More Laplace transform practice problems

Problem 1.1. Define $f : [0, \infty) \to \mathbb{R}$ via

$$f(t) = e^t(\mathcal{U}(t-1) - \mathcal{U}(t-2)). \tag{1.1}$$

What is the Laplace transform of f?

Problem 1.2. Define $f : [0, \infty) \to \mathbb{R}$ via

$$f(t) = \begin{cases} 0, & 0 \leq t < \pi \\ t - \pi, & \pi \leq t < 2\pi \\ 0, & t \geq 2\pi. \end{cases} \tag{1.2}$$

What is the Laplace transform of f?

Problem 1.3. Define $f : [0, \infty) \to \mathbb{R}$ via

$$f(t) = \begin{cases} t, & 0 \leq t < 1 \\ 1, & t \geq 1 \end{cases} \tag{1.3}$$

What is the Laplace transform of f?

Problem 1.4. Define $F : (3, \infty) \to \mathbb{R}$ via

$$F(s) = \frac{10e^{-s}}{s^2 - 5s + 6} + \frac{2}{s^2 - 2s + 5}. \tag{1.4}$$

What is the inverse Laplace transform of F?

Problem 1.5. Define $F : (0, \infty) \to \mathbb{R}$ via

$$F(s) = \frac{3s^2 + 4s + 1}{(s+1)(s^2 + 2s + 5)}. \tag{1.5}$$

What is the inverse Laplace transform of F?

Problem 1.6. Define $F : (4, \infty) \to \mathbb{R}$ via

$$F(s) = e^{-3s} \frac{s+1}{s^2 - 8s + 20}. \tag{1.6}$$

What is the inverse Laplace transform of F?

Problem 1.7. Consider a forced undamped mass-spring system modeled via the IVP

$$\begin{cases} y''(t) + 4y(t) = f(t), \ t \geq 0 \\ y(0) = -1, \ y'(0) = 4, \end{cases} \tag{1.7}$$

a) Find an expression for a solution $y : [0, \infty) \to \mathbb{R}$ for any reasonable forcing function $f : [0, \infty) \to \mathbb{R}$.

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b) Write down the solution for $f(t) = 2\delta(t-\pi)$.

Thus

$$y(t) = \mathcal{U}(t-\pi)\sin 2(t-\pi) + 2\sin 2t - \cos 2t,\ t \geq 0. \tag{1.8}$$

Problem 1.8. Define the function $f : [0,\infty) \to \mathbb{R}$ via

$$f(t) = e^t \int_0^t \sin\tau \cos(t-\tau)\, d\tau. \tag{1.9}$$

What is the Laplace transform of f?

Problem 1.9. Consider a forced undamped mass-spring system modeled via the IVP

$$\begin{cases} y''(t) + 2y(t) = \begin{cases} 0, & 0 \leq t < 2 \\ (t-2)e^{-3(t-2)}, & t \geq 2. \end{cases} \\ y(0) = y'(0) = 0. \end{cases} \tag{1.10}$$

If $y : [0,\infty) \to \mathbb{R}$ is a solution modeling the system and Y is its Laplace transform, what is $Y(0)$?

Problem 1.10. Consider a forced undamped mass-spring system modeled via the IVP

$$\begin{cases} y''(t) + y(t) = \begin{cases} t, & 0 \leq t < 2 \\ 3, & t \geq 2. \end{cases} \\ y(0) = y'(0) = 0. \end{cases} \tag{1.11}$$

Find a solution $y : [0,\infty) \to \mathbb{R}$ modeling the system.

Problem 1.11. Suppose a mass-spring system is modeled via

$$\begin{cases} x''(t) + 2x'(t) + 5x(t) = f(t), & t \geq 0 \\ x(0) = x'(0) = 0, \end{cases} \tag{1.12}$$

where δ is the Dirac delta and $\mathcal{U}$ is the unit step function and $f : [0,\infty) \to \mathbb{R}$ is defined via

$$f(t) = \begin{cases} 5, & 0 \leq t < \pi \\ \delta(t-3\pi) + \delta(t-4\pi), & t \geq \pi. \end{cases} \tag{1.13}$$

Find a solution $x : [0,\infty) \to \mathbb{R}$ describing the behavior of the system for $t \geq 0$.