Additional final practice problems

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Problem 1.1. Consider the initial value problem

$$\begin{cases} e^t y'(t) + 2e^t y(t) = 3e^{2t}, \ t \in \mathbb{R} \\ y(0) = 0. \end{cases} \tag{1.1}$$

a) Classify the equation by order.
b) Classify the equation by linearity. Is it linear or nonlinear?
c) Use an appropriate method to find a solution to the initial value problem. You may skip the verification step.
d) What is the maximal interval of existence of the solution?
e) Is the solution identified in part c) unique?

Problem 1.2. Consider the differential equation

$$y'(t) = 2t - y(t),\ t \in \mathbb{R}. \tag{1.2}$$

a) Verify that the function

$$y(t) = 2(t-1),\ t \in \mathbb{R} \tag{1.3}$$

is a solution on the interval $J = \mathbb{R}$. Please don't present your work backwards, instead calculate y' and the right-hand side of the equation for the given y separately and conclude that they are equal to each other.

b) Sketch the directional field associated to the equation by identifying the isoclines corresponding to $m = 0, \pm 1, \pm 2$. You're welcome to add more isoclines to the sketch to improve the accuracy of the sketch. Note that $2t - y = m \Longleftrightarrow y = 2t - m$, so the isoclines are straight lines with slope 2. Here is a very rough sketch for parts c) and d). Note that the line $y(t) = 2(t-1)$ is both an isocline and a solution curve, so no other solution curve can cross it.

c) Sketch the solution curve y_1 passing through $(0,0)$ and the solution curve y_2 passing through $(0,2)$ on top of the directional field you sketched in part b). Please make sure that your solution curves match the underlying directional field, and also you sketch them for enough t's so that the global behavior of the solution curves are easy to visualize.

d) Is it possible for the solutions curves y_1 and y_2 from part c) to ever cross? Please explain your reasoning and justify this part rigorously.

e) Is it possible for $y_1(2) \leq 2$? Please explain your reasoning and justify this part rigorously.

Problem 1.3. Consider the initial value problem

$$\begin{cases} y'(x) = 6x(y(x)-1)^{2/3}, \ x \in \mathbb{R} \\ y(0) = 1. \end{cases} \tag{1.4}$$

a) Verify that the function y_1 defined via

$$y_1(x) = 1, \ x \in \mathbb{R} \tag{1.5}$$

is a constant solution to the initial value problem on the interval $J = \mathbb{R}$. As with all verification problems, please do not present you work backwards. Also, please do not forget to check the initial condition.

b) Verify that the function y_2 defined via

$$y_2(x) = 1 + x^6, \ x \in \mathbb{R} \tag{1.6}$$

is a solution to the initial value problem on the interval $J = \mathbb{R}$. As with all verification problems, please do not present you work backwards. Also, please do not forget to check the initial condition.

c) Parts b) and c) show that solutions to the given initial value problem are not unique. Explain why this does not violate the conclusions of the existence and uniqueness theorem.

d) Classify all points $(t_0, y_0) \in \mathbb{R}^2$ for which if $y(t_0) = y_0$ is the specified initial condition (instead of $y(0) = 1$), the existence and uniqueness of solutions is guaranteed.

e) Find the unique solution to the initial value problem

$$\begin{cases} y'(x) = 6x(y(x)-1)^{2/3}, \ x \in \mathbb{R} \\ y(0) = 2. \end{cases} \tag{1.7}$$

You may skip the verification step. What is the maximal interval of existence of the solution? Hint: the antiderivative of $u^{-2/3}$ is $3u^{1/3}$.

Problem 1.4. Consider the differential equation

$$xy'(x) + 6y(x) = 3x(y(x))^{4/3}, \ x \in \mathbb{R}. \tag{1.8}$$

a) Classify the equation by linearity. Is it linear or nonlinear? No justification required.
b) Does the equation admit any constant solutions?
c) Use an appropriate method to find the solution to the initial value problem

$$\begin{cases} xy'(x) + 6y(x) = 3x(y(x))^{4/3}, \ x \in \mathbb{R}, \\ y(1) = -1. \end{cases} \tag{1.9}$$

You may skip the verification step.
d) What is the maximal interval of existence for the solution in part c)?
e) What happens if we change the initial condition to $y(1) = 0$? Does a solution exist and is it unique?

Problem 1.5. Suppose a constant coefficient linear differential equation admits the general solution

$$y(x) = c_1 e^x \cos x + c_2 e^x \sin x,\ x \in \mathbb{R} \tag{1.10}$$

where c_1, c_2 are arbitrary.

a) What are the roots of the characteristic equation associated to the differential equation?
b) Find a constant coefficient differential equation that admits this general solution.

Problem 1.6. Consider the variable coefficient initial value problem

$$\begin{cases} 5y''(x) + 12xy'(x) + 25x^2y(x) = 0, \ x \in \mathbb{R} \\ y(1) = 0 \\ y'(1) = 0. \end{cases} \tag{1.11}$$

a) Find a solution to the initial value problem over the interval $I = \mathbb{R}$.
b) Justify carefully and rigorously why the solution you found in part a) is the only solution to the initial value problem over the interval $I = \mathbb{R}$.

Problem 1.7. Consider the mass-spring system modeled via the homogeneous linear differential equation

$$x''(t) + \gamma x'(t) + 4x(t) = 0, \ t \in \mathbb{R}. \tag{1.12}$$

a) Find value(s) of γ for which the system is critically damped. For the equation to be critically damped we require

$$\gamma^2 - 16 = 0 \implies \gamma = 4. \tag{1.13}$$

We need to exclude the case that $\gamma = -4$ because in a damped mass-spring system, the damping constant is assumed to be positive.

b) Find the largest sub-interval I of $(0, \infty)$ such that if $\gamma \in I$, then the system is overdamped.

c) Suppose $\gamma = 2$. What is the quasi-period T of the solution?

d) Suppose $\gamma = 4$, and an external force is present in the system and the forced damped mass-spring system is modeled via

$$x''(t) + 4x'(t) + 4x(t) = 32e^{2t}, \ t \in \mathbb{R}. \tag{1.14}$$

Find the general solution to the system.

Problem 1.8. Consider the 2nd order differential equation

$$x^2y''(x) + 3xy'(x) - 3y(x) = 0,\ x > 0. \tag{1.15}$$

You are given that y_1 defined via

$$y_1(x) = x, x > 0 \tag{1.16}$$

is a solution to the homogeneous equation. Use the method of reduction of order to find a second linearly independent solution y_2 to the equation over the interval $I = (0, \infty)$. You do not need to check the independence of y_1, y_2, nor verify that y_2 is a solution.

Problem 1.9. Suppose a mass-spring system is modeled via

$$x''(t) + \beta x'(t) + 4x(t) = \cos \omega t, \ t \in \mathbb{R}. \tag{1.17}$$

where $\beta \geq 0, \omega > 0$.

a) Identify the parameters β, ω for which pure resonance occurs.
b) In the case of part a) where resonance occurs, use the method of undetermined coefficients to find a particular solution to the system.
c) Suppose $\beta > 0$ and $x(0) = x'(0) = 0$. Would a sizable change in the initial conditions, either in the initial position or the initial velocity, result in a sizable change in the behavior of the system in the long run? Please briefly explain why or why not.

Problem 1.10. Suppose the general homogeneous solution to the variable coefficient equation

$$x^2y''(x) + xy'(x) - y(x) = 1,\ x > 0 \tag{1.18}$$

is

$$y_h(x) = c_1y_1(x) + c_2y_2(x) = c_1x + c_2x^{-1},\ x > 0, \tag{1.19}$$

where c_1, c_2 are arbitrary.

a) Find the Wronskian $W(y_1, y_2)$ defined for $x > 0$.
b) Use the variation of parameters formula to find a particular solution to the equation. Note that the coefficient in from of the highest order term y'' is x^2, not 1.

Problem 1.11. Consider the eigenvalue problem

$$\begin{cases} y''(x) + \lambda y(x) = 0, \ x \in (0, \pi) \\ y(0) = 0, \ y(\pi) = 0. \end{cases} \tag{1.20}$$

Find the positive eigenvalues associated to this problem.

Problem 1.12. Suppose a mass-spring system is modeled via

$$\begin{cases} x''(t) + x(t) = f(t), & t \geq 0 \\ x(0) = x'(0) = 0, \end{cases} \tag{1.21}$$

where δ is the Dirac delta and $\mathcal{U}$ is the unit step function and $f : [0, \infty) \to \mathbb{R}$ is defined via

$$f(t) = \begin{cases} \delta(t - \pi), & 0 \leq t < 2\pi \\ 1, & t \geq 2\pi. \end{cases} \tag{1.22}$$

a) Write f in terms of the unit step function $\mathcal{U}(\cdot - 2\pi)$. (Note: $\delta(t - \pi) = 0$ for $t \geq 2\pi$).
b) Use the Laplace transform to find a solution x describing the behavior of the system for $t \geq 0$.