Review Part 1

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Part 1. **Concepts to review**

a) Classification of equations
- Order of an equation.
- ODE vs PDE.
- Linear vs nonlinear.

b) Definition of solutions
- Requirements for a function to be a solution
- Maximum interval of existence for an initial value problem: this is the largest possible interval on which the solution is defined and it also contains the initial point at which the initial condition is specified.

c) First order linear equations
- These are equations of the form $a_1(x)y'(x) + a_0 y(x) = g(x), x \in I$. For initial value problems, one specifies $y(x_0) = y_0$ for some $x_0 \in I$.
- If a_1, a_0, g are continuous on I and $a_1(x) \neq 0$ for all $x \in I$, then the existence and uniqueness of a solution y to an initial value problem on I is guaranteed.
- Note that here y exists on the entirety of I, not just a sub-interval.
- A special case of this is when one consider equations of the form $y'(x) + a_0(x)y(x) = g(x)$. If a_0, g are continuous on $\mathbb{R}$, then a unique solution to an initial value problem over $\mathbb{R}$ exists globally over $\mathbb{R}$.
- Can be solved explicitly using the method of integrating factors.

d) Structure of solutions to first order linear equations
- The general solution to the inhomogeneous problem is any particular solution plus the general solution to the homogeneous problem.
- The homogeneous part of the solutions can be scaled, the inhomogeneous part of the solution cannot be scaled.
- The method of integrating factors recovers both the homogeneous solution and the inhomogeneous solution simultaneously.

e) General first order equations
- The existence and uniqueness theorems guarantees the (local, not global) existence and uniqueness of solutions to initial value problems.
- If uniqueness failed for an initial value problem, the conditions of the theorem must not have been met.

f) Separable equations
- After identifying the constant functions, an implicit solution can usually be found via "separating the variables."
- Sometimes explicit solutions and intervals of existence can be found if the implicit relation is simple.
- Standard techniques: manipulating identities involving ln, removing absolute values by working over a smaller interval and absorbing the $\pm$ sign into the arbitrary constant coming from the constant of integration.

g) Autonomous equations
- Special type of separable equations.
- Solution curves are strictly monotone.
- Non-constant solution curves converge to constant solutions either going forward in time or backwards in time.
- One can perform stability analysis on the critical points (which correspond to constant functions) by doing a simple sign analysis.

h) Bernoulli equations
- Can be solved explicitly using a substitution method.
- By introducing a suitable function $v = v(y)$, one may write down a first order linear equation in v and solve for v, and use it to recover y.

i) Differential equations with homogeneous functions
- Can be solved implicitly (sometimes explicitly) using a substitution method.
- By introducing a suitable function $v = v(x, y)$, one may write down a separable equation for v and solve for v,and use it to recover y.
- Important to also look for constant solutions in v when solving the corresponding separable equation.

j) General strategies for solving ODEs
- When one is manipulating an equation, one is identifying candidate solutions.

- The idea is to identify the correct "form" of the solution by making a series of forward implications, at the price of potentially working over smaller and smaller sub-intervals.
- Once a correct candidate form has been identified, one can proceed to verify that the candidate solution is a genuine solution and also identify the maximal interval of existence.
- The maximal interval of existence in general depends on the initial condition specified.

k) Directional fields and the method of isoclines
- Method for sketching solution curves without solving for them analytically.
- Identifying the isoclines can help sketch out the directional field must faster than computing the linear elements point-by-point.
- Directional fields for autonomous equations are easiest to sketch: the slopes are always constant along each horizontal line.

l) Exact differential equations
- These are equations of the form $M(x, y(x)) + N(x, y(x))y'(x) = 0, x \in I$.
- Necessary and sufficient condition for the equation to be exact in a region: $M_y(x, y) = N_x(x, y)$ in some region R.
- If an equation is exact, an implicit solution of the form $F(x, y(x)) = C, x \in I$ can be recovered by using $F_x = M$ and $F_y = N$.

m) Linear independence and the Wronskian

a) Linear combination: if $f_1, \ldots, f_k$ are k functions, the function f defined via $f = c_1 f_1 + \ldots + c_k f_k$ where $c_1, \ldots, c_k$ are k real numbers is a linear combination of the functions $f_1, \ldots, f_k$.

b) Linear independence is a concept from linear algebra that encodes the notion of "non-redundancy" when we consider all possible linear combinations that can be built from a set of functions.

c) The linear independence of functions is encoded in an object called the Wronskian.

d) Under certain conditions, the vanishing of the Wronskian over an interval I implies the linear dependence of a set of functions. If the Wronskian is non-zero at any point $t_0 \in I$ then the set of functions is linearly independent.

e) If the functions in question are solutions to linear homogeneous differential equations over an interval I, then the Wronskian is either never zero or always equal to zero over I.

Part 2. First order linear equations

Problem 1.1. Consider the initial value problem

$$\begin{cases} y'(x) + 4xy(x) = x^3 e^{x^2},\ x \in \mathbb{R} \\ y(0) = -1. \end{cases} \tag{1.1}$$

a) Find a candidate solution to the initial value problem.
b) Verify that the candidate solution from the previous part is a solution on $J = \mathbb{R}$.
c) Is the solution to the initial value problem unique?

Part a). Since this is a first order linear equation, we can use the method of integrating factors. We may choose an integrating factor μ to be

$$\mu(x) = \exp\left(\int 4x\ dx\right) = e^{2x^2},\ x \in \mathbb{R}. \tag{1.2}$$

Then upon multiplying both sides of the original equation by μ, we find that

$$\frac{d}{dx}[e^{2x^2} y(x)] = x^3 e^{3x^2},\ x \in \mathbb{R}. \tag{1.3}$$

We note that by performing integration by parts,

$$\int x^3 e^{3x^2}\ dx = \frac{1}{6}\int x^2(6x)e^{3x^2}\ dx = \frac{1}{6}\left(x^2 e^{3x^2} - \int 2xe^{3x^2}\ dx\right) = \frac{1}{6}\left(x^2 e^{3x^2} - \frac{e^{3x^2}}{3} + C\right). \tag{1.4}$$

Therefore we find that

$$y(x) = \frac{e^{x^2}}{18}\left(3x^2 - 1\right) + Ce^{-2x^2}. \tag{1.5}$$

If $y(0) = -1$, then we require

$$y(0) = -\frac{1}{18} + C = -1 \implies C = \frac{-17}{18}. \tag{1.6}$$

□

Part b). To verify that y defined via

$$y(x) = \frac{e^{x^2}}{18}(3x^2 - 1) - \frac{17}{18}e^{-2x^2},\ x \in \mathbb{R} \tag{1.7}$$

is a solution to the initial value problem on $J = \mathbb{R}$, we first note that

$$y(0) = -\frac{1}{18} - \frac{17}{18} = -1. \tag{1.8}$$

Then, we note that

$$y'(x) = \frac{e^{x^2}}{18}(6x + 2x(3x^2 - 1)) + \frac{17}{18}(4x)e^{-2x^2} = \frac{e^{x^2}}{18}(6x^3 + 4x) + \frac{17}{18}(4x)e^{-2x^2} \tag{1.9}$$

We may then verify that

$$y'(x) + 4xy(x) = \frac{e^{x^2}}{18}(6x^3 + 4x) + \frac{17}{18}(4x)e^{-2x^2} + \frac{e^{x^2}}{18}(12x^3 - 4x) - \frac{17}{18}(4x)e^{-2x^2} \tag{1.10}$$

$$= \frac{e^{x^2}}{18}(6x^3 + 4x) + \frac{e^{x^2}}{18}(12x^3 - 4x) + \underbrace{\frac{17}{18}(4x)e^{-2x^2} - \frac{17}{18}(4x)e^{-2x^2}}_{=0} \tag{1.11}$$

$$= \frac{18x^3 e^{x^2}}{18} = x^3 e^{x^2},\ x \in \mathbb{R}. \tag{1.12}$$

Therefore the candidate solution from part a) is a solution over $\mathbb{R}$. □

Part c). Yes. We note that the equation is in the form of

$$y'(x) + a_0(x)y(x) = f(x),\ x \in \mathbb{R}, \tag{1.13}$$

for

$$a_0(x) = 4x,\ f(x) = x^3 e^{x^2},\ x \in \mathbb{R}. \tag{1.14}$$

Since a_0, f are both continuous over $\mathbb{R}$, by the existence and uniqueness theorem for first order linear differential equations, the solution is unique. $\square$

Problem 1.2. Consider the initial value problem

$$\begin{cases} xy'(x) + 3y(x) = x^3, \ x \in \mathbb{R} \\ y(1) = 10. \end{cases} \tag{1.15}$$

a) Find a candidate solution via the method of integrating factors. You may assume that the candidate solution solves the equation without verifying.
b) What is the maximal interval of existence of the solution? Explain your reasoning.
c) Is the solution that you found unique?
d) What happens if we change the initial condition to $y(0) = 1$?

Part a). Since the initial condition is specified at $x = 1$, we may restrict to the interval $I = (0, \infty)$ for now and study the equation

$$y'(x) + \frac{3}{x}y(x) = x^2, \ x > 0. \tag{1.16}$$

An integrating factor for this equation can be chosen to be

$$\mu(x) = \exp\left(\int \frac{3}{x}\, dx\right) = \exp\left(3 \ln |x|\right) = \exp\left(\ln x^3\right) = x^3, \ x > 0. \tag{1.17}$$

Therefore upon multiplying both sides of the equation by μ, we arrive at the equation

$$\frac{d}{dx}[x^3 y(x)] = x^5, \ x > 0. \tag{1.18}$$

Then via direct integration,

$$x^3 y(x) = \frac{x^6}{6} + C, \ x \in \mathbb{R}, \tag{1.19}$$

where C is an arbitrary constant. This implies that

$$y(x) = \frac{x^3}{6} + Cx^{-3}, x > 0. \tag{1.20}$$

If $y(1) = 10$, then we must have

$$y(1) = \frac{1}{6} + C = 10 \implies C = \frac{59}{6}. \tag{1.21}$$

Thus a candidate solution to the initial value problem is

$$y(x) = \frac{x^3}{6} + \frac{59}{6}x^{-3}, \ x > 0. \tag{1.22}$$

□

Part b). From part a) we see that the maximal interval of existence is $J = (0, \infty)$, since it is the largest interval that contains 1 and on which the function y is a solution to the initial value problem. Note that we cannot extend J past 0 since the function x^{-3} is singular at $x = 0$. □

Part c). Yes. If we restrict to the interval $I = (0, \infty)$, then y solving the original equation is equivalent to y solving the equation

$$y'(x) - \frac{3}{x}y(x) = x^2, \ x \in I. \tag{1.23}$$

Since a_0, f defined via

$$a_0(x) = -\frac{3}{x}, \ f(x) = x^2, \ x \in I \tag{1.24}$$

are continuous over I, by the existence and uniqueness theorem for first order linear equations we know that the function identified in the previous parts is the unique solution to the initial value problem.

□

Part d). We note that if the initial condition is $y(0) = 1$, then we cannot include any contributions from the homogeneous solutions as they are singular at $x = 0$. Furthermore, the particular solution

$$y_p(0) = \frac{0^3}{6} = 0 \neq 1, \tag{1.25}$$

therefore we see that there are no solutions to the initial value problem. □

Problem 1.3. Consider the initial value problem

$$\begin{cases} y'(t) - \frac{2}{(t+1)(t-1)} y(t) = t - 1, & t \in I \\ y(t_0) = 0 \end{cases} \tag{1.26}$$

where $t_0 \in I$ is and I is an unspecified interval.

a) At which points t_0 is the existence and uniqueness of a solution not guaranteed?
b) Suppose the interval is $I = (-1, 1)$. Find an analytic expression for the solution to the initial value problem where $t_0 = 0$. You may skip the verification step. (Note: $|t-1| = -(t-1)$ if $t \in (-1, 1)$.)

Part a). Consider the functions a_0, f defined via

$$a_0(t) = \frac{2}{(t+1)(t-1)}, \ t \neq \pm 1 \tag{1.27}$$

and

$$f(t) = t - 1, \ t \in \mathbb{R}. \tag{1.28}$$

Since a_0, f are both continuous away from the points $t = \pm 1$ but a_0 is not continuous when $t = \pm 1$, we see that the existence and uniqueness of solutions is not guaranteed when $t_0 = \pm 1$. Therefore the existence and uniqueness of solutions is guaranteed on the intervals $(-\infty, -1), (-1, 1)$, and $(1, \infty)$. $\square$

Part b). Since this is a first order linear equation, we can solve it using the method of integrating factors. We may choose an integrating factor μ as

$$\begin{aligned} \mu(t) = \exp\left(-2 \int \frac{1}{(t+1)(t-1)} \, dt\right) &= \exp\left(\int \frac{1}{t+1} - \frac{1}{t-1} \, dt\right) \\ &= \exp\left(|t+1| - |t-1|\right) = \exp\left(\ln\left|\frac{t+1}{t-1}\right|\right) = \frac{t+1}{-(t-1)}, \ t \in I. \end{aligned} \tag{1.29}$$

Note: since integrating factors are agnostic to scaling factors, choosing

$$\mu(t) = \frac{t+1}{t-1}, \ t \in I \tag{1.30}$$

is fine too. For the sake of simplicity we will choose the latter one. Then

$$\frac{d}{dt}\left(\frac{t+1}{t-1} y(t)\right) = t + 1, \tag{1.31}$$

which implies that

$$y(t) = \frac{t-1}{t+1}\left(\frac{t^2}{2} + t + C\right), \ t \in I. \tag{1.32}$$

where C is arbitrary. $\square$

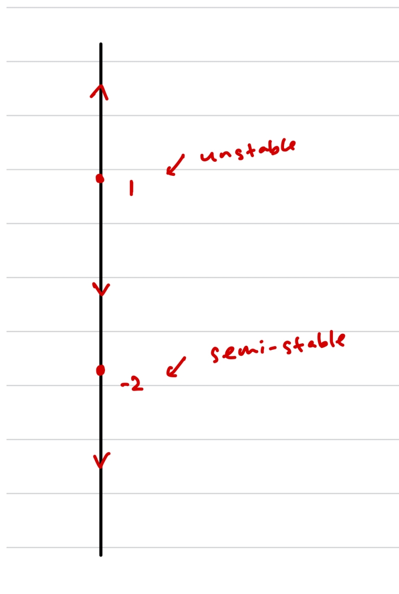
Part 3. Autonomous equations and stability analysis

Problem 1.4. Consider the differential equation

$$y'(x) = (y(x) - 1)(y(x) + 2)^2((y(x))^2 + 1), \ x \in \mathbb{R}. \tag{1.33}$$

a) Draw the one-dimensional phase portrait for this differential equation.
b) Let $y(t)$ be the solution satisfying the initial condition $y(0) = 0$. Can the value of y ever be less than -2? Why or why not?

Part a). We note that since $(y + 2)^2(y^2 + 1) \geq 0$ for all $y \in \mathbb{R}$, the sign of $(y - 1)(y + 2)^2(y^2 + 1)$ only depends on the linear factor $y - 1$. This leads to the following sketch for the one-dimensional phase portrait.



□

Part b). We note that since -2 is a critical point, $y(x) = -2$ for $x \in \mathbb{R}$ is a constant solution to the equation. Since different solution curves cannot cross due to uniqueness, it is not possible for any solution curve to start above -2 at $x = 0$ (note that $y(0) = 0 > -2$) and then fall below -2. □

Part 4. **Separable equations**

Problem 1.5. Consider the differential equation

$$y'(x) = \frac{x((y(x))^2+1)}{2y(x)},\ x \in \mathbb{R}. \tag{1.34}$$

a) Does the equation admit any constant solutions?
b) Find the general candidate implicit solution to the equation. You do not need to specify the interval of existence.

Part a). We note that the function h defined via

$$h(y) = \frac{y^2+1}{2y},\ y \in \mathbb{R} \setminus \{0\} \tag{1.35}$$

does not admit any critical points. In other words, no constant value of y makes h equal to zero, therefore the equation does not admit any constant solutions. □

Part b). Note that if y is a solution to the original equation, then

$$\frac{2y(x)}{(y(x))^2+1}y'(x) = x,\ x \in \mathbb{R}. \tag{1.36}$$

Thus we have the identity

$$\int \frac{2y}{y^2+1}\, dy = \int x\, dx. \tag{1.37}$$

This implies that

$$\ln(y(x)^2+1) = \frac{x^2}{2} + C,\ x \in I \tag{1.38}$$

is the general implicit solution to the equation over some interval I, and C is an arbitrary constant. □

Problem 1.6. Consider the differential equation

$$y'(x) = (y(x))^8 e^{-x^4}, \ x \in \mathbb{R} \tag{1.39}$$

a) Does the equation admit any constant solutions?
b) What is one solution y satisfying $y(0) = 0$? Is this solution unique? (Hint: you don't need to solve for the general solution if you did part a) correctly.)

Part a). Yes, the equation admits the constant solution $y(x) = 0$ for all $x \in \mathbb{R}$. □

Part b). Clearly, the zero function identified in the previous part is a solution satisfying the initial conditon $y(0) = 0$. Note that if we define the function f via

$$f(x, y) = y^8 e^{-x^4}, \ (x, y) \in \mathbb{R}^2, \tag{1.40}$$

we have

$$\frac{\partial f}{\partial y}(x, y) = 8y^7 e^{-x^4}, \ (x, y) \in \mathbb{R}^2. \tag{1.41}$$

Here we see that f and $\frac{\partial f}{\partial y}$ are continuous on $\mathbb{R}^2$, therefore the zero solution must also be the unique solution to the initial value problem. □

Problem 1.7. Consider the initial value problem

$$\begin{cases} y'(x) = \frac{-\sin(x)}{2y(x)}, \ x \in \mathbb{R} \\ y(0) = \frac{1}{\sqrt{2}}. \end{cases} \tag{1.42}$$

a) Find a candidate implicit solution to the initial value problem. You do not need to specify the interval of existence.
b) Identify a candidate explicit solution to the initial value problem and identify the maximal interval of existence.

Part a). First note that the equation does not admit any constant solutions. If y is a solution to the initial value problem, then

$$2y(x)y'(x) = -\sin(x), \ x \in \mathbb{R}. \tag{1.43}$$

This implies that

$$(y(x))^2 = \cos(x) + C, \ x \in I \tag{1.44}$$

for some interval I. If $y(0) = \frac{1}{\sqrt{2}}$, then

$$\frac{1}{2} = 1 + C \implies C = -\frac{1}{2}. \tag{1.45}$$

Therefore the implicit solution to the initial value problem is

$$(y(x))^2 = \cos(x) - \frac{1}{2}, \ x \in I. \tag{1.46}$$

□

b). We note that since $y(0) > 0$, we can write y explicitly as

$$y(x) = \sqrt{\cos(x) - \frac{1}{2}}, \ x \in I \tag{1.47}$$

for some interval I. Since $y(x)$ cannot be equal to 0 for the original equation to be well-defined, we require

$$\cos(x) - \frac{1}{2} > 0 \iff \cos(x) > \frac{1}{2}, \ x \in I. \tag{1.48}$$

The largest interval for which this is true that contains 0 is $I = (-\pi/3, \pi/3)$. □

Part 5. Bernoulli differential equations

Problem 1.8. Solve the initial value problem

$$\begin{cases} y'(x) = y(x)(x(y(x))^3 - 1), \ x \in \mathbb{R} \\ y(0) = 3^{1/3}. \end{cases} \tag{1.49}$$

What is the maximal interval of existence of the solution?

Solution. We note that if y is a solution to the equation, then

$$y'(x) + y(x) = x(y(x))^4, \ x \in \mathbb{R}. \tag{1.50}$$

This is a Bernoulli differential equation, and we use the substituion

$$v(x) = (y(x))^{-3}, \ x \in I \tag{1.51}$$

on an interval for which $y(x) \neq 0$ for all $x \in I$. Then

$$v'(x) = -3(y(x))^{-4}, \ x \in I \tag{1.52}$$

and also y satisfies the equation

$$(y(x))^{-4}y'(x) + y^{-3}(x) = x, \ x \in I. \tag{1.53}$$

Therefore v satisfies the equation

$$-\frac{1}{3}v'(x) + v(x) = x, \ x \in I, \tag{1.54}$$

or

$$v'(x) - 3v(x) = -3x, x \in I. \tag{1.55}$$

Therefore v satisfies a first order linear equation and we may choose an integrating factor μ via

$$\mu(x) = \exp\left(\int -3 \ dx\right) = e^{-3x}, \ x \in I. \tag{1.56}$$

Thus

$$\frac{d}{dx}[e^{-3x}v(x)] = -3xe^{-3x}, \ x \in I. \tag{1.57}$$

Thus

$$e^{-3x}v(x) = \int x(-3e^{-3x}) \ dx = xe^{-3x} - \frac{e^{-3x}}{-3} + C, \ x \in I \tag{1.58}$$

where C is an arbitrary constant. Therefore

$$v(x) = x + \frac{1}{3} + Ce^{3x}, \ x \in I. \tag{1.59}$$

If $y(0) = 3^{1/3}$, then $v(0) = (y(0))^{-3} = \frac{1}{3}$. Thus

$$v(0) = \frac{1}{3} + Ce^{3x} = \frac{1}{3} \implies C = 0. \tag{1.60}$$

Thus a candidate solution to the initial value problem is

$$y(x) = \left(x + \frac{1}{3}\right)^{-1/3}, \ x \in I. \tag{1.61}$$

Here we see that the maximal interval of existence is $J = (-1/3, \infty)$. $\square$

Part 6. **Differential equations with homogeneous functions**

Problem 1.9. Solve the initial value problem

$$\begin{cases} (3x+y(x))y'(x) = x+3y(x),\ x\in\mathbb{R} \\ y(1)=1. \end{cases} \tag{1.62}$$

What is the maximal interval of existence of the solution?

Solution. We note that if y is a solution to the initial value problem, we may assume that there exists an interval I not containing 0 for which

$$y'(x) = \frac{x+3y(x)}{3x+y(x)} = \frac{1+3\frac{y(x)}{x}}{3+\frac{y(x)}{x}},\ x\in I. \tag{1.63}$$

If we define the function v via

$$v(x) = \frac{y(x)}{x},\ x\in I, \tag{1.64}$$

then $y = xv$ and

$$y'(x) = v(x) + xv'(x),\ x\in I. \tag{1.65}$$

Therefore v satisfies the separable equation

$$v(x) + xv'(x) = \frac{1+3v}{3+v},\ x\in I. \tag{1.66}$$

This implies that

$$xv'(x) = \frac{1-(v(x))^2}{v(x)+3} = \frac{(1-v(x))(1+v(x))}{v+3},\ x\in I. \tag{1.67}$$

Here we note that $v(x) = \pm 1$ for $x\in I$ are constant solutions to this equation, and if $y(1)=1$, then $v(1) = 1/1 = 1$. Thus the constant solution $v(x)=1$ satisfies the initial condition on v, and $y(x)=x, x\in I$ is a candidate solution for the initial value problem for y.

We note that if $y(x) = x,\ x\in\mathbb{R}$, then

$$(3x+y(x))y'(x) = (3x+x)(1) = 4x \text{ and } x+3y(x) = x+3x = 4x. \tag{1.68}$$

Therefore $y(x)=x, x\in\mathbb{R}$ is a candidate solution to the initial value problem with its maximal interval of existence being $J=\mathbb{R}$. We also note that if we define the function f via

$$f(x,y) = \frac{x+3y}{3x+y},\ (x,y)\in\mathbb{R} \text{ such that } 3x+y\neq 0 \tag{1.69}$$

then

$$\frac{\partial f}{\partial y}(x,y) = \frac{(3x+y)(3)-(x+3y)(1)}{(3x+y)^2},\ (x,y)\in\mathbb{R} \text{ such that } 3x+y\neq 0, \tag{1.70}$$

and we see that f, $\frac{\partial f}{\partial y}$ are continuous at any point away from the line $3x+y=0$. Since $3(1)+1 = 4\neq 0$, the initial point $(1,1)$ does not lie on this line, so by the existence and uniqueness theorem the solution $y(x)=x, x\in\mathbb{R}$ we identified is the unique solution to the initial value problem.

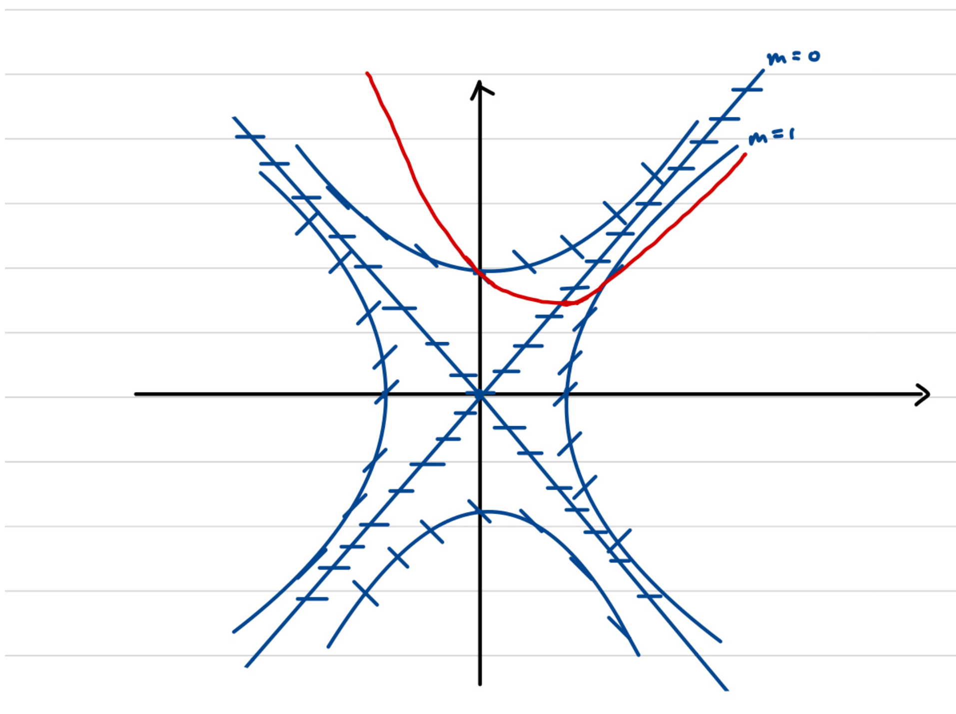
□

Part 7. Isoclines and directional fields

Problem 1.10. Sketch the solution curve to the initial value problem

$$\begin{cases} y'(x) = x^2 - (y(x))^2, \ x \in \mathbb{R} \\ y(0) = 1. \end{cases} \tag{1.71}$$

Solution. We note that $x^2 - y^2 = m$ are hyperbolas with vertices lying on $(\pm\sqrt{m}, 0)$ for $m > 0$, hyperbolas with vertices lying on $(0, \pm\sqrt{m})$ for $m < 0$, and $m = 0$ corresponds to the lines $y = \pm x$. Here is a very rough sketch:



□

Part 8. **Exact differential equations**

Problem 1.11. Find an implicit solution to the initial value problem

$$\begin{cases} (e^x + y(x)) + (2 + x + y(x)e^{y(x)})y'(x) = 0, \ x \in \mathbb{R} \\ y(0) = 1. \end{cases} \tag{1.72}$$

You do not need to specify the interval of existence.

Solution. Define M, N via

$$M(x, y) = e^x + y, \ N(x, y) = 2 + x + ye^y, \ (x, y) \in \mathbb{R}^2. \tag{1.73}$$

Then

$$M_y(x, y) = 1, \ N_x(x, y) = 1, \ (x, y) \in \mathbb{R}^2. \tag{1.74}$$

Therefore the equation is exact. Thus there exists a function F with $F_x = M, F_y = N$. We note that $F_x = M$ implies

$$F(x, y) = \int e^x + y \, dx + g(y) = e^x + xy + g(y), \ (x, y) \in \mathbb{R}^2, \tag{1.75}$$

and thus

$$F_y(x, y) = x + g'(y) = N(x, y) = 2 + x + ye^y, \ (x, y) \in \mathbb{R}^2. \tag{1.76}$$

Thus we must have

$$g'(y) = ye^y + 2 \implies g(y) = ye^y - e^y + 2y + C, \tag{1.77}$$

where C is an arbitrary constant. Thus

$$F(x, y(x)) = e^x + xy(x) + y(x)e^{y(x)} - e^{y(x)} + 2y(x) = C, \ x \in I \tag{1.78}$$

is an implicit solution to the equation over some interval $x \in I$, where C is an arbitrary constant. Since $y(0) = 1$, we require

$$C = e^0 + 0(1) + 1e^1 - e^1 + 2(1) = 3. \tag{1.79}$$

Thus an implicit solution to the initial value problem is

$$e^x + xy(x) + y(x)e^{y(x)} - e^{y(x)} + 2y(x) = 3, \ x \in I. \tag{1.80}$$

□

Part 9. Wronskian and linear independence

Problem 1.12. Are the functions y_1, y_2 defined via

$$y_1(x) = \cos(\ln x),\ y_2(x) = \sin(\ln x),\ x > 0 \tag{1.81}$$

linearly independent over $I = (0, \infty)$?

Solution. We calculate

$$W(y_1, y_2)(x) = \det \begin{pmatrix} \cos(\ln x) & \sin(\ln x) \\ -\frac{\sin(\ln x)}{x} & \frac{\cos(\ln x)}{x} \end{pmatrix} = \frac{1}{x}\left(\cos^2(\ln x) + \sin^2(\ln x)\right) = \frac{1}{x} \neq 0,\ x > 0. \tag{1.82}$$

Therefore the two functions are linearly independent over I.

□