Reference sheet

Calculus

a) Quadratic formula: the roots of the quadratic polynomial $ax^2 + bx + c, a \neq 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \ a \in \mathbb{R} \setminus \{0\}, b, c \in \mathbb{R}.$$

b) Exponential and logarithmic functions, assuming $b \in (0, \infty) \setminus \{1\}, x \in (0, \infty), y \in \mathbb{R}$:

- $\log_b x = y \iff b^y = x$
- $\ln x = \log_e x$, where $e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$
- $\log_b b^x = x$ and $b^{\log_b x} = x$

c) Laws of logarithms: assuming $b \in (0, \infty) \setminus \{1\}, x, y \in (0, \infty), \alpha \in \mathbb{R}$:

- $\log_b(xy) = \log_b x + \log_b y$
- $\log_b \frac{x}{y} = \log_b x - \log_b y$
- $\log_b x^\alpha = \alpha \log_b x$

d) Inverse trigonometric functions:

- $y = \arcsin x \iff x = \sin y, \ -1 \leq x \leq 1, \ -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
- $y = \arccos x \iff x = \cos y, \ -1 \leq x \leq 1, \ 0 \leq y \leq \pi$
- $y = \arctan x \iff x = \tan y, \ x \in \mathbb{R}, \ -\frac{\pi}{2} < y < \frac{\pi}{2}$
- $y = \operatorname{arccot} x \iff x = \cot y, \ x \in \mathbb{R}, \ 0 < y < \pi$
- $y = \operatorname{arcsec} x \iff x = \sec y, \ x \in (-\infty, -1] \cup [1, \infty), \ y \in [0, \pi/2) \cup (\pi/2, \pi]$
- $y = \operatorname{arccsc} x \iff x = \csc y, \ x \in (-\infty, -1] \cup [1, \infty), \ y \in [-\pi/2, 0) \cup (0, \pi/2]$

e) Trigonometric identities

- Pythagorean theorem:

$$\sin^2 x + \cos^2 x = 1, \ x \in \mathbb{R}.$$

As a result we also have

$$1 + \cot^2 x = \csc^2 x, \ x \in \mathbb{R} \setminus \{x \mid \sin x = 0\} \text{ and } \tan^2 x + 1 = \sec^2 x, \ x \in \mathbb{R} \setminus \{x \mid \cos x = 0\}.$$

- Angle addition and subtraction:

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta,$$

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta, \ \alpha, \beta \in \mathbb{R}.$$

- Double angle formulas:

$$\sin 2\theta = 2\sin\theta\cos\theta,$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta,\ \theta \in \mathbb{R}.$$

- Half angle formulas:

$$\sin\frac{\theta}{2} = \operatorname{sgn}\left(\sin\frac{\theta}{2}\right)\sqrt{\frac{1-\cos\theta}{2}} \implies \sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2},$$
$$\cos\frac{\theta}{2} = \operatorname{sgn}\left(\cos\frac{\theta}{2}\right)\sqrt{\frac{1+\cos\theta}{2}} \implies \cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2},\ \theta \in \mathbb{R}.$$

- Product to sum formulas: for $a, b, \in \mathbb{R}$,

$$\sin(ax)\sin(bx) = \frac{1}{2}\left[\cos((a-b)x) - \cos((a+b)x)\right]$$
$$\sin(ax)\cos(bx) = \frac{1}{2}\left[\sin((a-b)x) + \sin((a+b)x)\right]$$
$$\cos(ax)\cos(bx) = \frac{1}{2}\left[\cos((a-b)x) + \cos((a+b)x)\right],\ x \in \mathbb{R}.$$

f) Derivatives

1) Exponential and logarithmic functions, assuming $b \in (0,\infty) \setminus \{1\}$:
   - $\frac{d}{dx}\left(b^x\right) = \ln b \cdot b^x,\ x \in \mathbb{R}.$
   - If $f : I \to \mathbb{R}$ is differentiable, then $\frac{d}{dx}\left(b^{f(x)}\right) = \ln b \cdot b^{f(x)} \cdot f'(x),\ x \in I.$
   - $\frac{d}{dx}\left(\log_b |x|\right) = \frac{1}{\ln b} \cdot \frac{1}{x},\ x \in \mathbb{R} \setminus \{0\}.$

2) Trigonometric functions:
   - $\frac{d}{dx}(\sin x) = \cos x,\ \frac{d}{dx}(\cos x) = -\sin x,\ x \in \mathbb{R}.$
   - $\frac{d}{dx}(\tan x) = \sec^2 x,\ \frac{d}{dx}(\sec x) = \sec x \tan x,\ x \in \mathbb{R} \setminus \{x \in \mathbb{R} \mid \cos x = 0\}.$
   - $\frac{d}{dx}(\cot x) = -\csc^2 x,\ \frac{d}{dx}(\csc x) = -\csc x \cot x,\ x \in \mathbb{R} \setminus \{x \in \mathbb{R} \mid \sin x = 0\}.$

3) Inverse trigonometric functions:
   - $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}},\ \frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}},\ x \in (-1,1).$
   - $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2},\ \frac{d}{dx}(\operatorname{arccot} x) = -\frac{1}{1+x^2},\ x \in \mathbb{R}.$
   - $\frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{|x|\sqrt{x^2-1}},\ \frac{d}{dx}(\operatorname{arccsc} x) = -\frac{1}{|x|\sqrt{x^2-1}},\ x \in (-\infty,-1) \cup (1,\infty).$

4) Absolute value:

- $\dfrac{d}{dx}|x| = \dfrac{x}{|x|} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases} = \operatorname{sgn} x,\ x \in \mathbb{R} \setminus \{0\}.$
- If $f : I \to \mathbb{R}$ is differentiable, then $\dfrac{d}{dx}|f(x)| = \dfrac{f(x)}{|f(x)|} f'(x),\ x \in I \setminus \{x \mid f(x) = 0\}.$

g) Anti-derivatives (C denotes an arbitrary real constant in the identities to follow):

- $\displaystyle\int x^\alpha\, dx = \frac{x^{\alpha+1}}{\alpha+1} + C,\ \alpha \neq -1,\ x \in \mathbb{R} \setminus \{0\}$ if $\alpha < -1,\ x \in \mathbb{R}$ otherwise
- $\displaystyle\int \frac{1}{x}\, dx = \ln|x| + C = \begin{cases} \ln x + C_1, & x > 0 \\ \ln(-x) + C_2, & x < 0, \end{cases} \quad C_1, C_2 \in \mathbb{R}.$
- $\displaystyle\int a^x\ \ dx = \frac{a^x}{\ln a},\ x \in \mathbb{R}, a \in (0, \infty) \setminus \{1\}$
- $\displaystyle\int \tan x\, dx = \ln|\sec x| + C_n = -\ln|\cos x| + C_n,\ x \in \left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right),\ n \in \mathbb{Z}$
- $\displaystyle\int \cot x\, dx = \ln|\sin x| + C_n = -\ln|\csc x| + C_n,\ x \in (n\pi, (n+1)\pi),\ n \in \mathbb{Z}$
- $\displaystyle\int \sec x\, dx = \ln|\sec x + \tan x| + C_n,\ x \in \left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right),\ n \in \mathbb{Z}$
- $\displaystyle\int \csc x\, dx = -\ln|\csc x + \cot x| + C_n,\ x \in (n\pi, (n+1)\pi),\ n \in \mathbb{Z}$
- $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \arcsin\frac{x}{a} + C,\ x \in (-a, a),\ a \in (0, \infty)$
- $\displaystyle\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a}\arctan\frac{x}{a} + C,\ x \in \mathbb{R}$
- $\displaystyle\int \frac{1}{x\sqrt{x^2 - a^2}}\, dx = \frac{1}{a}\operatorname{arcsec}\left|\frac{x}{a}\right| + C,\ x \in (-\infty, -a) \cup (a, \infty),\ a \in (0, \infty)$

h) Integration formulas

- Change of variables: if $f : I \to \mathbb{R}$ is continuous and $g : [a, b] \to I$ is differentiable and $g' : (a, b) \to \mathbb{R}$ is continuous, then
$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$
- Integration by parts: if $f, g : [a, b] \to \mathbb{R}$ are differentiable, then
$$\int_a^b f'(x)g(x)\, dx = f(x)g(x)\Big|_{x=a}^{x=b} - \int_a^b f(x)g'(x)\, dx.$$
- Symmetry: if $f : [-a, a] \to \mathbb{R}$ is continuous and even, then $\displaystyle\int_{-a}^a f(x)\, dx = 2\int_0^a f(x)\, dx$; if $f : [-a, a] \to \mathbb{R}$ is continuous and odd, then $\displaystyle\int_{-a}^a f(x)\, dx = 0$.

Table of Laplace transforms

f in the t-domain	$F = \mathcal{L}\left\{f\right\}$ in the s-domain
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$t^n, n \in \mathbb{Z}$	$\frac{n!}{s^{n+1}}$
$t^\alpha, \alpha > -1$	$\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$y'(t)$	$sY(s) - y(0)$
$y''(t)$	$s^2Y(s) - sy(0) - y'(0)$
$y^{(n)}(t)$	$s^nY(s) - s^{n-1}y(0) - \ldots - sy^{(n-2)}(0) - y^{(n-1)}(0)$
$e^{at}f(t)$	$F(s-a)$
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$
$f(t)\mathcal{U}(t-a)$	$e^{-as}\mathcal{L}\{f(t+a)\}$
$\mathcal{U}(t-a)$	$\frac{e^{-as}}{s}$
$\delta(t)$	1
$\delta(t-t_0),\ t_0 > 0$	e^{-st_0}
$t^nf(t)$	$(-1)^n\frac{d^n}{ds^n}F(s)$
$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\, d\tau$	$\mathcal{L}\left\{F\right\}(s) \cdot \mathcal{L}\left\{G\right\}(s)$

Table of ansatzes for the method of undetermined coefficients

f	ansatz y_p
Polynomial $P_m(x) = a_0 + \ldots + a_m x^m$	Polynomial $x^s(A_0 + \ldots + A_m x^m)$
$a\cos kx + b\sin kx$ (one of a or b can be 0)	$x^s(A\cos kx + B\sin kx)$
$e^{rx}(a\cos kx + b\sin kx)$	$x^s e^{rx}(A\cos kx + B\sin kx)$
$P_m(x)e^{rx}$	$x^s(A_0 + \ldots + A_m x^m)e^{rx}$
$P_m(x)(a\cos kx + b\sin kx)$	$x^s\left[(A_0 + \ldots + A_m x^m)\cos kx + (B_0 + \ldots + B_m x^m)\sin kx\right]$
$P_m(x)e^{rx}(a\cos kx + b\sin kx)$	$x^s\left[(A_0 + \ldots + A_m x^m)e^{rx}\cos kx + (B_0 + \ldots + B_m x^m)e^{rx}\sin kx\right]$

Table of solutions to $\boldsymbol{X}' = A\boldsymbol{X} \in \mathbb{R}^2$

Eigenvalues of A	General solution $\boldsymbol{X}$
$\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq \lambda_2$	$c_1 e^{\lambda_1 t}\boldsymbol{v}_1 + c_2 e^{\lambda_2 t}\boldsymbol{v}_2$
$\lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1 = \lambda_2$	$\begin{cases} c_1 e^{\lambda t}\boldsymbol{v}_1 + c_2 t e^{\lambda t}\boldsymbol{v}_2, & \text{if } \boldsymbol{v}_1, \boldsymbol{v}_2 \text{ are linearly independent} \\ c_1 e^{\lambda t}\boldsymbol{v} + c_2(t e^{\lambda t}\boldsymbol{v} + e^{\lambda t}\boldsymbol{w}), & \boldsymbol{w} \text{ is a generalized eigenvector associated to } \boldsymbol{v} \end{cases}$
$\lambda = \alpha + i\beta$ $(\beta \neq 0)$ $\boldsymbol{p} = \boldsymbol{B}_1 + i\boldsymbol{B}_2$	$c_1 e^{\alpha t}\left[\cos(\beta t)\boldsymbol{B}_1 - \sin(\beta t)\boldsymbol{B}_2\right] + c_2 e^{\alpha t}\left[\cos(\beta t)\boldsymbol{B}_2 + \sin(\beta t)\boldsymbol{B}_1\right]$ or $c_1 \mathrm{Re}(e^{\lambda t}\boldsymbol{p}) + c_2 \mathrm{Im}(e^{\lambda t}\boldsymbol{p})$

The Matrix Exponential

Method of computation	e^{tA}
Diagonalization $A = PDP^{-1}$	$e^{tA} = Pe^{tD}P^{-1}$
Matrix-valued Laplace transform	$e^{tA} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}(t)$

Inverse of 2×2 matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{if} \quad ad - bc \neq 0.$$

Stability classification in 2D

Eigenvalues	Real Parts	Classification
Real, distinct	Both negative	Stable node
Real, distinct	Both positive	Unstable node
Real, distinct	Opposite signs	Saddle point
Real, repeated	Negative	Degenerate stable node
Real, repeated	Positive	Degenerate unstable node
Purely imaginary	Zero	Center
Complex	Negative	Stable spiral point
Complex	Positive	Unstable spiral point

Linearization of planar systems

Planar system	Linearization
$\boldsymbol{X}_0$ critical point	$\boldsymbol{X}'(t) = A(\boldsymbol{X}(t) - \boldsymbol{X}_0)$
$\begin{cases} x'(t) &= P(x(t), y(t)), \\ y'(t) &= Q(x(t), y(t)),\ t \in \mathbb{R} \end{cases}$	$A = \begin{pmatrix} \partial_x P(x_0, y_0) & \partial_y P(x_0, y_0) \\ \partial_x Q(x_0, y_0) & \partial_y Q(x_0, y_0) \end{pmatrix}$

If $\boldsymbol{X}_0$ is a stable node, stable spiral point, unstable spiral point, unstable node, or a saddle point for the linear system, then we adopt the same classification system for the nonlinear system. The trajectories for the nonlinear system will have the same general geometric features as the trajectories of the linear system locally, in a neighborhood of the critical point.

Fourier series

Function and associated series	Fourier coefficients
$f : [-L, L] \to \mathbb{R}$, Fourier series $f(x) \sim \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right]$	$a_0 = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\,dx$ $a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)\,dx,\ n \geq 1$ $b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)\,dx,\ n \geq 1$
$f : [0, L] \to \mathbb{R}$, Fourier series $f(x) \sim \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2n\pi}{L}x\right) + b_n \sin\left(\frac{2n\pi}{L}x\right)\right]$	$a_0 = \dfrac{2}{L}\displaystyle\int_{0}^{L} f(x)\,dx$ $a_n = \dfrac{2}{L}\displaystyle\int_{0}^{L} f(x)\cos\left(\frac{2n\pi}{L}x\right)\,dx,\ n \geq 1$ $b_n = \dfrac{2}{L}\displaystyle\int_{0}^{L} f(x)\sin\left(\frac{2n\pi}{L}x\right)\,dx,\ n \geq 1$
$f : [0, L] \to \mathbb{R}$, half-range Fourier cosine series $f(x) \sim \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$	$a_0 = \dfrac{2}{L}\displaystyle\int_{0}^{L} f(x)\,dx$ $a_n = \dfrac{2}{L}\displaystyle\int_{0}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)\,dx,\ n \geq 1$
$f : [0, L] \to \mathbb{R}$, half-range Fourier sine series $f(x) \sim \displaystyle\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$	$b_n = \dfrac{2}{L}\displaystyle\int_{0}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)\,dx,\ n \geq 1$

Partial differential equations

PDE	Explicit solutions formulas
Heat equation, homogeneous Dirichlet boundary conditions $\begin{cases} u_t = k u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$	$u(x,t) = \displaystyle\sum_{n=1}^{\infty} A_n \exp\left(-k\frac{n^2\pi^2}{L^2}t\right)\sin\left(\frac{n\pi}{L}x\right)$
Wave equation, homogeneous Dirichlet boundary conditions $\begin{cases} u_{tt} = \alpha^2 u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$	$u(x,t) = \displaystyle\sum_{n=1}^{\infty} \left(A_n \cos\frac{\alpha n\pi}{L}t + B_n \sin\frac{\alpha n\pi}{L}t\right)\sin\left(\frac{n\pi}{L}x\right)$