Homework 1 Solutions

DUE: SATURDAY, JANUARY 25, 11:59PM

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Problem 1.1. Please carefully read through Section 5 of the syllabus. For your convenience, I have included the text below. Then, sign and date on the bottom of page 3.

5. Policies and guidelines

5.1. Academic integrity.

Academic integrity is important for ensuring that you are getting the greatest learning benefit from the course, and that everyone's grades accurately reflect their learning outcomes. Personal integrity is important for reasons that extend far beyond the confines of this course.

The integrity policy for this course boils down to the following principles.

(1) Your conduct in the course should stem from an earnest desire to learn.
(2) The work you submit should be an accurate reflection of your own understanding and effort.
(3) You should clearly disclose all help received and resources used.
(4) You should not attempt to gain an unfair advantage over your classmates.

More specific academic integrity policies are as follows.

- **Collaboration.** Not only is collaboration between you and your peers allowed, it is in fact encouraged. Difficult problems in the real world are often solved through collaboration between people from various backgrounds. Learning how to communicate your ideas clearly and effectively to others will benefit you in the long run.

  With this in mind, you may discuss homework problems and quizzes with others, with the following caveats.
  - Any notes, recordings or other records created during collaborative sessions must be destroyed immediately after the session concludes and before anything to be submitted for credit is written.
  - You may not share your work with others, or attempt to view other students' work.
  - You must clearly cite the full names of anyone you discussed problem sheet questions with in the relevant text box on Canvas.
  - **Everything that you submit must be of your own.**
- **Reference materials.** You may use notes, textbooks and websites for reference purposes on homework assignments and quizzes; 'reference purposes' refers to things like looking up a definition, formula or theorem statement. Any use of reference materials on homework submissions must be clearly cited in the relevant text box on Canvas. You may not use non-reference materials such as a solution manual.
- **Computer assistance**. At this stage you need to do all the calculations by hand to understand the mechanics behind the concepts. You will not be getting any benefit if you allow a machine do all the work for you. As a result the use of computational tools (e.g. Wolfram Alpha) or any form of AI assistance (e.g. ChatGPT) is prohibited. On homework submissions, *any computational details must also be explicitly recorded*. If no computational details are provided, it will be up to the discretion of the grader to determine whether the work is sufficient.
- **Midterms and the final exam.** Midterms and the final exam must be taken individually during the allocated time and without external resources. In particular, during the midterms and the final exam:
  - You must stop working immediately when instructed to do so, and may not modify your solutions in any way after this point.
  - You may not attempt to view other students' solutions.
  - You may not use notes, calculators, computational tools or electronic devices.
  - You may not attempt to communicate with other students.
- **Seeking and accessing solutions.** You may not seek solutions to questions that were written by others, e.g. using a solution manual or by using a search engine. If you accessed a solution to a homework, test or final exam question (or a very similar one), you must notify me immediately.
- **Distributing course materials.** You may not distribute course materials or solutions outside the course, e.g. to other students, via group chats, or on third-party websites. This applies even after the course has ended.
- **Abuse of regrade requests.** Regrade requests should be used only for correcting grading errors or requesting additional feedback. They should not be used for other purposes, e.g. to request a higher score. See Section 5.6 for more information.
- **Honesty.** You should not falsify or withhold information in an attempt to gain an unfair advantage in the course, e.g. to attempt to obtain an extension on an assessment, or to cover up an academic integrity violation during an investigation.

Academic integrity violations usually occur for one of two reasons.

- **Misunderstanding.** Please take the time to make sure you understand this policy through-and-through, and do not be afraid to ask for clarification if you are not sure about anything.
- **Desperation.** If you ever feel desperate enough to cheat, there is probably a better way—please reach out for support, and use the resources listed in Section **??**.

More information about academic integrity at Carnegie Mellon can be found in the following places:

- **The Word student handbook**
  `https://www.cmu.edu/student-affairs/theword/`
- **Office of Community Standard and Integrity (OCSI)**
  `https://www.cmu.edu/student-affairs/ocsi/`

If you are suspected of an academic integrity violation, an investigation will take place to determine whether a violation occurred. If you are found to have committed a violation, it will be reported to OCSI and course-level action will apply. **The default course-level action for an academic integrity violation is an R grade in the course**, but more lenient action may be considered depending on the nature of the violation and conduct during the investigation.

5.2. **Conduct and community standards.**

When interacting with others in the course, remember that we are all here with a common goal: to learn and share our knowledge. Please be polite and respectful when interacting with each other and with the course staff, adhere to Carnegie Mellon's community standards, and do what you can to make sure that everyone feels safe, included and welcome in the course community.

See also: `https://www.cmu.edu/student-affairs/theword/community-standards/`.

5.3. **Attendance and absence.**

It is important that you attend lectures and recitations unless you have a good reason to be absent, such as due to illness or an emergency situation.

Attendance will not be monitored, but if you are absent from a lecture or recitation then you should catch up on what you missed as soon as possible. Reading slides to catch up is not usually as effective as being present. Missing one or two lectures or recitations is not the end of the world, but habitual absence is not a good idea.

See Section 5.5 for more information about absence from a test or the final exam.

5.4. **Work submission.**

All work for homework assignments should be submitted to Gradescope by the posted deadline. You are expected to manage your time in order to meet the work submission deadlines.

It is your responsibility to ensure that your work is successfully submitted to Gracdescope on time. In particular, you should check to make sure the upload was successful, that the files you upload are not corrupted, and that your work is legible. **Please do not email me your work, all work must be submitted through Gradescope.**

Homework may be submitted up to 3 hours late without penalty. This 'grace period' is intended for resolving unexpected technological issues. Work submitted after this time will not be accepted for credit. You are strongly advised to submit your work by the posted deadline, and not wait until the end of the grace period to start submitting your work.

5.5. **Flexibility.**

Sometimes circumstances arise that prevent you from uploading your work on time. There is flexibility built into the course to deal with such circumstances:

- Enough scores are dropped from the course average computation that you can theoretically miss two weeks of work without a catastrophic impact on your grade. **Please note that the dropped scores are in lieu of make-ups and extensions.**
- There is a grace period for online work submission to deal with unforeseen technological difficulties: see Section 5.4.

If an unforeseen circumstance that is out of your control causes you to miss the final exam— please contact me immediately.

If something comes up in the semester that makes it likely that you have difficulty keeping up with the course, please contact me and your academic advisor as soon as possible.

5.6. **Regrade requests.**

We will grade a lot of your work throughout the semester, but it is entirely possible that we will make an error from time to time. If we do, please submit a regrade request through Gradescope. The procedure is as follows.

- All regrade requests should be submitted via Gradescope - no email or verbal requests.
- Separate regrade requests should be submitted for separate questions.
- Regrade requests are due within 48 hours after feedback being released on Gradescope; for the final exam, this window is shortened to 24 hours.

Your work will be regraded from scratch according to the same rubric—because of this, please note that **it is possible for your score to increase, decrease or remain the same**.

Some scenarios where submitting a regrade request **would** be appropriate are:

- The grader overlooked some of your work, e.g. they only graded the first of two pages.
- The grader misread your solution.
- You did not receive full credit for a problem sheet solution and no feedback was left.
- The grader entered a score incorrectly on Gradescope.

You can also use the regrade request feature on Gradescope to ask the grader to elaborate on their feedback. If you want to do this, make it clear in your 'regrade request' that you are not actually requesting regrades.

Some scenarios where submitting a regrade request **would not** be appropriate are:

- You disagree with how points are allocated in the rubric.
- You need to raise your score by a few points in order to achieve a certain grade.

If you are not sure whether something is grounds for a regrade request, ask me prior to submitting your request.

By signing below, I acknowledge that I have read, understand, and agree to the policies outlined in Section 5 of the course syllabus.

Signature

Date

Problem 1.2 (Classification based on order).

Classify the following equations by their order and explain how you reached that conclusion by pointing out the highest order term(s) in the equation.

Then classify the equations based on whether they are ordinary differential equations or partial differential equations, and explain how you reached that conclusion by explicitly stating the underlying independent variables.

1)
$$e^{x^2} y'(x) + y(x) = x^5, \quad x \in \mathbb{R}^{\cdot} \tag{1.1}$$

2)
$$y''(x) + x^3 y(x) = 0, \quad x \in \mathbb{R}^{\cdot} \tag{1.2}$$

3)
$$y^{(5)}(t) + y(t) = \sin(t^4), \quad t \in \mathbb{R}. \tag{1.3}$$

4)
$$\partial_{tt} u(x,t) = \partial_{xx} u(x,t), \text{ for all } x \in \mathbb{R}, t \in \mathbb{R}. \tag{1.4}$$

Note that the notation $y^{(n)}$ denotes the n-th ordered derivative of the function y with respect to the variable x.

1) $e^{x^2} y'(x)$ is the highest ordered term appearing in the equation, therefore the equation is first ordered. The equation is an ordinary differential equation since the only independent variable is x.
2) $y''(x)$ is the highest ordered term appearing in the equation, therefore the equation is second ordered. The equation is an ordinary differential equation since the only independent variable is x.
3) $y^{(5)}(x)$ is the highest ordered term appearing in the equation, therefore the equation is fifth ordered. The equation is an ordinary differential equation since the only independent variable is t.
4) We note that $\partial_{tt} u(x,t)$ and $\partial_{xx} u(x,t)$ are the highest ordered term appearing in the equation, therefore the equation is second ordered. The equation is a partial differential equation since there are two independent variables, x and t.

Problem 1.3 (Verification of solutions)**.**

Verify that function $y : \mathbb{R} \to \mathbb{R}$ defined via $y(x) = \sin(x)$ is a solution to the differential equation

$$y''(x) + y(x) = 0, \; x \in \mathbb{R}. \tag{1.5}$$

Solution. For all $x \in \mathbb{R}$, we have $y'(x) = \cos(x)$ and $y''(x) = -\sin(x)$. Therefore the function $y : \mathbb{R} \to \mathbb{R}$ defined via $y(x) = \sin(x)$ satsifies

$$y''(x) + y(x) = -\sin(x) + \sin(x) = 0 \text{ for all } x \in \mathbb{R}. \tag{1.6}$$

This shows that y is a solution to the equation on the interval $J = \mathbb{R}$. $\square$

Problem 1.4. Consider the equation

$$y'(x) + 3x^2 y(x) = x^2 \text{ for all } x \in \mathbb{R}. \tag{1.7}$$

In this problem we aim to use the method of integrating factors to show that if y solves (1.7), then

$$y(x) = \frac{1}{3} + Ce^{-x^3} \text{ for all } x \in \mathbb{R} \tag{1.8}$$

for an arbitrary constant C.

a) Identify the integrating factor $\mu : \mathbb{R} \to \mathbb{R}$ corresponding to (1.7).
b) Write down the equation obtained by multiplying both sides of (1.7) by μ. Rewrite the LHS of the equation as the derivative of μy.
c) Integrate both sides and show that y must be given by (1.8).
d) Verify directly that the function y defined via (1.8) is a solution to (1.7) on the interval $J = \mathbb{R}$.

Part a). We can choose the integrating factor μ and define it via

$$\mu(x) = \exp\left(\int 3x^2 \, dx\right) = e^{x^3}, \ x \in \mathbb{R}. \tag{1.9}$$

□

Part b). Upon multiplying the equation by μ, we have

$$\frac{d}{dx}[e^{x^3} y(x)] = e^{x^3} y'(x) + 3x^2 e^{x^3} y(x) = x^2 e^{x^3}, \ x \in \mathbb{R} \tag{1.10}$$

□

Part c). Via direct integration we obtain

$$e^{x^3} y(x) = \int x^2 e^{x^3} \, dx = \frac{1}{3} e^{x^3} + C, \ x \in \mathbb{R}, \tag{1.11}$$

where C is an arbitrary constant. Therefore if y is a solution to the equation, then

$$y(x) = \frac{1}{3} + \frac{C}{e^{x^3}} = \frac{1}{3} + Ce^{-x^3}, \ x \in \mathbb{R}, \tag{1.12}$$

where C is arbitrary. □

Part d). If y is defined via (1.8), then

$$y'(x) = -3Cx^2 e^{-x^3}, \ x \in \mathbb{R}. \tag{1.13}$$

Then for all $x \in \mathbb{R}$,

$$y'(x) + 3x^2 y(x) = -3Cx^2 e^{-x^3} + 3x^2 \left(\frac{1}{3} + Ce^{-x^3}\right) \tag{1.14}$$

$$= -3Cx^2 e^{-x^3} + x^2 + 3Cx^2 e^{-x^3} = x^2. \tag{1.15}$$

Therefore we have verified that y given via (1.8) is a solution on the interval $J = \mathbb{R}$. □

Problem 1.5. Consider the equation

$$x^2y'(x) + x(x+2)y(x) = e^x \text{ for all } x > 0. \tag{1.16}$$

a) Use the method of integrating factors to identify the general candidate solution to (1.16).
b) Verify that the function you found in the previous part is a solution to (1.16) on the interval $J = (0, \infty)$.

Part a). Suppose that y is a solution to the given equation on the interval $I = (0, \infty)$. Then

$$y'(x) + \frac{x(x+2)}{x^2}y(x) = \frac{e^x}{x^2},\ x > 0. \tag{1.17}$$

We may then choose an integrating factor μ via

$$\begin{aligned}\mu(x) &= \exp\left(\int \frac{x(x+2)}{x^2}\,dx\right) = \exp\left(\int 1 + \frac{2}{x}\,dx\right) &\text{(1.18)}\\ &= \exp\left(x + 2\ln|x|\right) = \exp(x)\exp\ln|x|^2 = x^2e^x,\ x > 0. &\text{(1.19)}\end{aligned}$$

Then upon multiplying both sides of (1.17) by μ, we find that

$$\frac{d}{dx}[x^2e^xy(x)] = x^2e^xy'(x) + x(x+2)e^xy(x) = e^{2x},\ x > 0. \tag{1.20}$$

Upon direct integration we find that

$$x^2e^xy(x) = \frac{1}{2}e^{2x} + C,\ x > 0. \tag{1.21}$$

where C is an arbitrary constant. Therefore if y solves the equation, then

$$y(x) = \frac{1}{2}x^{-2}e^x + Cx^{-2}e^{-x}\ x > 0, \tag{1.22}$$

where C is arbitrary. □

Part b). Now we verify that y defined via (1.22) is a solution to the equation. We note that

$$y'(x) = \frac{1}{2}(-2x^{-3} + x^{-2})e^x + C(-2x^{-3} - x^{-2})e^{-x},\ x > 0. \tag{1.23}$$

Therefore for all $x > 0$,

$$\begin{aligned}x^2y'(x) + x(x+2)y(x) &= \frac{1}{2}(-2x^{-1}+1)e^x + C(-2x^{-1}-1)e^{-x} + x(x+2)\left(\frac{1}{2}x^{-2}e^x + Cx^{-2}e^{-x}\right) &\text{(1.24)}\\ &= \frac{1}{2}(-2x^{-1}+1)e^x + \underbrace{x(x+2)}_{=x^2+2x}\left(\frac{1}{2}x^{-2}e^x\right) + C(-2x^{-1}-1)e^{-x} + \underbrace{x(x+2)}_{=x^2+2x}\left(Cx^{-2}e^{-x}\right) &\text{(1.25)}\\ &= \underbrace{\frac{1}{2}(-2x^{-1}+1)e^x + \frac{1}{2}(1+2x^{-1})e^x}_{=(1/2)e^x+(1/2)e^x} + \underbrace{C(-2x^{-1}-1)e^{-x} + C(1+2x^{-1})e^{-x}}_{=0} &\text{(1.26)}\\ &= e^x. &\text{(1.27)}\end{aligned}$$

Therefore y defined via (1.22) is a solution to the original equation on the interval $J = (0, \infty)$. □

Problem 1.6. Consider the initial value problem

$$\begin{cases} (x+1)y'(x) + y(x) = x\ln x, \quad x > 0 \\ y(1) = 10. \end{cases} \tag{1.28}$$

a) Use the method of integrating factors to identify the general candidate solution to the IVP (1.28).
b) Identify the maximal interval of existence J of the solution in part a).
c) Verify that the function you found in part a) is a solution to (1.28) on the interval J.

Part a). If y is a solution to the IVP for $x > 0$, then

$$y'(x) + \frac{1}{x+1}y(x) = \frac{x\ln x}{x+1},\ x > 0. \tag{1.29}$$

We may choose an integrating factor μ via

$$\mu(x) = \exp\left(\int \frac{1}{x+1}\,dx\right) = \exp\ln|x+1| = |x+1| = x+1,\ x > 0. \tag{1.30}$$

Therefore y satisfies

$$\frac{d}{dx}[(x+1)y(x)] = (x+1)y'(x) + y(x) = x\ln x,\ x > 0. \tag{1.31}$$

Upon direct integration (using integration by parts for the RHS) we find that

$$(x+1)y(x) = \int x\ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx = \frac{x^2\ln x}{2} - \frac{x^2}{4} + C,\ x > 0. \tag{1.32}$$

Since y is a solution to the IVP, $y(1) = 10$, therefore

$$(1+1)(10) = \frac{1^2\ln 1}{2} - \frac{1^2}{4} + C. \tag{1.33}$$

Here we find that

$$C = 20 + \frac{1}{4} = \frac{81}{4}. \tag{1.34}$$

Therefore the candidate solution to the IVP is

$$y(x) = \frac{2x^2\ln x - x^2 + 81}{4(x+1)},\ x > 0. \tag{1.35}$$

□

Part c). Now we note that if y is defined via (1.35), then

$$(x+1)y(x) = \frac{2x^2\ln x - x^2 + 81}{4},\ x > 0. \tag{1.36}$$

Thus for all $x > 0$,

$$(x+1)y'(x) + y(x) = \frac{d}{dx}[(x+1)y(x)] = \frac{d}{dx}\left[\frac{2x^2\ln x - x^2 + 81}{4}\right] \tag{1.37}$$

$$= \frac{1}{4}\left(2(2x\ln x + x) - 2x\right) = \frac{4x\ln x}{4} = x\ln x. \tag{1.38}$$

Furthermore,

$$y(1) = \frac{2(1)^2\ln 1 - (1)^2 + 81}{4(1+1)} = \frac{80}{8} = 10. \tag{1.39}$$

Therefore y defined via (1.35) is a solution to the IVP on the interval $J = (0, \infty)$. □

Problem 1.7. Consider the following situation:

- A vat initially holds 100 ℓ of pure water.
- A salt solution is added to the vat at a rate of 2 ℓ/min. The concentration of salt is 10 g/ℓ.
- The vat is drained off at a rate of 3 ℓ/min until the vat is empty.

Let $V(t)$ denote the volume of the mixture in the vat at time $t > 0$ until some terminal time T^*, where $V(T^*) = 0$. Let $I = (0, T^*)$ and let $Q(t)$ denote the amount of salt in the vat at time $t \in I$, and define the concentration of salt at time $t \in I$ by the ratio $\frac{Q(t)}{V(t)}$.

a) What is $V(0)$? Write down an equation for $V'(t)$, the rate of change of the volume of the mixture at time $t \in I$. What is the volume of the mixture in the vat at time $t \in I$? What is the terminal time T^* at which the vat becomes empty?

b) What is $Q(0)$? Write down an equation for $Q'(t)$, the rate of change of the amount of salt in the vat at time $t \in I$.

c) Solve the initial value problem in part b) and find an expression for the amount of salt $Q(t)$ in the vat at time $t \in I$.

d) Show that the concentration of salt at time t is given by the expression

$$\frac{Q(t)}{V(t)} = 10 - \frac{1}{10^3} V(t)^2 \text{ for all } t \in I. \tag{1.40}$$

What is the concentration of salt as $t \to T^*$?

Part a). Since we are given at initial time there is 100 l of pure water, $V(0) = 0$. Since the solution is added at a rate of 2 l/min and drained at a rate of 3 l/min, V' satisfies

$$V'(t) = 2 - 3 = -1, \; t \in I. \tag{1.41}$$

This implies that

$$V(t) = -t + C, \; t \in I \tag{1.42}$$

for some constant C. Using the initial condition $V(0) = 100$ we find that $C = 100$. Therefore

$$V(t) = 100 - t, t \in (0, 100). \tag{1.43}$$

The terminal time $T^* = 100$. □

Part b). Since we are given that at initial time the water is pure water, we can infer that $Q(0) = 0$. According to the problem,

$$Q'(t) = (2)10 - (3)\frac{Q(t)}{V(t)} = 20 - 3\frac{Q(t)}{100 - t}, \; t \in I. \tag{1.44}$$

Therefore Q solves the initial value problem

$$\begin{cases} Q'(t) + \frac{3}{100-t} Q(t) = 20, \; t \in I \\ Q(0) = 0. \end{cases} \tag{1.45}$$

□

Part c). To solve the first order linear IVP we use the method of integrating factors. We may choose an integrating factor μ via

$$\mu(t) = \exp\left(\int \frac{3}{100 - t}\, dt\right) = \exp(-3 \ln |100 - t|) \tag{1.46}$$

$$= \exp(\ln |100 - t|^{-3}) = |100 - t|^{-3} = (100 - t)^{-3}, \; t \in I. \tag{1.47}$$

Then we find that

$$\frac{d}{dt}\left[(100 - t)^{-3} Q(t)\right] = 20(100 - t)^{-3}, \; t \in I. \tag{1.48}$$

Upon direct integration we find that

$$Q(t) = 10(100 - t) + C(100 - t)^3, \; t \in I, \tag{1.49}$$

for some constant C. Using $Q(0) = 0$ we find that

$$0 = 10 \times 100 + 100^3 C \implies C = -\frac{10^3}{10^6} = -\frac{1}{10^3}. \tag{1.50}$$

Therefore if Q is a solution to the IVP, then Therefore

$$Q(t) = 10V(t) - \frac{1}{10^3}V(t)^3, t \in I. \tag{1.51}$$

□

Part d). From part c) we see that

$$\frac{Q(t)}{V(t)} = 10 - \frac{1}{10^3}V(t)^2,\ t \in I. \tag{1.52}$$

It's clear that as $t \to T^*$, $V(t) \to 0$ and the quantity above goes to 10. □

Problem 1.8 (Structure of solutions to first order linear equations)**.**

Let I be an interval and let $a_0, a_1, f : I \to \mathbb{R}$ be continuous. Consider the general first ordered inhomogeneous linear equation

$$a_1(x)y'(x) + a_0(x)y(x) = f(x) \text{ for all } x \in I \tag{1.53}$$

with its associated homogeneous equation

$$a_1(x)y'(x) + a_0(x)y(x) = 0 \text{ for all } x \in I. \tag{1.54}$$

Define the map $L : C^1(I;\mathbb{R}) \to C^0(I;\mathbb{R})$ via

$$L(y) = a_1 \cdot y' + a_0 \cdot y. \tag{1.55}$$

One can easily use (1.55) to show that

$$L(y_1 + y_2) = L(y_1) + L(y_2) \tag{1.56}$$

and

$$L(cy) = cL(y) \tag{1.57}$$

for any arbitrary constant $c \in \mathbb{R}$. In this problem we will explore the consequences of these two properties.

a) Suppose $L(y)(x) = 0$ for all $x \in I$. In other words, y is a solution to the homogeneous equation. Use (1.57) to show that Cy is also a solution to the homogeneous equation for any constant C.
b) Suppose $L(y_1)(x) = f(x)$ for all $x \in I$ and $L(y_2)(x) = 0$ for all $x \in I$. In other words, y_1 is a solution to the inhomogeneous equation and y_2 is a solution to the homogeneous equation. Use (1.56) and (1.57) to show that $y_1 + cy_2$ is a solution to the inhomogeneous equation (1.53) for any constant c.
c) $L(y_1)(x) = f(x)$ for all $x \in I$ and $L(y_2)(x) = f(x)$ for all $x \in I$. In other words, y_1, y_2 are two solutions to the inhomogeneous equation. Use (1.56) and (1.57) to show that $y_1 - y_2$ is a solution to the homogeneous equation (1.54).
d) If f in (1.53) is a non-zero function and y is a solution to the inhomogeneous problem, why is it not possible for Cy to be a solution to the inhomogeneous problem for any constant C?

In summary, one can conclude that because the equation is linear,

a) Any scalar multiple of a homogeneous solution is also a homogeneous solution.
b) The sum of any inhomogeneous solution and any scalar multiple of a homogeneous solution is also an inhomogeneous solution.
c) The difference of two inhomogeneous solutions is always a homogeneous solution.
d) Inhomogeneous solutions cannot be scaled to produce another inhomogeneous solution.

Part a). If $L(y)(x) = 0$ for all $x \in I$, then by (1.57),

$$L(Cy)(x) = CL(y)(x) = C \cdot 0 = 0 \text{ for all } x \in I, \tag{1.58}$$

for any constant C. □

Part b). If $L(y_1)(x) = f(x)$ for all $x \in I$ and $L(y_2)(x) = 0$ for all $x \in I$, then by (1.56) and (1.57),

$$L(y_1 + cy_2)(x) = L(y_1)(x) + L(cy_2)(x) \tag{1.59}$$

$$= L(y_1)(x) + cL(y_2)(x) = f(x) + c \cdot 0 = f(x) \text{ for all } x \in I, \tag{1.60}$$

for any constant c. □

Part c). If $L(y_1)(x) = f(x)$ for all $x \in I$ and $L(y_2)(x) = f(x)$ for all $x \in I$, then

$$L(y_1 - y_2)(x) = L(y_1)(x) - L(y_2)(x) = f(x) - f(x) = 0 \text{ for all } x \in I. \tag{1.61}$$

□

Part d). If $L(y)(x) = f(x)$ for all $x \in I$, then

$$L(Cy)(x) = CL(y)(x) = Cf(x) \text{ for all } x \in I. \tag{1.62}$$

Since f is assumed to be non-zero, whenever f is nonzero $Cf = f$ if and only if $C = 1$. Therefore Cy cannot be a solution to the original inhomogeneous problem for any $C \neq 1$. □

Problem 1.9 (Verification of solution formula for first order linear equations)**.**

Let I be an interval and consider the differential equation

$$y'(x) + P(x)y(x) = f(x) \text{ for all } x \in I. \tag{1.63}$$

In lecture we have shown that if y is a solution to the differential equation, then $y : I \to \mathbb{R}$ is of the form

$$y(x) = (\mu(x))^{-1} \left(\int \mu(x) f(x)\, dx \right) + C(\mu(x))^{-1}, \tag{1.64}$$

where the integrating factor $\mu : I \to \mathbb{R}$ is defined via

$$\mu(x) = \exp\left(\int P(x)\, dx \right). \tag{1.65}$$

In this problem we complete the verification step. Argue as follows.

(1) Show that $\mu : I \to \mathbb{R}$ defined via (1.65) satisfies

$$\mu'(x) = P(x)\mu(x). \tag{1.66}$$

(2) Show that

$$(\mu(x))^{-1} = \exp\left(- \int P(x)\, dx \right) \text{ for all } x \in I. \tag{1.67}$$

Here μ^{-1} is notation for $\frac{1}{\mu}$.

(3) Use (1.67) to show that

$$\frac{d}{dx}\left((\mu(x))^{-1} \right) = -P(x)(\mu(x))^{-1} \text{ for all } x \in I. \tag{1.68}$$

By multiplying both sides of (1.68) by an arbitrary constant C, it also follows that

$$\frac{d}{dx}\left(C(\mu(x))^{-1} \right) = -P(x)(C(\mu(x))^{-1}) \text{ for all } x \in I. \tag{1.69}$$

(4) Use (1.69) to conclude $C\mu^{-1}$ is a solution to the homogeneous equation

$$y'(x) + P(x)y(x) = 0 \text{ for all } x \in I \tag{1.70}$$

for any constant C.

(5) Let y_p be defined via

$$y_p(x) = (\mu(x))^{-1} \left(\int \mu(x) f(x)\, dx \right) \text{ for all } x \in I. \tag{1.71}$$

Use the product rule and (1.68) to show that y_p is a solution to the inhomogeneous equation

$$y'(x) + P(x)y(x) = f(x) \text{ for all } x \in I. \tag{1.72}$$

(6) Conclude that y defined via (1.64) solves the inhomogeneous equation

$$y'(x) + P(x)y(x) = f(x) \text{ for all } x \in I. \tag{1.73}$$

Part 1). We note that if μ is defined via (1.65), then

$$\mu'(x) = \frac{d}{dx}\left[\int P(x)\, dx \right] \exp\left(\int P(x)\, dx \right) = P(x)\mu(x) \text{ for all } x \in I. \tag{1.74}$$

□

Part 2). We note that

$$(\mu(x))^{-1} = \frac{1}{\mu(x)} = \frac{1}{\exp\left(\int P(x)\, dx \right)} = \exp\left(- \int P(x)\, dx \right) \text{ for all } x \in I. \tag{1.75}$$

□

Part 3). By the previous part,

$$\frac{d}{dx}\left((\mu(x))^{-1}\right) = \frac{d}{dx}\left[-\int P(x)\,dx\right]\exp\left(-\int P(x)\,dx\right) = -P(x)(\mu(x))^{-1} \text{ for all } x \in I. \tag{1.76}$$

It then follows that for any constant C,

$$\frac{d}{dx}\left(C(\mu(x))^{-1}\right) = -P(x)(C(\mu(x))^{-1}) \text{ for all } x \in I. \tag{1.77}$$

□

Part 4). By rearranging (1.77), we see that

$$\frac{d}{dx}\left(C(\mu(x))^{-1}\right) + P(x)(C(\mu(x))^{-1}) = 0 \text{ for all } x \in I. \tag{1.78}$$

Therefore $C\mu^{-1}$ is a solution to the homogeneous equation $y' + Py = 0$. □

Part 5). If y_p is defined via (**??**), then

$$(y_p)'(x) = (\mu(x))^{-1}\frac{d}{dx}\left[\int \mu(x)f(x)\,dx\right] + \frac{d}{dx}\left[(\mu(x))^{-1}\right]\left(\int \mu(x)f(x)\,dx\right) \tag{1.79}$$

$$= \underbrace{(\mu(x))^{-1}\mu(x)}_{=1} f(x) - P(x)(\mu(x))^{-1}\left(\int \mu(x)f(x)\,dx\right),\ x \in I. \tag{1.80}$$

Then

$$(y_p)'(x) + P(x)y_p(x) = f(x)\underbrace{-P(x)(\mu(x))^{-1}\left(\int \mu(x)f(x)\,dx\right) + P(x)(\mu(x))^{-1}\left(\int \mu(x)f(x)\,dx\right)}_{=0} = f(x). \tag{1.81}$$

Therefore y_p is a solution to the inhomogeneous problem. □

Part 6). We note that by synthesizing the previous parts, if

$$y(x) = \underbrace{(\mu(x))^{-1}\left(\int \mu(x)f(x)\,dx\right)}_{:=y_p(x)} + \underbrace{C(\mu(x))^{-1}}_{:=y_h(x)} \text{ for all } x \in I, \tag{1.82}$$

then

$$y'(x) + P(x)y(x) = (y_p(x) + y_h(x))' + P(x)(y_p(x) + y_h(x)) \tag{1.83}$$

$$= \underbrace{((y_p)'(x) + P(x)y_p(x))}_{=f(x)} + \underbrace{((y_h)'(x) + P(x)y_h(x))}_{=0} \tag{1.84}$$

$$= f(x),\ x \in I. \tag{1.85}$$

Therefore the candidate solution y defined via (1.64), obtained through the method of integrating factors, is a solution to the original equation on the interval I. This completes the verification step. □