Homework 8

DUE: SUNDAY, MARCH 22, 11:59PM

If you completed this assignment through collaboration or consulted references, please list the names of your collaborators and the references you used below. Please refer to the syllabus for the course policy on collaborations and types of references that are allowed.

DEPARTMENT OF MATHEMATICAL SCIENCES, CARNEGIE MELLON UNIVERSITY
E-mail address: jkoganem@andrew.cmu.edu.
Date: March 17, 2025.

Problem 8.1. Consider the mass-spring system modeled via the initial value problem

$$\begin{cases} x''(t) + x(t) = \delta(t) + \delta(t-\pi) + \delta(t-2\pi) + \delta(t-3\pi), \ t \geq 0 \\ x(0) = x'(0) = 0. \end{cases} \tag{8.1}$$

One can think of the Dirac deltas as modeling a hammer striking the mass at times $t = 0, \pi, 2\pi, 3\pi$ with unit impulse.

a) Find a function $x : [0, \infty) \to \mathbb{R}$ modeling the behavior of the system.
b) Provide a rough sketch of the solution x and give a physical interpretation of the solution in terms of the hammer striking the mass at times $t = 0, \pi, 2\pi, 3\pi$.

Problem 8.2 (A fundamental reduction)**.** In this problem our goal is to show that every n-th order scalar equation can be converted into a first order $n \times n$ system, and vice versa. This shows that all equations that we have encountered in this class all have a common structure - they can all be studied as a first order system. If linear algebra were a prerequisite for this class, this would have been the natural starting point for studying ODEs.

For the sake of simplicity we will assume $n = 2$ for this problem, but this result can be easily generalized to any $n \in \mathbb{N}$.

a) Let $I \subseteq \mathbb{R}$ be an interval, $t_0 \in I$ and assume $x : I \to \mathbb{R}$ is a smooth function. Suppose x solves the initial value problem

$$\begin{cases} x''(t) = f(t, x(t)),\ t \in \mathbb{R} \\ x(t_0) = x_0, x'(t_0) = x_1 \end{cases} \tag{8.2}$$

for some function $f : \mathbb{R}^2 \to \mathbb{R}$. Consider the vector-valued function $\boldsymbol{X} : I \to \mathbb{R}^2$ defined via

$$\boldsymbol{X}(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix}. \tag{8.3}$$

Show that $\boldsymbol{X}$ satisfies the first order IVP

$$\begin{cases} \boldsymbol{X}'(t) = \begin{pmatrix} X_2(t) \\ f(t, X_1(t)) \end{pmatrix}, \quad t \in I \\ \boldsymbol{X}(t_0) = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}. \end{cases} \tag{8.4}$$

This shows that if x solves (8.7), then $\boldsymbol{X}$ solves (8.6). Next we show the converse.

b) Now suppose $\boldsymbol{X} : I \to \mathbb{R}$ is a vector-valued defined via

$$\boldsymbol{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \tag{8.5}$$

where $x, y : I \to \mathbb{R}$ are two smooth scalar-valued functions. Assume $\boldsymbol{X}$ solves the first order IVP

$$\begin{cases} \boldsymbol{X}'(t) = \begin{pmatrix} y(t) \\ f(t, x(t)) \end{pmatrix}, \quad t \in I \\ \boldsymbol{X}(t_0) = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}. \end{cases} \tag{8.6}$$

Show that $x = (\boldsymbol{X})_1$ satisfies the 2nd order scalar valued IVP

$$\begin{cases} x''(t) = f(t, x(t)),\ t \in \mathbb{R} \\ x(t_0) = x_0, x'(t_0) = x_1. \end{cases} \tag{8.7}$$

Problem 8.3 (Distinct real eigenvalues)**.** Consider the mass spring system modeled via the equation

$$x''(t) + 5x'(t) + 4x(t) = 0,\ t \in \mathbb{R}. \tag{8.8}$$

a) Find the general solution to the 2nd order scalar equation, and solve the IVP

$$\begin{cases} x''(t) + 5x'(t) + 4x(t) = 0,\ t \in \mathbb{R} \\ x(0) = 1, x'(0) = -1. \end{cases} \tag{8.9}$$

b) Convert (8.8) into a 2×2 linear system of the form

$$\frac{d}{dt}\boldsymbol{X}(t) = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{:=A} \boldsymbol{X}(t),\ a, b, c, d, t \in \mathbb{R} \tag{8.10}$$

where

$$\boldsymbol{X}(t) = \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix}. \tag{8.11}$$

c) Find the eigenvalues and a corresponding set of eigenvectors of the matrix A.

d) Write down the general solution to the linear system $\boldsymbol{X}' = A\boldsymbol{X}$ and solve the IVP

$$\begin{cases} \boldsymbol{X}'(t) = A\boldsymbol{X}(t), \quad t \in \mathbb{R} \\ \boldsymbol{X}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{cases} \tag{8.12}$$

What do you notice when you compare your answers in parts a) and d)?

Problem 8.4 (Complex eigenvalues)**.** Consider the mass spring system modeled via the equation

$$x''(t) + 4x'(t) + 5x(t) = 0, \ t \in \mathbb{R}. \tag{8.13}$$

a) Is the system underdamped, overdamped, or critically damped?
b) Convert (8.13) into a 2×2 linear system of the form

$$\frac{d}{dt}\boldsymbol{X}(t) = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{:=A}\boldsymbol{X}(t), \ a, b, c, d, t \in \mathbb{R} \tag{8.14}$$

where

$$\boldsymbol{X}(t) = \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix}. \tag{8.15}$$

c) Find the characteristic equation associated to (8.13) and also the characteristic polynomial associated to the matrix A. What do you notice when you compare them?
d) Find the eigenvalues and a corresponding set of eigenvectors of the matrix A.
e) Write down the general solution to the linear system $\boldsymbol{X}' = A\boldsymbol{X}$.

Problem 8.5 (Repeated eigenvalues)**.** Consider the 2×2 linear system

$$\begin{cases} x'(t) = y(t) \\ y'(t) = -x(t) + 2y(t), \quad t \in \mathbb{R}. \end{cases} \tag{8.16}$$

In matrix-vector form, this is the linear system

$$\boldsymbol{X}'(t) = A\boldsymbol{X}(t) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \ t \in \mathbb{R}. \tag{8.17}$$

a) Write down a 2nd order scalar equation for x and find the general solution for x.
b) Find the general solution to the first ordered system (8.17).
c) What do you notice when you compare your answers in parts a) and b)?