Continuos time derivation - risk averse firms

Cristina Lafuente

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Starting point:

$$J_t = \Delta\pi_t + (1-\Delta\delta)(1-\Delta\rho)\frac{u'(c_{t+\Delta})}{u'(c_t)}J_{t+\Delta} \tag{1}$$

Use the approximation $X_t \approx X_{t+\Delta} - \dot{X}_t\Delta$ on the LHS (left-hand-side):

$$J_{t+\Delta} - \dot{J}_t\Delta = \Delta\pi_t + (1-\Delta\delta)(1-\Delta\rho)\frac{u'(c_{t+\Delta})}{u'(c_t)}J_{t+\Delta}. \tag{2}$$

There are 2 general lessons for passing things to continuous time:

1. Try to pass everything not multiplied by a Δ to the left hand side before taking the limit
2. Make sure the limit of the left hand side is something of the sort $X_{t+\Delta} - X_t$ so it does not go to ∞ when we take the limit.

With this in mind, after expanding the second term on the RHS we obtain

$$J_{t+\Delta} - \dot{J}_t\Delta = \Delta\pi_t + (\Delta^2\delta\rho - \Delta\delta - \Delta\rho)\frac{u'(c_{t+\Delta})}{u'(c_t)}J_{t+\Delta} + \frac{u'(c_{t+\Delta})}{u'(c_t)}J_{t+\Delta}. \tag{3}$$

The last term on the RHS is the only thing not being multiplied by Δ, so we move it to the LHS. Similarly, the second term on the LHS is multiplied by Δ so we pass it to the RHS:

$$J_{t+\Delta} - \frac{u'(c_{t+\Delta})}{u'(c_t)}J_{t+\Delta} = \Delta\pi_t + (\Delta^2\delta\rho - \Delta\delta - \Delta\rho)\frac{u'(c_{t+\Delta})}{u'(c_t)}J_{t+\Delta} + \dot{J}_t\Delta. \tag{4}$$

Perfect. Before we divide by Δ, we can factor out $J_{t+\Delta}$ to obtain

$$J_{t+\Delta}\left(1 - \frac{u'(c_{t+\Delta})}{u'(c_t)}\right) = \Delta\pi_t + (\Delta^2\delta\rho - \Delta\delta - \Delta\rho)\frac{u'(c_{t+\Delta})}{u'(c_t)}J_{t+\Delta} + \dot{J}_t\Delta,$$

which can be nicely re-written as

$$\frac{J_{t+\Delta}}{u'(c_t)}(u'(c_t) - u'(c_{t+\Delta})) = \Delta\pi_t + (\Delta^2\delta\rho - \Delta\delta - \Delta\rho)\frac{u'(c_{t+\Delta})}{u'(c_t)}J_{t+\Delta} + \dot{J}_t\Delta. \tag{5}$$

The second term on the LHS is what we looking for: something of the form $X_{t+\Delta} - X_t$ that we can divide by Δ, take the limit and not end up in ∞.

Divide both sides by Δ and take the limit:

$$\lim_{\Delta\to 0}\frac{J_{t+\Delta}}{u'(c_t)}\frac{(u'(c_t) - u'(c_{t+\Delta}))}{\Delta} = \lim_{\Delta\to 0}\pi_t + (\Delta\delta\rho - \delta - \rho)\frac{u'(c_{t+\Delta})}{u'(c_t)}J_{t+\Delta} + \dot{J}_t. \tag{6}$$

From the definition of the derivative w.r.t. time and the chain rule, we know that the limit

$$\lim_{\Delta \to 0} \frac{(u'(c_t) - u'(c_{t+\Delta}))}{\Delta} = -u''(c_t)\dot{c}_t,$$

so taking the limit equation 6 becomes

$$J_t \frac{-u''(c_t)\dot{c}_t}{u'(c_t)} = \pi_t - (\delta + \rho)J_t + \dot{J}_t. \tag{7}$$

Here is the time to invoke the CRRA property $\frac{-u''(c_t)c_t}{u'(c_t)} = \gamma$ to rewrite the LHS as

$$J_t \gamma \frac{\dot{c}_t}{c_t} = \pi_t - (\delta + \rho)J_t + \dot{J}_t. \tag{8}$$

We can leave it here but it is nicer to rewrite this equation in a way we get one J_t on the LHS alone – and for that it is arguably more elegant to pass $(\delta + \rho)J_t$ term to the other side and divide the RHS by $(\delta + \rho)$. Then this last equation becomes

$$J_t = \frac{\pi_t - \gamma \frac{\dot{c}_t}{c_t} J_t + \dot{J}_t}{(\delta + \rho)}, \tag{9}$$

which is what is it coded in the solution file.

BONUS: Interpretation of this equation

Solving for J_t we obtain

$$J_t = \frac{\pi_t + \dot{J}_t}{(\delta + \rho + \gamma \frac{\dot{c}_t}{c_t})}. \tag{10}$$

Normally in the DMP model $J_t = \frac{\pi_t + \dot{J}_t}{(\delta + \rho)}$, or the value of an open job is equal to the flow of profits plus the increase in asset value of an open job, discounted by the discount factor ρ and job destruction rate δ.

Here the difference is that if consumption growth is not zero $\frac{\dot{c}_t}{c_t} \neq 0$ the discount factor has an extra term. This is because firms want to smooth consumption, so during recoveries from recessions (low starting c_t and J_t, $\dot{J}_t$; $\frac{\dot{c}_t}{c_t} \geq 0$) the value of a filled vacancy does not raise by as much as if firms did not care about smoothing consumption. Conversely, converging to steady state after a positive shock (high starting c_t and J_t; $\dot{J}_t$, $\frac{\dot{c}_t}{c_t} \leq 0$) the value of a filled vacancy does not fall by as much. Firms react more slowly to shocks of either sign.