Homework 9

DUE: SATURDAY, APRIL 12, 11:59PM

If you completed this assignment through collaboration or consulted references, please list the names of your collaborators and the references you used below. Please refer to the syllabus for the course policy on collaborations and types of references that are allowed.

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Problem 8.1 (IVPs with no forcing and homogeneous data)**.** Consider the initial value problem

$$\begin{cases} y^{(6)}(x) - 3y^{(3)}(x) + 2y''(x) + 10y'(x) + y(x) = 0, \ x \in \mathbb{R} \\ y^{(k)}(0) = 0, \ \text{for all } 0 \leq k \leq 5. \end{cases} \tag{8.1}$$

a) Verify that the zero solution y defined via $y(x) = 0$ for all $x \in \mathbb{R}$ is a solution to this IVP.
b) Use the uniqueness part of the existence and uniqueness theorem for higher order linear equations to show that the zero solution is the unique solution to this IVP. Make sure to verify that the conditions of the theorem are met.

Problem 8.2 (Convolutions and the Laplace transform)**.** Suppose a mass-spring system is modeled via the equation

$$\begin{cases} x''(t) + 4x'(t) + 3x(t) = f(t), \ t \geq 0, \\ x(0) = x'(0) = 0. \end{cases} \tag{8.2}$$

a) Recall that the *transfer function* associated to the system is a function $W : \mathbb{R} \to \mathbb{R}$ defined for which

$$X(s) = W(s)F(s), \ s > a \tag{8.3}$$

for an appropriate $a \in \mathbb{R}$, and $X = \mathcal{L}\{x\}, F = \mathcal{L}\{f\}$. What is the transfer function associated to (8.2)?

b) The *weight function* associated to the system is the function $w : [0, \infty) \to \mathbb{R}$ defined via $w = \mathcal{L}^{-1}\{W\}$. What is the weight function associated to (8.2)?

c) Recall that by the *convolution property*, the solution to the IVP can be written in terms of a convolution between the weight function w and the forcing function f. Write down an explicit integral representation of the solution $x : [0, \infty) \to \mathbb{R}$ for any reasonable function f.

Problem 8.3 (Variation of parameters formula/Duhamel's principle for inhomogeneous systems)**.** In this problem we explore the variation of parameters formula for $n \times n$ inhomogeneous systems. For the sake of simplicity we assume $n = 2$, but the result here can easily be generalized to any $n \in \mathbb{N}$.

Let $I \subseteq \mathbb{R}$ be an interval, $t_0 \in I$, $A : I \to \mathcal{M}_{2\times 2}(\mathbb{R})$ be a continuous matrix-valued function and $\boldsymbol{f} : I \to \mathbb{R}^2$ be a continuous vector-valued function. We would like to write down an explicit solution formula for the initial value problem

$$\begin{cases} \boldsymbol{X}'(t) - A(t)\boldsymbol{X}(t) = \boldsymbol{f}(t), \ t \in \mathbb{R} \\ \boldsymbol{X}(t_0) = \boldsymbol{X}_0. \end{cases} \tag{8.4}$$

From the existence and uniqueness theorem for linear systems we know that there exists two functions $\boldsymbol{X}_1, \boldsymbol{X}_2 : I \to \mathbb{R}^2$ such that

$$\boldsymbol{X}_1'(t) - A(t)\boldsymbol{X}_1(t) = \mathbf{0}_{2\times 1} \tag{8.5}$$

$$\boldsymbol{X}_2'(t) - A(t)\boldsymbol{X}_2(t) = \mathbf{0}_{2\times 1} \tag{8.6}$$

and

$$\det\left(\boldsymbol{X}_1(t_0) \;\middle|\; \boldsymbol{X}_2(t_0)\right) \neq 0. \tag{8.7}$$

a) Define the Wronskian $W(\boldsymbol{X}_1, \boldsymbol{X}_2) : I \to \mathbb{R}$ via

$$W(\boldsymbol{X}_1, \boldsymbol{X}_2)(t_0) = \det\left(\boldsymbol{X}_1(t) \;\middle|\; \boldsymbol{X}_2(t)\right). \tag{8.8}$$

One can show using some tools from linear algebra that

$$W(\boldsymbol{X}_1, \boldsymbol{X}_2)(t) = \det\Phi(t) = W(\boldsymbol{X}_1, \boldsymbol{X}_2)(t_0)\exp\left(\int_{t_0}^{t} \operatorname{tr} A(s)\, ds\right), \ t \in I \tag{8.9}$$

Use (8.8) and (8.9) to show that $W(\boldsymbol{X}_1, \boldsymbol{X}_2)$ is never equal to 0 on I. Conclude that $\boldsymbol{X}_1, \boldsymbol{X}_2$ are linearly independent over I.

b) Define the fundamental matrix $\Phi : I \to \mathcal{M}_{2\times 2}(\mathbb{R})$ associated to the system (8.4) via

$$\Phi(t) = \left(\boldsymbol{X}_1(t) \;\middle|\; \boldsymbol{X}_2(t)\right). \tag{8.10}$$

Verify by direct computation that Φ satisfies the matrix-valued equation

$$\Phi'(t) = A(t)\Phi(t), \ t \in I. \tag{8.11}$$

c) Verify by direct computation that

$$\boldsymbol{X}_h(t) = \Phi(t)\Phi^{-1}(t_0)\boldsymbol{X}_0, \ t \in I \tag{8.12}$$

is a solution to the homogeneous IVP

$$\begin{cases} \boldsymbol{X}'(t) - A(t)\boldsymbol{X}(t) = \mathbf{0}, \ t \in \mathbb{R} \\ \boldsymbol{X}(t_0) = \boldsymbol{X}_0. \end{cases} \tag{8.13}$$

d) Verify by direct computation that

$$\boldsymbol{X}(t) = \underbrace{\Phi(t)\Phi^{-1}(t_0)\boldsymbol{X}_0}_{\boldsymbol{X}_h(t)} + \underbrace{\Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\boldsymbol{f}(s)\, ds}_{\boldsymbol{X}_p(t)} \tag{8.14}$$

is a solution to the IVP (8.4).

Remark 8.4.

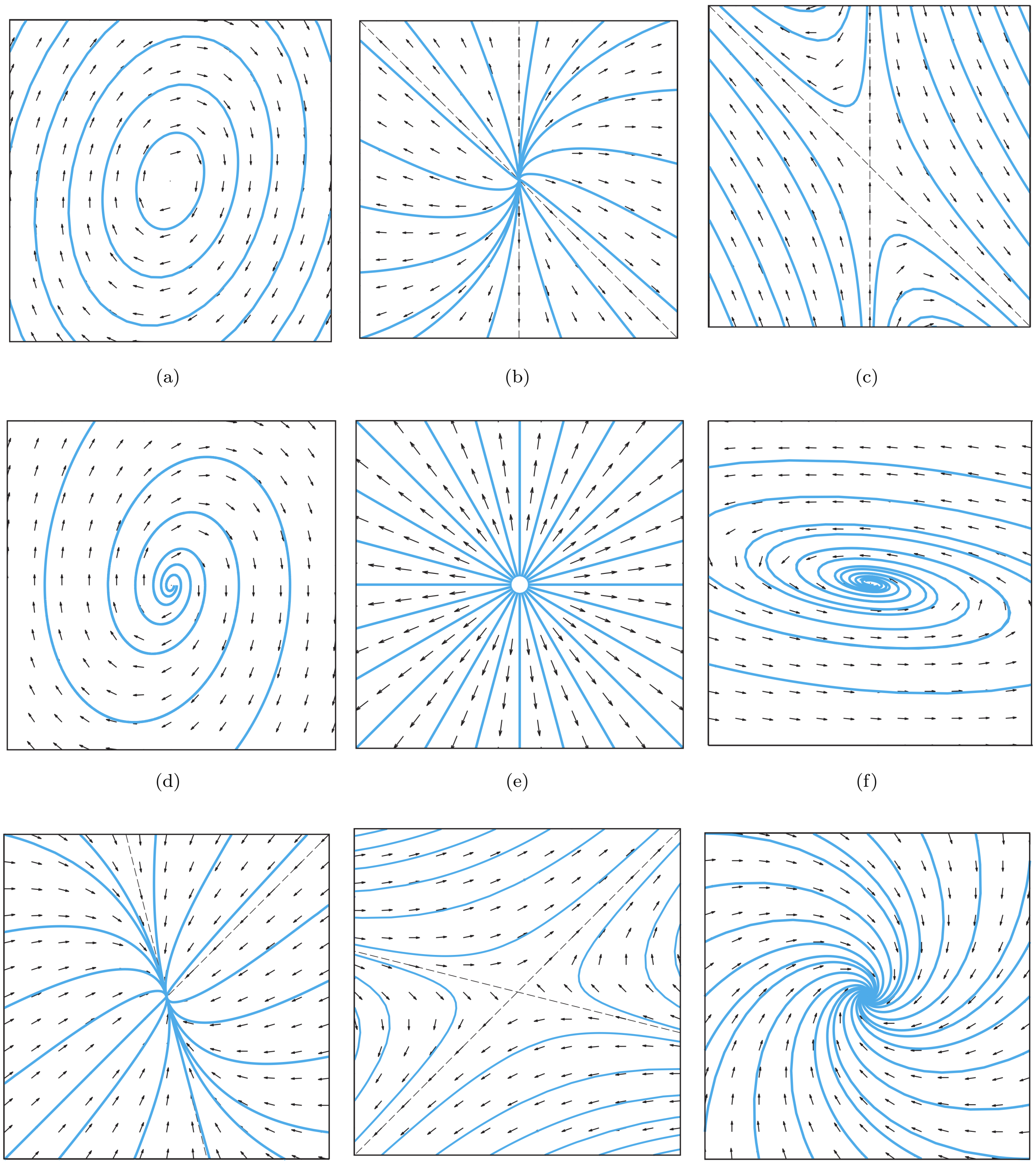
- Part a) shows that $\Phi(t)$ is invertible for all $t \in I$ as $\det\Phi(t) \neq 0$ for all $t \in I$, so Φ^{-1} is well-defined on I and therefore the variation of parameters formula (8.14) is well-defined for all $t \in I$.
- For part d), you may assume that the product rule and fundamental theorem of calculus hold for matrix and vector valued functions. The integral of a vector-valued function is defined term-wise: if $\boldsymbol{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ then

$$\int_{t_0}^{t} \boldsymbol{g}(s)\, ds = \begin{pmatrix} \int_{t_0}^{t} g_1(s)\, ds \\ \int_{t_0}^{t} g_2(s)\, ds \end{pmatrix}. \tag{8.15}$$

- By the uniqueness part of the existence and uniqueness theorem, (8.12) is the unique solution to the homogeneous IVP (8.13) and (8.14) is the unique solution to the inhomogeneous IVP (8.4). (8.14) is known as the variation of parameters formula or Duhamel's principle.
- The explicit solution formula shows that as long as we can find the general solution to the underlying homogeneous problem (which is typically difficult if $A(\cdot)$ depends on time), we can solve the inhomogeneous problem by computing the Duhamel term $\int_{t_0}^{t} \Phi^{-1}(s)\boldsymbol{f}(s)\, ds$. This is the essence of Duhamel's principle, which says that we can in principle solve any inhomogeneous problem as long as we can solve the associated homogeneous problem.
- Analogous to the situation for scalar equations, when $A(\cdot)$ is a variable-coefficient matrix the fundamental matrix Φ is in general difficult to identify. However, in the case when A is a constant coefficient matrix, Φ can be identified explicitly using the so-called *Jordan canonical form* of the matrix A, and the Jordan canonical form is intricately related to the eigenvalues and eigenvectors of the matrix A. This is the reason why we are able to identify the general solution of homogeneous systems explicitly by finding the eigenvalues and eigenvectors of the matrix A.

Problem 8.4 (Phase portraits of 2×2 systems)**.** The diagrams below are phase portraits of solutions $\boldsymbol{X}$ to the homogeneous system $\boldsymbol{X}' = A\boldsymbol{X}$. For each of the following diagrams, classify the eigenvalues of the matrix A and the origin by identifying the following.

a) Classify the eigenvalues of the matrix A by indicating whether the eigenvalues of A are real/complex and identifying the sign(s) of the real parts of the eigenvalues.
b) Classify the origin in terms of the geometry of the trajectories associated to the linear system, by identifying whether the origin is a node, saddle point, spiral, or center.
c) Classify the origin in terms of the stability of the trajectories. Is the origin a source or a sink? Is it a stable critical point or an unstable critical point? Is it asymptotically stable?



(a) (b) (c)

(d) (e) (f)

(g) (h) (i)

Problem 8.5 (Linearization of nonlinear planar systems)**.** Consider the nonlinear autonomous system

$$\begin{cases} x'(t) = (x(t))^2 + (y(t))^2 - 6 \\ y'(t) = (x(t))^2 - y(t), \qquad t \in \mathbb{R}. \end{cases} \tag{8.16}$$

a) Identify the critical points of the nonlinear system.
b) Write down the linearization of the nonlinear system around the constant/equilibrium solutions.
c) Classify the critical points in terms of their stability type.