Week 13

Problem 13.1 (Fourier series)**.** Consider the function $f : [0, \pi] \to \mathbb{R}$ defined via

$$f(x) = x^2. \tag{13.1}$$

Sketch a graph over $\mathbb{R}$ of the Fourier series of f over $[0, \pi]$, which is defined via

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(2nx) + b_n \sin(2nx)\right) \text{ for all } x \in \mathbb{R}, \tag{13.2}$$

for

$$a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos(2nx)\, dx,\ b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin(2nx)\, dx,\ n \in \mathbb{N}. \tag{13.3}$$

Solution. We note that the Fourier series of f is simply the periodic extension of f to $\mathbb{R}$, and it converges to the average of the left limit and the right limit at the points of discontinuity. In this case, the points of discontinuity are at multiples of π, since $f(0) = 0 \neq f(\pi) = \pi^2$. Therefore at the points of discontinuity, the Fourier series converges to $\frac{f(0)+f(\pi)}{2} = \frac{\pi^2}{2}$. □

Department of Mathematical Sciences, Carnegie Mellon University
E-mail address: jkoganem@andrew.cmu.edu.
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Problem 13.2 (Half-range Fourier series)**.** Consider the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = 1. \tag{13.4}$$

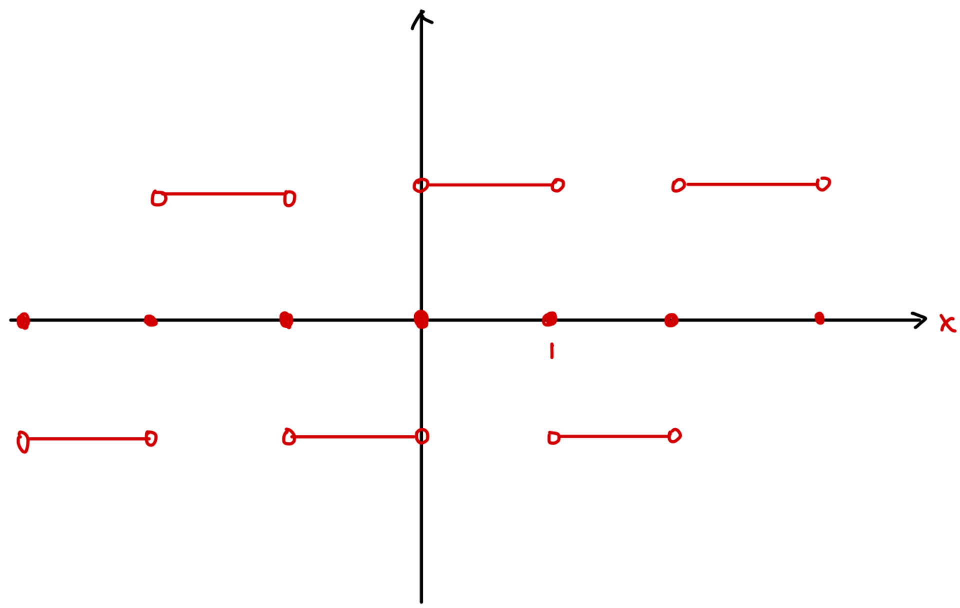
a) Sketch a graph over $\mathbb{R}$ of the half-range Fourier sine series of f, which is defined via

$$\sum_{n=1}^{\infty} b_n \sin(n\pi x) \text{ for all } x \in \mathbb{R}, \text{ where } b_n = 2\int_0^1 \sin(n\pi x)\,dx. \tag{13.5}$$

b) Sketch a graph over $\mathbb{R}$ of the half-range Fourier cosine series of f, which is defined via

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) \text{ for all } x \in \mathbb{R}, \text{ where } a_n = 2\int_0^1 f(x)\cos(n\pi x)\,dx,\ n \in \mathbb{N}. \tag{13.6}$$

Solution. The half-range Fourier sine series is the Fourier series of the odd extension of f into $(-1, 0)$. So we sketch out the periodic extension of the odd extension of f to $\mathbb{R}$, and also at the points of discontinuity the Fourier series converge to the average of the left limit and the right limit. This gives us the following sketch.



The Fourier cosine series is the Fourier series of the even extension of f into $(-1, 0)$; this is simply the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 1$. This can also be seen explicitly by calculating the coefficients a_n. From a simple calculation we find that

$$a_0 = 2 \text{ and } a_n = 0 \text{ for all } n \in \mathbb{Z}^+, \tag{13.7}$$

by using the orthogonality between the constant function and the cosine functions.

$\square$

Problem 13.3 (The heat equation)**.** Find the unique solution $u : [0, \pi] \times [0, \infty) \to \mathbb{R}$ as a *finite* linear combination of elementary functions satisfying

$$u_t = 4u_{xx}, \qquad x \in (0, \pi), t \geq 0, \tag{13.8}$$

$$u(x = 0, t) = 0 = u(\pi, t) \qquad t \geq 0 \tag{13.9}$$

$$u(x, t = 0) = \sin(2x) + \sin(5x), \qquad x \in (0, \pi). \tag{13.10}$$

Solution. Following the derivation from lecture, the unique solution to the heat equation is

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-4n^2 t} \sin(nx) \; x \in (0, \pi), t \geq 0, \tag{13.11}$$

where the coefficients A_n satisfy

$$u(x, 0) = \sum_{n=1}^{\infty} A_n \sin(nx) = \sin(2x) + \sin(5x), \; x \in (0, \pi). \tag{13.12}$$

Here we see that

$$A_n = \begin{cases} 1, & n = 2 \\ 1, & n = 5 \\ 0, & \text{otherwise.} \end{cases} \tag{13.13}$$

Therefore

$$u(x, t) = e^{-16t} \sin 2x + e^{-100t} \sin 5x, \; x \in (0, \pi), t \geq 0. \tag{13.14}$$

□

Problem 13.4 (The wave equation)**.** Find the unique solution $u : [0,\pi]\times[0,\infty)\to\mathbb{R}$ as a *finite* linear combination of elementary functions satisfying

$$u_{tt} = 9u_{xx}, \qquad x \in (0,\pi), t \geq 0 \tag{13.15}$$

$$u(x=0,t) = 0 = u(\pi,t) \qquad t \geq 0 \tag{13.16}$$

$$u(x,t=0) = \sin(3x) \qquad x \in (0,\pi) \tag{13.17}$$

$$u_t(x,t=0) = \sin(4x) \qquad x \in (0,\pi). \tag{13.18}$$

Solution. Following the derivation from lecture, the unique solution to the wave equation is

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \cos 3nt + B_n \sin 3nt)\sin nx,\ x \in (0,\pi), t \geq 0. \tag{13.19}$$

We then calculate

$$\partial_t u(x,t) = \sum_{n=1}^{\infty} (3nB_n \cos 3nt - 3nA_n \sin 3nt)\sin nx,\ x \in (0,\pi), t \geq 0. \tag{13.20}$$

Therefore the coefficients A_n, B_n satisfy

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin nx = \sin 3x,\ x \in (0,\pi) \tag{13.21}$$

and

$$\partial_t u(x,0) = \sum_{n=1}^{\infty} 3nB_n \sin nx = \sin 4x,\ x \in (0,\pi). \tag{13.22}$$

This shows that

$$A_n = \begin{cases} 1, & n = 3 \\ 0, & \text{otherwise} \end{cases} \tag{13.23}$$

and

$$B_n = \begin{cases} \frac{1}{12}, & n = 4 \\ 0, & \text{otherwise.} \end{cases} \tag{13.24}$$

Thus

$$u(x,t) = \cos 9t \sin 3x + \frac{1}{12}\sin 12t \sin 4x,\ x \in (0,\pi), t \geq 0. \tag{13.25}$$

□