Calculus reference sheet

a) Exponential and logarithmic functions, assuming $b \in (0,\infty) \setminus \{1\}, x \in (0,\infty), y \in \mathbb{R}$:

- $\log_b x = y \iff b^y = x$
- $\ln x = \log_e x$, where $e = \lim\limits_{n\to\infty}(1+\frac{1}{n})^n$
- $\log_b b^x = x$ and $b^{\log_b x} = x$

b) Laws of logarithms: assuming $b \in (0,\infty) \setminus \{1\}, x, y \in (0,\infty), \alpha \in \mathbb{R}$:

- $\log_b(xy) = \log_b x + \log_b y$
- $\log_b \frac{x}{y} = \log_b x - \log_b y$
- $\log_b x^\alpha = \alpha \log_b x$

c) Inverse trigonometric functions:

- $y = \arcsin x \iff x = \sin y,\ -1 \le x \le 1,\ -\frac{\pi}{2} \le y \le \frac{\pi}{2}$
- $y = \arccos x \iff x = \cos y,\ -1 \le x \le 1,\ 0 \le y \le \pi$
- $y = \arctan x \iff x = \tan y,\ x \in \mathbb{R},\ -\frac{\pi}{2} < y < \frac{\pi}{2}$
- $y = \operatorname{arccot} x \iff x = \cot y,\ x \in \mathbb{R},\ 0 < y < \pi$
- $y = \operatorname{arcsec} x \iff x = \sec y,\ x \in (-\infty,-1] \cup [1,\infty),\ y \in [0,\pi/2) \cup (\pi/2,\pi]$
- $y = \operatorname{arccsc} x \iff x = \csc y,\ x \in (-\infty,-1] \cup [1,\infty),\ y \in [-\pi/2,0) \cup (0,\pi/2]$

d) Trigonometric identities

- Pythagorean theorem:

$$\sin^2 x + \cos^2 x = 1,\ x \in \mathbb{R}.$$

  As a result we also have

$$1 + \cot^2 x = \csc^2 x,\ x \in \mathbb{R} \setminus \{x \mid \sin x = 0\}$$

$$\tan^2 x + 1 = \sec^2 x,\ x \in \mathbb{R} \setminus \{x \mid \cos x = 0\}$$

- Angle addition and subtraction:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta,$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta,\ \alpha,\beta \in \mathbb{R}.$$

- Double angle formulas:

$$\sin 2\theta = 2\sin\theta\cos\theta,$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta,\ \theta \in \mathbb{R}.$$

- Half angle formulas:

$$\sin\frac{\theta}{2} = \operatorname{sgn}\left(\sin\frac{\theta}{2}\right)\sqrt{\frac{1-\cos\theta}{2}} \implies \sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2},$$
$$\cos\frac{\theta}{2} = \operatorname{sgn}\left(\cos\frac{\theta}{2}\right)\sqrt{\frac{1+\cos\theta}{2}} \implies \cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2},\ \theta\in\mathbb{R}.$$

- Product to sum formulas: for $a, b, \in \mathbb{R}$,

$$\sin(ax)\sin(bx) = \frac{1}{2}\left[\cos((a-b)x) - \cos((a+b)x)\right]$$
$$\sin(ax)\cos(bx) = \frac{1}{2}\left[\sin((a-b)x) + \sin((a+b)x)\right]$$
$$\cos(ax)\cos(bx) = \frac{1}{2}\left[\cos((a-b)x) + \cos((a+b)x)\right],\ x\in\mathbb{R}.$$

e) Derivatives

1) Exponential and logarithmic functions, assuming $b \in (0,\infty)\setminus\{1\}$:
   - $\dfrac{d}{dx}(b^x) = \ln b \cdot b^x,\ x\in\mathbb{R}.$
   - If $f : I \to \mathbb{R}$ is differentiable, then $\dfrac{d}{dx}\left(b^{f(x)}\right) = \ln b\cdot b^{f(x)}\cdot f'(x),\ x\in I.$
   - $\dfrac{d}{dx}(\log_b|x|) = \dfrac{1}{\ln b}\cdot\dfrac{1}{x},\ x\in\mathbb{R}\setminus\{0\}.$

2) Trigonometric functions:
   - $\dfrac{d}{dx}(\sin x) = \cos x,\ \dfrac{d}{dx}(\cos x) = -\sin x,\ x\in\mathbb{R}.$
   - $\dfrac{d}{dx}(\tan x) = \sec^2 x,\ \dfrac{d}{dx}(\sec x) = \sec x\tan x,\ x\in\mathbb{R}\setminus\{x\in\mathbb{R}\mid \cos x = 0\}.$
   - $\dfrac{d}{dx}(\cot x) = -\csc^2 x,\ \dfrac{d}{dx}(\csc x) = -\csc x\cot x,\ x\in\mathbb{R}\setminus\{x\in\mathbb{R}\mid \sin x = 0\}.$

3) Inverse trigonometric functions:
   - $\dfrac{d}{dx}(\arcsin x) = \dfrac{1}{\sqrt{1-x^2}},\ \dfrac{d}{dx}(\arccos x) = -\dfrac{1}{\sqrt{1-x^2}},\ x\in(-1,1).$
   - $\dfrac{d}{dx}(\arctan x) = \dfrac{1}{1+x^2},\ \dfrac{d}{dx}(\operatorname{arccot} x) = -\dfrac{1}{1+x^2},\ x\in\mathbb{R}.$
   - $\dfrac{d}{dx}(\operatorname{arcsec} x) = \dfrac{1}{|x|\sqrt{x^2-1}},\ \dfrac{d}{dx}(\operatorname{arccsc} x) = -\dfrac{1}{|x|\sqrt{x^2-1}},\ x\in(-\infty,-1)\cup(1,\infty).$

4) Absolute value:
   - $\dfrac{d}{dx}|x| = \dfrac{x}{|x|} = \begin{cases}1, & x>0\\ -1, & x<0\end{cases} = \operatorname{sgn} x,\ x\in\mathbb{R}\setminus\{0\}.$
   - If $f : I \to \mathbb{R}$ is differentiable, then $\dfrac{d}{dx}|f(x)| = \dfrac{f(x)}{|f(x)|}f'(x),\ x\in I\setminus\{x\mid f(x)=0\}.$

f) Anti-derivatives (C denotes an arbitrary real constant in the identities to follow):

- $\displaystyle\int x^\alpha\, dx = \frac{x^{\alpha+1}}{\alpha+1} + C,\ \alpha \neq -1,\ x \in \mathbb{R}\setminus\{0\}$ if $\alpha < -1,\ x \in \mathbb{R}$ otherwise
- $\displaystyle\int \frac{1}{x}\, dx = \ln|x| + C = \begin{cases} \ln x + C_1, & x > 0 \\ \ln(-x) + C_2, & x < 0, \end{cases} \quad C_1, C_2 \in \mathbb{R}.$
- $\displaystyle\int a^x\ \ dx = \frac{a^x}{\ln a},\ x \in \mathbb{R}, a \in (0,\infty)\setminus\{1\}$
- $\displaystyle\int \tan x\, dx = \ln|\sec x| + C_n = -\ln|\cos x| + C_n,\ x \in \left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right),\ n \in \mathbb{Z}$
- $\displaystyle\int \cot x\, dx = \ln|\sin x| + C_n = -\ln|\csc x| + C_n,\ x \in (n\pi, (n+1)\pi),\ n \in \mathbb{Z}$
- $\displaystyle\int \sec x\, dx = \ln|\sec x + \tan x| + C_n,\ x \in \left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right),\ n \in \mathbb{Z}$
- $\displaystyle\int \csc x\, dx = -\ln|\csc x + \cot x| + C_n,\ x \in (n\pi, (n+1)\pi),\ n \in \mathbb{Z}$
- $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \arcsin\frac{x}{a} + C,\ x \in (-a, a),\ a \in (0,\infty)$
- $\displaystyle\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a}\arctan\frac{x}{a} + C,\ x \in \mathbb{R}$
- $\displaystyle\int \frac{1}{x\sqrt{x^2 - a^2}}\, dx = \frac{1}{a}\operatorname{arcsec}\left|\frac{x}{a}\right| + C,\ x \in (-\infty, -a) \cup (a, \infty),\ a \in (0,\infty)$

g) Integration formulas

- Change of variables: if $f : I \to \mathbb{R}$ is continuous and $g : [a,b] \to I$ is differentiable and $g' : (a,b) \to \mathbb{R}$ is continuous, then
$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$
- Integration by parts: if $f, g : [a,b] \to \mathbb{R}$ are differentiable, then
$$\int_a^b f'(x)g(x)\, dx = f(x)g(x)\Big|_{x=a}^{x=b} - \int_a^b f(x)g'(x)\, dx.$$
- Symmetry: if $f : [-a,a] \to \mathbb{R}$ is continuous and even, then $\displaystyle\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$; if $f : [-a,a] \to \mathbb{R}$ is continuous and odd, then $\displaystyle\int_{-a}^{a} f(x)\, dx = 0$.

h) Numerical integration

- Error bounds for the midpoint and trapezoid rules: let $f : [a,b] \to \mathbb{R}$ be a twice-differentiable function over the open interval (a,b). If $|f''(x)| \leq K$ for $x \in [a,b]$ (K can be chosen to be the maximum absolute value of the second derivative of f on $[a,b]$), then
$$\text{Absolute Error in } M_n \leq \frac{K(b-a)^3}{24n^2}.$$

We also have

$$\text{Absolute Error in } T_n \leq \frac{K(b-a)^3}{12n^2}.$$

- Simpson's rule: let $f : [a,b] \to \mathbb{R}$ be an integrable function over the interval $[a,b]$ and let $n \geq 1$ be some integer. Divide the interval $[a,b]$ into n (where n is even) equal-length subintervals $[x_{i-1}, x_i]$ ($i = 1, \ldots, n$) with width $\Delta x = (b-a)/n$. Then we can define the Simpson's rule approximation to $\int_a^b f(x)\,dx$ with n subintervals S_n via

$$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

- Error bound for Simpson's rule: Let $f : [a,b] \to \mathbb{R}$ be a four-times-differentiable function over the open interval (a,b). If $|f^{(4)}(x)| \leq K$ for $x \in [a,b]$ (K can be chosen to be the maximum absolute value of the fourth derivative of f on $[a,b]$), then

$$\text{Absolute Error in } S_n \leq \frac{K(b-a)^5}{180n^4}.$$

i) Sequences and series

- Partial sum: given a sequence $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$, the N-th partial sum s_N is defined via $s_N = \sum_{n=1}^{N} a_n = a_1 + \ldots + a_N$ for $N \geq 1$.
- Infinite series: given a sequence $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$, the infinite series $\sum_{n=1}^{\infty} a_n$ is defined as the limit of the sequence of partial sums $\{s_N\}_{N=1}^{\infty}$. If the limit exists, we say that the infinite series converges; otherwise, we say that the infinite series diverges.
- Divergence test: if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.
- Contrapositive of the divergence test: if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.
- Geometric series: the series $\sum_{n=1}^{\infty} ar^{n-1}$ converges to $\dfrac{a}{1-r}$ if $|r| < 1$ and diverges otherwise.
- Direct comparison test: suppose $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \subset \mathbb{R}$ are two sequences and $0 \leq a_n \leq b_n$ for all $n \geq N$ for some $N \in \mathbb{N}$.
  - If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
  - If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.
- Limit comparison test: suppose $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \subset \mathbb{R}$ are two eventually non-negative sequences.
  - If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ for some $L \in \mathbb{R}^+$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.
  - If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
  - If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

   - Otherwise, the test is inconclusive.

- Integral test: suppose $f : [1, \infty) \to \mathbb{R}$ is a continuous, positive, and decreasing function. If $a_n = f(n)$ for all $n \geq N$ for some $N \in \mathbb{N}$, then $\displaystyle\int_N^\infty f(x)\,dx$ and $\displaystyle\sum_{n=N}^\infty a_n$ either both converge or both diverge.
- Remainder estimate associated to the integral test: if a sequence $\{a_n\}_{n=1}^\infty \subset \mathbb{R}$ satisfies the hypotheses of the integral test ($a_n = f(n)$ for all $n \geq 1$ for a continuous, positive, and decreasing function) and the associated series converges to a number L, then the N-th remainder $R_N = L - s_N$ satisfies the estimate

$$\int_{N+1}^\infty f(x)\,dx \leq R_N \leq \int_N^\infty f(x)\,dx \text{ for all } N \geq 1.$$

- Alternating series test: suppose $\{b_n\}_{n=1}^\infty \subset \mathbb{R}$ is a sequence satisfying $0 \leq b_{n+1} \leq b_n$ for all $n \geq N$ for some $N \in \mathbb{N}$ and $\lim_{n\to\infty} b_n = 0$. Then the alternating series $\displaystyle\sum_{n=1}^\infty (-1)^n b_n$ and $\displaystyle\sum_{n=1}^\infty (-1)^{n+1} b_n$ both converge.
- Remainder estimate associated to the alternating series test: if a sequence $\{b_n\}_{n=1}^\infty \subset \mathbb{R}$ satisfies the hypotheses of the alternating series test ($0 \leq b_n \leq b_{n+1}$ for all $n \geq 1$ and $b_n \to 0$ as $n \to \infty$) and the associated alternating series converges to a number L, then the N-th remainder $R_N = L - s_N$ satisfies the estimate

$$|R_N| \leq b_{N+1} \text{ for all } N \geq 1.$$

- Ratio test: suppose $\{a_n\}_{n=1}^\infty \subset \mathbb{R}$ is a sequence with $a_n \neq 0$ for all $n \geq N$ for some $N \in \mathbb{N}$.
   - If $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ and $0 \leq L < 1$, then $\displaystyle\sum_{n=1}^\infty a_n$ converges absolutely.
   - If $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ and $L > 1$ or $L = \infty$, then $\displaystyle\sum_{n=1}^\infty a_n$ diverges.
   - Otherwise, the test is inconclusive.
- Root test: suppose $\{a_n\}_{n=1}^\infty \subset \mathbb{R}$.
   - If $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$ and $0 \leq L < 1$, then $\displaystyle\sum_{n=1}^\infty a_n$ converges absolutely.
   - If $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$ and $L > 1$ or $L = \infty$, then $\displaystyle\sum_{n=1}^\infty a_n$ diverges.
   - Otherwise, the test is inconclusive.
- p-test: the series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^p}$ converges if $p > 1$ and diverges otherwise.

j) Power series

1) Given a formal power series of the form $\sum_{n=0}^\infty a_n(x-a)^n$, the series either converges only for $x = a$ and diverges otherwise, only for $|x - a| < R$ for $R > 0$ (potentially also at $x = a \pm R$) and diverges otherwise, or for all $x \in \mathbb{R}$.
2) Some common power series representations centered at 0.

- $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots, \ |x| < 1$
- $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots, \ x \in \mathbb{R}$
- $\sin x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots, \ x \in \mathbb{R}$
- $\cos x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots, \ x \in \mathbb{R}$
- $\arctan x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1} = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots, \ |x| \le 1$
- $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{n+1}}{n+1} = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots, \ x \in (-1, 1]$
- For $k \in \mathbb{R}$, $(1+x)^k = \sum_{n=0}^{\infty} \dbinom{k}{n} x^n = 1 + kx + \dfrac{k(k-1)}{2!}x^2 + \dfrac{k(k-1)(k-2)}{3!}x^3 + \cdots, \ |x| < 1$

3) Term-by-term differentiation and integration: if an analytic function $f : I \to \mathbb{R}$ admits a power series representation $\sum_{n=0}^{\infty} a_n (x-a)^n$ centered at $a \in I$ with radius of convergence R, then the power series representation of f' centered at a is $\sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$ with the same radius of convergence and the power series representation of any antiderivative of f centered at a is $C + \sum_{n=0}^{\infty} a_n \dfrac{(x-a)^{n+1}}{n+1}$ with the same radius of convergence, for some constant $C \in \mathbb{R}$.

4) Cauchy product of two power series: suppose two analytic functions $f, g : I \to \mathbb{R}$ admit the power series representations $\sum_{n=0}^{\infty} c_n (x-a)^n$ and $\sum_{n=0}^{\infty} d_n (x-a)^n$ respectively, centered at a and over some interval $J \subseteq I$. Then for all $x \in J$, $f(x)g(x) = \left(\sum_{n=0}^{\infty} c_n (x-a)^n\right)\left(\sum_{n=0}^{\infty} d_n (x-a)^n\right) = \sum_{n=0}^{\infty} e_n (x-a)^n$ where $e_n = c_0 d_n + c_1 d_{n-1} + c_2 d_{n-2} + \cdots + c_{n-2} d_2 + c_{n-1} d_1 + c_n d_0 = \sum_{k=0}^{n} c_k d_{n-k}, \ n \ge 0$.

k) Taylor series

- If $f : I \to \mathbb{R}$ admits a local power series representation in a neighborhood J of $a \in I$, then $f : J \to \mathbb{R}$ is smooth and the coefficients of the power series representation $\sum_{n=0}^{\infty} a_n (x-a)^n$ are given by $a_n = f^{(n)}(a)/n!$ for $n \in \mathbb{N}$.
- If $f : I \to \mathbb{R}$ is a smooth function, then the Taylor series of f centered at $a \in I$ is defined as the formal power series $\sum_{n=0}^{\infty} \dfrac{f^{(n)}(a)}{n!}(x-a)^n$.
- For $N \in \mathbb{N}$, the N-th Taylor polynomial T_N associated to f centered at a is the N-th degree polynomial $T_N : \mathbb{R} \to \mathbb{R}$ defined via $T_N(x) = \sum_{n=0}^{N} \dfrac{f^{(n)}(a)}{n!}(x-a)^n$.
- For $N \in \mathbb{N}$, the N-th remainder R_N associated to f centered at a is a function $R_N : I \to \mathbb{R}$ defined via $R_N(x) = f(x) - T_N(x)$.
- The formal Taylor series associated to a smooth function $f : I \to \mathbb{R}$ converges to $f(x)$ for $x \in I$ if and only if $|R_N(x)| \to 0$ as $N \to \infty$ for $x \in I$.

- Taylor's remainder theorem: if $f : I \to \mathbb{R}$ is smooth and $R_N : I \to \mathbb{R}$ is the remainder associated to f centered at $a \in I$, then for each $N \in \mathbb{N}$ and $x \in I$, there exist c between a and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}. \tag{1.1}$$

Furthermore, if there exists a constant $M > 0$ for which $\left|f^{(N+1)}(x)\right| \leq M$ for all $x \in I$, then $|R_N(x)| \leq \dfrac{M}{(N+1)!}\,|x-a|^{N+1}$ for all $x \in I$.

l) Newton's method

- Newton's method is an iterative method that is used to approximate the roots of a differentiable function $f : \mathbb{R} \to \mathbb{R}$. One starts at an initial guess $x_0 \in \mathbb{R}$ and iteratively computes the sequence (if possible) $\{x_n\}_{n=0}^{\infty} \subset \mathbb{R}$ via the formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ for $n \geq 0$.

m) Arc length and surface area (in Cartesian coordinates)

- If a curve $\mathscr{C}$ is represented by the graph of a differentiable function $f : [a, b] \to \mathbb{R}$, then the arc length of $\mathscr{C}$ is defined as the definite integral $\displaystyle\int_a^b \sqrt{1 + (f'(x))^2}\, dx$.
- In the x-y plane, if a "nice" curve $\mathscr{C}$ is represented by the set $\{(x, y) \in \mathbb{R}^2 \mid y = f(x),\ a \leq x \leq b\}$ where $f : [a, b] \to \mathbb{R}$ is differentiable, then the surface area of the surface of revolution $\mathscr{S}$ obtained by rotating $\mathscr{C}$ about the x-axis is defined as the definite integral $\displaystyle\int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx$.
- In the x-y plane, if a "nice" curve $\mathscr{C}$ is represented by the set $\{(x, y) \in \mathbb{R}^2 \mid x = g(y),\ a \leq y \leq b\}$ where $g : [a, b] \to \mathbb{R}$ is differentiable, then the surface area of the surface of revolution $\mathscr{S}$ obtained by rotating $\mathscr{C}$ about the y-axis is defined as the definite integral $\displaystyle\int_a^b 2\pi g(y)\sqrt{1 + (g'(y))^2}\, dy$.

n) Differential equations

- A differential equation is an equation that involves an unknown function $y : I \to \mathbb{R}$ and its derivatives. A solution to a differential equation is a function $y : I \to \mathbb{R}$ that satisfies the equation.
- An order of a differential equation is the highest order of derivatives that appears in the equation.
- An initial value problem is a differential equation coupled with initial conditions. A solution to an initial value problem is a solution to the differential equation that also satisfies the initial conditions.
- A first-order separable differential equation is a differential equation that can be written in the form $y'(x) = f(x)g(y(x)), x \in I$ for some functions $f : I \to \mathbb{R}, g : J \to \mathbb{R}$.

o) Parametric equations

- If $x, y : I \to \mathbb{R}$ are continuous functions, then the curve $\mathscr{C}$ defined via $\mathscr{C} = \{(x, y) \in \mathbb{R}^2 \mid x = x(t), y = y(t), t \in I\}$ is said to be a parametric curve.
- If $x, y : I \to \mathbb{R}$ are differentiable, then

$$\frac{dy}{dx}(t) = \frac{y'(t)}{x'(t)} \text{ for all } t \in I \text{ such that } x'(t) \neq 0, \tag{1.2}$$

and

$$\frac{d^2y}{dx^2}(t) = \frac{\frac{d}{dt}\left[\frac{dy}{dx}(t)\right]}{x'(t)} \text{ for all } t \in I \text{ such that } x'(t) \neq 0. \tag{1.3}$$

- Suppose the graph of a non-negative function $f : I \to (0, \infty)$ is parametrically represented by the curve $\mathscr{C} = \{(x, y) \in \mathbb{R}^2 \mid x = x(t), y = y(t), t \in I\}$ and $x' : I \to \mathbb{R}$ is differentiable. Then the area of the region bounded by $\mathscr{C}$, the x-axis, and the vertical lines $x = x(a)$ and $x = x(b)$ is given by the definite integral $\int_a^b y(t)x'(t)\, dt$ or $\int_b^a y(t)x'(t)\, dt$.
- In the x-y plane, if a "nice" curve $\mathscr{C}$ is parametrically represented by the set $\mathscr{C} = \{(x, y) \in \mathbb{R}^2 \mid x = x(t), y = y(t), t \in [a, b]\}$, $\mathscr{C}$ is traversed exactly once as t increases from a to b, and $x, y : (a, b) \to \mathbb{R}$ are differentiable, then the arc length of $\mathscr{C}$ is defined as the definite integral $\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$.
- In the x-y plane, if a "nice" curve $\mathscr{C}$ is parametrically represented by the set $\mathscr{C} = \{(x, y) \in \mathbb{R}^2 \mid x = x(t), y = y(t), t \in I\}$ and $x, y : I \to \mathbb{R}$ are differentiable, then the surface area of the surface of revolution $\mathscr{S}_x$ obtained by rotating $\mathscr{C}$ about the x-axis is defined as the definite integral $\int_a^b 2\pi y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt$, and the surface area of the surface of revolution $\mathscr{S}_y$ obtained by rotating $\mathscr{C}$ about the y-axis is defined as the definite integral $\int_a^b 2\pi x(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt$.

p) Polar coordinates

- The polar coordinate system is a coordinate system in which each point P in the plane is determined by a tuple $(r, \theta) \in \mathbb{R}^2$, where $|r|$ denotes the distance from P to the origin and θ measures the angle between the polar axis (the positive x-axis) and the line segment connecting the origin to P.
- If $(r, \theta) \in \mathbb{R}^2$ is the polar representation of a point P in the plane, then the Cartesian representation of P is given by $(x, y) = (r\cos\theta, r\sin\theta) \in \mathbb{R}^2$.
- If $(x, y) \in \mathbb{R}^2$ is the Cartesian representation of a point P in the plane, then a possible polar representation (r, θ) of P is can be found by solving the system of equations $r^2 = x^2 + y^2$ and $\theta = \arctan(y/x)$ if $x \neq 0$.
- If $f : I \to \mathbb{R}$ is a continuous function and a "nice" curve $\mathscr{C}$ is represented in polar coordinates by the set $\mathscr{C} = \{(r, \theta) \in \mathbb{R}^2 \mid r = f(\theta), \theta \in I\}$, then the curve $\mathscr{C}$ is represented in Cartesian coordinates as a parametric curve $\mathscr{C} = \{(x, y) \in \mathbb{R}^2 \mid x = f(\theta)\cos\theta, y = f(\theta)\sin\theta, \theta \in I\}$.
- If $f : [\alpha, \beta] \to [0, \infty)$ is continuous with $0 < \beta - \alpha < 2\pi$, then the area of the region bounded by the curve $\mathscr{C} = \{(r, \theta) \in \mathbb{R}^2 \mid r = f(\theta), \theta \in [\alpha, \beta]\}$ and the radial lines $\theta = \alpha$ and $\theta = \beta$ is defined as the definite integral $\frac{1}{2}\int_\alpha^\beta (f(\theta))^2\, d\theta$.
- In polar coordinates, if a "nice" curve $\mathscr{C}$ is represented by the set $\mathscr{C} = \{(r, \theta) \in \mathbb{R}^2 \mid r = f(\theta), \theta \in [\alpha, \beta]\}$, $\mathscr{C}$ is traversed exactly once as θ increases from α to β, and $f : (\alpha, \beta) \to \mathbb{R}$ is differentiable, then the arc length of $\mathscr{C}$ is defined as the definite integral $\int_\alpha^\beta \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta$.