Recitation 3

Problem 1.1 (Existence and uniqueness)**.** Consider the initial value problem

$$\begin{cases} y'(t) = 5(y(t))^{\frac{4}{5}},\ t \in \mathbb{R} \\ y(2) = 1 \end{cases} \tag{1.1}$$

a) Find a candidate solution to the initial value problem. What is the maximal interval of existence of the solution?
b) Explain why there exists an interval $J = (a, b)$ with $a < 2 < b$ on which the solution in part (a) is the unique solution satisfying the given initial condition.
c) Is existence and uniqueness guaranteed for the initial condition were replaced with $y(0) = 0$?

Part a). First, we note that $y(t) = 0$ for all $t \in \mathbb{R}$ is a constant solution to the system, but it does not satisfy the initial value $y(2) = 1$. Suppose y is a solution that is not the zero solution to the initial value problem. Then there exists an interval I for which $y(t) \neq 0$ for all $t \in I$, and therefore y satisfies

$$(y(t))^{-4/5} y'(t) = 5,\ t \in I. \tag{1.2}$$

This implies the identity

$$\int y^{-4/5} y \, dy = \int 5 \, dt. \tag{1.3}$$

Thus we may conclude that

$$(y(t))^{1/5} = t + C, \tag{1.4}$$

where C is an arbitrary constant. Since $y(2) = 1$, we see that $C = -1$, therefore the candidate solution is

$$y(t) = (t-1)^5,\ t \in \mathbb{R}. \tag{1.5}$$

One may verify that this is a solution to the initial value problem, and the maximal interval of existence is $J = \mathbb{R}$. $\square$

Part b). We note that the equation is in the form of $y'(t) = f(t, y(t))$ for

$$f(t, y) = 5y^{4/5},\ (t, y) \in \mathbb{R}^2. \tag{1.6}$$

We may then calculate

$$\frac{\partial f}{\partial y}(t, y) = 4(y)^{-1/5},\ t \in \mathbb{R}, y \neq 0. \tag{1.7}$$

Since f and $\frac{\partial f}{\partial y}$ are continuous in a small neighborhood of $(2, 1)$, there exists an interval J containing $t_0 = 2$ for which the solution identified in part a) is unique.

$\square$

Part c). We note that f is continuous in any neighborhood of $(0, 0)$, but not $\frac{\partial f}{\partial y}$. This means that the existence of a solution is guaranteed, but not uniqueness. We can see this explicitly: $y(t) = 0$ for all $t \in \mathbb{R}$, is one solution to the IVP, but so is

$$y(t) = \begin{cases} 0,\ t \leq a \\ (t-a)^5, t > a \end{cases} \tag{1.8}$$

for any $a \geq 0$. $\square$

Department of Mathematical Sciences, Carnegie Mellon University
E-mail address: jkoganem@andrew.cmu.edu.
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Problem 1.2 (Curve sketching with isoclines)**.** Use the method of isoclines to draw the slope field for the differential equation

$$y'(t) = \frac{1}{4}t^2 + (y(t))^2 - 1, \ t \in \mathbb{R}. \tag{1.9}$$

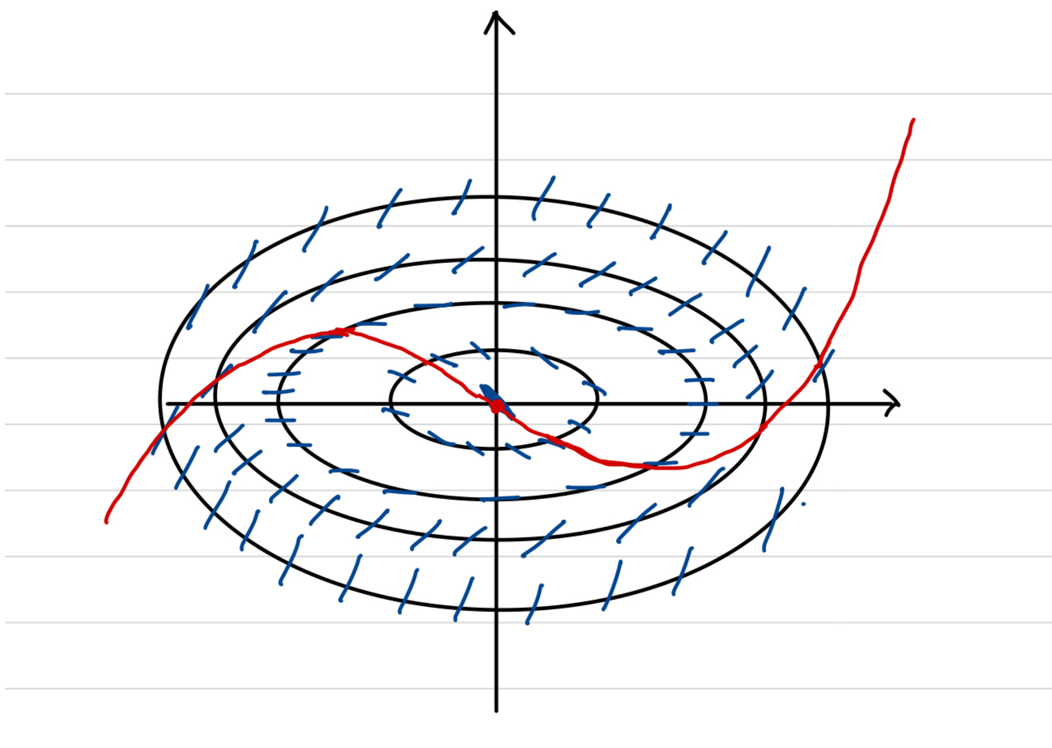
Sketch the solution curve that passes through the point $(0, 0)$.

Solution. We note that for any $m \in \mathbb{R}$,

$$\frac{1}{4}t^2 + y^2 - 1 = m \iff \frac{1}{4}t^2 + y^2 = m + 1. \tag{1.10}$$

So the isoclines for this equation are ellipses centered at the origin, and as the ellipses get larger the slopes increase. Also note that at $(0, 0)$, $y'(0) = -1$.

Here is a very rough sketch:



□

Problem 1.3 (Autonomous differential equations)**.** Consider the differential equation

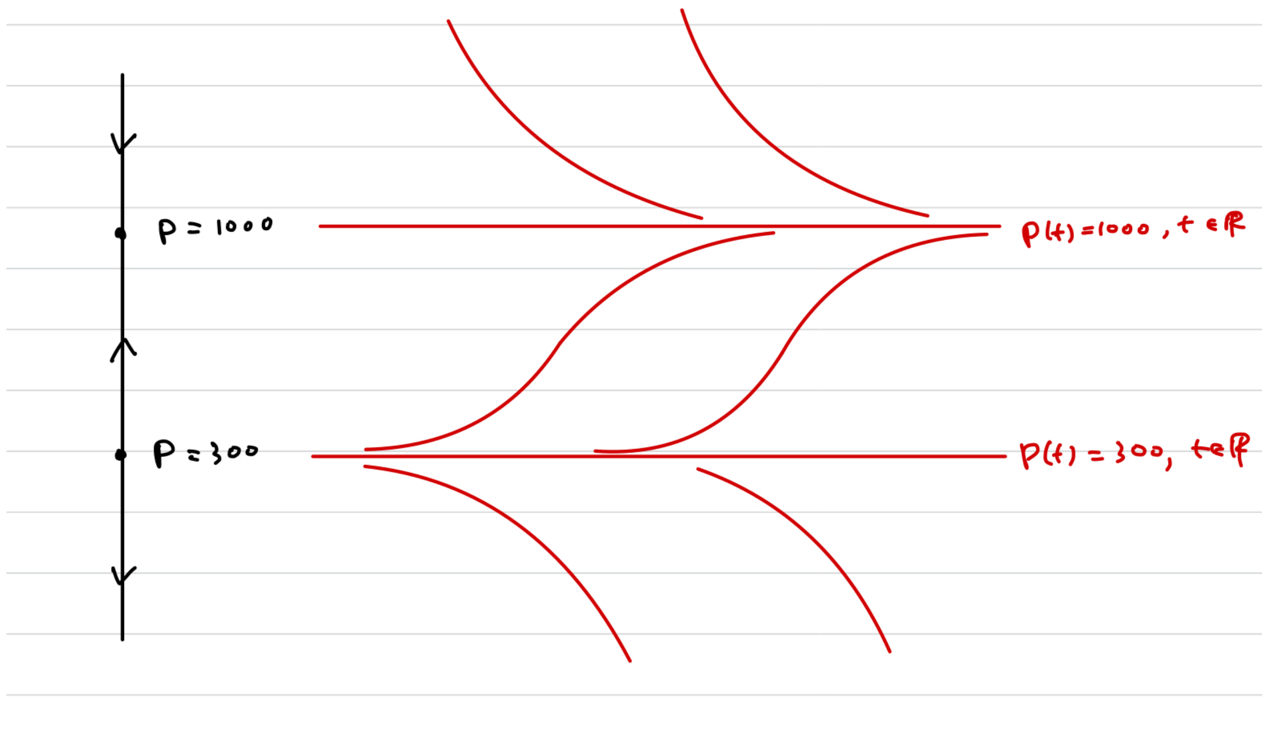
$$P'(t) = \left(\frac{P(t)}{300} - 1\right)\left(1 - \frac{P(t)}{1000}\right),\ t \in \mathbb{R}$$

which models the behavior of a certain population.

a) Draw the phase line for this differential equation. Indicate the stability of each equilibrium point.
b) Use information from the phase line to sketch some representative solution curves (i.e. curves in the t-P plane. Be sure to show each type of qualitative behavior.
c) What is the significance, in terms of the population dynamics, of the numbers 300 and 1000?

Part a). The equilibrium points are $P = 300$ and $P = 1000$. The right hand side is a quadratic in P, with the coefficient in front of the P^2 term being negative. Thus $P' > 0$ between the equilibrium points and $P' < 0$ away from them. Thus $P = 1000$ is stable and $P = 300$ is unstable. ☐

Part b). Here is a very rough sketch:



☐

Part c). If the initial population $P_0 < 300$ then the population dies out. If $300 < P_0 < 1000$ then the population will increase until it is arbitrarily close to an equilibrium population of $P = 1000$. If $P_0 > 1000$, then the population will decrease and eventually be arbitrarily close to an equilibrium population of $P = 1000$. ☐

Problem 1.4 (Berunoulli differential equations)**.** Consider the differential equation

$$y'(t) + y(t) = (y(t))^2,\ t \in \mathbb{R}. \tag{1.11}$$

(1) If $y(0) = 0$, is there a unique solution to the IVP?
(2) Find the general solution to the equation (not the IVP) on some interval I.

Part a. The equation can be written as

$$y'(t) = (y(t))^2 - y(t) = f(t, y(t)), \tag{1.12}$$

where

$$f(t, y) = y^2 - y,\ (t, y) \in \mathbb{R}^2. \tag{1.13}$$

Since

$$\frac{\partial f}{\partial y}(t, y) = 2y - 1,\ (t, y) \in \mathbb{R}^2, \tag{1.14}$$

we see that $f, \frac{\partial f}{\partial y}$ are continuous on $\mathbb{R}^2$. Therefore there exists a unique solution to the IVP. In fact, by inspection we see that $y(t) = 0$ for all $t \in \mathbb{R}$ is a trivial solution, and by uniqueness it is the unique solution to the IVP. □

Part b). This is a Bernoulli differential equation with $\alpha = 2$. We note that $y(t) = 0$ for all $t \in \mathbb{R}$ is a constant solution to the system. Suppose y is a solution that is not the zero solution. Then there exists an interval I for which $y(t) \neq 0$ for all $t \in I$. On this interval we may define the function v via

$$v(t) = (y(t))^{1-2} = (y(t))^{-1}, t \in I, \tag{1.15}$$

and thus

$$v'(t) = -(y(t))^{-2} y'(t), t \in I. \tag{1.16}$$

We note that y satisfies

$$(y(t))^{-2} y'(t) + (y(t))^{-1} = 1,\ t \in I, \tag{1.17}$$

therefore v satisfies the equation

$$-v'(t) + v(t) = 1,\ t \in I, \tag{1.18}$$

or

$$v'(t) - v(t) = -1. \tag{1.19}$$

From here we see that

$$\frac{d}{dt}[e^{-t} v(t)] = -e^{-t},\ t \in I, \tag{1.20}$$

thus

$$v(t) = 1 + Ce^t,\ t \in I, \tag{1.21}$$

where C is arbitrary. Thus the candidate general solutions are,

$$y(t) = 0,\ t \in \mathbb{R} \text{ or } y(t) = \frac{1}{1 + Ce^t},\ t \in I \tag{1.22}$$

for some interval I, and C is arbitrary.

One may verify that the second family of solutions are solutions to the differential equation on any interval I that avoids the value of t for which $1 + Ce^t = 0$. If $C \geq 0$, then the maximal interval of existence is $J = \mathbb{R}$. □

Problem 1.5 (ODE with homogeneous functions)**.** Consider the differential equation

$$t(y(t))^2 y'(t) = (y(t))^3 + t^3,\ t > 0. \tag{1.23}$$

Find the general candidate solution to the equation on some interval I.

Solution. First we note that this equation does not admit the zero solution. Suppose y is a solution. Then there exists an interval $I \subseteq (0, \infty)$ for which $y(t) \neq 0$ for all $t \in I$. Thus

$$y'(t) = \frac{y(t)}{t} + \frac{t^2}{(y(t))^2},\ t \in I. \tag{1.24}$$

Consider the function v defined on I via

$$v(t) = \frac{y(t)}{t},\ t \in I. \tag{1.25}$$

Then

$$y'(t) = v(t) + tv'(t),\ t \in I. \tag{1.26}$$

Thus v satisfies the differential equation

$$v(t) + tv'(t) = v(t) + \frac{1}{(v(t))^2},\ t \in I, \tag{1.27}$$

or

$$v'(t) = \frac{1}{t(v(t))^2},\ t \in I. \tag{1.28}$$

We note that this equation does not admit any constant solutions, thus we may rewrite the equation as

$$(v(t))^2 v'(t) = \frac{1}{t},\ t \in I \tag{1.29}$$

to arrive at the identity

$$\int v^2\, dv = \int \frac{1}{t}\, dt, \tag{1.30}$$

which implies that

$$\frac{1}{3}(v(t))^3 = \ln|t| + C = \ln t + C,\ t \in I, \tag{1.31}$$

where C is arbitrary. Thus

$$v(t) = (3\ln t + C)^{1/3},\ t \in I, C \in \mathbb{R}. \tag{1.32}$$

This implies that a candidate solution to the original equation is

$$y(t) = t(3\ln t + C)^{1/3},\ t \in I, \tag{1.33}$$

where C is arbitrary. $\square$