# Homework 7

DUE: SATURDAY, MARCH 15, 11:59PM

If you completed this assignment through collaboration or consulted references, please list the names of your collaborators and the references you used below. Please refer to the syllabus for the course policy on collaborations and types of references that are allowed.

Department of Mathematical Sciences, Carnegie Mellon University
*E-mail address*: jkoganem@andrew.cmu.edu.
*Date*: March 9, 2025.

**Problem 7.1** (Laplace transform of derivatives)**.** Suppose $y : [0, \infty)$ is a smooth function of exponential order satisfying the pointwise estimate

$$|y(t)| \leq Me^{at}, \tag{7.1}$$

for all $t > T$, for some constants $M, a, T > 0$. Consider its Laplace transform $Y : (a, \infty) \to \mathbb{R}$ defined via

$$Y(s) = \mathcal{L}\left\{y\right\}(s) = \int_0^\infty e^{-st} y(t)\, dt. \tag{7.2}$$

a) Use integration by parts to show that

$$\mathcal{L}\left\{y'\right\}(s) = sY(s) - y(0), \tag{7.3}$$

for all $s > a$.

b) With the additional assumption that $y'$ is also of exponential order satisfying the pointwise bound

$$|y'(t)| \leq Me^{at}, \tag{7.4}$$

for $t > T$, use integration by parts to show that

$$\mathcal{L}\left\{y''\right\}(s) = s^2 Y(s) - sy(0) - y'(0), \tag{7.5}$$

for all $s > a$.

**Problem 7.2** (Translation in frequency space).

a) Suppose $f : [0, \infty) \to \mathbb{R}$ is a continuous function of exponential order and its Laplace transform $F : (c, \infty)$ defined via is well-defined for some $c \in \mathbb{R}$. Show that for any $a \in \mathbb{R}$, we have the identity

$$\mathcal{L}\{e^{at} f(t)\} = F(s - a), \text{ for all } s > c + a. \tag{7.6}$$

b) Use part a) to find the inverse Laplace transform of

$$Y(s) = \frac{3s + 2}{s^2 - 2s + 10}, \ s > 1. \tag{7.7}$$

**Problem 7.3.**

a) Use the Laplace transform to solve the initial value problem

$$\begin{cases} x''(t) + 4x(t) = 1, & t \in [0, \infty) \\ x(0) = 1, x'(0) = 0. \end{cases} \tag{7.8}$$

b) Use the Laplace transform to solve the initial value problem

$$\begin{cases} y''(t) + 2y'(t) + 5y(t) = 0, & t \in [0, \infty) \\ y(0) = 2, y'(0) = -1 \end{cases} \tag{7.9}$$

**Problem 7.4** (Laplce transform of piecewise smooth functions). Define the function $g : [0, \infty) \to \mathbb{R}$ via

$$g(t) = \begin{cases} t, & 0 \leq t < 1 \\ 2 - t, & 1 \leq t < 2 \\ 0, & 2 \leq t. \end{cases} \tag{7.10}$$

Find $\mathcal{L}\{g(\cdot)\}$

**Problem 7.5.** Find a solution to the initial value problem

$$\begin{cases} x''(t) + 4x'(t) + 4x(t) = f(t), \ t \in [0, \infty) \\ x(0) = x'(0) = 0 \end{cases} \tag{7.11}$$

where $f : [0, \infty) \to \mathbb{R}$ is defined via

$$f(t) = \begin{cases} t, \ 0 \leq t < 2 \\ 0, \ t \geq 2. \end{cases} \tag{7.12}$$

**Problem 7.6.** Find a solution to the initial value problem

$$\begin{cases} x''(t) + 4x'(t) + 5x(t) = \delta(t-\pi) + \delta(t-2\pi),\ t \in [0,\infty) \\ x(0) = 0, x'(0) = 2. \end{cases} \tag{7.13}$$