Homework 2

DUE: SATURDAY, FEBRUARY 1, 2025, 11:59 PM

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Problem 1.1 (Separable equations)**.**

Consider the initial value problem

$$\begin{cases} e^{y(x)-x}y'(x) - e^{x-y(x)} = 0, \ x \in \mathbb{R} \\ y(\ln 2) = \frac{\ln 3}{2}. \end{cases} \tag{1.1}$$

Suppose y is a solution over $\mathbb{R}$.

a) Show that y satisfies a separable differential equation of the form

$$g(y(x))y'(x) = f(x), \ x \in \mathbb{R}, \tag{1.2}$$

for some functions f, g.

b) Proceed to identify a candidate implicit solution to the initial value problem in the form of

$$G(x, y(x)) = 0, \tag{1.3}$$

where x belongs to some interval I. You do not need to specify the interval I.

c) Suppose J is the maximal interval of existence of the candidate implicit solution to the IVP identified in part b). Show that it is not possible for $J = \mathbb{R}$ by considering what happens if $x = 0$.

Problem 1.2 (Structure of solutions to first order linear equations)**.**

Suppose the general solution to a first order linear differential equation

$$y'(x) + P(x)y(x) = f(x),\ x \in \mathbb{R} \tag{1.4}$$

is given by

$$y(x) = \sin(x) - 1 + Ce^{-\sin(x)},\ x \in \mathbb{R} \tag{1.5}$$

for an arbitrary constant C.

a) What is the general solution to the homogeneous problem

$$y'(x) + P(x)y(x) = 0? \tag{1.6}$$

b) Identify any two particular solutions (non-parametrized) to the inhomogeneous problem

$$y'(x) + P(x)y(x) = f(x),\ x \in \mathbb{R}. \tag{1.7}$$

c) What is P?

d) What is f?

Problem 1.3 (Autonomous equations and stability analysis)**.**

Consider the autonomous differential equation

$$y'(t) = y(t)(10 - y(t)),\ t \in \mathbb{R}. \tag{1.8}$$

a) Consider the function h defined via

$$h(y) = y(10 - y),\ y \in \mathbb{R}. \tag{1.9}$$

Identify the critical points of h (these are points where $h(y) = 0$) and draw a sign chart of h.

b) What are the constant solutions admitted by the equation?

c) Sketch a one-dimensional phase portrait corresponding to the equation and classify the critical points by stability found in part a).

d) Sketch a sample family of solution curves in the t-y plane.

e) Suppose y is a solution to the differential equation and $y(0) = 4$. Use the sketch in part c) to identify

$$\lim_{t\to\infty} y(t) \text{ and } \lim_{t\to-\infty} y(t). \tag{1.10}$$

Problem 1.4 (Bernoulli differential equations)**.**

Consider first order differential equations of the form

$$y'(x) + P(x)y(x) = Q(x)(y(x))^{\alpha},\ x \in \mathbb{R}, \tag{1.11}$$

where $\alpha \in \mathbb{R}$. If $\alpha = 0$ or $\alpha = 1$, then this is a linear equation, and we may identify the solution via the method of integrating factors. When $\alpha \neq 0, 1$, the equation is nonlinear, so the method of integrating factors cannot be applied. In this problem we explore how to reduce the original equation to a simpler equation via substitution.

Suppose $\alpha \neq 0, 1$. Note that if $\alpha > 0$, then the equation admits the zero solution $y(x) = 0,\ x \in \mathbb{R}$. Suppose y is a solution to the equation on $\mathbb{R}$ and there exists an interval I for which $y(x) \neq 0$ for all $x \in I$.

a) Show that y solves the equation

$$(y(x))^{-\alpha}y'(x) + P(x)(y(x))^{1-\alpha} = Q(x),\ x \in I. \tag{1.12}$$

b) Define the function v via

$$v(x) = (y(x))^{1-\alpha},\ x \in I. \tag{1.13}$$

Show that

$$\frac{1}{1-\alpha}v'(x) = (y(x))^{-\alpha}y'(x),\ x \in I. \tag{1.14}$$

c) Show that v solves the equation

$$v'(x) + (1-\alpha)P(x)v(x) = (1-\alpha)Q(x),\ x \in I. \tag{1.15}$$

d) What is the order of the equation (1.15)? Is it a linear equation?

e) Find the general candidate solution to the differential equation

$$y'(x) + xy(x) = x(y(x))^3,\ x \in \mathbb{R}. \tag{1.16}$$

Problem 1.5 (Differential equations with homogeneous functions)**.**

Consider the initial value problem

$$\begin{cases} xy'(x) = y(x) - \left(y(x) - x\right)^2, \ x \in \mathbb{R} \\ y(1) = 0. \end{cases} \tag{1.17}$$

Suppose y is a solution to the initial value problem.

a) Show that y satisfies the equation

$$y'(x) = \frac{y(x)}{x} - x\left(\frac{y(x)}{x} - 1\right)^2, \ x > 0. \tag{1.18}$$

b) Use part a) to identify a candidate solution to the initial value problem (1.17). What is a candidate for the maximal interval of existence J of the solution? (Hint: J is not $(0, \infty)$)

c) Verify that the candidate solution is a solution to (1.17) on the maximal interval of existence J identify in part b). Please make sure to verify the initial condition as well.

d) Repeat parts b) and c) when the initial condition is replaced with $y(1) = 1$. Show that in this case, the maximal interval of existence can be extended to $J = \mathbb{R}$. (Hint: when solving for separable equations, what is the first step that people tend to ignore?)

Problem 1.6 (Failure of uniqueness)**.**

Consider the initial value problem

$$\begin{cases} y'(t) = f(t, y(t)), \ t \in \mathbb{R} \\ y(2) = -1, \end{cases} \tag{1.19}$$

where

$$f(t, y) = \frac{-t + \sqrt{t^2 + 4y}}{2}, \ (t, y) \in \mathbb{R}^2. \tag{1.20}$$

a) Show that y_1 defined via

$$y_1(t) = -t + 1, \ t \in \mathbb{R} \tag{1.21}$$

is a solution to the initial value problem on the interval $[2, \infty)$, but not on the interval $(-\infty, 2)$.

b) Show that y_2 defined via

$$y_2(t) = \frac{-t^2}{4}, \ t \in \mathbb{R} \tag{1.22}$$

is a solution to the initial value problem on the interval $J = \mathbb{R}$.

c) Calculate $\frac{\partial f}{\partial y}$. What is the domain of this function as a function on $\mathbb{R}^2$?

d) Parts a) and b) show that the initial value problem (1.19) does not admit a unique solution. Why does this not violate the existence and uniqueness theorem for first order differential equations?

(Hint for the second part of part a): $\sqrt{x^2} = |x|$ for $x \in \mathbb{R}$)