Recitation 7

Differentiation in the s-domain

Proposition. Let $f : [0, \infty) \to \mathbb{R}$ be a smooth function of exponential order. Then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s) \tag{7.1}$$

for all s for which F is smooth and any $\mathbb{Z} \ni n \geq 0$.

Justification. By an application of the dominated convergence theorem (outside the scope of this class), the assumptions on f allows us to exchange the order of differentiation and integration and write

$$\frac{d^n}{ds^n} F(s) = \frac{d^n}{ds^n} \int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty \frac{d^n}{ds^n} e^{-st} f(t)\, dt = \int_0^\infty e^{-st} (-t)^n f(t)\, dt = (-1)^n \mathcal{L}\left\{t^n f(t)\right\}(s), \tag{7.2}$$

for all s such that F is smooth. □

Remark 7.1. This is saying that multiplying by powers polynomial terms in the t-domain correspond to differentiating in the s-domain, which is the analogue of the correspondence between taking derivatives in the t-domain and multiplying by polynomial terms in the s-domain.

Proposition. Under the same assumptions as the previous proposition,

$$\mathcal{L}^{-1}\left\{\frac{d^n}{ds^n} F(s)\right\} = (-1)^n t^n f(t). \tag{7.3}$$

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Problems

Problem 7.2. Find the unique solution to the IVP

$$\begin{cases} y''(t) + 4y(t) = \sin t, \ t \geq 0 \\ y(0) = 1, y'(0) = 1 \end{cases} \tag{7.4}$$

using the Laplace transform.

Solution. If y is a solution to the IVP, then

$$\mathcal{L}\left\{y''(t) + 4y(t)\right\} = \mathcal{L}\left\{\sin t\right\}, \tag{7.5}$$

for all appropriate values of s, where

$$\mathcal{L}\left\{y'' + 4y\right\} = s^2 Y(s) - s - 1 + 4Y(s) \tag{7.6}$$

$$\mathcal{L}\left\{\sin t\right\} = \frac{1}{s^2+1}. \tag{7.7}$$

This implies that

$$Y(s) = \frac{1}{(s^2+1)(s^2+4)} + \frac{s+1}{s^2+4}. \tag{7.8}$$

Using partial fraction decomposition we look for constants A, B, C, D such that

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} \tag{7.9}$$

for all $s \in \mathbb{R}$. It is then sufficient to find A, B, C, D satisfying

$$1 = (As+B)(s^2+4) + (Cs+D)(s^2+1) \tag{7.10}$$

$$= (A+C)s^3 + (B+D)s^2 + (4A+C)s + (4B+D) \tag{7.11}$$

for all $s \in \mathbb{R}$. This implies

$$\begin{cases} A + C &= 0 \\ B + D &= 0 \\ 4A + C &= 0 \\ 4B + D &= 1. \end{cases} \tag{7.12}$$

Here we find that $A = C = 0$ and $B = \frac{1}{3}, D = -\frac{1}{3}$. Thus

$$Y(s) = \frac{1}{3}\frac{1}{s^2+1} - \frac{1}{3}\frac{1}{s^2+4} + \frac{s}{s^2+4} + \frac{1}{s^2+4} \tag{7.13}$$

$$= \frac{1}{3}\frac{1}{s^2+1} - \frac{1}{6}\frac{2}{s^2+4} + \frac{s}{s^2+4} + \frac{1}{2}\frac{2}{s^2+4} \tag{7.14}$$

for appropriate values of s, and thus

$$y(t) = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t + \cos 2t + \frac{1}{2}\sin 2t \tag{7.15}$$

$$= \frac{1}{3}\sin t + \frac{1}{3}\sin 2t + \cos 2t \tag{7.16}$$

for $t \geq 0$. $\square$

Problem 7.3. Define $f : [0, \infty) \to \mathbb{R}$ via

$$f(t) = t \cos t. \tag{7.17}$$

Find the Laplace transform of f.

Solution. Using Proposition ,

$$\mathcal{L}\left\{t \cos t\right\}(s) = -\frac{d}{ds}\mathcal{L}\left\{\cos t\right\}(s) = -\frac{d}{ds}\frac{s}{s^2+1} = \frac{s^2-1}{(s^2+1)^2},\ s > 0. \tag{7.18}$$

$\square$

Problem 7.4. Define $f : [0, \infty) \to \mathbb{R}$ via

$$f(t) = t \sin t. \tag{7.19}$$

Find the Laplace transform of f.

Solution. Using Proposition ,

$$\mathcal{L}\left\{t \sin t\right\}(s) = -\frac{d}{ds}\mathcal{L}\left\{\sin t\right\}(s) = -\frac{d}{ds}\frac{1}{s^2+1} = \frac{2s}{(s^2+1)^2},\ s > 0. \tag{7.20}$$

□

Problem 7.5. Find

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}. \tag{7.21}$$

Solution. We note that

$$\begin{aligned}
\frac{1}{(s^2+1)^2} &= \frac{s^2+1-s^2}{(s^2+1)^2} \\
&= \frac{1}{s^2+1} - \frac{s^2}{(s^2+1)^2} \\
&= \frac{1}{s^2+1} - \frac{s^2-1+1}{(s^2+1)^2} \\
&= \frac{1}{s^2+1} - \frac{s^2-1}{(s^2+1)^2} - \frac{1}{(s^2+1)^2} \\
\implies 2\frac{1}{(s^2+1)^2} &= \frac{1}{s^2+1} + \frac{1-s^2}{(s^2+1)^2} \\
\implies \frac{1}{(s^2+1)^2} &= \frac{1}{2}\frac{1}{s^2+1} + \frac{1}{2}\underbrace{\frac{1-s^2}{(s^2+1)^2}}_{=\frac{d}{ds}\mathcal{L}\{\cos(t)\}}.
\end{aligned} \tag{7.22}$$

This implies that

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} &= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1-s^2}{(s^2+1)^2}\right\} \\
&= \frac{1}{2}\sin(t) - \frac{1}{2}t\cos(t).
\end{aligned} \tag{7.23}$$

$\square$

Problem 7.6. Find the unique solution to

$$\begin{cases} y''(t) + y(t) = \sin t, \ t \geq 0 \\ y(0) = y'(0) = 0. \end{cases} \tag{7.24}$$

using the Laplace transform.

Solution. If y is a solution to the IVP, then

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{\sin t\}, \tag{7.25}$$

or

$$(s^2 + 1)Y(s) = \frac{1}{s^2 + 1}, \tag{7.26}$$

which implies that

$$Y(s) = \frac{1}{(s^2 + 1)^2} \tag{7.27}$$

for appropriate values of t. By the previous problem,

$$y(t) = \frac{1}{2}\sin t - \frac{1}{2}t\cos t \tag{7.28}$$

for $t \geq 0$. □