

## Additional final practice problems

**Problem 1.1.** Consider the initial value problem

$$\begin{cases} e^t y'(t) + 2e^t y(t) = 3e^{2t}, & t \in \mathbb{R} \\ y(0) = 0. \end{cases} \quad (1.1)$$

- a) Classify the equation by order.

*Solution.* This is a first order equation. □

- b) Classify the equation by linearity. Is it linear or nonlinear?

*Solution.* This is a linear equation. □

- c) Use an appropriate method to find a solution to the initial value problem. You may skip the verification step.

*Solution.* If  $y$  is a solution to the IVP, then

$$y'(t) + 2y(t) = 3e^t, \quad t \in \mathbb{R}. \quad (1.2)$$

We can choose  $\mu(t) = e^{2t}, t \in \mathbb{R}$  to be an integrating factor. Therefore

$$\frac{d}{dt} [e^{2t}y(t)] = 3e^{3t}, \quad t \in \mathbb{R} \quad (1.3)$$

implying

$$e^{2t}y(t) = e^{3t} + C \implies y(t) = e^t + Ce^{-2t}, \quad t \in \mathbb{R} \quad (1.4)$$

and  $C$  is arbitrary. Since  $y(0) = 0$ , we must have  $0 = 1 + C \implies C = -1$ . Therefore a solution to the IVP is  $y(t) = e^t - e^{-2t}, t \in \mathbb{R}$ . □

- d) What is the maximal interval of existence of the solution?

*Solution.* The solution exists globally for all  $t \in \mathbb{R}$ , therefore the maximal interval of existence is  $\mathbb{R}$ . □

- e) Is the solution identified in part c) unique?

*Solution.* Yes. Since  $e^t, 2e^t, 3e^{2t}$  are all continuous over  $\mathbb{R}$  and  $e^t \neq 0$  for all  $t \in \mathbb{R}$ , by the existence and uniqueness theorem for first order linear equations, the solution identified in part c) must be the unique global solution on  $\mathbb{R}$ . □

**Problem 1.2.** Consider the differential equation

$$y'(t) = 2t - y(t), \quad t \in \mathbb{R}. \quad (1.5)$$

- a) Verify that the function

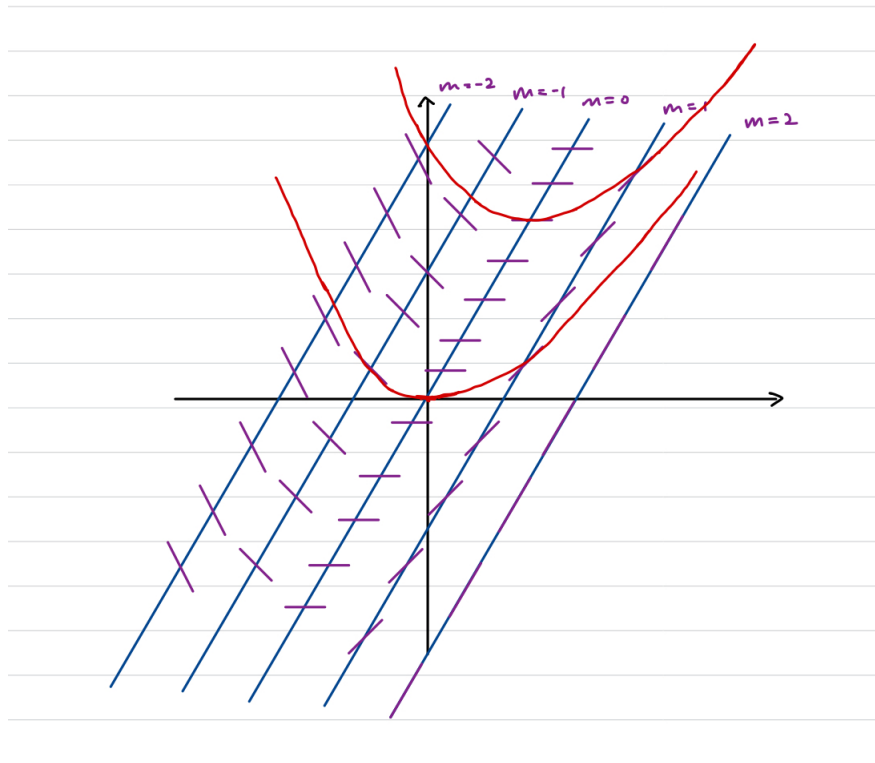
$$y(t) = 2(t - 1), \quad t \in \mathbb{R} \quad (1.6)$$

is a solution on the interval  $J = \mathbb{R}$ . Please don't present your work backwards, instead calculate  $y'$  and the right-hand side of the equation for the given  $y$  separately and conclude that they are equal to each other.

*Solution.* If  $y$  is defined via (1.6), then  $y'(t) = 2$  for all  $t \in \mathbb{R}$  and  $2t - y(t) = 2t - 2(t - 1) = 2t - 2t + 2 = 2$  for all  $t \in \mathbb{R}$ . Thus  $y$  is a solution to the differential equation.  $\square$

- b) Sketch the directional field associated to the equation by identifying the isoclines corresponding to  $m = 0, \pm 1, \pm 2$ . You're welcome to add more isoclines to the sketch to improve the accuracy of the sketch.
- c) Sketch the solution curve  $y_1$  passing through  $(0, 0)$  and the solution curve  $y_2$  passing through  $(0, 2)$  on top of the directional field you sketched in part b). Please make sure that your solution curves match the underlying directional field, and also you sketch them for enough  $t$ 's so that the global behavior of the solution curves are easy to visualize.

*Solution.* Note that  $2t - y = m \iff y = 2t - m$ , so the isoclines are straight lines with slope 2. Here is a very rough sketch for parts c) and d). Note that the line  $y(t) = 2(t - 1)$  is both an isocline and a solution curve, so no other solution curve can cross it.



$\square$

- d) Is it possible for the solutions curves  $y_1$  and  $y_2$  from part c) to ever cross? Please explain your reasoning and justify this part rigorously.

*Solution.* No. We note that the equation can be written as  $y'(t) + y(t) = 2t$  for all  $t \in \mathbb{R}$ . Since the constant function that takes the value 1 for all  $t \in \mathbb{R}$  and  $2t$  are continuous over  $\mathbb{R}$ , we have uniqueness globally for all time  $t$ . Therefore by uniqueness, solution curves cannot cross.  $\square$

- e) Is it possible for  $y_1(2) \leq 2$ ? Please explain your reasoning and justify this part rigorously.

*Solution.* Note that the straight line solution  $y$  identified in part a) satisfies  $y(2) = 2$ . Since  $y_1$  started at the point  $(0,0)$ , which is to the left of  $y$ , if  $y_1(2) \leq 2$  this means that  $y_1$  must cross  $y$ , therefore by uniqueness this is not possible.  $\square$

**Problem 1.3.** Consider the initial value problem

$$\begin{cases} y'(x) = 6x(y(x) - 1)^{2/3}, & x \in \mathbb{R} \\ y(0) = 1. \end{cases} \quad (1.7)$$

- a) Verify that the function  $y_1$  defined via

$$y_1(x) = 1, \quad x \in \mathbb{R} \quad (1.8)$$

is a constant solution to the initial value problem on the interval  $J = \mathbb{R}$ . As with all verification problems, please do not present you work backwards. Also, please do not forget to check the initial condition.

*Solution.* First we note that  $y_1(0) = 1$ , so it satisfies the initial condition. Also,  $y_1'(x) = 0$  for all  $x \in \mathbb{R}$  and  $6x(y_1(x) - 1)^{2/3} = 6x(1 - 1)^{2/3} = 0$  for all  $x \in \mathbb{R}$ . Therefore  $y_1$  is a solution to the IVP on  $\mathbb{R}$ .  $\square$

- b) Verify that the function  $y_2$  defined via

$$y_2(x) = 1 + x^6, \quad x \in \mathbb{R} \quad (1.9)$$

is a solution to the initial value problem on the interval  $J = \mathbb{R}$ . As with all verification problems, please do not present you work backwards. Also, please do not forget to check the initial condition.

*Solution.* First we note that  $y_2(0) = 1 + 0^6 = 1$ , so it satisfies the initial condition. Also,  $y_2'(x) = 6x^5$  and  $6x(y_2(x) - 1)^{2/3} = 6x(x^6)^{2/3} = 6x(x^4) = 6x^5$  for all  $x \in \mathbb{R}$ , therefore  $y_2$  is a solution to the IVP on  $\mathbb{R}$ .  $\square$

- c) Parts b) and c) show that solutions to the given initial value problem are not unique. Explain why this does not violate the conclusions of the existence and uniqueness theorem.

*Solution.* We note that if we define the function  $f(x, y) = 6x(y - 1)^{2/3}$ ,  $(x, y) \in \mathbb{R}^2$ , we have  $\frac{\partial f}{\partial y}(x, y) = 4x(y - 1)^{-1/3}$ ,  $(x, y) \in \mathbb{R}^2$  and  $y \neq 1$ . Since the initial condition is specified at  $y_0 = 1$  and  $\frac{\partial f}{\partial y}$  is undefined there, the existence and uniqueness theorem cannot be applied.  $\square$

- d) Classify all points  $(t_0, y_0) \in \mathbb{R}^2$  for which if  $y(t_0) = y_0$  is the specified initial condition (instead of  $y(0) = 1$ ), the existence and uniqueness of solutions is guaranteed.

*Solution.* By the calculations in the previous part  $f$  and  $\frac{\partial f}{\partial y}$  is continuous in a small rectangle around any point  $(t_0, y_0)$  for which  $y_0 \neq 1$ , and these are the points where the existence and uniqueness of solutions is guaranteed.  $\square$

- e) Find the unique solution to the initial value problem

$$\begin{cases} y'(x) = 6x(y(x) - 1)^{2/3}, & x \in \mathbb{R} \\ y(0) = 2. \end{cases} \quad (1.10)$$

You may skip the verification step. What is the maximal interval of existence of the solution? Hint: the antiderivative of  $u^{-2/3}$  is  $3u^{1/3}$ .

*Solution.* We note that this is a separable equation and we may rewrite the equation as

$$\frac{y'(x)}{(y(x) - 1)^{2/3}} = 6x, \quad x \in I \quad (1.11)$$

over some interval  $I$ . Then

$$3(y(x) - 1)^{1/3} = 3x^2 + C \implies y(x) = 1 + (x^2 + C)^3, \quad x \in I \quad (1.12)$$

and  $C$  is arbitrary. Since  $y(0) = 2$ , we see that  $C^3 = 1 \implies C = 1$ . So the unique solution is  $y(x) = 1 + (x^2 + 1)^3$  over the interval  $J = \mathbb{R}$ .  $\square$

**Problem 1.4.** Consider the differential equation

$$xy'(x) + 6y(x) = 3x(y(x))^{4/3}, \quad x \in \mathbb{R}. \quad (1.13)$$

- a) Classify the equation by linearity. Is it linear or nonlinear? No justification required.

*Solution.* This is a nonlinear equation.  $\square$

- b) Does the equation admit any constant solutions?

*Solution.* Yes, we can write the equation as  $xy'(x) = y(x)(3x(y(x))^{1/3} - 6)$ , so we see that  $y(x) = 0$  for all  $x \in \mathbb{R}$  is a constant solution.  $\square$

- c) Use an appropriate method to find the solution to the initial value problem

$$\begin{cases} xy'(x) + 6y(x) = 3x(y(x))^{4/3}, & x \in \mathbb{R}, \\ y(1) = -1. \end{cases} \quad (1.14)$$

You may skip the verification step.

*Solution.* This is a Bernoulli equation, and we use the substitution  $v(x) = (y(x))^{-1/3}$  over some interval  $I$ . Note that  $v'(x) = (-1/3)(y(x))^{-4/3}y'(x)$  over  $I$ . Following the homework problem, we divide both sides of the equation by  $(y(x))^{4/3}$  and arrive at

$$xy'(x)(y(x))^{-4/3} + 6(y(x))^{-1/3} = 3x, \quad x \in I. \quad (1.15)$$

This implies that

$$-3xv'(x) + 6v(x) = 3x, \quad x \in I. \quad (1.16)$$

Consider  $J = I \cap (0, \infty)$ . Then

$$v'(x) - \frac{2}{x}v(x) = -1, \quad x \in J. \quad (1.17)$$

We may choose an integrating factor to be  $\mu(x) = \exp(-2 \int \frac{1}{x} dx) = \exp \ln |x|^{-2} = x^{-2}, x \in J$ . Then

$$\frac{d}{dx} \left[ \frac{1}{x^2} v(x) \right] = -\frac{1}{x^2}, \quad x \in J. \quad (1.18)$$

Thus

$$\frac{1}{x^2} v(x) = \frac{1}{x} + C \implies v(x) = x + Cx^2, \quad x \in J. \quad (1.19)$$

Since  $v(1) = (y(1))^{-1/3} = (-1)^{-1/3} = -1$ , we see that  $1 + C = -1 \implies C = -2$ . Thus the solution to the initial value problem is

$$y(x) = (x - 2x^2)^{-3}, \quad x \in J, \quad (1.20)$$

for some interval  $J$ .  $\square$

- d) What is the maximal interval of existence for the solution in part c)?

*Solution.* We note that we need  $x - 2x^2 = x(1 - 2x) \neq 0 \iff x \neq 0, \frac{1}{2}$  for the solution identified in the previous part to be well-defined. Therefore the maximal interval of existence is  $J = (\frac{1}{2}, \infty)$ .  $\square$

- e) What happens if we change the initial condition to  $y(1) = 0$ ? Does a solution exist and is it unique?

*Solution.* We saw in part b)  $y(x) = 0$  for all  $x \in \mathbb{R}$  is a constant solution, which satisfies the initial condition  $y(1) = 0$ . So a solution exists. If we define

$$f(x, y) = \frac{1}{x} (3xy^{4/3} - 6y), \quad (x, y) \in \mathbb{R}^2, \quad (1.21)$$

the equation is in the form of  $y'(x) = f(x, y(x))$  and

$$\frac{\partial f}{\partial y}(x, y) = \frac{1}{x} (4xy^{1/3} - 6), \quad (x, y) \in \mathbb{R}^2, \quad (1.22)$$

and we see that  $f, \frac{\partial f}{\partial y}$  are both continuous over  $\mathbb{R}^2$  except at  $x = 0$ , but the initial value  $y(1) = -1$  is specified away from that point. Thus the constant solution  $y(x) = 0$  is the unique solution to the IVP.  $\square$

**Problem 1.5.** Suppose a constant coefficient linear differential equation admits the general solution

$$y(x) = c_1 e^x \cos x + c_2 e^x \sin x, \quad x \in \mathbb{R} \quad (1.23)$$

where  $c_1, c_2$  are arbitrary.

- a) What are the roots of the characteristic equation associated to the differential equation?

*Solution.* The roots of the characteristic equation are  $r_1 = 1 - i, r_2 = 1 + i$ . □

- b) Find a constant coefficient differential equation that admits this general solution.

*Solution.* The characteristic equation is

$$(r - (1 - i))(r - (1 + i)) = ((r - 1) + i)((r - 1) - i) = (r - 1)^2 - i^2 = r^2 - 2r + 1 + 1 = r^2 - 2r + 2, \quad r \in \mathbb{R}. \quad (1.24)$$

Therefore an equation that admits this general solution is

$$y''(x) - 2y'(x) + 2y(x) = 0, \quad x \in \mathbb{R}. \quad (1.25)$$

□

**Problem 1.6.** Consider the variable coefficient initial value problem

$$\begin{cases} 5y''(x) + 12xy'(x) + 25x^2y(x) = 0, & x \in \mathbb{R} \\ y(1) = 0 \\ y'(1) = 0. \end{cases} \quad (1.26)$$

- a) Find a solution to the initial value problem over the interval  $I = \mathbb{R}$ .

*Solution.* Note that the constant solution  $y : \mathbb{R} \rightarrow \mathbb{R}$  defined via  $y(x) = 0$  satisfies the equation and the initial conditions, therefore it is a solution to the IVP on  $I = \mathbb{R}$ .  $\square$

- b) Justify carefully and rigorously why the solution you found in part a) is the only solution to the initial value problem over the interval  $I = \mathbb{R}$ .

*Solution.* Note that the coefficients  $a_2, a_1, a_0 : \mathbb{R} \rightarrow \mathbb{R}$  defined via  $a_2(x) = 5, a_1(x) = 12x, a_0(x) = 25x^2$  are all continuous on  $\mathbb{R}$  and  $a_2(x) \neq 0$  for all  $x \in \mathbb{R}$ , therefore by the existence and unique theorem for linear equations, there exists a unique solution to the IVP on  $I = \mathbb{R}$ . By the uniqueness part of the theorem, the solution we identified in part a) is therefore the only solution to the IVP.  $\square$



**Problem 1.7.** Consider the mass-spring system modeled via the homogeneous linear differential equation

$$x''(t) + \gamma x'(t) + 4x(t) = 0, \quad t \in \mathbb{R}. \quad (1.27)$$

- a) Find value(s) of  $\gamma$  for which the system is critically damped.

*Solution.* For the equation to be critically damped we require

$$\gamma^2 - 16 = 0 \implies \gamma = 4. \quad (1.28)$$

We need to exclude the case that  $\gamma = -4$  because in a damped mass-spring system, the damping constant is assumed to be positive.  $\square$

- b) Find the largest sub-interval  $I$  of  $(0, \infty)$  such that if  $\gamma \in I$ , then the system is overdamped.

*Solution.* For the system to be overdamped we require

$$\gamma^2 - 16 = (\gamma - 4)(\gamma + 4) > 16. \quad (1.29)$$

This is equivalent to requiring  $\gamma > 4$ , so the largest sub-interval is  $(4, \infty)$ .  $\square$

- c) Suppose  $\gamma = 2$ . What is the quasi-period  $T$  of the solution?

*Solution.* Note that the general solution to the equation is

$$x(t) = c_1 e^{-t} \cos \sqrt{3}t + c_2 e^{-t} \sin \sqrt{3}t, \quad t \in \mathbb{R}, \quad (1.30)$$

where  $c_1, c_2$  are arbitrary. Therefore the quasi-period is

$$T = \frac{2\pi}{\sqrt{3}}. \quad (1.31)$$

$\square$

- d) Suppose  $\gamma = 4$ , and an external force is present in the system and the forced damped mass-spring system is modeled via

$$x''(t) + 4x'(t) + 4x(t) = 32e^{2t}, \quad t \in \mathbb{R}. \quad (1.32)$$

Find the general solution to the system.

*Solution.* We note that when  $\gamma = 4$ , the general homogeneous solution is

$$x_h(t) = c_1 e^{-2t} + c_2 t e^{-2t}, \quad t \in \mathbb{R}. \quad (1.33)$$

Therefore to recover the particular solution we may use the ansatz

$$x_p(t) = A e^{2t}, \quad t \in \mathbb{R} \quad (1.34)$$

and calculate

$$x'_p(t) = 2A e^{2t}, \quad x''_p(t) = 4A e^{2t}, \quad t \in \mathbb{R}. \quad (1.35)$$

Thus

$$x''_p(t) + 4x'_p(t) + 4x_p(t) = (4A + 8A + 4A)e^{2t} = 32e^{2t}, \quad t \in \mathbb{R} \implies A = 2. \quad (1.36)$$

Therefore the general solution to the equation is

$$x(t) = 2e^{2t} + c_1 e^{-2t} + c_2 t e^{-2t}, \quad t \in \mathbb{R}, \quad (1.37)$$

where  $c_1, c_2$  are arbitrary.  $\square$

**Problem 1.8.** Consider the 2nd order differential equation

$$x^2 y''(x) + 3xy'(x) - 3y(x) = 0, \quad x > 0. \quad (1.38)$$

You are given that  $y_1$  defined via

$$y_1(x) = x, \quad x > 0 \quad (1.39)$$

is a solution to the homogeneous equation. Use the method of reduction of order to find a second linearly independent solution  $y_2$  to the equation over the interval  $I = (0, \infty)$ . You do not need to check the independence of  $y_1, y_2$ , nor verify that  $y_2$  is a solution.

*Solution.* We use the ansatz  $y_2(x) = u(x)y_1(x), x > 0$  and calculate

$$y_2'(x) = u'(x)y_1(x) + u(x)y_1'(x) \quad (1.40)$$

$$y_2''(x) = u''(x)y_1(x) + 2u'(x)y_1'(x) + u(x)y_1''(x), \quad x > 0. \quad (1.41)$$

Thus if  $y_2$  is a solution, we have

$$\begin{aligned} x^2 y_2''(x) + 3xy_2'(x) - 3y_2(x) &= u''(x)(x^2 y_1(x)) + u'(x)(2x^2 y_1'(x) + 3xy_1(x)) + u(x) \underbrace{(x^2 y_1''(x) + 3xy_1'(x) - 3y_1(x))}_{=0} \\ &= x^3 u''(x) + 5x^2 u'(x) = 0, \quad x > 0. \end{aligned} \quad (1.42)$$

Thus  $w = u'$  satisfies the first order linear equation

$$w'(x) + \frac{5}{x}w(x) = 0, \quad x > 0. \quad (1.43)$$

An integrating factor for this equation is  $\mu(x) = x^5, x > 0$ . Therefore

$$\frac{d}{dx} [x^5 w(x)] = 0 \implies w(x) = \frac{C_1}{x^5} \implies u(x) = \frac{C}{x^4} + D. \quad (1.44)$$

Therefore by choosing  $C = 1, D = 0$ , we find that a second linearly independent solution is

$$y_2(x) = \frac{1}{x^3}, \quad x > 0. \quad (1.45)$$

□

**Problem 1.9.** Suppose a mass-spring system is modeled via

$$x''(t) + \beta x'(t) + 4x(t) = \cos \omega t, \quad t \in \mathbb{R}. \quad (1.46)$$

where  $\beta \geq 0, \omega > 0$ .

- a) Identify the parameters  $\beta, \omega$  for which pure resonance occurs.
- b) In the case of part a) where resonance occurs, use the method of undetermined coefficients to find a particular solution to the system.
- c) Suppose  $\beta > 0$  and  $x(0) = x'(0) = 0$ . Would a sizable change in the initial conditions, either in the initial position or the initial velocity, result in a sizable change in the behavior of the system in the long run? Please briefly explain why or why not.

*Solution.*

- a) Pure resonance occurs when there is no damping, so  $\beta = 0$ , and the forcing frequency must be equal to the natural frequency in absolute value, so  $\omega = \pm \sqrt{\frac{4}{1}} = \pm 2$ .

- b) We use the ansatz

$$x_p(t) = At \cos 2t + Bt \sin 2t, \quad t \in \mathbb{R}, \quad (1.47)$$

and calculate for all  $t \geq 0$ ,

$$x'_p(t) = (A + 2Bt) \cos 2t + (B - 2At) \sin 2t \quad (1.48)$$

$$x''_p(t) = (2B + 2B - 4At) \cos 2t + (-2A - 2A - 4Bt) \sin 2t. \quad (1.49)$$

Therefore

$$x''_p(t) + 4x_p(t) = (4B) \cos 2t + (-4A) \sin 2t = \cos 2t, \quad t \in \mathbb{R}. \quad (1.50)$$

This implies  $A = 0$  and  $B = \frac{1}{4}$ , therefore a particular solution to the system is

$$x_p(t) = \frac{1}{4}t \sin 2t, \quad t \in \mathbb{R}. \quad (1.51)$$

- c) No, because the initial conditions only affect the homogeneous solution, which decays exponentially with positive damping and becomes negligible in the long run. As a result, the solution converges at a steady-state solution which is the particular solution, and the particular solution is not affected by the initial conditions.

□

**Problem 1.10.** Suppose the general homogeneous solution to the variable coefficient equation

$$x^2 y''(x) + xy'(x) - y(x) = 1, \quad x > 0 \quad (1.52)$$

is

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 x + c_2 x^{-1}, \quad x > 0, \quad (1.53)$$

where  $c_1, c_2$  are arbitrary.

- a) Find the Wronskian  $W(y_1, y_2)$  defined for  $x > 0$ .
- b) Use the variation of parameters formula to find a particular solution to the equation. Note that the coefficient in front of the highest order term  $y''$  is  $x^2$ , not 1.

*Solution.*

- a) We calculate

$$W(y_1, y_2)(x) = \det \begin{pmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{pmatrix} = -x^{-1} - x^{-1} = -2x^{-1}, \quad x > 0. \quad (1.54)$$

- b) A particular solution is then given by

$$y_p(x) = -x \int \frac{x^{-1}x^{-2}}{-2x^{-1}} dx + x^{-1} \int \frac{xx^{-2}}{-2x^{-1}} dx = \frac{x}{2} \int x^{-2} dx - \frac{x^{-1}}{2} \int 1 dx = -\frac{1}{2} - \frac{1}{2} = -1, \quad x > 0. \quad (1.55)$$

□

**Problem 1.11.** Consider the eigenvalue problem

$$\begin{cases} y''(x) + \lambda y(x) = 0, & x \in (0, \pi) \\ y(0) = 0, & y(\pi) = 0. \end{cases} \quad (1.56)$$

Find the positive eigenvalues associated to this problem.

*Solution.* We note that if  $\lambda > 0$ , then the general solution to the equation is

$$y_h(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x), \quad x \in (0, \pi). \quad (1.57)$$

If  $y(0) = 0$ , we must have  $c_1 = 0$ . If  $y(\pi) = 0$ , then we must have

$$c_2 \sin \sqrt{\lambda}\pi = 0. \quad (1.58)$$

Since we are interested in finding non-trivial solutions, we may assume  $c_2 \neq 0$ . Then occurs whenever  $\sqrt{\lambda}$  is an integer, therefore we may parametrize the positive eigenvalues via

$$\lambda_n = n^2, \quad n = 1, 2, 3, \dots \quad (1.59)$$

□

**Problem 1.12.** Suppose a mass-spring system is modeled via

$$\begin{cases} x''(t) + x(t) = f(t), & t \geq 0 \\ x(0) = x'(0) = 0, \end{cases} \quad (1.60)$$

where  $\delta$  is the Dirac delta and  $\mathcal{U}$  is the unit step function and  $f : [0, \infty) \rightarrow \mathbb{R}$  is defined via

$$f(t) = \begin{cases} \delta(t - \pi), & 0 \leq t < 2\pi \\ 1, & t \geq 2\pi. \end{cases} \quad (1.61)$$

- Write  $f$  in terms of the unit step function  $\mathcal{U}(\cdot - 2\pi)$ . (Note:  $\delta(t - \pi) = 0$  for  $t \geq 2\pi$ ).
- Use the Laplace transform to find a solution  $x$  describing the behavior of the system for  $t \geq 0$ .

*Solution.*

- We note that

$$f(t) = \delta(t - \pi) + (1 - \delta(t - \pi))\mathcal{U}(t - 2\pi) = \delta(t - \pi) + \mathcal{U}(t - 2\pi), \quad t \geq 0. \quad (1.62)$$

- Assuming  $x$  is a solution and  $X = \mathcal{L}\{x\}$ , we then have

$$(s^2 + 1)X(s) = e^{-\pi s} + \frac{e^{-2\pi s}}{s} \quad (1.63)$$

for appropriate values of  $s$ . Then

$$X(s) = e^{-\pi s} \frac{1}{s^2 + 1} + e^{-2\pi s} \frac{1}{s(s^2 + 1)}. \quad (1.64)$$

We note that by using partial fraction decomposition, we should have

$$\frac{1}{s(s^2 + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1}, \quad s \neq 0 \quad (1.65)$$

or

$$1 = A(s^2 + 1) + (Bs + C)s, \quad s \in \mathbb{R}. \quad (1.66)$$

If  $s = 0$ , then  $A = 1$ . If  $s = 1$ , then  $B + C = -1$  and if  $s = -1$ , then  $B - C = -1$ . This implies  $B = -1$  and  $C = 0$ . Thus

$$\frac{1}{s(s^2 + 1)} = \frac{1}{s} - \frac{s}{s^2 + 1}, \quad s \neq 0. \quad (1.67)$$

Thus

$$X(s) = e^{-\pi s} \frac{1}{s^2 + 1} + e^{-2\pi s} \left( \frac{1}{s} - \frac{s}{s^2 + 1} \right) \quad (1.68)$$

for appropriate values of  $s$ . By taking the inverse Laplace transform, we find that

$$x(t) = \mathcal{U}(t - \pi) \sin t + \mathcal{U}(t - 2\pi)(1 - \cos(t - 2\pi)) = \mathcal{U}(t - \pi) \sin t + \mathcal{U}(t - 2\pi)(1 - \cos t), \quad t \geq 0. \quad (1.69)$$

□