Linearization problems solutions

Problem 1.1. Consider the nonlinear system of equations

$$\begin{cases} x'(t) &= 1 - 2x(t)y(t) \\ y'(t) &= 2x(t)y(t) - y(t), \ t \in \mathbb{R}. \end{cases} \tag{1.1}$$

Find the critical point(s) of the system and classify the critical(s) of the system by their stability type, if possible.

Solution. We look for $\boldsymbol{X}_0 = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ satisfying

$$\begin{cases} 1 - 2xy = 0 \\ 2xy - y = 0. \end{cases} \tag{1.2}$$

The second equation implies either $y = 0$ or $x = \frac{1}{2}$. If $y = 0$, we arrive at a contradiction using the first equation, so we must have $x = \frac{1}{2}$ and $y = 1$. The linearization of the system around $\boldsymbol{X}_0$ is $\boldsymbol{X}' = A(\boldsymbol{X} - \boldsymbol{X}_0)$ where

$$A = \begin{pmatrix} -2y\Big|_{\substack{x=1/2 \\ y=1}} & -2x\Big|_{\substack{x=1/2 \\ y=1}} \\ 2y\Big|_{\substack{x=1/2 \\ y=1}} & 2x-1\Big|_{\substack{x=1/2 \\ y=1}} \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 2 & 0 \end{pmatrix}. \tag{1.3}$$

Note that $\det A = 2 > 0$ and $\operatorname{tr} A = -2 < 0$, which implies that the real parts of the eigenvalues of A are negative. Thus we can conclude that $\boldsymbol{X}_0$ is an asymptotically stable critical point. $\square$

Problem 1.2. Consider the nonlinear system of equations

$$\begin{cases} x'(t) &= \alpha x(t) - \beta y(t) + (y(t))^2 \\ y'(t) &= \beta x(t) + \alpha y(t) - x(t)y(t), \ t \in \mathbb{R}, \end{cases} \tag{1.4}$$

where $\alpha, \beta \in \mathbb{R}$.

a) Show that

$$\boldsymbol{X}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.5}$$

is a critical point of the system.

b) Classify $\boldsymbol{X}_0$ in terms of its stability type, if possible, for
- $\alpha > 0$
- $\alpha < 0$
- $\alpha = 0$.

Solution. We note that if $x = 0, y = 0$, then

$$\begin{cases} \alpha x - \beta y + y^2 = 0 \\ \beta x + \alpha y - xy = 0, \end{cases} \tag{1.6}$$

Therefore $\boldsymbol{X}_0$ is a critical point of the system. The linearization of the system around $\boldsymbol{X}_0$ is the linear system $\boldsymbol{X}' = A(\boldsymbol{X} - \boldsymbol{X}_0)$ for

$$A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}. \tag{1.7}$$

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Date: December 14, 2023.

Note that $\det A = \alpha^2 + \beta^2 \geq 0$ and $\operatorname{tr} A = 2\alpha$. If $\alpha > 0$, then $\det A > 0$ and $\operatorname{tr} A > 0$, so the real parts of the eigenvalues are positive. In this case the origin is unstable. If $\alpha < 0$, then $\det A > 0$ and $\operatorname{tr} A < 0$, so the origin is an asymptotically stable critical point. If $\alpha = 0$, then $\operatorname{tr} A = 0$, so the real parts of the eigenvalues are 0, and in this case we need further tools to investigate the stability of the origin. □

Problem 1.3. Consider an undamped mass-spring system where the spring force is nonlinear

$$\begin{cases} mx''(t) + kx(t) + k_1 x^3(t) = 0,\ t \in \mathbb{R} \\ x(0) = x_0, v_0(0) = v_0. \end{cases} \tag{1.8}$$

Consider the special case when $m = 1$, $k = 1$ and $k_1 = -1$. Since the equation is nonlinear, finding non-zero explicit solutions can be difficult, but we can try to use the techniques we learned to study the behavior of solutions when the initial conditions are small in a certain sense.

a) Define $\boldsymbol{X} : \mathbb{R} \to \mathbb{R}^2$ via

$$\boldsymbol{X}(t) = \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix}. \tag{1.9}$$

Write down an autonomous first-order nonlinear differential equation for $\boldsymbol{X}$ of the form $\boldsymbol{X}'(t) = \boldsymbol{f}(\boldsymbol{X(t)}),\ t \in \mathbb{R}$.

b) Find the critical points of the system and classify the critical points by their stability type, if possible.

Solution. We note that

$$\boldsymbol{X}'(t) = \begin{pmatrix} x'(t) \\ x''(t) \end{pmatrix} = \begin{pmatrix} x'(t) \\ (x(t))^3 - x(t) \end{pmatrix},\ t \in \mathbb{R}. \tag{1.10}$$

If we define $y : \mathbb{R} \to \mathbb{R}$ via $y(t) = x'(t)$, then this is equivalent to the nonlinear system

$$\begin{cases} x'(t) = y(t) \\ y'(t) = (x(t))^3 - x(t). \end{cases} \tag{1.11}$$

Therefore the critical points $\boldsymbol{X}_0 = \begin{pmatrix} x \\ y \end{pmatrix}$ of the system satisfy

$$\begin{cases} y = 0 \\ x^3 - x = x(x^2 - 1) = 0. \end{cases} \tag{1.12}$$

Here we find that the critical points are

$$\boldsymbol{X}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ \boldsymbol{X}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \boldsymbol{X}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \tag{1.13}$$

The linearization of the nonlinear system around $\boldsymbol{X}_i$ for each $0 \leq i \leq 2$ is the linear system $\boldsymbol{X}' = A_i(\boldsymbol{X} - \boldsymbol{X}_i)$, where

$$A_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ A_1 = A_2 = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}. \tag{1.14}$$

Since $\det A_0 = 1 > 0$ and $\operatorname{tr} A_0 = 0$, the real parts of the eigenvalues of A_0 are 0, so we cannot classify the stability of $\boldsymbol{X}_0$ with the methods we have developed. Since $\det A_1 = \det A_2 < 0$, both $\boldsymbol{X}_1, \boldsymbol{X}_2$ are saddle points, therefore they are both unstable.

What we learn from this is that if $x_0 \approx \pm 1$ and $v_0 \approx 0$, then we expect some solutions to diverge away from the two critical points $\boldsymbol{X}_1, \boldsymbol{X}_2$ as $t \to \infty$. However, we cannot say much about the global behavior of the solutions, as linearization can only provide us with local information around critical points. If we found that the critical points were asymptotically stable, then we can conclude that all starting sufficiently close to the critical points would converge to them as $t \to \infty$. □