Homework 4

DUE: SATURDAY, FEBRUARY 15, 2025, 11:59PM

If you completed this assignment through collaboration or consulted references, please list the names of your collaborators and the references you used below. Please refer to the syllabus for the course policy on collaborations and types of references that are allowed.

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Problem 1.1. For each of the following equations, write down the general solution.

a) $y''(x) - 5y'(x) + 6y(x) = 0, x \in \mathbb{R}$

b) $y''(t) - 4y'(t) + 4y(t) = 0, t \in \mathbb{R}$

c) $y''(t) + 4y(t) = 0, t \in \mathbb{R}$

d) $y''(x) - 2y'(x) + 2y(x) = 0, x \in \mathbb{R}$

e) $y^{(4)}(x) - 2y''(x) + y(x) = 0,\ x \in \mathbb{R}$

Problem 1.2. For each of the following parts, find a constant coefficient equation over $\mathbb{R}$ whose general solution $y : \mathbb{R} \to \mathbb{R}$ is given below.

a) $y(x) = c_1 e^x + c_2 e^{5x}$, c_1, c_2 are arbitrary

b) $y(x) = c_1 e^{10x} + c_2 x e^{10x}$, c_1, c_2 are arbitrary

c) $y(x) = c_1 + c_2 e^{2x} \cos(5x) + c_3 e^{2x} \sin(5x)$, c_1, c_2, c_3 are arbitrary

Problem 1.3.

a) Verify that the functions $y_1, y_2 : (0, \infty) \to \mathbb{R}$ defined via

$$y_1(x) = x,\ y_2(x) = x \ln x \tag{1.1}$$

are linearly independent over the interval $I = (0, \infty)$.

b) Verify that the function $y : (0, \infty) \to \mathbb{R}$ defined via

$$y(x) = c_1 x + c_2 x \ln x, \tag{1.2}$$

where c_1, c_2 are two arbitrary constants, is a solution on the interval $I = (0, \infty)$ to the second order linear homogeneous equation

$$x^2 y''(x) - x y'(x) + y(x) = 0,\ x \in (0, \infty). \tag{1.3}$$

Problem 1.4. Verify that the functions $y_1, y_2 : (0, \infty) \to \mathbb{R}$ defined via

$$y_1(x) = x^3,\ y_2(x) = x^4 \tag{1.4}$$

form a fundamental set of solutions on the interval $I = (0, \infty)$ to the equation

$$x^2 y''(x) - 6xy'(x) + 12y(x) = 0. \tag{1.5}$$

Problem 1.5 (Spring-mass system)**.** Let $x : \mathbb{R} \to \mathbb{R}$ be a solution to the initial value problem

$$\begin{cases} x''(t) + \gamma x'(t) + x(t) = 0, \ t \in \mathbb{R}, \gamma \geq 0 \\ x(0) = 1, x'(0) = -2 \end{cases} \tag{1.6}$$

which models a spring-mass system.

a) For what value(s) of γ will this system be underdamped?
b) Sketch a graph showing the behavior of the solution to the initial value problem, assuming that γ is chosen so that the system is underdamped. You do not need to solve for the solution explicitly. How many times does this graph cross the t-axis?
c) For what value(s) of γ will this system be critically damped? At what time(s) will the mass pass through the equilibrium point?

Problem 1.6 (Reduction of order).

a) Verify that the function $y_1 : (0, \infty) \to \mathbb{R}$ defined via $y_1(x) = x^3$ is a solution to the 2nd order differential equation

$$x^2 y''(x) - 5xy'(x) + 9y(x) = 0, \ x > 0. \tag{1.7}$$

b) Use the method of reduction of order to identify a second linearly independent solution to (1.7).