Week 11

Problem 11.1 (Linear systems with initial conditions)**.** Find the unique solution to the IVP

$$\begin{cases} \boldsymbol{X}'(t) = \begin{pmatrix} -4 & 2 \\ 2 & -4 \end{pmatrix} \boldsymbol{X}(t), \quad t \in \mathbb{R} \\ \boldsymbol{X}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{cases} \tag{11.1}$$

Solution. Denoting

$$A = \begin{pmatrix} -4 & 2 \\ 2 & -4 \end{pmatrix}, \tag{11.2}$$

we note that

$$A + 2I = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \tag{11.3}$$

which has two rows that are multiples of each other, so $\lambda_1 = -2$ is an eigenvalue. Since $\operatorname{tr} A = -8$, the other eigenvalue is $\lambda_2 = -6$. One can also compute the characteristic polynomial directly:

$$p_A(\lambda) = \det(A - \lambda I) = (-4-\lambda)^2 - 4 = (\lambda+2)(\lambda+6), \ \lambda \in \mathbb{C}. \tag{11.4}$$

Here we have a pair of distinct real roots. An eigenvector $\boldsymbol{v}_1$ corresponding to $\lambda_1 = -2$ satisfies

$$(A + 2I)\boldsymbol{v}_1 = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{11.5}$$

This requires $v_1 - v_2 = 0$, therefore we can choose

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{11.6}$$

Likewise, an eigenvector corresponding to $\lambda_2 = -6$ satisfies

$$(A + 6I)\boldsymbol{v}_2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{11.7}$$

Therefore we can choose

$$\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{11.8}$$

The general solution to the equation is then

$$\boldsymbol{X}(t) = c_1 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-6t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \ t \in \mathbb{R}. \tag{11.9}$$

To solve for c_1, c_2 we use the initial conditions. We find that we must have

$$\begin{pmatrix} c_1 + c_2 \\ c_1 - c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{11.10}$$

Thus $c_1 = c_2 = \frac{1}{2}$, and the unique solution to the IVP is

$$\boldsymbol{X}(t) = \frac{1}{2} e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} e^{-6t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \ t \in \mathbb{R}. \tag{11.11}$$

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Problem 11.2 (Repeating eigenvalues)**.** Find the general solution to the system

$$\boldsymbol{X}'(t) = \begin{pmatrix} 3 & -1 \\ 1 & 5 \end{pmatrix} \boldsymbol{X}(t),\ t \in \mathbb{R}. \tag{11.12}$$

Solution. We first find the eigenvalues of the matrix. We compute

$$\det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & -1 \\ 1 & 5-\lambda \end{pmatrix} = (\lambda - 5)(\lambda - 3) + 1 = \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2. \tag{11.13}$$

Therefore we have $\lambda = 4$ as a repeating eigenvalue. Next we identify the eigenvectors by considering the equation

$$(A - 4I)\boldsymbol{v} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0}. \tag{11.14}$$

This requires $v_1 + v_2 = 0$, therefore we can choose the first eigenvector to be

$$\boldsymbol{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{11.15}$$

Note that we cannot find another eigenvector that is linearly independent of the one we just found, therefore we look for a generalized eigenvector $\boldsymbol{w}$ satisfying

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{11.16}$$

This requires $w_1 + w_2 = -1$, so we can choose

$$\boldsymbol{w} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \tag{11.17}$$

Therefore the general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^4 t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^4 t \left(t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right). \tag{11.18}$$

$\square$

Problem 11.3 (Claissification of critical points)**.** Consider the linear system

$$\boldsymbol{X}'(t) = A\boldsymbol{X}(t) = \begin{pmatrix} 5 & 5 \\ -8 & -7 \end{pmatrix} \boldsymbol{X}(t),\ t \in \mathbb{R}. \tag{11.19}$$

Classify the origin in terms of its stability and type.

Solution. To classify the origin we need to compute the eigenvalues of the coefficient matrix A. We compute

$$\det(A - \lambda I) = \det \begin{pmatrix} 5-\lambda & 5 \\ -8 & -7-\lambda \end{pmatrix} = (\lambda - 5)(\lambda + 7) + 40 = \lambda^2 + 2\lambda + 5 \tag{11.20}$$

This implies that

$$\lambda = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i. \tag{11.21}$$

Here we see that we have a pair of complex eigenvalues with negative real part, so the origin is a spiral point and it is also a sink (since the real part is negative, the spirals will spiral inward towards the origin). $\square$

Problem 11.4 (Claissification of critical points)**.** Consider the linear system

$$\boldsymbol{X}'(t) = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \boldsymbol{X}(t),\ t \in \mathbb{R}. \tag{11.22}$$

Classify the origin in terms of its stability and type.

Solution. We compute the eigenvalues by computing

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{pmatrix} = (\lambda - 1)^2 + 4 = \lambda^2 - 2\lambda + 5. \tag{11.23}$$

Then

$$\lambda = \frac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i. \tag{11.24}$$

Here the origin is a spiral point and also a source, since the real part of the eigenvalues is positive so the spiral trajectories spiral outward and away from the origin as $t \to \infty$. $\square$

Problem 11.5 (Claissification of critical points)**.** Consider the linear system

$$\boldsymbol{X}'(t) = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix} \boldsymbol{X}(t),\ t \in \mathbb{R} \tag{11.25}$$

where $\alpha \in \mathbb{R}$ is an unspecified parameter. Identify the value(s) of α for which the origin is a

a) stable node
b) unstable node
c) saddle point

Solution. By denoting

$$A = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}, \tag{11.26}$$

we note that its characteristic polynomial is

$$p_A(\lambda) = \det(A - \lambda I) = (\lambda - 1)^2 - \alpha^2 = (\lambda - 1 - \alpha)(\lambda - 1 + \alpha),\ \lambda \in \mathbb{C}. \tag{11.27}$$

Thus its eigenvalues are

$$\lambda_1 = 1 + \alpha, \lambda_2 = 1 - \alpha. \tag{11.28}$$

If we want the origin to be a stable node, we need $\lambda_1, \lambda_2 < 0$. Thus we require

$$1 + \alpha < 0 \implies \alpha < -1 \text{ and } 1 - \alpha < 0 \implies \alpha > 1. \tag{11.29}$$

Since both conditions cannot be satisfied simultaneously, the origin is never a stable node.

For the origin to be an unstable node, we need both eigenvalues to be positive. Thus we require

$$1 + \alpha > 0 \implies \alpha > -1 \text{ and } 1 - \alpha > 0 \implies \alpha < 1. \tag{11.30}$$

So here we see that for all $\alpha \in (-1, 1)$, the origin is an unstable node.

For the origin to be a saddle point, one of the eigenvalues must be positive and the other one must be negative. Thus we require

$$\lambda_1 > 0, \lambda_2 < 0 \text{ or } \lambda_1 < 0, \lambda_2 > 0. \tag{11.31}$$

In the first case we require

$$1 + \alpha > 0 \implies \alpha > -1 \text{ and } 1 - \alpha < 0 \implies \alpha > 1. \tag{11.32}$$

In the second case we require

$$1 + \alpha < 0 \implies \alpha < -1 \text{ and } 1 - \alpha > 0 \implies \alpha < 1. \tag{11.33}$$

Therefore for all $\alpha \in (1, \infty) \cup (-\infty, -1)$, the origin is a saddle point.

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