Homework 5

Due: Saturday, February 22, 11:59pm

If you completed this assignment through collaboration or consulted references, please list the names of your collaborators and the references you used below. Please refer to the syllabus for the course policy on collaborations and types of references that are allowed.

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Problem 5.1 (Ansatz for undetermined coefficients)**.** Determine an appropriate ansatz for a particular solution y_p to the following equations, without determining the values of the coefficients.

a) $y''(x) - 2y'(x) + 2y(x) = e^x \sin x,\ x \in \mathbb{R}$.
b) $y''(x) + 4y(x) = 3x \cos 2x,\ x \in \mathbb{R}$.
c) $y''(x) + 3y'(x) + 2y(x) = x(e^{-x} - e^{-2x}),\ x \in \mathbb{R}$.
d) $y^{(4)}(x) - 2y''(x) + y(x) = x^2 \cos x,\ x \in \mathbb{R}$.

Problem 5.2 (An inhomogeneous 2nd order equation)**.** Solve the initial value problem

$$\begin{cases} y''(x) - 4y(x) = 2e^{2x}, \ x \in \mathbb{R} \\ y(0) = 0 \\ y'(0) = \frac{9}{2}. \end{cases} \tag{5.1}$$

Problem 5.3 (An inhomogeneous 3rd order equation)**.** Consider the third order differential equation

$$y^{(3)}(x) + y''(x) = 3e^x + 4x^2,\ x \in \mathbb{R}. \tag{5.2}$$

Find the general solution to the equation.

Problem 5.4 (Mass-spring systems). A mass-spring system has the following properties: the mass is 2 kilograms, and the spring exerts a force of 6 Newtons (one Newton is equal to $1\mathrm{kg}\frac{\mathrm{m}}{\mathrm{s}^2}$) when the mass is displaced 2 meters from its equilibrium position, and a viscous force of 5 Newtons slows the system when the mass moves with velocity of 1 meter per second.

a) What is the spring constant k and the damping constant β?

b) Write down an equation of the form

$$x''(t) + 2\lambda x'(t) + \omega^2 x(t) = 0, \; t \in \mathbb{R} \tag{5.3}$$

that models the position of the mass.

c) Is the system underdamped, critically damped or overdamped?

d) If $x(0) = 1$ and $x'(0) = -9$, does the mass ever cross the equilibrium point at some finite time $t > 0$?

Problem 5.5 (Mass-spring systems)**.** Suppose a mass-spring system is modeled via the initial value problem

$$\begin{cases} 10x''(t) + 9x'(t) + 2x(t) = 0, \ t > 0 \\ x(0) = 0 \\ x'(0) = 5. \end{cases} \tag{5.4}$$

a) Find the solution to the initial value problem on the interval $I = (0, \infty)$. Is the system underdamped, critically damped, or overdamped?
b) Does the mass ever reach back to the equilibrium position for some finite time $t > 0$?
c) Identify the time intervals on which the mass is above the equilibrium point and below the equilibrium point.
d) Identify the time intervals on which the mass is moving away from the equilibrium point and towards the equilibrium point.
e) Suppose the unit of length is in meters. How far does the mass move to the bottom before it starts moving back towards the equilibrium point?
f) Sketch a rough graph of the position function x on the t-x axis.

Problem 5.6 (Pure resonance)**.** Consider a forced undamped mass-spring system modeled via the equation

$$mx''(t) + kx(t) = F_0 \cos \omega t, \ t \in \mathbb{R} \tag{5.5}$$

where $m, k, F_0, \omega \in \mathbb{R} \setminus \{0\}$ with $m, k > 0$. Define $\omega_0 > 0$ via

$$\omega_0 = \sqrt{\frac{k}{m}}. \tag{5.6}$$

a) Assuming $\omega \neq \omega_0$, use the method of undetermined coefficients to find a particular solution to (5.5). What happens if $\omega \to \omega_0$?
b) Now suppose $\omega = \omega_0$. Use the method of undetermined coefficients to find the general solution to (5.5). What happens when $t \to \infty$?