Review part 3

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Part 1. Concepts to review

a) Method of undetermined coefficients
   - If an equation is an inhomogeneous linear equation, then the general solution to the equation is
$$y(x) = y_p(x) + y_h(x),\ x \in I, \tag{1.1}$$
where y_p is any particular solution to the inhomogeneous equation and y_h is the general solution to the homogeneous equation.
   - To identify the particular solution, when the equation is a constant coefficient equation and the right hand side of the equation f is a polynomial function, an exponential function, sin or cos, or the product and linear combination of the aforementioned functions, then we may use the method of undetermined coefficients to find a particular solution.
   - We use an educated guess or an ansatz y_p that depends on some unknown constants, and we determine these constants by calculating $y_p', y_p'', y_p''', \ldots$ and assuming that y_p satisfies the equation.
   - The method is based on the observation that for these types of right hand side functions, f and all of its derivatives can be written as a finite linear combination of functions that are linearly independent, which forms the functions that we use in our educated guess.
   - Note: one should always solve for the general homogeneous solution y_h first, to ensure that there are no duplications between y_h and the educated guess y_p.
   - Therefore the method can be summarized as follows:
     1) First, check that the right hand side function f is of the appropriate type to use this method.
     2) Next, find the general homogeneous solution to the equation.
     3) Write down an initial ansatz y_p based on the structure of the f. Modify y_p if necessary by comparing it against the general homogeneous solution. If there are duplicate terms, multiply the corresponding part in the initial ansatz by an appropriate polynomial term to eliminate the overlap.
     4) Assume y_p is a solution to the inhomogeneous equation and use the equation to identify the undetermined coefficients. Since the functions used in the ansatz are linearly independent, if the final equality is true over $\mathbb{R}$ then the coefficients must match on both sides.

b) Method of variation of parameters
   - This method allows us to find a particular solution to the equation given that we know the general homogeneous solution to the equation.
   - The method applies to variable coefficients as well.
   - Variation of parameters formula for 2nd order equations: if we know the general homogeneous solution
$$y_h(x) = c_1 y_1(x) + c_2 y_2(x),\ x \in I, \tag{1.2}$$
then
$$y_p(x) = -y_1(x) \int \frac{y_2(x) f(x)}{W(y_1, y_2)(x)}\, dx + y_2(x) \int \frac{y_1(x) f(x)}{W(y_1, y_2)(x)}\, dx,\ x \in I \tag{1.3}$$
is a particular solution.

c) Pure resonance in mass-spring systems
   - For a mass spring system without damping and with a periodic external force, pure resonance occurs when the forcing frequency is equal to the natural frequency of the system. If the mass spring system is modeled via
$$mx''(t) + kx(t) = f(t),\ t \in \mathbb{R} \tag{1.4}$$
where $f(t) = F_0 \cos(\omega t)$ or $f(t) = F_0 \sin(\omega t)$, then this occurs exactly when
$$\omega = \omega_0 := \sqrt{\frac{k}{m}}. \tag{1.5}$$
This corresponds to the system admitting a paricular solution of the form
$$x(t) = At\cos(\omega t) + Bt\sin(\omega t),\ \mathbb{R}. \tag{1.6}$$
This means that the amplitude of the mass spring system increases over time.
   - If damping is introduced in the system, then pure resonance would never occur.

d) The Laplace transform
   - The Laplace transforms allows us to solve inhomogeneous equations where the forcing function is piecewise continuous and sometimes not a function in the classical sense.

- The Laplace transform transforms a differential equation into an algebraic equation.
- Since algebraic equations are easy to solve, we can recover the solution by finding an expression (in terms of the frequency variable s) for the Laplace transform of the unknown solution, and then recover the solution by taking the inverse Laplace transform.

2. Review of the Laplace transform

In this section we'll quickly review some basic facts related to the Laplace transform. For more details, read the previous set of notes.

Definition 2.1. We define the *Laplace transform* of a function $f : [0, \infty) \to \mathbb{R}$ to be

$$\mathcal{L}\{f\}(s) := \int_0^\infty e^{-st} f(t)\, dt. \tag{2.1}$$

Important Laplace transforms:

(1)

$$\mathcal{L}\{1\} = \frac{1}{s},\ s > 0 \tag{2.2}$$

(2)

$$\mathcal{L}\{t^\alpha\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}},\ \alpha > -1. \tag{2.3}$$

(3)

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a},\ s > a \tag{2.4}$$

(4)

$$\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2},\ s > 0. \tag{2.5}$$

(5)

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2},\ s > 0. \tag{2.6}$$

Important inverse transforms:

(1)

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1 \tag{2.7}$$

(2)

$$\mathcal{L}^{-1}\left\{\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}\right\} = t^\alpha \tag{2.8}$$

(3)

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \tag{2.9}$$

(4)

$$\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin(kt) \tag{2.10}$$

(5)

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos(kt) \tag{2.11}$$

Proposition 2.2 (Laplace transforms of derivatives)**.** *Let y be a smooth function satisfying appropriate growth conditions. Denote its Laplace transform $\mathcal{L}\{y\}$ by $Y(s)$. Then*

$$\mathcal{L}\{y^{(n)}(t)\} = s^n Y(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \ldots - sy^{(n-2)}(0) - y^{(n-1)}(0). \tag{2.12}$$

In particular,

$$\mathcal{L}\{y'\} = sY(s) - y(0) \tag{2.13}$$

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0). \tag{2.14}$$

Next we'd like to discuss operational properties of the Laplace transform. Before we do this we need to introduce the *unit step function* $\mathcal{U}(t)$, since piecewise continuous functions can be written explicitly in terms of the unit step function and we'll frequently compute the Laplace transform of piecewise defined functions. It'll also help us make sense of shifting functions $f(t)$ only defined on $t \geq 0$ ($f(t-a)$ is only defined for $t \geq a$ if $f(t)$ is only defined for $t \geq 0$; we want everything to be well-defined for $t \geq 0$).

Definition 2.3 (Unit step function)**.**

$$\mathcal{U}(t-a) := \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a. \end{cases} \tag{2.15}$$

Proposition 2.4.

$$1 - \mathcal{U}(t-a) := \begin{cases} 1, & 0 \leq t < a \\ 0, & t \geq a. \end{cases} \tag{2.16}$$

Proposition 2.5 (Shift)**.** *Given $f : [0, \infty) \to \mathbb{R}$,*

$$f(t-a)\mathcal{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ f(t-a), & t \geq a. \end{cases} \tag{2.17}$$

Notice that this function (unlike $f(t-a)$) is defined for all $t \geq 0$.

Proposition 2.6 (Truncation before a)**.**

$$f(t)\mathcal{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ f(t), & t \geq a. \end{cases} \tag{2.18}$$

Proposition 2.7 (Truncation after a)**.**

$$f(t)(1 - \mathcal{U}(t-a)) = \begin{cases} f(t), & 0 \leq t < a \\ 0, & t \geq a. \end{cases} \tag{2.19}$$

Proposition 2.8 (Piecewise function in terms of $\mathcal{U}(t)$)**.** *If*

$$\alpha(t) = \begin{cases} f(t), & 0 \leq t < a \\ g(t), & t \geq a, \end{cases} \tag{2.20}$$

then

$$\alpha(t) = f(t) + (g(t) - f(t))\mathcal{U}(t-a) \tag{2.21}$$

Proposition 2.9. *If*

$$\beta(t) = \begin{cases} f(t), & 0 \leq t < a \\ g(t), & a \leq t < b \\ h(t), & t \geq b, \end{cases} \tag{2.22}$$

then

$$\beta(t) = f(t) + (g(t) - f(t))\mathcal{U}(t-a) + (h(t) - g(t))\mathcal{U}(t-b). \tag{2.23}$$

Finally we can state the important operational properties of the Laplace transform. Very roughly speaking, the following propositions say that *translation in one domain (either t or s) corresponds to multiplying by an exponential function in the other domain.*

Proposition 2.10.

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a), \tag{2.24}$$

where $F(s) = \mathcal{L}\{f(t)\}$.

Corollary 2.11.

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t), \tag{2.25}$$

where $\mathcal{L}^{-1}\{F(s)\} = f(t)$.

Proposition 2.12.

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s) \tag{2.26}$$

Corollary 2.13.

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a). \tag{2.27}$$

Remark 2.14. This one is perhaps the trickiest one to apply. Notice that it's tempting to write $\mathcal{L}^{-1}\{F(s-a)\}\mathcal{U}(t-a)$ on the right hand side but it's wrong, since in general

$$\mathcal{L}^{-1}\{F(s-a)\} \neq f(t-a), \tag{2.28}$$

where $f(t) = \mathcal{L}^{-1}\{F(s)\}$, so what the corollary is saying is that one should calculate the inverse Laplace transform

$$f(t) = \mathcal{L}^{-1}\{F(s)\}, \tag{2.29}$$

first, and then shift it to obtain $f(t-a)$. Try this when you calculate $\mathcal{L}^{-1}\{\frac{e^{-as}}{s^2}\}$.

Corollary 2.15.

$$\mathcal{L}\{f(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}. \tag{2.30}$$

Corollary 2.16.

$$\mathcal{L}\{\mathcal{U}(t-a)\} = \frac{e^{-as}}{s}. \tag{2.31}$$

Part 2. Problems

2.1. Method of undetermined coefficients and resonance.

Exercise 2.1. Consider the following ODE:

$$y'' + 14y' + 58y = 0. \tag{2.32}$$

Is this system over/under/critically damped? Suppose an external force $F(t) = e^{-3t}$ is applied to the system. Find the solution of the resulting system with $y(0) = 0, y'(0) = 0$.

Solution. The characteristic polynomial associated to the ODE is $r^2 + 14r + 58 = 0$. The roots are given by

$$r = \frac{-14 \pm \sqrt{196 - 232}}{2} = -7 \pm 3i. \tag{2.33}$$

This tells us that the system is underdamped. The solution of the homogeneous equation is thus given by

$$y_h(x) = e^{-7t}\left(c_1 \cos(3t) + c_2 \sin(3t)\right). \tag{2.34}$$

To solve for the particular solution we use the method of undetermined coefficients; the appropriate ansatz is $y_p(x) = Ae^{-3t}$. Now we calculate

$$\begin{aligned} y_p &= Ae^{-3t} \\ y_p' &= -3Ae^{-3t} \\ y_p'' &= 9Ae^{-3t}. \end{aligned} \tag{2.35}$$

Then $e^{-3t} = y'' + 14y' + 58y = (9A - 42A + 58A)e^{-3t} \implies A = \frac{1}{9-42+58} = \frac{1}{25}$. So the general solution is given by

$$y(t) = y_h(t) + y_p(t) = e^{-7t}\left(c_1 \cos(3t) + c_2 \sin(3t)\right) + \frac{1}{25}e^{-3t}. \tag{2.36}$$

This implies that

$$y'(t) = e^{-7t}(-3c_1 \sin(3t) + 3c_2 \cos(3t) - 7c_1 \cos(3t) - 7c_2 \sin(3t)) - \frac{3}{25}e^{-3t} \tag{2.37}$$

Using the initial conditions $y(0) = 0$ and $y'(0) = 0$, we find

$$\begin{cases} c_1 = -\frac{1}{25} \\ 0 = 3c_2 - 7c_1 - \frac{3}{25} \implies c_2 = -\frac{4}{75}. \end{cases} \tag{2.38}$$

□

Exercise 2.2. Given

$$2x'' + 8x = 3\cos(\omega t), \tag{2.39}$$

use the method of undetermined coefficients to find a particular solution $x_p(t)$. For what values of ω will resonance occur? How can this be determined from the particular solution?

Solution. Since we're only interested in a particular solution we can assume that $x(0) = x'(0) = 0$ (the particular solution doesn't depend on the initial conditions). We still need to solve for the homogeneous solution to find the natural frequency of the system. The characteristic polynomial is given by $r^2 + 4$, therefore the roots are given by $r = \pm 2i$. Therefore the homogeneous solution is given by

$$x_h(t) = c_1\cos(2t) + c_2\sin(2t). \tag{2.40}$$

Assume that $\omega \neq 2$. To find the particular solution we use the ansatz given by $x = A\cos(\omega t) + B\sin(\omega t)$. Then

$$\begin{aligned} x &= A\cos(\omega t) + B\sin(\omega t) \\ x' &= B\omega\cos(\omega t) - A\omega\sin(\omega t) \\ x'' &= -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t). \end{aligned} \tag{2.41}$$

Thus we have

$$(-2A\omega^2 + 8A)\cos(\omega t) + (-2B\omega^2 + 8B)\sin(\omega t) = 3\cos(\omega t) \implies B = 0, A = \frac{3}{8 - 2\omega^2}. \tag{2.42}$$

Thus a particular solution is given by

$$x_p(t) = \frac{3}{8 - 2\omega^2}\cos(\omega t), \tag{2.43}$$

which is well-defined when $\omega \neq \pm 2$. We see that $|x_p(t)| \to \infty$ as $\omega \to \pm 2$, therefore resonance occurs when $\omega = \pm 2$, as expected. □

Exercise 2.3.

$$y'' + 4y' + 5y = 4\cos(t) - 4\sin(t). \tag{2.44}$$

Find the solution for $t > 0$ if $y(0) = 1, y'(0) = 1$. What is the long term behavior of the system? Identify the steady-state solution and the transient solution. What long-term difference would you notice if the initial conditions were different?

Solution. The characteristic polynomial is given by $r^2 + 4r + 5$, so the roots are $r = -2 \pm i$. The homogeneous solution is then given by

$$y_h(t) = e^{-2t}\left(c_1\cos(t) + c_2\sin(t)\right). \tag{2.45}$$

For the particular solution we use the ansatz $y = A\cos(t) + B\sin(t)$. Then

$$\begin{aligned} y &= A\cos(t) + B\sin(t) \\ y' &= B\cos(t) - A\sin(t) \\ y'' &= -A\cos(t) - B\sin(t). \end{aligned} \tag{2.46}$$

Thus

$$y'' + 4y' + 5y = (4A + 4B)\cos(t) + (4B - 4A)\sin(t) = 4\cos(t) - 4\sin(t) \implies A = 1, B = 0. \tag{2.47}$$

So the general solution is given by

$$y(t) = e^{-2t}\left(c_1\cos(t) + c_2\sin(t)\right) + \cos(t). \tag{2.48}$$

To solve for c_1, c_2 we use the initial conditions. We find that $c_1 = 0, c_2 = 1$. Therefore

$$y(t) = \underbrace{e^{-2t}\sin(t)}_{\to 0 \text{ as } t \to \infty} + \cos(t). \tag{2.49}$$

So the homogeneous solution is the transient solution and the particular solution $\cos(t)$ is the steady-state solution. From this we see that the initial conditions do not affect the long term behavior of the system, since the particular solution/the steady state solution doesn't depend on the initial conditions. □

Exercise 2.4. Consider the differential equation

$$x'' + x' + 4x = 4\cos(t) + 2\sin(t). \tag{2.50}$$

Find the solution satisfying $x(0) = 0, x'(0) = 0$. Identify the steady-state and transient solution.

Solution. The characteristic polynomial is given by $r^2 + r + 4 = 0$, where its roots are $r = \frac{-1\pm\sqrt{1-16}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{15}}{2}$. Thus the homogeneous solution is given by

$$x_h(t) = e^{-1/2t}\left(c_1 \cos\frac{\sqrt{15}}{2}t + c_2 \sin\frac{\sqrt{15}}{2}t\right). \tag{2.51}$$

The appropriate ansatz for the particular solution is $x = A\cos(t) + B\sin(t)$. Then

$$x'' + x' + 4x = (3A + B)\cos(t) + (3B - A)\sin(t) = 4\cos(t) + 2\sin(t) \implies A = 1, B = 1. \tag{2.52}$$

So the general solution is given by

$$x(t) = e^{-1/2t}\left(c_1 \cos\frac{\sqrt{15}}{2}t + c_2 \sin\frac{\sqrt{15}}{2}t\right) + \cos(t) + \sin(t) \tag{2.53}$$

Using the initial conditions we find that

$$\begin{cases} 0 = c_1 + 1 \implies c_1 = -1 \\ 0 = \frac{\sqrt{15}}{2}c_2 - \frac{1}{2}c_1 + 1 \implies c_2 = -\frac{3}{\sqrt{15}}. \end{cases} \tag{2.54}$$

Therefore the solution satisfying $x(0) = 0, x'(0) = 0$ is given by

$$x(t) = -e^{-1/2t}\left(\cos\frac{\sqrt{15}}{2}t + \frac{3}{\sqrt{15}}\sin\frac{\sqrt{15}}{2}t\right) + \cos(t) + \sin(t). \tag{2.55}$$

Here we see that

$$x(t) = \underbrace{e^{-1/2t}\left(c_1 \cos\left(\frac{\sqrt{15}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{15}}{2}t\right)\right)}_{\text{transient solution}} + \underbrace{\cos(t) + \sin(t)}_{\text{steady-state solution}}. \tag{2.56}$$

□

Exercise 2.5. Consider the differential equation

$$y'' + 2y' + 5y = \cos(\omega t). \tag{2.57}$$

Show that if $\omega = 2$, then the amplitude of the steady-state solution is $\frac{1}{\sqrt{17}}$.

Solution. The characteristic polynomial is $r^2 + 2r + 5$, so the roots are $-1 \pm 2i$. We use the ansatz $y = A\cos(t) + B\sin(t)$, then

$$\begin{aligned} y'' + 2y' + 5y &= (-4A + 4B + 5A)\cos(2t) + (-4B - 4A + 5B)\sin(2t) \\ &= (A + 4B)\cos(2t) + (B - 4A)\sin(2t) \implies B = 4A, A = \frac{1}{17}. \end{aligned} \tag{2.58}$$

Therefore the particular solution is given by

$$y = \frac{1}{17}\cos(t) + \frac{4}{17}\sin(t) \tag{2.59}$$

and the amplitude is given by

$$\sqrt{A^2 + B^2} = \sqrt{\frac{1 + 16}{17^2}} = \frac{1}{\sqrt{17}}. \tag{2.60}$$

$\square$

Exercise 2.6. Consider a mass-spring system described by

$$2x'' + 4x' + 4x = \cos(t). \tag{2.61}$$

What is the natural frequency and what is the pseudo-frequency? Find the steady-state solution and the amplitude.

Solution. The characteristic polynomial is $r^2 + 2r + 2 = (r+1)^2 + 1$, so the roots are $r = -1 \pm i$. Thus the homogeneous solution is given by

$$x_h(t) = e^{-t}\left(c_1 \cos(t) + c_2 \sin(t)\right). \tag{2.62}$$

The natural frequency is the frequency of the system without damping, which is $\omega = \sqrt{2}$. The pseudo-frequency is the frequency of the system with damping, which is 1. To find the steady-state/particular solution we use the ansatz $x = A\cos(t) + B\sin(t)$. Then

$$2x'' + 4x' + 4x = (-2A + 4B + 4A)\cos(t) + (-2B - 4A + 4B)\sin(t) = (2A + 4B)\cos(t) + (2B - 4A)\sin(t) = \cos(t)$$
$$\implies B = \frac{1}{5}, A = \frac{1}{10}. \tag{2.63}$$

Thus the steady-state solution is given by

$$x_p(t) = \frac{1}{10}\cos(t) + \frac{1}{5}\sin(t), \tag{2.64}$$

and the amplitude is given by

$$\sqrt{A^2 + B^2} = \sqrt{\frac{1+4}{100}} = \frac{\sqrt{5}}{10} = \frac{1}{2\sqrt{5}}. \tag{2.65}$$

$\square$

Exercise 2.7. Consider the equation

$$x'' + x' + x = \cos(\omega t), \tag{2.66}$$

Find an expression for the amplitude of the steady-state solution.

Solution. The characteristic polynomial is given by $r^2 + r + 1$, so the roots are $-1/2 \pm i\sqrt{3}/2$. So the steady-state solution will be given by the particular solution because of the presence of the exponential term in the homogeneous solution. Using the ansatz $x = A\cos(\omega t) + B\sin(\omega t)$ we find

$$x'' + x' + x = (-A\omega^2 + B\omega + A)\cos(\omega t) + (-B\omega^2 - A\omega + B)\sin(\omega t) = \cos(\omega t). \tag{2.67}$$

Then

$$\begin{cases} (1-\omega^2)A + \omega B & = 1 \\ -\omega A + (1-\omega^2)B & = 0 \end{cases} \implies \begin{cases} \omega(1-\omega^2)A + \omega^2 B & = \omega \\ -\omega(1-\omega^2)A + (1-\omega^2)^2 B & = 0 \end{cases} \tag{2.68}$$

This shows that

$$B = \frac{\omega}{\omega^2 + (1-\omega^2)^2}, A = \frac{\omega(1-\omega^2)^2}{\omega^2 + (1-\omega^2)^2}\frac{1}{\omega(1-\omega^2)} = \frac{1-\omega^2}{\omega^2 + (1-\omega^2)^2}. \tag{2.69}$$

Then the amplitude is given by

$$\sqrt{A^2 + B^2} = \sqrt{\frac{\omega^2 + (1-\omega^2)^2}{(\omega^2 + (1-\omega^2)^2)^2}} = \sqrt{\frac{1}{\omega^2 + (1-\omega^2)^2}}. \tag{2.70}$$

□

2.2. **Variation of parameters.**

Problem 2.8. Use the variation of parameters formula to find a particular solution to the differential equation

$$(x^2-1)y''(x) - 2xy'(x) + 2y(x) = x^2 - 1,\ x > 1, \tag{2.71}$$

given that the general homogeneous solution is

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 x + c_2(1+x^2),\ x > 1 \tag{2.72}$$

where c_1, c_2 are arbitrary.

Solution. We first compute

$$W(y_1, y_2)(x) = \det \begin{pmatrix} x & 1+x^2 \\ 1 & 2x \end{pmatrix} = 2x^2 - 1 - x^2 = x^2 - 1,\ x > 1. \tag{2.73}$$

Therefore by the variation of parameters formula (here $f(x) = \frac{x^2-1}{x^2-1} = 1$),

$$\begin{aligned} y_p(x) &= -y_1(x) \int \frac{y_2(x) f(x)}{W(y_1,y_2)(x)}\, dx + y_2(x) \int \frac{y_1(x) f(x)}{W(y_1,y_2)(x)}\, dx && (2.74) \\ &= -x \int \frac{1+x^2}{x^2-1}\, dx + (1+x^2) \int \frac{x}{x^2-1}\, dx && (2.75) \\ &= -x \int \frac{x^2-1}{x^2-1} + \underbrace{\frac{2}{x^2-1}}_{\substack{=(x-1)^{-1} \\ -(x+1)^{-1}}}\, dx + \frac{1+x^2}{2} \int \frac{2x}{x^2-1}\, dx && (2.76) \\ &= -x(x + \ln|x-1| - \ln|x+1|) + \frac{1+x^2}{2} \ln\left|x^2-1\right|,\ x > 1 && (2.77) \end{aligned}$$

is a particular solution to the equation. We can ignore the constants of integration here as the constants of integration simply contributes to adding multiples of y_1, y_2 to the particular solution above. $\square$

2.3. **The Laplace transform.**

Exercise 2.9. Solve the IVP

$$y'' + 2y' + 5y = 0, y(0) = 2, y'(0) = -1 \tag{2.78}$$

using the Laplace transform.

Solution. We have

$$\mathcal{L}\{y'' + 2y' + 5y\} = s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + 5Y(s) = (s^2 + 2s + 5)Y(s) - 2s - 3 \tag{2.79}$$

and $\mathcal{L}\{0\} = 0$. Therefore

$$Y(s) = \frac{2s+3}{s^2+2s+5} = \frac{2(s+1)+1}{(s+1)^2+4} \tag{2.80}$$

Now we calculate

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2+4}\right\} = e^{-t}\left(2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}\right) = e^{-t}\left(2\cos(2t) + \frac{1}{2}\sin(2t)\right). \tag{2.81}$$

□

Exercise 2.10. Use the Laplace transform to solve the IVP

$$x'' - 2x' + 10x = 0, x(0) = 3, x'(0) = 8. \tag{2.82}$$

Solution. We have

$$\mathcal{L}\{x'' - 2x' + 10x\} = s^2X(s) - sx(0) - x'(0) - 2(sX(s) - x(0)) + 10X(s) = (s^2 - 2s + 10)X(s) - 3s - 2. \tag{2.83}$$

Therefore

$$X(s) = \frac{3s+2}{(s-1)^2+9} = \frac{3(s-1)+5}{(s-1)^2+9}. \tag{2.84}$$

This implies that

$$x(t) = e^t\mathcal{L}^{-1}\left\{\frac{3s+5}{s^2+9}\right\} = e^t\left(3\cos(3t) + \frac{5}{3}\sin(3t)\right). \tag{2.85}$$

□

Exercise 2.11.

$$2x'' + 36x = \sin(\omega t). \tag{2.86}$$

For what values of ω will the system exhibit resonance? Use the Laplace transform method to find a particular solution in the case where ω is not the resonant frequency.

Solution. We use the initial conditions $x(0) = 0, x'(0) = 0$. The characteristic polynomial is given by

$$2r^2 + 36 = 0 \implies r = 0 \pm 3\sqrt{2}i. \tag{2.87}$$

Thus the system will exhibit resonance if $\omega = 3\sqrt{2}$. Now we assume that $\omega \neq 3\sqrt{2}$ and compute

$$\mathcal{L}\left\{2x'' + 36x\right\} = 2s^2 X(s) + 36X(s) = (2s^2 + 36)X(s), \mathcal{L}\left\{\sin(\omega t)\right\} = \frac{\omega}{s^2 + \omega^2}. \tag{2.88}$$

Then

$$X(s) = \frac{1}{2}\frac{\omega}{(s^2 + \omega^2)(s^2 + 18)} \tag{2.89}$$

Assuming $\omega^2 \neq 18$, we can decompose

$$\frac{\omega}{(s^2 + \omega^2)(s^2 + 18)} = \frac{As + B}{s^2 + \omega^2} + \frac{Cs + D}{s^2 + 18} \tag{2.90}$$

where we see that $A = C = 0$ and

$$\omega = (B + D)s^2 + (18B + \omega^2 D) \implies B = -D, B = \frac{\omega}{18 - \omega^2}. \tag{2.91}$$

Therefore

$$X(s) = \frac{B}{2}\left(\frac{1}{s^2 + \omega^2} - \frac{1}{s^2 + 18}\right) \implies x(t) = \frac{\omega}{2(18 - \omega^2)}\left(\frac{1}{\omega}\sin(\omega t) - \frac{1}{3\sqrt{2}}\sin(3\sqrt{2}t)\right). \tag{2.92}$$

□

Exercise 2.12. Let

$$g(t) = \begin{cases} t, & 0 \le t < 1 \\ 2-t, & 1 \le t < 2 \\ 0, & 2 \le t. \end{cases} \tag{2.93}$$

Find $\mathcal{L}\left\{g(t)\right\}$ and

$$\mathcal{L}^{-1}\left\{\frac{1-2e^{-(s-5)}+e^{-2(s-5)}}{s-5}\right\}. \tag{2.94}$$

Solution. We write

$$\begin{aligned} g(t) &= t + (2-t-t)\mathcal{U}(t-1) + (t-2)\mathcal{U}(t-2) \\ &= t - 2(t-1)\mathcal{U}(t-1) + (t-2)\mathcal{U}(t-2). \end{aligned} \tag{2.95}$$

Then

$$\begin{aligned} \mathcal{L}\left\{g(t)\right\} = \mathcal{L}\left\{t\right\} - 2\mathcal{L}\left\{(t-1)\mathcal{U}(t-1)\right\} + \mathcal{L}\left\{(t-2)\mathcal{U}(t-2)\right\} \\ = \frac{1}{s^2} - 2e^{-s}\frac{1}{s^2} + e^{-2s}\frac{1}{s^2}. \end{aligned} \tag{2.96}$$

For the second part we have

$$\mathcal{L}^{-1}\left\{\frac{1-2e^{-(s-5)}+e^{-2(s-5)}}{s-5}\right\} = e^{5t}\mathcal{L}^{-1}\left\{\frac{1-2e^{-s}+e^{-2s}}{s}\right\} = e^{5t}\left(1-2U(t-1)+U(t-2)\right). \tag{2.97}$$

$\square$

Exercise 2.13. Solve the IVP $x'' - 6x' + 8x = g(t), x(0) = 0, x'(0) = 0$ where

$$g(t) = \begin{cases} 8 & 0 \leq t < 1 \\ 16 - 8t & t \geq 1. \end{cases} \tag{2.98}$$

Solution. First we write

$$g(t) = 8 + (16 - 8t - 8)\mathcal{U}(t-1) = 8(1 - (t-1)\mathcal{U}(t-1)). \tag{2.99}$$

therefore

$$\mathcal{L}\{g(t)\} = 8(\mathcal{L}\{1\} - \mathcal{L}\{(t-1)\mathcal{U}(t-1)\} = 8\left(\frac{1}{s} - e^{-s}\frac{1}{s^2}\right). \tag{2.100}$$

We also have

$$\mathcal{L}\{x'' - 6x' + 8x\} = s^2X(s) - sx(0) - x'(0) - 6(sX(s) - x(0)) + 8X(s) = (s^2 - 6s + 8)X(s). \tag{2.101}$$

Therefore (after partial fraction decomposition)

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{2}{s-2} + \frac{1}{s-4} - e^{-s}\left(\frac{3/4}{s} + \frac{1}{s^2} - \frac{1}{s-2} + \frac{1/4}{s-4}\right)\right\} \\ &= 1 - 2e^{2t} + e^{4t} - \mathcal{U}(t-1)\left(\frac{3}{4} + (t-1) - e^{2(t-1)} + \frac{1}{4}e^{4(t-1)}\right). \end{aligned} \tag{2.102}$$

□

Exercise 2.14. Find the solution to the IVP

$$y'' + 16y = 4\mathcal{U}(t) - 4\mathcal{U}(t - \pi/4), y(0) = 0, y'(0) = 0. \tag{2.103}$$

Solution. We have

$$\mathcal{L}\left\{y'' + 16y\right\} = (s^2 + 16)Y(s), \quad \mathcal{L}\left\{4\mathcal{U}(t) - 4\mathcal{U}(t - \pi/4)\right\} = 4\left(\frac{1}{s} - \frac{e^{-\pi/4s}}{s}\right). \tag{2.104}$$

Therefore

$$Y(s) = 4\left(\frac{1}{s(s^2+16)} - e^{-\pi/4s}\frac{1}{s(s^2+16)}\right) = \frac{1}{4s} - \frac{s}{4(s^2+16)} - e^{-\pi/4s}\left(\frac{1}{4s} - \frac{s}{4(s^2+16)}\right). \tag{2.105}$$

Thus

$$y(t) = \frac{1}{4}\left[1 - \cos(4t) - \mathcal{U}(t - \pi/4)\left(1 - \cos\left(4\left(t - \frac{\pi}{4}\right)\right)\right)\right]. \tag{2.106}$$

□

Exercise 2.15. Compute $\mathcal{L}\{e^{at}g(t)\}$ with

$$g(t) = \begin{cases} 2t & 0 \leq t < 3 \\ 0 & t \geq 3. \end{cases} \tag{2.107}$$

Solution. Write $g(t) = 2t(1 - \mathcal{U}(t-3)) = 2t - 2t\mathcal{U}(t-3)$. Then using Corollary 2.15 and the fact that e^{3a} is a constant,

$$\begin{aligned} \mathcal{L}\left\{e^{at}g(t)\right\} = \mathcal{L}\left\{e^{at}2t\right\} - \mathcal{L}\left\{e^{at}\mathcal{U}(t-3)2t\right\} &= \frac{2}{(s-a)^2} - 2e^{-3s}\mathcal{L}\left\{e^{a(t+3)}(t+3)\right\} \\ &= \frac{2}{(s-a)^2} - 2e^{-3(s-a)}\left(\frac{1}{(s-a)^2} + \frac{3}{s-a}\right). \end{aligned} \tag{2.108}$$

□

Exercise 2.16. Use the Laplace transform to solve

$$x'' + 4x = 1, x(0) = 1, x'(0) = 0. \tag{2.109}$$

Solution. We have

$$\mathcal{L}\left\{x'' + 4x\right\} = s^2 X(s) - sx(0) - x'(0) + 4X(s) = (s^2+4) - s, \quad \mathcal{L}\left\{1\right\} = \frac{1}{s}. \tag{2.110}$$

Therefore

$$X(s) = \frac{1}{s(s^2+4)} + \frac{s}{s^2+4} = \frac{1}{4}\left(\frac{1}{s} - \frac{s}{s^2+4}\right) + \frac{s}{s^2+4}. \tag{2.111}$$

This implies that

$$x(t) = \frac{1}{4}\left(1 - \cos(2t)\right) + \cos(2t). \tag{2.112}$$

$\square$

Exercise 2.17. Use the Laplace transform to solve

$$y'' + 6y' + 13y = \mathcal{U}(t-4\pi)[12\cos(t) + 27\sin(t)], y(0) = 1, y'(0) = -1. \tag{2.113}$$

Solution. We calculate

$$\mathcal{L}\{y'' + 6y' + 13y\} = s^2Y(s) - sy(0) - y'(0) + 6(sY(s) - y(0)) + 13Y(s) = (s^2+6s+13)Y(s) - (s+5) \tag{2.114}$$

and

$$\mathcal{L}\{\mathcal{U}(t-4\pi)[12\cos(t) + 27\sin(t)]\} = e^{-4\pi s}\mathcal{L}\left\{12\underbrace{\cos(t+4\pi)}_{=\cos(t)} + 27\underbrace{\sin(t+4\pi)}_{=\sin(t)}\right\} = e^{-4\pi s}\left(12\frac{s}{s^2+1} + 27\frac{1}{s^2+1}\right). \tag{2.115}$$

Therefore

$$Y(s) = e^{-4\pi s}\frac{12s+27}{(s^2+1)(s^2+6s+13)} + \frac{s+5}{s^2+6s+13} = e^{-4\pi s}\left(\frac{1}{s^2+1} + \frac{2}{(s+3)^2+4}\right) + \frac{(s+3)+2}{(s+3)^2+4}. \tag{2.116}$$

Then

$$\begin{aligned} y(t) &= \mathcal{U}(t-4\pi)\sin(t-4\pi) + e^{-3t}\mathcal{L}^{-1}\left\{e^{-4\pi(s-3)}\frac{2}{s^2+4}\right\} + e^{-3t}\mathcal{L}^{-1}\left\{\frac{s+2}{s^2+4}\right\} \\ &= \mathcal{U}(t-4\pi)\sin(t) + e^{-3t+12\pi}\mathcal{U}(t-4\pi)\underbrace{\sin(2(t-4\pi))}_{=\sin(2t)} + e^{-3t}\left(\cos(2t) + \sin(2t)\right). \end{aligned} \tag{2.117}$$

□

Exercise 2.18. Use the Laplace transform to solve

$$x'' + 4x' + 8x = 11\cos(t) + 3\sin(t), x(0) = 1, x'(0) = -2. \tag{2.118}$$

Solution. We calculate

$$\mathcal{L}\{x'' + 4x' + 8x\} = s^2X(s) - sx(0) - x'(0) + 4(sX(s) - x(0)) + 8X(s) = (s^2 + 4s + 8)X(s) - s - 2 \tag{2.119}$$

and

$$\mathcal{L}\{11\cos(t) + 3\sin(t)\} = \frac{11s + 3}{s^2 + 1}. \tag{2.120}$$

Thus

$$X(s) = \frac{11s + 3}{(s^2 + 4s + 8)(s^2 + 1)} + \frac{s + 2}{(s + 2)^2 + 4} = \frac{s + 1}{s^2 + 1} - \frac{(s + 2) + 3}{(s + 2)^2 + 4} + \frac{s + 2}{(s + 2)^2 + 4} \tag{2.121}$$

and

$$x(t) = \cos(t) + \sin(t) - e^{-2t}\left(\cos(2t) + \frac{3}{2}\sin(2t)\right) + e^{-2t}\cos(2t) = \cos(t) + \sin(t) - \frac{3}{2}e^{2t}\sin(2t). \tag{2.122}$$

□

Exercise 2.19. For $n > 0$, let

$$f_n(t) = \begin{cases} 1 - \frac{t}{n} & 0 \le t < n \\ 0 & t \ge n. \end{cases} \tag{2.123}$$

Compute $F_n(s) = \mathcal{L}\{f_n(t)\}$ and $\lim_{n\to\infty} F_n(s)$.

Solution. We have

$$f_n(t) = \left(1 - \frac{t}{n}\right) + \frac{1}{n}(t-n)\mathcal{U}(t-n) \implies F_n(s) = \frac{1}{s} - \frac{1}{ns^2} + \frac{1}{n}\frac{e^{-ns}}{s^2}. \tag{2.124}$$

Then

$$F_n(s) \to \frac{1}{s} \tag{2.125}$$

as $n \to \infty$ (notice that $f_n \to 1$ pointwise). □

Exercise 2.20. Let f be the function defined by

$$f(t) = \begin{cases} t, & 0 \leq t < 1 \\ t-1, & 1 \leq t < 2 \\ 0, & t \geq 2. \end{cases} \tag{2.126}$$

Find the Laplace transform of f *using the definition* of the Laplace transform.

Solution. By definition

$$\mathcal{L}\left\{f(t)\right\}(s) = \int_0^1 e^{-st} t \, dt + \int_1^2 e^{-st}(t-1) \, dt = \int_0^1 e^{-st} t \, dt + \int_0^1 e^{-s(t+1)} t \, dt = (1 + e^{-s}) I(s) \tag{2.127}$$

where

$$I(s) = \int_0^1 e^{-st} t \, dt = t \frac{e^{-st}}{-s} \bigg|_0^1 + \frac{1}{s} \int_0^1 e^{-st} \, dt = \frac{-e^{-s}}{s} + \frac{e^{-st}}{-s^2} \bigg|_0^1 = \frac{-e^{-s}}{s} + \frac{1}{s^2} - \frac{e^{-s}}{s^2}. \tag{2.128}$$

□