Bounding the Integrality Gap of Transversal LPs

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1 Introduction

Recall that a graph H is a *minor* of G, if we can obtain H from G through a sequence of edge contractions and deletions, and vertex deletions. In the H*-transversal* problem (HTP) one is given a graph $G = (V, E)$, non-negative costs c_v, for all $v \in V$, and a graph H. The goal is to find a set $S \subseteq V$ such that $G[V \setminus S]$ has no H-minor. In the following, we let $\mathcal{H}$ be the set of vertex subsets of V whose induced subgraphs contain an H-minor; i.e., $S \in \mathcal{H}$ iff $G[S]$ has an H-minor. Consider the following natural LP relaxation of (HTP):

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & x(S) \geq 1 \quad \forall S \in \mathcal{H} \\ & x \geq \mathbb{0}. \end{aligned} \tag{$\mathrm{P_{HTP}}$}$$

It is not hard to see that the integrality gap of the above LP can be large, even in special cases. For example, it is large when H is a planar graph with at least one cycle as was argued in [5]. To see this, let G be an n-vertex graph with girth $\Omega(\log n)$ and treewidth $\Omega(n)$ (e.g., certain Ramanujan graphs [4]), and let H be a triangle. Letting $x_v = 1/\log n$ for all $v \in V$ is easily seen to yield a feasible solution for ($\mathrm{P_{HTP}}$) of value $n/\log n$. On the other hand, any integral solution for the given HTP instance has cost $\Omega(n)$ since H-minor free graphs have treewidth $O(1)$ [5].

2 Hitting even cycles in minor-closed graphs

Closely related to the above is the *even-cycle transversal* problem (ECT) where we are given a graph $G = (V, E)$, vertex costs $c_v \geq 0$ for all $v \in V$, and where the goal is to find a min-cost set $S \subseteq V$ such that $G[V \setminus S]$ has no even cycles. Let $\mathcal{C}$ be the set of vertex sets of even cycles in G. Then we can write the natural LP relaxation of ECT and its dual as follows:

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & x(S) \geq 1 \quad \forall S \in \mathcal{C} \\ & x \geq \mathbb{0}. \end{aligned} \tag{$\mathrm{P_{ECT}}$}$$

$$\begin{aligned} \max \quad & \mathbb{1}^T y \\ \text{s.t.} \quad & \sum_{S \in \mathcal{C}, v \in S} y_S \leq c_v \quad \forall v \in V \\ & y \geq \mathbb{0}. \end{aligned} \tag{$\mathrm{D_{ECT}}$}$$

Similar to the previous argument, we can show that ($\mathrm{P_{ECT}}$) has an integrality gap of $\Omega(\log n)$ in general. We suspect, however, that the LP has integrality gap $O(1)$ when G is from a minor-closed class of graphs. We will now show this for the special case where $c = \mathbb{1}$. We need the following result due to Fomin, Saurabh, and Thilikos [2].

Theorem 2.1. *Let $\mathcal{G}$ be a proper minor-closed graph class and let H be a planar graph. Then there is*

- *a feasible solution $U \subseteq V$ to HTP for G and H, and*
- *a collection $\mathcal{U}$ of pairwise disjoint vertex subsets of V each of which induces a subgraph of G with an H-minor,*

and $|U| \leq c|\mathcal{U}|$ for some constant $c(\mathcal{G}, H)$.

We apply Theorem 2.1 to the given graph G from some minor-closed graph class (e.g., planar), and choose H as the graph on two vertices with three parallel edges. The theorem provides us with disjoint sets of vertices $D_1, \dots, D_p$ such that H is a minor in $G[D_i]$, for all $i \in [p]$, and a set U of vertices such that $G[V \setminus U]$ has no H minor. We furthermore know that $|H| \leq cp$ for some constant c.

Note that $G[D_i]$ contains an H-minor, and hence there are vertices v_i and u_i in D_i, and $G[D_i]$ contains three internally vertex-disjoint v_i, u_i-paths. Clearly, some two of these paths together form an even cycle C_i. We conclude that letting $y_{C_i} = 1$ for all $i \in [p]$ yields a feasible solution for ($\mathrm{D_{ECT}}$). It is clear that U may not be a feasible even cycle transversal in G.

Recall that a *block* in G is an inclusion-wise maximal subgraph that is either a single vertex, a bridge-edge, or a 2-vertex connected subgraph. We then note that $G' := G[V \setminus U]$ is H-minor free, and thus a *cactus*; i.e., a graph in which every block is a simple cycle, or an edge. The *block graph* of G' is an acyclic bipartite graph with vertex set $B_1 \cup B_2$, where B_2 are the blocks of G', and B_1 are the cut-vertices. The block graph has an edge connecting each cut vertex with any of its incident blocks. Let $B'_2 \subseteq B_2$ be the set of block vertices that correspond to *even* cycles of G'. Also let $B'_1 \subseteq B_1$ be the cut vertices with at least two neighbours in B'_2. Let $\mathcal{B}$ be the subgraph of the block-graph induced by the vertices in $B'_1 \cup B'_2$. $\mathcal{B}$ is a forest, and we let $T_1, \dots, T_l$ be its trees. Let $V(T_i) = B'_{i,1} \cup B'_{i,2}$ such that $B'_{i,q} \subseteq B'_q$ for all $i \in [l]$, and $q \in \{1, 2\}$.

W.l.o.g., we assume that $T_1, .., T_j$ are those trees in $\mathcal{B}$ that have only a single node from B'_2 (if no such tree exists, we let $j = 0$). For T_i with $i > j$ choose a root $r \in B'_{i,1}$ and direct the edges of T_i away from r. For each node $a \in B'_{i,1}$, let $b(a) \in B'_{i,2}$ be an arbitrary descendant of a in T_i (such a node exists by the definition of B'_1).

In the following, we abuse notation mildly, and use $b(a)$, for some $a \in B'_{i,1}$, in place of the even cycle it represents in G'. We define a feasible solution $\bar{y}$ for ($\mathrm{D_{ECT}}$) by first letting $\bar{y}_{b(a)} = 1/2$ for all $a \in B'_{j+1,1} \cup \ldots \cup B'_{l,1}$. For $i \in \{1, \dots, j\}$, let C_i be the even cycle corresponding to the single B'_2-node in T_i. We then let $\bar{y}_{C_i} = 1$, for all $i \in \{1, \dots, j\}$. Let $\bar{y}_C = 0$ for all other even cycles. We claim that the $\bar{y}$ constructed is feasible for ($\mathrm{D_{ECT}}$). To see this, note that, by construction, no node $v \in V(G')$ is incident to more than two even cycles with positive $\bar{y}$ value. In fact, if v is incident to two even cycles C_1 and C_2 with positive $\bar{y}$ value, then $\bar{y}_{C_1} = \bar{y}_{C_2} = 1/2$.

For each $1 \leq i \leq j$, let a_i be an arbitrary vertex of C_i. Define

$$\bar{U} = \{a_1, \dots, a_j\} \cup \{a \in B'_{j+1,1} \cup \ldots \cup B'_{l,1} \,:\, \bar{y}_{b(a)} > 0\},$$

and note that $U \cup \bar{U}$ is a feasible solution for the even-cycle transversal problem. Thus, letting $x_v = 1$ for all $v \in U \cup \bar{U}$, and $x_v = 0$, otherwise, yields a feasible solution to ($\mathrm{P_{ECT}}$). The value of this solution is no more than $\max\{c, 2\}$ times the value of the feasible solution $y + \bar{y}$ for ($\mathrm{D_{ECT}}$). We obtain the following result.

Theorem 2.2. *Let G be chosen from some minor-closed family of graphs. Then* ($\mathrm{P_{ECT}}$) *has a constant integrality gap.*

3 Even cycles in planar graphs

In this section, we provide a constant-factor gap for ($\mathrm{P_{ECT}}$) in planar graphs in the case of general vertex costs. We will accomplish this by refining the argument given in [1] for the case of hitting *diamonds*.

A diamond is any sub-division of the graph consisting of three parallel edges. In [1], Fiorini et al. consider the problem of finding a minimum-cost diamond-transversal in a general graph. The authors show that the natural covering LP obtained from ($\mathrm{P_{ECT}}$) by replacing $\mathcal{C}$ by the set $\mathcal{D}$ of vertex sets of diamonds in G has an integrality gap of $O(\log n)$. The proof is constructive and uses a primal-dual algorithm using the natural LP and its dual.

The algorithm in [1] is natural and follows the well-known primal-dual strategy: start with a pair $x = y = \mathbb{0}$ of infeasible primal, and feasible dual solution. The algorithm iteratively modifies x, and y, maintaining the fact that x is $0, 1$, and y is dual feasible, and stops as soon as x is primal feasible. After applying a customary *reverse delete* step, the algorithm arrives at a minimally feasible solution $\bar{x} \leq x$, and the authors show that its total cost is bounded by $O(\log n)$ times the value of dual solution y.

Somewhat more specifically, in every step of the algorithm, where x is primal infeasible, we let X be the vertex set corresponding to x. The algorithm then carefully chooses a diamond D in $G[V \setminus X]$, and increases its dual variable y_D as much as possible, maintining dual feasibility. At this point, the dual packing constraint for a vertex $v \in V \setminus X$ becomes tight, and the algorithm sets $x_v = 1$. Once x is feasible for the LP, the algorithm comptes a *minimal* feasible solution $\bar{x} \leq x$, and the authors show that $y_D > 0$ only if $|D \cap \bar{X}| = O(\log n)$. This suffices to prove that $\bar{X}$ is an $O(\log n)$-approximate diamond hitting set.

In this section, we show that their algorithm can be simplified in the case of even cycles, and that it can be strengthened using the planarity of the underlying graph.

3.1 Ideas and a first attempt

A key first observation is captured in the following Lemma which is a special case of Kotzig's Theorem on Light Planar Subgraphs (e.g., see Section 3 of [3]).

Lemma 3.1. *A planar multigraph $G = (V, E)$ where every vertex has at least 3 distinct neighbours and no faces of length 2 contains 2 adjacent vertices whose degrees sum to at most 13.*

In particular, the above lemma has the following consequence:

Corollary 3.2. *A 2 connected planar graph G of minimum degree 3 and no faces of length 2 contains an even cycle with at most 11 edges.*

Hao's notes have a self-contained proof of the above two statements. From here, our algorithm follows the ideas provided in [1], simplifying and adapting to even cycles where possible.

Corollary 3.2 suggests the following natural algorithm: start with $X = \emptyset$, and $y = \mathbb{0}$. At any point in the algorithm where X is not feasible, compute the *1-compression* G_1 of $G[V \setminus X]$ as follows: as long as G has a node v with exactly two neighbours u and w in G, *contract* v; i.e., replace uv and vw by uw, and delete v. In the resulting graph G_1, all nodes have degree at least 3. Now find a cycle C_1 in G_1 of length at most 11 whose corresponding cycle C in G (which we will also call the *projection* of C_1 in G) is even. Increase y_C as much as possible, and add the newly tight vertices to X. Repeat the above until X is feasible, then run reverse delete, and obtain a minimally feasible set $\bar{X}$. Note that the minimality of $\bar{X}$ implies that, for all $v \in \bar{X}$, there is an even *witness* cycle C_v in G that that $C_v \cap \bar{X} = \{v\}$. More precisely, there is such a witness cycle C_v that is in the projection of the compressed residual graph *at the time* when v was chosen.

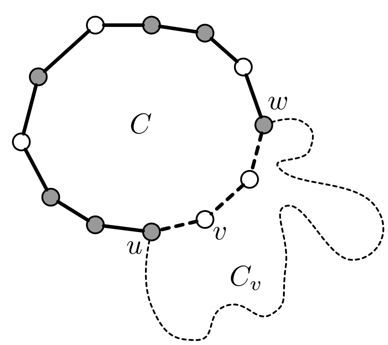

Figure 1: The figure shows even cycle C. White vertices are contracted during compression and have degree 2 in $G[V \setminus X]$, grey vertices have degree at least 3 in the same graph.

Lemma 3.3. *Suppose that the above algorithm terminates with feasible solution $\bar{X}$. Then the total cost of $\bar{X}$ is at most* 11 *times the value of the computed dual solution.*

Proof. Let us consider an even cycle C with $y_C > 0$, and let C_1 be the short cycle in the 1-compression G_1 of the graph $G[V \setminus X]$ at the time where y_C was increased. In Figure 1, white nodes have degree 2 in $G[V \setminus X]$, and grey nodes have degree at least three. Hence the cycle depicted has length 7 in G_1. Suppose that $v \in V(C) \cap \bar{X}$ is a node on C that was chosen by our algorithm for the final transversal.

Consider two adjacent nodes u and w in $V(C_1)$, and let P_{uw} be the corresponding path in G. Let us first assume that v is an internal (degree 2) node of P_{uw} for two adjacent nodes u, w of $V(C_1)$. In this case, note that the witness cycle C_v of v contains all nodes of P_{uw} including u and w themselves. Thus, if $\bar{X}$ contains v then it contains no other nodes from P_{uw}.

Now suppose that $v \in \bar{X} \cap C_1$ is node of degree at least three on C, and let u and w be the neighbours of v on C_1. Using the same argument as before, we see that no internal vertex of P_{uv} and P_{wv} can be in $\bar{X}$ in this case.

Given the above observations it is not hard to see that we therefore must have $|\bar{X} \cap C| \leq 11$. Hence, the solution $\bar{X}$ has cost no more than $11 \sum_C y_C$. $\square$

Unfortunately, the above algorithm does not always terminate. The reason is that we may not be able to find an even cycle whose 1-compression has at most 11 edges. Consider for example the graph depicted in Figure 2. This graph has many even cycles, albeit no short ones! Notice that the compression of this graph has faces of length 2, and hence Corollary 3.2 does not apply.

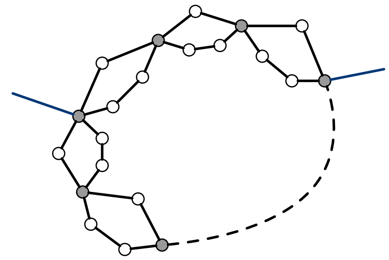

Figure 2: A graph in which every even cycle is long.

3.2 Dealing with graphs without even cycles of small compressed length

Consider even cycles on vertex sets S_1 and S_2, and let T be disjoint from $S_1 \cup S_2$. Define $a_v^{S_1,S_2,T} = 1$ for all $v \in (S_1 \cap S_2) \cup T$, $a_v^{S_1,S_2,T} = 1/2$ for $v \in S_1 \setminus S_2 \cup S_2 \setminus S_1$, and $a_v^{S_1,S_2,T} = 0$, otherwise. Note that the following inequality is dominated by the two original cover inequalities for sets S_1 and S_2:

$$\sum_v a_v^{S_1,S_2,T} x_v \geq 1.$$

Hence, we obtain a new pair of LPs that are equivalent to ($\mathrm{P_{ECT}}$) and its dual. Here, we abuse notation, and let $\mathcal{C}$ now be the set of triples (S_1, S_2, T) where S_1 and S_2 are (not necessarily disjoint nor different) vertex sets of even cycles in G, and T is a (not necessarily non-empty) set of vertices in $V \setminus (S_1 \cup S_2)$.

$$\begin{aligned} \min \quad & c^T x & (\mathrm{P_{ECT}}) \\ \text{s.t.} \quad & \sum_{v \in (S_1 \cap S_2) \cup T} a_v^{S_1,S_2,T} x_v \geq 1 \quad \forall (S_1, S_2, T) \in \mathcal{C} \\ & x \geq \mathbb{0}. \end{aligned}$$

$$\begin{aligned} \max \quad & \mathbb{1}^T y & (\mathrm{D_{ECT}}) \\ \text{s.t.} \quad & \sum_{(S_1,S_2,T) \in \mathcal{C}} a_v^{S_1,S_2,T} y_{S_1,S_2,T} \leq c_v \quad \forall v \in V \\ & y \geq \mathbb{0}. \end{aligned}$$

Note that the original cover inequalities for even cycles on vertex sets S are contained in the reformulation of ($\mathrm{P_{ECT}}$), by letting $S_1 = S_2 = S$, and $T = \emptyset$. For convenience we will write y_S in place of $y_{S,S,\emptyset}$ from here on.

Our algorithm maintains a pair (x, y) of (partial) primal, and dual solutions for the above pair of LPs; for convenience, we let X be the set of vertices for incidence vector is x. At any time, the algorithm will consider the 1-compression G_1 of the graph $G[V \setminus X]$. The algorithm works as before if G_1 contains a simple, short cycle (without parallel edges) whose projection in G has even-sized vertex set S. In this case we increase the dual variable y_S as described in the previous section as much as possible, adding newly tight vertices to set X.

Note that the same argument also works if there is a pair of vertices u and v that are connected by at least three edges e_1, e_2, and e_3 in G_1. In this case, two of these edges, say e_1 and e_2, project to an even length cycle S in G. We proceed as before.

From here on, we assume that G_1 has at most two edges connecting every pair of vertices. We also assume, w.l.o.g., that every edge of G_1 is contained in some cycle with even projection, and therefore, G_1 is 2-connected. Assume now that G_1 has cycles with even projection, but none that are short and simple. In this case, Corollary 3.2 implies that G_1 cannot be simple.

We obtain graph $\bar{G}_1$ from G_1 by replacing each pair of parallel edges connecting vertices u and v by a single *twin* edge uv. Note that each face in this new graph has length at least 3. Obtain the 2-compression G_2 of G by contracting degree-2 nodes in $\bar{G}_1$; see Figure 3. In the following, abusing notation slightly, we call edges created by contracting vertices in $\bar{G}_1$ *twin* (as their projection in G_1 must contain twin edges). Using G_2, we will now identify a certain even *cycle* whose dual variable we increase. We branch into two cases.

G_2 is simple. We first assume that any pair of vertices is connected by at most 1 edge in G_2. In this case, Corollary 3.2 implies that G_2 has an even cycle C_2 of length at most 11. Since this cycle did not exist in G_1, it must contain at least one twin edge.

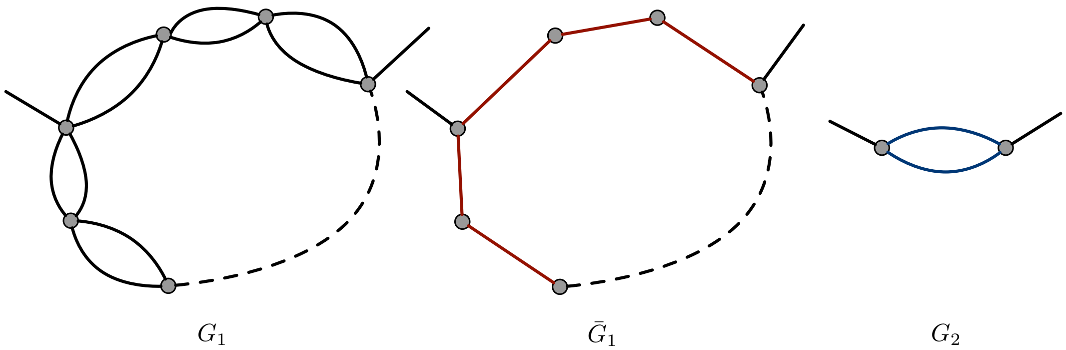

Figure 3: The figure shows 1 and 2-compression G_1 and G_2 of the graph shown in Figure 2. In $\bar{G}_1$ we replace parallel edges in G_1 by twin edges.

Each edge of C_2 projects to a well-defined subgraph of G which, following [1], we refer to as a *piece* of C_2. The piece corresponding to a non-twin edge of C_2 is a path whose internal nodes have degree 2 in $G[V \setminus X]$. The piece corresponding to twin edge uv of C_2 is a u, v-path in $\bar{G}_1$ with twin and non-twin edges. A twin-edge $u'v'$ on such a path corresponds to an induced subgraph of $G[V \setminus X]$ consisting of two internally vertex disjoint u', v'-paths S_1 and S_2 that, together, form an odd cycle. In summary, one now sees that the *block-graph* of the projection of twin edge uv of C_2 is a path in $G[V \setminus X]$. Moreover, the blocks in this subgraph are odd cycles; see Figure 4 for an example.

Let us focus on a piece P corresponding to a twin edge uv of C_2. Vertices $w \in V(P) \setminus \{u, v\}$ have degree 2 in G_1, or they are *cut-vertices* of P, and have degree 2 in $\bar{G}_1$. In Figure 4 we have coloured such vertices in red.

In the following, we keep track of the *slack* in the dual constraint of each vertex v for the current dual feasible solution for ($\mathrm{D}_{\mathrm{ECT}}$):

$$\bar{c}_v := c_v - \sum_{(S_1,S_2,T)\in\mathcal{C}} a_v^{S_1,S_2,T} y_{S_1,S_2,T}.$$

We now define a *canonical* subgraph of each piece P. This subgraph is induced by a set of vertices that we classify as *type-1* or *type-2*. If uv is a non-twin edge then we add all vertices of its projection to S_1. Now suppose that uv is a twin edge of G_2, and let P_1 be its projection in $\bar{G}_1$. All vertices of P_1 are added as type-1.

Let u' and v' be two neighbouring vertices on P_1 that are connected by twin edges e_1 and e_2 in $\bar{G}_1$ (see Figure 4). Let S_1 and S_2 be the paths in $G[V \setminus X]$ corresponding to e_1 and e_2, respectively. We let the residual cost $\bar{c}(S)$ of a path S be the smallest residual cost of any of its internal nodes. If the maximum

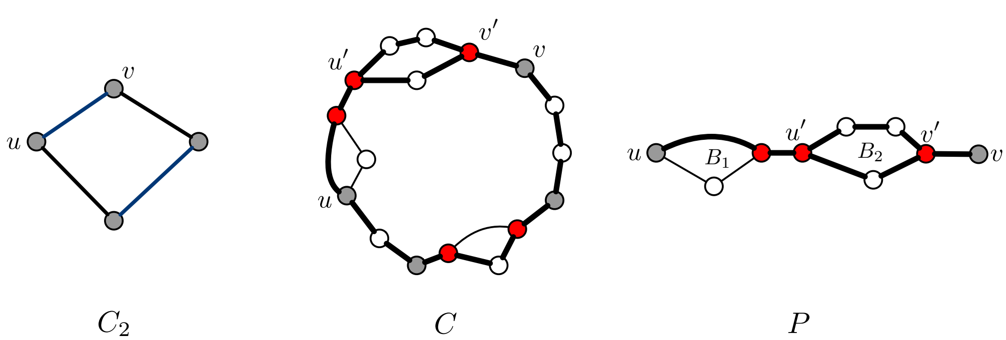

Figure 4: The first two figures show a short cycle C_2 in G_2, and its corresponding projection in G. The right figure above shows the piece of twin edge uv of C_2. Note the thick edges of cycle B_2 above: these edges are part of T.

among $\bar{c}(S_1)$ and $\bar{c}(S_2)$ is unique, then add the vertices of the maximizer as type-1. Otherwise label u' and v' type-1, and make all internal vertices of S_1 and S_2 type-2. We will call the vertices of S_1 and S_2 together with u' and v' a *type-2 cycle*.

Suppose first that the number of type-2 cycles over all pieces of C_2 is odd. In this case, pick an arbitrary such cycle with paths S_1 and S_2. Add the internal vertices of S_1 and S_2 to T, and *erase* their type. For any $v \in V$, we now let

$$a_v = \begin{cases} 1 & \text{if } v \text{ is type-1, or if } v \in T \\ 1/2 & \text{if } v \text{ is type-2} \\ 0 & \text{otherwise.} \end{cases}$$

The inequality

$$\sum_v a_v x_v \geq 1 \tag{$\circledast$}$$

can be seen to be part of ($\mathrm{P}_{\mathrm{ECT}}$). The algorithm now increase the dual variable $y_\circledast$ corresponding to inequality ($\circledast$) as much as possible, adding tight vertices to X.

G_2 **is not simple.** In this case, G_2 has two vertices u and v that are connected by at least two edges e_1 and e_2. By assumption $\bar{G}_1$ is a simple graph. Thus, at least one of e_1 and e_2, say e_1, is a twin edge. In this case we let C_2 be the cycle formed by e_1 and e_2 and proceed as above.

3.3 Analysis

The algorithm terminates at the first time when X is a feasible solution to ECT. We then obtain a minimal feasible solution $\bar{X}$ through a reverse-delete step. Let y be the final feasible dual solution computed by the algorithm. Our goal is to prove that the computed soluition is 11-approximate. For this it suffices to show that

$$\sum_v a_v^{S_1,S_2,T} \leq 11, \tag{1}$$

whenever $y_{S_1,S_2,T} > 0$.

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