Homework 10

DUE: SATURDAY, APRIL 19, 11:59PM

If you completed this assignment through collaboration or consulted references, please list the names of your collaborators and the references you used below. Please refer to the syllabus for the course policy on collaborations and types of references that are allowed.

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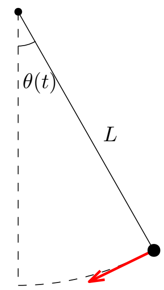
Problem 10.1 (Orientation of trajectories)**.** Let $\alpha \in \mathbb{R}$ and consider the first order systems $\boldsymbol{X}_1'(t) = A_1\boldsymbol{X}_1(t)$ and $\boldsymbol{X}_2'(t) = A_2\boldsymbol{X}_2(t)$, $t \in \mathbb{R}$, where

$$A_1 = \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix}, \quad A_2 = \begin{pmatrix} \alpha & -1 \\ 1 & \alpha \end{pmatrix}. \tag{10.1}$$

a) Show that for both systems, the origin is either a spiral point or a center. For each system, what value(s) of α is the origin a center?
b) Analyze the orientations of the trajectories in phase space for both systems. For which system is the orientation of the trajectories clockwise, and for which system is it counterclockwise? Does the orientation depend on the value of α?

Problem 10.2 (The nonlinear damped pendulum)**.**



The motion of a free pendulum experiencing air resistance is modeled by the nonlinear second ordered IVP

$$\begin{cases} L\theta''(t) + \beta\theta'(t) + g\sin(\theta(t)) = 0, \quad t \in \mathbb{R} \\ \theta(0) = \theta_0,\ \theta'(0) = v_0. \end{cases} \tag{10.2}$$

where $L > 0$ is the length of the rod, $\beta > 0$ is the damping constant indicating the strength of air resistance, $g > 0$ is the gravitational constant, and $\theta(t)$ represents the angle from the vertical to the pendulum at time t. Since the equation is nonlinear, finding explicit solution formulas can be difficult, but we can utilize the techniques we learned in this class to understand the behavior of the system when the initial angle and initial angular velocity are sufficiently small.

For the sake of simplicity we assume $L = \beta = g \neq 0$. Then the unknown function $\theta : \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{cases} \theta''(t) + \theta'(t) + \sin(\theta(t)) = 0, \quad t \in \mathbb{R} \\ \theta(0) = \theta_0,\ \theta'(0) = v_0. \end{cases} \tag{10.3}$$

a) Assume $\theta : \mathbb{R} \to \mathbb{R}$ is a solution to (10.3) and define $\boldsymbol{X} : \mathbb{R} \to \mathbb{R}^2$ via

$$\boldsymbol{X}(t) = \begin{pmatrix} \theta(t) \\ \theta'(t) \end{pmatrix}. \tag{10.4}$$

Write down a first order nonlinear system of the form

$$\begin{cases} \boldsymbol{X}'(t) = \boldsymbol{f}(\boldsymbol{X}(t)),\ t \in \mathbb{R} \\ \boldsymbol{X}(0) = \boldsymbol{X}_0. \end{cases} \tag{10.5}$$

b) Show that $\boldsymbol{X}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a critical point of (10.5). This should make physical sense as the pendulum should not move if it is at rest.

c) Use the linearization of the nonlinear system (10.5) to classify the stability of $\boldsymbol{X}_0$, and use this information to describe the long term behavior of solutions to the nonlinear equation (10.3) when the initial angle θ_0 and the initial angular velocity v_0 are both sufficiently small. To double-check that your classification is correct, you should ask yourself if the stability type you identified makes physical sense.

If you want to know what the trajectories associated to this nonlinear system looks like in phase space, Grant Sanderson (3Blue1Brown) made some nice visualizations in this video.

Problem 10.3 (The matrix-valued Laplace transform and the matrix exponential)**.** Consider the constant coefficient system $\boldsymbol{X}'(t) = A\boldsymbol{X}(t),\ t \in \mathbb{R}$, where

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{10.6}$$

a) Use the matrix-valued Laplace transform to compute the fundamental matrix given by $\Phi : \mathbb{R} \to \mathcal{M}_{2\times 2}(\mathbb{R})$ defined via $\Phi(t) = e^{tA}$. You can use the shortcut for finding the inverse of a 2×2 invertible matrix given in the notes.

b) Recall that the unique solution to the IVP

$$\begin{cases} \boldsymbol{X}'(t) = A\boldsymbol{X}(t),\ t \in \mathbb{R} \\ \boldsymbol{X}(0) = \boldsymbol{X}_0 \end{cases} \tag{10.7}$$

is given by $\boldsymbol{X} : \mathbb{R} \to \mathbb{R}^2$ defined via $\boldsymbol{X}(t) = \Phi(t)\boldsymbol{X}_0$. Use this to write down the solution to the IVP

$$\begin{cases} \boldsymbol{X}'(t) = A\boldsymbol{X}(t),\ t \in \mathbb{R} \\ \boldsymbol{X}(0) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \end{cases} \tag{10.8}$$

where c_1, c_2 are arbitrary constants.

c) Try solving the IVP (10.8) using the solution formula we learned previously in terms of the eigenvector(s)/generarized eigenvector(s) of A. How does this compare to the solution you found in part b)?

Problem 10.4 (The Fourier basis functions over $[0, L]$)**.**

Let $\mathbb{R} \ni L > 0$. In lecture, we claimed that the functions

$$\frac{1}{\sqrt{L}} \quad \text{and} \quad \sqrt{\frac{2}{L}} \cos\left(\frac{2\pi kx}{L}\right) \quad \text{and} \quad \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi kx}{L}\right),\ k \in \mathbb{Z}^+ = \{1, 2, 3, 4, \ldots\} \tag{10.9}$$

considered as functions over $[0, L]$ are orthonormal with respect to the L^2 inner product

$$\langle f, g \rangle = \int_0^L f(x)g(x)\ dx. \tag{10.10}$$

In this problem we aim to justify this claim. To introduce some notation, define $a_0, a_k, b_k : [0, L] \to \mathbb{R}$ via

$$a_0(x) = \frac{1}{\sqrt{L}}, \quad a_k(x) = \sqrt{\frac{2}{L}} \cos\left(\frac{2\pi kx}{L}\right), \quad b_k(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi kx}{L}\right),\ k \in \mathbb{Z}^+. \tag{10.11}$$

a) Show that $\langle a_0, a_0 \rangle = 1$ and $\langle a_0, a_k \rangle = \langle a_0, b_k \rangle = 0$ for all $k \in \mathbb{Z}^+$.

b) Show that

$$\langle a_m, a_n \rangle = \langle b_m, b_n \rangle = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \tag{10.12}$$

for all $m, n \in \mathbb{Z}^+$.

You may find the trigonometric identities given in section d) of the Calculus reference sheet to be helpful.

Problem 10.5 (The Fourier basis functions over $[-L, L]$)**.**

Let $\mathbb{R} \ni L > 0$. In this problem we repeat the same analysis as in the previous problem, but now we consider functions defined over $[-L, L]$. The L^2 inner product over $[-L, L]$ is given by

$$\langle f, g \rangle = \int_{-L}^{L} f(x)g(x)\, dx. \tag{10.13}$$

Define $a_0, a_k, b_k : [-L, L] \to \mathbb{R}$ via

$$a_0(x) = \frac{1}{\sqrt{2L}}, \quad a_k(x) = \sqrt{\frac{1}{L}} \cos\left(\frac{k\pi x}{L}\right), \quad b_k(x) = \sqrt{\frac{1}{L}} \sin\left(\frac{k\pi x}{L}\right),\ k \in \mathbb{Z}^+. \tag{10.14}$$

a) Show that $\langle a_0, a_0 \rangle = 1$ and $\langle a_0, a_k \rangle = \langle a_0, b_k \rangle = 0$ for all $k \in \mathbb{Z}^+$.

b) Show that

$$\langle a_m, a_n \rangle = \langle b_m, b_n \rangle = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \tag{10.15}$$

for all $m, n \in \mathbb{Z}^+$.

Problem 10.6 (Fourier series and periodic extensions)**.** Consider the function $f : (-\pi, \pi) \to \mathbb{R}$ defined via

$$f(x) = x. \tag{10.16}$$

a) Explain why the Fourier series of f over $(-\pi, \pi)$ reduces to a Fourier sine series (i.e. the constant term and the cosine terms vanish).
b) Note that by definition, the Fourier sine coefficients of a function over $(-\pi, \pi)$ are defined in terms of an integral over $(-\pi, \pi)$. Explain why here the Fourier sine coefficients of f can be computed via

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx, \ n \geq 1. \tag{10.17}$$

c) Show that

$$b_n = \frac{2(-1)^{n+1}}{n}, \ n \geq 1. \tag{10.18}$$

d) Sketch a graph over $\mathbb{R}$ of its Fourier sine series. Is this Fourier sine series a continuous function over $\mathbb{R}$?
e) Now consider the restriction of f to $[0, \pi)$, $g : [0, \pi) \to \mathbb{R}$ defined via

$$g(x) = x. \tag{10.19}$$

Consider an even reflection $\tilde{g} : (-\pi, \pi) \to \mathbb{R}$ of g to $(-\pi, 0]$ defined via

$$\tilde{g}(x) = \begin{cases} g(x), & x \in [0, \pi) \\ -g(-x), & x \in (-\pi, 0), \end{cases} \tag{10.20}$$

and consider the Fourier series associated to the reflected function $\tilde{g}$. Explain why in this case, the Fourier series of $\tilde{g}$ reduces to a Fourier cosine series (i.e. the sine terms vanish), and sketch a graph over $\mathbb{R}$ of this cosine series. Is this Fourier cosine series a continuous function over $\mathbb{R}$?

For the sketches, please be precise about the values of the corresponding series at the points of discontinuity.