Final practice problems

Problem 1.1 (Linear equations and the integrating factor). Find the general solution to the linear equation

$$x^2 y'(x) + 5xy(x) = \frac{e^{4x}}{x^3}, \ x > 0. \tag{1.1}$$

Problem 1.2 (Rate problems). A tank contains 130 liters of water and 50 grams of sugar. A solution containing a sugar concentration of $4e^{-t}$ g/L flows into the tank at a rate of 3 L/min, and the mixture in the tank flows out at a rate of 4 L/min. Let $Q(t)$ be the amount of sugar (in grams) in the tank at time t (in minutes). Write down an initial value problem for Q (without solving it explicitly).

Problem 1.3 (Autonomous equations, stability of critical points). Consider the autonomous equation

$$y'(x) = (1 - y(x))^3(y(x) + 2)(y(x) - 4), \ x \in \mathbb{R}. \tag{1.2}$$

a) Identify the critical points and the corresponding constant solutions to the equation.
b) Draw a one-dimensional phase portrait of the equation and determined the stability of the critical points.
c) Give a sketch of sample solution curves in the x-y plane.

Problem 1.4 (Existence and uniqueness of solutions). Consider the initial value problem

$$\begin{cases} y'(x) = \sqrt{y(x)}, \ x \geq 0 \\ y(0) = 0. \end{cases} \tag{1.3}$$

a) Does the IVP admit any constant solutions?
b) Does the IVP admit non-constant solutions? If so, why does this not violate the uniqueness part of the existence and uniqueness theorem?

Problem 1.5 (Separable equations, implicit solutions). Consider the equation

$$(y(x))^3 y'(x) = ((y(x))^4 + 1)\cos x, \ x \in \mathbb{R}. \tag{1.4}$$

a) Does the equation admit any constant solutions?
b) Find an implicit solution to the equation.

Problem 1.6 (Exact equations, implicit solutions). Consider the nonlinear equation

$$((y(x))^2 + 1) + (2xy(x) + 3(y(x))^2)y'(x) = 0, \ x \in \mathbb{R}. \tag{1.5}$$

a) Is the equation exact?
b) Find the general implicit solution to the equation.

Problem 1.7 (Bernoulli equations, constant solutions). Consider the initial value problem

$$\begin{cases} y'(x) - \frac{2}{x}y(x) = -x^2(y(x))^2, \ \ x > 0 \\ y(1) = \alpha. \end{cases} \tag{1.6}$$

Find a solution for the following values of α and state the maximal interval of existence of the solution.
a) $\alpha = 0$

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  b) $\alpha = 1$

Problem 1.8 (Equations with homogeneous functions). Find an implicit solution to the nonlinear equation

$$y'(x) = \frac{x^2 + 3(y(x))^2}{2xy(x)}, \quad x > 0. \tag{1.7}$$

Problem 1.9 (2nd order equations and mass-spring systems). Suppose a mass spring system is modeled via the equation

$$x''(t) + \beta x'(t) + 4x(t) = 0, \ t \in \mathbb{R}. \tag{1.8}$$

Identify the value(s) of β for which the system will be

  a) undamped
  b) underdamped
  c) critically damped
  d) overdamped

Problem 1.10 (Method of undetermined coefficients). Write down an appropriate ansatz (without solving for the coefficients) for the linear equation

$$y^{(4)}(t) - 2y^{(3)}(t) + 10y''(t) = t^2 + e^t \cos 2t, \ t \in \mathbb{R}. \tag{1.9}$$

Problem 1.11 (Variation of parameters). Find a particular solution to the equation

$$x^2 y''(x) + xy'(x) - y(x) = 600x^5, \ x > 0 \tag{1.10}$$

given that the general homogeneous solution to the equation is

$$y_h(x) = c_1 x + c_2 x^{-1}, \ x > 0 \tag{1.11}$$

where c_1, c_2 are arbitrary.

Problem 1.12 (Mass-spring systems and resonance). Suppose a mass-spring system is modeled via the IVP

$$\begin{cases} x''(t) + \beta x'(t) + 16x(t) = F_0 \sin \omega t, \quad t \in \mathbb{R} \\ x(0) = x'(0) = 0. \end{cases} \tag{1.12}$$

where $\beta \geq 0, \omega > 0$.

  a) Identify the parameters β, ω for which pure resonance occurs.
  b) In the case of part a) and $F_0 = 8$, find the unique solution to the IVP.
  c) Suppose $\beta = 8, \omega = 1$ and $F_0 = 16$. What is the (approximate, up to a negligible error) maximum amplitude of the mass-spring system in the long run?

Problem 1.13 (Eigenvalue problems). Consider the eigenvalue problem

$$\begin{cases} y''(x) + 2y'(x) + \lambda y(x) = 0, \ x \in (0, \pi) \\ y(0) = 0, \ y(\pi) = 0. \end{cases} \tag{1.13}$$

Parameterize the eigenvalues $\lambda > 1$ by $n = 1, 2, 3, \dots$ and list a corresponding set of eigenfunctions.

Problem 1.14 (The Laplace transform and equations with piecewise forcing). Suppose a mass-spring system is modeled via

$$\begin{cases} x''(t) + 4x(t) = f(t), & t \geq 0 \\ x(0) = x'(0) = 0, \end{cases} \tag{1.14}$$

and $f : [0, \infty) \to \mathbb{R}$ is defined via

$$f(t) = \begin{cases} 0, & 0 \leq t < 2\pi \\ \sin t, & t \geq 2\pi. \end{cases} \tag{1.15}$$

Use the Laplace transform to find a solution x describing the behavior of the system for $t \geq 0$

Problem 1.15 (The Laplace transform and impulse forces). Use the Laplace transform to find a solution to the equation

$$\begin{cases} x''(t) + x(t) = 1 + \delta(t - \pi), & t \geq 0 \\ x(0) = x'(0) = 0. \end{cases} \tag{1.16}$$

Problem 1.16 (The Laplace transform and convolutions). Write down a convolution integral solution to the IVP

$$\begin{cases} x''(t) + 8x'(t) + 16 = f(t), & t > 0 \\ x(0) = x'(0) = 0, \end{cases} \tag{1.17}$$

for any reasonable function $f : [0, \infty) \to \mathbb{R}$.

Problem 1.17 (Linear systems). Find the unique solution to the IVP

$$\begin{cases} \boldsymbol{X}'(t) = \begin{pmatrix} -4 & 2 \\ 2 & -4 \end{pmatrix} \boldsymbol{X}(t), & t \in \mathbb{R} \\ \boldsymbol{X}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{cases} \tag{1.18}$$

Problem 1.18 (Linear systems and stability of critical points). Consider the linear system

$$\boldsymbol{X}'(t) = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix} \boldsymbol{X}(t), \ t \in \mathbb{R} \tag{1.19}$$

where $\alpha \in \mathbb{R}$ is an unspecified parameter. Identify the value(s) of α for which the origin is a

    a) stable node
    b) unstable node
    c) saddle point

Problem 1.19 (Fourier series). Consider the function

$$f(x) = 1, \ x \in (0, 1). \tag{1.20}$$

Sketch a graph over $\mathbb{R}$ of the Fourier sine series of f, which is defined via

$$\sum_{n=1}^{\infty} b_n \sin(n\pi x) \text{ for all } x \in \mathbb{R}, \text{ where } b_n = 2 \int_0^1 \sin(n\pi x) \, dx. \tag{1.21}$$

Problem 1.20 (The heat equation). Find the unique solution $u : (0, \pi) \times [0, \infty) \to \mathbb{R}$ as a *finite* linear combination elementary functions satisfying

$$u_t = 4u_{xx}, \qquad\qquad\qquad x \in (0, \pi), t \geq 0, \qquad\qquad (1.22)$$
$$u(x = 0, t) = 0 = u(\pi, t) \qquad\qquad\qquad t \geq 0 \qquad\qquad (1.23)$$
$$u(x, t = 0) = \sin(2x) + \sin(5x), \qquad\qquad x \in (0, \pi). \qquad\qquad (1.24)$$

Problem 1.21 (The wave equation). Find the unique solution $u : (0, \pi) \times [0, \infty) \to \mathbb{R}$ as a *finite* linear combination elementary functions satisfying

$$u_{tt} = 9u_{xx}, \qquad\qquad\qquad x \in (0, \pi), t \geq 0 \qquad\qquad (1.25)$$
$$u(x = 0, t) = 0 = u(\pi, t) \qquad\qquad\qquad t \geq 0 \qquad\qquad (1.26)$$
$$u(x, t = 0) = \sin(3x) \qquad\qquad\qquad x \in (0, \pi) \qquad\qquad (1.27)$$
$$u_t(x, t = 0) = \sin(4x) \qquad\qquad\qquad x \in (0, \pi). \qquad\qquad (1.28)$$