Additional final practice problems

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Problem 1.1. Consider the initial value problem

$$\begin{cases} e^t y'(t) + 2e^t y(t) = 3e^{2t}, \ t \in \mathbb{R} \\ y(0) = 0. \end{cases} \tag{1.1}$$

a) Classify the equation by order.

   Solution. This is a first order equation. □

b) Classify the equation by linearity. Is it linear or nonlinear?

   Solution. This is a linear equation. □

c) Use an appropriate method to find a solution to the initial value problem. You may skip the verification step.

   Solution. If y is a solution to the IVP, then

$$y'(t) + 2y(t) = 3e^t, \ t \in \mathbb{R}. \tag{1.2}$$

   We can choose $\mu(t) = e^{2t}, t \in \mathbb{R}$ to be an integrating factor. Therefore

$$\frac{d}{dt}\left[e^{2t}y(t)\right] = 3e^{3t}, \ t \in \mathbb{R} \tag{1.3}$$

   implying

$$e^{2t}y(t) = e^{3t} + C \implies y(t) = e^t + Ce^{-2t}, \ t \in \mathbb{R} \tag{1.4}$$

   and C is arbitrary. Since $y(0) = 0$, we must have $0 = 1 + C \implies C = -1$. Therefore a solution to the IVP is $y(t) = e^t - e^{-2t}, t \in \mathbb{R}$.

   □

d) What is the maximal interval of existence of the solution?

   Solution. The solution exists globally for all $t \in \mathbb{R}$, therefore the maximal interval of existence is $\mathbb{R}$. □

e) Is the solution identified in part c) unique?

   Solution. Yes. Since $e^t, 2e^t, 3e^{2t}$ are all continuous over $\mathbb{R}$ and $e^t \neq 0$ for all $t \in \mathbb{R}$, by the existence and uniqueness theorem for first order linear equations, the solution identified in part c) must be the unique global solution on $\mathbb{R}$. □

Problem 1.2. Consider the differential equation

$$y'(t) = 2t - y(t), \; t \in \mathbb{R}. \tag{1.5}$$

a) Verify that the function

$$y(t) = 2(t-1), \; t \in \mathbb{R} \tag{1.6}$$

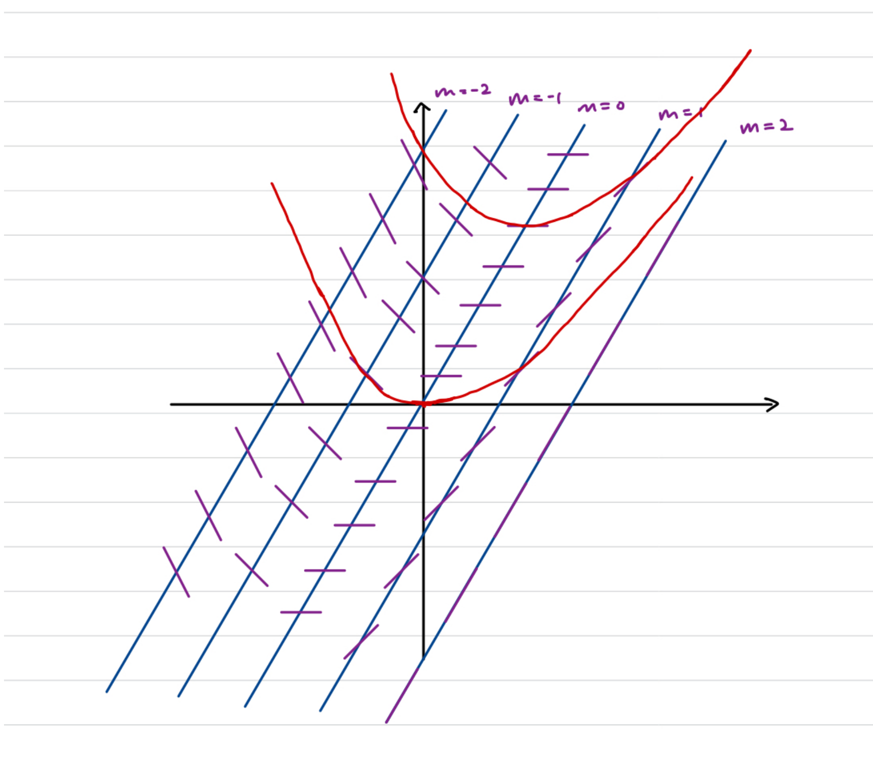
is a solution on the interval $J = \mathbb{R}$. Please don't present your work backwards, instead calculate y' and the right-hand side of the equation for the given y separately and conclude that they are equal to each other.

Solution. If y is defined via (1.6), then $y'(t) = 2$ for all $t \in \mathbb{R}$ and $2t - y(t) = 2t - 2(t-1) = 2t - 2t + 2 = 2$ for all $t \in \mathbb{R}$. Thus y is a solution to the differential equation. □

b) Sketch the directional field associated to the equation by identifying the isoclines corresponding to $m = 0, \pm 1, \pm 2$. You're welcome to add more isoclines to the sketch to improve the accuracy of the sketch.

c) Sketch the solution curve y_1 passing through $(0,0)$ and the solution curve y_2 passing through $(0,2)$ on top of the directional field you sketched in part b). Please make sure that your solution curves match the underlying directional field, and also you sketch them for enough t's so that the global behavior of the solution curves are easy to visualize.

Solution. Note that $2t - y = m \iff y = 2t - m$, so the isoclines are straight lines with slope 2. Here is a very rough sketch for parts c) and d). Note that the line $y(t) = 2(t-1)$ is both an isocline and a solution curve, so no other solution curve can cross it.



□

d) Is it possible for the solutions curves y_1 and y_2 from part c) to ever cross? Please explain your reasoning and justify this part rigorously.

Solution. No. We note that the equation can be written as $y'(t) + y(t) = 2t$ for all $t \in \mathbb{R}$. Since the constant function that takes the value 1 for all $t \in \mathbb{R}$ and $2t$ are continuous over $\mathbb{R}$, we have uniqueness globally for all time t. Therefore by uniqueness, solution curves cannot cross. □

e) Is it possible for $y_1(2) \le 2$? Please explain your reasoning and justify this part rigorously.

Solution. Note that the straight line solution y identified in part a) satisfies $y(2) = 2$. Since y_1 started at the point $(0, 0)$, which is to the left of y, if $y_1(2) \leq 2$ this means that y_1 must cross y, therefore by uniqueness this is not possible. $\square$

Problem 1.3. Consider the initial value problem

$$\begin{cases} y'(x) = 6x(y(x)-1)^{2/3}, \ x \in \mathbb{R} \\ y(0) = 1. \end{cases} \tag{1.7}$$

a) Verify that the function y_1 defined via

$$y_1(x) = 1, \ x \in \mathbb{R} \tag{1.8}$$

is a constant solution to the initial value problem on the interval $J = \mathbb{R}$. As with all verification problems, please do not present you work backwards. Also, please do not forget to check the initial condition.

Solution. First we note that $y_1(0) = 1$, so it satisfies the initial condition. Also, $y_1'(x) = 0$ for all $x \in \mathbb{R}$ and $6x(y_1(x)-1)^{2/3} = 6x(1-1)^{2/3} = 0$ for all $x \in \mathbb{R}$. Therefore y_1 is a solution to the IVP on $\mathbb{R}$. ☐

b) Verify that the function y_2 defined via

$$y_2(x) = 1 + x^6, \ x \in \mathbb{R} \tag{1.9}$$

is a solution to the initial value problem on the interval $J = \mathbb{R}$. As with all verification problems, please do not present you work backwards. Also, please do not forget to check the initial condition.

Solution. First we note that $y_2(0) = 1 + 0^6 = 1$, so it satisfies the initial condition. Also, $y_2'(x) = 6x^5$ and $6x(y_2(x)-1)^{2/3} = 6x(x^6)^{2/3} = 6x(x^4) = 6x^5$ for all $x \in \mathbb{R}$, therefore y_2 is a solution to the IVP on $\mathbb{R}$. ☐

c) Parts b) and c) show that solutions to the given initial value problem are not unique. Explain why this does not violate the conclusions of the existence and uniqueness theorem.

Solution. We note that if we define the function $f(x,y) = 6x(y-1)^{2/3}, (x,y) \in \mathbb{R}^2$, we have $\frac{\partial f}{\partial y}(x,y) = 4x(y-1)^{-1/3}, (x,y) \in \mathbb{R}^2$ and $y \neq 1$. Since the initial condition is specified at $y_0 = 1$ and $\frac{\partial f}{\partial y}$ is undefined there, the existence and uniqueness theorem cannot be applied. ☐

d) Classify all points $(t_0, y_0) \in \mathbb{R}^2$ for which if $y(t_0) = y_0$ is the specified initial condition (instead of $y(0) = 1$), the existence and uniqueness of solutions is guaranteed.

Solution. By the calculations in the previous part f and $\frac{\partial f}{\partial y}$ is continuous in a small rectangle around any point (t_0, y_0) for which $y_0 \neq 1$, and these are the points where the existence and uniqueness of solutions is guaranteed. ☐

e) Find the unique solution to the initial value problem

$$\begin{cases} y'(x) = 6x(y(x)-1)^{2/3}, \ x \in \mathbb{R} \\ y(0) = 2. \end{cases} \tag{1.10}$$

You may skip the verification step. What is the maximal interval of existence of the solution? Hint: the antiderivative of $u^{-2/3}$ is $3u^{1/3}$.

Solution. We note that this is a separable equation and we may rewrite the equation as

$$\frac{y'(x)}{(y(x)-1)^{2/3}} = 6x, \ x \in I \tag{1.11}$$

over some interval I. Then

$$3(y(x)-1)^{1/3} = 3x^2 + C \implies y(x) = 1 + \left(x^2 + C\right)^3, \ x \in I \tag{1.12}$$

and C is arbitrary. Since $y(0) = 2$, we see that $C^3 = 1 \implies C = 1$. So the unique solution is $y(x) = 1 + \left(x^2+1\right)^3$ over the interval $J = \mathbb{R}$. ☐

Problem 1.4. Consider the differential equation

$$xy'(x) + 6y(x) = 3x(y(x))^{4/3},\ x \in \mathbb{R}. \tag{1.13}$$

a) Classify the equation by linearity. Is it linear or nonlinear? No justification required.

Solution. This is a nonlinear equation. □

b) Does the equation admit any constant solutions?

Solution. Yes, we can write the equation as $xy'(x) = y(x)(3x(y(x))^{1/3} - 6)$, so we see that $y(x) = 0$ for all $x \in \mathbb{R}$ is a constant solution. □

c) Use an appropriate method to find the solution to the initial value problem

$$\begin{cases} xy'(x) + 6y(x) = 3x(y(x))^{4/3},\ x \in \mathbb{R}, \\ y(1) = -1. \end{cases} \tag{1.14}$$

You may skip the verification step.

Solution. This is a Bernoulli equation, and we use the substitution $v(x) = (y(x))^{-1/3}$ over some interval I. Note that $v'(x) = (-1/3)(y(x))^{-4/3}y'(x)$ over I. Following the homework problem, we divide both sides of the equation by $(y(x))^{4/3}$ and arrive at

$$xy'(x)(y(x))^{-4/3} + 6(y(x))^{-1/3} = 3x,\ x \in I. \tag{1.15}$$

This implies that

$$-3xv'(x) + 6v(x) = 3x,\ x \in I. \tag{1.16}$$

Consider $J = I \cap (0, \infty)$. Then

$$v'(x) - \frac{2}{x}v(x) = -1,\ x \in J. \tag{1.17}$$

We may choose an integrating factor to be $\mu(x) = \exp(-2\int \frac{1}{x}\, dx) = \exp \ln |x|^{-2} = x^{-2}, x \in J$. Then

$$\frac{d}{dx}\left[\frac{1}{x^2}v(x)\right] = -\frac{1}{x^2},\ x \in J. \tag{1.18}$$

Thus

$$\frac{1}{x^2}v(x) = \frac{1}{x} + C \implies v(x) = x + Cx^2,\ x \in J. \tag{1.19}$$

Since $v(1) = (y(1))^{-1/3} = (-1)^{-1/3} = -1$, we see that $1 + C = -1 \implies C = -2$. Thus the solution to the initial value problem is

$$y(x) = \left(x - 2x^2\right)^{-3},\ x \in J, \tag{1.20}$$

for some interval J. □

d) What is the maximal interval of existence for the solution in part c)?

Solution. We note that we need $x - 2x^2 = x(1 - 2x) \neq 0 \iff x \neq 0, \frac{1}{2}$ for the solution identified in the previous part to be well-defined. Therefore the maximal interval of existence is $J = (\frac{1}{2}, \infty)$. □

e) What happens if we change the initial condition to $y(1) = 0$? Does a solution exist and is it unique?

Solution. We saw in part b) $y(x) = 0$ for all $x \in \mathbb{R}$ is a constant solution, which satisfies the initial condition $y(1) = 0$. So a solution exists. If we define

$$f(x, y) = \frac{1}{x}\left(3xy^{4/3} - 6y\right),\ (x, y) \in \mathbb{R}^2, \tag{1.21}$$

the equation is in the form of $y'(x) = f(x, y(x))$ and

$$\frac{\partial f}{\partial y}(x, y) = \frac{1}{x}\left(4xy^{1/3} - 6\right),\ (x, y) \in \mathbb{R}^2, \tag{1.22}$$

and we see that $f, \frac{\partial f}{\partial y}$ are both continuous over $\mathbb{R}^2$ except at $x = 0$, but the initial value $y(1) = -1$ is specified away from that point. Thus the constant solution $y(x) = 0$ is the unique solution to the IVP. □

Problem 1.5. Suppose a constant coefficient linear differential equation admits the general solution

$$y(x) = c_1 e^x \cos x + c_2 e^x \sin x,\ x \in \mathbb{R} \tag{1.23}$$

where c_1, c_2 are arbitrary.

a) What are the roots of the characteristic equation associated to the differential equation?

Solution. The roots of the characteristic equation are $r_1 = 1 - i, r_2 = 1 + i$. □

b) Find a constant coefficient differential equation that admits this general solution.

Solution. The characteristic equation is

$$(r - (1 - i))(r - (1 + i)) = ((r - 1) + i)((r - 1) - i) = (r - 1)^2 - i^2 = r^2 - 2r + 1 + 1 = r^2 - 2r + 2,\ r \in \mathbb{R}. \tag{1.24}$$

Therefore an equation that admits this general solution is

$$y''(x) - 2y'(x) + 2y(x) = 0,\ x \in \mathbb{R}. \tag{1.25}$$

□

Problem 1.6. Consider the variable coefficient initial value problem

$$\begin{cases} 5y''(x) + 12xy'(x) + 25x^2y(x) = 0, \ x \in \mathbb{R} \\ y(1) = 0 \\ y'(1) = 0. \end{cases} \tag{1.26}$$

a) Find a solution to the initial value problem over the interval $I = \mathbb{R}$.

   Solution. Note that the constant solution $y : \mathbb{R} to \mathbb{R}$ defined via $y(x) = 0$ satisfies the equation and the intiial conditions, therefore it is a solution to the IVP on $I = \mathbb{R}$. □

b) Justify carefully and rigorously why the solution you found in part a) is the only solution to the initial value problem over the interval $I = \mathbb{R}$.

   Solution. Note that the coefficients $a_2, a_1, a_0 : \mathbb{R} \to \mathbb{R}$ defined via $a_2(x) = 5, a_1(x) = 12x, a_0(x) = 25x^2$ are all continuous on $\mathbb{R}$ and $a_2(x) \neq 0$ for all $x \in \mathbb{R}$, therefore by the existence and unique theorem for linear equations, there exists a unique solution to the IVP on $I = \mathbb{R}$. By the uniqueness part of the theorem, the solution we identified in part a) is therefore the only solution to the IVP. □

Problem 1.7. Consider the mass-spring system modeled via the homogeneous linear differential equation

$$x''(t) + \gamma x'(t) + 4x(t) = 0,\ t \in \mathbb{R}. \tag{1.27}$$

a) Find value(s) of γ for which the system is critically damped.

Solution. For the equation to be critically damped we require

$$\gamma^2 - 16 = 0 \implies \gamma = 4. \tag{1.28}$$

We need to exclude the case that $\gamma = -4$ because in a damped mass-spring system, the damping constant is assumed to be positive. □

b) Find the largest sub-interval I of $(0, \infty)$ such that if $\gamma \in I$, then the system is overdamped.

Solution. For the system to be overdamped we require

$$\gamma^2 - 16 = (\gamma - 4)(\gamma + 4) > 16. \tag{1.29}$$

This is equivalent to requiring $\gamma > 4$, so the largest sub-interval is $(4, \infty)$. □

c) Suppose $\gamma = 2$. What is the quasi-period T of the solution?

Solution. Note that the general solution to the equation is

$$x(t) = c_1 e^{-t} \cos \sqrt{3}t + c_2 e^{-t} \sin \sqrt{3}t,\ t \in \mathbb{R}, \tag{1.30}$$

where c_1, c_2 are arbitrary. Therefore the quasi-period is

$$T = \frac{2\pi}{\sqrt{3}}. \tag{1.31}$$

□

d) Suppose $\gamma = 4$, and an external force is present in the system and the forced damped mass-spring system is modeled via

$$x''(t) + 4x'(t) + 4x(t) = 32e^{2t},\ t \in \mathbb{R}. \tag{1.32}$$

Find the general solution to the system.

Solution. We note that when $\gamma = 4$, the general homogeneous solution is

$$x_h(t) = c_1 e^{-2t} + c_2 t e^{-2t},\ t \in \mathbb{R}. \tag{1.33}$$

Therefore to recover the particular solution we may use the ansatz

$$x_p(t) = Ae^{2t},\ t \in \mathbb{R} \tag{1.34}$$

and calculate

$$x_p'(t) = 2Ae^{2t},\ x_p''(t) = 4Ae^{2t},\ t \in \mathbb{R}. \tag{1.35}$$

Thus

$$x_p''(t) + 4x_p'(t) + 4x_p(t) = (4A + 8A + 4A)e^{2t} = 32e^{2t},\ t \in \mathbb{R} \implies A = 2. \tag{1.36}$$

Therefore the general solution to the equation is

$$x(t) = 2e^{2t} + c_1 e^{-2t} + c_2 t e^{-2t},\ t \in \mathbb{R}, \tag{1.37}$$

where c_1, c_2 are arbitrary. □

Problem 1.8. Consider the 2nd order differential equation

$$x^2 y''(x) + 3xy'(x) - 3y(x) = 0,\ x > 0. \tag{1.38}$$

You are given that y_1 defined via

$$y_1(x) = x, x > 0 \tag{1.39}$$

is a solution to the homogeneous equation. Use the method of reduction of order to find a second linearly independent solution y_2 to the equation over the interval $I = (0, \infty)$. You do not need to check the independence of y_1, y_2, nor verify that y_2 is a solution.

Solution. We use the ansatz $y_2(x) = u(x)y_1(x), x > 0$ and calculate

$$y_2'(x) = u'(x)y_1(x) + u(x)y_1'(x) \tag{1.40}$$

$$y_2''(x) = u''(x)y_1(x) + 2u'(x)y_1'(x) + u(x)y_1''(x),\ x > 0. \tag{1.41}$$

Thus if y_2 is a solution, we have

$$\begin{aligned} x^2 y_2''(x) + 3xy_2'(x) - 3y_2(x) &= u''(x)(x^2 y_1(x)) + u'(x)(2x^2 y_1'(x) + 3xy_1(x)) + u(x)\underbrace{(x^2 y_1''(x) + 3xy_1'(x) - 3y_1(x))}_{=0} \\ &= x^3 u''(x) + 5x^2 u'(x) = 0,\ x > 0. \end{aligned} \tag{1.42}$$

Thus $w = u'$ satisfies the first order linear equation

$$w'(x) + \frac{5}{x} w(x) = 0,\ x > 0. \tag{1.43}$$

An integrating factor for this equation is $\mu(x) = x^5, x > 0$. Therefore

$$\frac{d}{dx}\left[x^5 w(x)\right] = 0 \implies w(x) = \frac{C_1}{x^5} \implies u(x) = \frac{C}{x^4} + D. \tag{1.44}$$

Therefore by choosing $C = 1, D = 0$, we find that a second linearly independent solution is

$$y_2(x) = \frac{1}{x^3},\ x > 0. \tag{1.45}$$

□

Problem 1.9. Suppose a mass-spring system is modeled via

$$x''(t) + \beta x'(t) + 4x(t) = \cos \omega t,\ t \in \mathbb{R}. \tag{1.46}$$

where $\beta \geq 0, \omega > 0$.

a) Identify the parameters β, ω for which pure resonance occurs.
b) In the case of part a) where resonance occurs, use the method of undetermined coefficients to find a particular solution to the system.
c) Suppose $\beta > 0$ and $x(0) = x'(0) = 0$. Would a sizable change in the initial conditions, either in the initial position or the initial velocity, result in a sizable change in the behavior of the system in the long run? Please briefly explain why or why not.

Solution.

a) Pure resonance occurs when there is no damping, so $\beta = 0$, and the forcing frequency must be equal to the natural frequency in absolute value, so $\omega = \pm\sqrt{\frac{4}{1}} = \pm 2$.
b) We use the ansatz

$$x_p(t) = At\cos 2t + Bt\sin 2t,\ t \in \mathbb{R}, \tag{1.47}$$

and calculate for all $t \geq 0$,

$$x_p'(t) = (A + 2Bt)\cos 2t + (B - 2At)\sin 2t \tag{1.48}$$

$$x_p''(t) = (2B + 2B - 4At)\cos 2t + (-2A - 2A - 4Bt)\sin 2t. \tag{1.49}$$

Therefore

$$x_p''(t) + 4x_p(t) = (4B)\cos 2t + (-4A)\sin 2t = \cos 2t,\ t \in \mathbb{R}. \tag{1.50}$$

This implies $A = 0$ and $B = \frac{1}{4}$, therefore a particular solution to the system is

$$x_p(t) = \frac{1}{4}t\sin 2t,\ t \in \mathbb{R}. \tag{1.51}$$

c) No, because the initial conditions only affect the homogeneous solution, which decays exponentially with positive damping and becomes negligible in the long run. As a result, the solution converges at a steady-state solution which is the particular solution, and the particular solution is not affected by the initial conditions.

□

Problem 1.10. Suppose the general homogeneous solution to the variable coefficient equation

$$x^2 y''(x) + xy'(x) - y(x) = 1, \ x > 0 \tag{1.52}$$

is

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 x + c_2 x^{-1}, \ x > 0, \tag{1.53}$$

where c_1, c_2 are arbitrary.

a) Find the Wronskian $W(y_1, y_2)$ defined for $x > 0$.
b) Use the variation of parameters formula to find a particular solution to the equation. Note that the coefficient in from of the highest order term y'' is x^2, not 1.

Solution.

a) We calculate

$$W(y_1, y_2)(x) = \det \begin{pmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{pmatrix} = -x^{-1} - x^{-1} = -2x^{-1}, \ x > 0. \tag{1.54}$$

b) A particular solution is then given by

$$y_p(x) = -x \int \frac{x^{-1} x^{-2}}{-2x^{-1}} \, dx + x^{-1} \int \frac{x x^{-2}}{-2x^{-1}} \, dx = \frac{x}{2} \int x^{-2} \, dx - \frac{x^{-1}}{2} \int 1 \, dx = -\frac{1}{2} - \frac{1}{2} = -1, \ x > 0. \tag{1.55}$$

$\square$

Problem 1.11. Consider the eigenvalue problem

$$\begin{cases} y''(x) + \lambda y(x) = 0, \ x \in (0, \pi) \\ y(0) = 0, \ y(\pi) = 0. \end{cases} \tag{1.56}$$

Find the positive eigenvalues associated to this problem.

Solution. We note that if $\lambda > 0$, then the general solution to the equation is

$$y_h(x) = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x), \ x \in (0, \pi). \tag{1.57}$$

If $y(0) = 0$, we must have $c_1 = 0$. If $y(\pi) = 0$, then we must have

$$c_2 \sin \sqrt{\lambda} \pi = 0. \tag{1.58}$$

Since we are interested in finding non-trivial solutions, we may assume $c_2 \neq 0$. Then occurs whenever $\sqrt{\lambda}$ is an integer, therefore we may parametrize the positive eigenvalues via

$$\lambda_n = n^2, \ n = 1, 2, 3, \ldots \tag{1.59}$$

□

Problem 1.12. Suppose a mass-spring system is modeled via

$$\begin{cases} x''(t) + x(t) = f(t), & t \geq 0 \\ x(0) = x'(0) = 0, \end{cases} \tag{1.60}$$

where δ is the Dirac delta and $\mathcal{U}$ is the unit step function and $f : [0, \infty) \to \mathbb{R}$ is defined via

$$f(t) = \begin{cases} \delta(t - \pi), & 0 \leq t < 2\pi \\ 1, & t \geq 2\pi. \end{cases} \tag{1.61}$$

a) Write f in terms of the unit step function $\mathcal{U}(\cdot - 2\pi)$. (Note: $\delta(t - \pi) = 0$ for $t \geq 2\pi$).
b) Use the Laplace transform to find a solution x describing the behavior of the system for $t \geq 0$.

Solution.

a) We note that

$$f(t) = \delta(t - \pi) + (1 - \delta(t - \pi))\mathcal{U}(t - 2\pi) = \delta(t - \pi) + \mathcal{U}(t - 2\pi),\ t \geq 0. \tag{1.62}$$

b) Assuming x is a solution and $X = \mathcal{L}\{x\}$, we then have

$$(s^2 + 1)X(s) = e^{-\pi s} + \frac{e^{-2\pi s}}{s} \tag{1.63}$$

for appropriate values of s. Then

$$X(s) = e^{-\pi s}\frac{1}{s^2 + 1} + e^{-2\pi s}\frac{1}{s(s^2 + 1)}. \tag{1.64}$$

We note that by using partial fraction decomposition, we should have

$$\frac{1}{s(s^2 + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1},\ s \neq 0 \tag{1.65}$$

or

$$1 = A(s^2 + 1) + (Bs + C)s,\ s \in \mathbb{R}. \tag{1.66}$$

If $s = 0$, then $A = 1$. If $s = 1$, then $B + C = -1$ and if $s = -1$, then $B - C = -1$. This implies $B = -1$ and $C = 0$. Thus

$$\frac{1}{s(s^2 + 1)} = \frac{1}{s} - \frac{s}{s^2 + 1},\ s \neq 0. \tag{1.67}$$

Thus

$$X(s) = e^{-\pi s}\frac{1}{s^2 + 1} + e^{-2\pi s}\left(\frac{1}{s} - \frac{s}{s^2 + 1}\right) \tag{1.68}$$

for appropriate values of s. By taking the inverse Laplace transform, we find that

$$x(t) = \mathcal{U}(t - \pi)\sin t + \mathcal{U}(t - 2\pi)(1 - \cos(t - 2\pi)) = \mathcal{U}(t - \pi)\sin t + \mathcal{U}(t - 2\pi)(1 - \cos t),\ t \geq 0. \tag{1.69}$$

□

Problem 1.13 (The heat equation with inhomogeneous Dirichlet boundary conditions).

Consider the heat equation with *inhomogeneous* Dirichlet boundary conditions

$$\begin{cases} u_t(x,t) = 2u_{xx}(x,t) & x \in [0,\pi], t \geq \mathbb{R} \\ u(x=0,t) = 1, & t \geq 0 \\ u(x=\pi,t) = 1, & t \geq 0 \\ u(x,t=0) = f(x) = \sin(2x) + \sin(3x) + 1, & x \in [0,\pi]. \end{cases} \tag{1.70}$$

Find the unique solution to this system as a *finite* combination of elementary functions.

Solution. We apply the idea from the homework and look for a time-independent function $v : [0,\pi] \to \mathbb{R}$ such that $v_x x = 0$ and $v(0) = v(\pi) = 1$. We note that the function $v(x) = 1$ satisfies these conditions.

Suppose $u : [0,\pi] \times [0,\infty) \to \mathbb{R}$ is a solution to the system. Consider the ansatz $w : [0,\pi] \times [0,\infty) \to \mathbb{R}$ defined via $w(x,t) = u(x,t) - 1$, where $v(x) = 1$. Then w satisfies

$$\begin{cases} w_t(x,t) = 2w_{xx}(x,t) & x \in [0,\pi], t \geq \mathbb{R} \\ w(x=0,t) = 0, & t \geq 0 \\ w(x=\pi,t) = 0, & t \geq 0 \\ w(x,t=0) = \sin(2x) + \sin(3x), & x \in [0,\pi]. \end{cases} \tag{1.71}$$

From the solution formula we derived in lecture, we see that

$$w(x,t) = e^{-8t}\sin(2x) + e^{-18t}\sin(3x), \ x \in [0,\pi], t \geq 0. \tag{1.72}$$

Therefore the unique solution to the original system is

$$u(x,t) = e^{-8t}\sin(2x) + e^{-18t}\sin(3x) + 1, \ x \in [0,\pi], t \geq 0. \tag{1.73}$$

□

Problem 1.14 (The wave equation with inhomogeneous boundary conditions)**.**

Consider the wave equation with *inhomogeneous* Dirichlet boundary conditions of the form

$$\begin{cases} u_{tt}(x,t) = 9u_{xx}(x,t) & x \in [0,\pi], t \geq \mathbb{R} \\ u(0,t) = 0, & t \geq 0 \\ u(\pi,t) = \pi, & t \geq 0 \\ u(x,0) = f(x) = \sin(x) + \sin(2x) + x, & x \in [0,\pi] \\ u_t(x,0) = 0, & x \in [0,\pi]. \end{cases} \tag{1.74}$$

Find the unique solution to this system as a *finite* combination of elementary functions.

Solution. We apply the same idea as in the previous problem and look for a time-independent function $v : [0,\pi] \to \mathbb{R}$ such that $v_{xx} = 0$ and $v(0) = 0, v(\pi) = \pi$. We note that the function $v(x) = x$ satisfies these conditions.

We then consider the ansatz $w : [0,\pi] \times [0,\infty) \to \mathbb{R}$ defined via $w(x,t) = u(x,t) - x$. Then w satisfies

$$\begin{cases} w_{tt}(x,t) = 9w_{xx}(x,t) & x \in [0,\pi], t \geq \mathbb{R} \\ w(0,t) = 0, & t \geq 0 \\ w(\pi,t) = 0, & t \geq 0 \\ w(x,0) = \sin(x) + \sin(2x), & x \in [0,\pi] \\ w_t(x,0) = 0, & x \in [0,\pi]. \end{cases} \tag{1.75}$$

From the solution formula we derived in lecture, we see that

$$w(x,t) = \cos(3t)\sin x + \cos(6t)\sin(2x), \tag{1.76}$$

therefore the unique solution to the original system is

$$u(x,t) = \cos(3t)\sin x + \cos(6t)\sin(2x) + x, \; x \in [0,\pi], t \geq 0. \tag{1.77}$$

□