Final practice problems

Problem 1.1 (Linear equations and the integrating factor)**.** Find the general solution to the linear equation

$$x^2y'(x) + 5xy(x) = \frac{e^{4x}}{x^3},\ x > 0. \tag{1.1}$$

Solution. Since $x > 0$, we may rewrite the equation as

$$y'(x) + \frac{5}{x}y(x) = \frac{e^{4x}}{x^5},\ x > 0. \tag{1.2}$$

We then choose an integrating factor to be

$$\mu(x) = \exp\int \frac{5}{x}\,dx = \exp 5\ln x = \exp\ln x^5 = x^5,\ x > 0, \tag{1.3}$$

therefore if y is a solution to the original equation, then

$$\frac{d}{dx}[x^5y(x)] = e^{4x},\ x > 0 \tag{1.4}$$

This implies that

$$x^5y(x) = \frac{1}{4}e^{4x} + C,\ x > 0 \tag{1.5}$$

thus

$$y(x) = \frac{1}{4}e^{4x}x^{-5} + Cx^{-5},\ x > 0, \tag{1.6}$$

where $C \in \mathbb{R}$ is arbitrary. □

Problem 1.2 (Rate problems)**.** A tank contains 130 liters of water and 50 grams of sugar. A solution containing a sugar concentration of $4e^{-t}$ g/L flows into the tank at a rate of 3 L/min, and the mixture in the tank flows out at a rate of 4 L/min. Let $Q(t)$ be the amount of sugar (in grams) in the tank at time t (in minutes). Write down an initial value problem for Q (without solving it explicitly).

Solution. The basic idea behind rate problems is that the rate of change is the rate of change flowing in minus the rate of change flowing out. If we denote the volume of the mixture at time t to be $V(t)$, then V satisfies the IVP

$$\begin{cases} V'(t) = 3 - 4,\ t \geq 0 \\ V(0) = 130. \end{cases} \tag{1.7}$$

From here we find that

$$V(t) = 130 - t,\ 0 \leq t \leq 130. \tag{1.8}$$

Then Q satisfies the IVP

$$\begin{cases} Q'(t) = 4e^{-t} * 3 - \frac{Q(t)}{V(t)} * 4 = 12e^{-t} - \frac{4Q(t)}{130-t}, \quad 0 \leq t \leq 130. \\ Q(0) = 50. \end{cases} \tag{1.9}$$

□

Problem 1.3 (Autonomous equations, stability of critical points)**.** Consider the autonomous equation

$$y'(x) = (1 - y(x))^3(y(x) + 2)(y(x) - 4),\ x \in \mathbb{R}. \tag{1.10}$$

a) Identify the critical points and the corresponding constant solutions to the equation.
b) Draw a one-dimensional phase portrait of the equation and determined the stability of the critical points.
c) Give a sketch of sample solution curves in the x-y plane.

Solution. The critical points are $y = 1, -2, 4$ and the corresponding constant solutions are $y(x) = 1, -2, 4$ for all $x \in \mathbb{R}$. A rough sketch of the one-dimensional phase portrait and sample solution curves is given below. □

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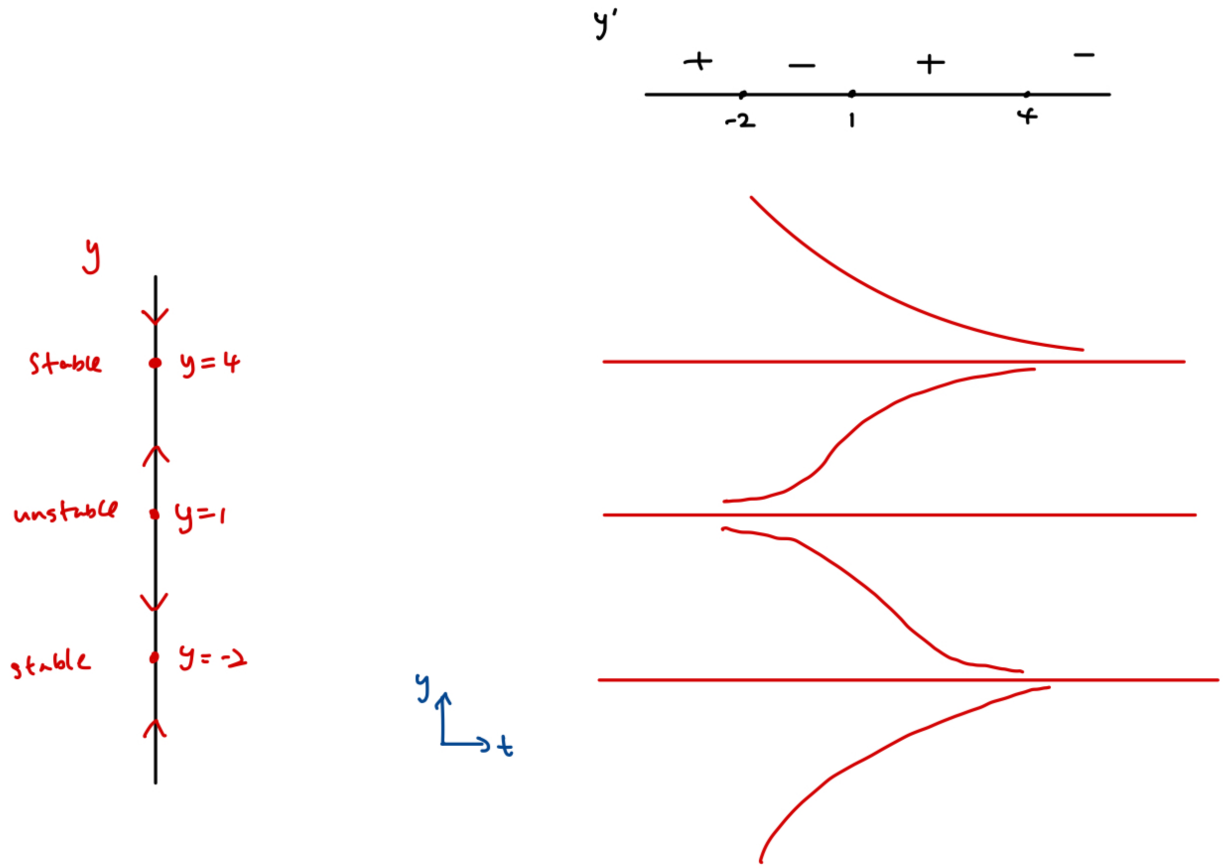


Problem 1.4 (Existence and uniqueness of solutions)**.** Consider the initial value problem

$$\begin{cases} y'(x) = \sqrt{y(x)}, \ x \geq 0 \\ y(0) = 0. \end{cases} \tag{1.11}$$

a) Does the IVP admit any constant solutions?
b) Does the IVP admit non-constant solutions? If so, why does this not violate the uniqueness part of the existence and uniqueness theorem?

Solution.

a) Yes, it admits the constant solution $y(x) = 0, x \geq 0$.
b) Assuming y is a solution and $y \not\equiv 0$, we have

$$(y(x))^{-1/2} y'(x) = 1 \implies \int y^{-1/2} \, dy = \int 1 dx. \tag{1.12}$$

This implies

$$2y^{1/2} = x + C_1 \implies y(x) = \left(\frac{x}{2} + C_2\right)^2, \ x \geq 0. \tag{1.13}$$

Using the initial condition we see that $C_2 = 0$, therefore

$$y(x) = \frac{x^2}{4}, \ x \geq 0 \tag{1.14}$$

is a non-constant solution to the equation. We note that if we write the equation as

$$y'(x) = f(x, y(x)), \ x \geq 0 \tag{1.15}$$

for $f(x,y) = y^{1/2},\ y \in \mathbb{R}$, we have

$$\frac{\partial f}{\partial y}(x,y) = \frac{1}{2\sqrt{y}},\ y \neq 0. \tag{1.16}$$

Since $\frac{\partial f}{\partial y}$ is not defined at $(0,0)$, it cannot be continuous in any neighborhood of it, therefore the existence and uniqueness theorem cannot be applied. This shows why the non-uniqueness of solutions in our problem does not violate the existence and uniqueness of solutions.

□

Problem 1.5 (Separable equations, implicit solutions)**.** Consider the equation

$$(y(x))^3 y'(x) = ((y(x))^4 + 1)\cos x,\ x \in \mathbb{R}. \tag{1.17}$$

a) Does the equation admit any constant solutions?
b) Find an implicit solution to the equation.

Solution.

a) No, we see that $y^4 + 1$ is never equal to 0, so the equation does not admit any constant solutions.
b) If y is a solution and we assume that y does not vanish over an interval I, then

$$\frac{(y(x))^3}{(y(x))^4+1} y'(x) = \cos x\ x \in I \implies \frac{1}{4}\int \frac{4y^3}{y^4+1}\,dy = \int \cos x\,dx. \tag{1.18}$$

From this we deduce that

$$\frac{1}{4}\ln(y^4+1) = \sin x + C, \tag{1.19}$$

which is an implicit solution to the equation.

□

Problem 1.6 (Exact equations, implicit solutions)**.** Consider the nonlinear equation

$$((y(x))^2 + 1) + (2xy(x) + 3(y(x))^2)y'(x) = 0,\ x \in \mathbb{R}. \tag{1.20}$$

a) Is the equation exact?
b) Find the general implicit solution to the equation.

Solution. For the solution to be exact, we write the equation in the form of $M(x,y) + N(x,y)y' = 0$ and check if $M_y = N_x$. We calculate

$$\partial_y(y^2+1) = 2y,\ \partial_x(2xy + 3y^2) = 2y, \tag{1.21}$$

therefore we see that the equation is exact.

To find the general implicit solution of the form $F(x,y) = C$, we note that we ought to have

$$\begin{cases} \partial_x F(x,y) = M(x,y) = y^2 + 1 \\ \partial_y F(x,y) = N(x,y) = 2xy + 3y^2,\ (x,y) \in \mathbb{R}^2. \end{cases} \tag{1.22}$$

Thus

$$F(x,y) = \int M(x,y)\,dx = xy^2 + x + g(y), \tag{1.23}$$

and we find that

$$\partial_y F(x,y) = 2xy + g'(y) = 2xy + 3y^2,\ (x,y) \in \mathbb{R}^2. \tag{1.24}$$

From here we see that $g(y) = y^3,\ y \in \mathbb{R}$, and therefore the general implicit solution is

$$F(x,y) = xy^2 + x + y^3 = C, \tag{1.25}$$

where C is arbitrary. □

Problem 1.7 (Bernoulli equations, constant solutions)**.** Consider the initial value problem

$$\begin{cases} y'(x) - \frac{2}{x}y(x) = -x^2(y(x))^2, \quad x > 0 \\ y(1) = \alpha. \end{cases} \tag{1.26}$$

Find a solution for the following values of α and state the maximal interval of existence of the solution.

a) $\alpha = 0$

b) $\alpha = 1$

Solution. We first identify the constant solutions. Note that if we write

$$y'(x) = \frac{2}{x}y(x) - x^2(y(x))^2 = y(x)\left(\frac{2}{x} - x^2 y(x)\right),\ x > 0, \tag{1.27}$$

we immediately see that $y(x) = 0, x > 0$ is a constant solution to the equation. Therefore when $\alpha = 0$, we find that the constant solution is the unique solution to the IVP with $\alpha = 0$. The maximal interval of existence is $(0, \infty)$.

For the case when $\alpha = 1$, we need to look for a non-constant solution to the equation. We note that this is a Bernoulli equation with $n = 2$, therefore we assume that y does not vanish over an interval I and consider the function

$$v(x) = (y(x))^{1-2} = (y(x))^{-1},\ x \in I. \tag{1.28}$$

Then $v'(x) = -(y(x))^{-2}y'(x), x \in I$, and the equation can be rewritten as

$$y'(x)(y(x))^{-2} - \frac{2}{x}(y(x))^{-1} = -x^2,\ x \in I. \tag{1.29}$$

This shows that v satisfies the first order linear equation

$$v'(x) + \frac{2}{x}v(x) = x^2,\ x \in I. \tag{1.30}$$

The integrating factor for this equation is $\mu(x) = x^2,\ x \in I$, and we find that

$$\frac{d}{dx}[x^2 v(x)] = x^4,\ x \in I. \tag{1.31}$$

This shows that

$$v(x) = \frac{x^3}{5} + \frac{C}{x^2},\ x \in I, \tag{1.32}$$

which shows that

$$y(x) = \frac{1}{\frac{x^3}{5} + \frac{C}{x^2}} = \frac{5x^2}{x^5 + C_2},\ x \in I \tag{1.33}$$

Using the initial condition $\alpha = 1$ we find that $1 + C_2 = 5 \implies C_2 = 4$, therefore the unique solution to the IVP is

$$y(x) = \frac{1}{\frac{x^3}{5} + \frac{C}{x^2}} = \frac{5x^2}{x^5 + 4},\ x \in I. \tag{1.34}$$

We note that $x^5 + 4 = 0$ when $x = (-4)^{1/5}$, so $x^5 + 4 \neq 0$ for $x \in (0, \infty)$. Therefore, the maximal interval of existence is $I = (0, \infty)$. □

Problem 1.8 (Equations with homogeneous functions)**.** Find an implicit solution to the nonlinear equation

$$y'(x) = \frac{x^2 + 3(y(x))^2}{2xy(x)},\ x > 0. \tag{1.35}$$

Solution. We first rewrite the right hand side as a function of y/x:

$$y'(x) = \frac{x^2 + 3(y(x))^2}{2xy(x)} = \frac{1}{2}\left(\frac{y(x)}{x}\right)^{-1} + \frac{3}{2}\frac{y(x)}{x},\ x > 0. \tag{1.36}$$

Next we consider the function v defined via $v(x) = \frac{y(x)}{x},\ x > 0$. Then $y(x) = xv(x) \implies y'(x) = v(x) + xv'(x), x > 0$. Thus v satisfies the equation

$$v(x) + xv'(x) = \frac{1}{2v(x)} + \frac{3}{2}v(x) \implies v'(x) = \frac{1}{2x}\left(\frac{1}{v(x)} + v(x)\right) = \frac{1}{2x}\frac{1 + (v(x))^2}{v(x)},\ x > 0. \tag{1.37}$$

This is a separable equation in v, thus

$$\frac{1}{2}\int \frac{2v}{v^2 + 1}\, dv = \int \frac{1}{2x}\, dx \implies \ln(v^2 + 1) = \frac{1}{2}\ln x + C,\ x > 0. \tag{1.38}$$

Thus an implicit solution to the original equation is

$$\ln\left(\frac{y^2}{x^2} + 1\right) = \frac{1}{2}\ln x + C,\ x > 0 \tag{1.39}$$

where C is arbitrary. □

Problem 1.9 (2nd order equations and mass-spring systems)**.** Suppose a mass spring system is modeled via the equation

$$x''(t) + \beta x'(t) + 4x(t) = 0, \ t \in \mathbb{R}. \tag{1.40}$$

Identify the value(s) of β for which the system will be

a) undamped
b) underdamped
c) critically damped
d) overdamped

Solution. If $\beta = 0$, then the system is undamped. The characteristic polynomial associated to the equation is

$$r^2 + \beta r + 4 = 0 \implies r = \frac{-\beta \pm \sqrt{\beta^2 - 16}}{2}. \tag{1.41}$$

Therefore the system is underdamped if $0 < \beta < 4$ (note that $\beta \geq 0$ by assumption), critically damped when $\beta = 4$, and overdamped when $\beta > 4$.

□

Problem 1.10 (Method of undetermined coefficients)**.** Write down an appropriate ansatz (without solving for the coefficients) for the linear equation

$$y^{(4)}(t) - 2y^{(3)}(t) + 10y''(t) = t^2 + e^t \cos 2t, \ t \in \mathbb{R}. \tag{1.42}$$

Solution. The characteristic equation associated to the equation is

$$r^4 - 2r^3 + 10r^2 = r^2(r^2 - 2r + 10) = 0. \tag{1.43}$$

Therefore the roots are $r = 0$ with multiplicity 2 and $r = \frac{2 \pm \sqrt{-36}}{2} = 1 \pm 3i$. This implies that the general homogeneous solution is

$$y_h(t) = c_1 + c_2 t + c_3 e^t \cos 3t + c_4 e^t \sin 3t, \ t \in \mathbb{R}. \tag{1.44}$$

With the given inhomogeneous right hand side, one writes down the initial guess

$$y_p(t) = At^2 + Bt + C + De^t \cos 2t + Ee^t \sin 2t, \ t \in \mathbb{R}, \tag{1.45}$$

though we note that our initial guess overlaps with the general homogeneous solution. We therefore use the modified ansatz

$$y_p(t) = t^2(At^2 + Bt + C) + De^t \cos 2t + Ee^t \sin 2t, \ t \in \mathbb{R}, \tag{1.46}$$

□

Problem 1.11 (Variation of parameters)**.** Find a particular solution to the equation

$$x^2 y''(x) + xy'(x) - y(x) = 600x^5, \ x > 0 \tag{1.47}$$

given that the general homogeneous solution to the equation is

$$y_h(x) = c_1 x + c_2 x^{-1}, \ x > 0 \tag{1.48}$$

where c_1, c_2 are arbitrary.

Solution. Since $x > 0$, the equation can be rewritten as

$$y''(x) + \frac{1}{x} y'(x) - \frac{1}{x^2} y(x) = 600x^3, \ x > 0. \tag{1.49}$$

We also note that if $y_1(x) = x, y_2(x) = x^{-1}, x > 0$, then

$$W(y_1, y_1)(x) = \det \begin{pmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{pmatrix} = -\frac{1}{x} - \frac{1}{x} = -2x^{-1}, \ x > 0. \tag{1.50}$$

Therefore by the variation of parameters formula,

$$\begin{aligned} y(x) &= -x \int \frac{x^{-1}(600x^3)}{-2x^{-1}} \, dx + x^{-1} \int \frac{x(600x^3)}{-2x^{-1}} \, dx && (1.51) \\ &= x \int 300x^3 \, dx - x^{-1} \int 300x^5 \, dx && (1.52) \\ &= 75x^5 - 50x^5 = 25x^5, \ x > 0. && (1.53) \end{aligned}$$

□

Problem 1.12 (Mass-spring systems and resonance)**.** Suppose a mass-spring system is modeled via the IVP

$$\begin{cases} x''(t) + \beta x'(t) + 16x(t) = F_0 \sin \omega t, \quad t \in \mathbb{R} \\ x(0) = x'(0) = 0. \end{cases} \tag{1.54}$$

where $\beta \geq 0, \omega > 0$.

a) Identify the parameters β, ω for which pure resonance occurs.
b) In the case of part a) and $F_0 = 8$, find the unique solution to the IVP.
c) Suppose $\beta = 8, \omega = 1$ and $F_0 = 16$. What is the (approximate, up to a negligible error) maximum amplitude of the mass-spring system in the long run?

Solution.

a) With purely sinusoidal forcing, pure resonance can only happen when there is no damping, so $\beta = 0$. The forcing frequency must also match the natural frequency, so we must have $\omega = 4$.
b) We use the ansatz

$$x_p(t) = At \cos 4t + Bt \sin 4t, \ t \in \mathbb{R}. \tag{1.55}$$

Then

$$x_p'(t) = (A + 4Bt) \cos 4t + (B - 4At) \sin 4t, \tag{1.56}$$
$$x_p''(t) = (4B + 4B - 16At) \cos 4t + (-4A - 4A - 16Bt) \sin 4t, \ t \in \mathbb{R}. \tag{1.57}$$

Thus

$$\begin{aligned} x_p''(t) + 16x_p(t) &= (8B - 16At + 16At) \cos 4t + (-8A - 16Bt + 16Bt) \sin 4t & (1.58)\\ &= 8B \cos 4t - 8A \sin 4t = 8 \sin 4t, \ t \in \mathbb{R}. & (1.59) \end{aligned}$$

This implies $B = 0, A = -1$. Therefore the general solution is

$$x(t) = c_1 \cos 4t + c_2 \sin 4t - t \cos 4t \ t \in \mathbb{R}. \tag{1.60}$$

If $x(0) = 0$, then $c_1 = 0$. Then

$$x'(t) = 4c_2 \cos 4t - \cos 4t + 4t \sin 4t, \ t \in \mathbb{R}. \tag{1.61}$$

If $x'(0) = 0$, then $4c_2 - 1 = 0 \implies c_2 = \frac{1}{4}$. Therefore the unique solution is

$$x(t) = \frac{1}{4} \sin 4t - t \cos t, \ t \in \mathbb{R}. \tag{1.62}$$

c) Since there is damping in the system, the homogeneous solution will become negligible in the long run, and the solution converges to the steady state solution. Therefore we look for a particular solution of the form

$$x_p(t) = A \cos t + B \sin t, \ t \in \mathbb{R}. \tag{1.63}$$

Then

$$x_p'(t) = B \cos t - A \sin t \tag{1.64}$$
$$x_p''(t) = -A \cos t - B \sin t, \ t \in \mathbb{R}. \tag{1.65}$$

Thus

$$\begin{aligned} x_p''(t) + 8x_p'(t) + 16x_p(t) &= (-A + 8B + 16A) \cos t + (-B - 8A + 16B) \sin t & (1.66)\\ &= (15A + 8B) \cos t + (-8A + 15B) \sin t = 16 \sin t, \ t \in \mathbb{R} & (1.67) \end{aligned}$$

Therefore we must have

$$\begin{cases} 15A + 8B = 0 \\ -8A + 15B = 16. \end{cases} \tag{1.68}$$

Thus

$$\begin{cases} 15 \times 8A + 64B = 0 \\ -8 \times 15A + 15^2 B = 16 \times 15. \end{cases} \tag{1.69}$$

This implies that

$$B = \frac{16 \times 15}{15^2 + 64}, \ A = -\frac{8}{15} \frac{16 \times 15}{15^2 + 64} = -\frac{8 \times 16}{15^2 + 64}. \tag{1.70}$$

The appropriate maximum amplitude is $\sqrt{A^2+B^2}$.

□

Problem 1.13 (Eigenvalue problems)**.** Consider the eigenvalue problem

$$\begin{cases} y''(x) + 2y'(x) + \lambda y(x) = 0,\ x \in (0,\pi) \\ y(0) = 0,\ y(\pi) = 0. \end{cases} \tag{1.71}$$

Parameterize the eigenvalues $\lambda > 1$ by $n = 1, 2, 3, \dots$ and list a corresponding set of eigenfunctions.

Solution. The characteristic equation is

$$r^2 + 2r + \lambda = 0 \implies r = \frac{-2 \pm \sqrt{4 - 4\lambda}}{2} = -1 \pm \sqrt{1-\lambda}. \tag{1.72}$$

If $\lambda > 1$, then $1 - \lambda < 0$, therefore the general solution is

$$y_h(x) = c_1 e^{-x} \cos\sqrt{\lambda - 1}x + c_2 e^{-x} \sin\sqrt{\lambda-1}x,\ x \in \mathbb{R}. \tag{1.73}$$

Using the boundary conditions we deduce that we must have

$$\sin\sqrt{\lambda-1}\pi = 0, \tag{1.74}$$

which means that we can parameterize the eigenvalues by

$$\sqrt{\lambda_n - 1} = n,\ n = 1, 2, 3, \dots \tag{1.75}$$

This is equivalent to parameterizing

$$\lambda_n = n^2 + 1,\ n = 1, 2, 3, \dots. \tag{1.76}$$

A corresponding set of eigenfunctions can be chosen to be

$$f_1(x) = e^{-x}\sin x,\ f_2(x) = e^{-x}\sin 2x,\ f_3(x) = e^{-x}\sin 3x, \dots \tag{1.77}$$

In general,

$$f_n(x) = e^{-x}\sin nx,\ n = 1, 2, 3, \dots \tag{1.78}$$

□

Problem 1.14 (The Laplace transform and equations with piecewise forcing)**.** Suppose a mass-spring system is modeled via

$$\begin{cases} x''(t) + 4x(t) = f(t), \quad t \geq 0 \\ x(0) = x'(0) = 0, \end{cases} \tag{1.79}$$

and $f : [0, \infty) \to \mathbb{R}$ is defined via

$$f(t) = \begin{cases} 0, & 0 \leq t < 2\pi \\ \sin t, & t \geq 2\pi. \end{cases} \tag{1.80}$$

Use the Laplace transform to find a solution x describing the behavior of the system for $t \geq 0$

Solution. We first write

$$f(t) = \mathcal{U}(t - 2\pi)\sin t,\ t \in \mathbb{R} \tag{1.81}$$

and calculate

$$\mathcal{L}\{f\} = \mathcal{L}\{\mathcal{U}(t-2\pi)\sin t\} = e^{-2\pi s}\mathcal{L}\{\sin(t + 2\pi)\} = e^{-2\pi s}\frac{1}{s^2+1},\ s > 0. \tag{1.82}$$

Thus upon taking the Laplace transform of both sides of the equation and using the initial conditions, we find that

$$X(s) = e^{-2\pi s}\frac{1}{(s^2+1)(s^2+4)},\ s > 0. \tag{1.83}$$

To simplify this expression we compute the partial fractional decomposition of the rational part of X:

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} \implies 1 = (As+B)(s^2+4) + (Cs+D)(s^2+1) \tag{1.84}$$

$$\implies 1 = (A+C)s^3 + (B+D)s^2 + (4A+C)s + (4B+D),\ s \in \mathbb{R}. \tag{1.85}$$

This implies that

$$\begin{cases} A + C = 0 \\ B + D = 0 \\ 4A + C = 0 \\ 4B + D = 1. \end{cases} \tag{1.86}$$

By routine algebra we find that

$$A = 0,\ B = \frac{1}{3},\ C = 0,\ D = -\frac{1}{3}. \tag{1.87}$$

Thus

$$X(s) = \frac{1}{3}e^{-2\pi s}\frac{1}{s^2+1} - \frac{1}{3}e^{-2\pi s}\frac{1}{s^2+4},\ s > 0. \tag{1.88}$$

This implies that

$$x(t) = \frac{1}{3}\mathcal{U}(t-2\pi)\sin(t-2\pi) - \frac{1}{6}\mathcal{U}(t-2\pi)\sin(2(t-2\pi)) \tag{1.89}$$

$$= \mathcal{U}(t-2\pi)\left(\frac{1}{3}\sin t - \frac{1}{6}\sin 2t\right),\ t \geq 0. \tag{1.90}$$

□

Problem 1.15 (The Laplace transform and impulse forces)**.** Use the Laplace transform to find a solution to the equation

$$\begin{cases} x''(t) + x(t) = 1 + \delta(t-\pi), \quad t \geq 0 \\ x(0) = x'(0) = 0. \end{cases} \tag{1.91}$$

Solution. If x is a solution, then

$$(s^2+1)X(s) = \frac{1}{s} + e^{-\pi s},\ s > 0, \tag{1.92}$$

thus

$$X(s) = \frac{1}{s(s^2+1)} + e^{-\pi s}\frac{1}{s^2+1},\ s > 0. \tag{1.93}$$

We simplify the first term by finding its partial fraction decomposition:

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} \implies 1 = A(s^2+1) + (Bs+C)s,\ s \in \mathbb{R}. \tag{1.94}$$

If $s = 0$, then $A = 1$. If $s = 1$, then $1 = 2 + B + C$; if $s = 2$, then $1 = 5 + 4B + 2C$. Thus we find that

$$\begin{cases} B + C = -1 \\ 2B + C = -2 \end{cases} \tag{1.95}$$

From this we find that $B = -1, C = 0$. Thus

$$X(s) = \frac{1}{s} - \frac{s}{s^2+1} + e^{-\pi s}\frac{1}{s^2+1},\ s > 0. \tag{1.96}$$

Upon taking the inverse Laplace transform we then find that

$$x(t) = 1 - \cos t + \mathcal{U}(t-\pi)\sin(t-\pi),\ t \geq 0. \tag{1.97}$$

□

Problem 1.16 (The Laplace transform and convolutions)**.** Write down a convolution integral solution to the IVP

$$\begin{cases} x''(t) + 8x'(t) + 16x(t) = f(t), \quad t > 0 \\ x(0) = x'(0) = 0, \end{cases} \tag{1.98}$$

for any reasonable function $f : [0,\infty) \to \mathbb{R}$.

Solution. We note that

$$X(s) = \frac{1}{(s+4)^2} F(s),\ s > a \tag{1.99}$$

for some $a \in \mathbb{R}$, and

$$w(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+4)^2}\right\} = \mathcal{L}^{-1}\left\{-\frac{d}{ds}\frac{1}{(s+4)}\right\} = t\mathcal{L}^{-1}\left\{\frac{1}{(s+4)}\right\} = te^{-4t}. \tag{1.100}$$

Therefore

$$x(t) = (w * f)(t) = \int_0^t w(\tau) f(t-\tau)\, d\tau = \int_0^t \tau e^{-4\tau} f(t-\tau)\, d\tau. \tag{1.101}$$

□

Problem 1.17 (Linear systems)**.** Find the unique solution to the IVP

$$\begin{cases} \boldsymbol{X}'(t) = \begin{pmatrix} -4 & 2 \\ 2 & -4 \end{pmatrix} \boldsymbol{X}(t), \quad t \in \mathbb{R} \\ \boldsymbol{X}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{cases} \tag{1.102}$$

Solution. Denoting

$$A = \begin{pmatrix} -4 & 2 \\ 2 & -4 \end{pmatrix}, \tag{1.103}$$

we note that

$$A + 2I = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \tag{1.104}$$

which has two rows that are multiples of each other, so $\lambda_1 = -2$ is an eigenvalue. Since $\operatorname{tr} A = -8$, the other eigenvalue is $\lambda_2 = -6$. One can also compute the characteristic polynomial directly:

$$p_A(\lambda) = \det(A - \lambda I) = (-4-\lambda)^2 - 4 = (\lambda+2)(\lambda+6),\ \lambda \in \mathbb{C}. \tag{1.105}$$

Here we have a pair of distinct real roots. An eigenvector $\boldsymbol{v}_1$ corresponding to $\lambda_1 = -2$ satisfies

$$(A+2I)\boldsymbol{v}_1 = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{1.106}$$

This requires $v_1 - v_2 = 0$, therefore we can choose

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{1.107}$$

Likewise, an eigenvector corresponding to $\lambda_2 = -6$ satisfies

$$(A+6I)\boldsymbol{v}_2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{1.108}$$

Therefore we can choose

$$\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{1.109}$$

The general solution to the equation is then

$$\boldsymbol{X}(t) = c_1 e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-6t}\begin{pmatrix} 1 \\ -1 \end{pmatrix},\ t \in \mathbb{R}. \tag{1.110}$$

To solve for c_1, c_2 we use the initial conditions. We find that we must have

$$\begin{pmatrix} c_1 + c_2 \\ c_1 - c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{1.111}$$

Thus $c_1 = c_2 = \frac{1}{2}$, and the unique solution to the IVP is

$$\boldsymbol{X}(t) = \frac{1}{2} e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} e^{-6t}\begin{pmatrix} 1 \\ -1 \end{pmatrix},\ t \in \mathbb{R}. \tag{1.112}$$

□

Problem 1.18 (Linear systems and stability of critical points)**.** Consider the linear system

$$\boldsymbol{X}'(t) = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix} \boldsymbol{X}(t),\ t \in \mathbb{R} \tag{1.113}$$

where $\alpha \in \mathbb{R}$ is an unspecified parameter. Identify the value(s) of α for which the origin is a

a) stable node
b) unstable node
c) saddle point

Solution. By denoting

$$A = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}, \tag{1.114}$$

we note that its characteristic polynomial is

$$p_A(\lambda) = \det(A - \lambda I) = (\lambda - 1)^2 - \alpha^2 = (\lambda - 1 - \alpha)(\lambda - 1 + \alpha),\ \lambda \in \mathbb{C}. \tag{1.115}$$

Thus its eigenvalues are

$$\lambda_1 = 1 + \alpha, \lambda_2 = 1 - \alpha. \tag{1.116}$$

If we want the origin to be a stable node, we need $\lambda_1, \lambda_2 < 0$. Thus we require

$$1 + \alpha < 0 \implies \alpha < -1 \text{ and } 1 - \alpha < 0 \implies \alpha > 1. \tag{1.117}$$

Since both conditions cannot be satisfied simultaneously, the origin is never a stable node.

For the origin to be an unstable node, we need both eigenvalues to be positive. Thus we require

$$1 + \alpha > 0 \implies \alpha > -1 \text{ and } 1 - \alpha > 0 \implies \alpha < 1. \tag{1.118}$$

So here we see that for all $\alpha \in (-1, 1)$, the origin is an unstable node.

For the origin to be a saddle point, one of the eigenvalues must be positive and the other one must be negative. Thus we require

$$\lambda_1 > 0, \lambda_2 < 0 \text{ or } \lambda_1 < 0, \lambda_2 > 0. \tag{1.119}$$

In the first case we require

$$1 + \alpha > 0 \implies \alpha > -1 \text{ and } 1 - \alpha < 0 \implies \alpha > 1. \tag{1.120}$$

In the second case we require

$$1 + \alpha < 0 \implies \alpha < -1 \text{ and } 1 - \alpha > 0 \implies \alpha < 1. \tag{1.121}$$

Therefore for all $\alpha \in (1, \infty) \cup (-\infty, -1)$, the origin is a saddle point.

□

Problem 1.19 (Fourier series)**.** Consider the function

$$f(x) = 1,\ x \in (0, 1). \tag{1.122}$$

Sketch a graph over $\mathbb{R}$ of the Fourier sine series of f, which is defined via

$$\sum_{n=1}^{\infty} b_n \sin(n\pi x) \text{ for all } x \in \mathbb{R},\ \text{where } b_n = 2\int_0^1 \sin(n\pi x)\, dx. \tag{1.123}$$

Solution. The Fourier sine series is the Fourier series of the odd extension of f into $(-1, 0)$. So we sketch out the periodic extension of the odd extension of f to $\mathbb{R}$, and also at the points of disconinuity the Fourier series converge to the average of the left limit and the right limit. This gives us the following sketch.

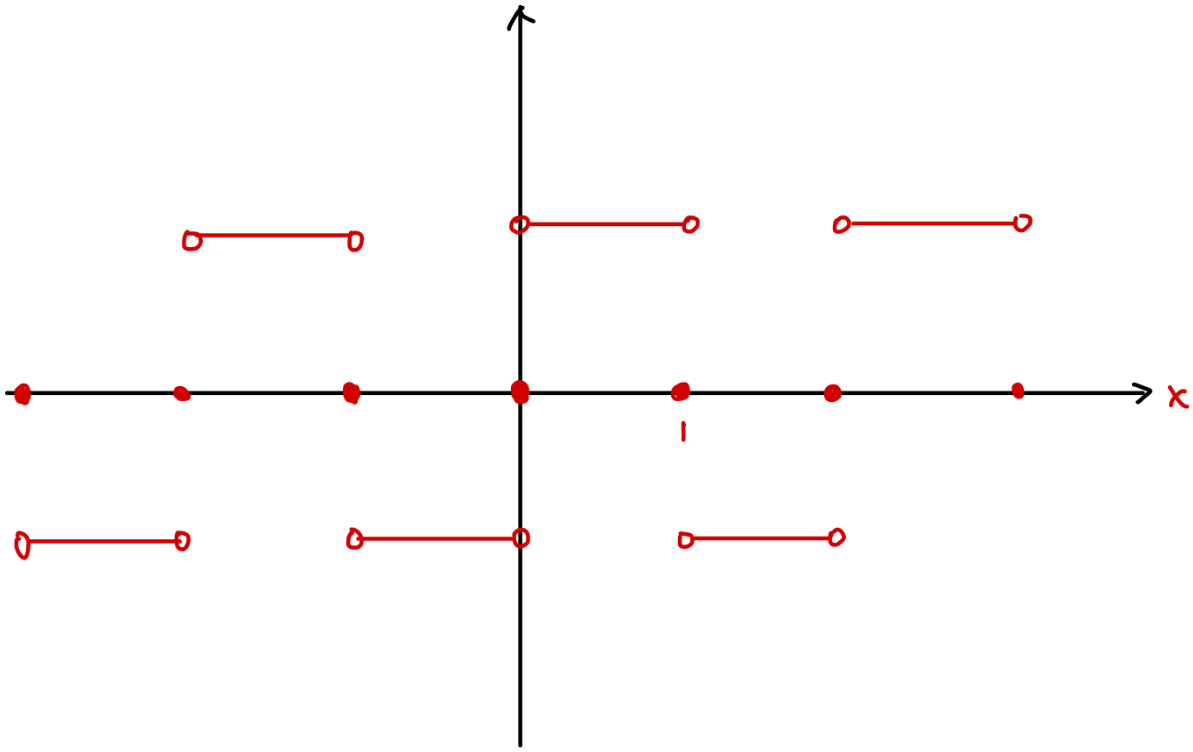
□

Problem 1.20 (The heat equation)**.** Find the unique solution $u : [0, \pi] \times [0, \infty) \to \mathbb{R}$ as a *finite* linear combination of elementary functions satisfying

$$u_t = 4u_{xx}, \qquad x \in (0, \pi), t \geq 0, \tag{1.124}$$

$$u(x = 0, t) = 0 = u(\pi, t) \qquad t \geq 0 \tag{1.125}$$

$$u(x, t = 0) = \sin(2x) + \sin(5x), \qquad x \in (0, \pi). \tag{1.126}$$



Solution. Following the derivation from lecture, the unique solution to the heat equation is

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-4n^2 t} \sin(nx) \; x \in (0,\pi), t \geq 0, \tag{1.127}$$

where the coefficients A_n satisfy

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(nx) = \sin(2x) + \sin(5x), \; x \in (0,\pi). \tag{1.128}$$

Here we see that

$$A_n = \begin{cases} 1, & n = 2 \\ 1, & n = 5 \\ 0, & \text{otherwise.} \end{cases} \tag{1.129}$$

Therefore

$$u(x,t) = e^{-16t} \sin 2x + e^{-100t} \sin 5x, \; x \in (0,\pi), t \geq 0. \tag{1.130}$$

□

Problem 1.21 (The wave equation)**.** Find the unique solution $u : [0,\pi] \times [0,\infty) \to \mathbb{R}$ as a *finite* linear combination of elementary functions satisfying

$$u_{tt} = 9u_{xx}, \qquad x \in (0,\pi), t \geq 0 \tag{1.131}$$

$$u(x=0,t) = 0 = u(\pi,t) \qquad t \geq 0 \tag{1.132}$$

$$u(x,t=0) = \sin(3x) \qquad x \in (0,\pi) \tag{1.133}$$

$$u_t(x,t=0) = \sin(4x) \qquad x \in (0,\pi). \tag{1.134}$$

Solution. Following the derivation from lecture, the unique solution to the wave equation is

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \cos 3nt + B_n \sin 3nt) \sin nx,\ x \in (0,\pi), t \geq 0. \tag{1.135}$$

We then calculate

$$\partial_t u(x,t) = \sum_{n=1}^{\infty} (3nB_n \cos 3nt - 3nA_n \sin 3nt) \sin nx,\ x \in (0,\pi), t \geq 0. \tag{1.136}$$

Therefore the coefficients A_n, B_n satisfy

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin nx = \sin 3x,\ x \in (0,\pi) \tag{1.137}$$

and

$$\partial_t u(x,0) = \sum_{n=1}^{\infty} 3nB_n \sin nx = \sin 4x,\ x \in (0,\pi). \tag{1.138}$$

This shows that

$$A_n = \begin{cases} 1, & n = 3 \\ 0, & \text{otherwise} \end{cases} \tag{1.139}$$

and

$$B_n = \begin{cases} \frac{1}{12}, & n = 4 \\ 0, & \text{otherwise.} \end{cases} \tag{1.140}$$

Thus

$$u(x,t) = \cos 9t \sin 3x + \frac{1}{12} \sin 12t \sin 4x,\ x \in (0,\pi), t \geq 0. \tag{1.141}$$

$\square$