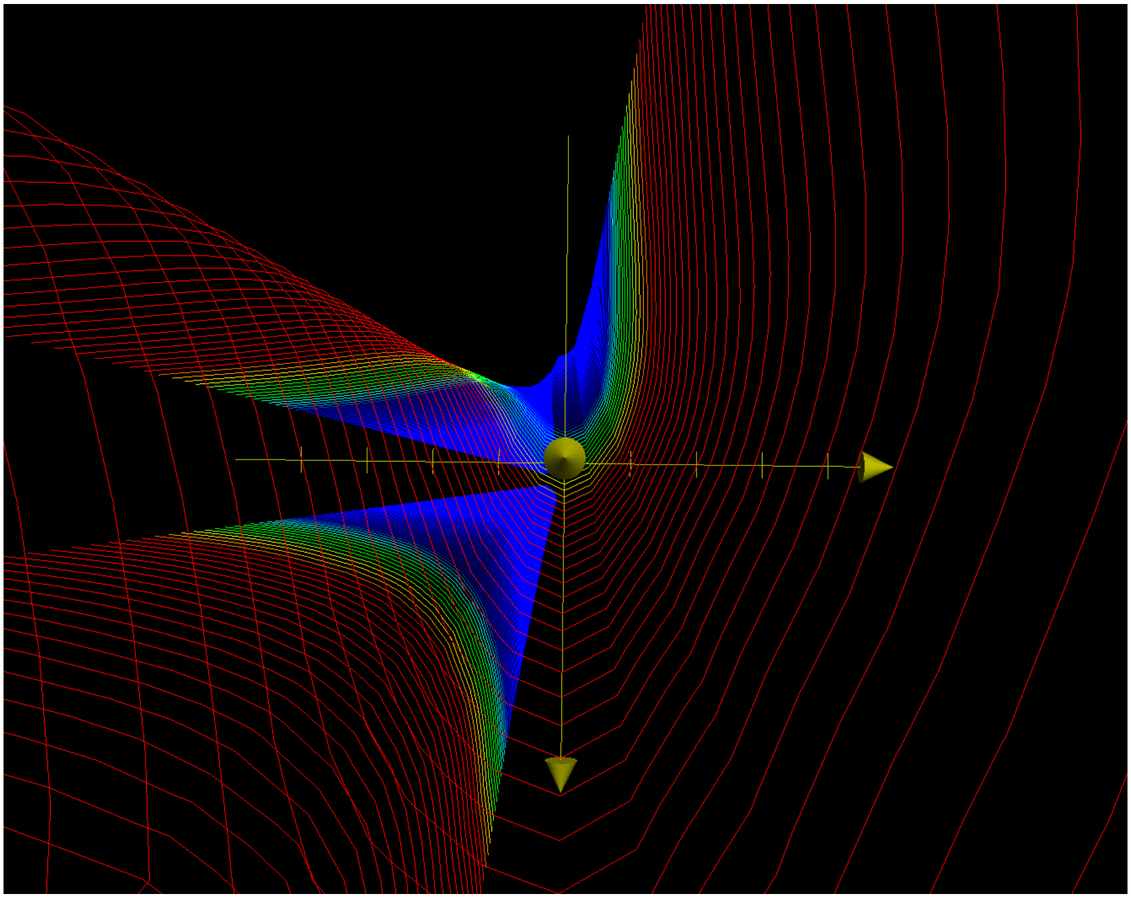
4 | Multi-Variable Slopes

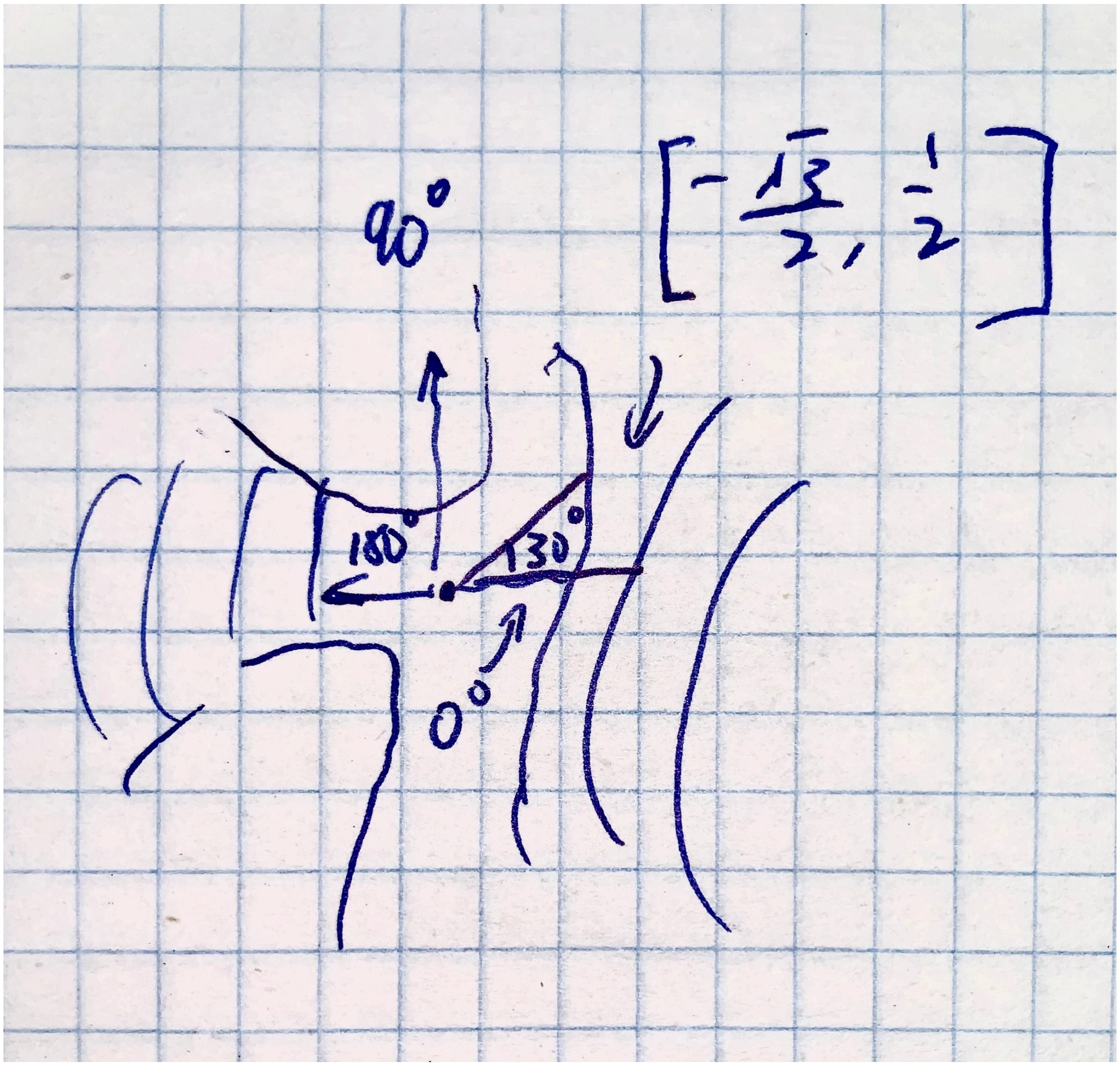
$$f(x,y) = 7x + 2x^2y^3 + 10y^2 \tag{19}$$

Plot of the manifold:



Plot of the contours:





$$f_x = 7 + 4xy^3 \tag{20}$$

$$f_y = 6x^2y^2 + 20y \tag{21}$$

4.1 | Steepness of function at $[-\frac{\sqrt{3}}{2}, \frac{1}{2}]$

The gradient at the $(1, 2)$, as a vector, is $[39, 64]$

4.1.1 | Steepness of the Function at Point

Projecting that onto the direction of $[-\frac{\sqrt{3}}{2}, \frac{1}{2}]$ would result in:

$$\begin{bmatrix} 39 \\ 64 \end{bmatrix} \cdot \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{64 - 39\sqrt{3}}{2} \tag{22}$$

Hence, $\frac{64-39\sqrt{3}}{2}$ is the "slope" of the function at $(1, 2)$ facing $[-\frac{\sqrt{3}}{2}, \frac{1}{2}]$. The angle of which the slope represents, therefore, would be $arctan(\frac{64-39\sqrt{3}}{2}) \approx -60.60°$.

4.2 | Optimizing for steepness

We want to figure an angle θ such that the gradient would be most steep, that is, a θ such that…

$$39cos\theta + 64sin\theta \tag{23}$$

is maximized. We know that there exists a θ_{max} such that this expression would be maximized, meaning that if there exists a critical point for this expression we would arrive as such an optima.

The derivative w.r.t. θ of the expression above is

$$-39sin\theta + 64cos\theta \tag{24}$$

Solving for a value of that expression at 0 to arrive at a critical point,

$$64cos\theta = 0 \tag{25}$$
$$\Rightarrow 64cos\theta = 39sin\theta \tag{26}$$
$$\Rightarrow \frac{64}{39} = \frac{sin\theta}{cos\theta} \tag{27}$$
$$\Rightarrow \frac{64}{39} = tan\theta \tag{28}$$
$$\Rightarrow \theta = arctan(\frac{64}{39}) \approx 58.64° \tag{29}$$

Hence, the angle at which the steepness would be optimized would be $58.64°$. Note also that the direction that the gradient is pointing is steepest.

4.3 | Optimizing for flatness

Similar to before, we want to figure an angle θ such that the value of the slope at that point would be 0, that is:

$$39cos\theta + 64sin\theta = 0 \tag{30}$$

Solving for the value of θ in this expression:

$$39cos\theta + 64sin\theta = 0 \tag{31}$$
$$\Rightarrow 64sin\theta = -39cos\theta \tag{32}$$
$$\Rightarrow tan\theta = \frac{-39}{64} \tag{33}$$
$$\Rightarrow \theta = arctan(\frac{-39}{64}) \approx -31.36° \tag{34}$$

Hence, the angle at which the steepness would be minimized would be $-31.36°$.

Note also that this is the vector for with the dot with the steepest vector is zero. That's "orthogonal", and hence the flattest angle.