Midterm solutions
21-260 Spring 2025

TIME: 50 MINUTES

Name: ______________________________

No electronic devices or notes allowed.

Please make sure to submit all **14 pages** and show all your work.

Date: April 4, 2025.

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Problem 1.1 (25pt)**.** Determine if the following statements are true or false. Please clearly mark the boxes under each statement, no explanations needed.

Just as a small hint, none of the problems below require any serious calculations.

a) (5pts) The equation $y''(x) + x^3 y(x) = 0,\ x \in \mathbb{R}$ is a second order non-linear equation.

☐ True

☒ False

Solution. This is modeled after Homework 1 Problem 2 and partially Homework 3 Problem 4. This equation is second ordered because the highest derivative is the second derivative of the solution, and it is linear because it is of the form $y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0$, where $a_0, a_1 : \mathbb{R} \to \mathbb{R}$ are continuous functions. ☐

b) (5pts) If the roots of the characteristic polynomial associated to a homogeneous fourth order constant coefficient linear differential equation over $\mathbb{R}$ are $r_1 = r_2 = 1$ and $r_3 = r_4 = -1$, then the general solution to the equation is given by

$$y(x) = c_1 e^x + c_2 x e^x + c_3 x^2 e^{-x} + c_4 x^3 e^{-x}, \quad x \in \mathbb{R}, \tag{1.1}$$

where $c_1, c_2, c_3, c_4 \in \mathbb{R}$ are arbitrary constants.

☐ True

☒ False

Solution. This is modeled after Homework 4 Problem 1, specifically part e). The genearl solution should be $y(x) = c_1 e^x + c_2 x e^x + c_3 e^{-x} + c_4 x e^{-x}, \quad x \in \mathbb{R}$ where $c_1, c_2, c_3, c_4 \in \mathbb{R}$ are arbitrary constants. ☐

c) (5pts) For the inhomogeneous equation

$$y''(x) + y(x) = \sin x,\ x \in \mathbb{R}, \tag{1.2}$$

one can use the ansatz $y_p : \mathbb{R} \to \mathbb{R}$ defined via $y_p(x) = A\cos x + B\sin x$ to find a particular solution to the equation, where $A, B \in \mathbb{R}$ are undetermined constants.

☐ True

☒ False

Solution. This is modeled after Homework 5 Problem 1, specifically part b). Since the general homogeneous solution is $y_h : \mathbb{R} \to \mathbb{R}$ given by $y_h(x) = c_1 \cos x + c_2 \sin x$, where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants, it is not possible for us to use the ansatz $y_p(x) = A\cos x + B\sin x$ to find a particular solution. ☐

d) (5pts) If $y : \mathbb{R} \to \mathbb{R}$ defined via $y(x) = \cos x + Ce^{\sin x}$ is the general solution to a first order linear differential equation, where $C \in \mathbb{R}$ is an arbitrary constant, then a particular solution to the equation is given by $y_p : \mathbb{R} \to \mathbb{R}$ defined via $y_p(x) = e^{\sin x}$.

☐ True

☒ False

Solution. This is modeled after Homework 2 Problem 2. Due to the structure of the solution, the general homogeneous solution $y_h : \mathbb{R} \to \mathbb{R}$ is given by $y_h(x) = Ce^{\sin x}$, where $C \in \mathbb{R}$ is an arbitrary constant, so a particular solution cannot just be a constant multiple of $e^{\sin(\cdot)}$. ☐

e) (5pts) If $y : \mathbb{R} \to \mathbb{R}$ is the unique solution to the initial value problem

$$\begin{cases} y'(t) = y(t)(y(t) - 5),\ t \in \mathbb{R} \\ y(0) = 10, \end{cases} \tag{1.3}$$

then $\lim_{t \to -\infty} y(t) = 5$ and $\lim_{t \to \infty} y(t) = 0$.

Hint: the equation is autonomous.

☐ True

☒ False

Solution. This is modeled after Homework 2 Problem 3. Since $g : \mathbb{R} \to \mathbb{R}$ given by $g(y) = y(y-5)$ is a quadratic polynomial in y with positive leading coefficient, we know that $g(y)$ is positive for $y > 5$. Therefore the unique solution passing through $(0, 10)$ must satisfy $\lim_{t \to -\infty} y(t) = 5$ and $\lim_{t \to \infty} y(t) = +\infty$ (note that this is a slight abuse of notation because the maximal interval of existence of y might not be $\mathbb{R}$, the more precise way to write this is that $\lim_{t \to \partial I^+} y(t) = +\infty$ where ∂I^+ is taken to be the right endpoint of the maximal of existence if y is bounded to the right, and taken to be $+\infty$ if the interval of existence I is unbounded to the right). $\square$

Problem 1.2. Determine if the following statements are true or false. Please clearly mark the boxes under each statement, no explanations needed.

The following problems will likely require some (very short) computations.

a) (5pts) An integrating factor for the first order equation

$$y'(x) + \frac{x+1}{x} y(x) = 0,\ x > 0 \tag{1.4}$$

is $\mu : (0,\infty) \to \mathbb{R}$ given by $\mu(x) = xe^x$.

☒ True

☐ False

Solution. This is modeled after Homework 1 Problem 5, specifically part a). We calculate $\mu(x) = \exp(\int \frac{x+1}{x}\, dx) = \exp(\int 1 + \frac{1}{x}\, dx) = \exp(x + \ln x) = \exp(x)\exp(\ln x) = xe^x$. ☐

b) (5pts) The functions $y_1, y_2 : (0,\infty) \to \mathbb{R}$ defined via

$$y_1(x) = x,\ y_2(x) = x \ln x \tag{1.5}$$

☒ True

☐ False

Solution. This is modeled after Homework 4 Problem 3, specifically part a). From a simple calculation we see that the Wronskian between the two functions is $W : (0,\infty) \to \mathbb{R}$ given by $W(x) = x$, therefore the two functions are linearly independent over the interval $I = (0,\infty)$. ☐

c) (5pts) Let $\omega \in \mathbb{R}$. For a mass-spring system modeled via the initial value problem

$$\begin{cases} x''(t) + 4x(t) = \sin(\omega t),\ t \in \mathbb{R} \\ x(0) = 0,\ x'(0) = 1, \end{cases} \tag{1.6}$$

the system exhibits resonance for $\omega = \pm 2$.

☒ True

☐ False

Solution. This is modeled after Homework 5 Problem 6. We learned that resonance occurs when the forcing frequency ω is equal to the natural frequency of the system (up to sign), which is given by $\sqrt{4} = 2$. Therefore the system exhibits resonance for $\omega = \pm 2$. ☐

d) (5pts) An equivalent first order system of the second order scalar equation $x''(t) + 2x'(t) + 3x(t) = 0, t \in \mathbb{R}$ is

$$\boldsymbol{X}'(t) = \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix} \boldsymbol{X}(t) \text{ for } \boldsymbol{X}(t) = \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix},\ t \in \mathbb{R}. \tag{1.7}$$

☒ True

☐ False

Solution. This is modeled after Homework 8 Problem 3. Using the equation we can write

$$\boldsymbol{X}'(t) = \begin{pmatrix} x'(t) \\ x''(t) \end{pmatrix} = \begin{pmatrix} x'(t) \\ -2x'(t) - 3x(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix} \boldsymbol{X}(t),\ t \in \mathbb{R}. \tag{1.8}$$

☐

e) (5pts) The equation $(x + y(x))^2 + (2xy(x) + x^2 - 1)y'(x) = 0,\ x \in \mathbb{R}$ is an exact differential equation.
Hint: if you forgot how to check, recall that under reasonable assumptions on a function $F : \mathbb{R}^2 \to \mathbb{R}$, we have

$$\frac{d}{dx} F(x, y(x)) = F_x(x, y(x)) + F_y(x, y(x)) y'(x) \text{ and } (F_x)_y = (F_y)_x. \tag{1.9}$$

☒ True

☐ False

Solution. This is modeled after Homework 3 Problem 2. Define $M, N : \mathbb{R}^2 \to \mathbb{R}$ via $M(x,y) = (x+y)^2$ and $N(x,y) = 2xy + x^2 - 1$. We can check that $M_y(x,y) = 2(x+y) = 2x + 2y,\ N_x(x,y) = 2y + 2x$, so the equation is exact. ☐

Problem 1.3 (10pts)**.** Consider the function $f : [0, \infty) \to \mathbb{R}$ defined via

$$f(t) = \begin{cases} e^t \sin t, & 0 \le t < 2\pi \\ 1 + e^t \sin t, & t \ge 2\pi. \end{cases} \tag{1.10}$$

a) (5pts) Write f in terms of the unit step function $U : \mathbb{R} \to \mathbb{R}$ defined via

$$U(t) = \begin{cases} 0, & t < 0 \\ 1, & t \ge 0. \end{cases} \tag{1.11}$$

b) (5pts) Use the previous part to find the Laplace transform $F : J \to \mathbb{R}$ of f defined via $F(s) = \mathcal{L}\{f\}(s)$. You do not need to state the domain J on which F is defined.

Solution. This is modeled after Homework 7 Problems 2, 4, and 5. We can write

$$f(t) = e^t \sin t + (1 + e^t \sin t - e^t \sin t)U(t - 2\pi) = e^t \sin t + U(t - 2\pi),\ t \ge 0. \tag{1.12}$$

Then

$$F(s) = \mathcal{L}\{f\}(s) = \mathcal{L}\{e^t \sin t\}(s) + \mathcal{L}\{U(t - 2\pi)\}(s) \tag{1.13}$$

$$= \frac{1}{(s-1)^2 + 1^2} + \frac{e^{-2\pi s}}{s},\ s > 1. \tag{1.14}$$

□

Problem 1.4 (15pts). Consider the eigenvalue problem

$$\begin{cases} y''(x) + \lambda y(x) = 0, \ x \in (0, \pi) \\ y'(0) = 0, \ y(\pi) = 0. \end{cases} \tag{1.15}$$

You can assume that all the eigenvalues of this problem are real-valued, as we only study real-valued solutions and equations in this course.

a) (5pts) Is $\lambda = 0$ an eigenvalue of this problem? Please briefly explain why or why not.
b) (5pts) Find all the positive eigenvalues $\lambda > 0$ of the problem.
c) (5pts) Determine a corresponding set of eigenfunctions to the positive eigenvalues.

It might be helpful to know that

$$\sin x = 0 \Longleftrightarrow x \in \{n\pi \mid n \in \mathbb{Z}\} = \{\dots, -\pi, 0, \pi, 2\pi, \dots\} \tag{1.16}$$

and

$$\cos x = 0 \Longleftrightarrow x \in \left\{\frac{2n-1}{2}\pi \mid n \in \mathbb{Z}\right\} = \{\dots, -\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi, \dots\}. \tag{1.17}$$

Solution. This is modeled after Homework 6 Problem 4.

a) If $\lambda = 0$, the general solution to the equation is

$$y(x) = c_1 + c_2 x, \quad x \in (0, \pi), \ c_1, c_2 \in \mathbb{R}. \tag{1.18}$$

The boundary conditions require

$$y'(0) = 0 \implies c_2 = 0 \text{ and } y(\pi) = 0 \implies c_1 = 0. \tag{1.19}$$

Therefore $\lambda = 0$ is not an eigenvalue of the problem.

b) If $\lambda > 0$, the general solution to the equation is

$$y(x) = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x), \quad x \in (0, \pi), \ c_1, c_2 \in \mathbb{R}. \tag{1.20}$$

The boundary conditions require

$$y'(0) = 0 \implies c_2 = 0 \text{ and } y(\pi) = 0 \implies c_1 \cos(\sqrt{\lambda}\pi) = 0. \tag{1.21}$$

Therefore the eigenvalues can be parameterized as

$$\lambda_n = \left(\frac{2n-1}{2}\right)^2, \ n = 1, 2, 3, \dots \tag{1.22}$$

c) A corresponding set of eigenfunctions are given by

$$y_n(x) = c_n \cos\left(\frac{2n-1}{2} x\right), \ n = 1, 2, 3, \dots \tag{1.23}$$

where $c_n \in \mathbb{R} \setminus \{0\}$ is an arbitrary constant.

$\square$

Problem 1.5 (25pts)**.** Let $\alpha \in \mathbb{R}$, and consider the initial value problem

$$\begin{cases} x''(t) + x(t) = \alpha\delta(t-\pi), & t \in (0,\infty) \\ x(0) = 0,\ x'(0) = \alpha, \end{cases} \tag{1.24}$$

where $\delta(\cdot)$ is the Dirac delta. Note that when $\alpha = 0$, the intial value problem reduces to

$$\begin{cases} x''(t) + x(t) = 0, & t \in (0,\infty) \\ x(0) = 0,\ x'(0) = 0. \end{cases} \tag{1.25}$$

For the first two parts, you're assuming $\alpha = 0$.

a) (5pts) If $\alpha = 0$, what is the unique solution $x : [0,\infty) \to \mathbb{R}$ to the initial value problem (1.24)?
b) (5pts) Use an appropriate theorem to explain why the solution you found in the previous part is unique, for $\alpha = 0$. Make sure to state and check that the conditions of the theorem are satisfied.

For parts c) through e), assume $\alpha \neq 0$.

c) (5pts) Use the Laplace transform to explain why (1.24) is equivalent to the initial value problem

$$\begin{cases} x''(t) + x(t) = \alpha\delta(t) + \alpha\delta(t-\pi), & t \in (0,\infty) \\ x(0) = 0,\ x'(0) = 0, \end{cases} \tag{1.26}$$

in the sense that both IVPs reduce to the same algebraic equation in the Laplace domain (the s-domain, frequency domain, transformed domain, whatever you want to call it).

For the next two parts you can take the previous part for granted.

d) (5pts) Find a solution $x : [0,\infty) \to \mathbb{R}$ to the initial value problem, in terms of the parameter α.
e) (5pts) Use the solution you found in part d) to show that $x(t) \geq 0$ for all $t \geq \pi$, for any value of $\alpha \in \mathbb{R}$.

Solution. This is modeled after Homework 8 Problem 1.

a) When $\alpha = 0$, there is no forcing and the initial conditions are homogeneous, so the unique solution is given by the zero solution $x(t) = 0$ for all $t \geq 0$.
b) We can use the exsitence and uniqueness theorem for linear second order equations to justfiy why the zero solution is unique. The theorem says that if the coefficients and forcing function are continuous functions, then a solution exists globally and is unique.

   The coefficients in this equation are constant functions and the forcing function is the zero function, which are continuous. Therefore the conditions of the theorem are satisfied and it follows that the zero solution is unique. Note that we cannot use the nonlinear existence and uniqueness theorem for scalar equations as we only stated the version for first order equations.
c) For the first IVP, we have

$$\mathcal{L}\{x''(t) + x(t)\}(s) = s^2X(s) - sx(0) - x'(0) + X(s) = (s^2+1)X(s) - \alpha,\ \mathcal{L}\{\alpha\delta(t-\pi)\}(s) = \alpha e^{-\pi s}. \tag{1.27}$$

   so the algebraic equation in the Laplace domain is given by

$$(s^2+1)X(s) - \alpha = \alpha e^{-\pi s} \tag{1.28}$$

   For the second IVP, we have

$$\mathcal{L}\{x''(t) + x(t)\}(s) = s^2X(s) - sx(0) - x'(0) + X(s) = (s^2+1)X(s),\ \mathcal{L}\{\alpha\delta(t) + \alpha\delta(t-\pi)\}(s) = \alpha + \alpha e^{-\pi s}. \tag{1.29}$$

   so the algebraic equation in the Laplace domain is given by

$$(s^2+1)X(s) = \alpha + \alpha e^{-\pi s} \tag{1.30}$$

   The two algebraic equations are equivalent.
d) We have

$$X(s) = \frac{\alpha + \alpha e^{-\pi s}}{s^2+1} = \frac{\alpha}{s^2+1} + \frac{\alpha e^{-\pi s}}{s^2+1}. \tag{1.31}$$

Therefore

$$x(t) = \alpha \sin t + \alpha \sin(t-\pi)U(t-\pi) = \begin{cases} \alpha \sin t, & 0 \leq t < \pi \\ \alpha(\sin t + \sin(t-\pi)), & t \geq \pi. \end{cases} \tag{1.32}$$

e) Since $\sin(t-\pi) = -\sin t$ for all $t \in \mathbb{R}$, it follows that $x(t) \geq 0$ for all $t \geq \pi$. This should make sense from the physical interpretation of the problem similar to what was asked on the homework.

$\square$