Homework 11 solutions

DUE: FRIDAY, APRIL 25, 11:59PM

If you completed this assignment through collaboration or consulted references, please list the names of your collaborators and the references you used below. Please refer to the syllabus for the course policy on collaborations and types of references that are allowed.

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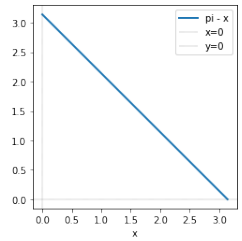
Problem 11.1 (Half-range Fourier series and the Fourier convergence theorem)**.**

Consider the function $f : [0, \pi] \to \mathbb{R}$ defined via $f(x) = \pi - x$. Here is the graph of f.



Recall that the half-range Fourier series for a function $f : [0, \pi] \to \mathbb{R}$ is obtained by extending the function f to the interval $[-\pi, \pi]$ via either an odd or even extension and considering the Fourier series of the extension defined over $[-\pi, \pi]$; by symmetry, the Fourier series over $[-L, L]$ reduces to either a Fourier sine or a Fourier cosine series, and the resulting series is what we refer to as a half-range Fourier series.

a) Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined via

$$g(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \text{ where } b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\ dx,\ n \geq 1. \tag{11.1}$$

Sketch the graph of g. Please make sure to indicate very clearly the values of g at multiples of π.

b) Consider the function $h : \mathbb{R} \to \mathbb{R}$ defined via

$$h(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \text{ where } a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\ dx,\ n \geq 0. \tag{11.2}$$

Sketch the graph of h. Please make sure to indicate very clearly the values of h at multiples of π.

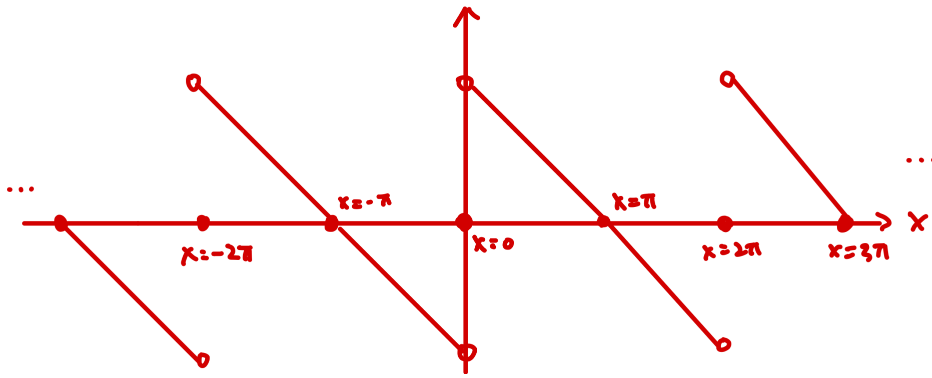
c) Note that $g(0) = 0$, and $f(0) = \pi$, yet in many textbooks one sees the expression

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) = g(x) \text{ for all } x \in [0, \pi] \tag{11.3}$$

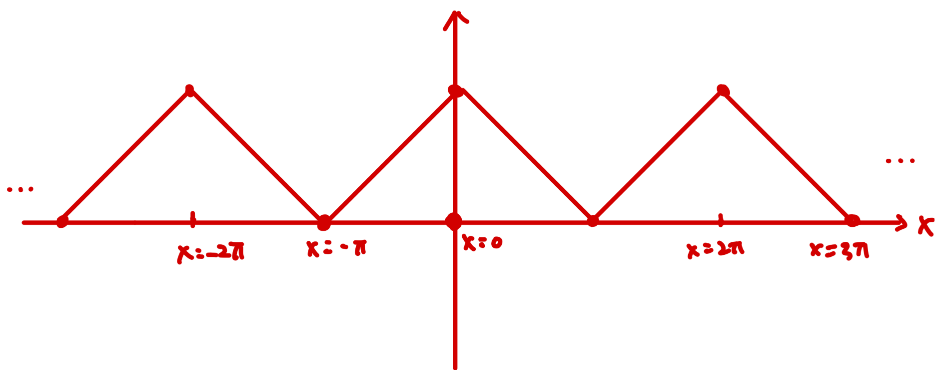
for b_n defined via (11.1). But clearly, $f(0) = \pi \neq 0 = g(0)$. Explain briefly how the pointwise "equality" at $x = 0$ in (11.3) can be explained from the point of view of the Fourier convergence theorem.

Solution.

a) We note that g is the half-range sine series of the function f defined over $[0, \pi]$. Therefore, we first extend the function f to the interval $[-\pi, \pi]$ via an odd extension, and then consider the periodic extension to $\mathbb{R}$ of the odd extension. This gives us the following sketch of g.



b) h is the half-range cosine series of the function f defined over $[0, \pi]$. Therefore, we first extend the function f to the interval $[-\pi, \pi]$ via an even extension, and then consider the periodic extension to $\mathbb{R}$ of the even extension. This gives us the following sketch of h.



c) The Fourier convergence theorem states that at points of discontinuity of the periodic extension $\tilde{f}$ of f, the Fourier series of f converges to the average of the left and right limits of $\tilde{f}$ at that point. In this case,

$$g(0) = \frac{1}{2}\left(\lim_{x\to 0^-} \tilde{f}(x) + \lim_{x\to 0^+} \tilde{f}(x)\right) = \frac{1}{2}(-\pi + \pi) = 0. \tag{11.4}$$

So while it is not equal to $f(0)$, it is equal to the average of the left and right limits of $\tilde{f}$ at $x = 0$. This is an example where the "equality" of Fourier series needs to be interpreted in the sense of the Fourier convergence theorem.

□

Problem 11.2 (An eigenvalue problem and Fourier series)**.**

Eigenvalue problems are an important class of problems, especially in the engineering and physics literature. In this problem we explore the connection between an eigenvalue problem and the basis functions that make up the Fourier series.

Consider the eigenvalue problem subject to periodic boundary conditions:

$$\begin{cases} y''(x) + \lambda y(x) = 0, \quad x \in [-\pi, \pi], \\ y(-\pi) = y(\pi), \\ y'(-\pi) = y'(\pi). \end{cases} \tag{11.5}$$

You can take for granted that the problem does not admit any negative eigenvalues.

a) Show that $\lambda = 0$ is an eigenvalue and identify a corresponding eigenfunction.
b) Identify and parametrize the positive eigenvalues by $n = 1, 2, 3, \ldots$
c) Identify and parametrize a corresponding set of eigenfunctions by $n = 1, 2, 3, \ldots$

You should find that the corresponding set of eigenfunctions for this problem coincide with the orthogonal family of basis functions that are utilized in the Fourier series over $[-\pi, \pi]$.

Solution.

a) We note that if $\lambda = 0$, then $y(x) = c_1 + c_2 x$ for some constants $c_1, c_2 \in \mathbb{R}$. The first boundary condition requires

$$c_1 - \pi c_2 = c_1 + \pi c_2 \implies c_2 = 0 \tag{11.6}$$

and the second boundary condition does not impose any further restrictions. Therefore any non-zero constant function is an eigenfunction corresponding to the eigenvalue $\lambda = 0$.

b) If $\lambda > 0$, then $y(x) = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)$ for some constants $c_1, c_2 \in \mathbb{R}$. The first boundary condition requires

$$c_1 \cos(-\sqrt{\lambda}\pi) + c_2 \sin(-\sqrt{\lambda}\pi) = c_1 \cos(\sqrt{\lambda}\pi) + c_2 \sin(\sqrt{\lambda}\pi)$$
$$\implies -c_2 \sin(\sqrt{\lambda}\pi) = c_2 \sin(\sqrt{\lambda}\pi) \implies c_2 = 0 \text{ or } \sin(\sqrt{\lambda}\pi) = 0. \tag{11.7}$$

The second boundary condition requires

$$c_2\sqrt{\lambda} \cos(-\sqrt{\lambda}\pi) - c_1\sqrt{\lambda} \sin(-\sqrt{\lambda}\pi) = c_2\sqrt{\lambda} \cos(\sqrt{\lambda}\pi) - c_1\sqrt{\lambda} \sin(\sqrt{\lambda}\pi)$$
$$\implies c_1 = 0 \text{ or } \sin(\sqrt{\lambda}\pi) = 0. \tag{11.8}$$

For both of these conditions to be satisfied, we must have either $c_2 = 0$ and $\sin(\sqrt{\lambda}\pi) = 0$ or $c_1 = 0$ and $\sin(\sqrt{\lambda}\pi) = 0$. In either case, the eigenvalues can be parametrized by

$$\lambda_n = n^2, \ n = 1, 2, 3, \ldots \tag{11.9}$$

c) By the previous part, for each $n \in \{1, 2, 3, ...\}$, any non-zero multiple of the functions $x \mapsto \cos(nx)$ and $x \mapsto \sin(nx)$ are eigenfunctions corresponding to the eigenvalue $\lambda_n = n^2$. These functions, together with the constant function, are exactly the functions that make up the Fourier series over $[-\pi, \pi]$.

□

Problem 11.3 (The heat equation with homogeneous Dirichlet boundary conditions)**.** Find the unique solution $u : [0, \pi] \times [0, \infty) \to \mathbb{R}$ satisfying

$$\begin{aligned} & u_t = 3u_{xx}, && x \in (0,\pi), t \geq 0, && (11.10)\\ & u(x=0,t) = 0 = u(x=\pi,t) && t \geq 0 && (11.11)\\ & u(x,t=0) = \sin(x) + 2\sin(2x) + 3\sin(3x), && x \in (0,\pi). && (11.12) \end{aligned}$$

You can use the solution formula derived in lecture directly.

Solution. Using the method of separation of variables, we find that

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-3n^2 t} \sin nx, \ x \in (0,\pi), t \geq 0. \tag{11.13}$$

To choose the coefficients A_n we note that we need

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin nx = \sin(x) + 2\sin(2x) + 3\sin(3x), \qquad x \in (0,\pi). \tag{11.14}$$

This implies we must choose

$$A_n = \begin{cases} 1, & n = 1 \\ 2, & n = 2 \\ 3, & n = 3 \\ 0, & \text{otherwise.} \end{cases} \tag{11.15}$$

Thus

$$u(x,t) = e^{-3t} \sin x + 2e^{-12t} \sin 2x + 3e^{-27t} \sin 3x, \ x \in (0,\pi), t \geq 0. \tag{11.16}$$

□

Problem 11.4 (The wave equation with homogeneous Dirichlet boundary conditions)**.** Find the unique solution $u : [0, \pi] \times [0, \infty) \to \mathbb{R}$ satisfying

$$u_{tt} = 9u_{xx}, \qquad x \in (0,\pi), t \geq 0 \tag{11.17}$$

$$u(x = 0, t) = 0 = u(x = \pi, t) \qquad t \geq 0 \tag{11.18}$$

$$u(x, t = 0) = \sin(2x) \qquad x \in (0,\pi) \tag{11.19}$$

$$u_t(x, t = 0) = \sin(3x) \qquad x \in (0,\pi). \tag{11.20}$$

You can use the solution formula derived in lecture directly.

Solution. Using the method of separation of variables, we find that

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \cos(3nt) + B_n \sin(3nt)) \sin nx, \ x \in (0,\pi), t \geq 0. \tag{11.21}$$

To choose the coefficients A_n, B_n we note that we must have

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin nx = \sin 2x, \tag{11.22}$$

$$u_t(x,0) = \sum_{n=1}^{\infty} 3nB_n \sin nx = \sin(3x), \ x \in (0,\pi). \tag{11.23}$$

Thus we must have

$$A_n = \begin{cases} 1, & n = 2 \\ 0, & \text{otherwise.} \end{cases}, \quad B_n = \begin{cases} \frac{1}{9}, & n = 3 \\ 0, & \text{otherwise.} \end{cases} \tag{11.24}$$

Thus

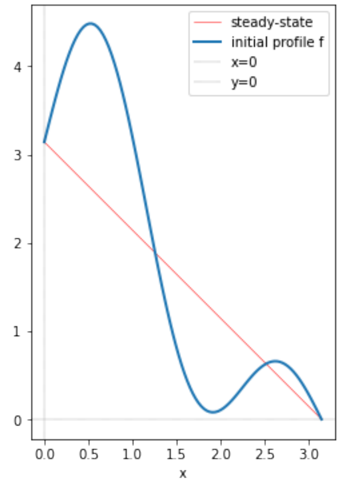
$$u(x,t) = \cos 6t \sin 2x + \frac{1}{9} \sin 9t \sin 3x, \ x \in (0,\pi), t \geq 0. \tag{11.25}$$

□

Problem 11.5 (The heat equation with inhomogeneous Dirichlet boundary conditions)**.**
Consider the heat equation with *inhomogeneous* Dirichlet boundary conditions of the form

$$\begin{cases} u_t(x,t) = 2u_{xx}(x,t) & x \in [0,\pi], t \geq \mathbb{R} \\ u(x=0,t) = \pi, & t \geq 0 \\ u(x=\pi,t) = 0, & t \geq 0 \\ u(x,t=0) = f(x) = \sin(2x) + \sin(3x) + \pi - x, & x \in [0,\pi]. \end{cases} \tag{11.26}$$

Below is a plot of the initial temperature profile f and the steady-state profile.



a) Find the unique time-independent function $v : [0,\pi] \to \mathbb{R}$ satisfying

$$\begin{cases} v_{xx}(x) = v''(x) = 0, & x \in [0,\pi] \\ v(0) = \pi, v(\pi) = 0. \end{cases} \tag{11.27}$$

b) Suppose $u : [0,\pi] \times [0,\infty) \to \mathbb{R}$ solves (11.26). Show that $w : [0,\pi] \times [0,\infty) \to \mathbb{R}$ defined via $w(x,t) = u(x,t) - v(x)$ satisfies the heat equation with homogeneous Dirichlet conditions

$$\begin{cases} w_t(x,t) = 2w_{xx}(x,t) & x \in [0,\pi], t \geq \mathbb{R} \\ w(0,t) = 0, & t \geq 0 \\ w(\pi,t) = 0, & t \geq 0 \\ w(x,0) = f(x) - v(x), & x \in [0,\pi] \end{cases} \tag{11.28}$$

c) Use parts a) and b) to find the unique solution to the system (11.26) as a *finite* combination of elementary functions.

d) What's the steady-state solution $y : [0,\pi] \to \mathbb{R}$ to (11.26)? In other words, what is

$$y(x) = \lim_{t\to\infty} u(x,t),\ x \in [0,\pi]? \tag{11.29}$$

e) Show that $y : [0,\pi] \to \mathbb{R}$ solves the 1D Laplace's equation

$$-\Delta y(x) = -\partial_{xx} y(x) = -y''(x) = 0,\ x \in (0,\pi). \tag{11.30}$$

This problem shows that to deal with inhomogeneous Dirichlet boundary conditions, one can "shift" the inhomogeneous boundary condition to the initial condition by creating an ansatz with an appropriate time-independent function v, and then recover the original solution by solving a reduced problem with homogeneous boundary conditions.

Solution.

a) From the equation we know that $v(x) = c_1 x + c_2$ for some constants $c_1, c_2 \in \mathbb{R}$. The boundary conditions imply that $c_2 = \pi$ and $c_1 = -1$. Thus

$$v(x) = \pi - x,\ x \in [0, \pi]. \tag{11.31}$$

b) Since v is time-independent and satisfies $\partial_{xx} v = 0$, we have

$$w_t(x,t) = u_t(x,t) = 2u_{xx}(x,t) = 2u_{xx}(x,t) - 2v_{xx}(x,t) = 2w_{xx}(x,t),\ x \in [0,\pi], t \geq 0. \tag{11.32}$$

Furthermore,

$$w(0,t) = u(0,t) - v(0) = u(0,t) - \pi = 0 \text{ and } w(\pi,t) = u(\pi,t) - v(\pi) = u(\pi,t) - 0 = 0,\ t \geq 0, \tag{11.33}$$

and

$$w(x,0) = u(x,0) - v(x) = f(x) - v(x) = \sin(2x) + \sin(3x),\ x \in [0,\pi]. \tag{11.34}$$

c) We note that since w satisfies the heat equation with homogeneous Dirichlet boundary conditions, we can use the solution formula derived in lecture to find that

$$w(x,t) = \sum_{n=1}^{\infty} A_n e^{-2n^2 t} \sin nx,\ x \in [0,\pi], t \geq 0, \tag{11.35}$$

where

$$A_n = \begin{cases} 1, & n = 2 \\ 1, & n = 3 \\ 0, & \text{otherwise.} \end{cases} \tag{11.36}$$

Therefore

$$w(x,t) = e^{-8t} \sin 2x + e^{-18t} \sin 3x,\ x \in [0,\pi], t \geq 0. \tag{11.37}$$

This implies that

$$u(x,t) = w(x,t) + v(x) = e^{-8t} \sin 2x + e^{-18t} \sin 3x + \pi - x,\ x \in [0,\pi], t \geq 0. \tag{11.38}$$

d) We note that as $t \to \infty$, the terms $e^{-8t} \sin 2x$ and $e^{-18t} \sin 3x$ vanish, therefore the steady-state solution is given by

$$y(x) = \lim_{t \to \infty} u(x,t) = \pi - x,\ x \in [0,\pi]. \tag{11.39}$$

e) Since y is an affine function, from a direct computation we have $-y''(x) = 0$ for all $x \in (0,\pi)$.

$\square$