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ASSET PRICING AND OPTIMAL PORTFOLIO CHOICE IN THE PRESENCE OF ILLIQUID DURABLE CONSUMPTION GOODS

BY SANFORD J. GROSSMAN AND GUY LAROQUE[1]

We analyze a model of optimal consumption and portfolio selection in which consumption services are generated by holding a durable good. The durable good is illiquid in that a transaction cost must be paid when the good is sold. It is shown that optimal consumption is not a smooth function of wealth; it is optimal for the consumer to wait until a large change in wealth occurs before adjusting his consumption. As a consequence, the consumption based capital asset pricing model fails to hold. Nevertheless, it is shown that the standard, one factor, market portfolio based capital asset pricing model does hold in this environment.

It is shown that the optimal durable level is characterized by three numbers (not random variables), say x, y, and z (where $x < y < z$). The consumer views the ratio of consumption to wealth (c/W) as his state variable. If this ratio is between x and z, then he does not sell the durable. If c/W is less than x or greater than z, then he sells his durable and buys a new durable of size S so that $S/W = y$. Thus y is his "target" level of c/W. If the stock market moves up enough so that c/W falls below x, then he sells his small durable to buy a larger durable. However, there will be many changes in the value of his wealth for which c/W stays between x and z, and thus consumption does not change.

Numerical simulations show that small transactions costs can make consumption changes occur very infrequently. Further, the effect of consumption transactions costs on the demand for risky assets is substantial.

KEYWORDS: Asset pricing, transaction cost, durable goods, consumption, permanent income hypothesis, stochastic control, liquidity, risk premium.

1. INTRODUCTION

WE ANALYZE A MODEL of optimal consumption and portfolio selection in which consumption services are generated by holding a durable good. The durable good is illiquid in that a transaction cost must be paid when the good is sold. It is shown that optimal consumption is not a smooth function of wealth; it is optimal for the consumer to wait until a large change in wealth occurs before adjusting his consumption. As a consequence, the consumption based capital asset pricing model (CCAPM) fails to hold. Nevertheless, it is shown that the standard, one factor, market portfolio based capital asset pricing model (CAPM) does hold in this environment.

In the standard model without transactions costs, and with additively separable utility, a consumer, at an optimum, will be indifferent between investing a dollar and consuming goods worth a dollar. This implies that the derivative of his indirect utility of wealth, say $V'(W)$ will equal his marginal utility of consumption $u'(c)$; call this the envelope condition. The CAPM is based upon noting that given two assets with returns r_i and r_j, the consumer must be indifferent about switching a dollar from one to the other, and this implies that $EV'(W)(r_i - r_j) = 0$. Roughly speaking, the CCAPM is derived from this equation by using the envelope condition. However, if it is costly to change the consumption flow, then

[1] We benefited from discussions with Yves Balasko.

it will no longer be the case that the envelope condition holds, and this breaks the link between the CCAPM and the CAPM.

A great deal of empirical evidence now exists in which the CCAPM is not only statistically rejected but also in which it is shown that the CAPM provides a better explanation for the observed risk premia on common stocks.[2] There are two sorts of inadequacies that are brought out by these studies. First, per capita consumption does not covary very much with stock returns, so a very high risk aversion is needed to explain the observed risk premia of stocks. Second, the envelope condition forces the same parameter to be used for both intertemporal substitution and risk aversion, while the data suggest that two parameters are needed.[3] Both of these difficulties are avoided by considering a model where consumption derives from illiquid durables.

The structure and results of this paper are as follows. Section 2 states the consumer's optimization problem. It is assumed that the level of consumption services can be changed only by selling the existing durable (e.g., car or house) and purchasing a new one. In selling the old durable, a transactions cost must be paid which is proportional to the value of the durable being sold (e.g., a commission to a real estate broker). This acts like a *fixed* cost in an optimal stopping problem. The consumer can invest in n risky assets and a risk free asset. There is no transactions cost involved in the purchase and sale of these financial assets.[4] The values of the risky assets follow a Brownian motion, and this is the source of randomness in the model. It is assumed that the consumer has a constant relative risk aversion utility function over durable services.

In the absence of a transactions cost, the consumer would choose his consumption to maintain it in a fixed proportion to his wealth; if the stock market rises, then he increases his consumption, while if it falls, he decreases his consumption. Needless to say, this is not an optimal policy in the presence of a (fixed) transactions cost. It is surely suboptimal for a person to sell his house on every day in which the stock market changes, if on each such sale he must pay a broker a 5% transactions fee. It is thus obvious that the covariance of a *single* consumer's instantaneous consumption changes and stock returns will be zero most of the time, and hence will not be a proper measure of asset riskiness.

Section 3 provides a characterization of the optimal policies. It is shown that the optimal durable level is characterized by three numbers (not random variables), say x, y, and z (where $x<y<z$). The consumer views the ratio of consumption to wealth (c/W) as his state variable. If this ratio is between x and z, then he does not sell the durable. If c/W is less than x or greater than z, then he sells his house and buys a new house of size S so that $S/W=y$. Thus y is his

[2] The CCAPM was rejected in tests performed in Hansen and Singleton (1982 and 1983). Mankiw and Shapiro (1984) compare the adequacy of the CCAPM with the CAPM and find that the latter performs better. The most favorable evidence for the CCAPM appears in Breeden, Gibbons, and Litzenberger (1986), where the unconditional form of the CCAPM is evaluated.

[3] These observations are based upon Grossman, Melino, and Shiller (1985).

[4] See Constantinides (1986) for results regarding proportional transaction costs in security trading and a survey of work on security transaction costs. See Harrison and Taylor (1978) and Richard (1977) for related work on the optimal stopping of a controlled diffusion.

"target" level of c/W. If the stock market moves up enough so that c/W falls below x, then he sells his small house to buy a larger house. However, there will be many changes in the value of his wealth for which c/W stays between x and z, and thus consumption does not change.

Section 3 also proves that the consumer will choose a portfolio of stocks which is mean-variance efficient. As a consequence, equilibrium in the stock market requires that all consumers hold the market portfolio. This, of course, implies that the standard capital asset pricing model gives the risk premia of financial assets, i.e., an asset's mean excess return is proportional to its covariance with the return on the market portfolio.

Finally, Section 3 discusses the extent to which transaction costs cause the consumer to act in a more risk averse manner with regard to his holdings of risky assets. It is shown that just after purchasing a new house, the consumer holds a smaller percentage of his wealth in risky assets than he would in the absence of transaction costs. However, if c/W is close to a value at which it is optimal to sell the durable, the consumer acts in a less risk averse manner relative to the no transactions cost case.

Section 4 presents numerical simulations of the model. It is shown that small transactions costs can make consumption changes occur very infrequently. Further, the effect of transactions costs on the demand for risky assets is substantial.

Section 5 contains conclusions.

The complete proofs are contained in Grossman-Laroque (1987), available from the authors on request. The thread of the argument is presented here, whenever possible in the main text, and for the longer developments in the Appendix.

2. STATEMENT OF THE CONSUMER'S OPTIMIZATION PROBLEM

We assume that consumption services can be obtained only from the possession of a durable physical asset K. This yields a continuous flow of services to its owner and depreciates at rate α over time, $\alpha \geqslant 0$.

We study a situation which departs in two main ways from the standard consumption model. First, the physical asset comes in bulk ("houses") of various sizes, and the services accruing to a consumer comes from the house he lives in (the good is indivisible once bought, and there is no rental market). Therefore, to change his consumption level beyond what is caused by depreciation, the consumer must sell his current house for a new one. Second, the market for houses may operate imperfectly, with transaction costs due to, e.g., costs of matching buyers and sellers. We model this imperfection by postulating that the selling price of the physical asset is a fraction $(1-\lambda)$ of its value, $0 \leqslant \lambda < 1$. The case $\lambda = 0$ corresponds to a perfect market. A similar description of the market for consumer durables has been used by Flemming (1969).

In addition to the durable good, the consumer can invest his wealth in a risk free asset and n risky assets. We take the durable good as the numeraire. We assume that the instantaneous return on the risk free asset is constant and given by r_f. Let $\hat{b}_{it}$ be the value of the ith risky asset (inclusive of accumulated dividends) at time t. We assume that $d\hat{b}_{it} = \hat{b}_{it}(\hat{\mu}_i\, dt + dw_{it})$ where $\underline{w}_t \equiv (w_{1t}, w_{2t}, \ldots, w_{nt})$ is an n dimensional Brownian motion without drift, and instantaneous positive definite covariance matrix Σ. Let $\underline{\hat{b}}_t = (\hat{b}_{1t}, \hat{b}_{2t}, \ldots, \hat{b}_{nt})$ and $\underline{\hat{\mu}} = (\hat{\mu}_1, \hat{\mu}_2, \ldots, \hat{\mu}_n)$. We assume that there are no transaction costs involved in buying or selling these financial assets.

If we let B_t, and $\underline{X}_t$, respectively denote the (dollar) amount of the risk free asset and the vector of risky assets chosen by the consumer at time t, then his total wealth Q_t satisfies

$$(2.1) \qquad Q_t \equiv K_t + B_t + \underline{X}_t \cdot \underline{l},$$

where $\underline{l}$ is a vector of ones. Let τ represent a time when the consumer sells his house. In any interval of time dt in which the consumer does not sell his house, his wealth evolves as

$$(2.2) \qquad dQ_t = -\alpha K_t\, dt + r_f B_t\, dt + \underline{X}_t \cdot (\underline{\hat{\mu}}\, dt + d\underline{w}_t);$$

see Karatzas et al. (1986) or Merton (1969). Note that we can define $d\underline{b}_t = (\underline{\hat{\mu}} - \underline{l}r_f)\, dt + d\underline{w}_t$ to be the vector of excess returns on the risky assets, $\underline{\mu} \equiv \underline{\hat{\mu}} - \underline{l}r_f$ as the vector of mean excess returns, and eliminate B_t from (2.1) and (2.2) to get

$$(2.3) \qquad dQ_t = -\alpha K_t\, dt + r_f(Q_t - K_t)\, dt + \underline{X}_t \cdot d\underline{b}_t \qquad \text{for} \qquad \tau \notin (t, t+dt).$$

If the consumer sells his house at time τ, then

$$(2.4) \qquad Q_\tau = Q_{\tau^-} - \lambda K_{\tau^-},$$

where Q_{τ^-} refers to the level of Q just before the house sale. Note that there is no transactions cost in purchasing a house, and λK_{τ^-} is the loss in selling a house of size K_{τ^-}.

We consider an infinitely lived consumer in the economy whose tastes are represented by the expected value of an intertemporally separable utility function $E\int_0^\infty e^{-\delta t} u(K_t)\, dt$ where $\delta > 0$ is the discount rate, and $K_t \geqslant 0$ is the quantity of durable good held at date t. The consumption service flow is taken to be proportional to the stock K_t.

Given initial conditions (Q_{0^-}, K_{0^-}), the problem of the consumer is to find nonanticipatory controls $(K_t, \underline{X}_t, t \geqslant 0)$ (i.e., where (K_t, X_t) depend only on the past values of $\underline{b}(t')$, $t' \leqslant t$), and nonanticipatory stopping times τ (i.e., he chooses a rule which determines for each time t whether he should sell his house as a function of all the information he has up to time t) which maximize expected utility subject to (2.3), (2.4) and a no bankruptcy constraint:

$$(2.5) \qquad Q_t - \lambda K_t \geqslant 0 \qquad \text{for all } t.$$

(If the consumer meets the constraint with equality at some date t, he is forced to sell his house at that date, and is left with a zero consumption from then on.) We also assume that the absolute value of the fraction of wealth invested in any asset is bounded.

Let $V(Q, K)$ be the supremum of the expected utility that the consumer can achieve, from the initial conditions (Q, K). We assume that the utility function exhibits constant relative risk aversion, i.e.:

$$u(K) = \frac{K^a}{a} \qquad \text{for some} \quad a < 1,\ a \neq 0,$$

and this enables us to reduce the problem from two state variables to a single state variable. The case of log utility (i.e., $a = 0$) can be analyzed as a separate case along identical lines as in the analysis to follow.

Let

$$\text{(2.6)} \qquad \mu \equiv \underline{\mu} \cdot \Sigma^{-1} \cdot \underline{\mu} / l \cdot \Sigma^{-1} \cdot \underline{\mu},$$

$$\sigma^2 \equiv \mu / l \cdot \Sigma^{-1} \cdot \underline{\mu}.$$

Note that $\sigma^2 > 0$, and we assume that μ and $r_f + \alpha$ are strictly positive. In the Appendix we prove the following theorem.

THEOREM 2.1: *Assume*

$$\beta = \delta - ar_f - \frac{\mu^2}{2\sigma^2} \frac{a}{1-a} > 0,$$

and let

$$\nu = \frac{1}{a(\alpha + r_f)} \left(\frac{(r_f + \alpha)(1-a)}{\beta} \right)^{1-a}.$$

Then $V(Q, K)$ is finite, and there exists a number $\nu_2 > 0$ such that

$$\nu Q^a \geqslant V(Q, K) \geqslant \frac{\nu_2 (Q - \lambda K)^a}{a}.$$

Furthermore, $V(Q, K)$ is homogenous of degree a in (Q, K) and does not increase when λ increases.

In all the following, we shall assume $\beta > 0$ (otherwise in the absence of transaction costs, when $\lambda = 0$, the consumer could achieve an infinite expected utility). The quantity ν in the above theorem is the utility the consumer gets when his initial wealth is equal to 1 and $\lambda = 0$. It is immediate that the consumer cannot gain from an increase in transaction costs, and therefore νQ^a provides an upper bound for $V(Q, K)$. On the other hand, the following strategy is always available to the consumer: sell the house now and invest all the proceeds in a new home to be kept forever without any intervention on the financial market. This gives the lower bound on $V(Q, K)$.

Now consider a consumer at date 0, with initial conditions (Q, K). If he decides to change houses immediately, he gets $\text{Sup}_c V(Q - \lambda K, c)$. Therefore, if $V(Q, K) > \text{Sup}_c V(Q - \lambda K, c)$ he will not change, while if the equality holds, he will change. Consequently, $V(Q, K)$ satisfies the following Bellman equation:

$$V(Q,K) = \operatorname*{Sup}_{c,\tau,(\underline{X}_t)} E\left[\int_0^\tau e^{-\delta t}\frac{K_t^a}{a}\,dt + e^{-\delta\tau}V(Q_\tau - \lambda K_\tau, c)\right] \tag{2.7}$$

where τ is the first stopping time from date 0.

Using the homogeneity of $V(Q, K)$ we make a change of variable which enables us to reduce the problem to one state variable y. Let:

$$y = \frac{Q}{K} - \lambda, \qquad \underline{x} = \left(\frac{1}{K}\right)\underline{X}, \tag{2.8}$$
$$h(y) = K^{-a}V(Q,K) = V(\lambda + y, 1),$$
$$\bar{\delta} = \delta + a\alpha, \qquad r = \alpha + r_f.$$

Substituting $K^a h(y)$ for $V(Q, K)$ into the Bellman equation gives:

$$K^a h(y) = \operatorname*{Sup}_{c,\tau,(\underline{x}_t)} E\left[\int_0^\tau e^{-\delta t}\frac{(Ke^{-\alpha t})^a}{a}\,dt + e^{-\delta\tau}c^a h\left(\frac{Q_\tau - \lambda K_{\tau^-}}{c} - \lambda\right)\right].$$

Let

$$M = \operatorname*{Sup}_{c}\left(\frac{Q_{\tau^-} - \lambda K_{\tau^-}}{c}\right)^{-a} h\left(\frac{Q_{\tau^-} - \lambda K_{\tau^-}}{c} - \lambda\right) = \operatorname*{Sup}_{y}(y+\lambda)^{-a}h(y),$$

since $Q_{\tau^+} = Q_{\tau^-} - \lambda K_{\tau^-}$. Divide through by K^a, a positive number, and use the fact that $K_\tau = Ke^{-\alpha\tau}$ to get

$$h(y) = \operatorname*{Sup}_{\tau,(\underline{x}_t)} E\left[\int_0^\tau \frac{e^{-\bar{\delta}t}}{a}\,dt + e^{-\bar{\delta}\tau}My_{\tau^-}^a\right]. \tag{2.9}$$

Since

$$dy = \frac{dQ}{K_t} - \frac{Q\,dK}{K_t^2},$$

we obtain:

$$dy = \underline{x}_t \cdot d\underline{b} + r(y_t + \lambda - 1)\,dt. \tag{2.10}$$

The no bankruptcy constraint (2.5) becomes:

$$y_t \geqslant 0 \qquad \text{all } t. \tag{2.11}$$

Finally, the definition of M is:

$$M = \operatorname*{Sup}_{y}(y+\lambda)^{-a}h(y). \tag{2.12}$$

By Theorem 2.1, we have

$$\operatorname*{Sup}_{y}(y+\lambda)^{-a}v(y+\lambda)^a \geqslant M \geqslant \operatorname*{Sup}_{y}(y+\lambda)^{-a}\frac{y^a}{a}v_2$$

and hence:

$$v \geqslant M \geqslant \frac{v_2}{a}. \tag{2.13}$$

Note that (2.9) is an optimal stopping and control problem where stopping occurs only once, and the payoff upon stopping in state y is My^a. If M satisfies (2.12) then the solution to this problem for $h(\cdot)$, and optimal policies, can immediately be used via (2.8) to obtain the function $V(Q, K)$ and the optimal policies for the original problem. Given M satisfying (2.13), we first study the optimization problem in (2.9) subject to (2.10) and (2.11).

An important feature of this type of stochastic control problem is that one cannot rule out situations where the consumer would keep all his wealth in the house and risk free asset, and invest nothing in the risky assets. In that case the variance of dy is equal to zero, and this singular control problem is much more difficult to handle than the traditional stochastic control problems where the variance of the change in the state variable is uniformly bounded away from zero. For the sake of brevity, we do not describe here in detail the approximation techniques that allow us to bypass this difficulty (see Grossman-Laroque (1987) for a complete proof using Krylov (1980)). It turns out that the only singularity that may occur here is at $y = 1 - \lambda$ (i.e. $Q = K$ and no investment in the risky asset), where the value function may not be differentiable. The value function otherwise satisfies the following standard property:

THEOREM 2.2: *Let M be exogenously fixed satisfying* (2.13), *and let $h(y; M)$ denote the solution to* (2.9)–(2.11) *for such a given M. Then suppressing the dependence of $h(y; M)$ on M for notational convenience, $h(y)$ is continuously differentiable* (*except possibly at $y = 1 - \lambda$*) *and*:

(*i*) *If $h(y) > y^aM$, then it is optimal to not stop* (*i.e.*, $\tau \neq 0$), *$h(y)$ is twice continuously differentiable except possibly at $y = 1 - \lambda$, and*

$$\text{(2.14a)} \quad \sup_{\underset{\sim}{x}} \left[\frac{h''(y)}{2} \operatorname{Var} dy + h'(y) E\, dy - \bar{\delta} h(y) + 1/a \right] = 0, \qquad \textit{where}$$

$$\text{(2.14b)} \quad \operatorname{Var} dy \equiv \underset{\sim}{x} \cdot \Sigma \cdot \underset{\sim}{x} \qquad \textit{and} \qquad E\, dy \equiv r(y + \lambda - 1) + \underset{\sim}{x} \cdot \underset{\sim}{\mu}.$$

(*ii*) *If $h(y) = y^aM$, then stop* (*i.e.*, $\tau = 0$) *and, in the interior of the set* $\{y \mid h(y) = y^aM\}$,

$$\text{(2.15)} \quad \sup_{\underset{\sim}{x}} \left[\frac{h''(y)}{2} \operatorname{Var} dy + h'(y) E\, dy - \bar{\delta} h(y) + 1/a \right] \leqslant 0.$$

The portfolio $\underset{\sim}{x}(y)$ is optimal at state y if it attains the supremum in (2.14).

Theorem 2.2 can be understood as follows. First, from (2.9) it is clear that $h(y) \geqslant My^a$ since it is always feasible to set $\tau = 0$. If $h(y) > My^a$ then this means that $\tau \neq 0$, so (roughly speaking) by continuity, there is a time t small enough so that we can ignore events where stopping occurs during $(0, t)$ and thus:

$$h(y) = \sup_{\underset{\sim}{x}} E\left[\int_0^t \frac{e^{-\bar{\delta}s}}{a} ds + e^{-\bar{\delta}t} h(y_t) \right],$$

and therefore, bringing the left-hand side to the right-hand side, and dividing by t:

$$0=\underset{t\to 0}{\operatorname{limit}}\ \sup_{\underline{X}} E\left[\frac{1}{t}\int_0^t \frac{e^{-\bar{\delta}s}}{a}\,ds+\frac{1}{t}\left(e^{-\bar{\delta}t}h(y_t)-h(y)\right)\right].$$

However, if we let $Z_t \equiv e^{-\bar{\delta}t}h(y_t)$, then

$$\lim_{t\to 0}\frac{1}{t}E\left(e^{-\bar{\delta}t}h(y_t)-h(y)\right)\equiv \lim_{t\to 0}\frac{EZ_t-Z_0}{t}\equiv E\,dZ,$$

where $E\,dZ$ at $t=0$ is found by Ito's Lemma to be

$$h'(y)E\,dy+\frac{1}{2}h''(y)\operatorname{Var}dy-\bar{\delta}h(y).$$

This, combined with

$$\lim_{t\to 0}\frac{1}{t}\int_0^t\frac{e^{-\bar{\delta}s}}{a}\,ds=1/a$$

gives (2.14). The theorem states that there will be regions of values for y where $h(y)>y^aM$ and no stopping is optimal, and other regions where stopping is optimal and $h(y)=y^aM$.

The theorem also states that $h(y)$ is continuously differentiable, except possibly at $y=1-\lambda$. This fact implies the "smooth pasting" condition which requires that at the boundary points of the set $\{y \mid h(y)>y^aM\}$, h is differentiable, and therefore:

(2.16) $\quad h'(y)=ay^{a-1}M.$

Note, on the other hand, that h is typically *not* twice continuously differentiable at these boundary points.

In the next section we show that there is only one connected region where $h(y)>y^aM$, and we characterize the optimal portfolio rules.

3. OPTIMAL PORTFOLIO AND CONSUMPTION RULES

We begin by showing that the consumer chooses a mean-variance efficient portfolio. This result does not require that the utility function exhibit constant relative risk aversion.[5]

[5] To see this, note that in any neighborhood of (Q,K) in which V is twice continuously differentiable and no stopping occurs, the Bellman equation is

$$0=\sup_{X}\Big\{u(K)-\delta V(Q,K)+\left(\underline{X}\cdot\underline{\mu}+r_f(Q-K)-\alpha K\right)V_1(Q,K) + \tfrac{1}{2}V_{11}(Q,K)\underline{X}\cdot\Sigma\cdot\underline{X}+V_2(Q,K)\alpha K\Big\}.$$

Clearly $V_1\geqslant 0$. If $V_1=0$, then this equation can only hold if $V_{11}=0$. In such a case any value of X is optimal. If $V_1>0$, then the equation requires that $V_{11}<0$ and the optimal portfolio satisfies $\underline{X}=(-V_1/V_{11})\Sigma^{-1}\cdot\underline{\mu}$.

THEOREM 3.1: *In a state y, where $h(y)$ is twice continuously differentiable and $h(y) > y^a M$, the consumer chooses a portfolio $\underline{x}(y)$ which maximizes*

$$\underline{x}\mu + \frac{h''(y)}{2h'(y)}\underline{x}\cdot\Sigma\cdot\underline{x},$$

and consequently

$$\underline{x}(y) = \frac{-h'(y)}{h''(y)}\Sigma^{-1}\cdot\underline{\mu}. \tag{3.1}$$

PROOF: This follows immediately from Theorem 2.2, by substituting (2.14b) into (2.14a) and maximizing with respect to $\underline{x}$.

Since all consumers hold risky assets in the same proportion (even if they have different utility functions and values of y), and since financial market clearing implies that the sum of their holdings must equal the value of the market portfolio, we obtain the standard (market portfolio based single factor) capital asset pricing model (CAPM). Let p_{it} be the total market value of asset i *exclusive* of accumulated dividends at time t, and $\underline{p}_t = (p_{1t} \cdots p_{nt})$; then we prove the following theorem.

THEOREM 3.2: *Let r_m be the instantaneous return on the market portfolio (i.e., a portfolio where the ith financial asset has weight $p_i/\sum_{j=1}^{n} p_j$); then a necessary condition for market clearing is*

$$\mu_i = \frac{\operatorname{Cov}\left(\dfrac{d\hat{b}_i}{\hat{b}_i}, r_m\right)}{\operatorname{Var}(r_m)}(Er_m - r_f) \qquad (i = 1, 2, \ldots, n).$$

PROOF: Recall that $\underline{x}(y) = (1/K)\underline{X}(y)$, so that (3.1) implies that $\underline{X}(y) = s(y, K)\Sigma^{-1}\underline{\mu}$, where $s(y, K)$ is a positive scalar which depends on the consumer's tastes, y and K. This can be summed over consumers to yield $\underline{p}_t = s_{2t}\Sigma^{-1}\underline{\mu}$, where s_{2t} is a positive scalar.

This implies that

$$\underline{\mu} = \frac{1}{s_{2t}}\Sigma\cdot\underline{p}_t \qquad \text{and therefore} \qquad s_{2t} = \underline{p}_t\cdot\Sigma\cdot\underline{p}_t/\underline{p}_t\cdot\underline{\mu},$$

and these two equations combine to give the result to be proved, since Σ is a matrix with (i, j) element equal to $\operatorname{Cov}(d\hat{b}_i/\hat{b}_i, d\hat{b}_j/\hat{b}_j)$.

Note that in any model where the CAPM holds, the vector of values for the n assets $\underline{p}_t$ is proportional to $\Sigma^{-1}\underline{\mu}$. Therefore the relative value of assets is deterministic. Recall that Σ is the covariance matrix of *returns*, and hence this matrix can be of full rank since it includes earnings (e.g., dividends) on the investments, even though the aggregate value invested in the vector of securities has only one dimension of uncertainty.

It may help some readers to imagine that we are providing a general equilibrium model for the following type of economy. The durable consumption good can be produced by firms utilizing n stochastic constant returns to scale technologies, and one technology with a sure constant productivity r_f. In particular, imagine that a firm can, without transactions cost, invest any amount of the durable good in technology i, and this will produce db_i/b_i more units of the durable good over the next instant of time. The technology is freely available to firms, but unavailable to consumers. The constant returns to scale assumption implies that the total supply of financial asset i (denoted by p_{it}) will always adjust to equal the aggregate demand of consumers. The mean returns and covariance matrix are technologically fixed. We make some further comments on general equilibrium in the Conclusions.

Theorem 3.2 allows us to change the control in (2.9), (2.10), (2.14), and (2.15) from the vector $\underline{x}$ to a scalar x which denotes the amount invested in the market portfolio. Note that (2.6) gives the mean rate of return μ and variance σ^2 of the market portfolio. Therefore, (3.1) becomes

$$x(y)=\frac{-h'(y)}{h''(y)}\frac{\mu}{\sigma^2}, \tag{3.2}$$

and $\operatorname{Var} dy = x^2\sigma^2$ and $E\,dy = r(y+\lambda-1)+x\mu$ in (2.14b).

In order to understand why the consumption based asset pricing model fails in our context it is necessary to further characterize the optimal consumption policy. The next theorem states that the optimal consumption policy is characterized by three numbers y_1, y^*, and y_2 with the property that $y_1 \leqslant y^* \leqslant y_2$ and if the state variable y is between y_1 and y_2, then the house is not sold; if y is not between y_1 and y_2, then the house is sold and a new house is purchased to bring the level of the state variable from y to y^*.

THEOREM 3.3: *If $M<(1-\lambda)^{-a}v$, there exist three numbers $y_1 \leqslant y^* \leqslant y_2$ such that $h(y)$ in Theorem 2.2 satisfies*

$$h(y) > My^a \quad \textit{if and only if} \quad y \in (y_1, y_2),$$

and

$$M=(y^*+\lambda)^{-a}h(y^*)=\sup_y (y+\lambda)^{-a}h(y). \tag{3.3}$$

If $M \geqslant (1-\lambda)^{-a}v$, then $h(y)=My^a$ for all y.

To understand this recall that $y=(Q/K)-\lambda$. If the financial assets rise in value, then Q will rise and eventually the consumer will feel sufficiently wealthy that he wants a larger house. When Q rises to the point where $y=y_2$, the consumer purchases a house $K^*(Q)$ with the property that his new ratio of wealth to housing $Q/K^*(Q)$ satisfies

$$Q/K^*(Q)-\lambda=y^*. \tag{3.4}$$

A similar effect appears on the downside. If wealth falls sufficiently, then the old house will be sold and a new house will be purchased which satisfies (3.4). Note that the size of the new house depends only on the total wealth Q. Note that if Q_{τ^-} was the level of wealth just before switching houses, then the Q in (3.4) is $Q_{\tau^-} - \lambda K_{\tau^-}$. We show later that the case $M \geqslant (1-\lambda)^a \nu$ never occurs.

PROOF OF THEOREM 3.3: We only give a sketch of the argument, leaving aside the difficulties associated with the singularity (again see Grossman-Laroque (1987) for the complete proof). Let:

$$G(h'', h', h, y) = -\frac{1}{2}\frac{\mu^2}{\sigma^2}\frac{h'^2}{h''} + r(y+\lambda-1)h' - \bar{\delta}h + \frac{1}{a}.$$

By (3.2) and Theorem 2.2, we have: $G(h''(y), h'(y), h(y), y)$ is equal to zero whenever $h(y) > y^a M$ and, otherwise, nonpositive (as long as $h''(y)$ is defined). Let $g(y)$ be the function obtained by substituting $h(y) = y^a M$ into G, i.e.,

$$g(y) = G\left[a(a-1)My^{a-2}, aMy^{a-1}, My^a, y\right].$$

It is immediate (using $aM > 0$ from (2.13)) that $g(y)$ increases for $0 \leqslant y \leqslant \bar{y}$, decreases for $\bar{y} \leqslant y$, where $\bar{y} = (1-\lambda)(1-a)r/\beta$. Furthermore, $g(\bar{y})$ is strictly positive for $M < (1-\lambda)^{-a}\nu$, nonpositive for $M \geqslant (1-\lambda)^{-a}\nu$. Consider first the case $M < (1-\lambda)^{-a}\nu$. The set $\{y | g(y) > 0\}$ is then a nonempty interval.

By Theorem (2.2)(ii) we have:

$$\{y | g(y) > 0\} \subseteq \{y | h(y) > y^a M\}.$$

To show that $\{y | h(y) > y^a M\}$ is also an interval, we proceed by contradiction. If not, there would exist (y_1', y_2') disjoint from $\{y | g(y) > 0\}$, with the smooth pasting condition holding at y_1' and y_2'. The function $h(y) - My^a$ would have a maximum on $[y_1', y_2']$, say at y', where:

$$h(y') > My'^a, \quad h'(y') = aMy'^{a-1},$$

$$h''(y') \leqslant a(a-1)My'^{a-2}.$$

Since G is increasing in its first argument and strictly decreasing in its third argument, this would imply $g(y') > 0$, a contradiction.

Finally, when $M \geqslant (1-\lambda)^{-a}\nu$, $h(y) = y^a M$ for all y. This can be seen by noting that $g(y) \leqslant 0$ for all y and if there existed a nonempty interval (y_1', y_2') where $h(y) > y^a M$, the same argument as above would lead to a y' with $g(y') > 0$, a contradiction. *Q.E.D.*

We are now in a position to give a complete mathematical description of the solution, using (2.14) and (3.2). Let M be given, let $h(y)$ be the value function in (2.9), and let y_1 and y_2 be as defined in Theorem 3.3. Then from Theorem 7, in Krylov (1980, p. 41), the triple $(h(y), y_1, y_2)$ is the unique solution of the following system (3.5)–(3.8):

$$(3.5) \qquad -\frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2\frac{(h'(y))^2}{h''(y)}+r(y+\lambda-1)h'(y)-\bar{\delta}h(y)+\frac{1}{a}=0$$

for $y\in[y_1,y_2],\ y\neq 1-\lambda$;

$$(3.6) \qquad h(y)\geqslant y^aM \qquad \text{for all } y;$$

and from (2.16)

$$(3.7) \qquad h(y_i)=y_i^aM \qquad \text{for} \quad i=1,2;$$

$$(3.8) \qquad h'(y_i)=ay_i^{a-1}M \qquad \text{for} \quad i=1,2.$$

Moreover, the solution of the complete problem is obtained for a value of M that satisfies

$$(3.9) \qquad M=(y^*+\lambda)^{-a}h(y^*)=\sup_y\,(y+\lambda)^{-a}h(y).$$

To understand this system, fix M and ignore (3.9). Equations (3.7) and (3.8) represent two free boundary conditions for the second order differential equation (3.5). Such a system can be understood by picking a guess for y_1, then using (3.7) and (3.8) to get $h(y_1)$ and $h'(y_1)$. This provides an initial condition to (3.5). Then (3.5) can be continued while $h(y)>y^aM$ until a point y_2 is found where $h(y_2)=y_2^aM$. If it is also the case that $h'(y_2)=ay_2^{a-1}M$, then we are done; otherwise choose a new value of y_1 and repeat. If a solution can be found, then for the fixed M, we will have a solution to (3.5)–(3.8): $h(y;M)$. This function is then substituted into (3.9) to yield the problem: find an M^* such that

$$(3.10) \qquad M^*=\sup_y\,(y+\lambda)^{-a}h(y;M^*).$$

The value for y^* is the y which attains the supremum in (3.10).

We do not have an analytic solution to (3.5)–(3.9); the next Section presents numerical simulations of the solution. In addition, in the Appendix we prove that as transaction costs increase, the interval over which no house sale occurs (y_1, y_2) grows:

THEOREM 3.4: *If* $(h(y), y_1, y_2, M)$ *solve* (3.5)–(3.9), *then* $M<(1-\lambda)^{-a}v$, y_1 *is a strictly decreasing function of* λ, *and* y_2 *is a strictly increasing function of* λ. *If* $M>1/(a\bar{\delta})$, *then* M *is a strictly decreasing function of* λ.

Note that $M=1/(a\bar{\delta})$ when it is optimal to never sell a house, starting from $y=y^*$, i.e. $y^*=1-\lambda$, $x(y^*)=0$ and neither y_1 nor y_2 are ever reached starting from $y=y^*$.

Though we cannot explicitly solve for $h(\cdot)$, substituting (3.2) into (3.5) and differentiating once can be used to show that $x(y)$ satisfies the following differential equation:

$$(3.11)\qquad \frac{\mu}{2}x'(y)x(y)+\left(r-\bar{\delta}-\frac{\mu^2}{2\sigma^2}\right)x(y)-\frac{\mu r}{\sigma^2}(y+\lambda-1)=0 \qquad \text{for} \quad y\in(y_1,y_2).$$

THEOREM 3.5: *Let*

$$(3.12)\qquad \gamma\equiv\left(\bar{\delta}+\frac{\mu^2}{2\sigma^2}-r\right)\Bigg/\sqrt{\left(\bar{\delta}+\frac{\mu^2}{2\sigma^2}-r\right)^2+2r\frac{\mu^2}{\sigma^2}}$$

(note that $0\leqslant\gamma\leqslant 1$*) and let* $\Theta_1<0,\Theta_2>0$ *be the two roots of the second degree equation*

$$(3.13)\qquad \Theta^2+2\left[\frac{r-\bar{\delta}}{\mu}-\frac{\mu}{2\sigma^2}\right]\Theta-2\frac{r}{\sigma^2}=0.$$

Then the optimal x *at* y, $x(y)$ *satisfies*:

$$(3.14a)\qquad x(y)\geqslant \operatorname{Max}\left(\Theta_1(y+\lambda-1),\Theta_2(y+\lambda-1)\right),$$

$$(3.14b)\qquad \left(x(y)-\Theta_1(y+\lambda-1)\right)^{1-\gamma}\left(x(y)-\Theta_2(y+\lambda-1)\right)^{1+\gamma}=c,$$

for some real number $c\geqslant 0$.

Furthermore:

$$\frac{x(y_i)}{y_i}\geqslant\frac{\mu}{(1-a)\sigma^2},\qquad i=1,2,$$

$$\frac{x(y^*)}{y^*}\leqslant\frac{\mu}{(1-a)\sigma^2}.$$

PROOF OF THEOREM 3.5: The first step is to prove the final inequalities. The property $h(y_i)=y_i^aM$, $h'(y_i)=ay_i^{a-1}M$, and $h''(y_i)\geqslant a(a-1)y_i^{a-2}M$, together with the definition of $x(y)$ in (3.2), give

$$\frac{x(y_i)}{y_i}\geqslant\frac{\mu}{(1-a)\sigma^2}.$$

Note that y^* maximizes $(y+\lambda)^{-a}h(y)$. The first and second order conditions associated with this maximization, again using (3.2), lead to

$$\frac{x(y^*)}{y^*}\leqslant\frac{\mu}{(1-a)\sigma^2}.$$

To complete the proof, one looks at the various solutions of (3.11). They are all of the shape (take logarithms and differentiate):

$$|x(y)-\Theta_1(y+\lambda-1)|^{1-\gamma}|x(y)-\Theta_2(y+\lambda-1)|^{1+\gamma}=c$$

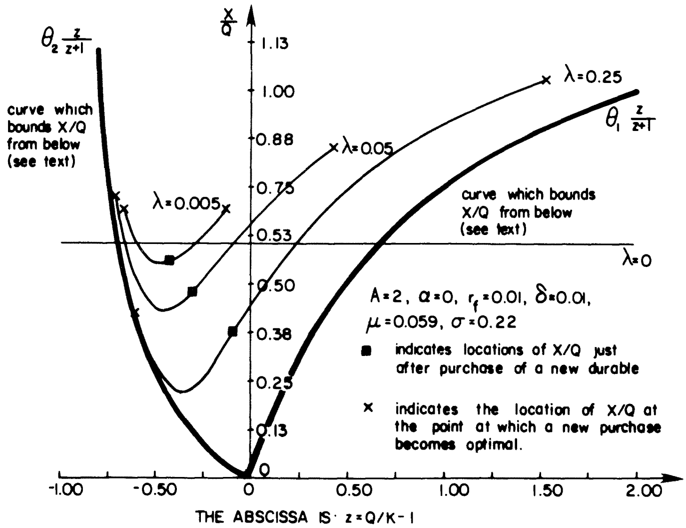

FIGURE 1.—X/Q plotted against $Q/K-1$ for various values of λ.

for some nonnegative c. The fact that $x(y) \geqslant \mu y/(1-a)\sigma^2$ for $y = y_1, y_2$ and $x(y) \leqslant \mu y/(1-a)\sigma^2$ for $y = y^*$, $y_1 < y^* < y_2$ implies that the solution of the problem satisfies (3.14a) and (3.14b). (For details, see Grossman-Laroque (1987).) *Q.E.D.*

To understand (3.14) make a change of variable to $z = y + \lambda - 1$ and $\bar{x}(z) = x(z + 1 - \lambda)$, so (3.14) becomes

$$(3.15) \qquad (\bar{x}(z) - \Theta_1 z)^{1-\gamma}(\bar{x}(z) - \Theta_2 z)^{1+\gamma} = c \qquad \text{for} \quad z \in (z_1, z_2)$$

where $z_1 = y_1 + \lambda - 1$ and $z_2 = y_2 + \lambda - 1$. As c goes to zero, the curve $\bar{x}(z)$ collapses onto the curve formed by two lines $\bar{x} = \Theta_1 z$ and $\bar{x} = \Theta_2 z$.

Note that $\bar{x}(z)/(z+1)$ represents the fraction of wealth invested in the risky asset, X/Q. Figure 1 plots this fraction $\bar{x}(z)/(z+1)$ against z for three values of λ, as well as the limit curves $\Theta_1 z/(z+1)$ and $\Theta_2 z/(z+1)$ which give the lower bounds for X/Q. Note that from (2.1) and (2.8), $z = (X+B)/K$, and thus unlike y, z is a description of the state independent of λ. Hence, the reaction of the $x(z)$ to a change in λ tells us how the holdings of risky assets change in a particular state when λ changes. Each curve is drawn only over that part of the state space for which z takes on realizations. That is, the consumer chooses y_1, y_2 for a particular λ, and therefore z is always between $y_1 + \lambda - 1 \equiv z_1$ and $y_2 + \lambda - 1 \equiv z_2$.

Note that the left-hand side of (3.15) is independent of λ (i.e., Θ_1, Θ_2, and γ do not depend on λ). Therefore, if we compare the solutions to two identical optimization problems, except that $\lambda = \lambda_a$ in one and $\lambda = \lambda_b$ in the other, then only c can be different across the two problems. If we denote the optimal value of $\bar{x}(z)$ by $x(z; \lambda)$ to indicate its dependence on λ, then the previous remarks imply

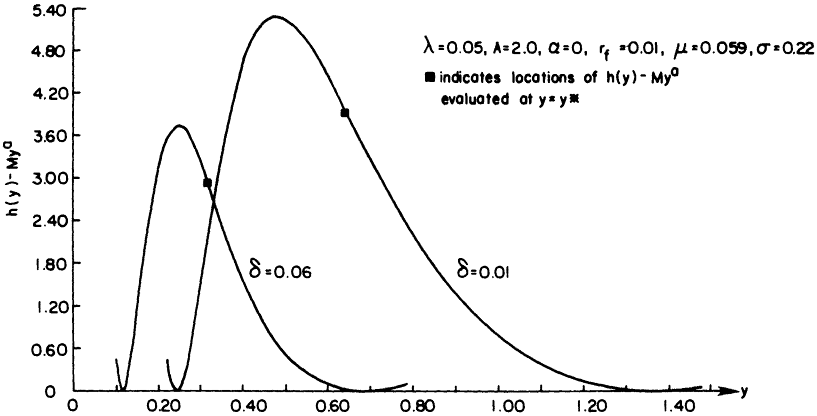

FIGURE 2.—$h(y) - My^a$ is plotted against y, where $y = Q/K - \lambda$.

that $\bar{x}(z; \lambda_a)$ is either everywhere above or everywhere below $\bar{x}(z; \lambda_b)$ for the range of z's where (3.15) holds. We do not know that c varies monotonically with λ.

This is illustrated in Figure 1 where $\lambda_a > \lambda_b$ and $\bar{x}(z; \lambda_a)$ lies everywhere below $\bar{x}(z, \lambda_b)$. Note, however, that there are values for $\bar{x}(\cdot; \lambda_a)$ which are *larger* than $\bar{x}(\cdot; \lambda_b)$. For example $x(z_{1a}; \lambda_a) > x(z; \lambda_b)$ for all $z \in [z_{1b}, z_{2b}]$, where z_{1a} is the value for z at which the consumer finds it optimal to sell his house when $\lambda = \lambda_a$, and z_{1b} is the corresponding value when $\lambda = \lambda_b$. Note that if $\lambda = \lambda_b$ then $z = z_{1a}$ is not an attainable point in the state space since the consumer sells his house at any state below z_{1b}, when $\lambda = \lambda_b$.

The above remarks indicate that the effect of λ on the holdings of risky assets is complex, and dependent on the particular state that the consumer is in. Roughly speaking, the existence of transactions costs makes the consumer more risk averse in the middle of his state space (i.e., near $y = y^*$) and less risk averse at the boundaries of his state space ($y = y_1$ or $y = y_2$). This can be understood by noting that the consumer's direct utility function My^a is replaced by his indirect utility function $h(y)$ for purposes of asset choice. Figure 2 illustrates the consequences of this by plotting $h(y) - My^a$. Note that the smooth pasting conditions assure that $h(y)$ is tangent to My^a at $y = y_1$ and $y = y_2$. As the Figure shows, this implies that $h(y)$ is less concave than My^a at these points. (The curve may not appear tangent at y_1, but it is tangent; the second derivative of $h(y) - My^a$ is quite large just to the right of y_1.) Clearly however, $h(y)$ must become more concave than My^a somewhere between y_1 and y_2.

To explore this further, note that

$$\frac{x(y)}{y} = \frac{X}{Q - \lambda X} \tag{3.16}$$

gives the proportion of marketable wealth invested in the risky asset. If $\lambda = 0$, then our model is equivalent to Merton's model where the proportion of wealth invested in the risky asset is

$$\frac{X}{Q} = \frac{\mu}{(1-a)\sigma^2}. \tag{3.17}$$

It therefore follows from Theorem 3.5 that the consumer behaves in a more risk averse manner just after purchasing a new house, and in a less risk averse manner just before purchasing a new house.

4. NUMERICAL SIMULATIONS

The boundary value problem (3.5)–(3.9) can be solved on a computer using roughly the technique described in Section 3 just after (3.9). We make the following assumptions about parameters: The mean and standard deviation of the annual excess return on the market portfolio are 5.9% and 22% respectively from Ibbotson and Sinquefeld (1982).

Since the durable good is the numeraire, the nominal inflation on durable goods should be subtracted from the nominal rate of interest to get the appropriate real rate, r_f. Ibbotson and Sinquefeld report an average short term nominal interest rate of 4.3% between 1953 and 1979. In the same period the nominal inflation in housing prices was 4.2% per year, and 2.5% per year for automobiles.[6] We thus, somewhat arbitrarily, set the risk free rate r_f (measured relative to the durable good numeraire) to be 1%.

Table I presents some numerical results for the case of no depreciation ($\alpha = 0$), $\delta = r_f$, and for various values of λ, and of $A \equiv 1 - a$ which is the coefficient of relative risk aversion. In order to define state variables independent of λ, we let

$$\bar{y} \equiv y + \lambda = Q/K. \tag{4.1}$$

Column 3 gives the left boundary $\bar{y}_1$, and the right boundary $\bar{y}_2$ of the no stopping region. It also gives the point $\bar{y}^*$ to which Q/K is brought after $\bar{y}$ hits a boundary point.

Column 4 presents the expected length of time between house purchases. Following Karlin and Taylor (1981, p. 192), let T_y be the length of time it takes to hit a boundary starting from y, and

$$V_a(y) = ET_y. \tag{4.2}$$

Then $V_a(\ )$ satisfies the following differential equation

$$-1 = V_a'(y)E\,dy + \tfrac{1}{2}V_a''(y)\operatorname{Var} dy, \tag{4.3}$$

with boundary conditions $V_a(y_1) = V_a(y_2) = 0$. This is solved numerically, and $V_a(y^*)$ appears in the table.

The fifth column presents the result of numerical calculations on the average holding of risky assets. The fraction of wealth invested in the risky asset is a function of the state y. This state is a renewal process which goes from y^* to a

[6] Economic Report of the President 1980, pp. 260–261. See Stambaugh (1982) for a discussion of real returns on various durables and assets.

TABLE I

SOME NUMERICAL SIMULATIONS

λ	A	$(\bar{y}_1, \bar{y}^*, \bar{y}_2)$	(Years) $E\tau$	$\frac{E(X/Q)}{E(\tau)}$	$\frac{E\left(\frac{X}{Q-\lambda K}\right)}{E(\tau)}$	$\lambda = 0$, X/Q	Prob. (buy-down)	Rate of Wealth Drift at y^*
.005	2.0	(.34, 0.58, 0.89)	11.47	0.584	0.587	0.610	.226	.0262
.25	2.0	(.40, 0.92, 2.54)	47.42	0.460	0.604	0.610	.022	.0293
.05	2.0	(.29, 0.70, 1.43)	28.94	0.530	0.570	0.610	.093	.0258
.05	1.75	(.27, 0.71, 1.56)	27.33	0.609	0.654	0.697	.100	.0313
.05	1.5	(.26, 0.76, 1.81)	25.64	0.717	0.768	0.813	.107	.0391
.05	1.1	(.26, 1.08, 3.16)	22.39	1.015	1.069	1.108	.121	.0622
.05	0.9	(.44, 2.56, 8.71)	20.46	1.303	1.335	1.354	.129	.0845
.005	0.9	(.76, 2.03, 4.38)	7.832	1.342	1.345	1.354	.261	.0840
.25	0.9	(.39, 3.62, 18.30)	35.08	1.220	1.342	1.354	.042	.0875
.08	2.0	(.30, 0.74, 1.65)	33.93	0.511	0.570	0.610	.069	.0262
.08	1.75	(.28, 0.76, 1.82)	32.17	0.587	0.654	0.697	.074	.0317
.08	1.5	(.26, 0.81, 2.12)	30.16	0.693	0.768	0.813	.082	.0397
.08	1.1	(.26, 1.17, 3.78)	26.49	0.989	1.068	1.108	.095	.0630
.08	0.9	(.40, 2.78, 10.56)	24.19	1.286	1.334	1.354	.103	.0849
.10	2.0	(.31, 0.77, 1.77)	36.48	0.501	0.572	0.610	.058	.0264
.10	1.75	(.29, 0.79, 1.97)	34.62	0.577	0.656	0.697	.063	.0321
.10	1.5	(.27, 0.85, 2.31)	32.49	0.680	0.770	0.813	.070	.0402
.10	1.1	(.27, 1.22, 4.14)	28.50	0.975	1.069	1.108	.083	.0635
.10	0.9	(.37, 2.90, 11.66)	26.34	1.276	1.334	1.354	.088	.0852

NOTES: $\alpha = 0$, $r_F = .01$, $\delta = 0.01$, $\mu = .059$, $\sigma = .22$.

100λ is the percent transactions cost of selling the old durable; $A \equiv 1 - a$ is the coefficient of relative risk aversion; $(\bar{y}_1, \bar{y}^*, \bar{y}_2)$ are the three values of Q/K which characterize optimal stopping; $E\tau$ is the expected length of time to reach either $\bar{y}_1$ or $\bar{y}_2$ starting from $\bar{y}^*$; $(E\tau)^{-1}EX/Q$ is the average fraction of wealth invested in the risky asset, using the steady state distribution under the optimal policy; $(E\tau)^{-1}E[X/(Q-\lambda K)]$ is the same as $(E\tau)^{-1}EX/Q$ except that "wealth" refers to the amount which is obtained net of transactions cost when all assets and durables are sold. $\lambda = 0$, X/Q is the value of X/Q in the absence of transactions cost. Prob(buy-down) is the prob that $\bar{y}_1$ is reached from $\bar{y}^*$ before $\bar{y}_2$ is reached; Rate of Drift at y^* is the expected rate of change in wealth evaluated at $y = y^*$.

boundary and then returns. Karlin and Taylor (1981, pp. 192, 261) show that the expected value of any function $f(\cdot)$ of the state, can be found by solving the differential equation

$$(4.4)\qquad -f(y) = V_b'(y)E\,dy + \tfrac{1}{2}V_b''(y)\operatorname{Var} y$$

for $V_b(\cdot)$ with boundary conditions $V_b(y_1) = V_b(y_2) = 0$. The average value of $f(\cdot)$ over renewal cycles is $V_b(y^*)/V_a(y^*)$. In column 5, the function presented is

$$(4.5)\qquad f(y) = \frac{X}{Q}(y) \equiv \frac{X}{K}\cdot\frac{K}{Q} \equiv \frac{x(y)}{y+\lambda},$$

which represents the fraction of total wealth invested in risky assets, and this is denoted by $E(X/Q)/E\tau$ to emphasize that it is the average of X/Q over renewal cycles.

The sixth column uses wealth net of the transactions cost of selling the durable to compute the average fraction of wealth held in the form of the risky asset. This is computed by setting $f(y) = x(y)/y$ in (4.4).

The seventh column gives the value of X/Q in the absence of transactions cost (i.e., in Merton's model), as it appears in (3.17).

The eighth column gives the probability that the consumer will reach the lower boundary before reaching the upper boundary, given that he starts from $y = y^*$. It thus gives the fraction of occasions that a change in the value of the stock market causes people to buy smaller houses, rather than larger houses.

The ninth (and final) column gives the rate of drift in wealth evaluated at $y = y^*$. It is computed as $(1/y)E\,dy = (\mu x(y) + (y + \lambda - 1)r)/y$.

Discussion of Table I: As is expected, a rise in transactions cost increases the average time between durable sales. The average time between durable sales is quite large even for very small transactions costs. Note that in this infinite horizon model, if risky assets were not held (i.e., if $\mu = 0$) and $r = \bar{\delta}$, then the consumer starting from y^* would never switch. He would invest all of his wealth in housing and consume the service flow. It is the uncertainty about stock returns and the upward drift in wealth when $\mu > 0$ that causes the boundaries to be hit.

In particular, with $\delta = .01$, $r = .01$, and $\mu = .059$ and the range of risk aversions being considered, the returns from saving in the form of financial assets are so large that the consumer chooses a lifetime consumption profile which (on average) drifts upward. To accomplish this he chooses a relatively small durable and relatively large financial investments with the property that his wealth drifts upward in the period between durable purchases.[7]

Hence, the major reason for durable sales is the upward drift in wealth. As the final column of the Table makes clear, the parameters chosen imply that wealth is expected to rise at a rate of, say, 6.22% for $A = 1.1$ and $\lambda = .05$ just after a new durable purchase has been made.

It should be emphasized that we are considering durable good sales caused only by changes in wealth; not caused by death, switching of jobs or spouses, or changes in family size. The point to realize is that changes in stock market wealth will be associated with consumption changes for an individual only when measured over *decades*; there is essentially no covariance between consumption changes and stock returns on a monthly or annual basis for realistic measures of transactions cost.

The Table shows that the average (over renewal cycles) fraction of wealth invested in risky assets, X/Q, falls as transactions costs rise. For $A = 2.0$, the average holdings of risky assets fall from .584 at $\lambda = .005$ to .460 at $\lambda = .25$; in each case it is substantially lower than the no transactions cost case where $\lambda = 0$.

The next column on the Table, labeled $(E\tau)^{-1}\bar{E}X/(Q - \lambda K)$, also computes the average (over renewal cycles) fraction of "wealth" invested in the risky asset, but "wealth" refers to the amount of money that would be realized if all assets and durables were liquidated. This measure of wealth depends on the level of transactions cost. The table shows that the average fraction invested in risky

[7] This is analogous to what would occur in a certainty mode with $r > \delta$. There, both consumption and wealth would "drift" upward at the same rate.

TABLE II

$\alpha = 0,\ r_f = .01,\ \delta = 0.02,\ \mu = .059,\ \sigma = .22$

λ	A	$(\bar{y}_1, \bar{y}^*, \bar{y}_2)$	(Years) $E\tau$	$\dfrac{E(X/Q)}{E(\tau)}$	$\dfrac{E\left(\frac{X}{Q-\lambda K}\right)}{E(\tau)}$	$\lambda = 0$ X/Q	Prob. (buy-down)	Rate of Wealth Drift at $\bar{y}^*$
.005	2.0	(.274, 0.465, 0.711)	11.86	0.576	0.583	0.610	.274	.0212
.25	2.0	(.375, 0.796, 2.162)	56.90	0.403	0.565	0.610	.050	.0245
.05	2.0	(.247, 0.576, 1.169)	32.06	0.504	0.552	0.610	.142	.0210
.05	1.75	(.226, 0.573, 1.243)	30.15	0.578	0.634	0.697	.149	.0257
.05	1.5	(.206, 0.581, 1.371)	28.04	0.678	0.745	0.813	.157	.0326
.05	1.1	(.179, 0.674, 1.935)	24.20	0.951	1.036	1.108	.171	.0533
.05	0.9	(.185, 0.909, 3.038)	21.95	1.206	1.292	1.354	.179	.0737
.005	0.9	(.262, 0.691, 1.480)	8.02	1.316	1.326	1.354	.302	.0724
.25	0.9	(.307, 1.369, 6.658)	39.81	1.017	1.292	1.354	.083	.0816

TABLE III

$\delta = 0.04,\ r_f = 0.01,\ \alpha = 0.00,\ \mu = 0.059,\ \sigma = 0.22$

A	λ	$(\bar{y}_1, \bar{y}^*, \bar{y}_2)$	(Years) $E\tau$	$\dfrac{E(X/Q)}{E(\tau)}$	$\dfrac{E\left(\frac{X}{Q-\lambda K}\right)}{E(\tau)}$	$\lambda = 0$ X/Q	Prob. (buy-down)	Rate of Wealth Drift at $\bar{y}^*$
2.0	.005	(.198, .338, 0.513)	12.58	0.562	0.571	0.610	.383	.0117
2.0	.25	(.344, .662, 1.753)	81.91	0.296	0.478	0.610	.208	.0154
2.0	.05	(.195, .443, 0.883)	37.95	0.454	0.517	0.610	.297	.0121
1.75	.05	(.175, .428, 0.911)	35.35	0.518	0.594	0.697	.301	.0154
1.5	.05	(.156, .415, 0.959)	32.43	0.603	0.697	0.813	.305	.0205
1.1	.05	(.125, .412, 1.153)	27.40	0.829	0.967	1.108	.311	.0365
0.9	.05	(.113, .440, 1.421)	24.43	1.032	1.200	1.354	.315	.0531
0.9	.005	(.118, .309, 0.652)	8.31	1.265	1.289	1.354	.393	.0504
0.9	.25	(.276, .736, 3.362)	50.61	0.706	1.144	1.354	.234	.0667

assets is not a monotone function of λ. When λ is large, "wealth" falls and this makes it appear as if a large fraction of "wealth" is invested in the risky asset.

It is interesting to note that, for a given λ, as the consumer becomes less risk averse (i.e., A falls) the average holdings of risky assets gets closer to the $\lambda = 0$ risky asset level. We understand this to be caused by the fact that when A falls the consumer holds more risky assets, and hence spends more time near the boundaries of his "no stopping" region. (Note that $E\tau$ falls as A falls.) The transactions cost causes him to be less risk averse near the boundaries than he would be if $\lambda = 0$, as noted in Section 3.

Discussion of Tables II, III, and IV: These tables consider the same parameters as Table I, except that δ is raised to 2%, 4%, and 6% in Tables II, III, and IV respectively. It may be thought that raising the discount rate lowers the average time between durable sales. The reasoning may be that when the discount rate is high, then the consumer should take advantage of a rise in wealth to increase his consumption sooner rather than later. A comparison of Table II and Table I shows that this is not the case. Instead, the dominant effect appears to be that a

TABLE IV

$\delta = 0.06,\ r_f = 0.01,\ \alpha = 0.00,\ \mu = 0.059,\ \sigma = 0.22$

A	λ	$(\bar{y}_1, \bar{y}^*, \bar{y}_2)$	(Years) $E\tau$	$\dfrac{E(X/Q)}{E(\tau)}$	$\dfrac{E\left(\frac{X}{Q-\lambda K}\right)}{E(\tau)}$	$\lambda = 0$ X/Q	Prob. (buy-down)	Rate of Wealth Drift at $\bar{y}^*$
2.0	.005	(.156,.268,0.404)	12.90	0.549	0.560	0.610	.501	.0027
2.0	.25	(.326,.590,1.535)	94.73	0.223	0.421	0.610	.512	.0076
2.0	.05	(.166,.369,0.730)	40.71	0.413	0.490	0.610	.506	.0042
1.75	.05	(.149,.352,0.741)	37.61	0.468	0.561	0.697	.502	.0063
1.5	.05	(.132,.336,0.764)	34.45	0.540	0.657	0.813	.497	.0096
1.1	.05	(.104,.314,0.860)	28.77	0.727	0.906	1.108	.487	.0213
0.9	.05	(.092,.312,0.983)	25.59	0.888	1.118	1.354	.480	.0343
0.9	.005	(.078,.205,0.427)	8.46	1.218	1.253	1.354	.488	.0297
0.9	.25	(.269,.568,2.476)	55.48	0.498	1.006	1.354	.469	.0500

rise in the discount rate to 2% lowers investment in financial assets (as can be seen by comparing ($\bar{y}_1$, $\bar{y}^*$, $\bar{y}_2$) across the two tables), and this lowers the rate of upward drift in wealth.

Table IV, in which $\delta = 6\%$, shows that the investment in financial assets falls to the point where the expected drift in wealth is almost zero just after a new durable is purchased. For such parameter values, the purchase of a durable, in the steady state, is due to variability in wealth rather than the drift in wealth. The expected time between purchases is very large, far larger than for the $\delta = 1\%$ of Table I. Thus, for an infinitely long lived consumer in a steady state (i.e., where wealth would not be expected to drift after a purchase), the variability of the stock market would not be correlated with durable purchase except at frequencies of many decades.

Discussion of Table V: The previous discussion concerned situations where the durable does not depreciate. If the durable depreciates rapidly, then there will be very frequent purchases. The first column of Table V gives various annual depreciation rates of the durable good from $\alpha = 0$ to $\alpha = .10$. Note that the average fraction of wealth invested in the risky asset becomes very close to its $\lambda = 0$ value when α is large. Further, it should be clear that consumption will be a more responsive function of wealth when rapid depreciation causes new purchases to occur very frequently.

5. CONCLUSIONS

In the model, it is optimal for consumers to have a target level y^* for the ratio $y = (Q - \lambda K)/K$ of liquid wealth to durable size. Further, there are two numbers y_1 and y_2 such that only if $y \leqslant y_1$ or $y \geqslant y_2$, will the consumer sell his current durable K to return y to the level y^*, by purchasing a new durable $K^*(Q)$ which satisfies $y^* = Q/K^*(Q) - \lambda$. We show that small costs of changing consumption levels will lead consumption to be insensitive to wealth for very long periods of time.

TABLE V

$\mu = 0.059,\ A = 1.1,\ \delta = 0.01,\ \sigma = 0.22,\ r_f = 0.01,\ \lambda = 0.05$

α	$(\bar{y}_1, \bar{y}^*, \bar{y}_2)$	(Years) $E\tau$	$\dfrac{E(X/Q)}{E(\tau)}$	$\dfrac{E\left(\frac{X}{Q-\lambda K}\right)}{E(\tau)}$	$\lambda = 0$ X/Q	Prob. (buy-down)	Rate of Wealth Drift at $\bar{y}^*$
.00	(0.26, 1.08, 3.16)	22.39	1.015	1.069	1.108	.121	.0622
.01	(0.54, 1.86, 4.90)	16.32	1.048	1.078	1.108	.114	.0714
.02	(0.84, 2.59, 6.46)	13.26	1.063	1.083	1.108	.099	.0811
.03	(1.14, 3.28, 7.93)	11.34	1.071	1.087	1.108	.084	.0908
.04	(1.46, 3.94, 9.32)	9.92	1.077	1.090	1.108	.071	.1006
.05	(1.77, 4.60, 10.67)	8.84	1.082	1.093	1.108	.059	.1105
.06	(2.08, 5.23, 12.01)	8.00	1.085	1.095	1.108	.049	.1203
.07	(2.40, 5.85, 13.29)	7.28	1.088	1.096	1.108	.041	.1302
.08	(2.73, 6.47, 14.55)	6.66	1.090	1.098	1.108	.034	.140
.09	(3.06, 7.07, 15.83)	6.16	1.092	1.099	1.108	.029	.150
.10	(3.37, 7.67, 17.09)	5.73	1.094	1.100	1.108	.023	.160

In addition, transactions costs for altering consumption do *not* change the conclusion that it is optimal for the consumer to hold a mean-variance efficient portfolio at all times. As a result, the standard capital asset pricing model (CAPM) holds for the consumer, but if his consumption flows derive from a stock of durables, then the consumption based capital asset price model (CCAPM) will not hold for the consumer.

The above remarks do not state that (1) the CCAPM will *not* hold in general equilibrium after aggregating over consumers or that (2) the CAPM *will* hold in general equilibrium. In Section 3, for illustrative purposes, we outlined a general equilibrium model with stochastic constant returns to scale in which, if all consumers are identical, then indeed the CCAPM will fail and the CAPM will hold. However, we have no reason to believe that this is an accurate model of reality.

We have not met with success in solving a general equilibrium model where the assets are in a fixed supply, and a production function is specified for durable goods, so that the price process is endogenous. Even with an exogenous price process as in Section 3, if there are many consumers the stationary distributions of the aggregate demand for new durables are quite difficult to characterize. For example, if all consumers have the same preferences, then they will choose the same barriers y_1 and y_2, and each of the consumers' state variables y will follow the same stochastic differential equation (2.10). If all consumers have the same y at time 0, then they will cross the barriers at the same time. This gives a set of stationary distributions where all consumers are alike. On the other hand, if there is a continuous cross-sectional distribution of y at time 0, then there will be something approximating a flow of arrivals at the barriers. This will generate an aggregate demand for a flow of durable goods production. Unfortunately, we are unable to characterize the long run cross-sectional distribution of y which evolves from a continuous cross-section at time 0.

One interesting fact is that starting from *any* cross-sectional distribution of consumer state variables at time 0, the cross-sectional distribution at time t (and thus the density of durable purchases) will depend on the history of the stock market between 0 and t. For example, if the stock market keeps going up, then consumers will be hitting the right-hand barrier y_2, and the cross-sectional distribution will shift away from y_1. This will lead to a spurt of new durable purchases. If the stock market then falls, there will be no new purchases or sales for a while. Of course, this implies that with many consumers the CCAPM will not hold.[8]

The above remarks motivate some interesting empirical implications. Suppose that the parameters of the model are such that after a new durable is purchased it is equally likely that the next purchase will be caused by (a) the stock market moving down so that y_1 is reached before y_2 or (b) the stock market moves up so that y_2 is reached before y_1. In case (a) the new purchase will be large, while in case (b) it will be small. That is, an implication of the model is that the new *size* of the durable purchased depends only on wealth, and to the extent that wealth changes are unpredictable, changes in the *size* of *new* durable purchases will be unpredictable. However, once a consumer makes a purchase, it is very unlikely that he will soon make another purchase.

The implications of the above two remarks at the macroeconomic level depend on the cross-sectional distribution of y and Q. Intuitively, if there has been a very large fraction of the population making new purchases (say because the stock market has risen to a very high level recently), then these same people will not soon be making new purchases, and thus the *number* of new durable purchases can be expected to fall. However, the *size* of the next purchase by each consumer who has just made a purchase is unpredictable. This will make it difficult to predict the change in the average *size* of new purchases. Therefore, by distinguishing the number of people making new purchases from the average size of each purchase, it appears to be possible to test predictions of this model which are absent from the standard model as well as from the quadratic adjustment cost model of the type studied in Bernanke (1985) and the convex adjustment cost model of Eichenbaum and Hansen (1985). Similar ideas have been used by Bar-Ilan and Blinder (1986) to test an illiquid durable goods model against the standard "permanent income" (i.e. no transactions cost) model as tested by Hall (1978) and Flavin (1981). Clearly, further work needs to be done on the aggregation problem before such tests can be made precise.

Another area in which further work is needed is in modelling the problems associated with multiple types of consumption goods, each of which has a possibly different transactions cost. In assuming one type of consumption good we ignore the issue of substitution between low transactions cost and high

[8] Caplin and Spulber (1985) consider a model with fixed transactions costs where aggregation over consumers causes the model to behave in many ways like a representative agent model *without* transactions cost. Unlike ours, in the Caplin and Spulber model the cross-sectional distribution at time t is independent of the history of the random shocks between 0 and t because random shocks are always of the same sign so it is as if the cross-sectional distribution of characteristics revolves in one direction around a circle, always maintaining the same density at each point.

transactions cost goods, and in particular that such substitution might lead a change in wealth to cause an excessive movement in the purchases of low transactions cost goods.

In the extreme case where the utility for nondurables (i.e., no transactions cost goods) is separable from durables, then the CCAPM will hold for nondurables. However, it should be recognized that many goods which our national income accounts consider to be nondurables actually have a large "durable" component. This is obvious for categories such as clothing and shoes. However, many nondurables are used in almost fixed proportions with durables, and this creates a transaction cost not dissimilar to that studied here. For example, changing the level of food consumption may have transactions cost if, for example, one has to find different friends with whom to go to better restaurants, or learn about better foods to buy in grocery stores. (Some people even live in areas for which eating better food would require them to change jobs and move out of town.) Changing the level of gasoline, or electricity might require selling an automobile, home, or changing a living or work location.

It should also be noted that there are alternative approaches to explaining the failure of the CCAPM. In particular, Constantinides (1988) and Epstein and Zin (1987) relax the assumptions of intertemporal additive separability, and/or Von Neumann-Morgenstern preferences. The approaches complement ours. It may be that preferences over consumption flows look quite different from standard Von Neumann-Morgenstern additively separable preferences exactly because the flows are generated by complex stocks of durables. These durables can include not only houses and automobiles, but also the stock of good memories associated with vacations, dinners, friends, or relatives.

We also ignore the possibility that there are various types of durable goods which are purchased at staggered dates by a given consumer. In such a situation the consumer may be purchasing some durable good almost every month. It would be interesting to know the correlation between consumption service flows and stock returns in such a model.

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APPENDIX

For the sake of brevity, we only give the proofs of Theorem 2.1 and Theorem 3.4. The reader is referred to Grossman-Laroque (1987) for more details. Further, to simplify notation, we consider the case of a single risky asset with drift $\mu + r_f$ and an instantaneous variance σ^2.

PROOF OF THEOREM 2.1: We begin with the last statement. Take any trajectory (Q_t, K_t, X_t), $t \geqslant 0$, satisfying the constraints (2.3), (2.4), and (2.5). Then, by homogeneity, for any $\nu > 0$, $(\nu Q_t, \nu K_t, \nu X_t)$, $t \geqslant 0$, also satisfies the constraints. Since the instantaneous utility function is homogeneous of degree

a in K, $V(Q_0, K_0)$ is homogeneous of degree a in the initial conditions (Q_0, K_0). Note also that any trajectory satisfying the constraints for λ_0 also satisfies the constraints for $\lambda < \lambda_0$. Therefore $V(Q, K)$ is nonincreasing in λ. We proceed with finding lower and upper bounds for $V(Q, K)$.

1. It is always possible for the agent to sell immediately his current durable good, and to invest all his wealth $Q - \lambda K$ into a house to be kept for all $t \geqslant 0$, without undertaking any financial operation in the future ($Q_t = K_t$, $X_t = 0$ for all $t \geqslant 0$). If $a > 0$, this gives:

$$V(Q, K) \geqslant \frac{(Q - \lambda K)^a}{a(\delta + a\alpha)},$$

and $\nu_2 \equiv (\delta + a\alpha)^{-1}$. If $a < 0$, then consider the strategy of setting $X_t \equiv 0$ and buying a new house every year. If $r > 0$ and $\lambda < 1$, then this strategy will do at least as well when $r = 0$ and $\lambda = 1$. Thus, assume that $Q - \lambda K$ is moved to an economy where henceforth $r = 0$ and $\lambda = 1$. In year n a fraction g_n of $Q - \lambda K$ is invested in housing,

$$g_n = e^{\delta n/2a}(1 - e^{\delta/2a}).$$

It is easily verified that $g_n \in (0,1)$ and $\sum_{n=0}^{\infty} g_n = 1$, since $a < 0$. It follows that setting $K(n) = (Q - \lambda K)g_n$ is feasible and yields a discounted utility of

$$\sum_{n=0}^{\infty} \int_n^{n+1} e^{-\delta t} u\big(g_n(Q - \lambda K)e^{-\alpha(t-n)}\big)\, dt = \frac{(Q - \lambda K)^a}{a}\nu_2, \qquad \text{where}$$

$$\nu_2 \equiv \frac{(1 - e^{\delta/2a})^a}{1 - e^{-\delta/2}} \cdot \frac{1 - e^{-\bar{\delta}}}{\bar{\delta}}.$$

2. When $\lambda = 0$, V does not depend on K, and Q becomes the only state variable, while K and X are control variables. It is thus identical to the standard model of consumption and portfolio choice as in Merton (1969). Since (2.1) and (2.3) are linear in (Q, K, X) and since

$$V(Q) = E\int_0^{\infty} e^{-\delta t K_t^a / a}\, dt,$$

the solution, if it exists, must be, by homogeneity, of the form $V(Q) = \nu Q^a$, where $\nu = V(1)$. This model has been studied by Karatzas, et al. (1986, p. 290) who show that (our) $\beta > 0$ is sufficient to ensure that ν is finite, where it is crucial to note that we require $Q_t \geqslant 0$, and if $Q_t = 0$, then the process is stopped and the consumer gets $u(0)$. The Bellman differential equation is

$$\text{(A.1)} \qquad \sup_{(K, X)} \left[\frac{\sigma^2 X^2}{2} V''(Q) + \big(r_F Q - (r_F + \alpha)K + \mu X\big)V'(Q) - \delta V(Q) + \frac{K^a}{a}\right] = 0.$$

Substituting $V(Q) = \nu Q^a$ into (A.1) yields optimal solutions

$$X = \frac{\mu Q}{\sigma^2(1 - a)}, \qquad K = \big[a(r_F + \alpha)\nu\big]^{1/a - 1} Q,$$

and a formula for ν:

$$\text{(A.2)} \qquad (r_F + \alpha)(1 - a)\big[a(r_F + \alpha)\nu\big]^{1/a - 1} = \beta.$$

If $\beta > 0$, then (A.2) can be solved for ν, and thus there is a solution to (A.1).

Since the consumer is obviously better off when $\lambda = 0$ than when $\lambda > 0$, it follows that $\nu Q^a \geqslant V(Q, K)$. *Q.E.D.*

Two preliminary lemmas are useful for the proof of Theorem 3.4.

LEMMA 1: *Consider the stochastic differential equation*

$$dy = x(y)\, db + r(y + \lambda - 1)\, dt$$

where $x(y)$ is a solution of (3.11) *satisfying* (3.14a), *for some initial condition y_0, in* (y_1, y_2).

If $y_0 < 1-\lambda$, then y reaches y_1 in finite time with a strictly positive probability.
If $y_0 > 1-\lambda$, then y reaches y_2 in finite time with a strictly positive probability.
If $y_0 = 1-\lambda$ and $x(y_0) > 0$, then y reaches either y_1 or y_2 in finite time with a strictly positive probability.

Proof of Lemma 1: It follows from a standard property of regular diffusion processes that if $y_0 < 1-\lambda$, there is an open interval containing y_0, say $(y_1, \bar{y}_1)$, on which $x(y) > \varepsilon > 0$. Therefore, the probability that y hits the boundaries of this interval in finite time is equal to one, and the probability that y hits y_1 before $\bar{y}_1$ is strictly positive (adapt, e.g., Karlin and Taylor (1981, Chapter 15)). A similar argument applies for y_2 when $y_0 > 1-\lambda$. *Q.E.D.*

To each M, $M < (1-\lambda)^{-a}\nu$, one can associate by Theorem 3.3 an interval $(y_1(M), y_2(M))$ in which the solution $h(y; M)$ of (2.9) under (2.10) and (2.11) is such that $h(y; M) > My^a$. Note that $y_1(M)$, $y_2(M)$, $h(y; M)$ are implicitly functions of λ. We write $h_\lambda(y; M)$ to make this dependence explicit. To go back to the original problem, an intermediate step is to study how $h_\lambda(y; M)/M$ varies with λ, holding M fixed.

Lemma 2: *For $M < (1-\lambda)^{-a}\nu$, and $a > 0$ (resp. $a < 0$), if y is in $(y_1(M), y_2(M))$ and $y \neq 1-\lambda$, then $h_\lambda(y; M)/M$ is strictly decreasing (resp. increasing) in M and strictly increasing (resp. decreasing) in λ. Furthermore, if y is in $(y_1(M)+\lambda, y_2(M)+\lambda)$ and $y \neq 1-\lambda$, then $h_\lambda(y-\lambda; M)/M$ is strictly decreasing (resp. increasing) in λ. Both of the above statements also hold at $y = 1-\lambda$ if $x(1-\lambda) \neq 0$.*

Proof of Lemma 2: We give the proof when a and M are positive. The case $a < 0$, $M < 0$ can be handled along the same lines. When $M > 0$, we have:

$$\frac{1}{M} h_\lambda(y; M) = \sup_{\tau, (x_t)} E\left[\int_0^\tau \frac{e^{-\bar{\delta} t}}{aM} dt + e^{-\bar{\delta}\tau} y_\tau^a\right],$$
$$dy_t = x_t\, db + r(y_t + \lambda - 1)\, dt,$$

where the supremum is taken over all the nonanticipatory strategies $\tau, (x_t)$.

Given y in $(y_1(M), y_2(M))$, and any random event w, let $(x_t(w), \tau^y(w))$ be the optimal nonanticipatory strategy followed by the consumer. $\tau^y(w)$ is strictly positive with probability 1. Therefore, if M is decreased to M', applying the same strategy leads to a higher value than $h_\lambda(y; M)/M$, since the first term in the expectation is strictly increasing and the second term is unchanged. Therefore:

$$h_\lambda(y; M')/M' > h_\lambda(y; M)/M.$$

Similarly, consider $\lambda' > \lambda$. For all y, $y \neq 1-\lambda$, if $x(1-\lambda) = 0$, by Lemma 1, $\tau^y(w)$ is finite with positive probability. Consider an event w such that $\tau^y(w) < +\infty$. Then applying the λ optimal strategy when λ' prevails gives a higher value to y_τ, and therefore the desired result. This shows that $h_\lambda(y, M)/M$ is strictly increasing for all y, $y \neq 1-\lambda$, if $x(1-\lambda) = 0$.

Finally, we study the function $h_\lambda(y-\lambda; M)$. By definition:

$$\text{(A.3)} \qquad \frac{1}{M} h_\lambda(y-\lambda; M) = \sup_{\tau, (x_t)} E\left[\int_0^\tau \frac{e^{-\bar{\delta} t}}{aM} dt + e^{-\bar{\delta}\tau} y_\tau^a\right],$$
$$dy_t = x_t\, db + r(y_t + \lambda - 1)\, dt,$$
$$y_0 = y - \lambda.$$

When λ decreases, the initial condition increases, but $(y_0 + \lambda)$ stays constant and, since the differential equation can be rewritten

$$d(y_t + \lambda) = x_t\, db + r(y_t + \lambda - 1)\, dt,$$

for any event w, $y_t(w, \lambda) + \lambda$ is equal to $y_t(w, \lambda') + \lambda'$. Therefore, for all y, such that $y \neq 1-\lambda$ if $x(1-\lambda) = 0$, we have for the λ optimal strategy starting at y:

$$\lambda' < \lambda \qquad \text{implies} \qquad y_\tau(w, \lambda') = y_\tau(w, \lambda) + \lambda - \lambda' > y_\tau(w, \lambda).$$

Consequently

$$\frac{1}{M}h_{\lambda'}(y-\lambda'; M) > \frac{1}{M}h_{\lambda}(y-\lambda; M),$$

since $\tau^{y}(w)$ is finite with strictly positive probability by Lemma 1. *Q.E.D.*

Finally, we have to determine how M varies with λ through equation (2.12), which can be rewritten

$$1 = \sup_{z} z^{-a} h_{\lambda}(z-\lambda; M)/M \tag{A.4}$$

when a is positive (sup is replaced with inf when $a<0$).

PROOF OF THEOREM 3.4: Let

$$H(\lambda, M) = \sup_{z} z^{-a} h_{\lambda}(z-\lambda; M)/M,$$

the sup being taken in some large enough compact interval $\{z \mid y^h+\lambda \geqslant z \geqslant \lambda\}$. When $a>0$, from (A.3), $H(\lambda, M)$ tends to $+\infty$ when M tends to zero. From Theorem 3.3, it is equal to $(y^h/(y^h+\lambda))^a$ for $M=(1-\lambda)^{-a}\nu$. It is continuous in M by Lemma 1 of Grossman-Laroque (1987) and by the theorem of the maximum. Furthermore, by Lemma 2, if the maximizer z^* is different from 1 or $x(1-\lambda)>0$, H is strictly decreasing in M and in λ. Therefore, for λ fixed, the solution in M of equation (A.4),

$$H(\lambda, M) = 1,$$

is unique and a strictly decreasing function of λ. This holds for any finite y^h (see Grossman-Laroque (1987) for the limit argument that allows $y_2 = +\infty$). If $z^*=1$ and $x(1-\lambda)=0$, $H(\lambda, M) = 1/a\bar{\delta}M$ and therefore $M = 1/\bar{\delta}a$.

By construction, again for $a>0$:

$$h_{\lambda}(y, M_{\lambda})/M_{\lambda} > y^a \quad \text{on} \quad (y_1, y_2).$$

By Lemma 2, the left-hand side is strictly increasing in λ (except perhaps at $y=1-\lambda$), which proves that the interval (y_1, y_2) increases with λ. *Q.E.D.*

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