Linearization problems solutions

Problem 1.1. Consider the nonlinear system of equations

$$\begin{cases} x'(t) &= 1 - 2x(t)y(t) \\ y'(t) &= 2x(t)y(t) - y(t), \ t \in \mathbb{R}. \end{cases} \tag{1.1}$$

Find the critical point(s) of the system and classify the critical(s) of the system by their stability type, if possible.

Problem 1.2. Consider the nonlinear system of equations

$$\begin{cases} x'(t) &= \alpha x(t) - \beta y(t) + (y(t))^2 \\ y'(t) &= \beta x(t) + \alpha y(t) - x(t)y(t), \ t \in \mathbb{R}, \end{cases} \tag{1.2}$$

where $\alpha, \beta \in \mathbb{R}$.

a) Show that

$$\boldsymbol{X}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.3}$$

is a critical point of the system.

b) Classify $\boldsymbol{X}_0$ in terms of its stability type, if possible, for
- $\alpha > 0$
- $\alpha < 0$
- $\alpha = 0$.

Problem 1.3. Consider an undamped mass-spring system where the spring force is nonlinear

$$\begin{cases} mx''(t) + kx(t) + k_1 x^3(t) = 0, \ t \in \mathbb{R} \\ x(0) = x_0, v_0(0) = v_0. \end{cases} \tag{1.4}$$

Consider the special case when $m = 1$, $k = 1$ and $k_1 = -1$. Since the equation is nonlinear, finding non-zero explicit solutions can be difficult, but we can try to use the techniques we learned to study the behavior of solutions when the initial conditions are small in a certain sense.

a) Define $\boldsymbol{X} : \mathbb{R} \to \mathbb{R}^2$ via

$$\boldsymbol{X}(t) = \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix}. \tag{1.5}$$

Write down an autonomous first-order nonlinear differential equation for $\boldsymbol{X}$ of the form $\boldsymbol{X}'(t) = \boldsymbol{f}(\boldsymbol{X(t)}), \ t \in \mathbb{R}$.

b) Find the critical points of the system and classify the critical points by their stability type, if possible.

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