Recitation 2

Problem 1.1. Solve the initial value problem

$$\begin{cases} xy'(x) + 3y(x) = x^3, \quad x > 0 \\ y(1) = 10. \end{cases} \tag{1.1}$$

What is the maximal interval of existence of the solution?

Solution. If y is a solution to the IVP, then

$$y'(x) + \frac{3}{x}y(x) = x^2 \text{ for all } x > 0. \tag{1.2}$$

We may choose the integrating factor μ to be $\mu(x) = \exp\left(\int \frac{3}{x}\right) = x^3, x > 0$. Then upon multiplying both sides of (1.2) by μ we arrive at

$$\frac{d}{dx}[x^3 y(x)] = x^3 y'(x) + 3x^2 y(x) = x^5, \ x > 0. \tag{1.3}$$

Then via direct integration we may deduce

$$x^3 y(x) = \frac{x^6}{6} + C, \ x > 0 \implies y(x) = \frac{x^3}{6} + Cx^{-3}, \ x > 0. \tag{1.4}$$

Now we use the initial condition to find C:

$$y(1) = \frac{1}{6} + C = 10 \implies C = \frac{59}{6}. \tag{1.5}$$

So the candidate solution corresponding to the initial condition $y(1) = 10$ is

$$y(x) = \frac{x^3}{6} + \frac{59}{6}x^{-3}, \ x > 0. \tag{1.6}$$

By the homework problem, we may skip the verification step. The maximal interval of existence is the interval $J = (0, \infty)$. □

Problem 1.2. Suppose we want to solve the problem

$$\begin{cases} xy'(x) + 3y(x) = x^3, \quad x \in \mathbb{R} \\ y(1) = \frac{1}{6}. \end{cases} \tag{1.7}$$

Is it possible to find a solution to the IVP with its maximal interval of existence being $J = \mathbb{R}$?

Solution. The answer is yes: if we take a look at (1.4), we see that we have a candidate solution y defined on $J = \mathbb{R}$ given by

$$y(x) = \frac{x^3}{6}, \ x \in \mathbb{R}. \tag{1.8}$$

We can verify that this is a solution directly. If y is defined via (1.8), then

$$xy'(x) + 3y(x) = x\left(\frac{x^2}{2}\right) + 3\left(\frac{x^3}{6}\right) = \frac{x^3}{2} + \frac{x^3}{2} = x^3, \ x \in \mathbb{R}. \tag{1.9}$$

Furthermore,

$$y(1) = \frac{1^3}{6} = \frac{1}{6}. \tag{1.10}$$

Therefore y is a solution to the IVP on $\mathbb{R}$ with its maximal interval of existence being $J = \mathbb{R}$. □

Problem 1.3. A vat initially (at $t = 0$) holds 100gal of pure water. A salt solution is added to the vat at a rate of 2gal/min. The concentration of this solution (in oz/gal) varies and is given by $c(t) = \frac{1}{2}t(50 - t)$. The (well mixed) solution is allowed to flow out of the vat at a rate of 4gal/min, and the vat becomes empty at some $T > 0$. Find an expression for the amount of salt in the vat as a function of t for $0 \le t < T$.

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Solution. Let $V(t)$ denote the amount of mixture in the vat for $t \geq 0$. $V(t)$ satisfies

$$V'(t) = r_{in} - r_{out} = 2 - 4,\ t > 0 \implies V(t) = 100 - 2t,\ t \geq 0. \tag{1.11}$$

This also tells us that nothing will be left in the vat at time $T = 50$. Let $Q(t)$ denote the amount of salt in the vat for $0 \leq t < 50$. Then Q satisfies

$$Q'(t) = r_{in}c(t) - r_{out}\frac{Q(t)}{V(t)} = t(50-t) - 4\frac{Q(t)}{100-2t},\ t \in (0, 50). \tag{1.12}$$

If If Q is a solution, then Q satisfies the first order linear equation

$$Q'(t) + \frac{4}{100-2t}Q(t) = t(50-t),\ t \in (0, 50). \tag{1.13}$$

We may choose an integrating factor to be

$$\mu(t) = \exp\left(\int \frac{4}{100-2t}\ dt\right) = (100-2t)^{-2},\ t \in (0, 50). \tag{1.14}$$

Now multiplying both sides of the linear equation by μ gives us

$$\frac{d}{dt}\left[(100-2t)^{-2}Q(t)\right] = (100-2t)^{-2}t(50-t) = \frac{1}{4}\frac{t}{50-t},\ t \in (0, 50). \tag{1.15}$$

Then via direct integration we find

$$(100-2t)^{-2}Q(t) = \frac{1}{4}\int -1 + \frac{50}{50-t}\ dt = \frac{1}{4}\left(-t - 50\ln(50-t)\right) + C,\ t \in (0, 50). \tag{1.16}$$

Using the intitial condition $Q(0) = 0$ we find that

$$0 = -\frac{1}{4}\left(50\ln 50\right) + C \implies C = \frac{25}{2}\ln 50. \tag{1.17}$$

So Q is given by

$$Q(t) = (100-2t)^2\left(-\frac{1}{4}(1 + 50\ln(50-t)) + \frac{25}{2}\ln 50\right),\ t \in [0, 50). \tag{1.18}$$

$\square$