Week 9

Problem 9.1 (Distinct eigenvalues)**.** Consider the IVP

$$\begin{cases} \boldsymbol{X}'(t) = A\boldsymbol{X}(t) = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \boldsymbol{X}(t), \ t \in \mathbb{R} \\ \boldsymbol{X}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}. \end{cases} \tag{9.1}$$

a) Find the unique solution to the IVP.
b) Define the fundamental matrix $\Phi : \mathbb{R} \to \mathcal{M}_{2\times 2}(\mathbb{R})$ associated to the system via

$$\Phi(t) = \left(\boldsymbol{X}_1(t) \mid \boldsymbol{X}_2(t)\right), \tag{9.2}$$

where $\{\boldsymbol{X}_1, \boldsymbol{X}_2\}$ is a fundamental set of solutions to the equation. Find a fundamental matrix Φ associated to the system.
c) Verify that the unique solution to the IVP can also be written as

$$\boldsymbol{X}(t) = \Phi(t)\Phi(0)^{-1}\boldsymbol{X}_0, \ t \in \mathbb{R}. \tag{9.3}$$

Solution. We note that

$$A + I = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \tag{9.4}$$

which has two rows that are multiples of each other, therefore -1 is an eigenvalue. Since the trace of A is $\operatorname{tr} A = 0 + 1 = 1$, the other eigenvalue is 2. Therefore the eigenvalues of the matrix are $\lambda_1 = 2, \lambda_2 = -1$. To find an eigenvector corresponding to λ_1, we look at the system

$$(A - \lambda_1 I)\boldsymbol{v}_1 = \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{9.5}$$

Here we require $2v_1 = v_2$, so we can choose $v_1 = 1, v_2 = 2$. This gives us the eigenvector

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{9.6}$$

Likewise, to find an eigenvector corresponding to λ_2 we study the system

$$(A - \lambda_2 I)\boldsymbol{v}_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{9.7}$$

This requires $v_1 + v_2 = 0$, therefore we can choose

$$\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{9.8}$$

Therefore the general solution to the equation is

$$\boldsymbol{X}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \ t \in \mathbb{R}, \tag{9.9}$$

where c_1, c_2 are arbitrary. To find the solution to the IVP we solve for c_1, c_2, and to do so we use the initial conditions. This requires

$$\boldsymbol{X}(0) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ 2c_1 - c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}. \tag{9.10}$$

Department of Mathematical Sciences, Carnegie Mellon University
E-mail address: jkoganem@andrew.cmu.edu.
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This implies $c_1 = \frac{5}{3}, c_2 = \frac{4}{3}$. Therefore the unique solution to the IVP is

$$\boldsymbol{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{5}{3} \underbrace{e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{=\boldsymbol{X}_1(t)} + \frac{4}{3} \underbrace{e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{=\boldsymbol{X}_2(t)}, \ t \in \mathbb{R}. \tag{9.11}$$

One fundamental matrix associated to the system is

$$\Phi(t) = \begin{pmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{pmatrix}, \ t \in \mathbb{R}. \tag{9.12}$$

We then calculate

$$\Phi(0)^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}^{-1} = \frac{1}{-3} \begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix} \tag{9.13}$$

and

$$\begin{aligned} \Phi(t)\Phi(0)^{-1}\boldsymbol{X}_0 &= \Phi(t) \left(-\frac{1}{3} \begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right) = \begin{pmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{pmatrix} \left(-\frac{1}{3} \begin{pmatrix} -5 \\ -4 \end{pmatrix} \right) \\ &= \begin{pmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{pmatrix} \begin{pmatrix} 5/3 \\ 4/3 \end{pmatrix} = \frac{5}{3} \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} + \frac{4}{3} \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} = \frac{5}{3} e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{4}{3} e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \ t \in \mathbb{R} \end{aligned} \tag{9.14}$$

which is the unique solution to the IVP. □

Problem 9.2 (Repeating eigenvalues)**.** Find the general solution to the constant coefficient linear system

$$\begin{cases} x'(t) &= 7x(t) - y(t) \\ y'(t) &= x(t) + 5y(t),\ t \in \mathbb{R}. \end{cases} \tag{9.15}$$

Solution. In matrix form this is the system

$$\boldsymbol{X}'(t) = A\boldsymbol{X}(t) = \begin{pmatrix} 7 & -1 \\ 1 & 5 \end{pmatrix} \boldsymbol{X}(t),\ t \in \mathbb{R}. \tag{9.16}$$

We calculate

$$p_A(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} 7-\lambda & -1 \\ 1 & 5-\lambda \end{pmatrix} = (\lambda - 7)(\lambda - 5) + 1 = \lambda^2 - 12\lambda + 36 = (\lambda - 6)^2 \tag{9.17}$$

This shows that $\lambda = 6$ is a repeated eigenvalue. The eigenvector $\boldsymbol{v}$ corresponding to $\lambda = 6$ satisfies $(A - \lambda I)\boldsymbol{v} = 0$, or

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{9.18}$$

So we can choose $\boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. From this calculation we see that we cannot find two linearly independent eigenvectors corresponding to λ, so we need to look for the generalized eigenvector $\boldsymbol{w}$ satisfying $(A - \lambda I)\boldsymbol{w} = \boldsymbol{v}$, or

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{9.19}$$

We can then choose $\boldsymbol{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then the general solution is given by

$$\boldsymbol{X}(t) = c_1 e^{6t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \left(t e^{6t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{6t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right),\ t \in \mathbb{R}. \tag{9.20}$$

□

Problem 9.3 (Complex eigenvalues)**.** Find the unique solution to the IVP

$$\begin{cases} \boldsymbol{X}'(t) = A\boldsymbol{X}(t) = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix} \boldsymbol{X}(t), \quad t \in \mathbb{R} \\ \boldsymbol{X}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}. \end{cases} \tag{9.21}$$

Solution. We first compute

$$p_A(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & 8 \\ -1 & -2-\lambda \end{pmatrix} = (\lambda - 2)(\lambda + 2) + 8 = \lambda^2 + 4,\ \lambda \in \mathbb{C}. \tag{9.22}$$

Therefore the eigenvalues of A are $\lambda = \pm 2i$. If $\lambda = 2i$, then we examine the system

$$(A - \lambda I)\boldsymbol{v} = \begin{pmatrix} 2-2i & 8 \\ -1 & -2-2i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{9.23}$$

This requires

$$(2-2i)v_1 + 8v_2 = 0, \tag{9.24}$$

so we may choose $v_1 = -8$ and $v_2 = 2 - 2i$. Therefore we found that one of the eigenvalues and a corresponding eigenvector are

$$\lambda = 0 + 2i \tag{9.25}$$

$$\boldsymbol{p} = \begin{pmatrix} -8 \\ 2 \end{pmatrix} + i \begin{pmatrix} 0 \\ -2 \end{pmatrix}. \tag{9.26}$$

The general solution to the system is then

$$\boldsymbol{X}(t) = c_1 \left(\cos 2t \begin{pmatrix} -8 \\ 2 \end{pmatrix} - \sin 2t \begin{pmatrix} 0 \\ -2 \end{pmatrix} \right) + c_2 \left(\cos 2t \begin{pmatrix} 0 \\ -2 \end{pmatrix} + \sin 2t \begin{pmatrix} -8 \\ 2 \end{pmatrix} \right),\ t \in \mathbb{R} \tag{9.27}$$

and c_1, c_2 are arbitrary. To find c_1, c_2 we use the initial conditions. We require

$$\boldsymbol{X}(0) = \begin{pmatrix} -8c_1 \\ 2c_1 - 2c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \implies c_1 = -\frac{1}{4},\ c_2 = \frac{1}{4}. \tag{9.28}$$

Thus the unique solution to the IVP is

$$\begin{aligned} \boldsymbol{X}(t) = \cos 2t \begin{pmatrix} 2 \\ -1/2 \end{pmatrix} - \sin 2t \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} + \cos 2t \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} + \sin 2t \begin{pmatrix} -2 \\ 1/2 \end{pmatrix} \\ = \cos 2t \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \sin 2t \begin{pmatrix} -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2\cos 2t - 2\sin 2t \\ -\cos 2t \end{pmatrix},\ t \in \mathbb{R}. \end{aligned} \tag{9.29}$$

□