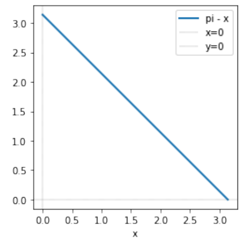
Homework 11

DUE: FRIDAY, APRIL 25, 11:59PM

If you completed this assignment through collaboration or consulted references, please list the names of your collaborators and the references you used below. Please refer to the syllabus for the course policy on collaborations and types of references that are allowed.

DEPARTMENT OF MATHEMATICAL SCIENCES, CARNEGIE MELLON UNIVERSITY
E-mail address: jkoganem@andrew.cmu.edu.
Date: April 19, 2025.

Problem 11.1 (Half-range Fourier series and the Fourier convergence theorem)**.**
Consider the function $f : [0, \pi] \to \mathbb{R}$ defined via $f(x) = \pi - x$. Here is the graph of f.



Recall that the half-range Fourier series for a function $f : [0, \pi] \to \mathbb{R}$ is obtained by extending the function f to the interval $[-\pi, \pi]$ via either an odd or even extension and considering the Fourier series of the extension defined over $[-\pi, \pi]$; by symmetry, the Fourier series over $[-L, L]$ reduces to either a Fourier sine or a Fourier cosine series, and the resulting series is what we refer to as a half-range Fourier series.

a) Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined via

$$g(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \text{ where } b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\ dx,\ n \geq 1. \tag{11.1}$$

Sketch the graph of g. Please make sure to indicate very clearly the values of g at multiples of π.

b) Consider the function $h : \mathbb{R} \to \mathbb{R}$ defined via

$$h(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \text{ where } a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\ dx,\ n \geq 0. \tag{11.2}$$

Sketch the graph of h. Please make sure to indicate very clearly the values of h at multiples of π.

c) Note that $g(0) = 0$, and $f(0) = \pi$, yet in many textbooks one sees the expression

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) = g(x) \text{ for all } x \in [0, \pi] \tag{11.3}$$

for b_n defined via (11.1). But clearly, $f(0) = \pi \neq 0 = g(0)$. Explain briefly how the pointwise "equality" at $x = 0$ in (11.3) can be explained from the point of view of the Fourier convergence theorem.

Problem 11.2 (An eigenvalue problem and Fourier series)**.**

Eigenvalue problems are an important class of problems, especially in the engineering and physics literature. In this problem we explore the connection between an eigenvalue problem and the basis functions that make up the Fourier series.

Consider the eigenvalue problem subject to periodic boundary conditions:

$$\begin{cases} y''(x) + \lambda y(x) = 0, \quad x \in [-\pi, \pi], \\ y(-\pi) = y(\pi), \\ y'(-\pi) = y'(\pi). \end{cases} \tag{11.4}$$

You can take for granted that the problem does not admit any negative eigenvalues.

a) Show that $\lambda = 0$ is an eigenvalue and identify a corresponding eigenfunction.
b) Identify and parametrize the positive eigenvalues by $n = 1, 2, 3, \ldots$
c) Identify and parametrize a corresponding set of eigenfunctions by $n = 1, 2, 3, \ldots$

You should find that the corresponding set of eigenfunctions for this problem coincide with the orthogonal family of basis functions that are utilized in the Fourier series over $[-\pi, \pi]$.

Problem 11.3 (The heat equation with homogeneous Dirichlet boundary conditions)**.** Find the unique solution $u : [0,\pi] \times [0,\infty) \to \mathbb{R}$ satisfying

$$u_t = 3u_{xx}, \qquad x \in (0,\pi), t \geq 0, \tag{11.5}$$

$$u(x=0,t) = 0 = u(x=\pi,t) \qquad t \geq 0 \tag{11.6}$$

$$u(x,t=0) = \sin(x) + 2\sin(2x) + 3\sin(3x), \qquad x \in (0,\pi). \tag{11.7}$$

You can use the solution formula derived in lecture directly.

Problem 11.4 (The wave equation with homogeneous Dirichlet boundary conditions)**.** Find the unique solution $u : [0, \pi] \times [0, \infty) \to \mathbb{R}$ satisfying

$$u_{tt} = 9u_{xx}, \qquad x \in (0, \pi), t \geq 0 \tag{11.8}$$

$$u(x = 0, t) = 0 = u(x = \pi, t) \qquad t \geq 0 \tag{11.9}$$

$$u(x, t = 0) = \sin(2x) \qquad x \in (0, \pi) \tag{11.10}$$

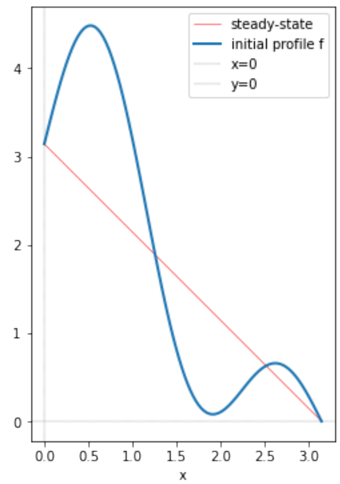
$$u_t(x, t = 0) = \sin(3x) \qquad x \in (0, \pi). \tag{11.11}$$

You can use the solution formula derived in lecture directly.

Problem 11.5 (The heat equation with inhomogeneous Dirichlet boundary conditions)**.**
Consider the heat equation with *inhomogeneous* Dirichlet boundary conditions of the form

$$\begin{cases} u_t(x,t) = 2u_{xx}(x,t) & x \in [0,\pi], t \geq \mathbb{R} \\ u(x=0,t) = \pi, & t \geq 0 \\ u(x=\pi,t) = 0, & t \geq 0 \\ u(x,t=0) = f(x) = \sin(2x) + \sin(3x) + \pi - x, & x \in [0,\pi]. \end{cases} \tag{11.12}$$

Below is a plot of the initial temperature profile f and the steady-state profile.



a) Find the unique time-independent function $v : [0,\pi] \to \mathbb{R}$ satisfying

$$\begin{cases} v_{xx}(x) = v''(x) = 0, & x \in [0,\pi] \\ v(0) = \pi, v(\pi) = 0. \end{cases} \tag{11.13}$$

b) Suppose $u : [0,\pi] \times [0,\infty) \to \mathbb{R}$ solves (11.12). Show that $w : [0,\pi] \times [0,\infty) \to \mathbb{R}$ defined via $w(x,t) = u(x,t) - v(x)$ satisfies the heat equation with homogeneous Dirichlet conditions

$$\begin{cases} w_t(x,t) = 2w_{xx}(x,t) & x \in [0,\pi], t \geq \mathbb{R} \\ w(0,t) = 0, & t \geq 0 \\ w(\pi,t) = 0, & t \geq 0 \\ w(x,0) = f(x) - v(x), & x \in [0,\pi] \end{cases} \tag{11.14}$$

c) Use parts a) and b) to find the unique solution to the system (11.12) as a *finite* combination of elementary functions.

d) What's the steady-state solution $y : [0,\pi] \to \mathbb{R}$ to (11.12)? In other words, what is

$$y(x) = \lim_{t\to\infty} u(x,t), \ x \in [0,\pi]? \tag{11.15}$$

e) Show that $y : [0,\pi] \to \mathbb{R}$ solves the 1D Laplace's equation

$$-\Delta y(x) = -\partial_{xx} y(x) = -y''(x) = 0, \ x \in (0,\pi). \tag{11.16}$$

This problem shows that to deal with inhomogeneous Dirichlet boundary conditions, one can "shift" the inhomogeneous boundary condition to the initial condition by creating an ansatz with an appropriate time-independent function v, and then recover the original solution by solving a reduced problem with homogeneous boundary conditions.