Homework 6

DUE: SATURDAY, MARCH 1, 11:59PM

If you completed this assignment through collaboration or consulted references, please list the names of your collaborators and the references you used below. Please refer to the syllabus for the course policy on collaborations and types of references that are allowed.

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Problem 6.1 (Variation of parameters)**.** Let P, Q, f be continuous function over an interval $I \subseteq \mathbb{R}$, and consider the 2nd order inhomogeneous equation

$$y''(x) + P(x)y'(x) + Q(x)y(x) = f(x),\ x \in I. \tag{6.1}$$

Suppose that the general homogeneous solution is given by

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x),\ x \in I, \tag{6.2}$$

where y_1, y_2 are two linearly independent twice-differential functions with continuous first and second derivatives, and c_1, c_2 are arbitrary. Define the Wronskian $W(y_1, y_2)$ via

$$W(y_1,y_2)(x) = \det \begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix} = y_1(x)y_2'(x) - y_2(x)y_1'(x),\ x \in I. \tag{6.3}$$

Recall that we have shown on Homework 3 that W never vanishes on I since y_1, y_2 are two linearly independent homogeneous solutions.

a) Verify that the function y_p defined via

$$y_p(x) = -y_1(x) \int \frac{y_2(x)f(x)}{W(x)}\, dx + y_2(x) \int \frac{y_1(x)f(x)}{W(x)}\, dx,\ x \in I \tag{6.4}$$

is a particular solution to the inhomogeneous equation (6.1).

b) Find a particular solution to the variable coefficient equation

$$x^2 y''(x) - 4xy'(x) + 6y(x) = x^3,\ x > 0 \tag{6.5}$$

given that the general homogeneous solution is

$$y_h(x) = c_1 x^2 + c_2 x^3,\ x > 0, \tag{6.6}$$

where c_1, c_2 are arbitrary.

Solution. If

$$y_p(x) = -y_1(x) \int \frac{y_2(x)f(x)}{W(x)}\, dx + y_2(x) \int \frac{y_1(x)f(x)}{W(x)}\, dx,\ x \in I, \tag{6.7}$$

then

$$y_p'(x) = -y_1'(x) \int \frac{y_2(x)f(x)}{W(x)}\, dx + y_2'(x) \int \frac{y_1(x)f(x)}{W(x)}\, dx \underbrace{- \frac{y_1(x)y_2(x)f(x)}{W(x)} + \frac{y_2(x)y_1(x)f(x)}{W(x)}}_{=0} \tag{6.8}$$

$$y_p''(x) = -y_1''(x) \int \frac{y_2(x)f(x)}{W(x)}\, dx + y_2''(x) \int \frac{y_1(x)f(x)}{W(x)}\, dx + \frac{(-y_1'(x)y_2(x) + y_1(x)y_2(x))f(x)}{\underbrace{W(x)}_{y_1(x)y_2'(x) - y_1'(x)y_2(x)}} \tag{6.9}$$

$$= -y_1''(x) \int \frac{y_2(x)f(x)}{W(x)}\, dx + y_2''(x) \int \frac{y_1(x)f(x)}{W(x)}\, dx + f(x) \tag{6.10}$$

for $x \in I$. Thus

$$\begin{aligned} y_p''(x) + P(x)y_p'(x) + Q(x)y_p(x) &= -(\underbrace{y_1''(x) + P(x)y_1'(x) + Q(x)y_1(x)}_{=0}) \int \frac{y_2(x)f(x)}{W(x)}\, dx \\ &\quad + (\underbrace{y_2''(x) + P(x)y_2'(x) + Q(x)y_2(x)}_{=0}) \int \frac{y_1(x)f(x)}{W(x)}\, dx + f(x) = f(x) \end{aligned} \tag{6.11}$$

for all $x \in I$. Therefore y_p is a particular solution to the equation. For part b), we apply the variation of parameters formula with $y_1(x) = x^2, y_2(x) = x^3, f(x) = \frac{x^3}{x^2} = x$ for $x > 0$ and calculate

$$W(y_1,y_2)(x) = \det \begin{pmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{pmatrix} = 3x^4 - 2x^4 = x^4,\ x > 0 \tag{6.12}$$

and

$$y_p(x) = -x^2 \int \frac{x^3 \cdot x}{x^4}\, dx + x^3 \int \frac{x^2 \cdot x}{x^4}\, dx = -x^3 + x^3 \ln x,\ x > 0. \tag{6.13}$$

Note that we may ignore the constants of integration as it'll simply contribute to adding a constant multiple of y_1 or y_2 to the particular solution above. $\square$

Problem 6.2. Consider a forced mass-spring system modeled via the system

$$\begin{cases} x''(t) + 4x'(t) + 5x(t) = 4\cos(t) - 4\sin(t), \ t \in \mathbb{R} \\ x(0) = 1, x'(0) = 1. \end{cases} \tag{6.14}$$

a) Find the unique solution satisfying the initial value problem.
b) Identify the steady-state solution and the transient solution.
c) Describe the long term behavior of the system.
d) What long-term difference would you notice if the initial conditions were different?

Solution.

a) The characteristic equation associated to the ODE is

$$r^2 + 4r + 5 = 0 \implies r = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i. \tag{6.15}$$

Therefore the general homogeneous solution is

$$x_h(t) = e^{-2t}(c_1 \cos t + c_2 \sin t), \ t \in \mathbb{R}, \tag{6.16}$$

where c_1, c_2 are arbitrary. To find a particular solution, we use the ansatz $x_p(t) = A\cos t + B\sin t$ for all $t \in \mathbb{R}$. We then calculate

$$x_p'(t) = B\cos t - A\sin t \tag{6.17}$$

$$x_p''(t) = -A\cos t - B\sin t, \ t \in \mathbb{R}. \tag{6.18}$$

Thus

$$x_p''(t) + 4x_p'(t) + 5x_p(t) = (4A + 4B)\cos t + (4B - 4A)\sin t = 4\cos t - 4\sin t, \tag{6.19}$$

therefore $B = 0, A = 1$. Therefore the general solution to the equation is

$$x(t) = e^{-2t}(c_1 \cos t + c_2 \sin t) + \cos t, \ t \in \mathbb{R}, \tag{6.20}$$

where c_1, c_2 are arbitrary. Then

$$x'(t) = e^{-2t}((-2c_1 + c_2)\cos t + (-2c_2 - c_1)\sin t) - \sin t, \ t \in \mathbb{R}. \tag{6.21}$$

Therefore if $x(0) = 1, x'(0) = 1$, then

$$1 = x(0) = c_1 + 1 \implies c_1 = 0 \tag{6.22}$$

$$1 = x'(0) = c_2. \tag{6.23}$$

Thus the unique solution to the IVP is

$$x(t) = e^{-2t}\sin t + \cos t, \ t \in \mathbb{R}. \tag{6.24}$$

b) The transient solution here is the homogeneous solution $x_h(t) = e^{-2t}\sin t$ and the steady-state solution is the particular solution $x_p(t) = \cos t$.
c) For large time, the contribution from the transient solution will be negligible, therefore the long term behavior of the system is dominated by the particular solution. In this case, we see that the system will behave almost like a simple harmonic oscillator.
d) Since the initial conditions only affect the homogeneous solution, and in this case the homogeneous solution is the transient solution, a change in the initial conditions will have a negligible effect on the long term dynamics of the system. In some sense, after a long period of time the system "forgets" the initial conditions, though rigorously speaking the initial conditions still affects the error term $x - x_p$ in the solution, but the magnitude of the error term is negligible.

□

Problem 6.3. Determine if the following boundary value problems admits a unique solution, infinitely many solutions or no solutions.

a)

$$\begin{cases} y''(x) + y(x) = 0, \ x \in (0, \pi) \\ y(0) = 0, \ y(\pi) = 1 \end{cases} \tag{6.25}$$

b)

$$\begin{cases} y''(x) + 3y(x) = 0, \ x \in (0, \pi) \\ y(0) = 0, \ y(\pi) = 0 \end{cases} \tag{6.26}$$

c)

$$\begin{cases} y''(x) + 4y(x) = 0, \ x \in (0, \pi) \\ y(0) = 0, \ y(\pi) = 0 \end{cases} \tag{6.27}$$

Solution.

a) We note that the general solution to the equation is

$$y(x) = c_1 \cos x + c_2 \sin x, \ x \in (0, \pi). \tag{6.28}$$

The boundary conditions requires

$$\begin{cases} 0 = y(0) = c_1 \\ 1 = y(\pi) = -c_1, \end{cases} \tag{6.29}$$

which is not possible. Therefore there are no solutions to this boundary value problem.

b) The general solution to the equation is

$$y(x) = c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x, \ x \in (0, \pi). \tag{6.30}$$

The boundary conditions require

$$\begin{cases} 0 = y(0) = c_1 \\ 0 = y(\pi) = c_2 \sin \sqrt{3}\pi \implies c_2 = 0. \end{cases} \tag{6.31}$$

Therefore the boundary value problem only admits the zero solution

$$y(x) = 0, \ x \in [0, \pi]. \tag{6.32}$$

c) The general solution to the equation is

$$y(x) = c_1 \cos 2x + c_2 \sin 2x, \ x \in (0, \pi). \tag{6.33}$$

The boundary conditions require

$$\begin{cases} 0 = y(0) = c_1 \\ 0 = y(\pi) = c_1 \end{cases} \tag{6.34}$$

Since there are no restrictions on c_2, the boundary value problem admits infinitely solutions of the form

$$y(x) = c_2 \sin 2x, \ x \in [0, \pi], \tag{6.35}$$

where $c_2 \in \mathbb{R}$ is arbitrary.

$\square$

Problem 6.4. Let $\mathbb{R} \ni L > 0$. Determine the eigenvalues and the eigenfunctions of the boundary value problem

$$\begin{cases} y''(x) + \lambda y(x) = 0, \ x \in (0, L) \\ y(0) = 0, \ y'(L) = 0. \end{cases} \tag{6.36}$$

Solution. If $\lambda = 0$, then the general solution to the equation is

$$y(x) = c_1 + c_2 x, \ x \in (0, L). \tag{6.37}$$

Thus

$$y'(x) = c_2, \quad x \in (0, L). \tag{6.38}$$

The boundary conditions require

$$0 = y(0) = c_1 \tag{6.39}$$

$$0 = y'(L) = c_2, \tag{6.40}$$

therefore $\lambda = 0$ is not an eigenvalue. If $\lambda < 0$, then the general solution is

$$y(x) = c_1 e^{-\sqrt{-\lambda}x} + c_2 e^{\sqrt{-\lambda}x}, \ x \in (0, L). \tag{6.41}$$

Thus

$$y'(x) = -c_1\sqrt{-\lambda}e^{-\sqrt{-\lambda}x} + c_2\sqrt{-\lambda}e^{\sqrt{-\lambda}x}, \ x \in (0, L). \tag{6.42}$$

The boundary conditions require

$$y(0) = c_1 + c_2 = 0 \tag{6.43}$$

$$y'(L) = -c_1\sqrt{-\lambda} + c_2\sqrt{-\lambda} = 0. \tag{6.44}$$

By routine algebra we find that $c_1 = c_2 = 0$, therefore the problem does not admit any negative eigenvalues. If $\lambda > 0$, then the general solution is

$$y(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x, \ x \in (0, L). \tag{6.45}$$

Thus

$$y'(x) = c_2\sqrt{\lambda}\cos\sqrt{\lambda}x - c_1\sqrt{\lambda}\sin\sqrt{\lambda}x, \ x \in (0, L). \tag{6.46}$$

The boundary conditions require

$$y(0) = c_1 = 0 \tag{6.47}$$

$$y'(L) = c_2\sqrt{\lambda}\cos\sqrt{\lambda}L = 0. \tag{6.48}$$

Since we want to identify non-zero solutions, we may assume that $c_2 \neq 0$, and therefore this implies that we must have

$$\cos\sqrt{\lambda}L \implies \sqrt{\lambda}L = \frac{(2n-1)\pi}{2}, \ n \in \mathbb{Z}. \tag{6.49}$$

Therefore we may parameterize λ with positive integers and write

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4L^2}, \ n \in \mathbb{Z}^+. \tag{6.50}$$

Therefore the eigenvalues of the problem are

$$\lambda_1 = \frac{\pi^2}{4L^2}, \ \lambda_2 = \frac{9\pi^2}{4L^2}, \ \lambda_3 = \frac{25\pi^2}{4L^2}, \ldots \tag{6.51}$$

with corresponding eigenfunctions

$$y_1(x) = \sin\frac{\pi}{2L}x, \ y_2(x) = \sin\frac{3\pi}{2L}x, \ y_3(x) = \sin\frac{5\pi}{2L}x, \ldots \tag{6.52}$$

for $x \in [0, \pi]$. □

Problem 6.5 (Laplace transform of $\cos kt$)**.** Let $k \in \mathbb{R} \setminus \{0\}$ and consider the function $f : [0, \infty) \to \mathbb{R}$ defined via $f(t) = \cos kt$. Consider the function

$$F(s) = \mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} \cos kt \; dt \tag{6.53}$$

defined for all values of $s > 0$ for which the improper integral converges.

a) Use the fact that $|\cos kt| \leq 1$ for all $t \in \mathbb{R}$ to show that

$$|F(s)| \leq \int_0^\infty e^{-st} \; dt \tag{6.54}$$

for all $s > 0$. You may use the fact that

$$\left| \int_0^\infty h(t) \; dt \right| \leq \int_0^\infty |h(t)| \; dt \tag{6.55}$$

for any continuous function $h : [0, \infty) \to \mathbb{R}$.

b) Show via direct computation that

$$\int_0^\infty e^{-st} < \infty \tag{6.56}$$

for all $s > 0$. Use parts a) and b) to conclude that F is well-defined for all $s > 0$.

c) Use integration by parts to derive the identity

$$F(s) = \frac{1}{s} - \frac{k^2}{s^2} F(s) \tag{6.57}$$

for all $s > 0$.

d) Conclude that

$$F(s) = \frac{s}{s^2 + k^2}, \; s > 0. \tag{6.58}$$

Solution.

a) We note that for all $s > 0$,

$$|F(s)| \leq \int_0^\infty \left| e^{-st} \cos kt \right| \; dt \leq \int_0^\infty \left| e^{-st} \right| |\cos kt| \; dt \leq \int_0^\infty e^{-st} \; dt. \tag{6.59}$$

b) For any $s > 0$,

$$\int_0^\infty e^{-st} \; dt = \lim_{\alpha \to \infty} \left. \frac{e^{-st}}{-s} \right|_{t=0}^{t=\alpha} = \lim_{\alpha \to \infty} \frac{1}{s} - \frac{e^{-\alpha s}}{s} = \frac{1}{s} < \infty. \tag{6.60}$$

Therefore for any $s > 0$,

$$|F(s)| < \infty. \tag{6.61}$$

This implies that F is well-defined for all $s > 0$.

c) For all $s > 0$, we have

$$\begin{aligned} F(s) &= \lim_{\alpha \to \infty} \int_0^\alpha e^{-st} \cos kt \; dt = \lim_{\alpha \to \infty} \left. \frac{e^{-st} \cos kt}{-s} \right|_{t=0}^{t=\alpha} - \frac{k}{s} \int_0^\alpha e^{-st} \sin kt \; dt \\ &= \frac{1}{s} - \frac{k}{s} \lim_{\alpha \to \infty} \int_0^\alpha e^{-st} \sin kt \; dt = \frac{1}{s} - \frac{k}{s} \left(\lim_{\alpha \to \infty} \left. \frac{e^{-st} \sin kt}{-s} \right|_{t=0}^{t=\alpha} + \frac{k}{s} \int_0^\infty e^{-st} \cos kt \; dt \right) \\ &= \frac{1}{s} - \frac{k^2}{s^2} F(s). \end{aligned} \tag{6.62}$$

Therefore

$$\left(1 + \frac{k^2}{s^2}\right) F(s) = \frac{1}{s} \implies F(s) = \frac{s}{s^2 + k^2}, \; s > 0. \tag{6.63}$$

□