Recitation 1

Problem 1.1 (Calculus review: substitution)**.** Find

$$\int \frac{t}{t^2-4}\,dt. \tag{1.1}$$

Solution. There are two ways to do this problem. The standard way is to use the substitution $u = t^2 - 4, du = 2tdt$ to write

$$\int \frac{t}{t^2-4}\,dt = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left|t^2-4\right| + C. \tag{1.2}$$

An equivalent but likely faster way is to use the identity

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C. \tag{1.3}$$

Then we note that since $(t^2-4)' = 2t$, we can manipulate the integral to make the derivative appear so that

$$\int \frac{t}{t^2-4}\,dt = \frac{1}{2}\underbrace{\int \frac{2t}{t^2-4}\,dt}_{\text{apply (1.3)}} = \frac{1}{2}\ln\left|t^2-4\right| + C. \tag{1.4}$$

□

Problem 1.2 (Calculus review: partial fraction decomposition)**.** Find

$$\int \frac{x}{(x+1)^2}\,dx \tag{1.5}$$

Solution. We note that since the linear factor $x + 1$ squared is of degree 2, without any specialized knowledge one would assume the general partial fraction decomposition is

$$\frac{x}{(x+1)^2} = \frac{A}{x+1} + \frac{Bx+C}{(x+1)^2}, \tag{1.6}$$

because the general "rule" is that the degree of the numerator should be one less than the degree of the denominator. However, there is a specialized theorem in this topic that guarantees that we can decompose

$$\frac{x}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \tag{1.7}$$

for some A, B to be determined. In other words, for all the terms involving powers of $x + 1$ in the denominator, we can assume that the numerator are all one degree less that just the linear factor $x + 1$. So for example, the theory guarantees that one can write

$$\frac{x}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} \tag{1.8}$$

even though $(x + 1)^2$ is of degree 2 and $(x + 1)^3$ is of degree 3.

To find A, B in (1.7) we manipulate the equation by multiplying both sides by $(x + 1)^2$ to arrive at

$$x = A(x+1) + B. \tag{1.9}$$

If $x = -1$, then we find that $B = -1$. If $x = 0$ then we find that $0 = A + B \implies A = 1$. Therefore

$$\frac{x}{(x+1)^2} = \frac{1}{x+1} - \frac{1}{(x+1)^2}. \tag{1.10}$$

Then

$$\int \frac{x}{(x+1)^2}\,dx = \int \frac{1}{x+1} - \frac{1}{(x+1)^2}\,dx = \ln|x+1| + (x+1)^{-1} + C. \tag{1.11}$$

□

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Problem 1.3. Find all solutions to $x + \sqrt{x} = 0$.

Solution. Upon first try most students would likely write something like this:

$$x + \sqrt{x} = 0 \implies x = -\sqrt{x} \implies x^2 = (-\sqrt{x})^2 = x \implies x^2 - x = 0 \implies x(x-1) = 0 \\ \implies x = 0 \text{ or } x = 1, \quad (1.12)$$

and then conclude that the solutions are $x = 0$ and $x = 1$. However, this isn't quite right, because what we've really shown above is that "if x solves $x + \sqrt{x} = 0$, then the candidate solutions are $x = 0$ or $x = 1$." In order to verify that the candidate solutions are actual solutions we actually need to evaluate the converse. We note that if $x = 0$, then $0 + \sqrt{0} = 0$, so $x = 0$ is a solution, but if $x = 1$, then $1 + \sqrt{1} = 2 \neq 0$, so $x = 1$ is not a solution. So the correct conclusion is that the only solution to the equation is $x = 0$. In this case we can write the conclusion as a bi-implication,

$$x + \sqrt{x} = 0 \text{ if and only if } x = 0. \quad (1.13)$$

Now if in (1.12) every forward implication is a bi-implication, then the verification step can be skipped, but as we can see in the example above when you naively manipulate equations sometimes the implication only goes one way (e.g. $x = 1$ implies $x^2 = 1$ but $x^2 = 1$ does not imply $x = 1$ since it's possible for $x = -1$).

The upshot is that when one naively manipulates equations and do not pay close attention to the direction of the implications, the final "answer" is really a candidate set of solutions to the original equation, and a verification step is necessary for us to find the actual solution set.

This idea applies when we solve for differential equations too: students are often taught to naively manipulate equations and they are done once they reach the "answer," but in reality the final "answer" is only a candidate solution set because one typically do not pay attention to the direction of the implications, and a final verification step is necessary for the argument to be mathematically and logically precise.

□

Problem 1.4 (Verification of solutions)**.** Verify that the function y defined via $y(x) = e^x$ for all $x \in \mathbb{R}$ is a solution to the differential equation

$$y'(x) - y(x) = 0 \quad (1.14)$$

for all $x \in \mathbb{R}$.

Proof. To verify that a function is a solution to a differential equation on an interval I, we simply check that when we substitute the function into the equation evaluated at all the values belonging to the interval I the equation remains valid. We note that for the given function y, $y'(x) = e^x$ for all $x \in \mathbb{R}$, therefore for all $x \in \mathbb{R}$ we have

$$y'(x) - y(x) = e^x - e^x = 0. \quad (1.15)$$

This shows that the given function y is a solution on the interval $I = \mathbb{R}$. □