Recursive Utility with Optimal Jump Intensity

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1 Basic recursion

The basic Kreps-Porteus recursion is

$$V_t = \left[(1-\beta_\Delta)\left(C_t^\alpha D_t^{1-\alpha}\right)^{1-\rho} + \beta_\Delta R_t^{1-\rho}\right]^{\frac{1}{1-\rho}}$$

where R_t is

$$R_t = \left(E_t\, V_{t+\Delta}^{1-\gamma}\right)^{\frac{1}{1-\gamma}}.$$

Notice that V_t is homogeneous of degree 1 in D_t.

$$\frac{V_t}{D_t} = \left[(1-\beta_\Delta)\left(\frac{C_t}{D_t}\right)^{\alpha(1-\rho)} + \beta_\Delta\left(\frac{R_t}{D_t}\right)^{1-\rho}\right]^{\frac{1}{1-\rho}}$$

with $c = C/D$ and

$$r_t = \frac{R_t}{D_t} = \left(E_t\left(\frac{V_{t+\Delta}}{D_{t+\Delta}}\frac{D_{t+\Delta}}{D_t}\right)^{1-\gamma}\right)^{\frac{1}{1-\gamma}}.$$

Writing $v = V/D$ we can rewrite the recursion as

$$1 = \left[(1-\beta_\Delta)\left(\frac{c_t^\alpha}{v_t}\right)^{(1-\rho)} + \beta_\Delta\left(\frac{r_t}{v_t}\right)^{1-\rho}\right]^{\frac{1}{1-\rho}}$$

and in log form

$$0 = \frac{1}{1-\rho}\log\left[(1-\beta_\Delta)\exp\left[(1-\rho)\left(\alpha\hat{c}_t - \hat{v}_t\right)\right] + \beta_\Delta\exp\left[(1-\rho)\left(\hat{r}_t - \hat{v}_t\right)\right]\right].$$

Develop the continuation value recursion

$$\hat{r}_t - \hat{v}_t = \frac{1}{1-\gamma}\log E_t\,\exp\left[(1-\gamma)\left(\hat{v}_{t+\Delta} - \hat{v}_t + \hat{d}_{t+\Delta} - \hat{d}_t\right)\right].$$

v only depends on time through a Brownian with jumps. Therefore v and $\hat{v}$ do too.

2 Optimal Jump Intensities.

Let

$$d\hat{v}_t = \hat{\mu}_t^v dt + \hat{\sigma}_t^v dZ_t + (\hat{v}_t^* - \psi - \hat{v}_t)\, dN_t.$$

where ψ is a scalar random variable with CDF F denoting the utility cost of moving. Instead d_t is a jump process following the same counter as $\hat{v}$ but with a different jump size

$$\mathrm{d}\hat{d}_t = \left(\hat{d}_t^* - \hat{d}_t\right) dN_t.$$

Finally, the recursion can be written as

$$0 = \delta \frac{\left(\frac{c_t^\alpha}{v_t}\right)^{1-\rho} - 1}{1-\rho} + \hat{\mu}_t^v + (1-\gamma)\frac{(\hat{\sigma}_t^v)^2}{2} + \frac{\lambda_t}{1-\gamma}\left(E_\psi \exp\left[(1-\gamma)\left(\hat{v}_t^* - \psi - \hat{v}_t + \hat{d}_t^* - \hat{d}_t\right)\right] - 1\right)$$

as $\rho \to 0$

$$\delta\hat{v}_t = \delta\alpha\hat{c}_t + \hat{\mu}_t^v + (1-\gamma)\frac{(\hat{\sigma}_t^v)^2}{2} + \frac{\lambda_t}{1-\gamma}\left(E_\psi \exp\left[(1-\gamma)\left(\hat{v}_t^* - \psi - \hat{v}_t + \hat{d}_t^* - \hat{d}_t\right)\right] - 1\right)$$

and as $\gamma \to 1$

$$\delta\hat{v}_t = \delta\alpha\hat{c}_t + \hat{\mu}_t^v + \lambda_t\left(\hat{v}_t^* - E_\psi\psi - \hat{v}_t + \hat{d}_t^* - \hat{d}_t\right)$$

Therefore the decision to adjust is optimally taken only if the utility shock ψ satisfies

$$y(\hat{v}) \equiv \hat{v}_t^* - \hat{v}_t + \hat{d}_t^* - \hat{d}_t > \psi$$

or equivalently

$$D_t^* v_t^* \geq \exp(\psi)\, D_t v_t$$
$$V_t^* \geq \exp(\psi)\, V_t$$

so ψ acts as a positive multiplicative utility shock to the current durable stock. In a sequence problem the decision would display as

$$\max\left\{V_t, \exp(-\psi)\, V_t^*\right\}$$

so it can also be interpreted as a negative utility shock to moving.

Operational HJB: From the equivalence of robustness to misspecification and recursive utility (with $\rho = 1$) we can write the HJB as a result of the following problem. Let $d\hat{v} = \hat{\mu}_t^v \hat{v} dt + \hat{\sigma}_t^v dZ_t + (\hat{v}_t^* - \psi - \hat{v}_t)\, dN_t$, then the HJB satisfies

$$\delta\hat{v} = \max_{c,\theta} \delta\alpha\hat{c} + \mathcal{A}\hat{v} + \min_{h} \left\{ \mathcal{B}\hat{v}h + \frac{1}{2}\frac{1}{\gamma-1}h^2 \right\}$$
$$+ \kappa F\left(y\left(\hat{v}\right)\right) E_\psi \left[\min_g g\left\{y\left(\hat{v}\right) - \psi\right\} + \frac{1}{\gamma-1}\left(1 - g + g\log g\right) | \psi < y\left(\hat{v}\right) \right]$$

where $y\left(\hat{v}\right) = \hat{v}_t^* - \hat{v}_t + \hat{d}_t^* - \hat{d}_t$ are the log gains from adjusting and $\kappa F\left(y\left(\hat{v}\right)\right) = \kappa P\left(\psi \leq y\left(\hat{v}\right)\right)$ is the optimal jump intensity at the current state. This incorporates the effects of optimal jump intensities.

$\hat{v}$ **diffusion.** The diffusion for $\hat{v}$ is

$$d\hat{v} = \mathcal{A}\hat{v}dt + \mathcal{B}\hat{v}dZ_t + \left(\hat{v}^* - \hat{v}\right) dN_t$$

where

$$\mathcal{A}\hat{v} = \left[rw + r^e\theta w - c - (1-\epsilon)(r+s)\right]\hat{v}' + \frac{1}{2}\left(\sigma\theta w\right)^2 \hat{v}''$$

and

$$\mathcal{B}\hat{v} = \sigma\theta w\hat{v}'$$

Therefore

$$\begin{aligned}
\delta\hat{v} = & \alpha\log c^* - \kappa F\left(y\left(\hat{v}\right)\right) E_\psi\left[g^*\left(\psi\right)\psi | \psi < y\left(\hat{v}\right)\right] \\
& + \frac{1}{2}\frac{1}{\gamma-1}\left(h^*\right)^2 + \frac{1}{\gamma-1}E_\psi\left[\left(1 - g^*\left(\psi\right) + g^*\left(\psi\right)\log g^*\left(\psi\right)\right) | \psi < y\left(\hat{v}\right)\right] \\
& + \left(\mu_w\left(w, c^*, \theta^*\right) + \sigma_w\left(w,\theta^*\right)h^*\right)\hat{v}' + \frac{\left[\sigma_w\left(w,\theta^*\right)\right]^2}{2}\hat{v}'' \\
& + \underbrace{\kappa F\left(y\left(\hat{v}\right)\right) E_\psi\left[g^*\left(\psi\right) | \psi < y\left(\hat{v}\right)\right]}_{\lambda_w} y\left(\hat{v}\right)
\end{aligned}$$

closed form solutions for h and g

$$h^* = (1-\gamma)\sigma\theta w v'$$
$$g^*\left(\psi\right) = \exp\left[(1-\gamma)(y - \psi)\right]$$