Homework 6

DUE: SATURDAY, MARCH 1, 11:59PM

If you completed this assignment through collaboration or consulted references, please list the names of your collaborators and the references you used below. Please refer to the syllabus for the course policy on collaborations and types of references that are allowed.

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Problem 6.1 (Variation of parameters)**.** Let P, Q, f be continuous function over an interval $I \subseteq \mathbb{R}$, and consider the 2nd order inhomogeneous equation

$$y''(x) + P(x)y'(x) + Q(x)y(x) = f(x),\ x \in I. \tag{6.1}$$

Suppose that the general homogeneous solution is given by

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x),\ x \in I, \tag{6.2}$$

where y_1, y_2 are two linearly independent twice-differential functions with continuous first and second derivatives, and c_1, c_2 are arbitrary. Define the Wronskian $W(y_1, y_2)$ via

$$W(y_1, y_2)(x) = \det \begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix} = y_1(x)y_2'(x) - y_2(x)y_1'(x),\ x \in I. \tag{6.3}$$

Recall that we have shown on Homework 3 that W never vanishes on I since y_1, y_2 are two linearly independent homogeneous solutions.

a) Verify that the function y_p defined via

$$y_p(x) = -y_1(x) \int \frac{y_2(x)f(x)}{W(x)}\, dx + y_2(x) \int \frac{y_1(x)f(x)}{W(x)}\, dx,\ x \in I \tag{6.4}$$

is a particular solution to the inhomogeneous equation (6.1).

b) Find a particular solution to the variable coefficient equation

$$x^2 y''(x) - 4xy'(x) + 6y(x) = x^3,\ x > 0 \tag{6.5}$$

given that the general homogeneous solution is

$$y_h(x) = c_1 x^2 + c_2 x^3,\ x > 0, \tag{6.6}$$

where c_1, c_2 are arbitrary.

Problem 6.2. Consider a forced mass-spring system modeled via the system

$$\begin{cases} x''(t) + 4x'(t) + 5x(t) = 4\cos(t) - 4\sin(t), \ t \in \mathbb{R} \\ x(0) = 1, x'(0) = 1. \end{cases} \tag{6.7}$$

a) Find the unique solution satisfying the initial value problem.
b) Identify the steady-state solution and the transient solution.
c) Describe the long term behavior of the system.
d) What long-term difference would you notice if the initial conditions were different?

Problem 6.3. Determine if the following boundary value problems admits a unique solution, infinitely many solutions or no solutions.

a)

$$\begin{cases} y''(x) + y(x) = 0, \ x \in (0, \pi) \\ y(0) = 0, \ y(\pi) = 1 \end{cases} \tag{6.8}$$

b)

$$\begin{cases} y''(x) + 3y(x) = 0, \ x \in (0, \pi) \\ y(0) = 0, \ y(\pi) = 0 \end{cases} \tag{6.9}$$

c)

$$\begin{cases} y''(x) + 4y(x) = 0, \ x \in (0, \pi) \\ y(0) = 0, \ y(\pi) = 0 \end{cases} \tag{6.10}$$

Problem 6.4. Let $\mathbb{R} \ni L > 0$. Determine the eigenvalues and the eigenfunctions of the boundary value problem

$$\begin{cases} y''(x) + \lambda y(x) = 0, \ x \in (0, L) \\ y(0) = 0, \ y'(L) = 0. \end{cases} \tag{6.11}$$

Problem 6.5 (Laplace transform of $\cos kt$)**.** Let $k \in \mathbb{R} \setminus \{0\}$ and consider the function $f : [0, \infty) \to \mathbb{R}$ defined via $f(t) = \cos kt$. Consider the function

$$F(s) = \mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} \cos kt \, dt \tag{6.12}$$

defined for all values of $s > 0$ for which the improper integral converges.

a) Use the fact that $|\cos kt| \leq 1$ for all $t \in \mathbb{R}$ to show that

$$|F(s)| \leq \int_0^\infty e^{-st} \, dt \tag{6.13}$$

for all $s > 0$. You may use the fact that

$$\left| \int_0^\infty h(t) \, dt \right| \leq \int_0^\infty |h(t)| \, dt \tag{6.14}$$

for any continuous function $h : [0, \infty) \to \mathbb{R}$.

b) Show via direct computation that

$$\int_0^\infty e^{-st} < \infty \tag{6.15}$$

for all $s > 0$. Use parts a) and b) to conclude that F is well-defined for all $s > 0$.

c) Use integration by parts to derive the identity

$$F(s) = \frac{1}{s} - \frac{k^2}{s^2} F(s) \tag{6.16}$$

for all $s > 0$.

d) Conclude that

$$F(s) = \frac{s}{s^2 + k^2}, \; s > 0. \tag{6.17}$$