More Laplace transform practice problems

Problem 1.1. Define $f : [0, \infty) \to \mathbb{R}$ via

$$f(t) = e^t(\mathcal{U}(t-1) - \mathcal{U}(t-2)). \tag{1.1}$$

What is the Laplace transform of f?

Solution. Note that

$$\mathcal{L}\left\{\mathcal{U}(t-1) - \mathcal{U}(t-2)\right\}(s) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s},\ s > 0. \tag{1.2}$$

Therefore

$$\mathcal{L}\left\{f\right\}(s) = \frac{e^{-(s-1)}}{s-1} - \frac{e^{-2(s-1)}}{s-1},\ s > 1. \tag{1.3}$$

□

Problem 1.2. Define $f : [0, \infty) \to \mathbb{R}$ via

$$f(t) = \begin{cases} 0, & 0 \le t < \pi \\ t - \pi, & \pi \le t < 2\pi \\ 0, & t \ge 2\pi. \end{cases} \tag{1.4}$$

What is the Laplace transform of f?

Solution. Note that

$$f(t) = 0 + (t - \pi - 0)\mathcal{U}(t-\pi) + (0 - (t-\pi))\mathcal{U}(t-2\pi) = (t-\pi)\mathcal{U}(t-\pi) - (t-\pi)\mathcal{U}(t-2\pi),\ t \ge 0. \tag{1.5}$$

Therefore

$$\mathcal{L}\left\{f\right\}(s) = e^{-\pi s}\mathcal{L}\left\{t\right\}(s) - e^{-2\pi s}\mathcal{L}\left\{t+\pi\right\}(s) = \frac{e^{-\pi s}}{s^2} - e^{-2\pi s}\left(\frac{1}{s^2} + \frac{\pi}{s}\right),\ s > 0 \tag{1.6}$$

□

Problem 1.3. Define $f : [0, \infty) \to \mathbb{R}$ via

$$f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases} \tag{1.7}$$

What is the Laplace transform of f?

Solution. Note that

$$f(t) = t + (1-t)\mathcal{U}(t-1),\ t \ge 0. \tag{1.8}$$

Therefore

$$\mathcal{L}\left\{f\right\}(s) = \frac{1}{s} + e^{-s}\mathcal{L}\left\{2-t\right\}(s) = \frac{1}{s} + e^{-s}\left(\frac{2}{s} - \frac{1}{s^2}\right),\ s > 0. \tag{1.9}$$

□

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Problem 1.4. Define $F : (3, \infty) \to \mathbb{R}$ via

$$F(s) = \frac{10e^{-s}}{s^2 - 5s + 6} + \frac{2}{s^2 - 2s + 5}. \tag{1.10}$$

What is the inverse Laplace transform of F?

Solution. Note that

$$\frac{1}{s^2 - 5s + 6} = \frac{1}{s-2} - \frac{1}{s-3},\ s \neq 2, 3. \tag{1.11}$$

and

$$\frac{2}{s^2 - 2s + 5} = \frac{2}{(s-1)^2 + 4},\ s \in \mathbb{R}. \tag{1.12}$$

Therefore

$$\mathcal{L}^{-1}\{F\}(t) = 10\mathcal{U}(t-1)\left(e^{2(t-1)} - e^{3(t-1)}\right) + e^t \sin 2t,\ t \geq 0. \tag{1.13}$$

□

Problem 1.5. Define $F : (0, \infty) \to \mathbb{R}$ via

$$F(s) = \frac{3s^2 + 4s + 1}{(s+1)(s^2 + 2s + 5)}. \tag{1.14}$$

What is the inverse Laplace transform of F?

Solution. Note that if one looks for a partial fraction decomposition of the form

$$\frac{3s^2 + 4s + 1}{(s+1)(s^2 + 2s + 5)} = \frac{A}{s+1} + \frac{Bs + C}{s^2 + 2s + 5} \tag{1.15}$$

for some $A, B, C \in \mathbb{R}$, which is equivalent to requiring

$$3s^2 + 4s + 1 = A(s^2 + 2s + 5) + (Bs + C)(s+1),\ s \in \mathbb{R}, \tag{1.16}$$

we find that if $s = -1$ then

$$3 - 4 + 1 = 0 = A(1 - 2 + 5) = 4A \implies A = 0. \tag{1.17}$$

This suggests that $s + 1$ divides into $3s^2 + 4s + 1$, so we look to factorize

$$3s^2 + 4s + 1 = (s+1)(3s+1), s \in \mathbb{R}. \tag{1.18}$$

Thus

$$F(s) = \frac{3s^2 + 4s + 1}{(s+1)(s^2 + 2s + 5)} = \frac{3s+1}{s^2 + 2s + 5} = \frac{3(s+1) - 2}{(s+1)^2 + 4},\ s > 0. \tag{1.19}$$

Thus

$$\mathcal{L}^{-1}\{F\}(t) = \frac{3}{2}e^{-t}\cos 2t - e^{-t}\sin 2t,\ t \geq 0. \tag{1.20}$$

□

Problem 1.6. Define $F : (4, \infty) \to \mathbb{R}$ via

$$F(s) = e^{-3s}\frac{s+1}{s^2 - 8s + 20}. \tag{1.21}$$

What is the inverse Laplace transform of F?

Solution. Note that

$$F(s) = e^{-3s} \frac{(s-4)+5}{(s-4)^2+4}, \tag{1.22}$$

and

$$\mathcal{L}^{-1}\left\{\frac{(s-4)+5}{(s-4)^2+4}\right\}(t) = e^{4t}\left(\cos 2t + \frac{5}{2}\sin 2t\right),\ t \geq 0. \tag{1.23}$$

Thus

$$\mathcal{L}^{-1}\{F\}(t) = \mathcal{U}(t-3)e^{4(t-3)}\left(\cos 2(t-3) + \frac{5}{2}\sin 2(t-3)\right),\ t \geq 0. \tag{1.24}$$

□

Problem 1.7. Consider a forced undamped mass-spring system modeled via the IVP

$$\begin{cases} y''(t) + 4y(t) = f(t),\ t \geq 0 \\ y(0) = -1,\ y'(0) = 4, \end{cases} \tag{1.25}$$

a) Find an expression for a solution $y : [0,\infty) \to \mathbb{R}$ for any reasonable forcing function $f : [0,\infty) \to \mathbb{R}$.
b) Write down the solution for $f(t) = 2\delta(t-\pi)$.

Solution. If y is a solution, then its Laplace transform Y satisfies

$$(s^2+4)Y(s) + s - 4 = F(s),\ s > a \tag{1.26}$$

for some $a \in \mathbb{R}$, where F is the Laplace transform of f. Thus

$$Y(s) = \frac{1}{s^2+4}F(s) + \frac{4-s}{s^2+4},\ s > a. \tag{1.27}$$

This implies that

$$y(t) = \frac{1}{2}(\sin(2\cdot) * f)(t) + 2\sin 2t - \cos 2t,\ t \geq 0, \tag{1.28}$$

where

$$(\sin(2\cdot) * f)(t) = \int_0^t f(\tau)\sin 2(t-\tau)\ d\tau. \tag{1.29}$$

If $f = 2\delta(t-\pi)$, then note that

$$\int_0^t f(\tau)\sin 2(t-\tau)\ d\tau = 2\int_0^t \delta(\tau-\pi)\sin 2(t-\tau)\ d\tau = \begin{cases} 0,\ 0 \leq t \leq \pi \\ 2\sin 2(t-\pi),\ t > \pi \end{cases} \tag{1.30}$$

$$= 2\mathcal{U}(t-\pi)\sin 2(t-\pi),\ t \geq 0. \tag{1.31}$$

Thus

$$y(t) = \mathcal{U}(t-\pi)\sin 2(t-\pi) + 2\sin 2t - \cos 2t,\ t \geq 0. \tag{1.32}$$

□

Problem 1.8. Define the function $f : [0,\infty) \to \mathbb{R}$ via

$$f(t) = e^t \int_0^t \sin\tau\cos(t-\tau)\ d\tau. \tag{1.33}$$

What is the Laplace transform of f?

Solution. Note that

$$\mathcal{L}\left\{\int_0^t \sin\tau\cos(t-\tau)\ d\tau\right\}(s) = \mathcal{L}\{\sin t\}(s)\mathcal{L}\{\cos t\}(s) = \frac{s}{(s^2+1)^2},\ s > 0. \tag{1.34}$$

Thus

$$\mathcal{L}\{f\}(s) = \frac{s-1}{((s-1)^2+1)},\ s > 1. \tag{1.35}$$

□

Problem 1.9. Consider a forced undamped mass-spring system modeled via the IVP

$$\begin{cases} y''(t) + 2y(t) = \begin{cases} 0, & 0 \le t < 2 \\ (t-2)e^{-3(t-2)}, & t \ge 2. \end{cases} \\ y(0) = y'(0) = 0. \end{cases} \tag{1.36}$$

If $y : [0, \infty) \to \mathbb{R}$ is a solution modeling the system and Y is its Laplace transform, what is $Y(0)$?

Solution. Note that if we define $f : [0, \infty) \to \mathbb{R}$ via

$$f(t) = \begin{cases} 0, & 0 \le t < 2 \\ (t-2)e^{-3(t-2)}, & t \ge 2. \end{cases} = \mathcal{U}(t-2)(t-2)e^{-3(t-2)}, \tag{1.37}$$

then

$$\mathcal{L}\{f\}(s) = e^{-2s}\mathcal{L}\left\{te^{-3t}\right\}(s) = e^{-2s}\frac{1}{(s+3)^2}, \ s > 0. \tag{1.38}$$

Thus if y is a solution and Y is its Laplace transform, then

$$Y(s) = e^{-2s}\frac{1}{(s+3)^2(s^2+2)}, \ s > 0. \tag{1.39}$$

Thus

$$Y(0) = \frac{1}{18}. \tag{1.40}$$

□

Problem 1.10. Consider a forced undamped mass-spring system modeled via the IVP

$$\begin{cases} y''(t) + y(t) = \begin{cases} t, & 0 \le t < 2 \\ 3, & t \ge 2. \end{cases} \\ y(0) = y'(0) = 0. \end{cases} \tag{1.41}$$

Find a solution $y : [0, \infty) \to \mathbb{R}$ modeling the system.

Solution. Note that if we define $f : [0, \infty) \to \mathbb{R}$ via

$$f(t) = \begin{cases} t, & 0 \le t < 2 \\ 3, & t \ge 2. \end{cases} = t + (3-t)\mathcal{U}(t-2) \tag{1.42}$$

then

$$\mathcal{L}\{f\}(s) = \frac{1}{s^2} + e^{-2s}\mathcal{L}\{5-t\}(s) = \frac{1}{s^2} + e^{-2s}\left(\frac{5}{s} - \frac{1}{s^2}\right), \ s > 0. \tag{1.43}$$

Thus if y is a solution and Y is its Laplace transform, then

$$Y(s) = \frac{1}{s^2(s^2+1)} + e^{-2s}\left(\frac{5}{s(s^2+1)} - \frac{1}{s^2(s^2+1)}\right). \tag{1.44}$$

We note that

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}, \ s > 0 \tag{1.45}$$

and

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}, \ s > 0. \tag{1.46}$$

Thus

$$y(t) = t - \sin t + \mathcal{U}(t-2)\left(5 - 5\cos(t-2) - (t-2) + \sin(t-2)\right), \ t \ge 0. \tag{1.47}$$

□

Problem 1.11. Suppose a mass-spring system is modeled via

$$\begin{cases} x''(t) + 2x'(t) + 5x(t) = f(t), & t \geq 0 \\ x(0) = x'(0) = 0, \end{cases} \tag{1.48}$$

where δ is the Dirac delta and $\mathcal{U}$ is the unit step function and $f : [0, \infty) \to \mathbb{R}$ is defined via

$$f(t) = \begin{cases} 5, & 0 \leq t < \pi \\ \delta(t - 3\pi) + \delta(t - 4\pi), & t \geq \pi. \end{cases} \tag{1.49}$$

Find a solution $x : [0, \infty) \to \mathbb{R}$ describing the behavior of the system for $t \geq 0$.

Solution. Note that

$$f(t) = 5 + (\delta(t - 3\pi) + \delta(t - 4\pi) - 5)\mathcal{U}(t - \pi) = 5 - 5\mathcal{U}(t - \pi) + \delta(t - 3\pi) + \delta(t - 4\pi),\ t \geq 0. \tag{1.50}$$

Thus

$$\mathcal{L}\left\{f\right\}(s) = \frac{5}{s} + e^{-\pi s}\frac{5}{s} + e^{-3\pi s} + e^{-4\pi s},\ s > 0. \tag{1.51}$$

Therefore if x is a solution and X is its Laplace transform, then

$$X(s) = \frac{5}{s(s^2 + 2s + 5)} - e^{-\pi s}\frac{5}{s(s^2 + 2s + 5)} + e^{-3\pi s}\frac{1}{s^2 + 2s + 5} + e^{-4\pi s}\frac{1}{s^2 + 2s + 5},\ s > 0. \tag{1.52}$$

Note that

$$\frac{5}{s(s^2 + 2s + 5)} = \frac{1}{s} + \frac{-s - 2}{s^2 + 2s + 5} = \frac{1}{s} - \frac{(s + 1) + 1}{(s + 1)^2 + 4},\ s > 0, \tag{1.53}$$

therefore

$$\begin{aligned} y(t) = &\left(1 - e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)\right) - \mathcal{U}(t - \pi)\left(1 - e^{-(t-\pi)}\left(\cos 2(t - \pi) + \frac{1}{2}\sin 2(t - \pi)\right)\right) \\ &+ \mathcal{U}(t - 3\pi)\left(\frac{1}{2}e^{-(t-3\pi)}\sin 2(t - 3\pi)\right) + \mathcal{U}(t - 4\pi)\left(\frac{1}{2}e^{-(t-4\pi)}\sin 2(t - 4\pi)\right),\ t \geq 0. \end{aligned} \tag{1.54}$$

□