Final practice problems

Problem 1.1 (Linear equations and the integrating factor)**.** Find the general solution to the linear equation

$$x^2y'(x) + 5xy(x) = \frac{e^{4x}}{x^3},\ x > 0. \tag{1.1}$$

Problem 1.2 (Rate problems)**.** A tank contains 130 liters of water and 50 grams of sugar. A solution containing a sugar concentration of $4e^{-t}$ g/L flows into the tank at a rate of 3 L/min, and the mixture in the tank flows out at a rate of 4 L/min. Let $Q(t)$ be the amount of sugar (in grams) in the tank at time t (in minutes). Write down an initial value problem for Q (without solving it explicitly).

Problem 1.3 (Autonomous equations, stability of critical points)**.** Consider the autonomous equation

$$y'(x) = (1 - y(x))^3(y(x) + 2)(y(x) - 4),\ x \in \mathbb{R}. \tag{1.2}$$

a) Identify the critical points and the corresponding constant solutions to the equation.
b) Draw a one-dimensional phase portrait of the equation and determined the stability of the critical points.
c) Give a sketch of sample solution curves in the x-y plane.

Problem 1.4 (Existence and uniqueness of solutions)**.** Consider the initial value problem

$$\begin{cases} y'(x) = \sqrt{y(x)},\ x \geq 0 \\ y(0) = 0. \end{cases} \tag{1.3}$$

a) Does the IVP admit any constant solutions?
b) Does the IVP admit non-constant solutions? If so, why does this not violate the uniqueness part of the existence and uniqueness theorem?

Problem 1.5 (Separable equations, implicit solutions)**.** Consider the equation

$$(y(x))^3y'(x) = ((y(x))^4 + 1)\cos x,\ x \in \mathbb{R}. \tag{1.4}$$

a) Does the equation admit any constant solutions?
b) Find an implicit solution to the equation.

Problem 1.6 (Exact equations, implicit solutions)**.** Consider the nonlinear equation

$$((y(x))^2 + 1) + (2xy(x) + 3(y(x))^2)y'(x) = 0,\ x \in \mathbb{R}. \tag{1.5}$$

a) Is the equation exact?
b) Find the general implicit solution to the equation.

Problem 1.7 (Bernoulli equations, constant solutions)**.** Consider the initial value problem

$$\begin{cases} y'(x) - \frac{2}{x}y(x) = -x^2(y(x))^2, \quad x > 0 \\ y(1) = \alpha. \end{cases} \tag{1.6}$$

Find a solution for the following values of α and state the maximal interval of existence of the solution.

a) $\alpha = 0$
b) $\alpha = 1$

Problem 1.8 (Equations with homogeneous functions)**.** Find an implicit solution to the nonlinear equation

$$y'(x) = \frac{x^2 + 3(y(x))^2}{2xy(x)},\ x > 0. \tag{1.7}$$

Problem 1.9 (2nd order equations and mass-spring systems)**.** Suppose a mass spring system is modeled via the equation

$$x''(t) + \beta x'(t) + 4x(t) = 0,\ t \in \mathbb{R}. \tag{1.8}$$

Identify the value(s) of β for which the system will be

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a) undamped
b) underdamped
c) critically damped
d) overdamped

Problem 1.10 (Method of undetermined coefficients)**.** Write down an appropriate ansatz (without solving for the coefficients) for the linear equation

$$y^{(4)}(t) - 2y^{(3)}(t) + 10y''(t) = t^2 + e^t \cos 2t, \ t \in \mathbb{R}. \tag{1.9}$$

Problem 1.11 (Variation of parameters)**.** Find a particular solution to the equation

$$x^2 y''(x) + xy'(x) - y(x) = 600x^5, \ x > 0 \tag{1.10}$$

given that the general homogeneous solution to the equation is

$$y_h(x) = c_1 x + c_2 x^{-1}, \ x > 0 \tag{1.11}$$

where c_1, c_2 are arbitrary.

Problem 1.12 (Mass-spring systems and resonance)**.** Suppose a mass-spring system is modeled via the IVP

$$\begin{cases} x''(t) + \beta x'(t) + 16x(t) = F_0 \sin \omega t, & t \in \mathbb{R} \\ x(0) = x'(0) = 0. \end{cases} \tag{1.12}$$

where $\beta \geq 0, \omega > 0$.

a) Identify the parameters β, ω for which pure resonance occurs.
b) In the case of part a) and $F_0 = 8$, find the unique solution to the IVP.
c) Suppose $\beta = 8, \omega = 1$ and $F_0 = 16$. What is the (approximate, up to a negligible error) maximum amplitude of the mass-spring system in the long run?

Problem 1.13 (Eigenvalue problems)**.** Consider the eigenvalue problem

$$\begin{cases} y''(x) + 2y'(x) + \lambda y(x) = 0, \ x \in (0, \pi) \\ y(0) = 0, \ y(\pi) = 0. \end{cases} \tag{1.13}$$

Parameterize the eigenvalues $\lambda > 1$ by $n = 1, 2, 3, \ldots$ and list a corresponding set of eigenfunctions.

Problem 1.14 (The Laplace transform and equations with piecewise forcing)**.** Suppose a mass-spring system is modeled via

$$\begin{cases} x''(t) + 4x(t) = f(t), & t \geq 0 \\ x(0) = x'(0) = 0, \end{cases} \tag{1.14}$$

and $f : [0, \infty) \to \mathbb{R}$ is defined via

$$f(t) = \begin{cases} 0, & 0 \leq t < 2\pi \\ \sin t, & t \geq 2\pi. \end{cases} \tag{1.15}$$

Use the Laplace transform to find a solution x describing the behavior of the system for $t \geq 0$

Problem 1.15 (The Laplace transform and impulse forces)**.** Use the Laplace transform to find a solution to the equation

$$\begin{cases} x''(t) + x(t) = 1 + \delta(t - \pi), & t \geq 0 \\ x(0) = x'(0) = 0. \end{cases} \tag{1.16}$$

Problem 1.16 (The Laplace transform and convolutions)**.** Write down a convolution integral solution to the IVP

$$\begin{cases} x''(t) + 8x'(t) + 16x(t) = f(t), & t > 0 \\ x(0) = x'(0) = 0, \end{cases} \tag{1.17}$$

for any reasonable function $f : [0, \infty) \to \mathbb{R}$.

Problem 1.17 (Linear systems)**.** Find the unique solution to the IVP

$$\begin{cases} \boldsymbol{X}'(t) = \begin{pmatrix} -4 & 2 \\ 2 & -4 \end{pmatrix} \boldsymbol{X}(t), \quad t \in \mathbb{R} \\ \boldsymbol{X}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{cases} \tag{1.18}$$

Problem 1.18 (Linear systems and stability of critical points)**.** Consider the linear system

$$\boldsymbol{X}'(t) = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix} \boldsymbol{X}(t),\ t \in \mathbb{R} \tag{1.19}$$

where $\alpha \in \mathbb{R}$ is an unspecified parameter. Identify the value(s) of α for which the origin is a

a) stable node
b) unstable node
c) saddle point

Problem 1.19 (Fourier series)**.** Consider the function

$$f(x) = 1,\ x \in (0,1). \tag{1.20}$$

Sketch a graph over $\mathbb{R}$ of the Fourier sine series of f, which is defined via

$$\sum_{n=1}^{\infty} b_n \sin(n\pi x) \text{ for all } x \in \mathbb{R},\ \text{where } b_n = 2\int_0^1 \sin(n\pi x)\, dx. \tag{1.21}$$

Problem 1.20 (The heat equation)**.** Find the unique solution $u : [0,\pi] \times [0,\infty) \to \mathbb{R}$ as a *finite* linear combination of elementary functions satisfying

$$u_t = 4u_{xx}, \qquad x \in (0,\pi), t \geq 0, \tag{1.22}$$

$$u(x=0,t) = 0 = u(\pi,t) \qquad t \geq 0 \tag{1.23}$$

$$u(x,t=0) = \sin(2x) + \sin(5x), \qquad x \in (0,\pi). \tag{1.24}$$

Problem 1.21 (The wave equation)**.** Find the unique solution $u : [0,\pi] \times [0,\infty) \to \mathbb{R}$ as a *finite* linear combination of elementary functions satisfying

$$u_{tt} = 9u_{xx}, \qquad x \in (0,\pi), t \geq 0 \tag{1.25}$$

$$u(x=0,t) = 0 = u(\pi,t) \qquad t \geq 0 \tag{1.26}$$

$$u(x,t=0) = \sin(3x) \qquad x \in (0,\pi) \tag{1.27}$$

$$u_t(x,t=0) = \sin(4x) \qquad x \in (0,\pi). \tag{1.28}$$