Recursive Utility with Optimal Jump Intensity

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1 Basic recursion

The basic Kreps-Porteus recursion is

$$V_t = \left[(1-\beta_\Delta)\left(C_t^\alpha D_t^{1-\alpha}\right)^{1-\rho} + \beta_\Delta R_t^{1-\rho}\right]^{\frac{1}{1-\rho}}$$

where R_t is

$$R_t = \left(E_t\, V_{t+\Delta}^{1-\gamma}\right)^{\frac{1}{1-\gamma}}.$$

Notice that V_t is homogeneous of degree 1 in D_t.

$$\frac{V_t}{D_t} = \left[(1-\beta_\Delta)\left(\frac{C_t}{D_t}\right)^{\alpha(1-\rho)} + \beta_\Delta\left(\frac{R_t}{D_t}\right)^{1-\rho}\right]^{\frac{1}{1-\rho}}$$

with $c = C/D$ and

$$r_t = \frac{R_t}{D_t} = \left(E_t\left(\frac{V_{t+\Delta}}{D_{t+\Delta}}\frac{D_{t+\Delta}}{D_t}\right)^{1-\gamma}\right)^{\frac{1}{1-\gamma}}.$$

Writing $v = V/D$ we can rewrite the recursion as

$$1 = \left[(1-\beta_\Delta)\left(\frac{c_t^\alpha}{v_t}\right)^{(1-\rho)} + \beta_\Delta\left(\frac{r_t}{v_t}\right)^{1-\rho}\right]^{\frac{1}{1-\rho}}$$

and in log form

$$0 = \frac{1}{1-\rho}\log\left[(1-\beta_\Delta)\exp\left[(1-\rho)\left(\alpha\hat{c}_t - \hat{v}_t\right)\right] + \beta_\Delta\exp\left[(1-\rho)\left(\hat{r}_t - \hat{v}_t\right)\right]\right].$$

Develop the continuation value recursion

$$\hat{r}_t - \hat{v}_t = \frac{1}{1-\gamma}\log E_t\,\exp\left[(1-\gamma)\left(\hat{v}_{t+\Delta} - \hat{v}_t + \hat{d}_{t+\Delta} - \hat{d}_t\right)\right].$$

v only depends on time through a Brownian with jumps. Therefore v and $\hat{v}$ do too.

2 Optimal Jump Intensities.

Let

$$d\hat{v}_t = \hat{\mu}_t^v dt + \hat{\sigma}_t^v dZ_t + \left(\hat{v}_t^* - \psi - \hat{v}_t\right) dN_t.$$

where ψ is a scalar random variable with CDF F denoting the utility cost of moving. Instead d_t is a jump process following the same counter as $\hat{v}$ but with a different jump size

$$\mathrm{d}\hat{d}_t = \left(\hat{d}_t^* - \hat{d}_t\right) dN_t.$$

Finally, the recursion can be written as

$$0 = \delta \frac{\left(\frac{c_t^\alpha}{v_t}\right)^{1-\rho} - 1}{1-\rho} + \hat{\mu}_t^v + (1-\gamma)\frac{\left(\hat{\sigma}_t^v\right)^2}{2} + \frac{\lambda_t}{1-\gamma}\left(E_\psi \exp\left[(1-\gamma)\left(\hat{v}_t^* - \psi - \hat{v}_t + \hat{d}_t^* - \hat{d}_t\right)\right] - 1\right)$$

as $\rho \to 0$

$$\delta\hat{v}_t = \delta\alpha\hat{c}_t + \hat{\mu}_t^v + (1-\gamma)\frac{\left(\hat{\sigma}_t^v\right)^2}{2} + \frac{\lambda_t}{1-\gamma}\left(E_\psi \exp\left[(1-\gamma)\left(\hat{v}_t^* - \psi - \hat{v}_t + \hat{d}_t^* - \hat{d}_t\right)\right] - 1\right)$$

and as $\gamma \to 1$

$$\delta\hat{v}_t = \delta\alpha\hat{c}_t + \hat{\mu}_t^v + \lambda_t\left(\hat{v}_t^* - E_\psi\psi - \hat{v}_t + \hat{d}_t^* - \hat{d}_t\right)$$

Therefore the decision to adjust is optimally taken only if the utility shock ψ satisfies

$$y(\hat{v}) \equiv \hat{v}_t^* - \hat{v}_t + \hat{d}_t^* - \hat{d}_t > \psi$$

or equivalently

$$\begin{aligned} D_t^* v_t^* &\geq \exp(\psi) D_t v_t \\ V_t^* &\geq \exp(\psi) V_t \end{aligned}$$

so ψ acts as a positive multiplicative utility shock to the current durable stock. In a sequence problem the decision would display as

$$\max\left\{V_t, \exp(-\psi) V_t^*\right\}$$

so it can also be interpreted as a negative utility shock to moving.

Operational HJB: From the equivalence of robustness to misspecification and recursive utility (with $\rho = 1$) we can write the HJB as a result of the following problem. Let $d\hat{v} = \hat{\mu}_t^v \hat{v} dt + \hat{\sigma}_t^v dZ_t + \left(\hat{v}_t^* - \psi - \hat{v}_t\right) dN_t$, then the HJB satisfies

$$
\begin{aligned}
\delta\hat{v} = & \max_{c,\theta} \alpha\hat{c} + \mathcal{A}\hat{v} + \min_{h}\left\{\mathcal{B}\hat{v}h + \frac{1}{2}\frac{1}{\gamma-1}h^2\right\} \\
& + \kappa F\left(y\left(\hat{v}\right)\right) E_\psi\left[\min_g g\left\{y\left(\hat{v}\right)-\psi\right\} + \frac{1}{\gamma-1}\left(1-g+g\log g\right)|\psi < y\left(\hat{v}\right)\right]
\end{aligned}
$$

where $y\left(\hat{v}\right) = \hat{v}_t^* - \hat{v}_t + \hat{d}_t^* - \hat{d}_t$ are the log gains from adjusting and $\kappa F\left(y\left(\hat{v}\right)\right) = \kappa P\left(\psi \leq y\left(\hat{v}\right)\right)$ is the optimal jump intensity at the current state. This incorporates the effects of optimal jump intensities.

$\hat{v}$ diffusion. The diffusion for $\hat{v}$ is

$$
d\hat{v} = \mathcal{A}\hat{v}dt + \mathcal{B}\hat{v}dZ_t + \left(\hat{v}^* - \hat{v}\right)dN_t
$$

where

$$
\mathcal{A}\hat{v} = \left[rw + r^e\theta w - c - \left(1-\epsilon\right)\left(r+s\right)\right]\hat{v}' + \frac{1}{2}\left(\sigma\theta w\right)^2\hat{v}''
$$

and

$$
\mathcal{B}\hat{v} = \sigma\theta w\hat{v}'
$$

Therefore

$$
\begin{aligned}
\delta\hat{v} = & \alpha\log c^* - \kappa F\left(y\left(\hat{v}\right)\right)E_\psi\left[g^*\left(\psi\right)\psi|\psi < y\left(\hat{v}\right)\right] \\
& + \frac{1}{2}\frac{1}{\gamma-1}\left(h^*\right)^2 + \frac{1}{\gamma-1}E_\psi\left[\left(1-g^*\left(\psi\right)+g^*\left(\psi\right)\log g^*\left(\psi\right)\right)|\psi < y\left(\hat{v}\right)\right] \\
& + \left(\mu_w\left(w,c^*,\theta^*\right) + \sigma_w\left(w,\theta^*\right)h^*\right)\hat{v}' + \frac{\left[\sigma_w\left(w,\theta^*\right)\right]^2}{2}\hat{v}'' \\
& + \underbrace{\kappa F\left(y\left(\hat{v}\right)\right)E_\psi\left[g^*\left(\psi\right)|\psi < y\left(\hat{v}\right)\right]}_{\lambda_w} y\left(\hat{v}\right)
\end{aligned}
$$

closed form solutions for h and g

$$
\begin{aligned}
h^* &= \left(1-\gamma\right)\sigma\theta w v' \\
g^*\left(\psi\right) &= \exp\left[\left(1-\gamma\right)\left(y-\psi\right)\right]
\end{aligned}
$$

Marginal valuation.

$$
\begin{aligned}
\delta\hat{u} = & \mathcal{A}\hat{u} + \left(\frac{\partial}{\partial w}\mu_w\left(w,c^*,\theta^*\right) + \frac{\partial}{\partial w}\sigma_w\left(w,\theta^*\right)h^*\right)\hat{u} + \sigma_w\left(w,\theta^*\right)\frac{\partial}{\partial w}\sigma_w\left(w,\theta^*\right)\hat{u}' + \mathcal{B}\hat{u}h^* + \frac{\partial}{\partial w}\sigma_w\left(w,\theta^*\right)\hat{u}h^* \\
& + \kappa F\left(y\left(\hat{v}\right)\right)E_\psi\left[g^*\left(\psi\right)\right]\left(\frac{1}{w-f+\epsilon} - \hat{u}\right)
\end{aligned}
$$

then

$$
\begin{aligned}
\left[\delta+\mathcal{J}(w)\,\overline{g}(w)\right]\hat{u} = {}& \mathcal{A}\hat{u}+\mathcal{B}\hat{u}h^{*}+\frac{\partial}{\partial w}\left(\mu_{w}\left(w,c^{*},\theta^{*}\right)+\sigma_{w}\left(w,\theta^{*}\right)h^{*}\right)\hat{u}+\frac{1}{2}\frac{\partial}{\partial w}\left[\sigma_{w}\left(w,\theta^{*}\right)\right]^{2}\hat{u}' \\
& +\mathcal{J}(w)\,\overline{g}(w)\,\frac{1}{w-f+\epsilon}
\end{aligned}
$$

then

$$
\begin{aligned}
\left[\delta+\mathcal{J}(w)\,\overline{g}(w)\right]\hat{u} = {}& \mathcal{A}\hat{u}+\mathcal{B}\hat{u}h^{*}+\left(r+r^{e}\theta^{*}+\theta^{*}\sigma h^{*}\right)\hat{u}+w\left(\sigma\theta^{*}\right)^{2}\hat{u}' \\
& +\mathcal{J}(w)\,\overline{g}(w)\,\frac{1}{w-f+\epsilon}
\end{aligned}
$$

then

$$
\left[\delta-r+\mathcal{J}(w)\,\overline{g}(w)\right]\hat{u} = \mathcal{J}(w)\,\overline{g}(w)\,\frac{1}{w-f+\epsilon}+\mathcal{A}\hat{u}+\mathcal{B}\hat{u}h^{*}
$$

If jump adjustment is just about ψ and not $\mathcal{J}$ itself then

$$
\left[\delta-r+\mathcal{J}(w)\right]\hat{u} = \mathcal{J}(w)\,\frac{1}{w-f+\epsilon}+\mathcal{A}\hat{u}+\mathcal{B}\hat{u}h^{*}
$$

3 Marginal Valuation to Recover F

Dear Lars, thanks to the robustness concerns approach I think that I can produce an inverse mapping between jump intensities and the primitives of switching costs $\{\kappa, F\}$. This is of interest because it makes the estimation algorithm for the primitives much simpler.

Recall that Aleksei solves this inverse mapping problem using the marginal valuation in an environment with constant drift and volatility and scalar continuation values after jumps (state-independent continuation values).

My framework is considerably harder, it has drift and volatility control and the jump continuation value $V^{\ell}(w)$ is state-dependent. An inverse mapping cannot be found with CRRA utility. However, with a particular case of a robustness concerns problem an inverse mapping can be found. The insight is that in this case, the relevant jump value is the jump of logarithms and this simplifies tractable marginal valuations a lot! The following short note shows this carefully.

I think this might be a potentially nice contribution too.

3.1 Value function

Let $y(w)$ denote the value of adjusting given by

$$y(w) \equiv \hat{v}(w^*) - \hat{v}(w) + \log(w - f + \epsilon) - \log(w^* + \epsilon)$$

where f, ϵ are parameters and w^* is the optimal reinjection point satisfying

$$w^* = \text{argmax}_w \left[\hat{v}(w) - \log(w + \epsilon)\right].$$

Let μ_w, σ_w be the drift of the wealth to durable ratio

$$\mu_w(w, c, \theta) = \left[rw + r^e \theta w - c - (1-\epsilon)(r+s)\right], \qquad \sigma_w(w, \theta) = \sigma w \theta$$

where r^e is the risk premium, θ is the risky portfolio share, c is non-durable consumption and $(1-\epsilon)(r+s)$ is the mortgage payment term.

Then the value function satisfies the HJB equation

$$\begin{aligned}\delta \hat{v}(w) = {} & \alpha \log c^* + \left(\mu_w(w, c^*, \theta^*) + \sigma_w(w, \theta^*) h^*\right) \hat{v}'(w) + \frac{\left[\sigma_w(w, \theta^*)\right]^2}{2} \hat{v}''(w) + \frac{1}{2}\frac{1}{\gamma - 1}(h^*)^2 \\ & + \mathcal{J}^*(w) E_\psi \left[g^*(\psi)\{y(w) - \psi\} + \frac{1}{\gamma - 1}\left(1 - g^*(\psi) + g^*(\psi)\log g^*(\psi)\right) | \psi < y(w)\right]\end{aligned}$$

where $c^*, \theta^*, h^*, g^*, \mathcal{J}^*$ are policy functions and $\mathcal{J}(w) = \kappa F(y(w))$. FOC are

$$\begin{aligned}
c: &\quad \frac{\alpha}{c^*} - \hat{v}'(w) = 0 \\
h: &\quad h^* = (1-\gamma)\sigma\theta w v' \\
g: &\quad g^*(\psi) = \exp\left[(1-\gamma)(y - \psi)\right] \\
\theta: &\quad (r^e w + \sigma w h^*)\hat{v}'(w) + (\sigma w)^2 \theta^* \hat{v}''(w) = 0
\end{aligned}$$

multiply the FOC for θ by θ^*/w

$$\left(r^e\theta+\sigma h^*\theta\right)\hat{v}'\left(w\right)+\left(\sigma\theta^*\right)^2 w\hat{v}''\left(w\right)=0, \tag{1}$$

this computation will become handy when computing marginal valuations because it is related to the derivative with respect to w.

3.2 Marginal Valuation

Take the derivative with respect to w

$$\begin{aligned}\delta\hat{v}'\left(w\right)=&\left(\mu_w\left(w,c^*,\theta^*\right)+\sigma_w\left(w,\theta^*\right)h^*\right)\hat{v}''\left(w\right)+\frac{\left[\sigma_w\left(w,\theta^*\right)\right]^2}{2}\hat{v}'''\left(w\right)\\&+\left(r+r^e\theta+\sigma\theta^*h^*\right)\hat{v}'\left(w\right)+w\left(\theta^*\sigma\right)^2\hat{v}''\left(w\right)\\&+\mathcal{J}^*\left(w\right)E_\psi\left[g^*\left(\psi\right)\left(-\hat{v}'\left(w\right)+\frac{1}{w-f+\epsilon}\right)|\psi<y\left(w\right)\right]\end{aligned}$$

where the derivatives with respect to policies $c^*,\theta^*,h^*,g^*,\mathcal{J}^*$ do not appear because of the Envelope Theorem. Using equation 1 we can simplify the second line of the marginal value function and obtain

$$\begin{aligned}\delta\hat{v}'\left(w\right)=&\left(\mu_w\left(w,c^*,\theta^*\right)+\sigma_w\left(w,\theta^*\right)h^*\right)\hat{v}''\left(w\right)+\frac{\left[\sigma_w\left(w,\theta^*\right)\right]^2}{2}\hat{v}'''\left(w\right)\\&+r\hat{v}'\left(w\right)\\&+\mathcal{J}^*\left(w\right)E_\psi\left[g^*\left(\psi\right)\left(-\hat{v}'\left(w\right)+\frac{1}{w-f+\epsilon}\right)|\psi<y\left(w\right)\right]\end{aligned}$$

Finally, let $\hat{v}'\equiv u$ and $\overline{g}\left(y\left(w\right)\right)\equiv E_\psi\left[g^*\left(\psi\right)|\psi<y\left(w\right)\right]$ then rearranging and following Hansen and Souganidis (2025), we have

$$\begin{aligned}\left(\delta-r+\mathcal{J}^*\left(w\right)\overline{g}\left(y\left(w\right)\right)\right)u\left(w\right)=&\,\mathcal{J}^*\left(w\right)\overline{g}\left(y\left(w\right)\right)\frac{1}{w-f+\epsilon}\\&+\left(\mu_w\left(w,c^*,\theta^*\right)+\sigma_w\left(w,\theta^*\right)h^*\right)u'\left(w\right)+\frac{\left[\sigma_w\left(w,\theta^*\right)\right]^2}{2}u''\left(w\right).\end{aligned}$$

It is important to note that there is no contribution from jump sensitivities because in my model jump intensities are optimally chosen, so their derivatives disappear because of the Envelope Theorem. This is an insight of problems with the so called Impulse Hamiltonian (Alvarez, Lippi and Souganidis (2023)).

Quasi-Inverse Mapping. I call the following a quasi-inverse mapping because the mapping between jump intensities and primitives for costs κ, F depends on $\mathcal{J}^*\left(w\right)$ and a given jump distortion function $\overline{g}$. However, this is useful since it will illustrate the procedure for the actual inverse mapping below.

Given two functions $\mathcal{J}^*\left(w\right)$, $\overline{g}\left(y\left(w\right)\right)$ we can compute the marginal values $u\left(w\right)$ through finite differences because a guess for u is all what is needed to compute optimal policies c^*, θ^*, h^*. With the marginal value u one can recover $w^* = \operatorname{argmax}_w \left[\hat{v}(w) - \log(w+\epsilon)\right]$ using a marginal condition

$$u(w^*) - \frac{1}{w^* + \epsilon} = 0.$$

With w^* at hand one can compute $y(w)$ the value of adjusting

$$y(w) \equiv \hat{v}(w^*) - \hat{v}(w) + \log(w - f + \epsilon) - \log(w^* + \epsilon) = \int_w^{w^*} u(w)\, dw + \log(w - f + \epsilon) - \log(w^* + \epsilon)$$

and invert the jump intensity

$$\mathcal{J}^*(w) = \kappa F(y(w))$$

to recover F, κ.

Actual Inverse Mapping. The procedure above does not create an inverse mapping because of the need to guess two functions at the same time $\mathcal{J}^*(w)$, $\overline{g}(y(w))$. In order to create an actual inverse mapping between $\mathcal{J}^*(w)$ and κ, F, I can choose to solve a **different problem** in which $g^*(\psi)$ distortions are misspeficication concerns over the distribution F of costs ψ and not over the jump intensity $\mathcal{J}^*(w)$. Namely, choosing g such that $\overline{g} = E_\psi\left[g^*(\psi) \mid \psi < y(w)\right] = 1$. In this case the marginal valuation equation is simply

$$(\delta - r + \mathcal{J}^*(w))\, u(w) = \mathcal{J}^*(w) \frac{1}{w - f + \epsilon} + \left(\mu_w(w, c^*, \theta^*) + \sigma_w(w, \theta^*)\, h^*\right) u'(w) + \frac{\left[\sigma_w(w, \theta^*)\right]^2}{2} u''(w)$$

and it can be solved without any input from F, κ given $\mathcal{J}^*$.