Week 12

Problem 12.1. Let $\mathbb{R} \ni L > 0$. Recall that the L^2 inner product on an interval $I \subseteq \mathbb{R}$ is defined via

$$\langle f, g\rangle = \int_I f(x)g(x)\,dx. \tag{12.1}$$

The *norm* (analogue of the length of a vector in $\mathbb{R}^n$) of a function $f \in L^2(I;\mathbb{R})$ is defined as

$$\|f\|_{L^2} = \sqrt{\langle f, f\rangle} = \sqrt{\int_I f(x)^2\,dx}. \tag{12.2}$$

a) Show that for any constants $c_1, c_2 \in \mathbb{R}$, we have

$$\langle c_1 f, c_2 g\rangle = c_1 c_2 \langle f, g\rangle. \tag{12.3}$$

b) Show that if $\|f\|_{L^2} = c \neq 0$, then $\left\|\frac{1}{c} f\right\|_{L^2} = 1$.

Solution. For the first item, we note that by definition,

$$\langle c_1 f, c_2 g\rangle = \int_I (c_1 f(x))(c_2 g(x))\,dx = c_1 c_2 \int_I f(x)g(x)\,dx = c_1 c_2 \langle f, g\rangle. \tag{12.4}$$

For the second item, we can use the first item to show that

$$\left\|\frac{1}{c} f\right\|_{L^2}^2 = \left\langle \frac{1}{c} f, \frac{1}{c} f\right\rangle = \frac{1}{c^2}\langle f, f\rangle = \frac{1}{c^2}\|f\|_{L^2}^2 = \frac{1}{c^2} c^2 = 1 \implies \left\|\frac{1}{c} f\right\|_{L^2} = 1. \tag{12.5}$$

$\square$

Remark 12.2. Using some techniques from real analysis, one can show that $\|f\|_{L^2} = 0$ if and only if f is the zero function over I if we assume that f is continuous, so the second item is saying that for any non-zero continuous function, one can divide by its norm to obtain a function with norm 1. This process is known as *normalizing* the function, and the resulting function is known as the *normalization* of the original function.

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Date: April 16, 2025.

Problem 12.3. Let $\mathbb{R} \ni L > 0$. Recall that the L^2 inner product on an interval $I \subseteq \mathbb{R}$

$$\langle f, g \rangle = \int_I f(x)g(x)\, dx, \tag{12.6}$$

A set of functions $\{f_n\}_{n=1}^{\infty} \subset L^2(I; \mathbb{R})$ is said to be *orthogonal* over I if $\langle f_m, f_n \rangle = 0$ for $m \neq n$, and is said to be *orthonormal* over I if

$$\langle f_m, f_n \rangle = \begin{cases} 1, & \text{if } m = n, \\ 0, & \text{if } m \neq n. \end{cases} \tag{12.7}$$

Show that the set

$$\{\sin x, \sin 2x, \sin 3x, \ldots\} \subset L^2([0, \pi]; \mathbb{R}) \tag{12.8}$$

is orthogonal but not orthonormal. How can we make it orthonormal?

Solution. Let $m, n \in \mathbb{Z}^+$. If $m \neq n$, we compute

$$\begin{aligned} \langle \sin(m\cdot), \sin(n\cdot) \rangle &= \int_0^{\pi} \sin(mx)\sin(nx)\, dx \\ &= \frac{1}{2}\int_0^{\pi} \left(\cos((m-n)x) - \cos((m+n)x)\right)\, dx = \frac{1}{2}\left(\frac{\sin((m-n)x)}{m-n} - \frac{\sin((m+n)x)}{m+n}\right)\Bigg|_0^{\pi} = 0, \end{aligned} \tag{12.9}$$

so the set is orthogonal. If $m = n$, then

$$\langle \sin(m\cdot), \sin(n\cdot) \rangle = \int_0^{\pi} \sin^2(mx)\, dx = \frac{1}{2}\int_0^{\pi} (1 - \cos(2mx))\, dx = \frac{1}{2}\left(x - \frac{\sin(2mx)}{2m}\right)\Bigg|_0^{\pi} = \frac{\pi}{2}, \tag{12.10}$$

so the set is not orthonormal. $\square$

Remark 12.4. The calculation above shows that

$$\|\sin(m\cdot)\|_{L^2}^2 = \frac{\pi}{2} \iff \|\sin(m\cdot)\|_{L^2} = \sqrt{\frac{\pi}{2}} \quad \forall m \in \mathbb{Z}^+, \tag{12.11}$$

which means that we can normalize sine functions to obtain

$$\left\| \sqrt{\frac{2}{\pi}} \sin(m\cdot) \right\|_{L^2} = 1 \quad \forall m \in \mathbb{Z}^+. \tag{12.12}$$

This means that the set

$$\left\{ \sqrt{\frac{2}{\pi}} \sin(m\cdot) \right\}_{m=1}^{\infty} = \left\{ \sqrt{\frac{2}{\pi}} \sin x, \sqrt{\frac{2}{\pi}} \sin 2x, \sqrt{\frac{2}{\pi}} \sin 3x, \ldots \right\} \subset L^2([0, \pi]; \mathbb{R}) \tag{12.13}$$

is orthonormal.

Problem 12.5. Recall that for a function $f \in L^2([0,2\pi];\mathbb{R})$, its Fourier series is given by

$$\frac{a_0}{2} + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x) + \dots, \quad x \in [0,2\pi], \tag{12.14}$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx, \quad n \geq 0, \tag{12.15}$$

and

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx, \quad n \geq 1. \tag{12.16}$$

Consider the function $f : [0,2\pi] \to \mathbb{R}$ defined via

$$f(x) = 2 + \sin(2x) + \cos(3x). \tag{12.17}$$

What are the Fourier coefficients of f?

Solution. Since f is already a finite linear combination of sine and cosine functions, we can read off the Fourier coefficients directly without any computations. We see immediately that

$$a_n = \begin{cases} 1, & n = 0, \\ 3, & n = 3, \\ 0, & \text{otherwise}, \end{cases} \tag{12.18}$$

and

$$b_n = \begin{cases} 2, & n = 2, \\ 0, & \text{otherwise}. \end{cases} \tag{12.19}$$

□

Problem 12.6. Consider the function $f : [0, 2\pi] \to \mathbb{R}$ defined via

$$f(x) = \begin{cases} 0, & \text{if } x \in [0, \pi), \\ 1, & \text{if } x \in [\pi, 2\pi). \end{cases} \tag{12.20}$$

Find the Fourier coefficients of f. What does the Fourier series of f converge to at $x = \pi$?

Solution. We compute

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_\pi^{2\pi} 1\,dx = \frac{1}{\pi}(2\pi - \pi) = 1, \tag{12.21}$$

and

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_\pi^{2\pi} 1 \cdot \cos(nx)\,dx = \frac{1}{\pi}\left(\frac{\sin(nx)}{n}\right)\Bigg|_\pi^{2\pi} = 0,\ n \geq 1, \tag{12.22}$$

and

$$\begin{aligned} b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx &= \frac{1}{\pi}\int_\pi^{2\pi} 1 \cdot \sin(nx)\,dx \\ &= \frac{1}{\pi}\left(-\frac{\cos(nx)}{n}\right)\Bigg|_\pi^{2\pi} = -\frac{1}{n\pi}(\cos(2n\pi) - \cos(n\pi)),\ n \geq 1. \end{aligned} \tag{12.23}$$

We note that

$$\cos(2n\pi) = 1 \text{ and } \cos(n\pi) = (-1)^n,\ n \in \mathbb{Z}^+, \tag{12.24}$$

therefore

$$b_n = -\frac{1}{n\pi}(1 - (-1)^n) = \begin{cases} 0, & n \text{ even}, \\ -\frac{2}{n\pi}, & n \text{ odd}. \end{cases} \tag{12.25}$$

Therefore the Fourier series of f is

$$\frac{1}{2} + \sum_{n \text{ odd}} \left(-\frac{2}{n\pi}\right)\sin(nx) = \frac{1}{2} - \frac{2}{\pi}\sum_{n \text{ odd}} \frac{1}{n}\sin(nx). \tag{12.26}$$

By the Fourier convergence theorem, the Fourier series converges to the average of the left limit and right limit at $x = 1$, which is $\frac{1}{2}$. This also follows directly from (12.26), since the sine terms vanish at $x = \pi$.

□