Recitation 6

Problem 6.1. Suppose a forced mass-spring system is modeled by

$$\begin{cases} x''(t) + 6x'(t) + 13x(t) = 10\sin 5t, \ t \in \mathbb{R} \\ x(0) = x'(0) = 0. \end{cases} \tag{6.1}$$

a) Find the unique solution to the initial value problem.
b) Identify the transient solution and the steady-state solution.
c) Describe the long term behavior of the system.
d) Does the long term behavior of the system change if the initial conditions were different?

Solution.

a) The characteristic equation is

$$r^2 + 6r + 13 = 0 \implies r = \frac{-6 \pm \sqrt{36-52}}{2} = -3 \pm 2i. \tag{6.2}$$

Thus the general homogeneous solution is

$$x_h(t) = c_1 e^{-3t}\cos 2t + c_2 e^{-3t}\sin 2t, \ t \in \mathbb{R}. \tag{6.3}$$

To find a particular solution we can use the method of undetermined coefficients. By using the ansatz $x_p(t) = A\cos 5t + B\sin 5t$, we calculate for all $t \in \mathbb{R}$,

$$x_p'(t) = 5B\cos 5t - 5A\sin 5t \tag{6.4}$$

$$x_p''(t) = -25A\cos 5t - 25B\sin 5t, \tag{6.5}$$

thus for all $t \in \mathbb{R}$,

$$x_p''(t) + 6x_p'(t) + 13x_p(t) = (-25A + 30B + 13A)\cos 5t + (-25B - 30A + 13B)\sin 5t = 10\sin 5t. \tag{6.6}$$

This implies

$$\begin{cases} -12A + 30B &= 0 \\ -30A - 12B &= 10. \end{cases} \tag{6.7}$$

Through routine algebra we find that

$$A = \frac{-25}{87}, \ B = \frac{-10}{87}. \tag{6.8}$$

Thus the general solution to the equation is

$$x(t) = x_h(t) + x_p(t) = c_1 e^{-3t}\cos 2t + c_2 e^{-3t}\sin 2t - \frac{25}{87}\cos 5t - \frac{10}{87}\sin 5t, t \in \mathbb{R} \tag{6.9}$$

and c_1, c_2 are arbitrary. We note that

$$x'(t) = e^{-3t}(-3c_1\cos 2t - 3c_2\sin 2t - 2c_1\sin 2t + 2c_2\cos 2t) + \frac{125}{87}\sin 5t - \frac{50}{87}\cos 5t, t \in \mathbb{R}. \tag{6.10}$$

Therefore if $x(0) = 0$ and $x'(0) = 0$, this implies

$$\begin{cases} c_1 - \frac{25}{87} = 0 \implies c_1 = \frac{25}{87} \\ -3c_1 + 2c_2 - \frac{50}{87} = 0 \implies c_2 = \frac{125}{174}. \end{cases} \tag{6.11}$$

Thus the unique solution to the IVP is

$$x(t) = \frac{25}{87}e^{-3t}\cos 2t + \frac{125}{174}e^{-3t}\sin 2t - \frac{25}{87}\cos 5t - \frac{10}{87}\sin 5t, t \in \mathbb{R}. \tag{6.12}$$

b) The transient solution is the general homogeneous solution as $|x_h(t)| \to 0$ as $t \to \infty$. The steady-state solution is the particular solution x_p.

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c) Since the transient solution vanishes as $t \to \infty$, we have $|x(t) - x_p(t)| \to 0$ as $t \to \infty$, which means that in the long term the behavior of the system is largely described by the particular solution (will a small error term).

d) We note that the initial conditions only affects constants appearing in the general homogeneous solution and not the particular solution, therefore the initial conditions do not really change the long term behavior of the system as the behavior is mostly described by the particular solution (it will affect the error term, but the error term is negligible in magnitude).

$\square$

Problem 6.2. Identify the eigenvalues and the associated eigenfunctions to

$$\begin{cases} y''(x) + \lambda y(x) = 0,\ x \in (0, \pi) \\ y'(0) = 0,\ y'(\pi) = 0. \end{cases} \tag{6.13}$$

Solution. We consider a few cases. If $\lambda = 0$, then the general solution is given by

$$y(x) = c_1 + c_2 x,\ x \in (0, \pi). \tag{6.14}$$

Therefore

$$y'(x) = c_2,\ x \in (0, \pi). \tag{6.15}$$

If $y'(0) = 0, y'(\pi) = 0$, then $c_2 = 0$ and there are no restriction on c_1. Therefore $\lambda = 0$ is an eigenvalue corresponding to constant functions of the form

$$y(x) = c_1, x \in (0, \pi), c_1 \neq 0. \tag{6.16}$$

If $\lambda < 0$, then the general solution is

$$y(x) = c_1 e^{-\sqrt{-\lambda}x} + c_2 e^{\sqrt{-\lambda}x},\ x \in (0, \pi). \tag{6.17}$$

Then

$$y'(x) = -\sqrt{-\lambda}c_1 e^{-\sqrt{-\lambda}x} + \sqrt{-\lambda}c_2 e^{\sqrt{-\lambda}x},\ x \in (0, \pi). \tag{6.18}$$

Therefore if $y'(0) = 0, y'(\pi) = 0$, then we require

$$\begin{cases} y'(0) = -\sqrt{-\lambda}c_1 + \sqrt{-\lambda}c_2 = 0 \\ y'(\pi) = -\sqrt{-\lambda}e^{-\sqrt{-\lambda}\pi}c_1 + \sqrt{-\lambda}e^{\sqrt{-\lambda}\pi}c_2 = 0. \end{cases} \tag{6.19}$$

This implies (upon multiplying the first equation by $-e^{-\sqrt{-\lambda}\pi}$ and adding the two equations)

$$c_2(\underbrace{-\sqrt{-\lambda}e^{-\sqrt{-\lambda}\pi} + \sqrt{-\lambda}e^{\sqrt{-\lambda}\pi}}_{\neq 0}) = 0 \implies c_2 = 0 \implies c_1 = 0. \tag{6.20}$$

So this problem does not admit negative eigenvalues. If $\lambda > 0$ then the general solution is

$$y(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x,\ x \in (0, \pi). \tag{6.21}$$

This implies

$$y'(x) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}x,\ x \in (0, \pi). \tag{6.22}$$

Therefore if $y'(0) = 0$ and $y'(\pi) = 0$, then this implies $c_2 = 0$ and

$$-c_1\sqrt{\lambda}\sin\sqrt{\lambda}\pi = 0. \tag{6.23}$$

In order for the problem to admit non-zero solutions we need $c_1 \neq 0$ and

$$\sqrt{\lambda}\pi = n\pi,\ n \in \mathbb{Z}. \tag{6.24}$$

Therefore we may enumerate the eigenvalues via

$$\lambda_n = n^2,\ n \in \mathbb{Z}^+. \tag{6.25}$$

Therefore the eigenvalues of this eigenvalue problem are

$$\lambda_0 = 0,\ \lambda_1 = 1,\ \lambda_2 = 4, \ldots \tag{6.26}$$

with corresponding eigenfunctions $y_k : (0, \pi) \to \mathbb{R}$ defined via

$$y_0(x) = 1,\ y_1(x) = \sin x,\ y_2(x) = \sin 2x, \ldots \tag{6.27}$$

$\square$

Problem 6.3. Use the definition of the Laplace transform to find the Laplace transform of the function $f : [0, \infty) \to \mathbb{R}$ defined via $f(t) = t$.

Solution. By definition, the Laplace transform of t (which we will adopt an abuse of notation and denote as $\mathcal{L}\{t\}$) is the function of s

$$\mathcal{L}\{t\}(s) = \int_0^\infty te^{-st}\, dt \tag{6.28}$$

defined over all values of s for which the improper integral converges. First we note that if $s = 0$, then the improper integral diverges. We may then assume $s \neq 0$ and calculate using integration by parts,

$$\int_0^\infty te^{-st}\, dt = \lim_{\gamma\to\infty} \int_0^\gamma te^{-st}\, dt = \lim_{\gamma\to\infty} \left(\frac{te^{-st}}{-s}\bigg|_{t=0}^{t=\gamma} - \int_0^\gamma \frac{e^{-st}}{-s}\, dt \right) = \lim_{\gamma\to\infty} \left(\frac{\gamma e^{-s\gamma}}{-s} - 0 - \frac{e^{-s\gamma}}{s^2} + \frac{1}{s^2} \right) \tag{6.29}$$

We note that if $s < 0$, then the limit does not exist as $e^{-s\gamma} \to \infty$ as $\gamma \to \infty$. On the other hand, if $s > 0$, then $e^{-s\gamma} \to 0$ and $\gamma e^{-s\gamma} \to 0$ (this you need to justify using L'hopital's rule) as $\gamma \to \infty$. Therefore for $s > 0$, we find that

$$\mathcal{L}\{t\}(s) = \frac{1}{s^2}. \tag{6.30}$$

Thus the Laplace transform of t is defined for $s > 0$ and is equal to $\frac{1}{s^2}$. □