Fluxes: advective and diffusive

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June 15, 2008

Introduction

Understanding how to quantify fluxes is an important part of this course and oceanography in general. Here I will attempt to give a bit more information on how to do this.

The Conservation of Volume

Suppose we are interested in the dynamics of a body of water, say a rectangular swimming pool. The pool has a limited number of entrances and exits. If the pool has straight side walls how fast does the water rise in the pool if there is a flow q into the pool and the surface area is A?

Here the answer is relatively easy. The speed at which water rises in the pool is $w = q/A$, the volume transport divided by the surface area. Here we have applied the conservation of volume:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\sum_i q_i + \sum_j s_j. \tag{1}$$

where V is the volume we are studying, q_i are individual fluxes out of (positive) or into (negative) the volume. The last term is the sum of the sources (positive) and sinks (negative) in the volume, and is included for completeness. For the conservation of volume of water, a sink could be evaporation, for instance.

Formally, this should really be conservation of mass in the volume. In which case:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \mathrm{d}V = -\sum_i \rho_i q_i + \sum_j \rho_j s_j. \tag{2}$$

where the integration over the volume is summing all the mass in the volume. However, the density of water only changes by a few percent, so equation (1) is a very good approximation.

Calculating volume transport

Volume transports only arise because water is moving from point a to point b. Sometimes we are given the volume transport (in $\mathrm{m}^3\,\mathrm{s}^{-1}$), but sometimes we want to report the velocity. i.e. in the swimming pool example above we wanted to know how fast the water was rising (i.e. in ms^{-1}). In the simple case:

$$q_i = A_i u_i \tag{3}$$

where A_i is the area the volume flux is flowing through, and u_i is a velocity *perpendicular* to the area. In vector calculus we write this as:

$$q_i = \int_A \mathbf{u} \cdot \mathrm{d}\mathbf{A} \tag{4}$$

where the dot product makes it clear that the flux is perpendicular to the area.

Practice questions

Suppose the water level of a straight-sided pool is observed to rise at 0.01 m/s, and has a surface area of 50 m^2. There is one hose filling it with speed 4 m/s and cross-sectional area 0.01m^2, and a second hose filling it with a cross sectional area of 0.02m^2. What must the flow speed be in the second hose?

Suppose a pipe has a diameter of 0.1 m and a flow speed of 1 m/s. If the pipe narrows to 0.05 m what is the flow speed at that part of the pipe (in steady state)?

Conservation of a substance

The conservation of a substance (or heat) has essentially the same terms:

$$\overbrace{\frac{\mathrm{d}}{\mathrm{dt}} \int_V S\,\mathrm{d}V}^{\text{change of amount of stuff}} = \overbrace{\sum_i q_S}^{\text{transport of stuff in}} + \overbrace{\sum_j s_S}^{\text{internal sources}} \quad . \tag{5}$$

Here the substance, S, is expressed in stuff-per-volume. i.e. $\mathrm{g\,m^{-3}}$. We often discuss the *mean concentration in a volume* as $\overline{S} = \int_V S\ \mathrm{d}V/V$, in which case the equation simplifies to:

$$V \frac{\mathrm{d}\overline{S}}{\mathrm{d}t} = \sum_i q_S + \sum_j s_S \tag{6}$$

The *transport* of stuff q_S is expressed in units of stuff per time (i.e.

Flux

We often know the "flux" of a material and the area it is fluxing across, so the *transport* is often expressed as the product of the flux and area A. i.e.

$$q_S = A\ F_S \tag{7}$$

where the flux is stuff-per-second-per-area (i.e. $\mathrm{g\,m^{-2}\,s^{-1}}$).

Using the concentration and volume transport to calculate advective transport and flux

In fluid mechanics we often measure the concentration of a substance (S) separately from the volume transport q or velocity u. For instance if Chlorine is mixed in a vat at a concentration of $S = 200\ \mathrm{g/m^3}$ and flows through a pipe at $q_v = 0.01\mathrm{m^3 s^{-1}}$ then the rate of chlorine transport into the pool is:

$$q_S = Sq_v \tag{8}$$

and has units $\mathrm{gs^{-1}}$. This is called an *advective transport* of chlorine.

If instead we knew the velocity of the water u and the concentration, then we might have expressed this as a *flux* of chlorine:

$$F_S = Su \tag{9}$$

Because velocity is a vector, this flux is more properly written as a vector:

$$\mathrm{F}_S = S\mathrm{u} \tag{10}$$

and the flux flows in the same direction as the water. Note that the units of the flux are $\mathrm{g\,s^{-1}\,m^{-2}}$

This can all get even more confusing if the concentration is expressed as parts per thousand, or some other volume unit like that (i.e. mL/L). This can be dealt with in the same ways, but it requires some extra care with the units.

Diffusive flux

As we discussed, the diffusive flux of a substance is governed by Fick's law. Diffusive fluxes arise because of gradients in the concentration of a substance:

$$F_D = -K\frac{\mathrm{d}S}{\mathrm{d}z} \tag{11}$$

where K is the diffusivity of the tracer S. this flux is in the positive-z direction and has units of $\mathrm{g\,s^{-1}\,m^{-2}}$ just like an advective flux. In general, we express this flux as a vector dependent on the 3-D gradient of the tracer:

$$\mathbf{F}_D = -K\nabla S \tag{12}$$

When included in a conservation equation, the diffusive flux is used just like an advective flux. In order to get a transport it needs to be multiplied by the area it passes through.

Practice questions

Suppose there is a vat with $200\ \mathrm{g/m^3}$ solution of chlorine being fed into the pool through a hose with diameter 0.1 m at a rate fast enough to replace evaporative losses in the pool of $0.01\ \mathrm{m^3/s}$. What is the transport of chlorine into the pool?

If the average concentration of chlorine in the pool is initially 50 $\mathrm{g/m^3}$ and 1000 s later is measured to be 55 $\mathrm{g/m^3}$, what is the volume of the pool?

Suppose at mid depth in a pool with cross-sectional area 20 $\mathrm{m^2}$, there is a vertical gradient of Chlorine concentration of $dC/dz = 2\ \mathrm{g\,m^{-4}}$. Suppose that the top half of the pool has a mean concentration of $C = 150\mathrm{gm^{-3}}$ initially and $C = 140\mathrm{gm^{-3}}$ 1000 seconds later. How deep is the pool if the diffusivity is $K = 10^{-6}\ \mathrm{m^2s^{-1}}$? What if it is $K = 10^{-3}\ \mathrm{m^2s^{-1}}$? Assume the gradient doesn't change appreciably between measurements.

Suppose at mid depth in a pool with cross-sectional area 20 $\mathrm{m^2}$, there is a vertical gradient of Chlorine concentration of $dC/dz = 2\ \mathrm{g\,m^{-4}}$, and a measured Chlorine concentration of $C = 100\ \mathrm{g\,m^{-3}}$. Suppose we also measure a vertical velocity at this location of $w = 0.1\ \mathrm{m\,s^{-1}}$. If the bottom half of the pool is to stay in steady state, what must the diffusivity coefficient be?