Homework 5

DUE: SATURDAY, FEBRUARY 22, 11:59PM

If you completed this assignment through collaboration or consulted references, please list the names of your collaborators and the references you used below. Please refer to the syllabus for the course policy on collaborations and types of references that are allowed.

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Problem 5.1 (Ansatz for undetermined coefficients)**.** Determine an appropriate ansatz for a particular solution y_p to the following equations, without determining the values of the coefficients.

a) $y''(x) - 2y'(x) + 2y(x) = e^x \sin x,\ x \in \mathbb{R}$.
b) $y''(x) + 4y(x) = 3x \cos 2x,\ x \in \mathbb{R}$.
c) $y''(x) + 3y'(x) + 2y(x) = x(e^{-x} - e^{-2x}),\ x \in \mathbb{R}$.
d) $y^{(4)}(x) - 2y''(x) + y(x) = x^2 \cos x,\ x \in \mathbb{R}$.

Solution.

a) We first compute the general homogeneous solution. We note that the characteristic equation is $r^2 - 2r + 2 = 0 \implies r = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$. So the general homogeneous solution is $y_h(x) = c_1 e^x \cos x + c_2 e^x \sin x, x \in \mathbb{R}$ and c_1, c_2 are arbitrary. Thus our ansatz should be $y_p(x) = x(A \cos x + B \sin x)e^x, x \in \mathbb{R}$ to avoid duplications with y_h.
b) The homogeneous solution here is $y_h(x) = c_1 \cos 2x + c_2 \sin 2x, x \in \mathbb{R}$. Thus our ansatz should be $y_p(x) = x(Ax + B) \cos 2x + x(Cx + D) \sin 2x, x \in \mathbb{R}$.
c) The characteristic equation here is $r^2 + 3r + 2 = (r + 1)(r + 2) = 0 \implies r = -1, -2$, and therefore the general homogeneous solution is $y_h(x) = c_1 e^{-x} + c_2 e^{-2x}, x \in \mathbb{R}$. Therefore the ansatz here should be $y_p(x) = x(Ax + B)e^{-x} + x(Cx + D)e^{-2x}, x \in \mathbb{R}$.
d) The characteristic equation here is $r^4 - 2r^2 + 1 = (r^2 - 1)^2 = 0 \implies r = -1, 1$ with multiplicity 2. Therefore the general homogeneous solution is $y_h(x) = c_1 e^{-x} + c_2 x e^{-x} + c_3 e^x + c_4 x e^x, x \in \mathbb{R}$. An ansatz for this problem would be $y_p(x) = (Ax^2 + Bx + C) \cos x + (Cx^2 + Dx + E) \sin x$.

$\square$

Problem 5.2 (An inhomogeneous 2nd order equation)**.** Solve the initial value problem

$$\begin{cases} y''(x) - 4y(x) = 2e^{2x},\ x \in \mathbb{R} \\ y(0) = 0 \\ y'(0) = \frac{9}{2}. \end{cases} \tag{5.1}$$

Solution. We first identify the general homogeneous solution, which is the general solution to the homogeneous equation

$$y''(x) - 4y(x) = 0,\ x \in \mathbb{R}. \tag{5.2}$$

The characteristic equation is

$$r^2 - 4 = (r-2)(r+2) = 0, \tag{5.3}$$

therefore the general homogeneous solution is

$$y_h(x) = c_1 e^{-2x} + c_2 e^{2x},\ x \in \mathbb{R}, \tag{5.4}$$

where c_1, c_2 are arbitrary. To find a particular solution to the inhomogeneous equation we use the method of undetermined coefficients. We note that since the right hand side of the original equation is $2e^{2x}$ is e^{2x} is in the fundamental set of solutions for the homogeneous problem, we must use the ansatz $y_p = Axe^{2x}$. Then for all $x \in \mathbb{R}$,

$$y_p(x) = e^{2x}(Ax) \tag{5.5}$$

$$y_p'(x) = e^{2x}(2Ax + A) \tag{5.6}$$

$$y_p''(x) = e^{2x}(2(2Ax + A) + 2A) = e^{2x}(4Ax + 4A). \tag{5.7}$$

Thus if y_p is a solution to the inhomogeneous equation, we must have

$$y_p''(x) - 4y_p(x) = e^{2x}(4Ax + 4A - 4Ax) = 4Ae^{2x} = 2e^{2x},\ x \in \mathbb{R} \implies A = \frac{1}{2}. \tag{5.8}$$

Therefore the general solution to the inhomogeneous problem is

$$y(x) = \frac{1}{2}xe^{2x} + c_1 e^{-2x} + c_2 e^{2x},\ x \in \mathbb{R}, \tag{5.9}$$

where c_1, c_2 are arbitrary. This implies that

$$y'(x) = \frac{1}{2}e^{2x}(2x+1) - 2c_1 e^{-2x} + 2c_2 e^{2x},\ x \in \mathbb{R}. \tag{5.10}$$

Therefore if $y(0) = 0$ and $y'(0) = \frac{9}{2}$, we must have

$$c_1 + c_2 = 0 \tag{5.11}$$

$$\frac{1}{2} - 2c_1 + 2c_2 = \frac{9}{2}. \tag{5.12}$$

This implies that

$$c_1 = -c_2 \text{ and } 4c_2 = 4 \implies -c_1 = c_2 = 1. \tag{5.13}$$

Therefore the solution to the initial value problem is

$$y(x) = \frac{1}{2}xe^{2x} - e^{-2x} + e^{2x},\ x \in \mathbb{R}. \tag{5.14}$$

$\square$

Problem 5.3 (An inhomogeneous 3rd order equation)**.** Consider the third order differential equation

$$y^{(3)}(x) + y''(x) = 3e^x + 4x^2,\ x \in \mathbb{R}. \tag{5.15}$$

Find the general solution to the equation.

Solution. We begin by finding the general solution to the homogeneous equation. The characteristic equation is

$$r^3 + r^2 = r^2(r+1) = 0 \implies r = 0, -1 \tag{5.16}$$

where $r = 0$ is a repeating root. Therefore the general homogeneous solution is

$$y_h(x) = c_1 + c_2 x + c_3 e^{-x},\ x \in \mathbb{R} \tag{5.17}$$

and c_1, c_2, c_3 are arbitrary. We then use the ansatz

$$y_p(x) = Ae^x + x^2(Bx^2 + Cx + D) = Ae^x + (Bx^4 + Cx^3 + Dx^2),\ x \in \mathbb{R} \tag{5.18}$$

and calculate for all $x \in \mathbb{R}$,

$$y_p'(x) = Ae^x + 4Bx^3 + 3Cx^2 + 2Dx \tag{5.19}$$

$$y_p''(x) = Ae^x + 12Bx^2 + 6Cx + 2D \tag{5.20}$$

$$y_p^{(3)}(x) = Ae^x + 24Bx + 6C \tag{5.21}$$

Thus

$$y_p^{(3)}(x) + y_p''(x) = 2Ae^x + 12Bx^2 + (24B + 6C)x + (2D + 6C) = 3e^x + 4x^2 \text{ for all } x \in \mathbb{R}. \tag{5.22}$$

Thus

$$A = \frac{3}{2},\ B = \frac{1}{3},\ C = -\frac{4}{3},\ D = 4. \tag{5.23}$$

Therefore the general solution to the equation is

$$y(x) = c_1 + c_2 x + c_3 e^{-x} + \frac{3}{2}e^x + \frac{1}{3}x^4 - \frac{4}{3}x^3 + 4x^2,\ x \in \mathbb{R} \tag{5.24}$$

where c_1, c_2, c_3 are arbitrary. □

Problem 5.4 (Mass-spring systems)**.** A mass-spring system has the following properties: the mass is 2 kilograms, and the spring exerts a force of 6 Newtons (one Newton is equal to $1\mathrm{kg}\frac{\mathrm{m}}{\mathrm{s}^2}$) when the mass is displaced 2 meters from its equilibrium position, and a viscous force of 5 Newtons slows the system when the mass moves with velocity of 1 meter per second.

a) What is the spring constant k and the damping constant β?
b) Write down an equation of the form
$$x''(t) + 2\lambda x'(t) + \omega^2 x(t) = 0,\ t \in \mathbb{R} \tag{5.25}$$
that models the position of the mass.
c) Is the system underdamped, critically damped or overdamped?
d) If $x(0) = 1$ and $x'(0) = -9$, does the mass ever cross the equilibrium point at some finite time $t > 0$?

Solution. Based on the problem, $k = 3\frac{\mathrm{kg}}{\mathrm{s}^2}$ and $\beta = 5\frac{\mathrm{kg}}{\mathrm{s}}$. Therefore the mass spring system is modeled via the equation
$$x''(t) + \frac{5}{2}x'(t) + \frac{3}{2}x(t) = 0,\ t \in \mathbb{R}. \tag{5.26}$$
The characteristic equation associated to this differential equation is
$$2r^2 + 5r + 3 = (2r+3)(r+1) = 0, \tag{5.27}$$
therefore the roots are $r = -1, -\frac{3}{2}$. This tells us that the system is overdamped and the general solution is
$$x(t) = c_1 e^{-t} + c_2 e^{-3/2t},\ t \in \mathbb{R}. \tag{5.28}$$
We note that this implies
$$x'(t) = -c_1 e^{-t} - \frac{3}{2}c_2 e^{-3/2t},\ t \in \mathbb{R}. \tag{5.29}$$
Therefore if $x(0) = 1, x'(0) = -9$, we must have
$$c_1 + c_2 = 1 \tag{5.30}$$
$$-c_1 - \frac{3}{2}c_2 = -9. \tag{5.31}$$
This immediately implies that
$$-\frac{1}{2}c_2 = -8 \implies c_2 = 16 \implies c_1 = -15. \tag{5.32}$$
Therefore the solution to the initial value problem with $x(0) = 1, x'(0) = -9$ is
$$x(t) = -15e^{-t} + 16e^{-3/2t} = e^{-3/2t}(16 - 15e^{1/2t}),\ t \in \mathbb{R}. \tag{5.33}$$
If $x(t) = 0$, then we must have
$$16 - 15e^{1/2t} = 0 \iff e^{1/2t} = \frac{16}{15} \iff t = 2\ln\frac{16}{15} > 0 \tag{5.34}$$
since $\frac{16}{15} > 1$, so here we see that the mass crosses the equilibrium point once for some finite time $t > 0$.

□

Problem 5.5 (Mass-spring systems)**.** Suppose a mass-spring system is modeled via the initial value problem

$$\begin{cases} 10x''(t) + 9x'(t) + 2x(t) = 0,\ t > 0 \\ x(0) = 0 \\ x'(0) = 5. \end{cases} \tag{5.35}$$

a) Find the solution to the initial value problem on the interval $I = (0, \infty)$. Is the system underdamped, critically damped, or overdamped?
b) Does the mass ever reach back to the equilibrium position for some finite time $t > 0$?
c) Identify the time intervals on which the mass is above the equilibrium point and below the equilibrium point.
d) Identify the time intervals on which the mass is moving away from the equilibrium point and towards the equilibrium point.
e) Suppose the unit of length is in meters. How far does the mass move to the bottom before it starts moving back towards the equilibrium point?
f) Sketch a rough graph of the position function x on the t-x axis.

Solution. The characteristic equation associated to the differential equation is

$$10r^2 + 9r + 2 = (5r + 2)(2r + 1) = 0, \tag{5.36}$$

therefore $r = -\frac{2}{5}, -\frac{1}{2}$. Therefore the system is overdamped and the general solution is

$$x(t) = c_1 e^{-2/5t} + c_2 e^{-1/2t},\ t \in \mathbb{R}. \tag{5.37}$$

This implies that

$$x'(t) = -\frac{2}{5} c_1 e^{-2/5t} - \frac{1}{2} c_2 e^{-1/2t},\ t \in \mathbb{R}. \tag{5.38}$$

Therefore if $x(0) = 0$ and $x'(0) = 5$, we must have

$$c_1 + c_2 = 0 \tag{5.39}$$

$$-\frac{2}{5} c_1 - \frac{1}{2} c_2 = 5. \tag{5.40}$$

This implies that

$$c_2 = -c_1 \text{ and } -\frac{1}{10} c_2 = 5 \implies -c_1 = c_2 = -50. \tag{5.41}$$

Therefore the solution to the initial value problem is

$$x(t) = 50e^{-2/5}t - 50e^{-1/2t} = 50e^{-1/2t}(e^{1/10t} - 1),\ t \in \mathbb{R}. \tag{5.42}$$

From here we see that $x(t) = 0$ if

$$e^{1/10t} - 1 = 0 \iff e^{1/10t} = 1 \iff t = 0. \tag{5.43}$$

So we see that the mass never reach back to the equilibrium for any positive time.

We note that the mass is below the equilibrium point when $x > 0$ and above the equilibrium point when $x < 0$. Since $50e^{-1/2t} > 0$ for all $t \in \mathbb{R}$ and $e^{1/10t} > 1$ for all $t > 0$, we see that the mass stays below the equilibrium point for all $t > 0$.

Since the mass is always below the equilibrium point, the mass is moving away from the equilibrium point when the velocity is positive and moving towards the equilibrium point when the velocity is negative. We calculate

$$x'(t) = 50e^{-1/2t}\left(\frac{1}{10}e^{1/10t} - \frac{1}{2}(e^{1/10t} - 1)\right) = 25e^{-1/2t}\left(-\frac{4}{5}e^{1/10t} + 1\right). \tag{5.44}$$

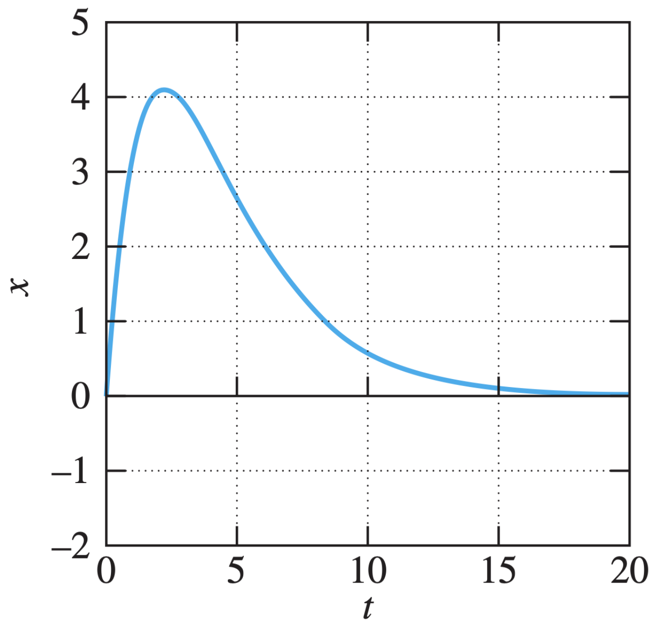
Here $x'(t) = 0$ when

$$-\frac{4}{5}e^{1/10t} + 1 = 0 \Longleftarrow e^{1/10t} = \frac{5}{4} \iff t = 10\ln\frac{5}{4} > 0. \tag{5.45}$$

Since the initial velocity is positive, it's not hard to see that $x' > 0$ up to time $10\ln\frac{5}{4}$ and $x' < 0$ after time $10\ln\frac{5}{4}$. Therefore the mass is moving away from the equilibrium on the interval $(0, 10\ln\frac{5}{4})$ and towards the equilibrium point on the interval $(10\ln\frac{5}{4}, \infty)$.

The only critical points in consideration here is $t = 0$ and $t = 10\ln\frac{5}{4}$, and since the initial velocity is positive and $x(t) > 0$ for all $t > 0$, the absolute maximum of x is $x(10\ln\frac{5}{4})$. So it moves $x(10\ln\frac{5}{4})$ meters to the bottom before it starts moving back to the equilibrium point.

Here is a rough sketch of the graph of x:



$\square$

Problem 5.6 (Pure resonance)**.** Consider a forced undamped mass-spring system modeled via the equation

$$mx''(t) + kx(t) = F_0 \cos \omega t, \ t \in \mathbb{R} \tag{5.46}$$

where $m, k, F_0, \omega \in \mathbb{R} \setminus \{0\}$ with $m, k > 0$. Define $\omega_0 > 0$ via

$$\omega_0 = \sqrt{\frac{k}{m}}. \tag{5.47}$$

a) Assuming $\omega \neq \omega_0$, use the method of undetermined coefficients to find a particular solution to (5.46). What happens if $\omega \to \omega_0$?
b) Now suppose $\omega = \omega_0$. Use the method of undetermined coefficients to find the general solution to (5.46). What happens when $t \to \infty$?

Solution. We note that the general homogeneous solution is

$$x_h(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0, \ t \in \mathbb{R}. \tag{5.48}$$

If $\omega \neq \omega_0$, then we may use the ansatz

$$x_p(t) = A \cos \omega t + B \sin \omega t, \ t \in \mathbb{R}. \tag{5.49}$$

We then calculate

$$x_p'(t) = B\omega \cos \omega t - A\omega \sin \omega t \tag{5.50}$$

$$x_p''(t) = -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t, \ t \in \mathbb{R}. \tag{5.51}$$

Thus

$$x_p''(t) + \omega_0^2 x_p(t) = (-A\omega^2 + A\omega_0^2) \cos \omega t + (-B\omega^2 + B\omega_0^2) \sin \omega t = \frac{F_0}{m} \cos \omega t \text{ for all } t \in \mathbb{R}. \tag{5.52}$$

Thus we must have

$$A = \frac{F_0}{m(\omega_0^2 - \omega^2)}, \ B = 0. \tag{5.53}$$

Therefore a particular solution is

$$x_p(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t, \ t \in \mathbb{R}. \tag{5.54}$$

If $\omega = \omega_0$, then we need to modify our initial ansatz and use

$$x_p(t) = At \cos \omega t + Bt \sin \omega t, \ t \in \mathbb{R}. \tag{5.55}$$

Thus

$$x_p'(t) = (B\omega t + A) \cos \omega t + (-A\omega t + B) \sin \omega t, \tag{5.56}$$

$$x_p''(t) = (-A\omega^2 t + B\omega + B\omega) \cos \omega t + (-B\omega^2 t - A\omega - A\omega) \sin \omega t \tag{5.57}$$

$$= (-A\omega^2 t + 2B\omega) \cos \omega t + (-B\omega^2 t - 2A\omega) \sin \omega t \tag{5.58}$$

for $t \in \mathbb{R}$. Thus

$$x_p''(t) + \omega^2 x_p(t) = (-A\omega^2 t + 2B\omega + A\omega^2 t) \cos \omega t + (-B\omega^2 t - 2A\omega + B\omega^2 t) \sin \omega t \tag{5.59}$$

$$= 2B\omega \cos \omega t - 2A\omega \sin \omega t = \frac{F_0}{m} \cos \omega t, \ t \in \mathbb{R}. \tag{5.60}$$

Thus $A = 0$ and $B = \frac{F_0}{2m\omega}$. Therefore a particular solution is

$$x_p(t) = \frac{F_0}{2m\omega} t \sin \omega t, \ t \in \mathbb{R} \tag{5.61}$$

Note that

$$-\frac{F_0}{2m\omega} t \leq x_p(t) \leq \frac{F_0}{2m\omega} t \tag{5.62}$$

for all $t \in \mathbb{R}$ and $\frac{F_0}{2m\omega} t \to \infty$ as $t \to \infty$. Therefore as $t \to \infty$, we see that the amplitude grows without bound. $\square$