Homework 3 solutions

DUE: SATURDAY, FEBRUARY 8, 2025, 11:59PM

If you completed this assignment through collaboration or consulted references, please list the names of your collaborators and the references you used below. Please refer to the syllabus for the course policy on collaborations and types of references that are allowed.

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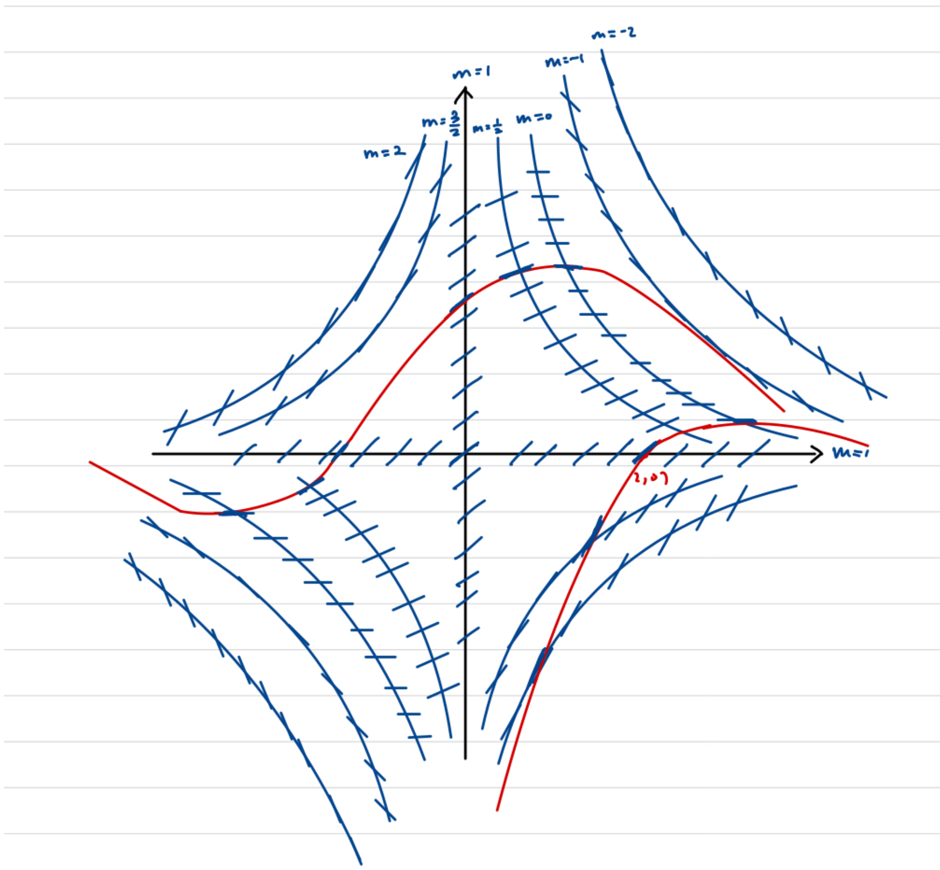
Problem 3.1 (Isoclines)**.** Consider the differential equation

$$y'(t) = 1 - ty(t),\ t \in \mathbb{R}. \tag{3.1}$$

a) Determine the general form of the isoclines associated to the equation, with slope m. (Note: you need to consider $m = 1$ as a separate case)
b) Sketch the directional field associated to the equation by using part a). You should at the very least sketch the isoclines corresponding to $m = 0, \pm 1, \pm 2$.
c) Sketch the solution curve y_1 passing through the point $(0, 2)$, and also the solution curve y_2 passing through the point $(2, 0)$.
d) Is it possible for the two curves y_1 and y_2 to ever cross, either going forward in time (the variable t) or backwards in time? Explain why or why not.

Part a). If $m = 1$, then $1 - ty = 1 \iff ty = 0$. This implies that the isoclines corresponding to slope 1 are the lines $y = 0$ and $t = 0$. If $m \neq 1$, then $1 - ty = m \iff ty = 1 - m \implies y = \frac{1-m}{t}, t \neq 0$. ☐

Parts b) and c). Here is a very rough sketch:



☐

Part d). No. If we define the function f via

$$f(t, y) = 1 - ty,\ (t, y) \in \mathbb{R}^2. \tag{3.2}$$

We note that

$$\frac{\partial f}{\partial y} = -t,\ (t, y) \in \mathbb{R}^2. \tag{3.3}$$

Since f and $\partial f / \partial y$ are both continuous functions on $\mathbb{R}^2$, we see that solutions to any initial value problem must be unique. Therefore by uniqueness, the solution curves cannot cross. ☐

Problem 3.2 (Exact differential equations)**.** Find an implicit solution to the initial value problem

$$\begin{cases} (x+y(x))^2 + (2xy(x) + x^2 - 1)y'(x) = 0, \ x \in \mathbb{R} \\ y(1) = 1. \end{cases} \tag{3.4}$$

You do not need to specify the interval of existence.

Solution. We first check that the equation is exact. Define M, N via

$$M(x,y) = (x+y)^2, \ N(x,y) = 2xy + x^2 - 1, \ (x,y) \in \mathbb{R}^2. \tag{3.5}$$

Then

$$M_y = 2(x+y), \ N_x = 2y + 2x = 2(x+y), \ (x,y) \in \mathbb{R}^2. \tag{3.6}$$

This shows that the equation is exact. To recover a function F such that $F_x = M$ and $F_y = N$, we can start from M and note that

$$F(x,y) = \int (x+y)^2 \ dx + g(y) = \int x^2 + 2xy + y^2 \ dx + g(y) = \frac{x^3}{3} + x^2 y + xy^2 + g(y), \ (x,y) \in \mathbb{R}^2. \tag{3.7}$$

Then

$$F_y(x,y) = x^2 + 2xy + g'(y) = N(x,y) = 2xy + x^2 - 1, \ (x,y) \in \mathbb{R}^2. \tag{3.8}$$

This implies that we must have

$$g'(y) = -1 \implies g(y) = -y + C. \tag{3.9}$$

Thus an implicit solution to the equation is given by

$$F(x, y(x)) = \frac{x^3}{3} + x^2 y + xy^2 - y = C, \ x \in I \tag{3.10}$$

for an arbitrary constant C over some interval I. □

Problem 3.3 (Wronskian and linear independence)**.** Show that the following sets of functions are linearly independent over the interval $I = \mathbb{R}$ by computing the Wronskian between them.

a) $y_1(x) = e^{r_1 x}, y_2(x) = e^{r_2 x}, x \in \mathbb{R}$, for any $r_1 \neq r_2 \in \mathbb{R}$.
b) $y_1(x) = e^{rx}, y_2(x) = xe^{rx}, x \in \mathbb{R}$, for any $r \in \mathbb{R}$.
c) $y_1(x) = e^{ax}\cos(bx), y_2(x) = e^{ax}\sin(bx), x \in \mathbb{R}$, for any $\mathbb{R} \ni a, b \neq 0$.

Solution. For a), we calculate

$$W(y_1, y_2)(x) = \det \begin{pmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{pmatrix} = (r_2 - r_1)e^{(r_1+r_2)x} \neq 0 \text{ for all } x \in \mathbb{R} \tag{3.11}$$

if $r_1 \neq r_2$. Since the Wronskian never vanishes, the two functions are linearly independent over $\mathbb{R}$. For b), we calculate

$$W(y_1, y_2)(x) = \det \begin{pmatrix} e^{rx} & xe^{rx} \\ re^{rx} & e^{rx}(1+rx) \end{pmatrix} = (1+rx)e^{2rx} - rxe^{2rx} = e^{2rx} \neq 0 \text{ for all } x \in \mathbb{R}. \tag{3.12}$$

Since the Wronskian never vanishes, the two functions are linearly independent over $\mathbb{R}$. For c), we calculate

$$W(y_1, y_2)(x) = \det \begin{pmatrix} e^{ax}\cos(bx) & e^{ax}\sin(bx) \\ e^{ax}(-b\sin(bx) + a\cos(bx)) & e^{ax}(b\cos(bx) + a\sin(bx)) \end{pmatrix} \tag{3.13}$$

$$= e^{2ax}\left(b\cos^2(bx) + a\sin(bx)\cos(bx) + b\sin^2(bx) - a\sin(bx)\cos(bx)\right) = be^{2ax} \neq 0 \tag{3.14}$$

as long as $b \neq 0$. Since the Wronskian never vanishes, the two functions are linearly independent over $\mathbb{R}$.

□

Problem 3.4 (Wronskians of solutions to second order linear homogeneous differential equations)**.** Consider the second order linear homogeneous differential equation

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0,\ x \in I, \tag{3.15}$$

where I is an interval on which a_2, a_1, a_0 are continuous and $a_2(x) \neq 0$ for all $x \in I$.

Suppose y_1, y_2 are two C^2 solutions to the equation and define the Wronskian $W : I \to \mathbb{R}$ via

$$W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x). \tag{3.16}$$

a) Calculate W' and show that W satisfies the differential equation

$$a_2(x)W'(x) = y_1(x)(a_2(x)y_2''(x)) - y_2(x)(a_2(x)y_1''(x)),\ x \in I. \tag{3.17}$$

b) Use (3.17) to show that W satisfies

$$a_2(x)W'(x) = -a_1(x)W(x),\ x \in I. \tag{3.18}$$

c) Show that

$$W(x) = C\exp\left(-\int \frac{a_1(x)}{a_2(x)}\,dx\right),\ x \in I \tag{3.19}$$

where C is an arbitrary constant.

d) Conclude that in this setting, either $W(x) = 0$ for all $x \in I$ or $W(x) \neq 0$ for all $x \in I$. In other words, either it vanishes everywhere on I or it does not vanish anywhere on I.

Part a). We calculate

$$W'(x) = y_1'(x)y_2'(x) + y_1(x)y_2''(x) - y_2'(x)y_1'(x) - y_2(x)y_1''(x) = y_1(x)y_2''(x) - y_2(x)y_1''(x),\ x \in I. \tag{3.20}$$

Therefore

$$a_2(x)W'(x) = y_1(x)(a_2(x)y_2''(x)) - y_2(x)(a_2(x)y_1''(x)),\ x \in I. \tag{3.21}$$

□

Part b). We note that by (3.17),

$$a_2(x)y_i''(x) = -a_1(x)y_i'(x) - a_0(x)y_i(x),\ x \in I, i = 1, 2. \tag{3.22}$$

Therefore

$$\begin{aligned} a_2W'(x) &= y_1(x)(-a_1(x)y_2'(x) - a_0(x)y_2(x)) - y_2(x)(-a_1(x)y_1'(x) - a_0(x)y_1(x)) && (3.23)\\ &= -a_1(x)\underbrace{(y_1(x)y_2'(x) - y_2(x)y_1'(x))}_{=W(x)} - a_0(x)\underbrace{(y_1(x)y_2(x) - y_2(x)y_1(x))}_{=0} && (3.24)\\ &= -a_1(x)W(x),\ x \in I. && (3.25) \end{aligned}$$

□

Part c). The previous part and the assumption that $a_2(x) \neq 0$ for all $x \in I$ shows us that W satisfies the first order homogeneous linear equation

$$W'(x) + \frac{a_1(x)}{a_2(x)}W(x) = 0,\ x \in I. \tag{3.26}$$

We may choose an integrating factor μ to be

$$\mu(x) = \exp\left(\int \frac{a_1(x)}{a_2(x)}\,dx\right),\ x \in I. \tag{3.27}$$

Thus upon multiplying both sides of the linear equation by μ, we have

$$\frac{d}{dx}[\mu(x)W(x)] = 0,\ x \in I. \tag{3.28}$$

Therefore

$$W(x) = C\exp\left(-\int \frac{a_1(x)}{a_2(x)}\,dx\right),\ x \in I, \tag{3.29}$$

where C is an arbitrary constant. □

Part d). If $C = 0$, then $W(x) = 0$ for lal $x \in I$. Otherwise, if $C \neq 0$, then $W(x) \neq 0$ for all $x \in I$ since $\exp\left(-\int \frac{a_1(x)}{a_2(x)}\,dx\right) > 0$ for all $x \in I$. Therefore W either vanishes uniformly on I or it never vanishes on I. □

Problem 3.5. For each of the following initial value problems, use an appropriate method to find a solution and state the maximal interval of existence. You may skip the verification steps.

a)

$$\begin{cases} y'(t) = y(t)e^t, \ t \in \mathbb{R} \\ y(0) = 2e. \end{cases} \tag{3.30}$$

b)

$$\begin{cases} x^2 y'(x) - (y(x))^2 - xy(x) = 0, \ x \in \mathbb{R} \\ y(1) = 2. \end{cases} \tag{3.31}$$

c)

$$\begin{cases} xy'(x) = 3y(x) + 2x^4, \ x \in \mathbb{R} \\ y(1) = 0 \end{cases} \tag{3.32}$$

Solution. Part a). Note that we may rewrite the equation as

$$y'(t) - e^t y(t) = 0, \ t \in \mathbb{R}, \tag{3.33}$$

therefore we may recover the solution via the method of integrating factors. An integrating factor for this equation is

$$\mu(t) = \exp\left(-\int e^t \, dt\right) = \exp(-e^t), \ t \in \mathbb{R}. \tag{3.34}$$

Thus

$$\frac{d}{dt}[\exp(-e^t)y(t)] = 0 \implies y(t) = C\exp(e^t), \ t \in \mathbb{R}. \tag{3.35}$$

If $y(0) = 2e$, then $2e = C\exp(e^0) = Ce^1 \implies C = 2$. Therefore the solution to the initial value problem is

$$y(t) = 2\exp(e^t), \ t \in \mathbb{R}. \tag{3.36}$$

The maximal interval of existence is $\mathbb{R}$. ☐

Part b). Note that we may rewrite the equation as

$$x^2 y'(x) - xy(x) = (y(x))^2, x \in \mathbb{R} \tag{3.37}$$

which is in the form of a Bernoulli equation. If y is a solution to the equation, we may assume that $y \neq 0$ on some interval I and define the function v via

$$v(x) = (y(x))^{1-2} = (y(x))^{-1}, \ x \in I. \tag{3.38}$$

Then

$$v'(x) = -(y(x))^{-2} y'(x), \ 'x \in I. \tag{3.39}$$

We note that the original equation may be written as

$$x^2 (y(x))^{-2} y'(x) - x(y(x))^{-1} = 1, \ x \in I \tag{3.40}$$

therefore v satisfies the first order linear equation

$$-x^2 v'(x) - xv(x) = 1, \ x \in I. \tag{3.41}$$

Then on $J = I \cap (0, \infty)$, v satisfies the equation

$$v'(x) + \frac{1}{x} v(x) = -\frac{1}{x^2}, \ x \in J. \tag{3.42}$$

We may choose an integrating factor to be $\mu(x) = \exp(\int \frac{1}{x} \, dx) = x, x \in J$. Thus

$$\frac{d}{dx}[xv(x)] = -\frac{1}{x} \implies xv(x) = -\ln|x| = \ln\frac{1}{x} + C, \ x \in J. \tag{3.43}$$

Therefore

$$v(x) = \frac{1}{x}\ln\frac{1}{x} + \frac{C}{x}, \ x \in J \tag{3.44}$$

where C is arbitrary. If $y(1) = 2$, then $v(1) = (y(1))^{-1} = \frac{1}{2}$, thus

$$\frac{1}{2} = \frac{1}{1}\ln\frac{1}{1} + \frac{C}{1} = C. \tag{3.45}$$

Thus the solution to initial value problem is

$$y(x) = \left(\frac{1}{x}\ln\frac{1}{x} + \frac{1}{2x}\right)^{-1},\ x \in J. \tag{3.46}$$

Note that if $x > 0$,

$$\frac{1}{x}\ln\frac{1}{x} + \frac{1}{2x} = 0 \Longleftrightarrow \ln\frac{1}{x} = -\frac{1}{2} \Longleftrightarrow x = e^{1/2} > 1. \tag{3.47}$$

The maximal interval of existence is $(0, e^{1/2})$. □

Part c). We note that we may write the equation as

$$xy'(x) - 3y(x) = 2x^4,\ x \in \mathbb{R}. \tag{3.48}$$

If y is a solution, then

$$y'(x) - \frac{3}{x}y(x) = 2x^3,\ x > 0. \tag{3.49}$$

An integrating factor for this equation is $\mu(x) = \exp(\int -\frac{3}{x}\,dx) = \frac{1}{x^3}$. Thus

$$\frac{d}{dx}\left[\frac{1}{x^3}y(x)\right] = 2 \implies y(x) = 2x^4 + Cx^3,\ x > 0. \tag{3.50}$$

where C is arbitrary. If $y(1) = 0$, then $0 = 2 + C \implies C = -2$. Thus the solution to the initial value problem is

$$y(x) = 2x^4 - 2x^3,\ x \in \mathbb{R}, \tag{3.51}$$

and the maximal interval of existence is $(-\infty, \infty)$. □

□

Problem 3.6 (Wronskian)**.** Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are two smooth functions, and we are given that the Wronskian of f and g is given by

$$W(f,g)(t) = -3e^{4t} \text{ and } f(t) = 4e^{2t},\ t \in \mathbb{R}. \tag{3.52}$$

What is g?

Solution. We note that

$$-3e^{4t} = W(f,g)(t) = \det\begin{pmatrix} 4e^{2t} & g(t) \\ 8e^{2t} & g'(t) \end{pmatrix} = 4e^{2t}g'(t) - 8e^{2t}g(t),\ t \in \mathbb{R}. \tag{3.53}$$

Thus g satisfies the first order linear equation

$$g'(t) - 2g(t) = -\frac{3}{4}e^{2t},\ t \in \mathbb{R}. \tag{3.54}$$

We may choose an integrating factor to be $\mu(t) = \exp(\int -2\ dt) = e^{-2t}, t \in \mathbb{R}$. Thus

$$\frac{d}{dt}\left[e^{-2t}g(t)\right] = \frac{-3}{4} \implies g(t) = \frac{-3}{4}te^{2t} + Ce^{2t},\ t \in \mathbb{R}. \tag{3.55}$$

where C is arbitrary. $\square$