# Homework 3

DUE: SATURDAY, FEBRUARY 8, 2025, 11:59PM

If you completed this assignment through collaboration or consulted references, please list the names of your collaborators and the references you used below. Please refer to the syllabus for the course policy on collaborations and types of references that are allowed.

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*Date*: February 1, 2025.

**Problem 3.1** (Isoclines)**.** Consider the differential equation

$$y'(t) = 1 - ty(t),\ t \in \mathbb{R}. \tag{3.1}$$

a) Determine the general form of the isoclines associated to the equation, with slope $m$. (Note: you need to consider $m = 1$ as a separate case)
b) Sketch the directional field associated to the equation by using part a). You should at the very least sketch the isoclines corresponding to $m = 0, \pm 1, \pm 2$.
c) Sketch the solution curve $y_1$ passing through the point $(0, 2)$, and also the solution curve $y_2$ passing through the point $(2, 0)$.
d) Is it possible for the two curves $y_1$ and $y_2$ to ever cross, either going forward in time (the variable $t$) or backwards in time? Explain why or why not.

**Problem 3.2** (Exact differential equations)**.** Find an implicit solution to the initial value problem

$$\begin{cases} (x+y(x))^2 + (2xy(x) + x^2 - 1)y'(x) = 0, \ x \in \mathbb{R} \\ y(1) = 1. \end{cases} \tag{3.2}$$

You do not need to specify the interval of existence.

**Problem 3.3** (Wronskian and linear independence)**.** Show that the following sets of functions are linearly independent over the interval $I = \mathbb{R}$ by computing the Wronskian between them.

a) $y_1(x) = e^{r_1 x}, y_2(x) = e^{r_2 x}, x \in \mathbb{R}$, for any $r_1 \neq r_2 \in \mathbb{R}$.
b) $y_1(x) = e^{rx}, y_2(x) = xe^{rx}, x \in \mathbb{R}$, for any $r \in \mathbb{R}$.
c) $y_1(x) = e^{ax} \cos(bx), y_2(x) = e^{ax} \sin(bx), x \in \mathbb{R}$, for any $\mathbb{R} \ni a, b \neq 0$.

**Problem 3.4** (Wronskians of solutions to second order linear homogeneous differential equations)**.** Consider the second order linear homogeneous differential equation

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0,\ x \in I, \tag{3.3}$$

where $I$ is an interval on which $a_2, a_1, a_0$ are continuous and $a_2(x) \neq 0$ for all $x \in I$.

Suppose $y_1, y_2$ are two $C^2$ solutions to the equation and define the Wronskian $W : I \to \mathbb{R}$ via

$$W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x). \tag{3.4}$$

a) Calculate $W'$ and show that $W$ satisfies the differential equation

$$a_2(x)W'(x) = y_1(x)(a_2(x)y_2''(x)) - y_2(x)(a_2(x)y_1''(x)),\ x \in I. \tag{3.5}$$

b) Use (3.5) to show that $W$ satisfies

$$a_2(x)W'(x) = -a_1(x)W(x),\ x \in I. \tag{3.6}$$

c) Show that

$$W(x) = C \exp\left(-\int \frac{a_1(x)}{a_2(x)}\, dx\right),\ x \in I \tag{3.7}$$

where $C$ is an arbitrary constant.

d) Conclude that in this setting, either $W(x) = 0$ for all $x \in I$ or $W(x) \neq 0$ for all $x \in I$. In other words, either it vanishes everywhere on $I$ or it does not vanish anywhere on $I$.

**Problem 3.5.** For each of the following initial value problems, use an appropriate method to find a solution and state the maximal interval of existence. You may skip the verification steps.

a)

$$\begin{cases} y'(t) = y(t)e^t, \ t \in \mathbb{R} \\ y(0) = 2e. \end{cases} \tag{3.8}$$

b)

$$\begin{cases} x^2 y'(x) - (y(x))^2 - xy(x) = 0, \ x \in \mathbb{R} \\ y(1) = 2. \end{cases} \tag{3.9}$$

c)

$$\begin{cases} xy'(x) = 3y(x) + 2x^4, \ x \in \mathbb{R} \\ y(1) = 0 \end{cases} \tag{3.10}$$

**Problem 3.6** (Wronskian)**.** Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are two smooth functions, and we are given that the Wronskian of $f$ and $g$ is given by

$$W(f, g)(t) = -3e^{4t} \text{ and } f(t) = 4e^{2t},\ t \in \mathbb{R}. \tag{3.11}$$

What is $g$?