Homework 7

DUE: SATURDAY, MARCH 15, 11:59PM

If you completed this assignment through collaboration or consulted references, please list the names of your collaborators and the references you used below. Please refer to the syllabus for the course policy on collaborations and types of references that are allowed.

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Problem 7.1 (Laplace transform of derivatives)**.** Suppose $y : [0, \infty)$ is a smooth function of exponential order satisfying the pointwise estimate

$$|y(t)| \leq Me^{at}, \tag{7.1}$$

for all $t > T$, for some constants $M, a, T > 0$. Consider its Laplace transform $Y : (a, \infty) \to \mathbb{R}$ defined via

$$Y(s) = \mathcal{L}\{y\}(s) = \int_0^\infty e^{-st} y(t)\, dt. \tag{7.2}$$

a) Use integration by parts to show that

$$\mathcal{L}\{y'\}(s) = sY(s) - y(0), \tag{7.3}$$

for all $s > a$.

b) With the additional assumption that y' is also of exponential order satisfying the pointwise bound

$$|y'(t)| \leq Me^{at}, \tag{7.4}$$

for $t > T$, use integration by parts to show that

$$\mathcal{L}\{y''\}(s) = s^2 Y(s) - sy(0) - y'(0), \tag{7.5}$$

for all $s > a$.

Solution. We note that by using integration by parts and the fact that y, y' are of exponential order, we have

$$\mathcal{L}\{y'\}(s) = \lim_{\alpha \to \infty} e^{-st} y(t)\Big|_{t=0}^{t=\alpha} + s \int_0^\alpha y(t) e^{-st}\, dt = sY(s) - y(0) \tag{7.6}$$

and

$$\mathcal{L}\{y''\}(s) = \lim_{\alpha \to \infty} \left(e^{-st} y'(t)\Big|_{t=0}^{t=\alpha} + s \int_0^\alpha y'(t) e^{-st}\, dt \right) = s(sY(s) - y(0)) - y'(0) = s^2 Y(s) - sy(0) - y'(0), \tag{7.7}$$

for all $s > a$. □

Problem 7.2 (Translation in frequency space)**.**

a) Suppose $f : [0, \infty) \to \mathbb{R}$ is a continuous function of exponential order and its Laplace transform $F : (c, \infty)$ defined via is well-defined for some $c \in \mathbb{R}$. Show that for any $a \in \mathbb{R}$, we have the identity

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a), \text{ for all } s > c + a. \tag{7.8}$$

b) Use part a) to find the inverse Laplace transform of

$$Y(s) = \frac{3s+2}{s^2 - 2s + 10}, \ s > 1. \tag{7.9}$$

Solution. For part a), we note that by definition

$$\mathcal{L}\left\{e^{at} f(t)\right\}(s) = \int_0^\infty e^{-st} e^{at} f(t)\, dt = \int_0^\infty e^{-(s-a)t} f(t)\, dt = \mathcal{L}\left\{f\right\}(s-a) = F(s-a), \tag{7.10}$$

for all $s - a > c \Longleftrightarrow s > c + a$. For part b, we note that we may write

$$Y(s) = \frac{3(s-1)+5}{(s-1)^2 + 3^2}, \ s > 1, \tag{7.11}$$

therefore

$$\begin{aligned} \mathcal{L}^{-1}\left\{Y\right\}(t) &= 3\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+3^2}\right\}(t) + \frac{5}{3}\mathcal{L}^{-1}\left\{\frac{3}{(s-1)^2+3^2}\right\}(t) \\ &= 3e^t \mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\}(t) + \frac{5}{3}e^t \mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}(t) = 3e^t \cos 3t + \frac{5}{3} e^t \sin 3t, \ t \geq 0. \end{aligned} \tag{7.12}$$

$\square$

Problem 7.3.

a) Use the Laplace transform to solve the initial value problem

$$\begin{cases} x''(t) + 4x(t) = 1, & t \in [0, \infty) \\ x(0) = 1, x'(0) = 0. \end{cases} \tag{7.13}$$

b) Use the Laplace transform to solve the initial value problem

$$\begin{cases} y''(t) + 2y'(t) + 5y(t) = 0, & t \in [0, \infty) \\ y(0) = 2, y'(0) = -1 \end{cases} \tag{7.14}$$

Solution.

a) If x is a solution satisfying appropriate growth conditions to the IVP on $t \in [0, \infty)$, then $X = \mathcal{L}\{x\}$ solves the algebraic equation

$$\mathcal{L}\{x'' + 4x\}(s) = s^2 X(s) - s + 4X(s) = \mathcal{L}\{1\}(s) = \frac{1}{s} \tag{7.15}$$

for all $s > a$ for some constant a. Therefore

$$X(s) = \frac{1}{s(s^2+4)} + \frac{s}{s^2+4} = \frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4} + \frac{s}{s^2+4}, \ s > a. \tag{7.16}$$

Thus

$$x(t) = \mathcal{L}^{-1}\{X\} = \frac{1}{4} - \frac{1}{4}\cos 2t + \cos 2t = \frac{1}{4} + \frac{3}{4}\cos 2t, \ t \geq 0. \tag{7.17}$$

b) If y is a solution to the IVP satisfying appropriate growth conditions, then $Y = \mathcal{L}\{y\}$ satisfies the algebraic equation

$$s^2 Y(s) - 2s + 1 + 2(sY(s) - 2s) + 5Y(s) = 0, \text{ for all } s > a \tag{7.18}$$

for some constant $a \in \mathbb{R}$. Therefore

$$Y(s) = \frac{2s+3}{s^2+2s+5} = \frac{2(s+1)+1}{(s+1)^2+4}, \ s > a. \tag{7.19}$$

Now we calculate

$$\begin{aligned} y(t) = \mathcal{L}^{-1}\{Y\}(t) = e^{-t}\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2+4}\right\} \\ = e^{-t}\left(2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}\right) = e^{-t}\left(2\cos(2t) + \frac{1}{2}\sin(2t)\right), \ t \geq 0. \end{aligned} \tag{7.20}$$

□

Problem 7.4 (Laplce transform of piecewise smooth functions)**.** Define the function $g : [0, \infty) \to \mathbb{R}$ via

$$g(t) = \begin{cases} t, & 0 \le t < 1 \\ 2 - t, & 1 \le t < 2 \\ 0, & 2 \le t. \end{cases} \tag{7.21}$$

Find $\mathcal{L}\{g(\cdot)\}$

Solution. We note that we can write g as

$$g(t) = t + (2 - t - t)\mathcal{U}(t-1) + (0 - (2-t))\mathcal{U}(t-2) = t - 2(t-1)\mathcal{U}(t-1) + (t-2)\mathcal{U}(t-2),\ t \ge 0. \tag{7.22}$$

Therefore using the fact that $\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\}(s) = e^{-as}\mathcal{L}\{f(t)\}$, we have

$$\mathcal{L}\{g\}(s) = \frac{1}{s^2} - 2e^{-s}\mathcal{L}\{t\} + e^{-2s}\mathcal{L}\{t\} = \frac{1}{s^2}\left(1 - 2e^{-s} + e^{-2s}\right),\ s > 0. \tag{7.23}$$

$\square$

Problem 7.5. Find a solution to the initial value problem

$$\begin{cases} x''(t) + 4x'(t) + 4x(t) = f(t),\ t \in [0, \infty) \\ x(0) = x'(0) = 0 \end{cases} \tag{7.24}$$

where $f : [0, \infty) \to \mathbb{R}$ is defined via

$$f(t) = \begin{cases} t,\ 0 \le t < 2 \\ 0,\ t \ge 2. \end{cases} \tag{7.25}$$

Solution. We first write

$$f(t) = t + (0 - t)\mathcal{U}(t-2) = t - \mathcal{U}(t-2)t,\ t \ge 0. \tag{7.26}$$

Then assuming x is a solution, its Laplace transform $X = \mathcal{L}\{x\}$ satisfies

$$(s^2 + 4s + 4)X(s) = \frac{1}{s^2} - e^{-2s}\mathcal{L}\{t+2\} = \frac{1}{s^2} - e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right), \tag{7.27}$$

for appropriate values of s. One can also calculate $\mathcal{L}\{U(t-2)t\}$ by writing

$$\mathcal{L}\{U(t-2)t\}(s) = \mathcal{L}\{U(t-2)(t-2) + 2\mathcal{U}(t-2)\} = e^{-2s}\frac{1}{s^2} + 2\frac{e^{-2s}}{s},\ s > 0. \tag{7.28}$$

Then

$$X(s) = \frac{1}{s^2(s+2)^2} - e^{-2s}\left(\frac{1}{s^2(s+2)^2} + \frac{2}{s(s+2)^2}\right), \tag{7.29}$$

for appropriate values of s. We then look for partial fraction decompositions of the form

$$\frac{1}{s^2(s+2)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} + \frac{D}{s+2} \tag{7.30}$$

and

$$\frac{2}{s(s+2)^2} = \frac{E}{s} + \frac{F}{s+2} + \frac{G}{(s+2)^2}. \tag{7.31}$$

Through routine algebra we then find that

$$\begin{aligned} X(s) = {} & -\frac{1}{4}\frac{1}{s} + \frac{1}{4}\frac{1}{s^2} + \frac{1}{4}\frac{1}{s+2} + \frac{1}{4}\frac{1}{(s+2)^2} \\ & - e^{-2s}\left(-\frac{1}{4}\frac{1}{s} + \frac{1}{4}\frac{1}{s^2} + \frac{1}{4}\frac{1}{s+2} + \frac{1}{4}\frac{1}{(s+2)^2} + \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{s+2} - \frac{1}{(s+2)^2}\right), \end{aligned} \tag{7.32}$$

for appropriate values of s. Thus

$$\begin{aligned} x(t) = {} & -\frac{1}{4} + \frac{1}{4}t + \frac{1}{4}e^{-2t} + \frac{1}{4}te^{-2t} \\ & - \mathcal{U}(t-2)\left(-\frac{1}{4} + \frac{1}{4}(t-2) + \frac{1}{4}e^{-2(t-2)} + \frac{1}{4}(t-2)e^{-2(t-2)} + \frac{1}{2} - \frac{1}{2}e^{-2(t-2)} - (t-2)e^{-2(t-2)}\right) \end{aligned} \tag{7.33}$$

□

Problem 7.6. Find a solution to the initial value problem

$$\begin{cases} x''(t) + 4x'(t) + 5x(t) = \delta(t-\pi) + \delta(t-2\pi), \ t \in [0, \infty) \\ x(0) = 0, x'(0) = 2. \end{cases} \tag{7.34}$$

Solution. If x is a solution, then $X = \mathcal{L}\{x\}$ satisfies

$$(s^2 + 4s + 5)X(s) - 2 = e^{-\pi s} + e^{-2\pi s} \tag{7.35}$$

for appropriate values of s. Thus

$$X(s) = \frac{2}{(s+2)^2 + 1} + e^{-\pi s}\frac{1}{(s+2)^2 + 1} + e^{-2\pi s}\frac{1}{(s+2)^2 + 1}, \tag{7.36}$$

for appropriate values of s. Thus

$$x(t) = 2e^{-2t}\sin t + \mathcal{U}(t-\pi)e^{-2(t-\pi)}\sin(t-\pi) + \mathcal{U}(t-2\pi)e^{-2(t-2\pi)}\sin(t-2\pi), \ t \geq 0. \tag{7.37}$$

□