

More Laplace transform practice problems

Problem 1.1. Define $f : [0, \infty) \rightarrow \mathbb{R}$ via

$$f(t) = e^t(\mathcal{U}(t-1) - \mathcal{U}(t-2)). \quad (1.1)$$

What is the Laplace transform of f ?

Solution. Note that

$$\mathcal{L}\{\mathcal{U}(t-1) - \mathcal{U}(t-2)\}(s) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}, \quad s > 0. \quad (1.2)$$

Therefore

$$\mathcal{L}\{f\}(s) = \frac{e^{-(s-1)}}{s-1} - \frac{e^{-2(s-1)}}{s-1}, \quad s > 1. \quad (1.3)$$

□

Problem 1.2. Define $f : [0, \infty) \rightarrow \mathbb{R}$ via

$$f(t) = \begin{cases} 0, & 0 \leq t < \pi \\ t - \pi, & \pi \leq t < 2\pi \\ 0, & t \geq 2\pi. \end{cases} \quad (1.4)$$

What is the Laplace transform of f ?

Solution. Note that

$$f(t) = 0 + (t - \pi - 0)\mathcal{U}(t - \pi) + (0 - (t - \pi))\mathcal{U}(t - 2\pi) = (t - \pi)\mathcal{U}(t - \pi) - (t - \pi)\mathcal{U}(t - 2\pi), \quad t \geq 0. \quad (1.5)$$

Therefore

$$\mathcal{L}\{f\}(s) = e^{-\pi s}\mathcal{L}\{t\}(s) - e^{-2\pi s}\mathcal{L}\{t + \pi\}(s) = \frac{e^{-\pi s}}{s^2} - e^{-2\pi s}\left(\frac{1}{s^2} + \frac{\pi}{s}\right), \quad s > 0 \quad (1.6)$$

□

Problem 1.3. Define $f : [0, \infty) \rightarrow \mathbb{R}$ via

$$f(t) = \begin{cases} t, & 0 \leq t < 1 \\ 1, & t \geq 1 \end{cases} \quad (1.7)$$

What is the Laplace transform of f ?

Solution. Note that

$$f(t) = t + (1 - t)\mathcal{U}(t - 1), \quad t \geq 0. \quad (1.8)$$

Therefore

$$\mathcal{L}\{f\}(s) = \frac{1}{s^2} + e^{-s}\mathcal{L}\{1 - (t + 1)\}(s) = \frac{1}{s} - e^{-s}\frac{1}{s^2}, \quad s > 0. \quad (1.9)$$

□

Problem 1.4. Define $F : (3, \infty) \rightarrow \mathbb{R}$ via

$$F(s) = \frac{10e^{-s}}{s^2 - 5s + 6} + \frac{2}{s^2 - 2s + 5}. \quad (1.10)$$

What is the inverse Laplace transform of F ?

Solution. Note that

$$\frac{1}{s^2 - 5s + 6} = \frac{1}{s - 2} - \frac{1}{s - 3}, \quad s \neq 2, 3. \quad (1.11)$$

and

$$\frac{2}{s^2 - 2s + 5} = \frac{2}{(s - 1)^2 + 4}, \quad s \in \mathbb{R}. \quad (1.12)$$

Therefore

$$\mathcal{L}^{-1}\{F\}(t) = 10\mathcal{U}(t - 1) \left(e^{2(t-1)} - e^{3(t-1)} \right) + e^t \sin 2t, \quad t \geq 0. \quad (1.13)$$

□

Problem 1.5. Define $F : (0, \infty) \rightarrow \mathbb{R}$ via

$$F(s) = \frac{3s^2 + 4s + 1}{(s + 1)(s^2 + 2s + 5)}. \quad (1.14)$$

What is the inverse Laplace transform of F ?

Solution. Note that if one looks for a partial fraction decomposition of the form

$$\frac{3s^2 + 4s + 1}{(s + 1)(s^2 + 2s + 5)} = \frac{A}{s + 1} + \frac{Bs + C}{s^2 + 2s + 5} \quad (1.15)$$

for some $A, B, C \in \mathbb{R}$, which is equivalent to requiring

$$3s^2 + 4s + 1 = A(s^2 + 2s + 5) + (Bs + C)(s + 1), \quad s \in \mathbb{R}, \quad (1.16)$$

we find that if $s = -1$ then

$$3 - 4 + 1 = 0 = A(1 - 2 + 5) = 4A \implies A = 0. \quad (1.17)$$

This suggests that $s + 1$ divides into $3s^2 + 4s + 1$, so we look to factorize

$$3s^2 + 4s + 1 = (s + 1)(3s + 1), \quad s \in \mathbb{R}. \quad (1.18)$$

Thus

$$F(s) = \frac{3s^2 + 4s + 1}{(s + 1)(s^2 + 2s + 5)} = \frac{3s + 1}{s^2 + 2s + 5} = \frac{3(s + 1) - 2}{(s + 1)^2 + 4}, \quad s > 0. \quad (1.19)$$

Thus

$$\mathcal{L}^{-1}\{F\}(t) = 3e^{-t} \cos 2t - e^{-t} \sin 2t, \quad t \geq 0. \quad (1.20)$$

□

Problem 1.6. Define $F : (4, \infty) \rightarrow \mathbb{R}$ via

$$F(s) = e^{-3s} \frac{s + 1}{s^2 - 8s + 20}. \quad (1.21)$$

What is the inverse Laplace transform of F ?

Solution. Note that

$$F(s) = e^{-3s} \frac{(s - 4) + 5}{(s - 4)^2 + 4}, \quad (1.22)$$

and

$$\mathcal{L}^{-1}\left\{\frac{(s - 4) + 5}{(s - 4)^2 + 4}\right\}(t) = e^{4t} \left(\cos 2t + \frac{5}{2} \sin 2t \right), \quad t \geq 0. \quad (1.23)$$

Thus

$$\mathcal{L}^{-1}\{F\}(t) = \mathcal{U}(t-3)e^{4(t-3)}\left(\cos 2(t-3) + \frac{5}{2}\sin 2(t-3)\right), \quad t \geq 0. \quad (1.24)$$

□

Problem 1.7. Consider a forced undamped mass-spring system modeled via the IVP

$$\begin{cases} y''(t) + 4y(t) = f(t), & t \geq 0 \\ y(0) = -1, & y'(0) = 4, \end{cases} \quad (1.25)$$

- a) Find an expression for a solution $y : [0, \infty) \rightarrow \mathbb{R}$ for any reasonable forcing function $f : [0, \infty) \rightarrow \mathbb{R}$.
 b) Write down the solution for $f(t) = 2\delta(t - \pi)$.

Solution. If y is a solution, then its Laplace transform Y satisfies

$$(s^2 + 4)Y(s) + s - 4 = F(s), \quad s > a \quad (1.26)$$

for some $a \in \mathbb{R}$, where F is the Laplace transform of f . Thus

$$Y(s) = \frac{1}{s^2 + 4}F(s) + \frac{4-s}{s^2 + 4}, \quad s > a. \quad (1.27)$$

This implies that

$$y(t) = \frac{1}{2}(\sin(2\cdot) * f)(t) + 2\sin 2t - \cos 2t, \quad t \geq 0, \quad (1.28)$$

where

$$(\sin(2\cdot) * f)(t) = \int_0^t f(\tau) \sin 2(t - \tau) \, d\tau. \quad (1.29)$$

If $f = 2\delta(t - \pi)$, then note that

$$\int_0^t f(\tau) \sin 2(t - \tau) \, d\tau = 2 \int_0^t \delta(\tau - \pi) \sin 2(t - \tau) \, d\tau = \begin{cases} 0, & 0 \leq t \leq \pi \\ 2\sin 2(t - \pi), & t > \pi \end{cases} \quad (1.30)$$

$$= 2\mathcal{U}(t - \pi) \sin 2(t - \pi), \quad t \geq 0. \quad (1.31)$$

Thus

$$y(t) = \mathcal{U}(t - \pi) \sin 2(t - \pi) + 2\sin 2t - \cos 2t, \quad t \geq 0. \quad (1.32)$$

□

Problem 1.8. Define the function $f : [0, \infty) \rightarrow \mathbb{R}$ via

$$f(t) = e^t \int_0^t \sin \tau \cos(t - \tau) \, d\tau. \quad (1.33)$$

What is the Laplace transform of f ?

Solution. Note that

$$\mathcal{L}\left\{\int_0^t \sin \tau \cos(t - \tau) \, d\tau\right\}(s) = \mathcal{L}\{\sin t\}(s)\mathcal{L}\{\cos t\}(s) = \frac{s}{(s^2 + 1)^2}, \quad s > 0. \quad (1.34)$$

Thus

$$\mathcal{L}\{f\}(s) = \frac{s-1}{((s-1)^2 + 1)}, \quad s > 1. \quad (1.35)$$

□

Problem 1.9. Consider a forced undamped mass-spring system modeled via the IVP

$$\begin{cases} y''(t) + 2y(t) = \begin{cases} 0, & 0 \leq t < 2 \\ (t-2)e^{-3(t-2)}, & t \geq 2. \end{cases} \\ y(0) = y'(0) = 0. \end{cases} \quad (1.36)$$

If $y : [0, \infty) \rightarrow \mathbb{R}$ is a solution modeling the system and Y is its Laplace transform, what is $Y(0)$?

Solution. Note that if we define $f : [0, \infty) \rightarrow \mathbb{R}$ via

$$f(t) = \begin{cases} 0, & 0 \leq t < 2 \\ (t-2)e^{-3(t-2)}, & t \geq 2. \end{cases} = \mathcal{U}(t-2)(t-2)e^{-3(t-2)}, \quad (1.37)$$

then

$$\mathcal{L}\{f\}(s) = e^{-2s} \mathcal{L}\{te^{-3t}\}(s) = e^{-2s} \frac{1}{(s+3)^2}, \quad s > 0. \quad (1.38)$$

Thus if y is a solution and Y is its Laplace transform, then

$$Y(s) = e^{-2s} \frac{1}{(s+3)^2(s^2+2)}, \quad s > 0. \quad (1.39)$$

Thus

$$Y(0) = \frac{1}{18}. \quad (1.40)$$

□

Problem 1.10. Consider a forced undamped mass-spring system modeled via the IVP

$$\begin{cases} y''(t) + y(t) = \begin{cases} t, & 0 \leq t < 2 \\ 3, & t \geq 2. \end{cases} \\ y(0) = y'(0) = 0. \end{cases} \quad (1.41)$$

Find a solution $y : [0, \infty) \rightarrow \mathbb{R}$ modeling the system.

Solution. Note that if we define $f : [0, \infty) \rightarrow \mathbb{R}$ via

$$f(t) = \begin{cases} t, & 0 \leq t < 2 \\ 3, & t \geq 2. \end{cases} = t + (3-t)\mathcal{U}(t-2) \quad (1.42)$$

then

$$\mathcal{L}\{f\}(s) = \frac{1}{s^2} + e^{-2s} \mathcal{L}\{5-t\}(s) = \frac{1}{s^2} + e^{-2s} \left(\frac{5}{s} - \frac{1}{s^2} \right), \quad s > 0. \quad (1.43)$$

Thus if y is a solution and Y is its Laplace transform, then

$$Y(s) = \frac{1}{s^2(s^2+1)} + e^{-2s} \left(\frac{5}{s(s^2+1)} - \frac{1}{s^2(s^2+1)} \right). \quad (1.44)$$

We note that

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}, \quad s > 0 \quad (1.45)$$

and

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}, \quad s > 0. \quad (1.46)$$

Thus

$$y(t) = t - \sin t + \mathcal{U}(t-2) (5 - 5 \cos(t-2) - (t-2) + \sin(t-2)), \quad t \geq 0. \quad (1.47)$$

□

Problem 1.11. Suppose a mass-spring system is modeled via

$$\begin{cases} x''(t) + 2x'(t) + 5x(t) = f(t), & t \geq 0 \\ x(0) = x'(0) = 0, \end{cases} \quad (1.48)$$

where δ is the Dirac delta and \mathcal{U} is the unit step function and $f : [0, \infty) \rightarrow \mathbb{R}$ is defined via

$$f(t) = \begin{cases} 5, & 0 \leq t < \pi \\ \delta(t - 3\pi) + \delta(t - 4\pi), & t \geq \pi. \end{cases} \quad (1.49)$$

Find a solution $x : [0, \infty) \rightarrow \mathbb{R}$ describing the behavior of the system for $t \geq 0$.

Solution. Note that

$$f(t) = 5 + (\delta(t - 3\pi) + \delta(t - 4\pi) - 5)\mathcal{U}(t - \pi) = 5 - 5\mathcal{U}(t - \pi) + \delta(t - 3\pi) + \delta(t - 4\pi), \quad t \geq 0. \quad (1.50)$$

Thus

$$\mathcal{L}\{f\}(s) = \frac{5}{s} + e^{-\pi s} \frac{5}{s} + e^{-3\pi s} + e^{-4\pi s}, \quad s > 0. \quad (1.51)$$

Therefore if x is a solution and X is its Laplace transform, then

$$X(s) = \frac{5}{s(s^2 + 2s + 5)} - e^{-\pi s} \frac{5}{s(s^2 + 2s + 5)} + e^{-3\pi s} \frac{1}{s^2 + 2s + 5} + e^{-4\pi s} \frac{1}{s^2 + 2s + 5}, \quad s > 0. \quad (1.52)$$

Note that

$$\frac{5}{s(s^2 + 2s + 5)} = \frac{1}{s} + \frac{-s - 2}{s^2 + 2s + 5} = \frac{1}{s} - \frac{(s + 1) + 1}{(s + 1)^2 + 4}, \quad s > 0, \quad (1.53)$$

therefore

$$\begin{aligned} y(t) = & \left(1 - e^{-t} \left(\cos 2t + \frac{1}{2} \sin 2t\right)\right) - \mathcal{U}(t - \pi) \left(1 - e^{-(t-\pi)} \left(\cos 2(t - \pi) + \frac{1}{2} \sin 2(t - \pi)\right)\right) \\ & + \mathcal{U}(t - 3\pi) \left(\frac{1}{2} e^{-(t-3\pi)} \sin 2(t - 3\pi)\right) + \mathcal{U}(t - 4\pi) \left(\frac{1}{2} e^{-(t-4\pi)} \sin 2(t - 4\pi)\right), \quad t \geq 0. \end{aligned} \quad (1.54)$$

□