Week 10

Problem 10.1.

Consider the linear ordinary differential equation

$$t^2 y'(t) + t^3 y(t) = t^4,\ t \in \mathbb{R}. \tag{10.1}$$

What is the order of this equation?

Solution. Since the highest ordered derivative appearing in the equation is the first derivative, this is a first order equation. □

Problem 10.2.

Consider the first order linear ordinary differential equation

$$y'(t) + \cos(t) y(t) = 0,\ t \in \mathbb{R}. \tag{10.2}$$

What is one possible candidate for the integrating factor μ?

Solution. We can choose

$$\mu(t) = \exp\left(\int \cos t \, dt\right) = e^{\sin t},\ t \in \mathbb{R}. \tag{10.3}$$

□

Problem 10.3.

Find the general solution to the differential equation

$$2e^{2t} g'(t) + 4e^{2t} g(t) = 6e^{4t},\ t \in \mathbb{R}. \tag{10.4}$$

You may skip the verification step.

Solution. Note that if g is a solution then

$$g'(t) + 2g(t) = 3e^{2t},\ t \in \mathbb{R}, \tag{10.5}$$

therefore by choosing $\mu(t) = e^{2t}, t \in \mathbb{R}$, we have

$$\frac{d}{dt}\left(e^{2t} g(t)\right) = 3e^{4t},\ t \in \mathbb{R}. \tag{10.6}$$

Thus

$$g(t) = \frac{3}{4} e^{2t} + Ce^{-2t},\ t \in \mathbb{R} \tag{10.7}$$

and C is arbitrary. □

Problem 10.4.

Verify that the functions y_1, y_2 defined via

$$y_1(x) = e^{2x} \cos x,\ y_2(x) = e^{2x} \sin x,\ x \in \mathbb{R} \tag{10.8}$$

form a fundamental set of solutions on the interval $I = \mathbb{R}$ to the equation

$$y''(x) - 4y'(x) + 5y(x) = 0,\ x \in \mathbb{R}. \tag{10.9}$$

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Solution. We first check that y_1, y_2 satisfy the equation. We note that for all $x \in \mathbb{R}$,

$$y_1'(x) = e^{2x}(2\cos x - \sin x) \tag{10.10}$$
$$y_1''(x) = e^{2x}(4\cos x - 2\sin x - 2\sin x - \cos x) = e^{2x}(3\cos x - 4\sin x) \tag{10.11}$$
$$y_2'(x)' = e^{2x}(\cos x + 2\sin x) \tag{10.12}$$
$$y_2''(x) = e^{2x}(2\cos x + 4\sin x - \sin x + 2\cos x) = e^{2x}(4\cos x + 3\sin x). \tag{10.13}$$

Then for all $x \in \mathbb{R}$,

$$y_1''(x) - 4y_1'(x) + 5y_1(x) = e^{2x}(3\cos x - 4\sin x - 8\cos x + 4\sin x + 5\cos x) = 0, \tag{10.14}$$
$$y_2''(x) - 4y_2'(x) + 5y_2(x) = e^{2x}(4\cos x + 3\sin x - 4\cos x - 8\sin x + 5\sin x) = 0. \tag{10.15}$$

Therefore y_1, y_2 are solutions to the equation. Furthermore,

$$\begin{aligned} W(y_1, y_2)(x) = \det \begin{pmatrix} e^{2x}\cos x & e^{2x}\sin x \\ e^{2x}(2\cos x - \sin x) & e^{2x}(\cos x + 2\sin x) \end{pmatrix} \\ = e^{4x}(\cos^2 x + 2\sin x\cos x - 2\sin x \cos x + \sin^2 x) = e^{4x} \neq 0 \text{ for all } x \in \mathbb{R}. \end{aligned} \tag{10.16}$$

Therefore y_1, y_2 are linearly independent over $\mathbb{R}$. This shows that they form a fundamental set of solutions for the equation. □

Problem 10.5.

Find a constant coefficient linear differential equation that has

$$y(x) = c_1 e^{-x} + c_2 e^{-2x} + c_3 x e^{-2x} \text{ for all } x \in \mathbb{R} \text{ and } c_1, c_2, c_3 \text{ are arbitrary} \tag{10.17}$$

as the general solution.

Solution. The roots to the characteristic equation would be $r = -1, -2, -2$. Therefore the characteristic equation associated to the differential equation is

$$\begin{aligned} (r - (-1))(r - (-2))^2 = (r+1)(r+2)^2 = (r+1)(r^2 + 4r + 4) \\ = r^3 + 4r^2 + 4r + r^2 + 4r + 4 = r^3 + 5r^2 + 8r + 4. \end{aligned} \tag{10.18}$$

Therefore the constant coefficient equation

$$y^{(3)}(x) + 5y''(x) + 8y'(x) + 4y(x) = 0, \ x \in \mathbb{R} \tag{10.19}$$

would have (10.17) as the general solution. □

Problem 10.6.

The general solution to the homogeneous linear equation

$$y''(x) + 2y'(x) + y(x) = 0, \ x \in \mathbb{R} \tag{10.20}$$

is

$$y_h(x) = c_1 e^{-x} + c_2 x e^{-x}, \ x \in \mathbb{R}. \tag{10.21}$$

Find the general solution to the inhomogeneous linear equation

$$y''(x) + 2y'(x) + y(x) = 8e^x, \ x \in \mathbb{R}. \tag{10.22}$$

Solution. We use the ansatz

$$y_p(x) = Ae^x, \ x \in \mathbb{R}. \tag{10.23}$$

Then for all $x \in \mathbb{R}$,

$$y_p'(x) = Ae^x \tag{10.24}$$
$$y_p''(x) = Ae^x. \tag{10.25}$$

Thus if y_p is a particular solution, then

$$y_p''(x) + 2y_p'(x) + y_p(x) = 4Ae^x = 8e^x \implies A = 2. \tag{10.26}$$

Therefore the general solution is

$$y(x) = y_p(x) + y_h(x) = 2e^x + c_1 e^{-x} + c_2 x e^{-x}, \ x \in \mathbb{R}, \tag{10.27}$$

where c_1, c_2 are arbitrary. $\square$

Problem 10.7.

For what value(s) of ω do pure resonances occur for a mass-spring system modeled via

$$2x''(t) + 3x(t) = \cos \omega t,\ t \in \mathbb{R}? \tag{10.28}$$

Solution. Resonance occurs when the forcing frequency is equal to the natural frequency of the system, so when

$$\omega = \sqrt{\frac{3}{2}}. \tag{10.29}$$

$\square$

Problem 10.8.

Suppose $f : [0, \infty) \to \mathbb{R}$ is a continuous function for which

$$\mathcal{L}\{f\}(s) = \frac{2s+1}{s^2+16},\ s > 0. \tag{10.30}$$

Find f. You may use the fact that for any $k \in \mathbb{R}$,

$$\mathcal{L}\{\sin kt\}(s) = \frac{k}{s^2+k^2},\ s > 0 \tag{10.31}$$

$$\mathcal{L}\{\cos kt\}(s) = \frac{s}{s^2+k^2},\ s > 0. \tag{10.32}$$

Solution. We note that

$$\mathcal{L}\{f\}(s) = 2\frac{s}{s^2+16} + \frac{1}{4}\frac{4}{s^2+16},\ s > 0. \tag{10.33}$$

Therefore

$$f(t) = 2\cos 4t + \frac{1}{4}\sin 4t,\ t \geq 0. \tag{10.34}$$

$\square$

Problem 10.9.

Find the inverse Laplace transform of the function $F : (0, \infty) \to \mathbb{R}$ defined via

$$F(s) = e^{-2s}\frac{1}{(s+1)^2+4}. \tag{10.35}$$

Solution. Note that

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\}(t) = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+4}\right\}(t) = \frac{1}{2}e^{-t}\sin 2t,\ t \geq 0. \tag{10.36}$$

Therefore

$$\mathcal{L}^{-1}\{F\}(t) = \mathcal{U}(t-2)\frac{1}{2}e^{-(t-2)}\sin 2(t-2),\ t \geq 0. \tag{10.37}$$

$\square$

Problem 10.10.

Consider the system of differential equations

$$\begin{cases} x'(t) = x(t) + y(t) \\ y'(t) = x(t) + y(t),\ t \in \mathbb{R}. \end{cases} \tag{10.38}$$

Write the system in matrix-vector form

$$\boldsymbol{X}'(t) = A\boldsymbol{X}(t),\ t \in \mathbb{R} \tag{10.39}$$

for

$$\boldsymbol{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, t \in \mathbb{R}, \tag{10.40}$$

and also identify the eigenvalues of A.

Solution. We can write the system as

$$\boldsymbol{X}'(t) = \begin{cases} x(t) + y(t) \\ x(t) + y(t) \end{cases} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \boldsymbol{X}(t),\ t \in \mathbb{R}. \tag{10.41}$$

Note that A is not invertible, so 0 is an eigenvalue. Since $\operatorname{tr} A = 2$, the other eigenvalue is 2. We can also see this from examining the characteristic polynomial, which is

$$p_A(\lambda) = \det(A - \lambda I) = \begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 1^2 = (\lambda - 2)(\lambda),\ \lambda \in \mathbb{C}. \tag{10.42}$$

$\square$