Homework 10

DUE: SATURDAY, APRIL 19, 11:59PM

If you completed this assignment through collaboration or consulted references, please list the names of your collaborators and the references you used below. Please refer to the syllabus for the course policy on collaborations and types of references that are allowed.

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Problem 10.1 (Orientation of trajectories)**.** Let $\alpha \in \mathbb{R}$ and consider the first order systems $\boldsymbol{X}_1'(t) = A_1\boldsymbol{X}_1(t)$ and $\boldsymbol{X}_2'(t) = A_2\boldsymbol{X}_2(t)$, $t \in \mathbb{R}$, where

$$A_1 = \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix}, \quad A_2 = \begin{pmatrix} \alpha & -1 \\ 1 & \alpha \end{pmatrix}. \tag{10.1}$$

a) Show that for both systems, the origin is either a spiral point or a center. For each system, what value(s) of α is the origin a center?
b) Analyze the orientations of the trajectories in phase space for both systems. For which system is the orientation of the trajectories clockwise, and for which system is it counterclockwise? Does the orientation depend on the value of α?

Solution.

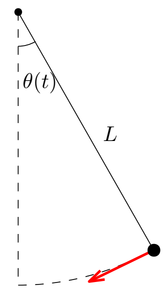
a) We note that
$$\det(A_1 - \lambda I) = \det(A_2 - \lambda I) = (\alpha - \lambda)^2 + 1 \implies \lambda = \alpha \pm i. \tag{10.2}$$
This shows that the eigenvalues of both systems are complex conjugates with real part α and imaginary part 1. Thus, the origin is a spiral point if $\alpha \neq 0$ and a center if $\alpha = 0$.
b) To analyze the orientation of the trajectories we analyze the nullclines for both systems.
We first analyze the case when $\alpha = 0$ for the first system.
 1) The x-nullcline is $y = 0$ (the x-axis) and $x' > 0$ for $y > 0$ and $x' < 0$ for $y < 0$. This means that trajectories move to the right above the x-axis and to the left below the x-axis.
 2) The y-nullcline is $x = 0$ (the y-axis) and $y' > 0$ for $x < 0$ and $y' < 0$ for $x > 0$. This means that trajectories move up to the left of the y-axis and down to the right of the y-axis.

This means that the trajectories are oriented clockwise around the origin. A similar analysis can be performed for the second system to show that the trajectories are oriented counterclockwise around the origin.
Next we analyze the case when $\alpha \neq 0$ for the first system.
 1) The x-nullcline is $y = -\alpha x$ and $x' > 0$ for $y > -\alpha x$ and $x' < 0$ for $y < -\alpha x$. This means that trajectories move to the right above the line $y = -\alpha x$ and to the left below the line if $\alpha > 0$, and the opposite if $\alpha < 0$.
 2) The y-nullcline is $y = \frac{1}{\alpha}x$ and $y' > 0$ for $x < \alpha y$ and $y' < 0$ for $x > \alpha y$. This means that trajectories move up to the left of the line $y = \frac{1}{\alpha}x$ and down to the right of the line if $\alpha > 0$, and the opposite if $\alpha < 0$.

This implies that the trajectories are oriented clockwise around the origin. A similar analysis can be performed for the second system to show that the trajectories are oriented counterclockwise around the origin.
The analysis above shows that the orientation of the trajectories does not depend on the value of α.

□

Problem 10.2 (The nonlinear damped pendulum).



The motion of a free pendulum experiencing air resistance is modeled by the nonlinear second ordered IVP

$$\begin{cases} L\theta''(t) + \beta\theta'(t) + g\sin(\theta(t)) = 0, & t \in \mathbb{R} \\ \theta(0) = \theta_0,\ \theta'(0) = v_0. \end{cases} \tag{10.3}$$

where $L > 0$ is the length of the rod, $\beta > 0$ is the damping constant indicating the strength of air resistance, $g > 0$ is the gravitational constant, and $\theta(t)$ represents the angle from the vertical to the pendulum at time t. Since the equation is nonlinear, finding explicit solution formulas can be difficult, but we can utilize the techniques we learned in this class to understand the behavior of the system when the initial angle and initial angular velocity are sufficiently small.

For the sake of simplicity we assume $L = \beta = g \neq 0$. Then the unknown function $\theta : \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{cases} \theta''(t) + \theta'(t) + \sin(\theta(t)) = 0, & t \in \mathbb{R} \\ \theta(0) = \theta_0,\ \theta'(0) = v_0. \end{cases} \tag{10.4}$$

a) Assume $\theta : \mathbb{R} \to \mathbb{R}$ is a solution to (10.4) and define $\boldsymbol{X} : \mathbb{R} \to \mathbb{R}^2$ via

$$\boldsymbol{X}(t) = \begin{pmatrix} \theta(t) \\ \theta'(t) \end{pmatrix}. \tag{10.5}$$

Write down a first order nonlinear system of the form

$$\begin{cases} \boldsymbol{X}'(t) = \boldsymbol{f}(\boldsymbol{X}(t)),\ t \in \mathbb{R} \\ \boldsymbol{X}(0) = \boldsymbol{X}_0. \end{cases} \tag{10.6}$$

b) Show that $\boldsymbol{X}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a critical point of (10.6). This should make physical sense as the pendulum should not move if it is at rest.

c) Use the linearization of the nonlinear system (10.6) to classify the stability of $\boldsymbol{X}_0$, and use this information to describe the long term behavior of solutions to the nonlinear equation (10.4) when the initial angle θ_0 and the initial angular velocity v_0 are both sufficiently small. To double-check that your classification is correct, you should ask yourself if the stability type you identified makes physical sense.

If you want to know what the trajectories associated to this nonlinear system looks like in phase space, Grant Sanderson (3Blue1Brown) made some nice visualizations in this video.

Solution.

a) Using (10.4), we have

$$\boldsymbol{X}'(t) = \begin{pmatrix} \theta'(t) \\ \theta''(t) \end{pmatrix} = \begin{pmatrix} 0 \cdot \theta(t) + 1 \cdot \theta'(t) \\ 0 \cdot \theta(t) - 1 \cdot \theta'(t) - \sin(\theta(t)) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \boldsymbol{X}(t) + \begin{pmatrix} 0 \\ -\sin(\theta(t)) \end{pmatrix}. \tag{10.7}$$

b) We note that

$$\boldsymbol{f}(\boldsymbol{X}_0) = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \boldsymbol{X}_0 + \begin{pmatrix} 0 \\ -\sin(0) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{10.8}$$

which shows that $\boldsymbol{X}_0$ is a critical point of (10.6). This makes physical sense as the pendulum should not move if it is at rest.

c) We note that the by defining $x : I \to \mathbb{R}$ via $x(t) = \theta(t)$ and $y : I \to \mathbb{R}$ via $y(t) = \theta'(t)$, we can rewrite the system (10.6) as

$$\begin{cases} x'(t) = y(t) =: P(x(t), y(t)) \\ y'(t) = -y(t) - \sin(x(t)) =: Q(x(t), y(t)). \end{cases} \tag{10.9}$$

We note that the linearization of the system (10.6) at the critical point $\boldsymbol{X}_0$ is then given by $\boldsymbol{X}' = A\boldsymbol{X}$ for

$$A = \begin{pmatrix} \partial_x P(0,0) & \partial_y P(0,0) \\ \partial_x Q(0,0) & \partial_y Q(0,0) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}. \tag{10.10}$$

We note that $\det(A - \lambda I) = (-\lambda)(-1-\lambda) + 1 = \lambda^2 + \lambda + 1$, so the eigenvalues of A are $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. This means that $\boldsymbol{X}_0$ is an asymptotically stable spiral point, which makes sense as one would expect the pendulum to return to equilibrium due to air resistance/damping.

$\square$

Problem 10.3 (The matrix-valued Laplace transform and the matrix exponential)**.** Consider the constant coefficient system $\boldsymbol{X}'(t) = A\boldsymbol{X}(t),\ t \in \mathbb{R}$, where

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{10.11}$$

a) Use the matrix-valued Laplace transform to compute the fundamental matrix given by $\Phi : \mathbb{R} \to \mathcal{M}_{2\times 2}(\mathbb{R})$ defined via $\Phi(t) = e^{tA}$. You can use the shortcut for finding the inverse of a 2×2 invertible matrix given in the notes.

b) Recall that the unique solution to the IVP

$$\begin{cases} \boldsymbol{X}'(t) = A\boldsymbol{X}(t),\ t \in \mathbb{R} \\ \boldsymbol{X}(0) = \boldsymbol{X}_0 \end{cases} \tag{10.12}$$

is given by $\boldsymbol{X} : \mathbb{R} \to \mathbb{R}^2$ defined via $\boldsymbol{X}(t) = \Phi(t)\boldsymbol{X}_0$. Use this to write down the solution to the IVP

$$\begin{cases} \boldsymbol{X}'(t) = A\boldsymbol{X}(t),\ t \in \mathbb{R} \\ \boldsymbol{X}(0) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \end{cases} \tag{10.13}$$

where c_1, c_2 are arbitrary constants.

c) Try solving the IVP (10.13) using the solution formula we learned previously in terms of the eigenvector(s)/generarized eigenvector(s) of A. How does this compare to the solution you found in part b)?

Solution.

a) We first compute

$$sI - A = \begin{pmatrix} s-1 & -1 \\ 0 & s-1 \end{pmatrix} \implies (sI-A)^{-1} = \frac{1}{(s-1)^2}\begin{pmatrix} s-1 & 1 \\ 0 & s-1 \end{pmatrix} = \begin{pmatrix} \frac{1}{s-1} & \frac{1}{(s-1)^2} \\ 0 & \frac{1}{s-1} \end{pmatrix}. \tag{10.14}$$

Therefore

$$\Phi(t) = e^{tA} = \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\} = \begin{pmatrix} \mathcal{L}^{-1}\left(\frac{1}{s-1}\right) & \mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}\right) \\ 0 & \mathcal{L}^{-1}\left(\frac{1}{s-1}\right) \end{pmatrix} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}. \tag{10.15}$$

b) By part a), the solution to the IVP (10.13) $\boldsymbol{X} : \mathbb{R} \to \mathbb{R}^2$ is given by

$$\boldsymbol{X}(t) = \Phi(t)\boldsymbol{X}_0 = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1e^t + c_2te^t \\ c_2e^t \end{pmatrix}. \tag{10.16}$$

c) We note that since A is upper triangular, A admits the repeating eigenvalue $\lambda = 1$. One can choose a corresponding eigenvector and a generalized eigenvector is given by

$$\boldsymbol{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{10.17}$$

Therefore the solution to the IVP (10.13) $\boldsymbol{X} : \mathbb{R} \to \mathbb{R}^2$ is given by

$$\boldsymbol{X}(t) = c_1e^t\boldsymbol{v} + c_2e^t(t\boldsymbol{v} + \boldsymbol{w}) = \begin{pmatrix} c_1e^t \\ 0 \end{pmatrix} + \begin{pmatrix} c_2te^t \\ c_2e^t \end{pmatrix} = \begin{pmatrix} c_1e^t + c_2te^t \\ c_2e^t \end{pmatrix}. \tag{10.18}$$

This coincides with the solution we found in part b).

$\square$

Problem 10.4 (The Fourier basis functions over $[0, L]$).

Let $\mathbb{R} \ni L > 0$. In lecture, we claimed that the functions

$$\frac{1}{\sqrt{L}} \quad \text{and} \quad \sqrt{\frac{2}{L}}\cos\left(\frac{2\pi kx}{L}\right) \quad \text{and} \quad \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi kx}{L}\right),\ k \in \mathbb{Z}^+ = \{1, 2, 3, 4, \ldots\} \tag{10.19}$$

considered as functions over $[0, L]$ are orthonormal with respect to the L^2 inner product

$$\langle f, g \rangle = \int_0^L f(x)g(x)\ dx. \tag{10.20}$$

In this problem we aim to justify this claim. To introduce some notation, define $a_0, a_k, b_k : [0, L] \to \mathbb{R}$ via

$$a_0(x) = \frac{1}{\sqrt{L}}, \quad a_k(x) = \sqrt{\frac{2}{L}}\cos\left(\frac{2\pi kx}{L}\right), \quad b_k(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi kx}{L}\right),\ k \in \mathbb{Z}^+. \tag{10.21}$$

a) Show that $\langle a_0, a_0 \rangle = 1$ and $\langle a_0, a_k \rangle = \langle a_0, b_k \rangle = 0$ for all $k \in \mathbb{Z}^+$.

b) Show that

$$\langle a_m, a_n \rangle = \langle b_m, b_n \rangle = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \tag{10.22}$$

for all $m, n \in \mathbb{Z}^+$.

You may find the trigonometric identities given in section d) of the Calculus reference sheet to be helpful.

Solution.

a) We calculate

$$\langle a_0, a_0 \rangle = \int_0^L \frac{1}{\sqrt{L}} \cdot \frac{1}{\sqrt{L}}\ dx = \int_0^L \frac{1}{L}\ dx = 1, \tag{10.23}$$

$$\langle a_0, a_k \rangle = \int_0^L \frac{1}{\sqrt{L}} \cdot \sqrt{\frac{2}{L}}\cos\left(\frac{2\pi kx}{L}\right)\ dx = \frac{\sqrt{2}}{L}\frac{L}{2\pi k}\sin\left(\frac{2\pi kx}{L}\right)\Bigg|_0^L = 0 \tag{10.24}$$

and

$$\langle a_0, b_k \rangle = \int_0^L \frac{1}{\sqrt{L}} \cdot \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi kx}{L}\right)\ dx = \frac{\sqrt{2}}{L}\left(-\frac{L}{2\pi k}\cos\left(\frac{2\pi kx}{L}\right)\right)_0^L = 0, \tag{10.25}$$

for $k \in \mathbb{Z}^+$.

b) If $m = n$, we note that by using the trigonometric identity $\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}, \theta \in \mathbb{R}$, we have

$$\langle a_m, a_n \rangle = \frac{2}{L}\int_0^L \cos^2\left(\frac{2\pi kx}{L}\right)\ dx = \frac{1}{L}\left(\int_0^L 1\ dx + \int_0^L \cos\frac{4\pi kx}{L}\ dx\right) = \frac{1}{L}(L + 0) = 1. \tag{10.26}$$

Using a similar identify and calculation one finds $\langle b_m, b_n \rangle = 1$ as well for $m = n$. If $m \neq n$, then we use the identity $\cos(a)\cos(b) = \frac{1}{2}\left(\cos(a+b) + \cos(a-b)\right),\ a, b \in \mathbb{R}$ to find

$$\begin{aligned}\langle a_m, a_n \rangle = \frac{2}{L}\int_0^L \cos\left(\frac{2\pi mx}{L}\right)\cos\left(\frac{2\pi nx}{L}\right)\ dx \\ = \frac{1}{L}\left(\int_0^L \cos\left(\frac{2\pi(m+n)x}{L}\right) + \cos\left(\frac{2\pi(m-n)x}{L}\right) dx\right) = 0,\end{aligned} \tag{10.27}$$

and following a similar calculation one finds that $\langle b_m, b_n \rangle = 0$ for $m \neq n$.

□

Problem 10.5 (The Fourier basis functions over $[-L, L]$)**.**

Let $\mathbb{R} \ni L > 0$. In this problem we repeat the same analysis as in the previous problem, but now we consider functions defined over $[-L, L]$. The L^2 inner product over $[-L, L]$ is given by

$$\langle f, g\rangle = \int_{-L}^{L} f(x)g(x)\, dx. \tag{10.28}$$

Define $a_0, a_k, b_k : [-L, L] \to \mathbb{R}$ via

$$a_0(x) = \frac{1}{\sqrt{2L}}, \quad a_k(x) = \sqrt{\frac{1}{L}} \cos\left(\frac{k\pi x}{L}\right), \quad b_k(x) = \sqrt{\frac{1}{L}} \sin\left(\frac{k\pi x}{L}\right),\ k \in \mathbb{Z}^+. \tag{10.29}$$

a) Show that $\langle a_0, a_0\rangle = 1$ and $\langle a_0, a_k\rangle = \langle a_0, b_k\rangle = 0$ for all $k \in \mathbb{Z}^+$.
b) Show that

$$\langle a_m, a_n\rangle = \langle b_m, b_n\rangle = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \tag{10.30}$$

for all $m, n \in \mathbb{Z}^+$.

Solution.

a) Similar to the previous problem, we calculate

$$\langle a_0, a_0\rangle = \int_{-L}^{L} \frac{1}{\sqrt{2L}} \cdot \frac{1}{\sqrt{2L}}\, dx = \int_{-L}^{L} \frac{1}{2L}\, dx = 1, \tag{10.31}$$

$$\langle a_0, a_k\rangle = \frac{1}{L\sqrt{2}} \int_{-L}^{L} \cos\left(\frac{k\pi x}{L}\right)\, dx = 0 \tag{10.32}$$

and

$$\langle a_0, b_k\rangle = \frac{1}{L\sqrt{2}} \int_{-L}^{L} \sin\left(\frac{k\pi x}{L}\right)\, dx = 0, \tag{10.33}$$

for $k \in \mathbb{Z}^+$.

b) The calculation for $m \neq n$ is completely analogous to the previous problem, the only thing that differs is the normalization factor for $m = n$. If $m = n$, we calculate

$$\langle a_m, a_n\rangle = \frac{1}{L} \int_{-L}^{L} \cos^2\left(\frac{k\pi x}{L}\right)\, dx = \frac{1}{2L}\left(\int_{-L}^{L} 1\, dx + \int_{-L}^{L} \cos\left(\frac{2k\pi x}{L}\right)\, dx\right) = \frac{1}{2L}(2L + 0) = 1, \tag{10.34}$$

and using a similar identify and calculation one finds $\langle b_m, b_n\rangle = 1$ as well for $m = n$.

□

Problem 10.6 (Fourier series and periodic extensions)**.** Consider the function $f : (-\pi, \pi) \to \mathbb{R}$ defined via

$$f(x) = x. \tag{10.35}$$

a) Explain why the Fourier series of f over $(-\pi, \pi)$ reduces to a Fourier sine series (i.e. the constant term and the cosine terms vanish).
b) Note that by definition, the Fourier sine coefficients of a function over $(-\pi, \pi)$ are defined in terms of an integral over $(-\pi, \pi)$. Explain why here the Fourier sine coefficients of f can be computed via

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx, \; n \geq 1. \tag{10.36}$$

c) Show that

$$b_n = \frac{2(-1)^{n+1}}{n}, \; n \geq 1. \tag{10.37}$$

d) Sketch a graph over $\mathbb{R}$ of its Fourier sine series. Is this Fourier sine series a continuous function over $\mathbb{R}$?
e) Now consider the restriction of f to $[0, \pi)$, $g : [0, \pi) \to \mathbb{R}$ defined via

$$g(x) = x. \tag{10.38}$$

Consider an even reflection $\tilde{g} : (-\pi, \pi) \to \mathbb{R}$ of g to $(-\pi, 0]$ defined via

$$\tilde{g}(x) = \begin{cases} g(x), & x \in [0, \pi) \\ g(-x), & x \in (-\pi, 0), \end{cases} \tag{10.39}$$

and consider the Fourier series associated to the reflected function $\tilde{g}$. Explain why in this case, the Fourier series of $\tilde{g}$ reduces to a Fourier cosine series (i.e. the sine terms vanish), and sketch a graph over $\mathbb{R}$ of this cosine series. Is this Fourier cosine series a continuous function over $\mathbb{R}$?

For the sketches, please be precise about the values of the corresponding series at the points of discontinuity.

Solution.

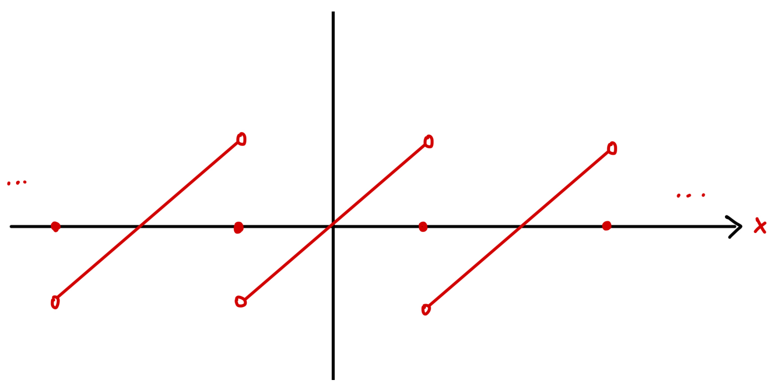
a) Since f is an odd function, the Fourier series of f over $(-\pi, \pi)$ reduces to a Fourier sine series as the constant term and the cosine terms vanish due to symmetry.

b) Since f and the sine functions are both odd functions, the product between them is an even function over $(-\pi, \pi)$. This means that the inner product between them can be computed via symmetry as an integral of $(0, \pi)$, which is why the Fourier sine coefficients can be computed via an integral over $(0, \pi)$ instead of $(-\pi, \pi)$.

c) We calculate for $n \geq 1$,

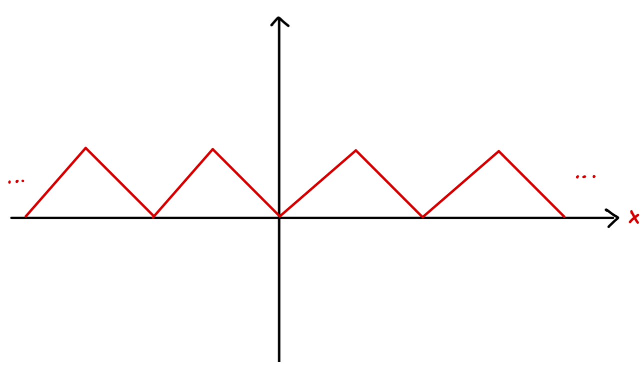
$$b_n = \frac{2}{\pi}\int_0^\pi x\sin nx\,dx = \frac{2}{\pi}\left(-\frac{x\cos nx}{n}\bigg|_0^\pi + \frac{1}{n}\int_0^\pi \cos nx\,dx\right) = \frac{2}{\pi}\left(-\frac{x\cos nx}{n}\bigg|_0^\pi + \frac{\sin nx}{n^2}\right)\bigg|_0^\pi$$
$$= \frac{2}{\pi}\left(-\frac{\pi\cos n\pi}{n}\right) = \frac{2(-1)^{n+1}}{n}. \quad (10.40)$$

d) Note that by the Fourier convergence theorem, at the points of discontinuity of the periodic extension, the Fourier series converges to the average of the left limit and the right limit. Therefore we have the following sketch over $\mathbb{R}$.



Clearly, this is not a continuous function over $\mathbb{R}$.

e) Similarly, we have the following sketch over $\mathbb{R}$ for the odd extension of g.



In this case, the half-range Fourier cosine series of f converges to a continuous function over $\mathbb{R}$.

□