Homework 2

DUE: SATURDAY, FEBRUARY 1, 2025, 11:59 PM

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Date: February 2, 2025.

Problem 1.1 (Separable equations)**.**

Consider the initial value problem

$$\begin{cases} e^{y(x)-x}y'(x) - e^{x-y(x)} = 0, \ x \in \mathbb{R} \\ y(\ln 2) = \frac{\ln 3}{2}. \end{cases} \tag{1.1}$$

Suppose y is a solution over $\mathbb{R}$.

a) Show that y satisfies a separable differential equation of the form

$$g(y(x))y'(x) = f(x), \ x \in \mathbb{R}, \tag{1.2}$$

for some functions f, g.

b) Proceed to identify a candidate implicit solution to the initial value problem in the form of

$$G(x, y(x)) = 0, \tag{1.3}$$

where x belongs to some interval I. You do not need to specify the interval I.

c) Suppose J is the maximal interval of existence of the candidate implicit solution to the IVP identified in part b). Show that it is not possible for $J = \mathbb{R}$ by considering what happens if $x = 0$.

Part a). If y is a solution to the original equation, then

$$e^{y(x)-x}y'(x) = e^{x-y(x)}, \ x \in \mathbb{R} \implies e^{2y(x)}y'(x) = e^{2x}, \ x \in \mathbb{R}. \tag{1.4}$$

□

Part b). First we note that the equation does not admit any constant solutions. Then via direct integration we find that we have the identity

$$\frac{1}{2}e^{2y} = \frac{1}{2}e^{2x} + C. \tag{1.5}$$

We assume that this implies that there exists an interval I containing $\ln 2$ for which y is a differentiable function of x and

$$e^{2y(x)} - e^{2x} = C, \ x \in I. \tag{1.6}$$

We note that if y satisfies $y(\ln 2) = \ln 3/2$, then

$$C = e^{\ln 3} - e^{2\ln 2} = 3 - 4 = -1. \tag{1.7}$$

Thus the candidate implicit solution is

$$e^{2y(x)} - e^{2x} + 1 = 0, \ x \in I \tag{1.8}$$

for some interval I containing $\ln 2$. □

Part c). We note that if the maximal interval of existence contains 0, then we must have the identity

$$e^{2y(0)} - e^{2(0)} + 1 = 0 \implies e^{2y(0)} = 0. \tag{1.9}$$

However, since the exponential function is always positive, this is not possible. Therefore J cannot contain 0. □

Problem 1.2 (Structure of solutions to first order linear equations)**.**

Suppose the general solution to a first order linear differential equation

$$y'(x) + P(x)y(x) = f(x),\ x \in \mathbb{R} \tag{1.10}$$

is given by

$$y(x) = \sin(x) - 1 + Ce^{-\sin(x)},\ x \in \mathbb{R} \tag{1.11}$$

for an arbitrary constant C.

a) What is the general solution to the homogeneous problem

$$y'(x) + P(x)y(x) = 0? \tag{1.12}$$

b) Identify any two particular solutions (non-parametrized) to the inhomogeneous problem

$$y'(x) + P(x)y(x) = f(x),\ x \in \mathbb{R}. \tag{1.13}$$

c) What is P?
d) What is f?

Part a). From the structure of the general solution to the inhomogeneous problem, we see that the general solution to the homogeneous problem is

$$y_h(x) = Ce^{-\sin(x)},\ x \in \mathbb{R} \tag{1.14}$$

for an arbitrary constant C. This is because this is the only part of the general solution of the inhomogeneous problem that can be scaled. □

Part b). Likewise, we see that one particular solution is

$$y_p(x) = \sin(x) - 1,\ x \in \mathbb{R}, \tag{1.15}$$

since this is the part of the general solution to the inhomogeneous problem that cannot be scaled. We can also add copies of the homogeneous solution to form another particular solution. For example, by setting $C = 1$, we can produce another particular solution

$$y_p(x) = \sin(x) - 1 + e^{-\sin(x)}. \tag{1.16}$$

□

Part c). Since $y(x) = e^{-\sin(x)}, x \in \mathbb{R}$ is a solution to the homogeneous problem, we have

$$y'(x) = -\cos(x)e^{-\sin(x)} = -\cos(x)y(x),\ x \in \mathbb{R}. \tag{1.17}$$

Therefore

$$P(x) = \cos(x),\ x \in \mathbb{R}. \tag{1.18}$$

□

Part d). Since $y(x) = \sin(x) - 1, x \in \mathbb{R}$ is a solution to the inhomogeneous problem, we have

$$y'(x) + \cos(x)y(x) = \cos(x) + \sin(x)\cos(x) - \cos(x) = \sin(x)\cos(x),\ x \in \mathbb{R}. \tag{1.19}$$

Therefore

$$f(x) = \sin(x)\cos(x),\ x \in \mathbb{R}. \tag{1.20}$$

□

Problem 1.3 (Autonomous equations and stability analysis)**.**

Consider the autonomous differential equation

$$y'(t) = y(t)(10 - y(t)),\ t \in \mathbb{R}. \tag{1.21}$$

a) Consider the function h defined via

$$h(y) = y(10 - y),\ y \in \mathbb{R}. \tag{1.22}$$

Identify the critical points of h (these are points where $h(y) = 0$) and draw a sign chart of h.

b) What are the constant solutions admitted by the equation?

c) Sketch a one-dimensional phase portrait corresponding to the equation and classify the critical points by stability found in part a).

d) Sketch a sample family of solution curves in the t-y plane.

e) Suppose y is a solution to the differential equation and $y(0) = 4$. Use the sketch in part c) to identify

$$\lim_{t\to\infty} y(t) \text{ and } \lim_{t\to-\infty} y(t). \tag{1.23}$$

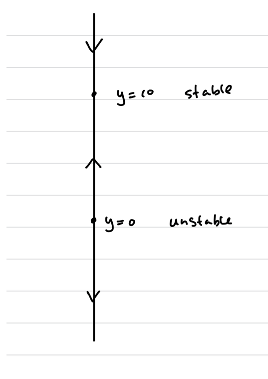
Part a). The critcal points of h are $y = 0$ and $y = 10$. Since the leading coefficient of the quadratic term in h is negative, we can deduce that h is negative when $y < 0$ and $y > 10$, and h is positive when $0 < y < 10$. □

Part b). The constant solutions admitted by the equation are

$$y(t) = 0,\ t \in \mathbb{R} \text{ and } y(t) = 10,\ t \in \mathbb{R}. \tag{1.24}$$

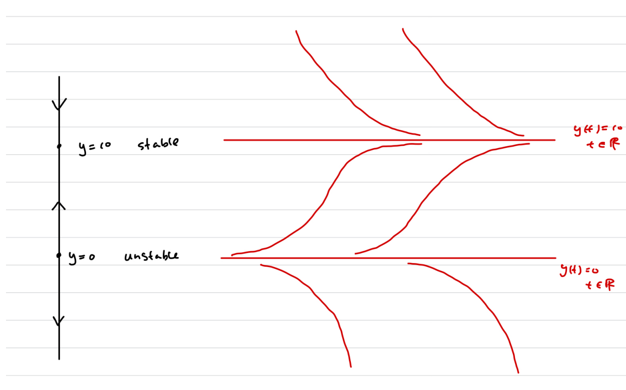
□

Part c). Here is a very rough sketch:



We see that the critical point $y = 10$ is stable, whereas the critical point $y = 0$ is unstable. □

Part d). Here is a very rough sketch:



□

Part e). From the previous parts we see that if $y(0) = 4$, since $0 < 4 < 10$, we may conclude that

$$\lim_{t\to\infty} y(t) = 10, \ \lim_{t\to-\infty} y(t) = 0. \tag{1.25}$$

□

Problem 1.4 (Bernoulli differential equations)**.**

Consider first order differential equations of the form

$$y'(x) + P(x)y(x) = Q(x)(y(x))^{\alpha},\ x \in \mathbb{R}, \tag{1.26}$$

where $\alpha \in \mathbb{R}$. If $\alpha = 0$ or $\alpha = 1$, then this is a linear equation, and we may identify the solution via the method of integrating factors. When $\alpha \neq 0, 1$, the equation is nonlinear, so the method of integrating factors cannot be applied. In this problem we explore how to reduce the original equation to a simpler equation via substitution.

Suppose $\alpha \neq 0, 1$. Note that if $\alpha > 0$, then the equation admits the zero solution $y(x) = 0,\ x \in \mathbb{R}$. Suppose y is a solution to the equation on $\mathbb{R}$ and there exists an interval I for which $y(x) \neq 0$ for all $x \in I$.

a) Show that y solves the equation

$$(y(x))^{-\alpha}y'(x) + P(x)(y(x))^{1-\alpha} = Q(x),\ x \in I. \tag{1.27}$$

b) Define the function v via

$$v(x) = (y(x))^{1-\alpha},\ x \in I. \tag{1.28}$$

Show that

$$\frac{1}{1-\alpha}v'(x) = (y(x))^{-\alpha}y'(x),\ x \in I. \tag{1.29}$$

c) Show that v solves the equation

$$v'(x) + (1-\alpha)P(x)v(x) = (1-\alpha)Q(x),\ x \in I. \tag{1.30}$$

d) What is the order of the equation (1.30)? Is it a linear equation?
e) Find the general candidate solution to the differential equation

$$y'(x) + xy(x) = x(y(x))^3,\ x \in \mathbb{R}. \tag{1.31}$$

Part a). Since $y(x) \neq 0$ for $x \in I$, we may divide the original equation by y^{α} to deduce that

$$(y(x))^{-\alpha}y'(x) + P(x)(y(x))^{1-\alpha} = Q(x),\ x \in I. \tag{1.32}$$

□

Part b). By the power rule and chain rule,

$$v'(x) = (1-\alpha)(y(x))^{-\alpha}y'(x),\ x \in I, \tag{1.33}$$

and by assumption $\alpha \neq 1$, therefore

$$\frac{1}{1-\alpha}v'(x) = (y(x))^{-\alpha}y'(x),\ x \in I. \tag{1.34}$$

□

Part c). We note that via substitution, we see that v solves the equation

$$\frac{1}{1-\alpha}v'(x) + P(x)v(x) = Q(x),\ x \in I. \tag{1.35}$$

Therefore v also solves the equation

$$v'(x) + (1-\alpha)P(x)v(x) = (1-\alpha)Q(x),\ x \in I. \tag{1.36}$$

□

Part d). We see that v solves a first order linear equation, since the equation is of the form

$$a_1(x)v'(x) + a_0(x)v(x) = g(x),\ x \in I. \tag{1.37}$$

□

Part e). We follow the idea outlined in the previous steps. We suppose y is a solution to the original equation and there exists an interval I on which $y(x) \neq 0$ for all $x \in I$. Then we define the function v via

$$v(x) = (y(x))^{-2},\ x \in I. \tag{1.38}$$

We may readily calculate

$$v'(x) = -2(y(x))^{-3}y'(x),\ x \in I \tag{1.39}$$

y also satisfies the eqaution

$$(y(x))^{-3}y'(x) + x(y(x))^{-2} = x,\ x \in I. \tag{1.40}$$

Therefore v satisfies

$$-\frac{1}{2}v'(x) + xv(x) = x,\ x \in I, \tag{1.41}$$

or

$$v(x) - 2xv(x) = -2x,\ x \in I. \tag{1.42}$$

Via the method of integrating factors we see that

$$\frac{d}{dx}[e^{-x^2}v(x)] = -2xe^{-x^2},\ x \in I, \tag{1.43}$$

Thus

$$v(x) = 1 + Ce^{x^2},\ x \in I \tag{1.44}$$

and C is an arbitrary constant. Thus the general candidate solutions to the original equation are

$$y(x) = \pm\sqrt{\frac{1}{1 + Ce^{x^2}}},\ x \in J \tag{1.45}$$

where C is arbitrary, over some interval J. $\square$

Problem 1.5 (Differential equations with homogeneous functions)**.**

Consider the initial value problem

$$\begin{cases} xy'(x) = y(x) - (y(x) - x)^2\,,\ x \in \mathbb{R} \\ y(1) = 0. \end{cases} \tag{1.46}$$

Suppose y is a solution to the initial value problem.

a) Show that y satisfies the equation

$$y'(x) = \frac{y(x)}{x} - x\left(\frac{y(x)}{x} - 1\right)^2\,,\ x > 0. \tag{1.47}$$

b) Use part a) to identify a candidate solution to the initial value problem (1.46). What is a candidate for the maximal interval of existence J of the solution? (Hint: J is not $(0, \infty)$)

c) Verify that the candidate solution is a solution to (1.46) on the maximal interval of existence J identify in part b). Please make sure to verify the initial condition as well.

d) Repeat parts b) and c) when the initial condition is replaced with $y(1) = 1$. Show that in this case, the maximal interval of existence can be extended to $J = \mathbb{R}$. (Hint: when solving for separable equations, what is the first step that people tend to ignore?)

Part a). We note that since $x > 0$, if y solves the original equation then

$$y'(x) = \frac{y(x)}{x} - \frac{1}{x}(y(x) - x)^2 = \frac{y(x)}{x} - \frac{1}{x}\left(x\left(\frac{y(x)}{x} - 1\right)\right)^2 = \frac{y(x)}{x} - x\left(\frac{y(x)}{x} - 1\right)^2,\ x > 0. \tag{1.48}$$

□

Part b). Consider the function v defined via

$$v(x) = \frac{y(x)}{x},\ x > 0. \tag{1.49}$$

Then

$$y'(x) = v(x) + xv'(x),\ x > 0. \tag{1.50}$$

Thus v satisfies the equation

$$v(x) + xv'(x) = v(x) - x(v(x) - 1)^2,\ x > 0, \tag{1.51}$$

which implies that v satisfies

$$v'(x) = -(v(x) - 1)^2,\ x > 0. \tag{1.52}$$

First we note that $v(x) = 1, x > 0$ is a constant solution to the system, but it does not satisfy the initial condition $v(1) = \frac{y(1)}{1} = 0$. If v is a solution, we may assume that $v \neq 1$ in an interval $I \subseteq (0, \infty)$ containing 1, therefore v satisfies

$$-\frac{1}{(v(x) - 1)^2}v'(x) = 1,\ x \in I \tag{1.53}$$

This leads us to the identity

$$(v - 1)^{-1} = x + C,\ (x, v) \in \mathbb{R}^2, C \in \mathbb{R}. \tag{1.54}$$

We assume that this implies that there exists an interval $J \subseteq I$ on which v is a differentiable function of x and we may write

$$(v(x) - 1)^{-1} = x + C,\ x \in J, C \in \mathbb{R}. \tag{1.55}$$

Since we require $v(1) = 0$, we need $C = (0 - 1)^{-1} - 1 = -2$. Thus a candidate solution to the initial value problem (1.46) is

$$y(x) = \frac{x}{x - 2} + x,\ x \in J \tag{1.56}$$

for some interval J. A candidate for the maximal interval of existence is $J = (-\infty, 2)$. □

Part c). First note that

$$y(1) = \frac{1}{1-2} + 1 = -1 + 1 = 0. \tag{1.57}$$

We may also calculate

$$xy'(x) = x\frac{x-2-x}{(x-2)^2} + x = \frac{-2x}{(x-2)^2} + x,\ x \in (-\infty, 2). \tag{1.58}$$

Also,

$$y(x) - (y(x) - x)^2 = \frac{x}{x-2} + x - \left(\frac{x}{x-2}\right)^2 = \frac{x(x-2) - x^2}{(x-2)^2} + x = \frac{-2x}{(x-2)^2} + x,\ x \in (-\infty, 2). \tag{1.59}$$

Therefore y is a solution to the IVP (1.46) with the maximal interval of existence being $J = (-\infty, 2)$. $\square$

Part d). Note that in this case $v(1) = \frac{y(1)}{1} = 1$, so the constant solution $v(x) = 1$ satisfies the initial condition, and this is the unique solution to the separable equation in v. Thus $y(x) = x, x \in \mathbb{R}$ is a candidate solution to the original initial value problem. We can check this directly by noting that $y(1) = 1$, and

$$xy'(x) = x(1) = x,\ x \in \mathbb{R}, \tag{1.60}$$

and

$$y(x) - (y(x) - x)^2 = x - (x - x)^2 = x,\ x \in \mathbb{R}. \tag{1.61}$$

Thus $y(x) = x$ is a solution to the initial value problem with the maximal interval of existence being $J = \mathbb{R}$.

$\square$

Problem 1.6 (Failure of uniqueness)**.**

Consider the initial value problem

$$\begin{cases} y'(t) = f(t, y(t)), \ t \in \mathbb{R} \\ y(2) = -1, \end{cases} \tag{1.62}$$

where

$$f(t, y) = \frac{-t + \sqrt{t^2 + 4y}}{2}, \ (t, y) \in \mathbb{R}^2. \tag{1.63}$$

a) Show that y_1 defined via

$$y_1(t) = -t + 1, \ t \in \mathbb{R} \tag{1.64}$$

is a solution to the initial value problem on the interval $[2, \infty)$, but not on the interval $(-\infty, 2)$.

b) Show that y_2 defined via

$$y_2(t) = \frac{-t^2}{4}, \ t \in \mathbb{R} \tag{1.65}$$

is a solution to the initial value problem on the interval $J = \mathbb{R}$.

c) Calculate $\frac{\partial f}{\partial y}$. What is the domain of this function as a function on $\mathbb{R}^2$?

d) Parts a) and b) show that the initial value problem (1.62) does not admit a unique solution. Why does this not violate the existence and uniqueness theorem for first order differential equations?

(Hint for the second part of part a): $\sqrt{x^2} = |x|$ for $x \in \mathbb{R}$)

Part a). We first note that $y_1(2) = -2 + 1 = -1$. Next we note that

$$(y_1)'(t) = -1, \ t \in \mathbb{R} \tag{1.66}$$

and

$$\frac{-t + \sqrt{t^2 + 4(y_1(t))}}{2} = \frac{-t + \sqrt{t^2 - 4t + 4}}{2} = \frac{-t + \sqrt{(t-2)^2}}{2} = \frac{-t + |t-2|}{2}, \ t \in (2, \infty). \tag{1.67}$$

Since $t \in (2, \infty)$, $|t - 2| = t - 2$, therefore

$$\frac{-t + \sqrt{t^2 + 4(y_1(t))}}{2} = \frac{-t + t - 2}{2} = -1, \ t \in (2, \infty). \tag{1.68}$$

This shows that y_1 is a solution to the initial value problem on the interval $(2, \infty)$. On the other hand, if $t \in (-\infty, 2)$, then $|t - 2| = -(t - 2)$, then

$$\frac{-t + \sqrt{t^2 + 4(y_1(t))}}{2} = \frac{-t - t + 2}{2} = -t + 1 \neq (y_1)'(t), \ t \in (-\infty, 2). \tag{1.69}$$

Therefore y_1 is not a solution on the interval $(-\infty, 2)$. □

Part b). First note that $y_2(2) = -4/4 = -1$. We then note that

$$(y_2)'(t) = \frac{-t}{2}, \ t \in \mathbb{R} \tag{1.70}$$

and

$$\frac{-t + \sqrt{t^2 + 4(y_2(t))}}{2} = \frac{-t}{2}, \ t \in \mathbb{R}. \tag{1.71}$$

Therefore y_2 is a solution to the IVP over the interval $J = \mathbb{R}$ □

Part c). We calculate

$$\frac{\partial f}{\partial y}(t, y) = \frac{1}{\sqrt{t^2 + 4y}}, \ (t, y) \in \mathbb{R}^2. \tag{1.72}$$

From here we see that the domain of this function is

$$D = \{(t, y) \in \mathbb{R}^2 \mid t^2 + 4y > 0\}. \tag{1.73}$$

□

Part d). We note that the existence and uniqueness theorem can only be applied if f and $\frac{\partial f}{\partial y}$ is continuous in a rectangle containing the point $(2, -1)$. However, we note that since $2^2 + 4(-1) = 0$, this point is not in the domain of $\frac{\partial f}{\partial y}$, so $\frac{\partial f}{\partial y}$ cannot be continuous in any rectangle containing $(2, -1)$. As a result, the existence and uniqueness theorem cannot be applied. Therefore the failure of uniqueness in this problem does not violate the theorem as the conditions of the theorem are not met. $\square$