Recitation 8

Problem 8.1. Assuming that the eigenvalues of the following problem are positive, determine the eigenvalues of

$$\begin{cases} y''(x) + \lambda y(x) = 0, \quad x \in (-\pi, \pi) \\ y(-\pi) = y(\pi) = 0. \end{cases} \tag{8.1}$$

Solution. If $\lambda > 0$, then the general solution of the 2nd order linear equation is

$$y(x) = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x),\ x \in (-\pi, \pi). \tag{8.2}$$

The boundary conditions then require

$$0 = y(-\pi) = c_1 \cos(\sqrt{\lambda}\pi) - c_2 \sin(\sqrt{\lambda}\pi) \tag{8.3}$$

$$0 = y(\pi) = c_1 \cos(\sqrt{\lambda}\pi) + c_2 \sin(\sqrt{\lambda}\pi). \tag{8.4}$$

We note that this requires

$$\begin{cases} c_1 \cos\sqrt{\lambda}\pi = 0 \\ c_2 \sin\sqrt{\lambda}\pi = 0. \end{cases} \tag{8.5}$$

We note both equations cannot hold simultaneously if $c_1, c_2 \neq 0$. So either $c_1 = 0, c_2 \neq 0$ or $c_1 \neq 0, c_2 = 0$, and we exclude the case $c_1 = c_2 = 0$ since we are interested in non-zero solutions. In the first case, we require

$$\sin\sqrt{\lambda}\pi = 0, \tag{8.6}$$

which implies that $\sqrt{\lambda}$ must be an integer; in the second case, we require

$$\cos\sqrt{\lambda}\pi = 0, \tag{8.7}$$

which implies that $\sqrt{\lambda}$ must be an odd integer multiple of $\frac{1}{2}$. Therefore we may parametrize the positive eigenvalues via

$$\lambda_n = \left(\frac{n}{2}\right)^2 = \frac{n^2}{4},\ n = 1, 2, 3, 4, \ldots \tag{8.8}$$

□

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Problem 8.2. For $a > 0$, find a function x that satisfies

$$\begin{cases} x'(t) = \delta(t-a), & t \geq 0 \\ x(0) = 0. \end{cases} \tag{8.9}$$

Solution. Taking the Laplace transform gives us

$$sX(s) = e^{-as}, \tag{8.10}$$

for appropriate values of s. Then

$$X(s) = \frac{e^{-as}}{s} \tag{8.11}$$

for appropriate values of s. This implies that

$$x(t) = \mathcal{U}(t-a),\ t \geq 0. \tag{8.12}$$

From this we can also see why the Dirac delta is the "derivative" of the unit step function, but both the differential equation and the relationship between δ and $\mathcal{U}$ has to be understood in the distributional sense, since the step function is not differentiable at the point of discontinuity in the classical sense. □

Problem 8.3. Find the inverse Laplace transform of $F : (0, \infty) \to \infty$ defined via

$$F(s) = \frac{s+2}{s^2 + 4s + 5}. \tag{8.13}$$

Solution. We note that we can write

$$F(s) = \frac{s+2}{(s+2)^2 + 1}, \tag{8.14}$$

therefore

$$f(t) = \mathcal{L}^{-1}\left\{F\right\}(t) = e^{-2t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}(t) = e^{-2t}\cos t,\ t \geq 0 \tag{8.15}$$

$\square$

Problem 8.4. Find the Laplace transform of $f : \mathbb{R} \to R$ defined via

$$f(t) = \begin{cases} 0, & 0 \leq t < 1 \\ te^t, & t \geq 1 \end{cases} \tag{8.16}$$

Solution. We can write

$$f(t) = te^t \mathcal{U}(t-1). \tag{8.17}$$

Therefore

$$\mathcal{L}\left\{f\right\}(s) = e^{-s}\mathcal{L}\left\{(t+1)e^{t+1}\right\} = e^{-s+1}(\mathcal{L}\left\{te^t\right\} + \mathcal{L}\left\{e^t\right\}) = e^{-s+1}\left(\frac{1}{(s-1)^2} + \frac{1}{s-1}\right),\ s > 1. \tag{8.18}$$

$\square$