Homework 4

DUE: SATURDAY, FEBRUARY 15, 2025, 11:59PM

If you completed this assignment through collaboration or consulted references, please list the names of your collaborators and the references you used below. Please refer to the syllabus for the course policy on collaborations and types of references that are allowed.

Department of Mathematical Sciences, Carnegie Mellon University
E-mail address: jkoganem@andrew.cmu.edu.
Date: February 18, 2025.

Problem 4.1. For each of the following equations, write down the general solution.

a) $y''(x) - 5y'(x) + 6y(x) = 0, x \in \mathbb{R}$
b) $y''(t) - 4y'(t) + 4y(t) = 0, t \in \mathbb{R}$
c) $y''(t) + 4y(t) = 0, t \in \mathbb{R}$
d) $y''(x) - 2y'(x) + 2y(x) = 0, x \in \mathbb{R}$
e) $y^{(4)}(x) - 2y''(x) + y(x) = 0,\ x \in \mathbb{R}$

Solution.

a) The characteristic equation associated to the equation is $r^2 - 5r + 6 = (r-2)(r-3) = 0$, therefore $r = 2$ or $r = 3$. Since we have distinct real roots, the general solution is
$$y(x) = c_1 e^{2x} + c_2 e^{3x},\ x \in \mathbb{R}, \tag{4.1}$$
where c_1, c_2 are arbitrary.

b) The characteristic equation associated to the equation is $r^2 - 4r + 4 = (r-2)^2 = 0$, therefore $r = 2$ is a repeated root. Since we have a repeated real root, the general solution is
$$y(t) = c_1 e^{2t} + c_2 t e^{2t},\ t \in \mathbb{R}, \tag{4.2}$$
where c_1, c_2 are arbitrary.

c) The characteristic equation associated to the equation is $r^2 + 4 = (r-2i)(r+2i) = 0$, therefore $r = -2i$ or $r = 2pi$. Since we have a pair of complex roots, the general solution is
$$y(t) = c_1 \cos(2t) + c_2 \sin(2t),\ t \in \mathbb{R}, \tag{4.3}$$
where c_1, c_2 are arbitrary.

d) The characteristic equation associated to the equation is $r^2 - 2r + 2 = 0$, therefore $r = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$. Since we have a pair of complex roots, the general solution is
$$y(x) = c_1 e^x \cos x + c_2 e^x \sin x,\ x \in \mathbb{R}, \tag{4.4}$$
where c_1, c_2 are arbitrary.

e) The characteristic equation associated to the equation is $r^4 - 2r^2 + 1 = (r^2 - 1)^2 = 0$, therefore $r = \pm 1$ are repeated roots. Since we have a repeated real roots, the general solution is
$$y(x) = c_1 e^x + c_2 x e^x + c_3 e^{-x} + c_4 x e^{-x},\ x \in \mathbb{R}, \tag{4.5}$$

$\square$

Problem 4.2. For each of the following parts, find a constant coefficient equation over $\mathbb{R}$ whose general solution $y : \mathbb{R} \to \mathbb{R}$ is given below.

a) $y(x) = c_1 e^x + c_2 e^{5x}$, c_1, c_2 are arbitrary
b) $y(x) = c_1 e^{10x} + c_2 x e^{10x}$, c_1, c_2 are arbitrary
c) $y(x) = c_1 + c_2 e^{2x} \cos(5x) + c_3 e^{2x} \sin(5x)$, c_1, c_2, c_3 are arbitrary

Solution.

a) The roots of the characteristic equation should be 1 and 5, so the characteristic equation can be
$$(r-1)(r-5) = r^2 - 6r + 5. \tag{4.6}$$
Therefore the equation
$$y''(x) - 6y'(x) + 5y(x) = 0,\ x \in \mathbb{R} \tag{4.7}$$
would admit the given general solution.

b) The characteristic polynomial should have $r = 10$ as a double root, so the characteristic equation can be
$$(r-10)^2 = r^2 - 20r + 100 \tag{4.8}$$
Therefore the equation
$$y''(x) - 20y'(x) + 100y(x) = 0,\ x \in \mathbb{R} \tag{4.9}$$
would admit the given general solution.

c) The roots of the characteristic equation should be 0 and $2 \pm 5i$, so the characteristic equation can be
$$r(r-(2-5i))(r-(2+5i)) = r(r^2 - 4r + 29) = r^3 - 4r^2 + 29r \tag{4.10}$$
Therefore the equation
$$y'''(x) - 4y''(x) + 29y'(x) = 0,\ x \in \mathbb{R} \tag{4.11}$$
would admit the given general solution.

□

Problem 4.3.

a) Verify that the functions $y_1, y_2 : (0,\infty) \to \mathbb{R}$ defined via
$$y_1(x) = x,\ y_2(x) = x\ln x \tag{4.12}$$
are linearly independent over the interval $I = (0,\infty)$.

b) Verify that the function $y : (0,\infty) \to \mathbb{R}$ defined via
$$y(x) = c_1 x + c_2 x \ln x, \tag{4.13}$$
where c_1, c_2 are two arbitrary constants, is a solution on the interval $I = (0,\infty)$ to the second order linear homogeneous equation
$$x^2 y''(x) - xy'(x) + y(x) = 0,\ x \in (0,\infty). \tag{4.14}$$

Part a). We calculate
$$W(y_1,y_2)(x) = \det \begin{pmatrix} x & x\ln x \\ 1 & \ln x + 1 \end{pmatrix} = x(\ln x + 1) - x\ln x = x > 0,\ x \in I. \tag{4.15}$$
Thus y_1, y_2 are independent over I. □

Part b). We calculate
$$y'(x) = c_1 + c_2(\ln x + 1),\ x > 0 \tag{4.16}$$
$$y''(x) = c_2\left(\frac{1}{x}\right),\ x > 0. \tag{4.17}$$
Thus
$$x^2 y''(x) - xy'(x) + y(x) = c_2 x - c_1 x - c_2(x\ln x + x) + c_1 x + c_2 x \ln x = 0,\ x > 0. \tag{4.18}$$
Therefore (4.13) defines a solution to the homogeneous equation. In fact, since x and $x\ln x$ are linearly independent on I, from the theory of 2nd order homogeneous linear equations we know that (4.13) actually gives us the general solution to the equation on I. □

Problem 4.4. Verify that the functions $y_1, y_2 : (0, \infty) \to \mathbb{R}$ defined via

$$y_1(x) = x^3, \; y_2(x) = x^4 \tag{4.19}$$

form a fundamental set of solutions on the interval $I = (0, \infty)$ to the equation

$$x^2 y''(x) - 6xy'(x) + 12y(x) = 0. \tag{4.20}$$

Solution. We first check that y_1, y_2 are linearly independent. We note that

$$W(y_1, y_2)(x) = \det \begin{pmatrix} x^3 & x^4 \\ 3x^2 & 4x^3 \end{pmatrix} = 4x^6 - 3x^6 = x^6 \neq 0 \text{ for all } x \in I. \tag{4.21}$$

Therefore y_1, y_2 are linearly independent on I. Next we check that y_1, y_2 solve the homogeneous equation. We note that for $x \in I$,

$$y_1'(x) = 3x^2 \tag{4.22}$$

$$y_1''(x) = 6x \tag{4.23}$$

$$y_2'(x) = 4x^3 \tag{4.24}$$

$$y_2''(x) = 12x^2. \tag{4.25}$$

Thus

$$x^2 y_1''(x) - 6xy_1'(x) + 12y_1(x) = 6x^3 - 18x^3 + 12x^3 = 0, \; x \in I \tag{4.26}$$

$$x^2 y_2''(x) - 6xy_2'(x) + 12y_2(x) = 12x^4 - 24x^4 + 12x^4 = 0, \; x \in I. \tag{4.27}$$

Therefore y_1, y_2 form a fundamental set of solutions over the interval $I = (0, \infty)$. $\square$

Problem 4.5 (Spring-mass system)**.** Let $x : \mathbb{R} \to \mathbb{R}$ be a solution to the initial value problem

$$\begin{cases} x''(t) + \gamma x'(t) + x(t) = 0, \ t \in \mathbb{R}, \gamma \geq 0 \\ x(0) = 1, x'(0) = -2 \end{cases} \tag{4.28}$$

which models a spring-mass system.

a) For what value(s) of γ will this system be underdamped?
b) Sketch a graph showing the behavior of the solution to the initial value problem, assuming that γ is chosen so that the system is underdamped. You do not need to solve for the solution explicitly. How many times does this graph cross the t-axis?
c) For what value(s) of γ will this system be critically damped? At what time(s) will the mass pass through the equilibrium point?

Solution. The characteristic equation associated to the differential equation is $r^2 + \gamma r + 1$, which implies that the roots are $r = -\gamma/2 \pm \sqrt{\gamma^2 - 4}/2$. So the system is underdamped whenever $\gamma^2 - 4 < 0$ and $\gamma > 0$, or when $0 < \gamma < 2$.

Below is a rough sketch of the solution to the IVP. Note that the initial position is at $x = 1$ and the initial velocity is negative.

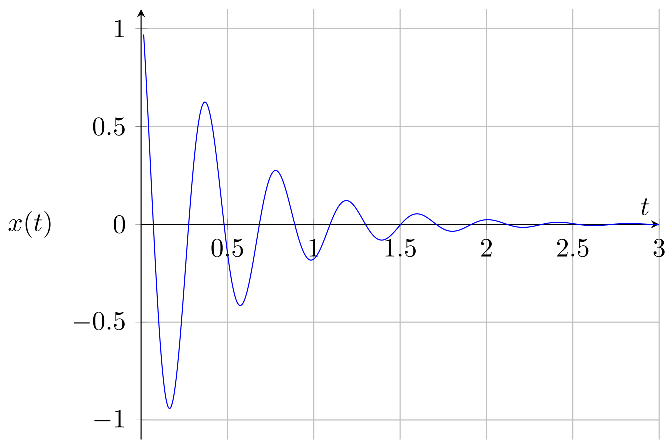

FIGURE 1. $x(t)$ in an underdamped system

Since the system is underdamped, the periodic part of the solution guarantees that the graph crosses the t-axis infinitely many times.

The system is critically damped if $\gamma^2 - 4 = 0$ and $\gamma > 0$, or when $\gamma = 2$. Then the general solution is given by

$$x(t) = e^{-t}\left(c_1 + c_2 t\right), \ t \in \mathbb{R}. \tag{4.29}$$

Using the initial conditions we find that

$$1 = x(0) = c_1, -2 = x'(0) = c_2 - c_1 \implies c_2 = -1. \tag{4.30}$$

Therefore the solution to the initial value problem is

$$x(t) = e^{-t}\left(1 - t\right), \ t \in \mathbb{R}. \tag{4.31}$$

This shows that the mass passes through the equilibrium exactly once, when $t = 1$. □

Problem 4.6 (Reduction of order)**.**

a) Verify that the function $y_1 : (0, \infty) \to \mathbb{R}$ defined via $y_1(x) = x^3$ is a solution to the 2nd order differential equation

$$x^2 y''(x) - 5xy'(x) + 9y(x) = 0, \ x > 0. \tag{4.32}$$

b) Use the method of reduction of order to identify a second linearly independent solution to (4.32).

Solution. We note that

$$y_1'(x) = 3x^2, \ x > 0 \tag{4.33}$$
$$y_1''(x) = 6x, \ x > 0. \tag{4.34}$$

Thus

$$x^2 y_1''(x) - 5xy_1'(x) + 9y_1(x) = 6x^3 - 15x^3 + 9x^3 = 0, \ x > 0, \tag{4.35}$$

therefore y_1 is a solution the equation.

To identify a second linearly independent solution, we use the ansatz $y_2(x) = u(x)y_1(x), x > 0$, where u is an unknown function. Then

$$y_2'(x) = u'(x)y_1(x) + u(x)y_1'(x), \ x > 0 \tag{4.36}$$
$$y_2''(x) = u''(x)y_1(x) + 2u'(x)y_1'(x) + u(x)y_1''(x), \ x > 0. \tag{4.37}$$

Assuming that y_2 is a solution to the equation, we then have

$$0 = x^2(u''(x)y_1(x) + 2u'(x)y_1'(x) + u(x)y_1''(x)) - 5x(u'(x)y_1(x) + u(x)y_1'(x)) + 9u(x)y_1(x) \tag{4.38}$$
$$= x^2 y_1(x)u''(x) + (2x^2 y_1'(x) - 5xy_1(x))u'(x) + \underbrace{(x^2 y_1''(x) - 5xy_1'(x) + 9y_1(x))}_{=0} u(x), \ x > 0. \tag{4.39}$$

Thus u satisfies the equation

$$x^5 u''(x) + (6x^4 - 5x^4)u'(x) = x^5 u''(x) + x^4 u'(x) = 0, \ x > 0. \tag{4.40}$$

If we define the function w via $w(x) = u'(x), x > 0$, then w satisfies the first order linear equation

$$w'(x) + \frac{1}{x} w(x) = 0, \ x > 0. \tag{4.41}$$

We may choose an integrating factor to be $\mu(x) = \exp(\int \frac{1}{x} \, dx) = x, x > 0$. Thus

$$\frac{d}{dx}[xw(x)] = 0 \implies w(x) = \frac{C}{x}, \ x > 0. \tag{4.42}$$

Then

$$u(x) = \int \frac{C}{x} \, dx = C \ln|x| + D = C \ln x + D, \ x > 0, \tag{4.43}$$

where C, D are arbitrary. Thus

$$y_2(x) = u(x)y_1(x) = Cx^3 \ln x + Dx^3, \ x > 0. \tag{4.44}$$

We may choose $C = 1, D = 0$ to find that a second linearly independent solution to the equation is $y_2(x) = x^3 \ln x, x > 0$. $\square$