Review Part 2

Department of Mathematical Sciences, Carnegie Mellon University
E-mail address: jkoganem@andrew.cmu.edu.
Date: January 8, 2025.

Part 1. Concepts to review

a) Linear independence and the Wronskian
- A set of functions $\{f_1, \dots, f_k\}$ is said to be linearly independent over an interval I
$$\text{if } c_1 f_1(x) + \dots + c_k f_k(x) = 0 \text{ for all } x \in I \implies c_1 = \dots = c_k. \tag{1.1}$$
- If $f_1, \dots, f_k$ are all differentiable over an interavl I with $k-1$ continuous derivatives, then the Wronskian of $f_1, \dots, f_k$ is defined via
$$W(f_1, \dots, f_k)(x) = \det \begin{pmatrix} f_1(x) & \dots & f_k(x) \\ f_1'(x) & \dots & f_k'(x) \\ \vdots & \ddots & \vdots \\ f_1^{(k-1)}(x) & \dots & f_k^{(k-1)}(x) \end{pmatrix}, \ x \in I. \tag{1.2}$$
If $W(f_1, \dots, f_k)(x) \neq 0$ for some $x \in I$, then $f_1, \dots, f_k$ are linearly independent.
- If we further assume that $f_1, \dots, f_k$ are analytic, then $W(f_1, \dots, f_k)(x) = 0$ for all $x \in I$ implies that $f_1, \dots, f_k$ are linearly dependent.

b) Higher order linear equations
- Existence and uniqueness theorem: suppose $a_n, \dots, a_0, g$ are continuous functions over an interval I and $a_n(x) \neq 0$ for all $x \in I$. Then there exists a unique solution with its interval of existence equal to I to the initial value problem
$$\begin{cases} a_n(x)y^{(n)}(x) + \dots + a_0(x)y(x) = g(x), \ x \in I \\ y^{(k)}(x_0) = y_k, \ 0 \leq k \leq n-1 \end{cases} \tag{1.3}$$
for all $x_0 \in I$.
- Consequences of uniqueness:
  1) If a solution to an IVP can be identified via inspection and uniqueness is guaranteed, then that solution is the unique solution to the IVP.
  2) Solution curves in the x-y cannot cross over any interval I for which uniqueness of solutions is guaranteed.
  3) The general homogeneous solution to an n-th order equation will be a linear combination of n linearly independent homogeneous solutions over I. In other words, if $g(x) = 0$ for all $x \in I$, then the general solution to the linear equation is
$$y_h(x) = c_1 y_1(x) + \dots c_n y_n(x), \ x \in I, \tag{1.4}$$
where $c_1, \dots, c_n$ are arbitrary. If y_h is a solution to an initial value problem, then the coefficients $c_1, \dots, c_n$ can be determined from the initial conditions.
  4) For this reason, we refer to the set $\{y_1, \dots, y_n\}$ as a fundamental set of solutions. This is a set that is linearly independent and all elements in the set are solutions to the homogeneous equation.
- Fundamental sets of solutions are not unique: we can scale the functions in a given fundamental set of solutions or take linearly combinations of functions and obtain new fundamental set of solutions.
- If $a_n, \dots, a_0$ are not constant functions, we say that the equation has variable coefficients. In this case there is no simple mechanism to identify $y_1, \dots, y_n$.
- If $a_n, \dots, a_0$ are constant functions, we say that the equation has constant coefficients. In this case there is a simple way to determine $y_1, \dots, y_n$.

c) Reduction of order
- In the variable coefficient case, if we happen to know all except one of the linearly independent solutions, we can recover the last one via the method of reduction of order.
- For us we only study the case for 2nd order equations. The idea is that if we know that y_1 is a solution to a 2nd order linear equation, we use the ansatz $y_2(x) = u(x)y_1(x)$ $x \in I$ and assume that y_2 is another solution to the differential equation. By substituting the ansatz back into the original equation, we may derive a differential equation in u.
- Useful identity: $(fg)'' = f''g + 2f'g' + fg''$.
- The differential equation will involve u'' and u', and the terms involving u will cancel. Thus by using the substitution $w = u'$, we arrive at a first order linear equation in w, which can be solved using the method of integrating factors.
- Upon solving for w, we can find u' via the relation $u' = w$, and recover u via direct integration.

- Once we have u (which will involve two constants of integration), we can set the constants of integration to be 1 and 0 respectively, and use the relation $y_2 = uy_1$ to recover the second linearly independent solution to the equation.

d) Constant coefficient linear equations
- Constant coefficient linear equations can be solved by identifying the roots of the characteristic equation associated to the equation.
- For 2nd order equations, you either have distinct real roots, repeated real roots, or a pair of conjugate complex roots.
- In general, $\lambda_1, \lambda_2, ..., \lambda_n$ are the roots to the characteristic equation.
  (a) If $\lambda_1, ..., \lambda_n$ are all real and they are all distinct, then the general solution is given by
  $$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + ... + c_n e^{\lambda_n x},\ x \in I. \tag{1.5}$$
  (b) If we have k repeated real roots, say $\lambda_1 = \lambda_2 = ... = \lambda_k$ (and $\lambda_{k+1}, ..., \lambda_n$ are real distinct roots), then
  $$\begin{aligned} y(x) = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_2 x} + c_3 x^2 e^{\lambda_3 x} ... + c_k x^{k-1} e^{\lambda_k x} \\ + c_{k+1} e^{\lambda_{k+1} x} + ... + c_n e^{\lambda_n x},\ x \in I. \end{aligned} \tag{1.6}$$
  (c) If $\lambda_i = \alpha + \beta i$ is a complex root of multiplicity 1, then $\lambda_j = \overline{\lambda_i} = \alpha - \beta i$ must also be a root. Then we replace $c_i e^{\lambda_i x} + c_j e^{\lambda_j x}$ (which are still valid as complex-valued solutions, but we are interested only in real-valued solutions) with $c_i e^{\alpha x} \cos(\beta x) + c_j e^{\alpha x} \sin(\beta x)$.
  (d) If $\lambda = \alpha + \beta i$ is a complex root of multiplicity k, then its conjugate $\overline{\lambda} = \alpha - \beta i$ must also be a root of multiplicity k. In this case the solution is the same as the solution given in case (2), except we replace the complex exponentials in (1.6) with sines and cosines, as in case (3).

e) Mass-spring systems
- Mass-spring systems with or without damping can be modeled via a 2nd order linear equation via Newton's second law. Under simple assumptions, these systems are modeled via constant coefficient linear equations.
- By convention, the position x of the mass is positive when it is below the equilibrium point and negative when it is above the equilibrium point. Equivalently, we choose the downward direction to be the positive direction and the upward direction to be the negative direction.
- The general equation that models a mass-spring system without external forces is
  $$mx''(t) + \beta x'(t) + kx(t) = 0,\ t \in I, \tag{1.7}$$
  where $k > 0$ is the spring constant and $\beta > 0$ is the damping constant. If $\beta = 0$, there is no damping.
- If the roots of the characteristic equation are complex roots, then the system is underdamped. This is when the damping force is small relative to the spring force. The solution in this case resembles the periodic nature of the undamped solution and we refer to them as quasi-periodic solutions.
- In the undamped and underdamped cases, the solution can be written in the form of
  $$y(t) = Ae^{-\lambda t} \sin(\sqrt{\omega^2 - \lambda^2} t + \varphi). \tag{1.8}$$
  It is often easier to analyze the solution to the system when it is written in this form.
- For quasi-periodic functions, if the quasi-period is T, then $\frac{x(t)}{x(t+T)}$ is constant. The natural log of this constant is the logarithmic decrement.
- If the roots of the characteristic equation are a pair of repeated real roots, then the system is critically damped. The solution in this case is not oscillatory and passes through the equilibrium point exactly once.
- If the roots of the characteristic polynomial are a pair of distinct real roots, then the system is overdamped. This happens when the damping force is large relative to the spring force. The solution in this case is not oscillatory and passes through the equilibrium point at most once.
- In all three cases, the presence of a decaying exponential in the solution ensures that the mass converges back to the equilibrium point as $t \to \infty$. We can also see this when we analyze the energy, the system is losing energy due to the presence of the damping force.

f) 2nd order inhomogeneous linear equations
- If the equation is inhomogeneous, the general solution to the equation is
  $$y(x) = y_p(x) + y_h(x),\ x \in I, \tag{1.9}$$

where y_p is any particular solution to the inhomogeneous equation and y_h is the general solution to the homogeneous equation.

- To identify the particular method, when the right hand side of the equation is a polynomial, an exponential, sin or cos, or the product and linear combination of the aforementioned functions, then we may use the method of undetermined coefficients to find a particular solution.
- We use an educated guess or an ansatz y_p that depends on some unknown constants, and we determine these constants by calculating y_p', y_p'' and assuming that y_p satisfies the equation.
- We also solve for the general homogeneous solution y_h first, to ensure that there are no duplications between y_h and the educated guess y_p.

Part 2. Problems

Problem 1.1. Show that the functions

$$y_1(x) = x,\ y_2(x) = x^2,\ y_3(x) = x^3,\ x \in \mathbb{R} \tag{1.10}$$

are linearly independent over the interval $I = \mathbb{R}$.

Solution. We note that

$$\begin{aligned} W(y_1, y_2, y_3)(x) = \det \begin{pmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{pmatrix} &= x \det \begin{pmatrix} 2x & 3x^2 \\ 2 & 6x \end{pmatrix} - \det \begin{pmatrix} x^2 & x^3 \\ 2 & 6x \end{pmatrix} \\ &= x(12x^2 - 6x^2) - (6x^3 - 2x^3) = 2x^3 \neq 0 \text{ if } x \neq 0. \end{aligned} \tag{1.11}$$

Since $W(y_1, y_2, y_3)$ is non-zero for some $x \in \mathbb{R}$, y_1, y_2, y_3 are independent over $\mathbb{R}$.

□

Problem 1.2. Solve the initial value problem

$$\begin{cases} y''(x) - 6y'(x) + 25y(x) = 0, \ x \in \mathbb{R} \\ y(0) = 3 \\ y'(0) = 1. \end{cases} \tag{1.12}$$

Solution. The characteristic equation associated to the ODE is

$$r^2 - 6r + 25 = 0 \implies r = \frac{6 \pm \sqrt{36 - 100}}{2} = \frac{6 \pm 8i}{2} = 3 \pm 4i. \tag{1.13}$$

Thus the general solution to the equation is

$$y(x) = e^{3x}(c_1 \cos 4x + c_2 \sin 4x), \ x \in \mathbb{R}. \tag{1.14}$$

This implies

$$y'(x) = e^{3x}(-4c_1 \sin 4x + 4c_2 \cos 4x + 3c_1 \cos 4x + 3c_2 \sin 4x) \tag{1.15}$$

$$= e^{3x}\left((3c_1 + 4c_2)\cos 4x + (-4c_1 + 3c_2)\sin 4x\right), \ x \in \mathbb{R}. \tag{1.16}$$

Therefore if $y(0) = 3, y'(0) = 1$, then

$$c_1 = 3 \tag{1.17}$$

$$3c_1 + 4c_2 = 1, \tag{1.18}$$

This implies $c_1 = 3$ and $c_2 = -2$. Therefore the solution to the IVP is

$$y(x) = 3e^{3x} \cos 4x - 2e^{3x} \sin 2x, \ x \in \mathbb{R}. \tag{1.19}$$

□

Problem 1.3. Consider the 2nd order linear differential equation

$$(1-x^2)y''(x)+2xy'(x)-2y(x)=0,\ x\in(-1,1). \tag{1.20}$$

a) Verify that y_1 defined via $y_1(x)=x, x\in(-1,1)$ is a solution to the equation on the interval $I=(-1,1)$.
b) Use the method of reduction of order to derive a second linearly independent solution to the equation over the interval I.

Solution. We note that for all $x\in(-1,1)$,

$$y_1'(x)=1 \tag{1.21}$$
$$y_1''(x)=0. \tag{1.22}$$

Therefore

$$(1-x^2)y_1''(x)+2xy_1'(x)-2y_1(x)=2x-2x=0,\ x\in I. \tag{1.23}$$

Thus y_1 is a solution to the ODE on I. Next we assume $y_2(x)=u(x)y_1(x), x\in I$ is a solution to the equation. Then for all $x\in I$,

$$y_2'(x)=u'(x)y_1(x)+u(x)y_1'(x)=xu'(x)+u(x) \tag{1.24}$$
$$y_2''(x)=u''(x)y_1(x)+2u'(x)y_1'(x)+u(x)y_1''(x)=xu''(x)+2u'(x). \tag{1.25}$$

Since y_2 is assumed to be a solution, this implies

$$\begin{aligned}0&=(1-x^2)y_2''(x)+2xy_2'(x)-2y_2(x) &(1.26)\\ &=(1-x^2)(xu''(x)+2u'(x))+2x(xu'(x)+u(x))-2xu(x) &(1.27)\\ &=x(1-x^2)u''(x)+(2-2x^2+2x^2)u'(x) &(1.28)\\ &=x(1+x)(1-x)u''(x)+2u'(x),\ x\in I. &(1.29)\end{aligned}$$

Then if we define w via $w(x)=u'(x)$, then w satisfies the first order linear equation

$$w'(x)+\frac{2}{x(1+x)(1-x)}w(x)=0,\ x\in J \tag{1.30}$$

for $J=(0,1)$. We note that by partial fraction decomposition,

$$\frac{2}{x(1+x)(1-x)}=\frac{A}{x}+\frac{B}{1+x}+\frac{C}{1-x},\ x\in J, \tag{1.31}$$

which implies

$$2=A(1+x)(1-x)+Bx(1-x)+Cx(1+x),\ x\in\mathbb{R}. \tag{1.32}$$

If $x=0$, this implies $A=2$. If $x=1$, we find that $2C=2 \implies C=1$. If $x=-1$, we find that $-2B=2 \implies B=-1$. Therefore

$$\frac{2}{x(1+x)(1-x)}=\frac{2}{x}-\frac{1}{1+x}+\frac{1}{1-x},\ x\in J. \tag{1.33}$$

We may then choose an integrating factor to be

$$\begin{aligned}\mu(x)&=\exp\left(\int \frac{2}{x}-\frac{1}{1+x}+\frac{1}{1-x}\,dx\right)\\ &=\exp(2\ln x-\ln(1+x)-\ln(1-x))=\exp\ln\frac{x^2}{(1+x)(1-x)}=\frac{x^2}{(1+x)(1-x)}=,\ x\in J. &(1.34)\end{aligned}$$

Thus

$$\frac{d}{dx}\left(\frac{2}{(1+x)(1-x)}w(x)\right)=0 \implies w(x)=C\frac{1-x^2}{x^2}=C\left(x^{-2}-1\right),\ x\in J, \tag{1.35}$$

where C is arbitrary. This implies that

$$u(x)=\int w(x)\,dx=C\left(-x^{-1}-x\right)+D, \tag{1.36}$$

where C,D are arbitrary. If we choose $C=-1$ and $D=0$, then

$$y_2(x)=\left(x^{-1}+x\right)x=1+x^2,\ x\in\mathbb{R}. \tag{1.37}$$

□

Problem 1.4. Consider the differential equation

$$y^{(3)}(x) + 3y''(x) - 54y(x) = 0,\ x \in \mathbb{R}. \tag{1.38}$$

a) Verify that $y(x) = e^{3x}, x \in \mathbb{R}$ is a solution to the equation over the interval $I = \mathbb{R}$.
b) Find the general solution to the homogeneous equation.

Solution. For all $x \in \mathbb{R}$ we have

$$y'(x) = 3e^{3x},\ y''(x) = 9e^{3x},\ y^{(3)}(x) = 27e^{3x}. \tag{1.39}$$

Therefore

$$y^{(3)}(x) + 3y''(x) - 54y(x) = e^{3x}(27 + 27 - 54) = 0,\ x \in \mathbb{R}. \tag{1.40}$$

Therefore y is a solution to the equation. This also implies that $r - 3$ is a factor of the characteristic polynomial. This implies that

$$r^3 + 3r^2 - 54 = (r - 3)q(r), \tag{1.41}$$

and q can be found via long division to be

$$q(r) = r^2 + 6r + 18. \tag{1.42}$$

This implies that the other two roots are $r = \frac{-6\pm\sqrt{36-72}}{2} = -3 \pm 3i$. Therefore the general solution to the homogeneous solution is

$$y(x) = c_1 e^{3x} + c_2 e^{-3x}\cos 3x + c_3 e^{-3x}\sin 3x,\ x \in \mathbb{R}, \tag{1.43}$$

where c_1, c_2, c_3 are arbitrary. □

Problem 1.5. Find a constant coefficient linear equation that has

$$y(x) = (c_1x^2 + c_2x + c_3)\cos(2x) + (c_4x^2 + c_5x + c_6)\sin(2x),\ x \in \mathbb{R} \tag{1.44}$$

as the general solution over the interval $I = \mathbb{R}$.

Solution. From the general solution we see that $r = \pm 2i$ is a root of multiplicity 3 in the characteristic equation, therefore the characteristic equation is

$$(r - 2i)^3(r + 2i)^3 = (r^2 + 4)^3 = r^6 + 12r^4 + 48r^2 + 64 = 0. \tag{1.45}$$

Thus an equation that admits y as a general solution is

$$y^{(6)}(x) + 12y^{(4)}(x) + 48y''(x) + 64y(x) = 0,\ x \in \mathbb{R}. \tag{1.46}$$

$\square$

Problem 1.6. Suppose a mass spring system is modeled via the initial value problem

$$\begin{cases} x''(t) + 4x'(t) + 5x(t) = 0, \ x \in \mathbb{R}, \\ x(0) = 1 \\ x'(0) = 1. \end{cases} \tag{1.47}$$

a) Find the solution to the initial value problem.
b) Write the solution in the form of

$$x(t) = Ae^{-\lambda t} \sin\left(\frac{2\pi}{T}t + \varphi\right), \ t \in \mathbb{R} \tag{1.48}$$

for some $\varphi \in [0, 2\pi]$. Find λ, A, T, φ explicitly.
c) What is the significance of the functions $x_1(t) = Ae^{-\lambda t}$ and $x_2(t) = -Ae^{-\lambda t}$? You may draw a very rough sketch of the graphs of x, x_1, x_2 to explain.
d) What is the significance of the parameter T?
e) Calculate the logarithmic decrement: $\ln \frac{x(t)}{x(t+T)}$ for any time $t \in \mathbb{R}$.
f) Find the time $t_* > 0$ where it is the first positive time the mass passes through the equilibrium.
g) What is the maximum displacement of the mass in the interval $t \in (0, t_*)$?

Solution. The characteristic equation is

$$r^2 + 4r + 5 = 0 \implies r = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm i. \tag{1.49}$$

Thus the general solution to the ODE is

$$x(t) = e^{-2t}(c_1 \cos t + c_2 \sin t), \ t \in \mathbb{R}. \tag{1.50}$$

This implies

$$x'(t) = e^{-2t}(-c_1 \sin t + c_2 \cos t - 2c_1 \cos t - 2c_2 \sin t) = e^{-2t}\left((-2c_1 + c_2)\cos t + (-c_1 - 2c_2)\sin t\right), \ t \in \mathbb{R}. \tag{1.51}$$

Therefore if $x(0) = 1, x'(0) = 1$, we must have

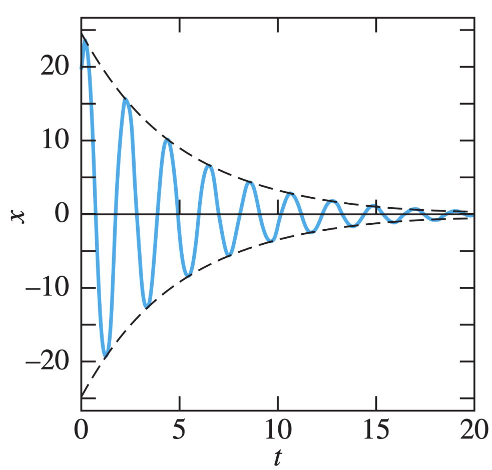
$$c_1 = 1 \tag{1.52}$$

$$-2c_1 + c_2 = 1 \implies c_2 = 3. \tag{1.53}$$

Therefore we can write the solution as

$$x(t) = Ae^{-\lambda t} \sin\left(\frac{2\pi}{T}t + \varphi\right), \ t \in \mathbb{R}, \tag{1.54}$$

where $A = \sqrt{1^1 + 3^2} = \sqrt{10}, \lambda = 2, T = 2\pi, \varphi = \tan^{-1}\frac{1}{3}$. Since $x_2(t) \le x(t) \le x_1(t)$ for all $t \in \mathbb{R}$, x_1, x_2 form the "envelope curves" that control the behavior of x; see the figure below.



The parameter T is the quasi-period, which is the time-distance between two successive local maximum or local minimum of x. For any $t \in \mathbb{R}$ we may calculate

$$\ln \frac{x(t)}{x(t+T)} = \ln A \frac{e^{-\lambda t} \sin\left(\frac{2\pi}{T} t + \varphi\right)}{e^{-\lambda(t+T)} \sin\left(\frac{2\pi}{T}(t+T) + \varphi\right)} = \ln e^{\lambda T} = \lambda T = 4\pi. \tag{1.55}$$

We note that

$$x(t) = 0 \Longleftrightarrow \sin\left(\frac{2\pi}{T} t + \varphi\right) = 0 \Longleftrightarrow \frac{2\pi}{T} t + \varphi = k\pi,\ k \in \mathbb{Z} \Longleftrightarrow t = k\pi - \varphi. \tag{1.56}$$

Since $\varphi \in (0, \frac{\pi}{2})$, the first time that this occurs correspond to when $k = 1$, so $t_* = \pi - \tan^{-1}\frac{1}{3}$. To calculate the maximum displacement we note that

$$x'(t) = Ae^{-2t}\left(\cos(t+\varphi) - 2\sin(t+\varphi)\right) = 0 \tag{1.57}$$

when

$$\tan(t+\varphi) = \frac{1}{2} \Longleftrightarrow t + \varphi = \tan^{-1}\frac{1}{2} + k\pi,\ k \in \mathbb{Z} \Longleftrightarrow t = \left(\tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{3}\right) + k\pi. \tag{1.58}$$

Since $\tan^{-1}$ is an increasing function, $\tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{3}$ is positive but less than π, therefore the first positive time when $x' = 0$ is when $k = 0$. Thus the maximum displacement of the mass is $x(t_m)$ where $t_m = \tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{3}$. $\square$

Problem 1.7. Solve the initial value problem

$$\begin{cases} y''(x) + 4y'(x) + 5y(x) = 35e^{-4x}, x \in \mathbb{R} \\ y(0) = -3 \\ y'(0) = 1. \end{cases} \tag{1.59}$$

Solution. First we find the general solution to the homogeneous equation. The characteristic equation is

$$r^2 + 4r + 5 = 0 \implies r = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm i. \tag{1.60}$$

Therefore the general homogeneous solution is

$$y_h(x) = c_1 e^{-2x} \cos x + c_2 e^{-2x} \sin x,\ x \in \mathbb{R}. \tag{1.61}$$

Next we look for a particular solution to the inhomogeneous equation. Our initial ansatz is

$$y_p(x) = Ae^{-4x},\ x \in \mathbb{R}, \tag{1.62}$$

and since there are no duplicate terms in y_p when compared to y_h, this is a valid ansatz. We then compute

$$y_p'(x) = e^{-4x}(-4A),\ y_p''(x) = e^{-4x}(16A),\ x \in \mathbb{R}. \tag{1.63}$$

Thus

$$y_p''(x) + 4y_p'(x) + 5y_p(x) = e^{-4x}(16A - 16A + 5A) = 35e^{-4x} \implies A = 7. \tag{1.64}$$

Thus the general solution to the inhomogeneous equation is

$$y(x) = 7e^{-4x} + e^{-2x}(c_1 \cos x + c_2 \sin x),\ x \in \mathbb{R}. \tag{1.65}$$

Then

$$y'(x) = -28e^{-4x} + e^{-2x}(-c_1 \sin x + c_2 \cos x - 2c_1 \cos x - 2c_2 \sin x) \tag{1.66}$$

$$= -28e^{-4x} + e^{-2x}((-2c_1 + c_2) \cos x + (-c_1 - 2c_2) \sin x),\ x \in \mathbb{R}. \tag{1.67}$$

Therefore if $y(0) = -3, y'(0) = 1$, we must have

$$7 + c_1 = -3 \implies c_1 = -10 \tag{1.68}$$

$$-28 - 2c_1 + c_2 = 1 \implies 9. \tag{1.69}$$

Therefore the solution to the IVP is

$$y(x) = 7e^{-4x} - 10e^{-2x} \cos x + 9e^{-2x} \sin x,\ x \in \mathbb{R}. \tag{1.70}$$

□

Problem 1.8. Solve the initial value problem

$$\begin{cases} 25y''(x) + 45y'(x) + 99y(x) = 0 \\ y(0) = 0 \\ y'(0) = 0. \end{cases} \tag{1.71}$$

Solution. We note that $y(x) = 0, x \in \mathbb{R}$ is a constant solution to the IVP. We note that since the equation is a homogeneous constant coefficient linear equation, the conditions of the existence and uniqueness theorem are satisfied, therefore by the uniqueness part of the theorem $y(x) = 0, x \in \mathbb{R}$ is the unique solution to the IVP. $\square$