A 7 Approximation for Hitting the Even Cycles of a Planar Graph

1 Introduction

In this short note we present a constant-factor approximation algorithm for the *even-cycle transversal* (ECT) problem: Given a planar graph $G = (V, E)$ and weights $w : V \to \mathbb{R}_+$ find a minimum weight set $H \subset V$ such that $C \cap H \neq \emptyset$ for all even cycles C in G. The problem of hitting all cycles has a 2 approximation [1] and a PTAS [2] . [4] consider for planar graphs, a more general problem where we are given a set of cycles $\mathcal{C}$ satisfying some "uncrossing property" (see [4]) is given and we wish to find a minimum weight set $H \subset V$ of vertices, such that $C \cap H \neq \emptyset$ for all $C \in \mathcal{C}$. They give a 3 approximation algorithm for this problem in planar graphs and propose an improved approximation using a so called "pocket oracle" which they claim is a 9/4 approximation. [5] show that the approximation of the pocket oracle proposed in [4] is not 9/4, but 18/7 instead and also give a 2.4 approximation algorithm using a new "three-pocket oracle". The approach presented here uses the primal-dual framework of Goemans and Williamson [3, 4].

2 Even cycles

In the following, we fix an embedding of graph G. For a cycle M in G, we let $f(M)$ be the faces in the interior region of M. Given a family $\mathcal{C}$ of cycles, a cycle $M \in \mathcal{C}$ is *face-minimal* (with respect to $\mathcal{C}$) if there is no $M' \in \mathcal{C}$ with $f(M') \subsetneq f(M)$. In the following, we also abuse notation slightly, and use $A \subset B$ as a short-hand for graph A being a subgraph of graph B. The following lemma captures a key property for our algorithm.

Lemma 2.1. *Let C be a face minimal even cycle of our graph, then C contains at most 2 faces of our graph.*

Proof. Let $F_1, F_2, , ..F_l$ be the faces of C assume for a contradiction $l \geq 3$ then $|E(C)| = |E(F_1)| + |E(F_2)| + .. + |E(F_l)| \mod 2$. Thus if l is odd one of the faces of C must be even. If $l \geq 4$ is even, then remove a path $P \subset E$ bordering 2 faces of C in $G \backslash P$, C has $l-1$ faces $F'_1, F'_2, ..., F'_{l-1}$, $l - 1 \geq 3$ is odd and one of the F'_i is even. This cycle is strictly contained in C contradiction. □

The above lemma easily implies that the set of face-minimal even cycles can be found efficiently by checking all even faces, and all adjoining faces in G. Suppose that our goal is to hit a set $\mathcal{C}$ of cycles. The algorithm presented here is then based on the following natural pair of linear programs.

$$\begin{aligned} \min \quad & \sum_{v \in V} w_x x_v && \text{(P)} \\ \text{s.t.} \quad & \sum_{v \in C} x_v \geq 1 \quad \forall C \in \mathcal{C} \\ & x \geq \mathbb{0} \end{aligned}$$

$$\begin{aligned} \max \quad & \sum_{C \in \mathcal{C}} y_C && \text{(D)} \\ \text{s.t.} \quad & \sum_{C:\ v \in C} y_C \leq w_v \quad \forall v \in V \\ & y \geq \mathbb{0} \end{aligned}$$

In the following we state the classicial primal-dual framework for feedback vertex set problems as was previously described by Goemans and Williamson [3, 4]. In the algorithm, we use $u \bullet M$ as a short for "*u is a vertex on cycle M*", and we use **violation**$(G, \mathcal{C}, S)$ to denote a call to a so called *violation oracle* that, given a graph G, cycles $\mathcal{C}$, and a partial solution S, returns a *minimal* collection of cycles that are not hit by S.

Algorithm 2.1: [4] Generic primal-dual algorithm for feedback vertex set problem given by $(G(V,E), w, \mathcal{C})$

$y = 0$
Let $w(y,u) = w(u) - \sum_{M \in \mathcal{M}: u \bullet M} y_M$
$S = \{u \in V : w(u) = 0\}$
while S is not a hitting set for $\mathcal{C}$ **do**
  $\mathcal{M} =$ **violation**$(G, \mathcal{C}, S)$.
  $c_{\mathcal{M}(u)} \leftarrow |M \in \mathcal{M} : u \bullet M|, \forall u \in V$.
  $\alpha \leftarrow \min_{u \in V \setminus S} |w(y,u)|/c_{\mathcal{M}(u)}$
  $y_M = y_M + \alpha$, for all $M \in \mathcal{M}$ $S \leftarrow \{u \in V : w(y,u) = 0\}$.
end while
return a minimal hitting set $H \subset S$ of $\mathcal{C}$.

In [4], the set $\mathcal{C}$ is the set of all directed cycles in a directed planar graph. For us, $\mathcal{C}$ is the set of even cycles in an undirected planar graph G. Given a partial hitting set $S \subseteq V$, our oracle **violation** will return a maximal set of face-minimal even cycles.

Lemma 2.2 ([4])**.** *Suppose that Algorithm 2.1 returns solution $F \subseteq V$, and that it generates minimally violated sets of cycles $\mathcal{M}_1, \ldots, \mathcal{M}_q$ during its execution. F is an α-approximate solution if*

$$\sum_{M \in \mathcal{M}_i} |F \cap M| \leq \alpha |\mathcal{M}_i|,$$

for all $1 \leq i \leq q$.

Consider an inclusion-wise minimal hitting set $H \subseteq V$. Then note that the minimality of H implies that, for each $h \in H$, there is at least one $M_h \in \mathcal{C}$ such that h is the only H-vertex on M_h. We call such a cycle M_h a *witness cycle* of h.

Definition 2.3. *For a hitting set $H \subset V$, $\mathcal{M}$ a set of face-minimal cycles of our graph, define the debit graph $B = (\mathcal{M} \cup H, E)$. Where $(M, h) \in E$ if node h is on cycle M.*

Like in [4], we will sometimes think of our debit graph as drawn (embedded in the plane) with the node v_M for cycle M (in debit graph) located at the center of the cycle M (in G) (nodes h where they are in the original graph G). In [4] for the case of all faces the debit graph is planar we will show this also happens to be the case here. Notice that $\sum_{C \in \mathcal{M}} |H \cap C| = |E(B)|$.

The following is an easy 3 approximation for GW in planar graphs if each witness cycle was incremented in our iteration.

Lemma 2.4. *For a bipartite planar graph $G = (A, B)$ such that for each node $b \in B$ is incident to a node of degree 1 in A then $|E(G)| \leq 3|A|$.*

Proof. Let for each $b \in B$ choose a node $a_b \in A$ of degree 1 and denote $A' := \{a_b \quad b \in B\}$ and consider $G' = G[(A \backslash A' \cup B)$ applying Euler's formula (for bipartite planar graphs) to G' we get $E(G') \leq 2|A \backslash A' \cup B|$. Since $E(G) = E(G') + |A'| \leq |A'| + 2|A \backslash A' \cup B| \leq 2|A| + |A'| \leq 3|A|$. $\square$

Corollary 2.5. *In Lemma 2.4 if at most k nodes of B in G=(A,B) are not incident to nodes of degree 1 then $|E(G)| \leq (3 + 3k/|A|)|A|$.*

Proof. Construct G' from B by adding k nodes of deg 1 to the A side of G. $|E(G')| \leq 3(|A| + k) \leq (3 + 3k/|A|)|A|$ $\square$

Note that this gives an easy 3 apx for instances of planar FVS where all witness cycles of our hitting set, for each iteration is a cycle incremented in our iteration. In the following arguments we will think of paths and cycles as being subgraphs and for graph H and path P use the convention $H \backslash P := H \backslash ((P \backslash \{u\}) \backslash \{v\})$ that is we delete the interior of the path from H.

Given a planar graph G let us call an even cycle C of G face minimal, if no other even cycle of G is contained in the (finite) region bounded by C. Let us define two cycles to be crossing the same as in Goemans/ Williamson.

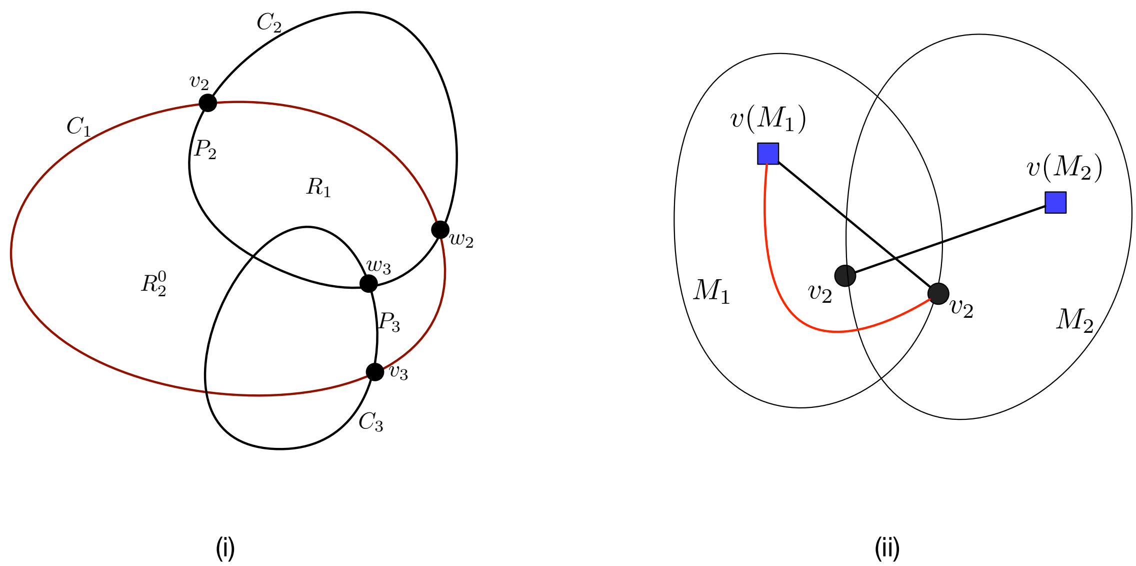


Figure 1: Figure (i) illustrates crossing even cycles as discussed in Theorem 2.7. Figure (ii) illustrates the fact that the debit graph for our oracle is planar.

Definition 2.6. *We say that cycles A_1 and A_2* cross *if the set of common faces is a proper subset of the faces of A_1 and A_2. E.g., in Figure 1.(i), cycles C_1 and C_2 cross.*

We are now ready to show that face minimal cycles can not cross arbitrarily.

Theorem 2.7. *Each face minimal even cycle crosses at most one other face minimal even cycle.*

Proof. Suppose that a face minimal even cycle C_1 crosses 2 other face minimal even cycles C_2, C_3. Let P_2 be a subpath of C_2 s.t. the internal nodes of P_2 are in the interior of C_1, and the endpoints of P_2 are on C_1. Clearly, P_2 divides the interior of C_1 into two regions, R_1 and R_2^0. Suppose, w.l.o.g., that C_3 intersects the interior of R_2^0. Let P_3 be a subpath of C_3 such that its internal nodes lie in the interior of R_2^0, and its endpoints lie on the boundary of R_2^0. Hence, P_3 subdivides R_2^0 into R_2 and R_3.

Now note that the sum of the lengths of R_1, R_2, and R_3 equals $|C_1| + 2(|P_2| + |P_3|)$, an even number by assumption. Hence, one of the R_i must be an even cycle, contradicting the face-minimality of C_1. (see figure 1.(i)) $\square$

An immediate consequence of the above theorem is that our debit graph is planar.

Corollary 2.8. *The debit graph for our oracle is planar.*

Proof. As in [4] given the planar embedding of our graph G, we draw (that is, we embed in the plane) our debit graph by placing a node $v(M)$ representing cycle M in the interior of the region bounded by M in G and draw the edges $(v(M), h)$ of our debit graph by drawing an edge from v_M to node h in G.

In this drawing the only potential pairs of crossing edges are of the form $(v(M_1), h_1)$, $(v(M_2), h_2)$ where M_1 and M_2 are two crossing even cycles (see Figure **??**.(ii)). However, since (by Theorem **??**) each face minimal even cycle crosses at most one other face minimal even cycle. For such a crossing pair, we can *detour* (see Figure **??**.(ii)) our edges so they don't cross. □

Theorem 2.9. *We may choose a family $\mathcal{A}$ of (even) witness cycles such that if two witness cycles $A_2, A_3 \in \mathcal{A}$ cross $A_1 \in \mathcal{A}$ then there are vertices $u, v \in A_2 \cap A_3$ such that $A_i = Q \cup Q_i$ for i=2,3 where Q, Q_2, Q_3 are u,v paths and every other cycle of $\mathcal{A}$ crossed by A_2 or A_3 intersects A_2, A_3 at u, v only.*

Proof. First let us prove that a cycle of $M \in \mathcal{M}$ cannot cross 2 cycles A_1, A_2 of $\mathcal{A}$. The proof is basically the same as the proof of theorem 2.7. Let P_1 be a subpath of A_1 lying in the region bounded by M s.t. the internal nodes of P_1 are in the interior of M and the endpoints of P_1 are on M. P_1 path divides M into two regions M_1, M_2^0. W.l.o.g., A_2 intersects the interior of M_2^0. Let P_2 be a subpath of A_2 such that its internal nodes lie in the interior of M_2^0, and its endpoints lie on the boundary of M_2^0. P_2 divides M_2^0 into 2 regions, which we call M_2 and M_3. Note that the sum of the lengths of M_1, M_2, and M_3 equals $|M| + 2(|P_1| + |P_2|)$, which is even. So one of the M_i is even contradicting face-minimality of M.
Next we introduce the uncrossing algorithm of [4] on our witness cycles.

Definition 2.10. *[4] For a set of witness cycles $\mathcal{A}$ any two even cycles $A_1, A_2 \in \mathcal{A}$ that cross, we define the uncrossing operation: Let $P_1, P_2, ..., P_l, Q_1, Q_2, .., Q_l$ be a series of internally disjoint paths such that P_i (resp Q_i) are subpaths of A_1 (resp A_2) for each i, P_i has the same endpoints as Q_i and , all P_i (resp Q_i) lies in the interior of the region bounded by A_2 (resp A_1). (Put simply A_1, A_2 cross each other at P_i, Q_i see figures 2 3.) If for some $S \subset [l]$, if both of $A_1' := A_1 \cup (\cup_{i \in S} Q_i) \backslash (\cup_{i \in S} P_i), \quad A_2' := A_2 \cup (\cup_{i \in S} P_i) \backslash (\cup_{i \in S} Q_i)$ are even, and contain exactly one hit node then replace A_1, A_2 with A_1', A_2' in $\mathcal{A}$, that is we "uncross" the specified P_i, Q_i. (See figure 2.) Otherwise, if for some i, $P_i \cup Q_i \in \mathcal{C}$ is even, and contains exactly one hit node and there is an even cycle C in $(A_1 \cup A_2 \backslash P_i) \backslash Q_i$ containing exactly one hit node, define $A_1' = P_i \cup Q_i, \quad A_2' = C$. (See figure 3.) Replace A_1, A_2 by A_1', A_2' in $\mathcal{A}$. (Otherwise we will say that A_1, A_2 cannot be uncrossed and the operation does nothing.)*

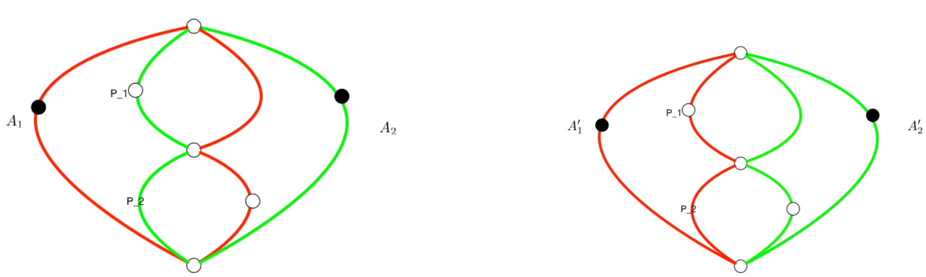


Figure 2: Example of uncrossing with $A_1' := A_1 \cup (\cup_{i \in S} Q_i) \backslash (\cup_{i \in S} P_i), \quad A_2' := A_2 \cup (\cup_{i \in S} P_i) \backslash (\cup_{i \in S} Q_i)$

One can adapt the proof of lemma 4.2 in [4] to show that this kind of uncrossing action will eventually terminate. Further the uncrossing action does not increase the number of crossing pairs. $(|(A_1, A_2) \in \mathcal{A} \times \mathcal{A} \quad A_1 \text{ crosses } A_2|)$ Thus starting with any set of witness cycles $\mathcal{A}'$, with the fewest

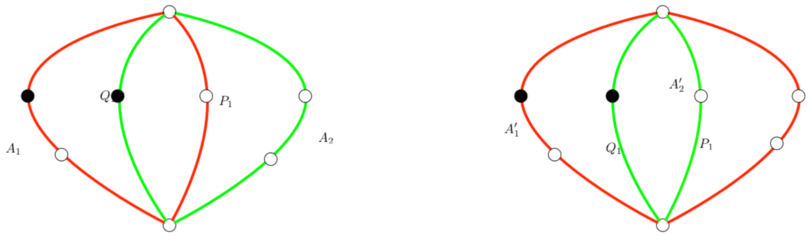
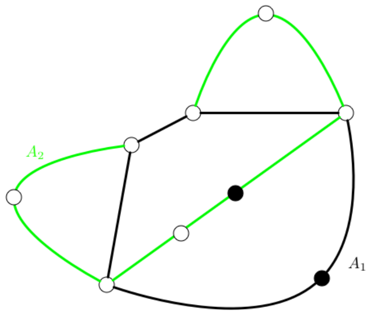


Figure 3: Example of uncrossing with $A'_1 = P_i \cup Q_i, \quad A'_2 = C$



number of crossing pairs, using repeated applications of the uncrossing algorithm in definition 2.10, we can construct a set of witness cycles $\mathcal{A}$, such that the uncrossing procedure of definition 2.10 can not be applied to any two cycles of $\mathcal{A}$ and also has the fewest number of crossing pairs.
Let us also introduce another uncrossing procedure

Definition 2.11. *If A_1, A_2 cross and A_2 consists of internally disjoint paths $Q, R_1, P_1, R_2, P_2.., P_l, R_l$ in that order with P_i, lies outside A_1 R_i lie on A_1 Q lies inside A_1 for each P_i, let B_i be the portion of A_1 connecting the endpoints of P_i such that $P_i \cup B_i$ does not contain A_1 if it is possible to replace a subset of P_i with B_i and still have an even witness cycle, we do so.*

Lemma 2.12. *Let A_1 , A_2 be two crossing witness cycles. Then there does not exist 2 subpaths P_1, P_2 of A_2 lying in the region bounded by A_1, (in our embedding of G in the plane) such that P_1, P_2 are internally disjoint and each intersects A_1 at their endpoints , put simply, A_2 does not cross A_1 twice. (see figure 4) Further if Q_1 is a subpath of A_2 with endpoints a_1, b_1 on A_2 internally disjoint from A_1, then Q_1 has different parity than the paths between a_1, b_1 in A_1.*

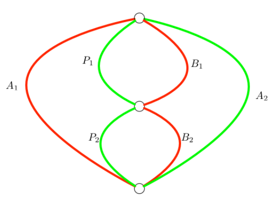


Figure 4

Proof. First suppose for a contradiction that P_1, P_2 are two such paths, choose P_1, P_2 so that there is a path in A_1 internally disjoint from A_2 connecting an endpoint a_1 of P_1 with an endpoint b_2 of P_2. (That is P_1, P_2 are two "consecutive" crossings) Choose paths $B_1, B_2 \subset A_1$ internally disjoint connecting the endpoints of P_1, P_2 respectively. (see figure 4) Now we may uncross either P_1, B_1 , P_2, B_2 or both unless P_1, B_1 or P_2, B_2 has exactly one hit node. WLOG P_1 contains the hit node of A_2.

If P_2, B_2 does not contain a hit node. In this case, P_2, B_2 must be of different parity. (otherwise $P_2 \cup B_2$ is an even cycle not hit) Thus the cycles $(A_2 \backslash P_1) \cup B_1$, $\quad ((A_2 \backslash P_1) \cup B_1 \backslash P_2) \cup B_2$ are of different parity, so one must be even, but neither contains a hit node contradiction. (See figure 5) Thus the hit node of A_1 lies on B_2. If the closed walk $W := ((A_1 \backslash B_1) \backslash B_2) \Delta (A_2 \backslash P_1) \backslash P_2)$ is

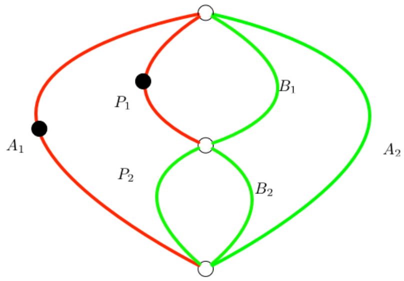


Figure 5: One of the green cycles is even and not hit.

not a cycle, then it contains an odd cycle C'. Then $(A_2 \cup B_1 \backslash P_1), (A_2 \cup B_1 \backslash P_1) \Delta C'$ are cycles of different parity, and are not hit, which is a contradiction. (see figure 6) So W is a cycle and has

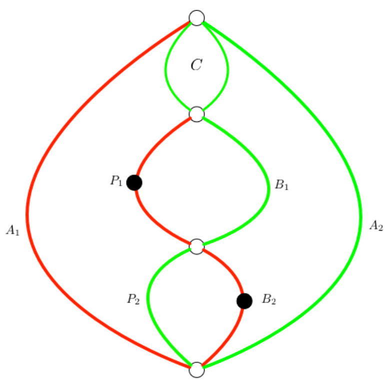


Figure 6: One of the green cycles is even and not hit which shows that A_1, A_2 cannot have "too much intersection" so to speak.

length $|E(A_1)| + E(A_2) - |E(P_1 \cup B_1)| - |E(P_2 \cup B_2)| - 2|E(A_1) \cup E(A_2)|$. As $|E(P_i \cup B_i)|$ are both odd we have this cycle is even, but not hit, a contradiction. (see figure 7)

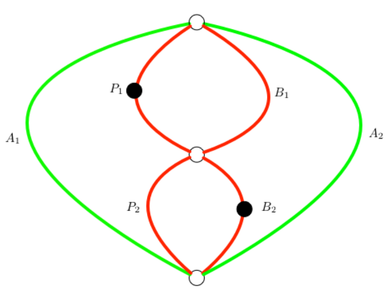


Figure 7: Green cycle above is even and not hit.

The last point (that P_i, B_i are different parity) is because otherwise one of the hit nodes of A_1, A_2 must lie on the even cycle $P_i \cup B_i$. W.l.o.g the hit node of A_2 lies on P_1 then $A_2 \cup B_1 \backslash P_1$

$\square$

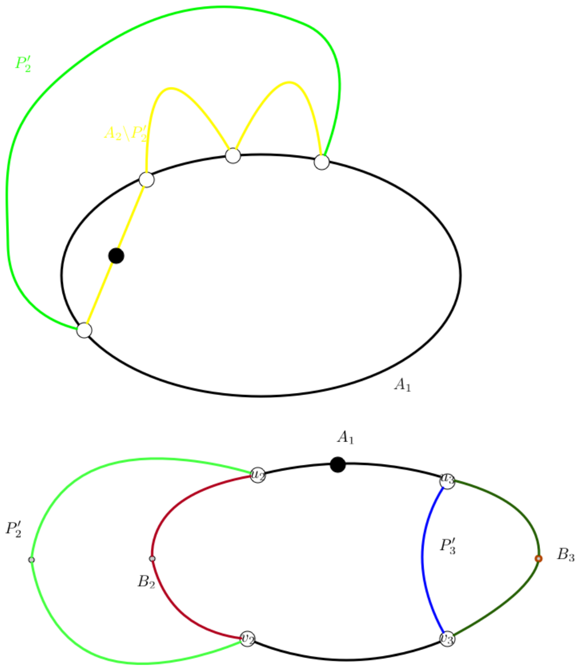


Figure 8

Suppose for a contradiction, that for some $A_1, A_2, A_3 \in \mathcal{A}$,$A_1$ crosses A_2 and A_3 let P_2, P_3 be the portions of A_2, A_3 lying in A_1 let u_2, v_2 (resp u_3, v_3) be the endpoints of P_2 resp P_3. If P_i does not contain a hit node, then set $P_i' = P_i$ otherwise, let P_i' be the intersection of the boundary of $A_1 \cup A_i$ and A_i. If $P_i' = P_i$ define B_i to be the portion of A_1 lying in A_i, otherwise define B_i to be the portion of A_1 lying in the interior of a region of a subgraph of $A_1 \cup P_i'$. (see figure **??**)

Let us prove the following structural lemma which will be used later.

Lemma 2.13. *(P_i', B_i as defined above) It cannot be that P_2', P_3', are internally disjoint and B_2, B_3 are either internally disjoint, or one is a subpath of the other.*

Proof. Assume otherwise, since P_2', P_3', B_2, B_3 are internally disjoint, $(A_1 \cup P_2' \cup P_3') \backslash (B_1 \cup B_2)$ is an even cycle. So the hit node of A_1 does not lie on B_2 or B_3. Letting u_i, v_i be the ends of P_i' let us denote by C_i the subpath of A_1 from u_i to v_i not containing the hit node. We claim that for $i = 2, 3$ the cycle $D_i := (A_i \cup B_i) \backslash P_i$ does not cross any other cycle of $\mathcal{A}$. Assume the contrary, that D_i crosses $W \in \mathcal{A}$. Let $W = W_1 \cup .. \cup W_t$ be a decomposition of W into internally disjoint paths which are internally disjoint from D_i. W.l.o.g W_1 does not contain the hit node of W. Let a, b be the endpoints of W_1 and let P be a path between a, b in D_i. Then by Lemma 2.12, W_1, P are different parity so $D_i \backslash P_i \cup W_1$ is an even cycle that is not hit contradiction. This proves that replacing C_i or $A_1 \backslash C_i$ with P_i' in A_1 decreases the number of crossings. If B_2, B_3 are internally disjoint, note one of $A' := A_1 \cup P_2' \backslash B_2$, $A" := A_1 \cup P_3' \backslash B_3$, $A"'_1 := A_1 \cup P_2' \cup P_3' \backslash (B_2 \cup B_3)$, is even and contains at most one hit node. If $B_2 \subset B_3$ note one of $A' := A_1 \cup P_2' \backslash B_2$, $A" := A_1 \cup P_3' \backslash B_3$, $A"'_1 := (A_1 \backslash B_3) \cup P_3 \cup P_2 \cup (B_3 \backslash B_2)$ is even and contains at most one hit node. In both cases replacing A_1 with $A_1', A"_1$, or $A"'_1$ in $\mathcal{A}$ yields a family of witness cycles with fewer crossing pairs. This is a contradiction. (see figure 9) $\square$

Lemma 2.14. *It cannot be the case that u_2, v_2 lies on the same side of u_3, v_3. (that is they lie on the same subpath of A_1 that u_3, v_3 divides A_1 into)*

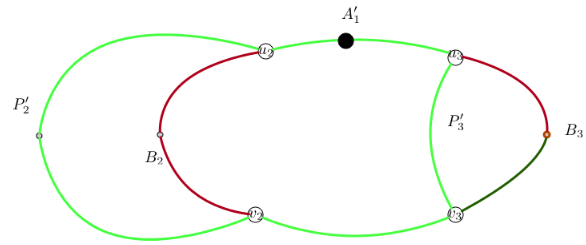


Figure 9

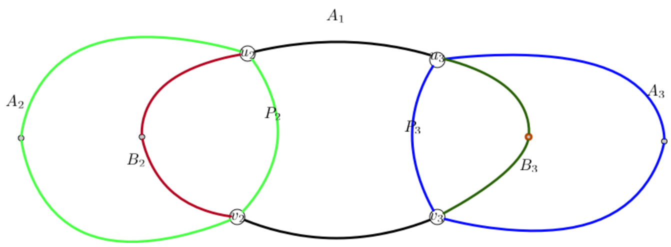


Figure 10: Case 1)

Proof. Thus by lemma 2.13 P_2', P_3' are not internally disjoint, then Suppose for a contradiction, there are nodes r, r' of P_3' such that there is subpath R_2 of P_2 connecting r, r'. Let R_3 be the subpath of P_3' between r,r' in P_3'. Then R_2, R_3 have different parity. W.l.o.g. the hit node of A_1 does not lies on B_3, then the cycles $P_3' \cup B_3$, $(P_3' \backslash R_3) \cup B_3 \cup R_2$ have different parity but no hit nodes, which is a contradiction. (see figure 11) Thus $(P_3' \Delta P_2') \cup B_2 \cup B_3$ is an even cycle (parity

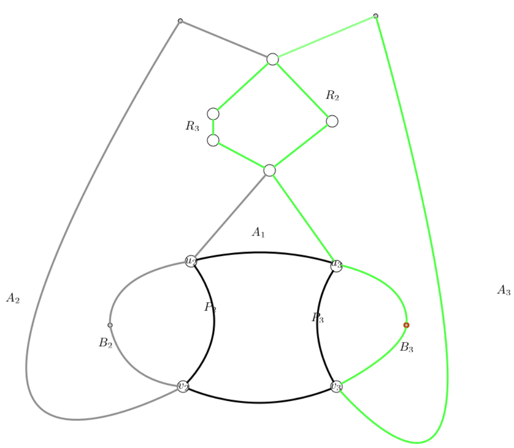


Figure 11: One of the green cycles is even and not hit

follows because P_i', B_i are different parity) thus the hit node of A_1 lies on $B_2 \cup B_3$. W.l.o.g. it lies on B_2. (see figure 12) Let r be the first common vertex of P_2', P_3' and r' be the last common vertex (when viewing P_i' to go from v_i to u_i) Denote by F_i' the subpath of P_i' from v_i to r and H_i' the subpath of P_i' from r' to u_i; by H the subpath of A_1 from u_2 to u_3 and F the subpath of A_1 from v_2 to v_3 not hitting u_2. Now notice that the sum of the lengths of the cycles $F_2 \cup F_3 \cup F$ and $H_2 \cup H_3 \cup H$ is even and are not both trivial by assumption. Since the hit node of A_1 lies on B_2, both $F_2 \cup F_3 \cup F$, $H_2 \cup H_3 \cup H$ are odd so one of $B_3 \cup P_3', B_3 \cup P_3' \Delta (F_2 \cup F_3 \cup F)$ is even and not hit contradiction.

□

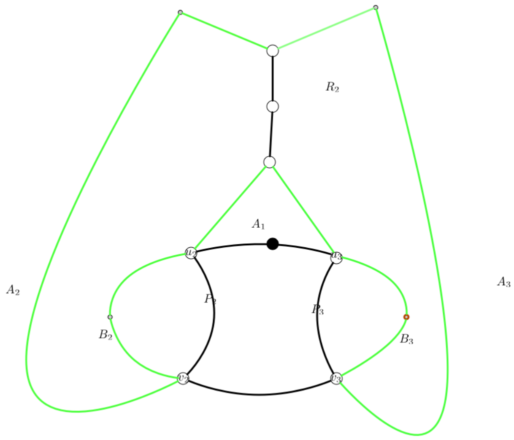


Figure 12

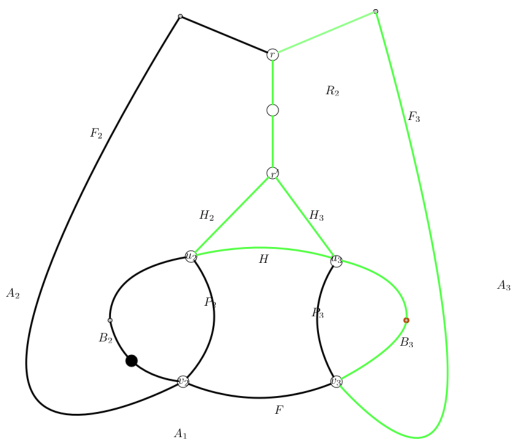
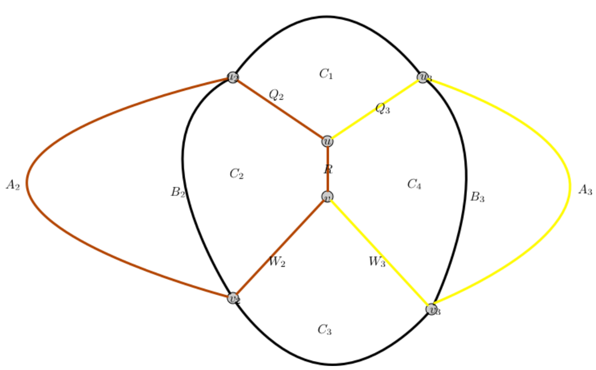


Figure 13

We consider the following cases:

Case 2 P_2, P_3 intersect.



Case 2 a) u_2, v_2 (can be thought of to) lie on the same side of v_3, u_3, then let P_2, P_3 intersect at u, v that is there exists a path $R \subset P_i$ connecting u, v. Again choose subpaths B_2, B_3 of A_1 with B_i connecting u_i to v_i and B_2, B_3 internally disjoint. By lemma 2.13 one of $(= 2, 3)$ $A_i \backslash P_i$ contains a hit node. Define Q_i to be the path from u to u_i, and W_i the path from v to v_i so $(A_2\backslash P_2) \cup W_2 \cup W_3 \cup Q_2 \cup Q_3 \cup B_3$, $\;(A_2\backslash P_2) \cup W_2 \cup W_3 \cup Q_2 \cup Q_3 \cup (A_3\backslash B_3)$. Thus w.l.o.g. P_2 contains no hit node denote by R the path from u to v in P_2 . Let C_1, C_2, C_3, C_4

be the faces of $A_1 \cup P_2 \cup Q_3 \cup W_3$ in counter clockwise order with C_2 the portion bounded by B_2, P_2 . Since R is part of P_2 , C_2 is odd (lemma 2.12) so either C_1 and $C_4 \Delta C_2 \Delta C_3$ are both even, or $C_1 \Delta C_2$ and $C_3 \Delta C_4$ are both even hence the hit node of A_3 lies on $Q_3 \cup W_3$. WLOG the hit node of A_3 lies on Q_3. If the hit node of A_1 lies on C_3 then $A_3 \cup (A_1 \Delta C_3 \Delta C_2)$, $A_3 \cup (A_1 \Delta C_3)$ are different parity and not hit if the hit node of A_1 lies on C_2 then C_3 is odd and $A_3 \cup (A_1 \Delta C_2 \Delta C_3)$, $A_3 \cup (A_1 \Delta C_2)$ are different parity and not hit contradiction.

Case 2b) u_2, v_2 lie on different sides of v_3, u_3. WLOG u_2 lies inside A_3 and v_2 lies outside A_2. Let u be the first vertex P_3 intersects P_2 and v be the last (when considering P_3 to go from u_3 to v_3). WLOG u lies between u_2 and v in P_2. Denote by R_i the portion of P_i between u_i and u and Q_i the portion of P_i from v to v_i. WLOG suppose that v_2 lies inside A_3 let w be the first vertex of A_3 in the path from v_2 to u_2 in $A_2 \backslash P_2$. Let L_2, L_3 be the paths from v_2 to w and u_3 to w respectively. Let T_i be the path from u to v in P_i we claim $T_3 = T_2$. Assume for a contradiction let u', v' be nodes on T_2 such that the path T_3' in T_3 between u', v' in T_3 is internally disjoint from T_2 let T_2' be the path from u', v' in P_2.

First if T_2', T_3' have the same parity, then one of them T_j contains a hit node, but then $(A_j \backslash T_j) \cup T_l$ where $l \in \{2,3\} \backslash j$ is an even cycle so T_l contains a hit node as well. If $B_2 \backslash B_3 \cup R_3 \cup Q_2$ is even, then so is $(A_1 \backslash (B_2 \backslash B_3)) \cup R_3 \cup Q_2$ and the hit node of A_1 cannot hit both. Thus $B_2 \backslash B_3 \cup R_3 \cup Q_2$ is odd. Then $A_2 \backslash (P_2 \cup L_2) \cup L_3 \cup R_3 \cup Q_2$ is of different parity as $A_2 \backslash (P_2 \cup L_2) \cup L_3 \cup (B_2 \backslash B_3), A_2 \backslash (P_2 \cup L_2) \cup L_3 \cup (A_2 \backslash (B_2 \backslash B_3))$, and not both of these can contain the hit node of A_1 contradiction.

Thus T_2', T_3' have different parity if one of these WLOG T_2 contains a hit node, then if T_3 does not contain a hit node, then $Q_2 \cup T_3 \cup R_2 \cup B_2$ or $Q_2 \cup T_3 \cup R_2 \cup (A_1 \backslash B_2)$ is an even cycle with no hit node contradiction, otherwise like in the above paragraphs, $B_2 \backslash B_3 \cup R_3 \cup Q_2$ is odd, as $(A_2 \backslash (P_2 \cup L_2)) \cup L_3 \cup (B_2 \backslash B_3), (A_2 \backslash (P_2 \cup L_2)) \cup L_3 \cup (A_2 \backslash (B_2 \backslash B_3))$, are different parity and not both of these can contain the hit node of A_1 contradiction. One of each of the following pairs of cycles $(B_2 \cap B_3) \cup R_3 \cup R_3 \cup T_i$; $(A_1 \backslash (B_2 \cap B_3)) \cup R_3 \cup R_3 \cup T_i$; $(A_1 \backslash (B_2 \cup B_3)) \cup Q_2 \cup Q_3 \cup T_i$; $(B_2 \cup B_3) \cup Q_2 \cup Q_3 \cup T_i$; $(A_3 \backslash T_3) \cup T_i$; $(A_2 \backslash T_2) \cup T_i$ this implies (*) one of R_2, R_3 is hit, one of Q_2, Q_3 is hit, one of Q_i, R_i is hit.

If $(A_2 \backslash L_2) \cup L_3 \cup (B_2 \backslash B_3)$ is even, so is $(A_2 \backslash L_2) \cup L_3 \cup (A_1 \backslash (B_2 \backslash B_3))$ and not both cycles can be hit while satisfying (*) so $(A_2 \backslash L_2) \cup L_3 \cup (B_2 \backslash B_3)$ (and likewise $(A_3 \backslash L_3) \cup L_2 \cup (B_3 \backslash B_2)$) is odd. Then for $i = 2,3$, $l = \{2,3\} \backslash l$ either $A_i \backslash L_i \cup L_j \cup R_j \cup Q_i$ is even, or both $(A_1 \backslash (B_i \backslash B_l)) \cup R_l \cup Q_i$, $(B_i \backslash B_l) \cup R_l \cup Q_i$ are, or that one of Q_2, R_3 and one of Q_3, R_2 is hit, combining with (*) we see this is not possible.

Let C_i be as in the diagram below,if C_1 is odd, then C_2, C_3, C_4 are all even, (lemma 2.12) so each contains a hit node, then replacing A_1 with C_4 uncrosses A_1 contradiction , if C_1 is even, then so is C_5 and C_4, C_3, C_2 are odd, so $(A_2 \backslash (L_2 \cup P_2)) \cup L_3 \cup (B_3 \backslash B_2), (A_3 \backslash (L_3 \cup P_3)) \cup L_2 \cup (B_2 \backslash B_3)$ are even cycles, thus 2 of the hit nodes of A_1, A_2, A_3 lie on these cycles and only one can lie on C_5 so replace A_1 by C_5 .

□

Proposition 2.15. *We may 2 color the witness cycles $\mathcal{A}$ so that witness cycles of the same color do not cross. Let us label the witness cycles with color 1 $\mathcal{A}_1$ respectively $\mathcal{A}_2$.*

2.1 Direct Goemans Approach

The following result is basically proven in [4]

Theorem 2.16. *[4] Let $\mathcal{C}'$ be a set of cycles of a graph G' let $\mathcal{M}'$ be the set of face minimal cycles of $\mathcal{C}'$ let H be a minimal hitting set, suppose that the debit graph $B(\mathcal{M} \cup H)$ is planar and we can choose a laminar family of witness cycles for the nodes of H . Then $\sum_{M \in \mathcal{M}'} |H \cap M| \leq 3|\mathcal{M}|$.*

Using 2.9 and the following proposition, let us partition the set of witness cycles as follows:since each $m \in \mathcal{M}$ intersects at most one witness cycles W(M) one could partition the debit graph B as follows,
1) $B[\mathcal{M} \cup \mathcal{A}_1] \backslash W(\mathcal{M})$
2) $B[\mathcal{M} \cup \mathcal{A}_2] \backslash W(\mathcal{M})$
3) $B[\mathcal{M} \cup W(\mathcal{M})]$

$B[M \cup \mathcal{A}_1] \backslash W(\mathcal{M})$ and $B[M \cup \mathcal{A}_2] \backslash W(M)$ satisfy the properties of Goemans / Williamson Theorem 2.16 (by choosing $\mathcal{C}' = W(M) \cup \mathcal{A}_i$ and choosing $\mathcal{A}_i$ as our laminar family of witness cycles) thus the number of edges is bounded as such i.e. $E[B[\mathcal{M} \cup \mathcal{A}_i] \backslash W(M)] \leq 3|\mathcal{M}|$
$E[B[\mathcal{M} \cup W(\mathcal{M})] \leq |\mathcal{M}|$ (only edges $(M, W(M))$ $M \in \mathcal{M}$) So the total number of edges in $E[B[\mathcal{M} \cup W(\mathcal{M}) \leq |\mathcal{M}|$ thus $|E[B[\mathcal{M} \cup \mathcal{A}]]| \leq 7|\mathcal{M}|$

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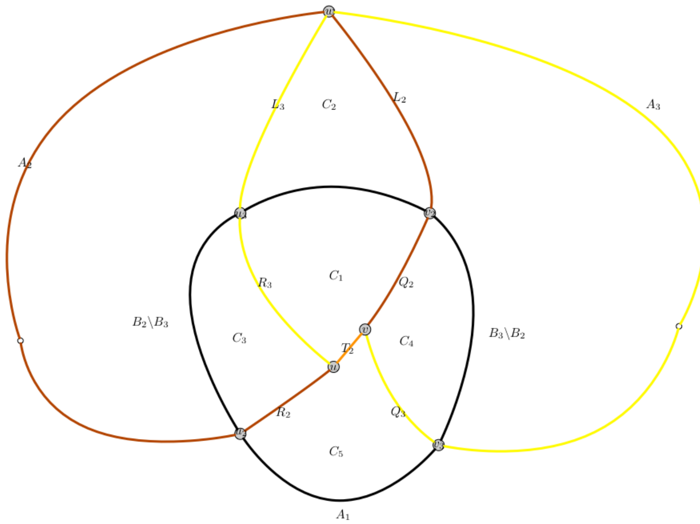

Figure 14: Diagram for case 2b) u_2, v_2 lie on different sides of v_3, u_3