Homework 8

DUE: SUNDAY, MARCH 23, 11:59PM

If you completed this assignment through collaboration or consulted references, please list the names of your collaborators and the references you used below. Please refer to the syllabus for the course policy on collaborations and types of references that are allowed.

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Date: March 18, 2025.

Problem 8.1. Consider the mass-spring system modeled via the initial value problem

$$\begin{cases} x''(t) + x(t) = \delta(t) + \delta(t-\pi) + \delta(t-2\pi) + \delta(t-3\pi), \ t \geq 0 \\ x(0) = x'(0) = 0. \end{cases} \tag{8.1}$$

One can think of the Dirac deltas as modeling a hammer striking the mass at times $t = 0, \pi, 2\pi, 3\pi$ with unit impulse.

a) Find a function $x : [0,\infty) \to \mathbb{R}$ modeling the behavior of the system.
b) Provide a rough sketch of the solution x and give a physical interpretation of the solution in terms of the hammer striking the mass at times $t = 0, \pi, 2\pi, 3\pi$.

Solution.

a) Assuming $x : [0,\infty)$ is a solution to the IVP, taking the Laplace transform allows us the transform the differential equation for the unknown function x in the t-domain to the algebraic equation for $X = \mathcal{L}\{x\}$ in the s-domain, which is

$$s^2X(s) + X(s) = 1 + e^{-\pi s} + e^{-2\pi s} + e^{-3\pi s}, \tag{8.2}$$

for appropriate values of s. Then

$$X(s) = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1} + \frac{e^{-2\pi s}}{s^2+1} + \frac{e^{-3\pi s}}{s^2+1} \tag{8.3}$$

for appropriate values of s. We note that the last three terms are modulations in the s-domain, which would correspond to translations plus truncation in the t-domain. By taking the inverse Laplace transform, we find that

$$x(t) = \sin t + \mathcal{U}(t-\pi)\sin(t-\pi) + \mathcal{U}(t-2\pi)\sin(t-2\pi) + \mathcal{U}(t-3\pi)\sin(t-3\pi), \ t \geq 0. \tag{8.4}$$

b) We note that since $\sin(t-\pi) = \sin(t-3\pi) = -\sin t, \sin(t-2\pi) = \sin t$ for all $t \in \mathbb{R}$, we can write x more explicitly in terms of the piecewise function

$$x(t) = \begin{cases} \sin t, & 0 \leq t < \pi \\ 0, & \pi \leq t < 2\pi \\ \sin t, & 2\pi \leq t < 3\pi, \\ 0, & t \geq 3\pi. \end{cases} \tag{8.5}$$

Therefore a rough sketch for x would be a sinusoidal wave over $[0,\pi]$ and $[2\pi, 3\pi]$ and zero elsewhere.

The physical interpretation of the solution is that the first hammer strike at $t = 0$ hits the mass from above equilibrium, which makes the mass move below equilibrium, and the second hammer strike at $t = \pi$ hits the mass right as it returns to equilibrium. Since the mass is returning to equilibrium traveling upward, which is in the direction opposite to the second hammer strike, the second hammer strike stops the motion of the mass until the mass gets struck again by the third blow at $t = 2\pi$. Similar to the second hammer strike, the last hammer strike at $t = 3\pi$ strikes the mass right as it returns to equilibrium, stopping the motion of the mass.

□

Problem 8.2 (A fundamental reduction)**.** In this problem our goal is to show that every n-th order scalar equation can be converted into a first order $n \times n$ system, and vice versa. This shows that all equations that we have encountered in this class all have a common structure - they can all be studied as a first order system. If linear algebra were a prerequisite for this class, this would have been the natural starting point for studying ODEs.

For the sake of simplicity we will assume $n = 2$ for this problem, but this result can be easily generalized to any $n \in \mathbb{N}$.

a) Let $I \subseteq \mathbb{R}$ be an interval, $t_0 \in I$ and assume $x : I \to \mathbb{R}$ is a smooth function. Suppose x solves the initial value problem

$$\begin{cases} x''(t) = f(t, x(t)), \ t \in \mathbb{R} \\ x(t_0) = x_0, x'(t_0) = x_1 \end{cases} \tag{8.6}$$

for some function $f : \mathbb{R}^2 \to \mathbb{R}$. Consider the vector-valued function $\boldsymbol{X} : I \to \mathbb{R}^2$ defined via

$$\boldsymbol{X}(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix}. \tag{8.7}$$

Show that $\boldsymbol{X}$ satisfies the first order IVP

$$\begin{cases} \boldsymbol{X}'(t) = \begin{pmatrix} X_2(t) \\ f(t, X_1(t)) \end{pmatrix}, \quad t \in I \\ \boldsymbol{X}(t_0) = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}. \end{cases} \tag{8.8}$$

This shows that if x solves (8.11), then $\boldsymbol{X}$ solves (8.10). Next we show the converse.

b) Now suppose $\boldsymbol{X} : I \to \mathbb{R}$ is a vector-valued defined via

$$\boldsymbol{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \tag{8.9}$$

where $x, y : I \to \mathbb{R}$ are two smooth scalar-valued functions. Assume $\boldsymbol{X}$ solves the first order IVP

$$\begin{cases} \boldsymbol{X}'(t) = \begin{pmatrix} y(t) \\ f(t, x(t)) \end{pmatrix}, \quad t \in I \\ \boldsymbol{X}(t_0) = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}. \end{cases} \tag{8.10}$$

Show that $x = (\boldsymbol{X})_1$ satisfies the 2nd order scalar valued IVP

$$\begin{cases} x''(t) = f(t, x(t)), \ t \in \mathbb{R} \\ x(t_0) = x_0, x'(t_0) = x_1. \end{cases} \tag{8.11}$$

Solution.

a) Note that $X_1 = x$ and $X_2 = x'$. Then

$$\boldsymbol{X}'(t) = \begin{pmatrix} X_1'(t) \\ X_2'(t) \end{pmatrix} = \begin{pmatrix} x'(t) \\ x''(t) \end{pmatrix} = \begin{pmatrix} X_2(t) \\ f(t, X_1(t)) \end{pmatrix}, \ t \in I \tag{8.12}$$

and

$$\boldsymbol{X}(t_0) = \begin{pmatrix} x(t_0) \\ x'(t_0) \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}. \tag{8.13}$$

Therefore X satisfies the desired IVP.

b) We note that

$$\boldsymbol{X}'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ f(t, x(t)) \end{pmatrix}, \ t \in I, \tag{8.14}$$

therefore

$$x''(t) = y'(t) = f(t, x(t)), \ t \in I. \tag{8.15}$$

Furthermore, $x(t_0) = x_0, \ x'(t_0) = y(t_0) = x_1$, therefore x satisfies the desired 2nd order scalar equation.

□

Problem 8.3 (Distinct real eigenvalues)**.** Consider the mass spring system modeled via the equation

$$x''(t) + 5x'(t) + 4x(t) = 0,\ t \in \mathbb{R}. \tag{8.16}$$

a) Find the general solution to the 2nd order scalar equation, and solve the IVP

$$\begin{cases} x''(t) + 5x'(t) + 4x(t) = 0,\ t \in \mathbb{R} \\ x(0) = 1, x'(0) = -1. \end{cases} \tag{8.17}$$

b) Convert (8.16) into a 2×2 linear system of the form

$$\frac{d}{dt}\boldsymbol{X}(t) = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{:=A} \boldsymbol{X}(t),\ a, b, c, d, t \in \mathbb{R} \tag{8.18}$$

where

$$\boldsymbol{X}(t) = \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix}. \tag{8.19}$$

c) Find the eigenvalues and a corresponding set of eigenvectors of the matrix A.
d) Write down the general solution to the linear system $\boldsymbol{X}' = A\boldsymbol{X}$ and solve the IVP

$$\begin{cases} \boldsymbol{X}'(t) = A\boldsymbol{X}(t), \quad t \in \mathbb{R} \\ \boldsymbol{X}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{cases} \tag{8.20}$$

What do you notice when you compare your answers in parts a) and d)?

Solution.

a) The characteristic equation associated to the 2nd order ODE is $r^2 + 5r + 4 = (r+1)(r+4) = 0$, therefore the general solution is

$$x(t) = c_1 e^{-t} + c_2 e^{-4t},\ t \in \mathbb{R}. \tag{8.21}$$

This implies

$$x'(t) = -c_1 e^{-t} - 4c_2 e^{-4t},\ t \in \mathbb{R}. \tag{8.22}$$

Using the initial conditions we find that we must have

$$\begin{cases} c_1 + c_2 = 1 \\ -c_1 - 4c_2 = -1 \end{cases} \tag{8.23}$$

This immediately implies $c_1 = 1, c_2 = 0$. Therefore the unique solution to the IVP is

$$x(t) = e^{-t},\ t \in \mathbb{R}. \tag{8.24}$$

b) We calculate

$$\boldsymbol{X}'(t) = \begin{pmatrix} x'(t) \\ x''(t) \end{pmatrix} = \begin{pmatrix} 0x(t) + 1x'(t) \\ -4x(t) - 5x'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix} \boldsymbol{X}(t),\ t \in \mathbb{R}. \tag{8.25}$$

c) The characteristic polynomial associated to A is

$$p_A(\lambda) \det \begin{pmatrix} -\lambda & 1 \\ -4 & -5-\lambda \end{pmatrix} = \lambda(\lambda+5) + 4 = \lambda^2 + 5\lambda + 4,\ \lambda \in \mathbb{C}. \tag{8.26}$$

Therefore the eigenvalues of A are $\lambda_1 = -1, \lambda_2 = -4$. To find a corresponding set of eigenvectors we look for non-zero vectors satisfying $A\boldsymbol{v} = \lambda\boldsymbol{v}$ or $(A - \lambda I)\boldsymbol{v} = 0$. If $\lambda = -1$, then

$$A - (-1)I = \begin{pmatrix} 1 & 1 \\ -4 & -4 \end{pmatrix}, \tag{8.27}$$

therefore if $(A + I)\boldsymbol{v} = \boldsymbol{0}$ for a vector

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{8.28}$$

then we must have $v_1 + v_2 = 0$. Therefore we may choose

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{8.29}$$

If $\lambda = -4$, then

$$A - (-4)I = \begin{pmatrix} 4 & 1 \\ -4 & -1 \end{pmatrix}, \tag{8.30}$$

therefore $(A + 4I)\boldsymbol{v} = \boldsymbol{0}$ we require $4v_1 + v_2 = 0$. Therefore we may chooses

$$\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}. \tag{8.31}$$

Thus the general solution to the system is

$$\boldsymbol{X}(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} 1 \\ -4 \end{pmatrix},\ t \in \mathbb{R}. \tag{8.32}$$

Using the initial conditions we find that we must have

$$\begin{pmatrix} c_1 + c_2 \\ -c_1 - 4c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \implies c_1 = 1, c_2 = 0. \tag{8.33}$$

Therefore the unique solution to the IVP is

$$\boldsymbol{X}(t) = e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix},\ t \in \mathbb{R}. \tag{8.34}$$

We see here that $X_1(t) = e^{-t}, t \in \mathbb{R}$ is the solution to the 2nd order scalar equation we found in part a).

$\square$

Problem 8.4 (Complex eigenvalues)**.** Consider the mass spring system modeled via the equation

$$x''(t) + 4x'(t) + 5x(t) = 0,\ t \in \mathbb{R}. \tag{8.35}$$

a) Is the system underdamped, overdamped, or critically damped?
b) Convert (8.35) into a 2×2 linear system of the form

$$\frac{d}{dt}\boldsymbol{X}(t) = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{:=A} \boldsymbol{X}(t),\ a, b, c, d, t \in \mathbb{R} \tag{8.36}$$

where

$$\boldsymbol{X}(t) = \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix}. \tag{8.37}$$

c) Find the characteristic equation associated to (8.35) and also the characteristic polynomial associated to the matrix A. What do you notice when you compare them?
d) Find the eigenvalues and a corresponding set of eigenvectors of the matrix A.
e) Write down the general solution to the linear system $\boldsymbol{X}' = A\boldsymbol{X}$.

Solution.

a) The characteristic equation associated to the 2nd order equation is $r^2 + 4r + 5 = 0 \implies r = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$. Therefore the system is underdamped.
b) We calculate

$$\boldsymbol{X}'(t) = \begin{pmatrix} x'(t) \\ x''(t) \end{pmatrix} = \begin{pmatrix} 0x(t) + 1x'(t) \\ -5x(t) - 4x'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -5 & -4 \end{pmatrix} \boldsymbol{X}(t),\ t \in \mathbb{R}. \tag{8.38}$$

c) The characteristic polynomial associated to A is

$$p_A(\lambda) = \det \begin{pmatrix} -\lambda & 1 \\ -5 & -4-\lambda \end{pmatrix} = \lambda(\lambda + 4) + 5 = \lambda^2 + 4\lambda + 5,\ \lambda \in \mathbb{C}. \tag{8.39}$$

Therefore we see that the characteristic equation associated to the 2nd order scalar equation and characteristic polynomial associated to the matrix A are the same.

d) The eigenvalues of A are $\lambda_1 = -2 - i, \lambda_2 = -2 + i$. Since complex eigenvectors come in pairs, we can just look for an eigenvector corresponding to one of the two eigenvalues, say $\lambda = -2 - i$. We note that if

$$(A - \lambda I)\boldsymbol{v} = \begin{pmatrix} 2+i & 1 \\ -5 & -2+i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{8.40}$$

then we must have

$$(2 + i)v_1 + v_2 = 0. \tag{8.41}$$

Thus we can choose

$$\boldsymbol{v}_1 = \begin{pmatrix} -1 \\ 2+i \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{8.42}$$

We may then choose an eigenvector associated to $\lambda_2 = -2 + i$ to be

$$\boldsymbol{v}_2 = \overline{\boldsymbol{v}_1} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \tag{8.43}$$

e) From the previous part we found that the eigenvalues of A are

$$\lambda = \alpha \pm i\beta = -2 \mp i \tag{8.44}$$

and a corresponding set of eigenvectors are

$$\boldsymbol{p} = \boldsymbol{B}_1 \pm i\boldsymbol{B}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \pm i \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{8.45}$$

If we use the solution formula

$$\boldsymbol{X}(t) = c_1 e^{\alpha t}(\boldsymbol{B}_1 \cos \beta t - \boldsymbol{B}_2 \sin \beta t) + c_2 e^{\alpha t}(\boldsymbol{B}_2 \cos \beta t + \boldsymbol{B}_1 \sin \beta t), \tag{8.46}$$

then we use

$$\alpha = -2,\ \beta = -1,\ \boldsymbol{B}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix},\ \boldsymbol{B}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{8.47}$$

Therefore the general solution is

$$\boldsymbol{X}(t) = c_1 e^{\alpha t}(\boldsymbol{B}_1 \cos\beta t - \boldsymbol{B}_2 \sin\beta t) + c_2 e^{\alpha t}(\boldsymbol{B}_2 \cos\beta t + \boldsymbol{B}_1 \sin\beta t) \tag{8.48}$$

$$= c_1 e^{-2t}\left(\cos t \begin{pmatrix}-1\\2\end{pmatrix} + \sin t \begin{pmatrix}0\\1\end{pmatrix}\right) + c_2 e^{-2t}\left(\cos t \begin{pmatrix}0\\1\end{pmatrix} - \sin t \begin{pmatrix}-1\\2\end{pmatrix}\right). \tag{8.49}$$

We can also use the real and complex parts of the general complex solution to find the general real solution:

$$\begin{aligned}\boldsymbol{X}(t) &= c_1 \Re(e^{\lambda_1 t}\boldsymbol{v}_1) + c_2 \Im(e^{\lambda_1 t}\boldsymbol{v}_1) = c_1 e^{-2t}\Re\left(e^{-it}\boldsymbol{v}_1\right) + c_2 e^{-2t}\Im(e^{-it}\boldsymbol{v}_1)\\
&= c_1 e^{-2t}\Re\left((\cos t - i\sin t)\left(\begin{pmatrix}-1\\2\end{pmatrix} + i\begin{pmatrix}0\\1\end{pmatrix}\right)\right) + c_2 e^{-2t}\Im\left((\cos t - i\sin t)\left(\begin{pmatrix}-1\\2\end{pmatrix} + i\begin{pmatrix}0\\1\end{pmatrix}\right)\right)\\
&= c_1 e^{-2t}\left(\cos t\begin{pmatrix}-1\\2\end{pmatrix} + \sin t\begin{pmatrix}0\\1\end{pmatrix}\right) + c_2 e^{-2t}\left(\cos\begin{pmatrix}0\\1\end{pmatrix} - \sin t\begin{pmatrix}-1\\2\end{pmatrix}\right),\ t \in \mathbb{R}\end{aligned} \tag{8.50}$$

$\square$

Problem 8.5 (Repeated eigenvalues)**.** Consider the 2×2 linear system

$$\begin{cases} x'(t) = y(t) \\ y'(t) = -x(t) + 2y(t), \quad t \in \mathbb{R}. \end{cases} \tag{8.51}$$

In matrix-vector form, this is the linear system

$$\boldsymbol{X}'(t) = A\boldsymbol{X}(t) = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \ t \in \mathbb{R}. \tag{8.52}$$

a) Write down a 2nd order scalar equation for x and find the general solution for x.
b) Find the general solution to the first ordered system (8.52).
c) What do you notice when you compare your answers in parts a) and b)?

Solution.

a) We calculate

$$x''(t) = y'(t) = -x(t) + 2y(t) = -x(t) + 2x'(t), \ t \in \mathbb{R}. \tag{8.53}$$

Therefore x satisfies the 2nd order scalar equation

$$x''(t) - 2x'(t) + x(t) = 0, \ t \in \mathbb{R}. \tag{8.54}$$

The characteristic equation for this equation is $r^2 - 2r + 1 = (r-1)^2 = 0$. Therefore the general solution is

$$x(t) = c_1 e^t + c_2 te^t, \ t \in \mathbb{R}. \tag{8.55}$$

b) The trace of the matrix A is 2 and the determinant of it is 1. Since the trace is the sum of eigenvalues and the determinant is the product, by inspection we see that the matrix A admits $\lambda = 1$ as a repeating eigenvalue. We then look for a nonzero vector $\boldsymbol{v}$ satisfying

$$(A - \lambda I)\boldsymbol{v} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{8.56}$$

This requires $-v_1 + v_2 = 0$, so we can choose

$$\boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{8.57}$$

Note that here we cannot find another linearly independent eigenvector corresponding to the eigenvalue $\lambda = 1$, so we look for a generalized eigenvalue $\boldsymbol{w}$ satisfying

$$(A - \lambda I)\boldsymbol{w} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{8.58}$$

which requires $-w_1 + w_2 = 1$. Here we may choose

$$\boldsymbol{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{8.59}$$

Therefore the general solution to the first order system is

$$\boldsymbol{X}(t) = c_1 e^{\lambda t}\boldsymbol{v} + c_2 e^{\lambda t}(t\boldsymbol{v} + \boldsymbol{w}) = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^t \left(t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} c_1 e^t + c_2 te^t \\ c_1 e^t + c_2(te^t + e^t) \end{pmatrix}, \ t \in \mathbb{R}. \tag{8.60}$$

Here we see that $x = X_1$ and $x' = X_2$ match the solution in part a), as expected.

□