Recitation 4

Problem 1.1 (Initial value problem)**.** Consider the initial value problem

$$\begin{cases} y''(x) - 3y'(x) + 2y(x) = 0, \ x \in \mathbb{R} \\ y(0) = a \\ y'(0) = b, \end{cases} \tag{1.1}$$

where $a, b \in \mathbb{R}$.

(1) Find the solution to the initial value problem (depending on a, b).
(2) What is the maximal interval of existence of the solution?
(3) Do solutions exist for all a, b? Are solutions unique?

Solution. The associated characteristic equation is

$$r^2 - 3r + 2 = (r-1)(r-2) = 0, \tag{1.2}$$

so its roots are $r = 1, r = 2$. This means that the general solution to the equation is given by

$$y(t) = c_1 e^x + c_2 e^{2x}, \ x \in \mathbb{R}. \tag{1.3}$$

To identify the solution to the IVP, we need to identify the coefficients c_1, c_2 using the initial conditions. We have

$$y(0) = c_1 e^0 + c_2 e^0 = c_1 + c_2 \tag{1.4}$$

$$y'(0) = c_1 e^0 + 2c_2 e^0 = c_1 + 2c_2, \tag{1.5}$$

so for y to be a solution to the IVP we must have

$$c_1 + c_2 = a \tag{1.6}$$

$$c_1 + 2c_2 = b. \tag{1.7}$$

This implies that $c_2 = b - a, c_1 = 2a - b$. So the solution to the IVP is

$$y(t) = (2a - b)e^x + (b - a)e^{2x}, \ x \in \mathbb{R}. \tag{1.8}$$

The maximal interval of existence is the interval $J = \mathbb{R}$. We note that by the existence and uniqueness theorem for higher order linear equations, as long as the coefficients and the right hand side of the equation are continuous functions over $\mathbb{R}$, and the leading coefficient never vanishes, then the associated IVP admits a unique global solution (meaning maximal interval of existence is $\mathbb{R}$) for all $t_0 \in \mathbb{R}$ if the initial conditions are specified at $t = t_0$. Since the equation in consideration is a constant coefficient homogeneous equation, the conditions of the theorem are satisfied, therefore at time $t = 0$, we always have a unique global solution regardless of the values of a, b. □

Department of Mathematical Sciences, Carnegie Mellon University
E-mail address: jkoganem@andrew.cmu.edu.
Date: January 28, 2025.

Problem 1.2 (A third order equation)**.** Find the general solution to

$$\dddot{x}(t) - \ddot{x}(t) - 4x(t) = 0,\ t \in \mathbb{R}. \tag{1.9}$$

Solution. The associated characteristic equation is

$$r^3 - r^2 - 4 = 0. \tag{1.10}$$

By inspection, $r = 2$ is a root to this polynomial. Then via long division we see that

$$r^3 - r^2 - 4 = (r-2)(r^2 + r + 2) = 0. \tag{1.11}$$

Therefore the roots of this polynomial are

$$\begin{aligned} r_1 &= 2 \\ r_2 &= \frac{-1 - \sqrt{1-8}}{2} = -\frac{1}{2} - i\frac{\sqrt{7}}{2} \\ r_3 &= -\frac{1}{2} + i\frac{\sqrt{7}}{2}. \end{aligned} \tag{1.12}$$

So the general solution is given by

$$x(t) = c_1 e^{2t} + c_2 e^{-1/2t} \cos\left(\frac{\sqrt{7}}{2}t\right) + c_3 e^{-1/2t} \sin\left(\frac{\sqrt{7}}{2}t\right),\ t \in \mathbb{R}. \tag{1.13}$$

$\square$

Problem 1.3 (A second order equation)**.** Find the solution to the initial value problem

$$\begin{cases} y''(x) + y'(x) + 2y(x) = 0, \ x \in \mathbb{R} \\ y(0) = y'(0) = 0. \end{cases} \tag{1.14}$$

Solution. We note that the zero solution y defined via $y(x) = 0$ for all $x \in \mathbb{R}$ is a solution to the IVP. We note that since this is a homogeneous constant coefficient linear equation, the coefficients and the right hand side of the equation are constant functions (hence continuous) and the leading coefficient (as a function) never vanishes, therefore by the existence and uniqueness theorem the zero solution must be unique. Therefore the unique solution to the IVP is the zero solution. $\square$

Problem 1.4 (Mass-spring systems)**.** Consider a mass-spring system modeled via the equation

$$x''(t) + 2x'(t) + 10x(t) = 0, \ t \in \mathbb{R}. \tag{1.15}$$

Is the system underdamped, critically damped or overdamped?

Solution. The associated characteristic equation is given by $r^2 + 2r + 10 = 0 \implies r = \frac{-2 \pm \sqrt{4-40}}{2} = -1 \pm 3i$. Therefore the system is underdamped and the general solution is

$$x(t) = e^{-t}(c_1 \cos(3t) + c_2 \sin(3t)), \ t \in \mathbb{R}. \tag{1.16}$$

$\square$

Problem 1.5 (Critical damping)**.** Consider a mass-spring system modeled via the initial value problem

$$\begin{cases} x''(t) + 8x'(t) + 16x(t) = 0, \ t \in \mathbb{R} \\ x(0) = x_0 \\ x'(0) = v_0, \end{cases} \tag{1.17}$$

where $x_0, v_0 \in \mathbb{R}$ denote the initial position and velocity of the mass at time $t = 0$.

(1) Find the solution to the initial value problem.
(2) What happens to the solution x as $t \to \infty$? What does this mean physically?
(3) Suppose $v_0 = 0$. Is it possible for the mass to ever cross the equilibrium point for some positive time $t > 0$?
(4) Suppose $x_0 > 0$. Derive a condition on the initial velocity v_0 so that the mass passes through the equilibrium point at some positive time $t > 0$.

Part (1). The associated characteristic equation is $r^2 + 8r + 16 = (r+4)^2 = 0$, which implies that we have $r = -4$ as a double root and that the system is critically damped. This implies that the general solution is

$$\begin{cases} x(t) &= c_1 e^{-4t} + c_2 t e^{-4t}, \ t \in \mathbb{R} \\ x'(t) &= -4c_1 e^{-4t} + c_2 e^{-4t} - 4c_2 t e^{4t}, \ t \in \mathbb{R}. \end{cases} \tag{1.18}$$

Applying the initial conditions we find that

$$\begin{cases} x_0 &= x(0) = c_1 \implies c_1 = x_0 \\ v_0 &= x'(0) = -4c_1 + c_2 \implies c_2 = v_0 + 4x_0. \end{cases} \tag{1.19}$$

Thus the solution to the initial value problem is

$$x(t) = x_0 e^{-4t} + (v_0 + 4x_0) t e^{-4t}, \ t \in \mathbb{R}. \tag{1.20}$$

□

Part (2). We note that since

$$\lim_{t\to\infty} e^{-4t} = 0, \ \lim_{t\to\infty} t e^{-4t} = 0, \tag{1.21}$$

we have

$$\lim_{t\to\infty} x(t) = 0. \tag{1.22}$$

This means that the mass approaches the equilibrium point $x = 0$ as $t \to \infty$. □

Part (3). In the case where we have zero initial velocity, i.e. $v_0 = 0$, the solution becomes

$$x(t) = x_0 e^{-4t} + 4x_0 t e^{-4t} = x_0(e^{-4t} + 4te^{-4t}) = x_0(1+4t)e^{-4t}, \ t \in \mathbb{R}. \tag{1.23}$$

Assuming that the initial displacement $x_0 \neq 0$, $x(t) \neq 0$ for all t, since $e^{-4t} > 0, 1 + 4t > 1$ for all $t > 0$. This means that the system never passes through the equilibrium point even though the mass is approaching the equilibrium as $t \to \infty$. □

Part (4). If $x_0 > 0$, we see that

$$x(t) = 0 \iff x_0 + (v_0 + 4x_0)t = 0 \iff t = -\frac{x_0}{v_0 + 4x_0}. \tag{1.24}$$

In order for this to be achieved at a positive time t we must have $v_0 + 4x_0 < 0 \implies v_0 < -4x_0$. □