Recitation 5

Problem 1.1. Use the method of undetermined coefficients to find the general solution to the 2nd order equation

$$y''(x) + y(x) = 2x\sin x,\ x \in \mathbb{R}. \tag{1.1}$$

Solution. The characteristic equation associated to the homogeneous equation is

$$r^2 + 1 = 0 \implies r = \pm i. \tag{1.2}$$

Therefore the general homogeneous solution is

$$y_h(x) = c_1 \cos x + c_2 \sin x,\ x \in \mathbb{R} \text{ and } c_1, c_2 \text{ are arbitrary.} \tag{1.3}$$

We use the ansatz $y_p(x) = x(Ax+B)\cos x + x(Cx+D)\sin x = (Ax^2+Bx)\cos x + (Cx^2+Dx)\sin x, x \in \mathbb{R}$ for the particular solution, and calculate

$$\begin{aligned}
y_p'(x) &= (2Ax + B + Cx^2 + Dx)\cos x + (2Cx + D - Ax^2 - Bx)\sin x & (1.4)\\
&= (Cx^2 + (2A+D)x + B)\cos x + (-Ax^2 + (2C-B)x + D)\sin x, & (1.5)\\
y_p''(x) &= (2Cx + (2A+D) - Ax^2 + (2C-B)x + D)\cos x & (1.6)\\
&\quad + (-2Ax + (2C-B) - Cx^2 - (2A+D)x - B)\sin(x) & (1.7)\\
&= (-Ax^2 + (4C-B)x + (2A+2D))\cos x - (Cx^2 + (4A+D)x + (2B-2C))\sin x,\ x \in \mathbb{R}. & (1.8)
\end{aligned}$$

Therefore

$$y_p''(x) + y_p(x) = (4Cx + (2A+2D))\cos x - ((4Ax + (2B-2C))\sin x = 2x\sin x,\ x \in \mathbb{R}. \tag{1.9}$$

This implies that

$$\begin{cases} 4C = 0 \\ 2A + 2D = 0 \\ 4A = -2 \\ 2B - 2C = 0. \end{cases} \tag{1.10}$$

From this we deduce that $B = C = 0$ and $A = -\frac{1}{2}, D = \frac{1}{2}$. Thus the general solution to the equation is

$$y(x) = y_h(x) + y_p(x) = c_1\cos x + c_2 \sin x - \frac{1}{2}x^2\cos x + \frac{1}{2}x\sin x,\ x \in \mathbb{R} \text{ and } c_1, c_2 \text{ are arbitrary.} \tag{1.11}$$

□

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Problem 1.2. Use the method of variation of parameters to find the general solution of the 2nd order equation

$$y''(x) - 4y'(x) + 4y(x) = 2e^{2x}, \; x \in \mathbb{R}. \tag{1.12}$$

Solution. The general homogeneous solution to the equation is

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 e^{2x} + c_2 x e^{2x}, \; x \in \mathbb{R} \text{ and } c_1, c_2 \text{ are arbitrary.} \tag{1.13}$$

The Wronskian

$$W(y_1, y_2)(x) = \det \begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix} = \det \begin{pmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x}(1+2x) \end{pmatrix} = e^{4x}, \; x \in \mathbb{R}. \tag{1.14}$$

Therefore via the variation of parameters formula, a particular solution is given by

$$y_p(x) = -y_1(x) \int \frac{y_2(x) f(x)}{W(y_1, y_2)(x)} \, dx + y_2(x) \int \frac{y_1(x) f(x)}{W(y_1, y_2)(x)} \, dx \tag{1.15}$$

$$= -e^{2x} \int \frac{xe^{2x}(2e^{2x})}{e^{4x}} \, dx + xe^{2x} \int \frac{e^{2x} 2e^{2x}}{e^{4x}} \, dx = -x^2 e^{2x} + 2x^2 e^{2x} = x^2 e^{2x}, \; x \in \mathbb{R}. \tag{1.16}$$

Therefore the general solution to the equation is

$$y(x) = y_h(x) + y_p(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 e^{2x} + c_2 x e^{2x} + x^2 e^{2x}, \; x \in \mathbb{R} \text{ and } c_1, c_2 \text{ are arbitrary.} \tag{1.17}$$

$\square$

Problem 1.3. Find the general solution of the equation

$$y''(x) + y(x) = \tan x,\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{1.18}$$

Solution. Note that for this problem we cannot apply the method of undetermined coefficients, so we have to use the method of variation of parameters. The general homogeneous solution is

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 \cos x + c_2 \sin x,\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \text{ and } c_1, c_2 \text{ are arbitrary.} \tag{1.19}$$

The Wronskian W is

$$W(y_1, y_2)(x) = \det \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} = \cos^2 x + \sin^2 x = 1,\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{1.20}$$

Therefore

$$\begin{aligned}
y_p(x) &= -y_1(x) \int \frac{y_2(x) f(x)}{W(y_1, y_2)(x)}\, dx + y_2(x) \int \frac{y_1(x) f(x)}{W(y_1, y_2)(x)}\, dx && (1.21)\\
&= -\cos x \int \frac{\sin x \tan x}{1}\, dx + \sin x \int \frac{\cos x \tan x}{1}\, dx && (1.22)\\
&= -\cos x \int \frac{\sin^2 x}{\cos x}\, dx + \sin x \int \sin x\, dx && (1.23)\\
&= -\cos x \int \frac{1 - \cos^2 x}{\cos x}\, dx + \sin x(-\cos x) && (1.24)\\
&= -\cos x \int \sec x - \cos x\, dx - \sin x \cos x && (1.25)\\
&= -\cos x(\ln|\sec x + \tan x| - \sin x) - \sin x \cos x && (1.26)\\
&= -\cos x \ln|\sec x + \tan x|\,,\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). && (1.27)
\end{aligned}$$

Therefore the general solution is

$$y(x) = c_1 \cos x + c_2 \sin x - \cos x \ln|\sec x + \tan x|\,,\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \text{ and } c_1, c_2 \text{ are arbitrary.} \tag{1.28}$$

□

Problem 1.4. Suppose a damped mass-spring system is modeled via the inhomogeneous equation

$$mx''(t) + \beta x'(t) + kx(t) = F_0 \sin(\omega t),\ t \in \mathbb{R} \tag{1.29}$$

where $m, \beta, k, F_0, \omega > 0$.

a) Find a particular solution x_p to the equation.
b) What happens to the general homogeneous solution x_h as $t \to \infty$?
c) Use part b) to describe the behavior of the mass-spring system for large time.

Solution.

a) We use the method of undetermined coefficients and use the ansatz

$$x_p(t) = A\cos\omega t + B\sin\omega t,\ t \in \mathbb{R}. \tag{1.30}$$

Then

$$x_p'(t) = B\omega\cos\omega t - A\omega\sin\omega t \tag{1.31}$$

$$x_p''(t) = -A\omega^2\cos\omega - B\omega^2\sin\omega t,\ t \in \mathbb{R}. \tag{1.32}$$

Thus

$$mx_p''(t) + \beta x_p'(t) + kx_p(t) = (-mA\omega^2 + \beta B\omega + kA)\cos\omega t + (-mB\omega^2 - \beta A\omega + kB)\sin\omega t \tag{1.33}$$

$$= F_0\sin\omega t,\ t \in \mathbb{R}. \tag{1.34}$$

This implies that

$$\begin{cases} (k - m\omega^2)A + \beta\omega B &= 0 \\ -\beta\omega A + (k - m\omega^2)B &= F_0. \end{cases} \tag{1.35}$$

Through routine algebraic manipulations we then obtain

$$A = \frac{(k - m\omega^2)F_0}{(k - m\omega^2)^2 + \beta^2\omega^2},\ B = \frac{\beta\omega F_0}{(k - m\omega^2)^2 + \beta^2\omega^2}. \tag{1.36}$$

Note that these quantities are well-defined since $(k - m\omega^2)^2 + \beta^2\omega^2 \geq \beta^2\omega^2 > 0$ since $\beta, \omega > 0$. Therefore a particular solution to the equation is

$$x_p(t) = \frac{(k - m\omega^2)F_0}{(k - m\omega^2)^2 + \beta^2\omega^2}\cos\omega t + \frac{\beta\omega F_0}{(k - m\omega^2)^2 + \beta^2\omega^2}\sin\omega t,\ t \in \mathbb{R}. \tag{1.37}$$

We note that since

$$\left(\frac{(k - m\omega^2)F_0}{(k - m\omega^2)^2 + \beta^2\omega^2}\right)^2 + \left(\frac{\beta\omega F_0}{(k - m\omega^2)^2 + \beta^2\omega^2}\right)^2 = \frac{F_0^2((k - m\omega^2)^2 + \beta^2\omega^2)}{((k - m\omega^2)^2 + \beta^2\omega^2)^2} = \frac{F_0^2}{(k - m\omega^2)^2 + \beta^2\omega^2}, \tag{1.38}$$

we can write x_p as

$$x_p(t) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + \beta^2\omega^2}}\sin(\omega t + \alpha), \tag{1.39}$$

where α is defined via

$$\tan\alpha = \frac{(k - m\omega^2)F_0}{\beta\omega F_0} = \frac{(k - m\omega^2)}{\beta\omega}. \tag{1.40}$$

b) Since the damping constant is strictly positive, the homogeneous solution will vanish as $t \to \infty$ as the damping constant contributes to a decaying exponential term in the general homogeneous solution.
c) For large time, the contribution of the homogeneous solution will start to vanish (in this case we refer to the homogeneous solution as a *transient solution*), and therefore the behavior of the system will be mostly described by the particular solution we identified in part a). By examining the particular solution, we see that the mass-spring system will oscillate as $t \to \infty$.

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