Bounding the Integrality Gap of Transversal LPs

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1 Introduction

Recall that a graph H is a *minor* of G, if we can obtain H from G through a sequence of edge contractions and deletions, and vertex deletions. In the H-*transversal* problem (HTP) one is given a graph $G = (V, E)$, non-negative costs c_v, for all $v \in V$, and a graph H. The goal is to find a set $S \subseteq V$ such that $G[V \setminus S]$ has no H-minor. In the following, we let $\mathcal{H}$ be the set of vertex subsets of V whose induced subgraphs contain an H-minor; i.e., $S \in \mathcal{H}$ iff $G[S]$ has an H-minor. Consider the following natural LP relaxation of (HTP):

$$\begin{aligned} \min \quad & c^T x && \text{(P}_{\text{HTP}}\text{)} \\ \text{s.t.} \quad & x(S) \geq 1 \quad \forall S \in \mathcal{H} \\ & x \geq \mathbb{0}. \end{aligned}$$

It is not hard to see that the integrality gap of the above LP can be large, even in special cases. For example, it is large when H is a planar graph with at least one cycle as was argued in [5]. To see this, let G be an n-vertex graph with girth $\Omega(\log n)$ and treewidth $\Omega(n)$ (e.g., certain Ramanujan graphs [4]), and let H be a triangle. Letting $x_v = 1/\log n$ for all $v \in V$ is easily seen to yield a feasible solution for (P_{HTP}) of value $n/\log n$. On the other hand, any integral solution for the given HTP instance has cost $\Omega(n)$ since H-minor free graphs have treewidth $O(1)$ [5].

2 Hitting even cycles in minor-closed graphs

Closely related to the above is the *even-cycle transversal* problem (ECT) where we are given a graph $G = (V, E)$, vertex costs $c_v \geq 0$ for all $v \in V$, and where the goal is to find a min-cost set $S \subseteq V$ such that $G[V \setminus S]$ has no even cycles. Let $\mathcal{C}$ be the set of even cycles in G. In the following, we will sometimes use $C \in \mathcal{C}$ for the set of vertices of the corresponding cycle; the meaning will be clear from the context. The natural LP relaxation of ECT and its dual as follows:

$$\begin{aligned} \min \quad & c^T x && \text{(P}_{\text{ECT}}\text{)} \\ \text{s.t.} \quad & x(C) \geq 1 \quad \forall C \in \mathcal{C} \\ & x \geq \mathbb{0}. \end{aligned}$$

$$\begin{aligned} \max \quad & \mathbb{1}^T y && \text{(D}_{\text{ECT}}\text{)} \\ \text{s.t.} \quad & \sum_{C \in \mathcal{C}, v \in C} y_C \leq c_v \quad \forall v \in V \\ & y \geq \mathbb{0}. \end{aligned}$$

Similar to the previous argument, we can show that (P_{ECT}) has an integrality gap of $\Omega(\log n)$ in general. We suspect, however, that the LP has integrality gap $O(1)$ when G is from a minor-closed class of graphs. We will now show this for the special case where $c = \mathbb{1}$. We need the following result due to Fomin, Saurabh, and Thilikos [2].

Theorem 2.1. *Let $\mathcal{G}$ be a proper minor-closed graph class and let H be a planar graph. Then there is*

- *a feasible solution $U \subseteq V$ to HTP for G and H, and*

- *a collection $\mathcal{U}$ of pairwise disjoint vertex subsets of V each of which induces a subgraph of G with an H-minor,*

and $|U| \leq c|\mathcal{U}|$ for some constant $c(\mathcal{G}, H)$.

We apply Theorem 2.1 to the given graph G from some minor-closed graph class (e.g., planar), and choose H as the graph on two vertices with three parallel edges. The theorem provides us with disjoint sets of vertices $D_1, \ldots, D_p$ such that H is a minor in $G[D_i]$, for all $i \in [p]$, and a set U of vertices such that $G[V \setminus U]$ has no H minor. We furthermore know that $|H| \leq cp$ for some constant c.

Note that $G[D_i]$ contains an H-minor, and hence there are vertices v_i and u_i in D_i, and $G[D_i]$ contains three internally vertex-disjoint v_i, u_i-paths. Clearly, some two of these paths together form an even cycle C_i. We conclude that letting $y_{C_i} = 1$ for all $i \in [p]$ yields a feasible solution for ($\mathrm{D}_{\mathrm{ECT}}$). It is clear that U may not be a feasible even cycle transversal in G.

Recall that a *block* in G is an inclusion-wise maximal subgraph that is either a single vertex, a bridge-edge, or a 2-vertex connected subgraph. We then note that $G' := G[V \setminus U]$ is H-minor free, and thus a *cactus*; i.e., a graph in which every block is a simple cycle, or an edge. The *block graph* of G' is an acyclic bipartite graph with vertex set $B_1 \cup B_2$, where B_2 are the blocks of G', and B_1 are the cut-vertices. The block graph has an edge connecting each cut vertex with any of its incident blocks. Let $B'_2 \subseteq B_2$ be the set of block vertices that correspond to *even* cycles of G'. Also let $B'_1 \subseteq B_1$ be the cut vertices with at least two neighbours in B'_2. Let $\mathcal{B}$ be the subgraph of the block-graph induced by the vertices in $B'_1 \cup B'_2$. $\mathcal{B}$ is a forest, and we let $T_1, \ldots, T_l$ be its trees. Let $V(T_i) = B'_{i,1} \cup B'_{i,2}$ such that $B'_{i,q} \subseteq B'_q$ for all $i \in [l]$, and $q \in \{1, 2\}$.

W.l.o.g., we assume that $T_1, .., T_j$ are those trees in $\mathcal{B}$ that have only a single node from B'_2 (if no such tree exists, we let $j = 0$). For T_i with $i > j$ choose a root $r \in B'_{i,1}$ and direct the edges of T_i away from r. For each node $a \in B'_{i,1}$, let $b(a) \in B'_{i,2}$ be an arbitrary descendant of a in T_i (such a node exists by the definition of B'_1).

In the following, we abuse notation mildly, and use $b(a)$, for some $a \in B'_{i,1}$, in place of the even cycle it represents in G'. We define a feasible solution $\bar{y}$ for ($\mathrm{D}_{\mathrm{ECT}}$) by first letting $\bar{y}_{b(a)} = 1/2$ for all $a \in B'_{j+1,1} \cup \ldots \cup B'_{l,1}$. For $i \in \{1, \ldots, j\}$, let C_i be the even cycle corresponding to the single B'_2-node in T_i. We then let $\bar{y}_{C_i} = 1$, for all $i \in \{1, \ldots, j\}$. Let $\bar{y}_C = 0$ for all other even cycles. We claim that the $\bar{y}$ constructed is feasible for ($\mathrm{D}_{\mathrm{ECT}}$). To see this, note that, by construction, no node $v \in V(G')$ is incident to more than two even cycles with positive $\bar{y}$ value. In fact, if v is incident to two even cycles C_1 and C_2 with positive $\bar{y}$ value, then $\bar{y}_{C_1} = \bar{y}_{C_2} = 1/2$.

For each $1 \leq i \leq j$, let a_i be an arbitrary vertex of C_i. Define

$$\bar{U} = \{a_1, \ldots, a_j\} \cup \{a \in B'_{j+1,1} \cup \ldots \cup B'_{l,1} \,:\, \bar{y}_{b(a)} > 0\},$$

and note that $U \cup \bar{U}$ is a feasible solution for the even-cycle transversal problem. Thus, letting $x_v = 1$ for all $v \in U \cup \bar{U}$, and $x_v = 0$, otherwise, yields a feasible solution to ($\mathrm{P}_{\mathrm{ECT}}$). The value of this solution is no more than $\max\{c, 2\}$ times the value of the feasible solution $y + \bar{y}$ for ($\mathrm{D}_{\mathrm{ECT}}$). We obtain the following result.

Theorem 2.2. *Let G be chosen from some minor-closed family of graphs. Then* ($\mathrm{P}_{\mathrm{ECT}}$) *has a constant integrality gap.*

3 Even cycles in planar graphs

In this section, we provide a constant-factor gap for ($\mathrm{P}_{\mathrm{ECT}}$) in planar graphs in the case of general vertex costs. We will accomplish this by refining the argument given in [1] for the case of hitting *diamonds*.

A diamond is any sub-division of the graph consisting of three parallel edges. In [1], Fiorini et al. consider the problem of finding a minimum-cost diamond-transversal in a general graph. The authors show that the natural covering LP obtained from ($\mathrm{P}_{\mathrm{ECT}}$) by replacing $\mathcal{C}$ by the set $\mathcal{D}$ of vertex sets of diamonds in G has an integrality gap of $\Omega(\log n)$. The proof is constructive and uses a primal-dual algorithm using the natural LP and its dual.

The algorithm in [1] is natural and follows the well-known primal-dual strategy: start with a pair $x = y = \mathbb{0}$ of infeasible primal, and feasible dual solution. The algorithm iteratively modifies x, and y, maintaining the fact that x is $0, 1$, and y is dual feasible, and stops as soon as x is primal feasible. After applying a customary *reverse delete* step, the algorithm arrives at a minimally feasible solution $\bar{x} \leq x$, and the authors show that its total cost is bounded by $O(\log n)$ times the value of dual solution y.

Somewhat more specifically, in every step of the algorithm, where x is primal infeasible, we let X be the vertex set corresponding to x. The algorithm then carefully chooses a diamond D in $G[V \setminus X]$, and increases its dual variable y_D as much as possible, maintaining dual feasibility. At this point, the dual packing constraint for a vertex $v \in V \setminus X$ becomes tight, and the algorithm sets $x_v = 1$. Once x is feasible for the LP, the algorithm computes a *minimal* feasible solution $\bar{x} \leq x$, and the authors show that $y_D > 0$ only if $|D \cap \bar{X}| = O(\log n)$. This suffices to prove that $\bar{X}$ is an $O(\log n)$-approximate diamond hitting set.

In this section, we show that their algorithm can be simplified in the case of even cycles, and that it can be strengthened using the planarity of the underlying graph.

3.1 Ideas and a first attempt

A key first observation is captured in the following Lemma which is a special case of Kotzig's Theorem on Light Planar Subgraphs (e.g., see Section 3 of [3]). Let us call a vertex *heavy* if it has degree 6 or more and *light* otherwise.

Lemma 3.1. *A planar multigraph $G = (V, E)$ where every vertex has at least 3 distinct neighbours and no faces of length 2 contains 2 adjacent vertices whose degrees sum to at most 13. Further this bound is tight.*

Proof. Assume $G = (V, E)$ is an edge maximal counterexample. Define $d(v)$ as the degree of v in G. So G has minimum degree at least three and $d(u) + d(v) > 13$ for every edge uv. The choice of G implies that if there are any non-adjacent u and v with $d(u) + d(v) > 13$, then $G + uv$ is not planar. Now we assign a *charge* to each vertex: a vertex v gets charge $6 - d(v)$. The total charge $\sum_v (6 - d(v)) = 6|V| - 2|E|$, which is at least 12 using the fact that $|E| \leq 3|V| - 6$ by Euler's formula.

Now we *discharge*: a vertex v with a positive charge distributes its charge evenly among its neighbours. To clarify, the charge is sent evenly along all edges incident to v. If a neighbour is connected to v by multiple edges, then it gets additional charge for every edge. Since no charge is lost, there must still be vertices with a positive charge. Let v be a vertex with a positive charge. So v has a light neighbour. If v is light, then v and a light neighbour of v form a pair of vertices whose degrees sum to less than 13. So let us assume v is heavy.

Let $u_1, u_2, .., u_l$ be the neighbours of v in clockwise order. We claim that u_i, u_{i+1} cannot both have degrees less than 6 (where $u_{l+1} = u_1$). For the sake of contradiction assume that this is not true, then u_i, v, u_{i+1} are contained in a single face f, and u_i, u_{i+1} are not neighbours. Let w be the other neighbour of u_i in f; thus w is heavy. So adding an edge between v, w inside f would not create adjacent vertices whose degrees summed to less than 13 nor create a face of length 2, which contradicts our choice of G. Thus v does not have 2 consecutive light neighbours and thus has at most as many light neighbours as heavy neighbours. Let l, h be the number of light and heavy neighbours of v. Suppose v has a neighbour u of degree 3. The charge of v is at most $6 - l - h + l = 6 - h$. This is because v starts with charge at most $6 - l - h$ and receives charge at most 1 from each light neighbour and none from heavy neighbours. So v has at most 5 heavy neighbours and hence at most 10 neighbours. So u, v form adjacent vertices whose degrees sum to at most 13. If v has no neighbour of degree of exactly 3, then v has charge at most $6 - l - h + 0.5l$. This is because a vertex of degree 4 or more sends a charge of at most 0.5. So $0 < 6 - 0.5 - h \leq 6 - 1.5h$, so v has degree at most 7. Let u be a light neighbour of v, then u, v is a pair of adjacent vertices whose degrees sum to at most 13. A graph for which the sum of degrees of every edge is at least 13 is given by the stellated icosahedron, the graph obtained from a icosahedron by adding a vertex inside each face and edges between that vertex and the boundary of that face; see Figure 1. □

Corollary 3.2. *A 2 connected planar graph G of minimum degree 3 and no faces of length 2 contains a diamond with at most 12 edges.*

Proof. Consider the dual graph G^*. Since G contains no vertex of degree 2, G^* does not have a face of degree 2. Since G has no face of length 2, G^* has minimum degree 3. Thus G^* contains 2 adjacent vertices whose sum of degrees is at most 13. This in turn gives us two neighbouring faces whose lengths sum to at most 13 in G, which gives us a diamond of size at most 12. □

In particular, the above lemma has the following consequence:

Corollary 3.3. *A 2 connected planar graph G of minimum degree 3 and no faces of length 2 contains an even cycle with at most 11 edges.*

Proof. By lemma 3.1, there are 2 faces $f_1, f_2 \subset E$ of G whose total length is at most 13. If either f_1, f_2 is even it is an even cycle with at most 10 edges. Otherwise, $f_1 \Delta f_2$ is an even cycle with at most 11 edges. □

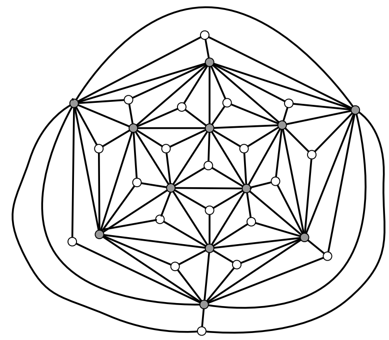


Figure 1

Hao's notes have a self-contained proof of the above two statements. From here, our algorithm follows the ideas provided in [1], simplifying and adapting to even cycles where possible.

Corollary 3.3 suggests the following natural algorithm: start with $X = \emptyset$, and $y = \mathbb{0}$. At any point in the algorithm where X is not feasible, compute the *1-compression* G_1 of $G[V \setminus X]$ as follows: as long as G has a node v of degree 2 with exactly two neighbours u and w in G, *contract* v; i.e., replace uv and vw by uw, and delete v. In the resulting graph G_1, all nodes have degree at least 3. Now find a cycle C_1 in G_1 of length at most 11 whose corresponding cycle C in G (which we will also call the *projection* of C_1 in G) is even. Increase y_C as much as possible, and add the newly tight vertices to X. Repeat the above until X is feasible, then run reverse delete, and obtain a minimally feasible set $\bar{X}$. Note that the minimality of $\bar{X}$ implies that, for all $v \in \bar{X}$, there is an even *witness* cycle C_v in G that that $C_v \cap \bar{X} = \{v\}$. More precisely, there is such a witness cycle C_v that is in the projection of the compressed residual graph *at the time* when v was chosen.

Lemma 3.4. *Suppose that the above algorithm terminates with feasible solution $\bar{X}$. Then the total cost of $\bar{X}$ is at most 11 times the value of the computed dual solution.*

Proof. Let us consider an even cycle C with $y_C > 0$, and let C_1 be the short cycle in the 1-compression G_1 of the graph $G[V \setminus X]$ at the time where y_C was increased. In Figure 2, white nodes have degree 2 in $G[V \setminus X]$, and grey nodes have degree at least three. Hence the cycle depicted has length 7 in G_1. Suppose that $v \in V(C) \cap \bar{X}$ is a node on C that was chosen by our algorithm for the final transversal.

Consider two adjacent nodes u and w in $V(C_1)$, and let P_{uw} be the corresponding path in G. Let us first assume that v is an internal (degree 2) node of P_{uw} for two adjacent nodes u, w of $V(C_1)$. In this case, note that the witness cycle C_v of v contains all nodes of P_{uw} including u and w themselves. Thus, if $\bar{X}$ contains v then it contains no other nodes from P_{uw}.

Now suppose that $v \in \bar{X} \cap C_1$ is node of degree at least three on C, and let u and w be the neighbours of v on C_1. Using the same argument as before, we see that no internal vertex of P_{uv} and P_{wv} can be in $\bar{X}$ in this case. Hence, we must have $|\bar{X} \cap C| \leq 11$, and the solution $\bar{X}$ has cost no more than $11 \sum_C y_C$. □

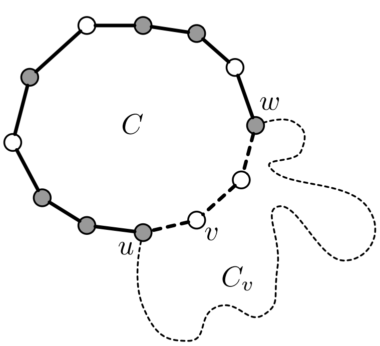


Figure 2: The figure shows even cycle C. White vertices are contracted during compression and have degree 2 in $G[V \setminus X]$, grey vertices have degree at least 3 in the same graph.

Unfortunately, the above algorithm does not always terminate. The reason is that we may not be able to find an even cycle whose 1-compression has at most 11 edges. Consider for example the graph depicted in Figure 3. This graph has many even cycles, albeit no short ones! Notice that the compression of this graph has faces of length 2, and hence Corollary 3.3 does not apply.

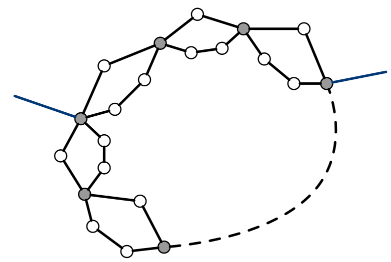


Figure 3: A graph in which every even cycle is long.

3.2 Dealing with graphs without even cycles of small compressed length

Consider vertex sets C'_1 and C'_2 that each contain the vertex set of an even cycle in G; i.e., for $j \in \{1,2\}$, there is an even cycle $C_j \in \mathcal{C}$ such that $C_j \subseteq C'_j$. Define $a_v^{C'_1,C'_2} = 1$ for all $v \in (C'_1 \cap C'_2)$, $a_v^{C',C'} = 1/2$ for v in the symmetric difference $C'_1 \setminus C'_2 \cup C'_2 \setminus C'_1$ of C'_1 and C'_2, and $a_v^{C'_1,C'_2} = 0$, otherwise. Note that the following inequality is dominated by the two original cover inequalities for sets C_1 and C_2:

$$\sum_v a_v^{C'_1,C'_2} x_v \geq 1.$$

Hence, we obtain a new pair of LPs that are equivalent to $(\mathrm{P_{ECT}})$ and its dual. Here, we abuse notation, and let $\mathcal{C}$ now be the set of pairs (C'_1, C'_2) where C'_1 and C'_2 are (not necessarily disjoint nor different) supersets of even cycles in G.

$$\begin{aligned} \min \quad & c^T x && (\mathrm{P_{ECT}}) \\ \text{s.t.} \quad & \sum_{v\in V} a_v^{C'_1,C'_2} x_v \geq 1 \quad \forall (C'_1,C'_2) \in \mathcal{C} \\ & x \geq \mathbb{0}. \end{aligned}$$

$$\begin{aligned} \max \quad & \mathbb{1}^T y && (\mathrm{D_{ECT}}) \\ \text{s.t.} \quad & \sum_{(C'_1,C'_2)\in\mathcal{C}} a_v^{C'_1,C'_2} y_{C'_1,C'_2} \leq c_v \quad \forall v \in V \\ & y \geq \mathbb{0}. \end{aligned}$$

Note that the original cover inequalities for even cycles C are contained in the reformulation of $(\mathrm{P_{ECT}})$, by letting $C'_1 = C'_2 = C$. For convenience we will write y_C in place of $y_{C,C}$ from here on.

Our algorithm maintains a pair (x, y) of (partial) primal, and dual solutions for the above pair of LPs; for convenience, we let X be the set of vertices corresponding to incidence vector x. At any time, the algorithm will consider the 1-compression G_1 of the graph $G[V \setminus X]$. The algorithm works as before if G_1 contains a simple, short cycle (i.e., a cycle without parallel edges, and length no more than 11) whose projection in G is an even-length cycle C. In this case we increase the dual variable y_C as described in the previous section as much as possible, adding newly tight vertices to set X.

Note that the same argument also works if there is a pair of vertices u and v that are connected by at least three edges e_1, e_2, and e_3 in G_1. In this case, two of these edges, say e_1 and e_2, project to an even length cycle C in G. We proceed as before.

From here on, we assume that G_1 has at most two edges connecting every pair of vertices. We also assume, w.l.o.g., that every edge of G_1 is contained in some cycle with even projection, and therefore, G_1 is 2-connected. Assume now that G_1 has cycles with even projection, but none that are short and simple. In this case, Corollary 3.3 implies that G_1 cannot be simple.

We obtain graph $\bar{G}_1$ from G_1 by replacing each pair of parallel edges connecting vertices u and v by a single *twin* edge uv. Note that each face in this new graph has length at least 3. Obtain the 2-compression G_2 of G by contracting degree-2 nodes in $\bar{G}_1$; see Figure 4. In the following, abusing notation slightly, we call edges created by contracting vertices in $\bar{G}_1$ *twin* (as their projection in G_1 must contain twin edges). Using G_2, we will now identify a certain even *cycle* whose dual variable we increase. We branch into two cases.

G_2 **is simple.** We first assume that any pair of vertices is connected by at most 1 edge in G_2. In this case, Corollary 3.3 implies that G_2 has an even cycle C_2 of length at most 11. Since G_1 does not have a short, simple cycle with even projection, it follows that C_2 must contain at least one twin edge.

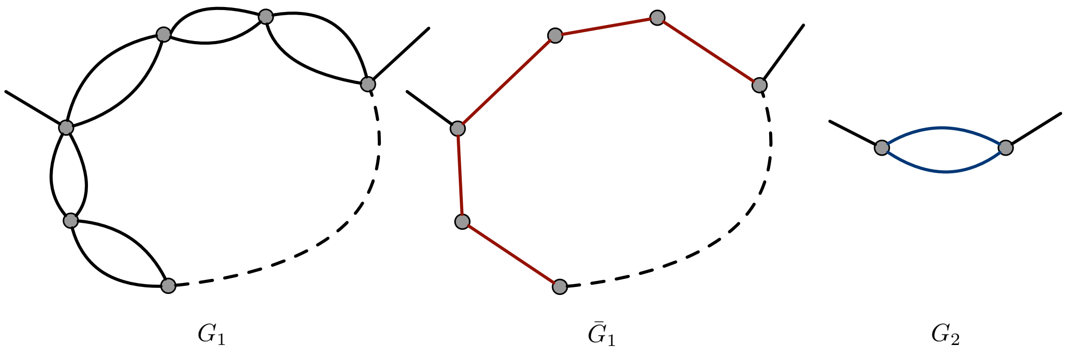


Figure 4: The figure shows 1 and 2-compression G_1 and G_2 of the graph shown in Figure 3. In $\bar{G}_1$ we replace parallel edges in G_1 by twin edges.

We now focus on an edge uv on C_2, and let $\bar{P}_1$, P_1, and P be its projections in graphs $\bar{G}_1$, G_1, and G, respectively. Following the notational conventions of [1], we say that P is the *piece* corresponding to uv. Note that the piece of a non-twin edge uv is a u,v-path whose internal nodes have degree 2 in $G[V \setminus X]$.

If uv is a twin edge of C_2, then the path $\bar{P}_1$ consists of twin, and non-twin edges. In turn, a twin edge $u'v'$ on $\bar{P}_1$ projects to a subgraph of $G[V \setminus X]$ that is induced by two internally vertex disjoint u', v'-paths S_1 and S_2. Furthermore, concatenating these two u', v'-paths yields an odd cycle.

In summary, one now sees that the *block-graph* of the projection of twin edge uv of C_2 is a path in $G[V \setminus X]$. Moreover, the blocks in this subgraph are odd cycles; see Figure 5 for an example.

Let us focus on a piece P corresponding to a twin edge uv of C_2. Vertices $w \in V(P) \setminus \{u, v\}$ have degree 2 in G_1, or they are *cut-vertices* of P, and have degree 2 in $\bar{G}_1$. In Figure 5 we have coloured such vertices in red. In the following, we keep track of the *slack* in the dual constraint of each vertex v for the current dual feasible solution for $(\mathrm{D_{ECT}})$:

$$\bar{c}_v := c_v - \sum_{(C'_1,C'_2)\in\mathcal{C}} a_v^{C'_1,C'_2} y_{C'_1,C'_2}.$$

We now define a *canonical* subgraph of each piece P. This subgraph will be the support of the inequality of $(\mathrm{P_{ECT}})$ whose dual variable we want to increase. The subgraph is induced by a set of vertices that we classify as *type-1 type-2*, or *type-3*.

For a non-twin edge uv of C_2 we let all vertices of the corresponding piece P be of type 1. Now consider a twin-edge uv of C_2. All vertices of P_1 are added as type-1. Let u' and v' be two neighbouring vertices on P_1 that are connected by twin edges e_1 and e_2 in G_1 (see Figure 5). Let S_1 and S_2 be the paths in $G[V \setminus X]$

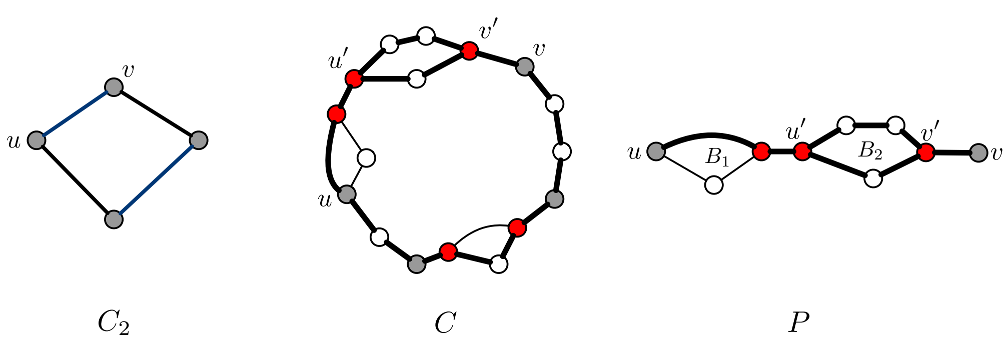


Figure 5: The first two figures show a short cycle C_2 in G_2, and its corresponding projection in G. The right figure above shows the piece of twin edge uv of C_2. Note the thick edges of cycle B_2 above: these are type-3 edges.

corresponding to e_1 and e_2, respectively. We let the residual cost $\bar{c}(S)$ of a path S be the smallest residual cost of any of its internal nodes. If the maximum among $\bar{c}(S_1)$ and $\bar{c}(S_2)$ is unique, then let the vertices of the maximizer be of type 1. Otherwise label u' and v' type-1, and make all internal vertices of S_1 and S_2 type-2. We will call the vertices of S_1 and S_2 together with u' and v' a *type-2 cycle* and call the two u', v' paths handles [1] in this case.

Suppose first that the number of type-2 cycles over all pieces of C_2 is odd. In this case, pick an arbitrary such cycle and label all its internal vertices type-3. We call the corresponding cycle a *type-3 cycle*. For any $v \in V$, we now let

$$a_v = \begin{cases} 1 & \text{if } v \text{ is type-1, or type-3} \\ 1/2 & \text{if } v \text{ is type-2} \\ 0 & \text{otherwise.} \end{cases}$$

We claim that the inequality

$$\sum_v a_v x_v \geq 1 \tag{$\circledast$}$$

is part of ($\mathrm{P}_{\mathrm{ECT}}$). To see this, let us assume that the number of type-1 vertices is even (the subsequent argument is easily adapted if this number is odd). Let $\bar{C}^1, \ldots, \bar{C}^{2q}$ be the even cardinality set of all type-2 cycles over all pieces of C_2. Furthermore, for $1 \leq i \leq 2q$, let $\bar{S}_1^i$ and $\bar{S}_2^i$ be the internal vertices of the two paths defining $\bar{C}^i$. Since $\bar{C}^i$ is odd by assumption it follows that $\bar{S}_1^i$ and $\bar{S}_2^i$ have different parity. By possibly renumbering, we may therefore assume that the parity of $\bar{S}_1^{2i}$ is different from that of $\bar{S}_1^{2i+1}$, for all $1 \leq i \leq q$.

Assume first that there are no type-3 vertices. In this case, we obtain two even cycles, by adding the set of type-1 vertices to

$$Q_j = \bigcup_{i=1}^{q} (\bar{S}_j^{2i} \cup \bar{S}_j^{2i+1}),$$

for $j = 1, 2$. Assume, on the other hand, that there are type-3 vertices, and let $S_1 \cup S_2$ be the corresponding partition of the internal vertices of the paths defined above. We now see that adding the set of type-3 vertices to Q_j yields the superset of an even cycle, for $j = 1, 2$. This now completes the argument that ($\circledast$) is part of $\mathrm{P}_{\mathrm{ECT}}$.

The algorithm increases the dual variable $y_{\circledast}$ corresponding to inequality ($\circledast$) as much as possible, while maintaining dual feasibility. We then add tight vertices to X. Note that our definition of a and inequality ($\circledast$) allows us to maintain the following invariant.

Invariant 3.5. *Consider a twin edge e_1 and e_2 in C_1 as defined above, and let S_1 and S_2 be the corresponding vertex sets in $G[V \setminus X]$. If $\bar{c}(S_1) \geq \bar{c}(S_2)$ before the increase of $y_{\circledast}$ then this will be true also after the increase, and until the end of the algorithm's execution.*

Let $S_1 \cup S_2$ be the set of internal vertices of a type-2 or type-3 cycle. Note that Invariant 3.5 implies that if the dual constraint of some $v_1 \in S_1$ becomes tight in a step of the algorithm, then there is a vertex $v_2 \in S_2$ whose dual constraint becomes tight at the same time. For such a tight cycle, we pick exactly two vertices v_1 and v_2 in S_1 and S_2, respectively, and add them to X. We will later refer to such an addition as a *type-2/3 addition.*

G_2 is not simple. In this case, G_2 has two vertices u and v that are connected by at least two edges e_1 and e_2. By assumption $\bar{G}_1$ is a simple graph. Thus, at least one of e_1 and e_2, say e_1, is a twin edge. In this case we let C_2 be the cycle formed by e_1 and e_2 and proceed as above.

When the algorithm terminates. The algorithm terminates at the first time when X is a feasible solution to ECT. We then obtain a minimal feasible solution $\bar{X}$ through a reverse-delete step. Suppose that reverse delete considers vertices v_1 and v_2 that were part of a type-2/3 addition in the algorithm. Note that if reverse delete decides to remove one of these vertices, then it will remove both.

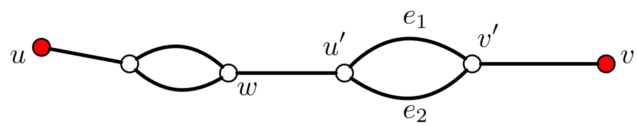


Figure 6: The figure shows the projection P_1 of a piece of an edge uv on an even cycle C_2 in G_2.

4 Analysis

Let y be the final feasible dual solution computed by the algorithm. Our goal is to prove that the computed solution is 12-approximate. Following the classic primal-dual argument, it suffices to show that

$$\sum_{v \in \bar{X}} a_v^{C'_1, C'_2} \leq 12, \tag{1}$$

whenever $y_{C'_1, C'_2} > 0$. The inequality follows from the argument used in Lemma 3.4 in the case where $C'_1 = C'_2 = C$ for some even cycle $C \in \mathcal{C}$ (e.g., this is the case when our algorithm found a simple, short, even cycle in G_1).

So let us consider the situation during the algorithm where no such short, simple, even cycle exists, but where $G[V \setminus X]$ still has even cycles. In this case, as described above, our algorithm computes the 2-compression G_2 of $G[V \setminus X]$ and finds a short cycle C_2 whose projection C in $G[V \setminus X]$ contains an even cycle. In the construction of inequality ($\circledast$), the algorithm selects type-1, type-2, and type-3 vertices in C. We will show that the sum of the a-coefficients of the internal vertices of $P \cap \bar{X}$ and any one of u and v is at most 1 if P has no type-3 vertices, and at most 2 otherwise. Inequality (1) then follows from the fact that at most one piece of C_2 has type-3 vertices by construction.

Consider first the case where uv is a non-twin edge of C_2, and recall that its piece P is a path whose internal vertices have degree 2. As in the proof of Lemma 3.4, at most one of the internal vertices of P can be in $\bar{X}$. Similarly, if one of P's endpoints is in $\bar{X}$ then none of the internal vertices can be in $\bar{X}$.

In the remaining part of the proof, we consider a twin edge uv on C_2 with projections P_1 in G_1, and P in $G[V \setminus X]$; we refer the reader to Figure 6 for an example.

Case 1: P has no type-3 vertices. As in the proof of Lemma 3.4, $\bar{X}$ has at most one of the internal nodes of P_1, and if there is such a node, then P contains no other node of $\bar{X}$. In this case, the sum of $a_v^{C'_1, C'_2}$ over the nodes of $P \cap \bar{X}$ is at most 1. So suppose that $\bar{X}$ has none of the internal vertices of P_1.

Focus, on a pair of twin edges e_1 and e_2, connecting vertices u' and v' of P_1, and as before, let S_1 and S_2 be the vertex sets of the projections of these edges. Suppose that the residual cost of S_1 is at least that of S_2. Then by Invariant 3.5, and by the way we perform reverse-delete, $\bar{X}$ contains a vertex from S_1 only if it also contains a vertex of S_2. If $S_1 \cup S_2$ forms a type-2/3 cycle, then a stronger condition holds: $\bar{X}$ contains either exactly two vertices $v_1 \in S_1$, and $v_2 \in S_2$, or $(S_1 \cup S_2) \cap \bar{X} = \emptyset$.

Let us first assume that $\bar{X}$ contains vertices from both S_1 and S_2. In this case, by the familiar witness cycle argument, we see that these two vertices are the only vertices in $P \cap \bar{X}$. If S_2 is in the support of ($\circledast$) then it must be that the projection of e_1 and e_2 forms a type-2 cycle. Thus, the sum of the a-coefficients of the two $\bar{X}$-vertices in $S_1 \cup S_2$ is 1 in this case. On the other hand if e_1 and e_2 do not form a type-2 cycle, then S_2 is not in the support of ($\circledast$), and hence the sum of a-coefficients of vertices in P is at most 1 in that case as well.

Let us now assume that for all cycles $S_1 \cup S_2$ of piece P, $\bar{X}$ contains at most one vertex from $S_1 \cup S_2$. Thus, if $\bar{X} \cap S_2 \neq \emptyset$ we must have $\bar{X} \cap S_1 = \emptyset$ for such a cycle. But this implies that we must have had $\bar{c}(S_1) > \bar{c}(S_2)$ at the time of increase, and hence $a(S_2) = 0$.

Case 2: P has type-3 vertices. Let S_1, S_2 be the vertex set of the type-3 cycle. By the same argument as before, $\bar{X}$ contains none or two vertices from $S_1 \cup S_2$. If it contains none, then the argument of the previous case applies and the sum of a coefficients of P vertices is at most 1. Otherwise, P contains exactly these two vertices from $\bar{X}$, and the a coefficients of P vertices sum to 2. Since at most one piece of C_2 can contain type-3 vertices, we now obtain the following theorem.

Theorem 4.1. *The above algorithm is* 12*-approximate.*

We can get an 11-approximation by using a stronger version of Corollary 3.3 that is specifically tunes for our application. In the following, let G_2 be the 2-compression of a given planar graph G. As before, we assume that G_2 is 2-connected.

By an abuse of notation, we will say that a cycle of G_2 is even if it contains a twin edge, or its projection in G is even.

Corollary 4.2. *Suppose that G_2 has no faces of length 2. Then either*
(a) it contains an even cycle without twin edges of length at most 11*, or*
(b) it contains a cycle C that contains a twin edge of length no more than 10*.*

Proof. If G_2 has a parallel edge, one of which is twin, then we get an even cycle with at most 3 pieces. By Lemma 3.1 we can find adjacent faces f_1, f_2 of G_2 whose lengths sum to at most 13. If one of the faces has a twin edge, then the corresponding cycle satisfies condition (b) above. Otherwise, the union of f_1 and f_2 contains an even cycle without twin edges of length at most 11, and hence (a) holds. □

One can improve the 11-approximation by noting that getting 11 pieces in our even cycle only happens in a very specific case. Further, we will modify our algorithm to return a cycle with 8 or fewer pieces if such a cycle exists. For the next part, let us modify which nodes were added to our ECT as follows. We remove any nodes that were not paid for by our blended diamond inequality, and if two internal nodes of a type 2 cycle were selected, we instead choose an end-node of that cycle and declare that that vertex is selected instead. This does not decrease $\sum_{v\in\bar{X}} a_v^{C_1',C_2'}$. Given an edge uv in G_2, let $p(u,v)$ denote the piece corresponding to uv in G.

Consider the discharging argument used in the proof of Lemma 3.1. We previously showed that in an edge maximal counterexample, there is a vertex v of positive charge and a neighbour u of v, such that the degrees of u and v sum to at most 13. That is, suppose G' is obtained from G by adding edges between heavy vertices. Let uv be an edge of G' with degrees of u, v summing to at most 13. If either u or v is incident to an edge of $G' \backslash G$, then u, v have degrees summing up to less than 13 in G. One can further see that either v has a neighbour of degree 3 or there is a neighbour u of v such that the degrees of u, v sum to at most 11 [3]. By analyzing the charge, if 2 or more of the light neighbours of v have degree at least 4, then we get $6 - l - h + (l-2) + 1 > 0$. So $5 > h$. h, l are as in the proof of Lemma 3.1. This is because at least 2 of the light neighbours of v contribute a charge of at most 1/2. Further, note we proved in Lemma 3.1 the node v has at most 5 heavy neighbours, so the only way for v to have degree 10 is to have heavy and light neighbours of v alternate in the clockwise order. Following the proof of 4.2, if we find 2 adjacent faces f_1, f_2 of G_2 whose lengths sum to at most 12, then we can find an even cycle with (i) 9 or fewer pieces, or (ii) with at most 10 pieces, none of which is twin. To sum up our above observations:

Lemma 4.3. *Suppose that G_2 has no faces of length 2. Then either*
(a) it contains an even cycle without twin edges of length at most 10*,*
(b) it contains a cycle C that contains a twin edge of length no more than 9*,*
(c) it contains an even cycle without twin edges of length 11 *that contains exactly 2 faces f_1, f_2, or*
(d) it contains an even cycle f_1 that contains a twin edge of length at most 10 that is also a face.

In (c), (d) f_1 has length 10 and f_2 length 3. f_1 has exactly 5 light neighbours and 5 heavy neighbours in G^* and they alternate in the clockwise order around f_1. Further, edges on both f_1 and a triangle face have both endpoints degree 3.

We now modify our algorithm to return either an even cycle with 8 pieces or one that is contained in the projection of once of the cases of Lemma 4.3.

Proof. We now prove the 10-approximation as follows. If we get case (a) or (b), the 10-approximation follows in the same manner as the previous 11-approximation. Suppose we are in case (c), denote the vertices of f_1 by $v_1, v_3, v_4, .., v_{11}$, and denote by v_2 the node of f_2 not on f_1. Let $v_i t v_{i+1}$ be a triangle face sharing an edge with f_1 that is not f_2. By assumption, this face has odd length in G.

Remark 4.4. *Let tv_iv_{i+1} be a triangle face of G_2 which has odd length in G with no twin edges. Let v_{i-1} be a neighbour of v_i, and v_{i+2} a neighbour of v_i. Suppose that v_i, v_{i+1} have no other neighbours. Then $p(v_{i-1}, v_i) \cup p(v_i, v_{i+1}) \cup p(v_{i+1}, v_{i+2})$ contains fewer than 3 hit nodes; see Figure 8c.*

Proof. Suppose that $p(v_{i-1}, v_i) \cup p(v_i, v_{i+1}) \cup p(v_{i+1}, v_{i+2})$ contains 3 hit nodes, let w be the "middle" hit node, then each of $p(v_{i-1}, v_i), p(v_{i+1}, v_{i+2})$ contains a hit node besides w. Let A_w be a witness cycle for w, and consider the subpath Q_w of A_w containing w and lying in $p(v_{i-1}, v_i) \cup p(v_i, v_{i+1}) \cup p(v_{i+1}, v_{i+2}) \cup p(v_{i+1}, t) \cup p(v_i, t)$ such a path cannot use nodes of $p(v_{i-1}, v_i) \backslash v_i, p(v_{i+1}, v_{i+2}) \backslash v_{i+1}$ and hence both ends of the path are t and so A_w is the projection of tv_iv_{i+1}. But by assumption this cycle is odd in G, which is a contradiction. □

Thus $p(v_{i-1}, v_i) \cup p(v_i, v_{i+1}) \cup p(v_i, v_{i+1})$ contains at most 2 hit nodes. The remaining 8 pieces have at most 8 hit nodes not counting v_{i-1}, v_{i+2}, which implies $\sum_{v \in X} a_v^{C_1', C_2'} x_v \leq 10$. Case (d) is similar, let G' be the graph obtained from G by deleting the interior nodes of one handle of a type 3 cycle. Let the even cycle be $v_1 v_2 ... v_{10}$. Let v_i, t, v_{i+1} be a triangle face of G_2. By remark 4.4 the pieces $p(v_{i-1}, v_i) \cup p(v_i, v_{i+1}) \cup p(v_i, v_{i+1})$ contains at most 2 hit nodes, so the remaining 7 pieces contain 7 hit nodes, not counting v_{i-1}, v_{i+2} which again proves $\sum_{v \in X} a_v^{C_1', C_2'} x_v \leq 10$. □

By being a bit more careful in our book-keeping, we can get a 9-approximation.

Lemma 4.5. *[3] A planar multigraph $G = (V, E)$ where every vertex has at least 3 neighbours embedded with no faces of length 2 contains an edge uv such that one of the following is true:*
1) u has degree 3 and if v has degree more than 6 it has degree at most $2a_1 + a_2 + 2c + d$ with $6 - (a_1 + a_2) - 1.5c - d - b > 0$. a_1 is the number of neighbours w of degree 3 for which the next neighbour of v in the clockwise order is heavy and a_2 is the number of other neighbours of degree 3. c is the number of neighbours of degree 4 or 5. b is the number of faces of length 4 or more containing v and both a light and heavy neighbour of v in G. d is the number of heavy neighbours w of v whose such that the next clockwise neighbour of v is also heavy; see Figure 7.
2) $deg(u) + deg(v) \leq 11$

Proof. (of part not in [3]) If G contains 2 adjacent light neighbours we are done. Otherwise, let G' be a triangulation of G by drawing edges between heavy vertices. Note that this is possible, since any face of length four or more contains 2 heavy nodes, and we can add that edge. Just as in [3], by max triangulation v cannot have 2 consecutive light neighbours in G'. In [3] it is proven their exists an edge uv, v a vertex

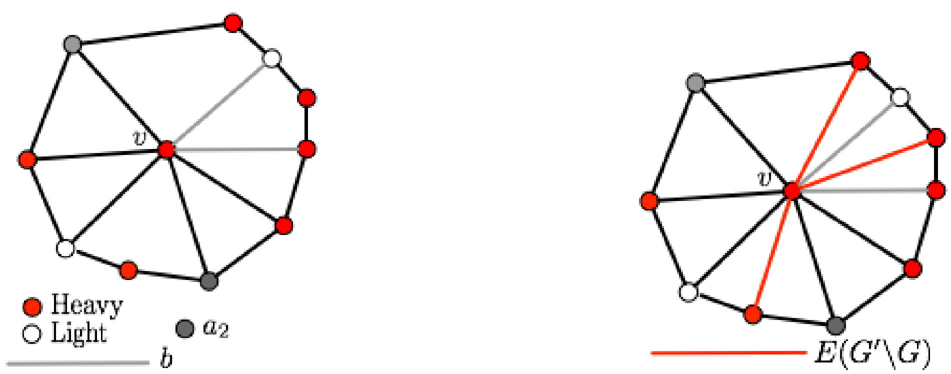


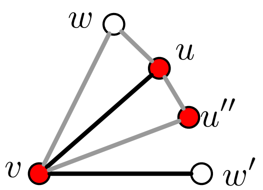
Figure 7: Example of a graph G and edges added in obtaining G'.

with positive charge such that either
1) u has degree 3 and $\deg_{G'}(v) \leq 10$, or
2) $\deg_{G'}(u) + \deg_{G'}(v) \leq 11$.

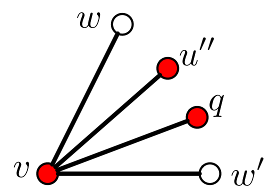
We claim v has degree at least $2a_1 + 2a_2 + 2c + d + b$ in G'. First let $f_1, f_2, .., f_l$ be the b-faces of G that is the faces that b is counting. Let E' be the set of edges of $\delta_{G'}(v)$ contained in the interior of some f_i and let E'' be the d-edges, that is the edges whose endpoint other than v is counted by d. In any f_i one neighbour u of v is light and the other u' is heavy and thus uu' is not an edge of G' and thus $\delta_{G'}(v)$ contains an edge in the interior of f_i. It thus suffices to prove that $|\delta_{G'}(v) \backslash E' \backslash E''| \geq 2a_1 + 2a_2 + 2c$. We claim that in $(G' \backslash E') \backslash E''$ there are no consecutive light neighbours of v. Assume for a contradiction that there were 2 such neighbours w, w' with w' next neighbour after w in $(G' \backslash E') \backslash E''$ in the clockwise order about v.

Suppose that there is a d neighbour u' between w, w' in G' let u'' be the last such d neighbour of v before w' in the clockwise order. Then there is a heavy non d -neighbour q of v in G which lies between u'', w'. So henceforth we assume that no d-neighbours lie between w, w' in the clockwise order about v in G'.

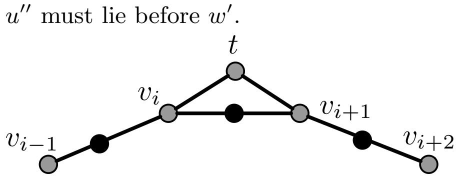
Let u' be the previous neighbour before w' in G'. Assume $vu \in E'$ so $u' \in f_j$ for some f_j then f_j must contain one of vw, vw'. This is because f_j contains 2 consecutive neighbours of v in G and contains the edge vu in its interior, and u lies between w, w' in the clockwise order. So the neighbours of v in f_j lie between w, w' in the clockwise order. Since one of them is light, that neighbour must be w or w'. Let w'' be the one of w, w' contained in f_j and let u'' be the other neighbour of v in G, then u'' lies between w, w' in the clockwise order, which is a contradiction. Hence there are no 2 consecutive light neighbours. Since each light edge is preceded and followed by a heavy edge hence $|\delta_{G'}(v) \backslash E' \backslash E''| \geq 2a_1 + 2a_2 + 2c$. Calculating charge thus gives us $6 - (a_1 + a_2) - 1.5c - d - b > 0$. □



(a) Here $w'' = w$. Since w, u are consecutive in G, u'' must lie before w'.



(b) If there is a heavy node between w, w' then there is a non-d neighbour heavy node.



(c) Since tv_iv_{i+1} is odd and v_i, v_{i+1} have exactly 3 neighbours, the hit node on v_iv_{i+1} cannot have a witness cycle.

Figure 8

When v and G are specified we will refer to a_1, a_2, b, d, c neighbours as those neighbours counted by a_1, a_2, b, d, cv respectively and a a_1, a_2, b, d, c edge as an edge between v and a a_1, a_2, b, d, c neighbour respectively.

Remark 4.6. *In case 1) of lemma 4.5 we have that v has degree at most $10 - a_2 - 2\lfloor 0.5c \rfloor - d - 2b$*

Proof. Using same variables as 4.5, $6 > (a_1+a_2)+1.5c+d+b$ so $5 \geq \lfloor (a_1+a_2)+1.5c+d+b \rfloor$ so $5-(a_1+a_2)-\lfloor 1.5c \rfloor - d - b \geq 0$ which implies $10 \geq 2(a_1+a_2)+2c+2\lfloor 0.5c \rfloor + 2d + 2b \geq \deg(v) + a_2 + 2\lfloor 0.5c \rfloor + d + 2b$. □

Lemma 4.7. *We can find an cycle C of G_2 even in G such that either:*
1) C is the union of 2 faces f_1, f_2 of G_2 with, f_1 length 3 and at most max$(9, 11 - a_2 - 2\lfloor 0.5c \rfloor - d - b)$ *pieces. Where a_2, c, d, b are as in lemma 4.5 for G_2^* with $v = f_1$ and C is disjoint from the interior of any double piece.*
2) C has at most 9 pieces and is disjoint from the interior of any double piece.
3) C has at most $10 - a_2 - 2\lfloor 0.5c \rfloor - d - b$ pieces. One piece may be double.
4) C has at most 7 pieces.
Where for 1),2) a_1, a_2, c, d, b, c_2 are as in lemma 4.5 with $v = v_2$ $u = f_1$ in the dual of G_2.

Proof. Apply lemma 4.5 to G_2^*. In case 1) of 4.5 we find a node u of degree 3 adjacent to a node v of degree at most $10 - a_2 - 2 * \lfloor 0.5c \rfloor - d - b$ in H^*. (a_2, c, d, b as in 4.5) . Let f_1, f_2 be the faces corresponding to v, u in G if f_1 or f_2 is even such an even cycle works otherwise $f_1 \cup f_2$ is an even cycle in G with at most $11 - a_2 - 2\lfloor 0.5c \rfloor - d - b$ pieces. Case 2) is similar. □

Proof. Let us consider each case of 4.7.

Case 1). If C has fewer than 9 pieces we are done. Otherwise let $v_1, v_2, .., v_l$ be the cycle C in G_2 with v_1, v_2, v_3 vertices belonging to a triangle face of G_2 we denote:

$$p'(v_i, v_j) = \begin{cases} \cup_{u=i}^{j-1} p(v_u, v_{u+1}) & \text{if } i < j \\ \cup_{u=j}^{l} p(v_u, v_{u+1}) \cup \cup_{u=1}^{i-1} p(v_u.v_{u+1}) & \text{otherwise} \end{cases} \tag{2}$$

$v_{l+i} = v_i$ in above equation.

Case 1a). v_1, v_2 are both incident to 3 faces of G_2. We claim that there are at most 3 hit nodes in $p(v_l, v_1) \cup p(v_1, v_2) \cup p(v_2, v_3) \cup p(v_3, v_4)$. Assume not then each of $p(v_l, v_1), p(v_1, v_2), p(v_2, v_3), p(v_3, v_4)$ contains a hit node, and one of $p(v_2, v_4)\backslash\{v_2\}, p(v_l, v_2)\backslash\{v_2\}$ contains 2 hit nodes. By symmetry assume $p(v_2, v_4)\backslash\{v_2\}$ contains 2 hit nodes. Let u be the node on $p(v_2, v_4)\backslash\{v_2\}$ closest to v_2. Now consider the witness cycle C_u of u and consider a subpath P of C_u lying in C and let P' be the portion of P starting at u not headed towards v_2 P' cannot go to v_4 since $p(v_2, v_4)\backslash\{v_2\}$ contains 2 hit nodes $p(v_3, v_4)\backslash v_3$ contains a hit node. So after reaching v_3 P' must head towards v_1 since $p(v_l, v_1), p(v_1, v_2)$ each contain a hit node P' hits another hit node besides u which is a contradiction. Let $f_1, f_2, a_1, a_2, b, c, d$ be as in lemma 4.7. So we have that $l \leq 11 - a_2 - 2\lfloor 0.5c \rfloor - d - 2b$ if $l \leq 10$ then since there are at most 3 nodes on $p(v_l, v_1) \cup p(v_1, v_2) \cup p(v_2, v_3) \cup p(v_3, v_4)$ there are at most 9 nodes on C. So $a_2 + 2\lfloor 0.5c \rfloor + d + 2b \leq 1$

From $l \leq 2a_1 + a_2 + 2c + d$ we get $a_1 \geq 4$ with 3 of the a_1 edges not being v_1v_3. Let v_rv_{r+1}, v_wv_{w+1} be the next and previous a_1 edges from v_1, v_2 on C in the clockwise order and let v_kv_{k+1} be the remaining a_1 edge so $\{v_{k-1}v_k, v_kv_{k+1}, v_{k+1}w_{k+2}\} \cap \{v_lv_1, v_1v_2, v_2v_3, v_3v_4\} = \emptyset$ by remark 4.4 we get $p'(v_{k-1}, v_{k+2}) \cup p'(v_l, v_4)$ contains at most 5 hit nodes. Since $C\backslash p'(v_{w-1}, v_{w+2}) \cup p'(v_l, v_4)$ is contained in 4 pieces we get that C contains at most 9 hit nodes. See Figure 9a.

Case 1b). Now suppose that one of v_1, v_3 is incident to more than 3 faces of G_2. Recall $l \leq 11 - a_2 - 2\lfloor 0.5c \rfloor - d - b, 2a_1 + a_2 + 2c + d + 1$. If $l \leq 9$ we are done otherwise so $c \leq 1, \quad a_2 + b + d \leq 1$ $a_1 \geq 3$ Thus we can find an a_1 edge v_rv_{r+1} distinct from v_1v_3 that is not proceeded or followed by a c edge. Since one of v_1, v_3 is incident to more than 3 faces of G_2 either one of v_lv_1, v_3v_4 is an a_2 or c edge or one of v_1, v_3 is a b face in G_2^* which implies that neither v_r or v_{r+1} can be a b-face in G_2^* . nor can $v_{r-1}v_r$ be an a_2 edge. Combined with v_rv_{r+1} is an a_1 edge we get that v_r, v_{r+1} both have exactly 3 neighbours and by remark 4.4 $p(v_{r-1}, v_{r+2})$ contains at most 2 hit nodes and hence C has at most 9 hit nodes.

Case 2). A 9-approximation is immediate.

Case 3). Let C_1 be the result of taking only the bottom handle of each (and cut nodes and end nodes) piece of C. Since every hit node v' on a top handle except the one on the special cycle (if it exists) is either not paid for, or appears as a pair v', w' where w' is on the bottom handle and the coefficients of v', w' in our blended diamond inequality are both 1/2, it suffices to prove that C_1 contains at most 8 hit nodes. Denote $m(u, v) = p(u, v) \cap C_1$ $m'(u, v) = p'(u, v) \cap C_1$. Where the edges of C in G_2 are $v_1, v_2, .., v_l$ note that if v_i, v_{i+1}, t_i is a triangle in G_2 then v_iv_{i+1} is a single piece in G and remark 4.4 holds for $m'(v_{i-1}, v_{i+2})$. From $l \leq 2a_1 + a_2 + 2c + d$ One can check that either $a_1 \geq 4$ $c \leq 1, \quad b = d = 0$ or $l \leq 9$, $a_2 + 2\lfloor 0.5c \rfloor + d + b \leq 1$.
Case 3a). $b = \lfloor 0.5c \rfloor = d = a_2 = 0$. Then there are at least 4 a_1 edges and since $b = 0$ each endpoint of an

(a) Case 1a.

(b) Case 3a).

Figure 9: Red and grey portion represent our even cycle. Consecutive red pieces contain at least one fewer hit node than number of cycles.

a_1 edge v_iv_{i+1} has exactly 3 neighbours. Thus we can find 2 a_1 edges v_iv_{i+1}, v_jv_{j+1} with $i \neq j - 2, j + 2$. $(v_{j+l} = v_j)$ Hence $m'(v_{i-1}, v_{i+2})$, $m'(v_{i-1}, v_{i+2})$ each contain at most 2 hit nodes and hence C_1 contains at most 8 hit nodes.

Case 3b). $b + d + a_2 + 2\lfloor 0.5c \rfloor \ = 1$ C_1 has 9 pieces. Here we wish to show there is an a_1 edge with both endpoints having exactly 3 neighbours. If $b = 1$, $a_2 = d = 0$ and from $l \leq 2a_1 + a_2 + c + d$ we get there are at least 3 a_1 edges and the endpoints and thus for at least 1 a_1 edge $v_i v_{i+1}$ v_i, v_{i+1} both (since they are not b nodes) have 3 neighbours. If $d = 1$ then $a_1 \geq 2$ and the endpoints of any a_1 edge have exactly 3 neighbours, since they are not b nodes.

Case 4). Our even cycle C has fewer than 7 pieces, then our inequality will count at most 8 hit nodes. □

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