Homework 9 solutions

DUE: SATURDAY, APRIL 12, 11:59PM

If you completed this assignment through collaboration or consulted references, please list the names of your collaborators and the references you used below. Please refer to the syllabus for the course policy on collaborations and types of references that are allowed.

Department of Mathematical Sciences, Carnegie Mellon University
E-mail address: jkoganem@andrew.cmu.edu.
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Problem 9.1 (IVPs with no forcing and homogeneous data)**.** Consider the initial value problem

$$\begin{cases} y^{(6)}(x) - 3y^{(3)}(x) + 2y''(x) + 10y'(x) + y(x) = 0, \ x \in \mathbb{R} \\ y^{(k)}(0) = 0, \ \text{for all } 0 \le k \le 5. \end{cases} \tag{9.1}$$

a) Verify that the zero solution y defined via $y(x) = 0$ for all $x \in \mathbb{R}$ is a solution to this IVP.
b) Use the uniqueness part of the existence and uniqueness theorem for higher order linear equations to show that the zero solution is the unique solution to this IVP. Make sure to verify that the conditions of the theorem are met.

Solution. We note that the zero solution $y(x) = 0$ satisfies $y^k(x) = 0$ for all $1 \le k \le 6$ and all $x \in \mathbb{R}$, so it trivially satisfies the equation and the initial conditions. Since this is a constant coefficient linear equation, the conditions of the existence and uniqueness theorem for higher order equations is also met since the coefficients and the right hand side of the equation (as constant functions over $\mathbb{R}$) are continuous and the leading coefficient (as a constant function over $\mathbb{R}$) never vanishes, therefore by the uniqueness part of the theorem, the zero solution is the unique solution to the initial value problem. $\square$

Problem 9.2 (Convolutions and the Laplace transform)**.** Suppose a mass-spring system is modeled via the equation

$$\begin{cases} x''(t) + 4x'(t) + 3x(t) = f(t), \ t \geq 0, \\ x(0) = x'(0) = 0. \end{cases} \tag{9.2}$$

a) Recall that the *transfer function* associated to the system is a function $W : \mathbb{R} \to \mathbb{R}$ defined for which

$$X(s) = W(s)F(s), \ s > a \tag{9.3}$$

for an appropriate $a \in \mathbb{R}$, and $X = \mathcal{L}\{x\}$, $F = \mathcal{L}\{f\}$. What is the transfer function associated to (9.2)?

b) The *weight function* associated to the system is the function $w : [0, \infty) \to \mathbb{R}$ defined via $w = \mathcal{L}^{-1}\{W\}$. What is the weight function associated to (9.2)?

c) Recall that by the *convolution property*, the solution to the IVP can be written in terms of a convolution between the weight function w and the forcing function f. Write down an explicit integral representation of the solution $x : [0, \infty) \to \mathbb{R}$ for any reasonable function f.

Solution.

a) The transfer function associated to this system is

$$W(s) = \frac{1}{s^2 + 4s + 3} = \frac{1}{(s+1)(s+3)} = \frac{1}{2}\frac{1}{s+1} - \frac{1}{2}\frac{1}{s+3}, \ s > -1. \tag{9.4}$$

b) The weight function associated to this system is

$$w(t) = \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{1}{s+1} - \frac{1}{2}\frac{1}{s+3}\right\} = \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t}, \ t \geq 0. \tag{9.5}$$

c) The solution $x : [0, \infty) \to \mathbb{R}$ is then given by a convolution of w and f:

$$x(t) = (w * f)(t) = \int_0^t w(\tau)f(t-\tau)\, d\tau = \int_0^t \left(\frac{1}{2}e^{-\tau} - \frac{1}{2}e^{-3\tau}\right) f(t-\tau)\, d\tau. \tag{9.6}$$

$\square$

Problem 9.3 (Variation of parameters formula/Duhamel's principle for inhomogeneous systems)**.** In this problem we explore the variation of parameters formula for $n \times n$ inhomogeneous systems. For the sake of simplicity we assume $n = 2$, but the result here can easily be generalized to any $n \in \mathbb{N}$.

Let $I \subseteq \mathbb{R}$ be an interval, $t_0 \in I$, $A : I \to \mathcal{M}_{2\times 2}(\mathbb{R})$ be a continuous matrix-valued function and $\boldsymbol{f} : I \to \mathbb{R}^2$ be a continuous vector-valued function. We would like to write down an explicit solution formula for the initial value problem

$$\begin{cases} \boldsymbol{X}'(t) - A(t)\boldsymbol{X}(t) = \boldsymbol{f}(t),\ t \in \mathbb{R} \\ \boldsymbol{X}(t_0) = \boldsymbol{X}_0. \end{cases} \tag{9.7}$$

From the existence and uniqueness theorem for linear systems we know that there exists two functions $\boldsymbol{X}_1, \boldsymbol{X}_2 : I \to \mathbb{R}^2$ such that

$$\boldsymbol{X}_1'(t) - A(t)\boldsymbol{X}_1(t) = \boldsymbol{0}_{2\times 1} \tag{9.8}$$

$$\boldsymbol{X}_2'(t) - A(t)\boldsymbol{X}_2(t) = \boldsymbol{0}_{2\times 1} \tag{9.9}$$

and

$$\det\left(\boldsymbol{X}_1(t_0) \;\middle|\; \boldsymbol{X}_2(t_0)\right) \neq 0. \tag{9.10}$$

a) Define the Wronskian $W(\boldsymbol{X}_1, \boldsymbol{X}_2) : I \to \mathbb{R}$ via

$$W(\boldsymbol{X}_1, \boldsymbol{X}_2)(t) = \det\left(\boldsymbol{X}_1(t) \;\middle|\; \boldsymbol{X}_2(t)\right). \tag{9.11}$$

One can show using some tools from linear algebra that

$$W(\boldsymbol{X}_1, \boldsymbol{X}_2)(t) = \det \Phi(t) = W(\boldsymbol{X}_1, \boldsymbol{X}_2)(t_0) \exp\left(\int_{t_0}^{t} \operatorname{tr} A(s)\, ds\right),\ t \in I \tag{9.12}$$

Use (9.11) and (9.12) to show that $W(\boldsymbol{X}_1, \boldsymbol{X}_2)$ is never equal to 0 on I. Conclude that $\boldsymbol{X}_1, \boldsymbol{X}_2$ are linearly independent over I.

b) Define the fundamental matrix $\Phi : I \to \mathcal{M}_{2\times 2}(\mathbb{R})$ associated to the system (9.7) via

$$\Phi(t) = \left(\boldsymbol{X}_1(t) \;\middle|\; \boldsymbol{X}_2(t)\right). \tag{9.13}$$

Verify by direct computation that Φ satisfies the matrix-valued equation

$$\Phi'(t) = A(t)\Phi(t),\ t \in I. \tag{9.14}$$

c) Verify by direct computation that

$$\boldsymbol{X}_h(t) = \Phi(t)\Phi^{-1}(t_0)\boldsymbol{X}_0,\ t \in I \tag{9.15}$$

is a solution to the homogeneous IVP

$$\begin{cases} \boldsymbol{X}'(t) - A(t)\boldsymbol{X}(t) = \boldsymbol{0},\ t \in \mathbb{R} \\ \boldsymbol{X}(t_0) = \boldsymbol{X}_0. \end{cases} \tag{9.16}$$

d) Verify by direct computation that

$$\boldsymbol{X}(t) = \underbrace{\Phi(t)\Phi^{-1}(t_0)\boldsymbol{X}_0}_{\boldsymbol{X}_h(t)} + \underbrace{\Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\boldsymbol{f}(s)\, ds}_{\boldsymbol{X}_p(t)} \tag{9.17}$$

is a solution to the IVP (9.7).

Remark 9.4.

- Part a) shows that $\Phi(t)$ is invertible for all $t \in I$ as $\det \Phi(t) \neq 0$ for all $t \in I$, so Φ^{-1} is well-defined on I and therefore the variation of parameters formula (9.17) is well-defined for all $t \in I$.
- For part d), you may assume that the product rule and fundamental theorem of calculus hold for matrix and vector valued functions. The integral of a vector-valued function is defined term-wise: if $\boldsymbol{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ then

$$\int_{t_0}^{t} \boldsymbol{g}(s)\, ds = \begin{pmatrix} \int_{t_0}^{t} g_1(s)\, ds \\ \int_{t_0}^{t} g_2(s)\, ds \end{pmatrix}. \tag{9.18}$$

- By the uniqueness part of the existence and uniqueness theorem, (9.15) is the unique solution to the homogeneous IVP (9.16) and (9.17) is the unique solution to the inhomogeneous IVP (9.7). (9.17) is known as the variation of parameters formula or Duhamel's principle.
- The explicit solution formula shows that as long as we can find the general solution to the underlying homogeneous problem (which is typically difficult if $A(\cdot)$ depends on time), we can solve the inhomogeneous problem by computing the Duhamel term $\int_{t_0}^{t} \Phi^{-1}(s)\boldsymbol{f}(s)\, ds$. This is the essence of Duhamel's principle, which says that we can in principle solve any inhomogeneous problem as long as we can solve the associated homogeneous problem.
- Analogous to the situation for scalar equations, when $A(\cdot)$ is a variable-coefficient matrix the fundamental matrix Φ is in general difficult to identify. However, in the case when A is a constant coefficient matrix, Φ can be identified explicitly using the so-called *Jordan canonical form* of the matrix A, and the Jordan canonical form is intricately related to the eigenvalues and eigenvectors of the matrix A. This is the reason why we are able to identify the general solution of homogeneous systems explicitly by finding the eigenvalues and eigenvectors of the matrix A.

Solution.

a) We note that by assumption,

$$W(\boldsymbol{X}_1, \boldsymbol{X}_2)(t_0) = \det\left(\boldsymbol{X}_1(t_0) \;\middle|\; \boldsymbol{X}_2(t_0)\right) =\neq 0, \tag{9.19}$$

and

$$\exp\left(\int_{t_0}^{t} \operatorname{tr} A(s)\, ds\right) \neq 0 \tag{9.20}$$

for all $t \in I$ since the exponential function never vanishes. Therefore $W(\boldsymbol{X}_1, \boldsymbol{X}_2)$ never vanishes on I, so $\boldsymbol{X}_1, \boldsymbol{X}_2$ are linearly independent on I.

b) We calculate

$$\Phi'(t) = \left(\boldsymbol{X}_1'(t) \;\middle|\; \boldsymbol{X}_2'(t)\right) = \left(A(t)\boldsymbol{X}_1(t) \;\middle|\; A(t)\boldsymbol{X}_2(t)\right) = A(t)\left(\boldsymbol{X}_1(t) \;\middle|\; \boldsymbol{X}_2(t)\right) = A(t)\Phi(t),\ t \in I. \tag{9.21}$$

c) We first check the initial condition. We note that

$$\boldsymbol{X}_h(t_0) = \Phi(t_0)\Phi^{-1}(t_0)\boldsymbol{X}_0 = I\boldsymbol{X}_0 = \boldsymbol{X}_0. \tag{9.22}$$

Next we calculate

$$\boldsymbol{X}_h'(t) = \Phi'(t)\Phi^{-1}(t_0)\boldsymbol{X}_0 = A(t)\Phi(t)\Phi^{-1}(t_0)\boldsymbol{X}_0 = A(t)\boldsymbol{X}_h(t),\ t \in I. \tag{9.23}$$

Therefore $\boldsymbol{X}_h$ is a solution to the homogeneous IVP.

d) We first verify the initial condition. We note that

$$\boldsymbol{X}(t_0) = \Phi(t_0)\Phi^{-1}(t_0)\boldsymbol{X}_0 + \Phi(t_0)\int_{t_0}^{t_0} \Phi^{-1}(s)\boldsymbol{f}(s)\, ds = I\boldsymbol{X}_0 + \boldsymbol{0} = \boldsymbol{X}_0. \tag{9.24}$$

Next we compute

$$\boldsymbol{X}'(t) = \Phi'(t)\Phi^{-1}(t_0)\boldsymbol{X}_0 + \Phi'(t)\int_{t_0}^{t} \Phi^{-1}(s)\boldsymbol{f}(s)\, ds + \Phi(t)\Phi^{-1}(t)\boldsymbol{f}(t) \tag{9.25}$$

$$= A(t)\left(\Phi(t)\Phi^{-1}(t_0) + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\boldsymbol{f}(s)\, ds\right) + I\boldsymbol{f}(t) \tag{9.26}$$

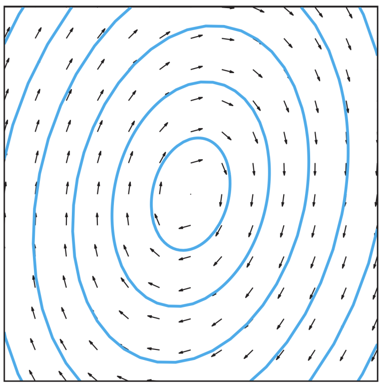
$$= A(t)\boldsymbol{X}(t) + \boldsymbol{f}(t),\ t \in I. \tag{9.27}$$

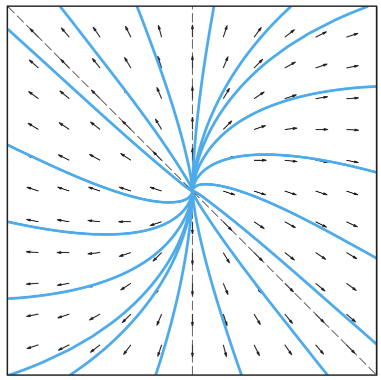
Therefore $\boldsymbol{X}$ satisfies the inhomogeneous IVP.

□

Problem 9.4 (Phase portraits of 2×2 systems)**.** The diagrams below are phase portraits of solutions $\boldsymbol{X}$ to the homogeneous system $\boldsymbol{X}' = A\boldsymbol{X}$. For each of the following diagrams, classify the eigenvalues of the matrix A and the origin by identifying the following.

a) Classify the eigenvalues of the matrix A by indicating whether the eigenvalues of A are real/complex and identifying the sign(s) of the real parts of the eigenvalues.
b) Classify the origin in terms of the geometry of the trajectories associated to the linear system, by identifying whether the origin is a node, saddle point, spiral, or center.
c) Classify the origin in terms of the stability of the trajectories. Is the origin a source or a sink? Is it a stable critical point or an unstable critical point? Is it asymptotically stable?


(a)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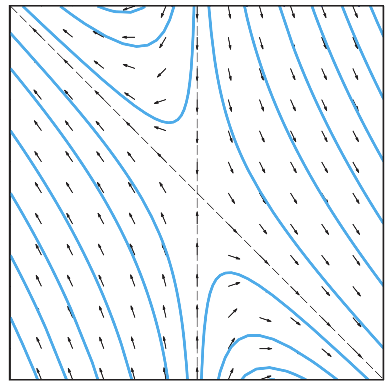
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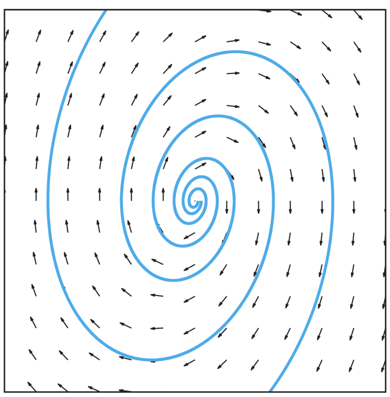
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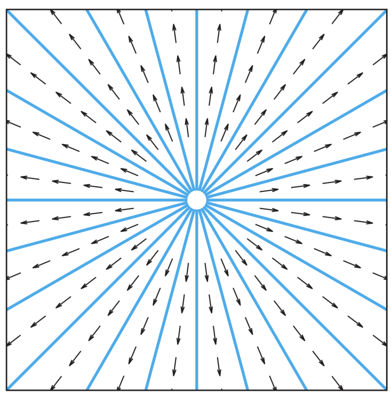
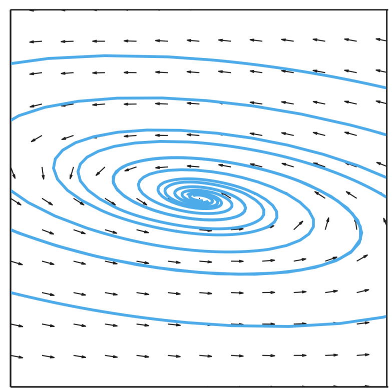

(e)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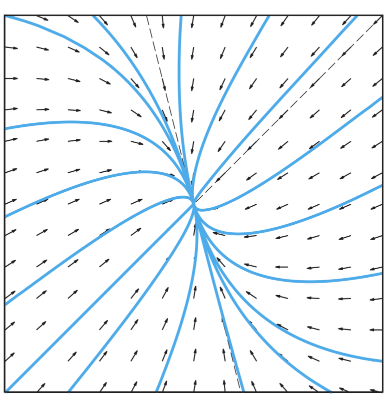
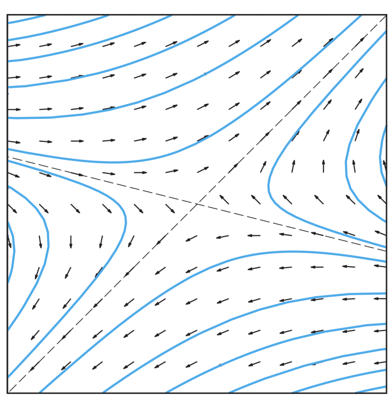
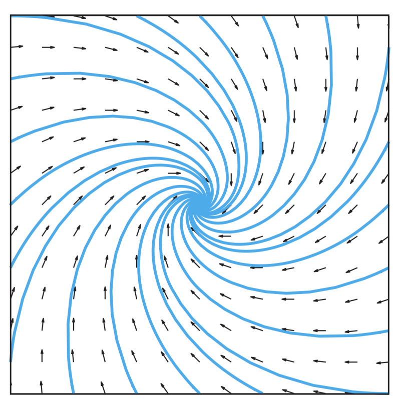

(g)


(h)


(i)

Solution.

(a)
 a) The eigenvalues of A are complex. The real parts of the eigenvalues are 0.
 b) The origin is a center.
 c) The origin is neither a source nor a sink. It is a stable critical point but not asymptotically stable.

(b)
 a) The eigenvalues of A are real. The real parts of the eigenvalues are positive.
 b) The origin is a node.
 c) The origin is source. It is an unstable critical point.

(c)
 a) The eigenvalues of A are real. The real parts of the two eigenvalues have opposite signs.
 b) The origin is a saddle point.
 c) The origin is neither a source nor a sink. It is an unstable critical point.

(d)
 a) The eigenvalues of A are complex. The real parts of the eigenvalues are positive.
 b) The origin is a spiral point.
 c) The origin is a source. It is an unstable critical point

(e)
 a) The eigenvalues of A are real. The real parts of the eigenvalues are positive.
 b) The origin is a node.
 c) The origin is a source. It is an unstable critical point.

(f)
 a) The eigenvalues of A are complex. The real parts of the eigenvalues are negative.
 b) The origin is a spiral point.
 c) The origin is a sink. It is a stable and asymptotically stable critical point.

(g)
 a) The eigenvalues of A are real. The real parts of the eigenvalues are negative.
 b) The origin is a node.
 c) The origin is a sink. It is an stable and asymptotically stable critical point.

(h)
 a) The eigenvalues of A are real. The real parts of the eigenvalues have opposite signs.
 b) The origin is a saddle point.
 c) The origin is neither a source nor a sink. It is an unstable critical point.

(i)
 a) The eigenvalues of A are complex. The real parts of the eigenvalues are negative.
 b) The origin is a spiral point.
 c) The origin is a sink. It is a stable and asymptotically stable critical point.

$\square$

Problem 9.5 (Linearization of nonlinear planar systems)**.** Consider the nonlinear autonomous system

$$\begin{cases} x'(t) = (x(t))^2 + (y(t))^2 - 6 \\ y'(t) = (x(t))^2 - y(t), \qquad\qquad t \in \mathbb{R}. \end{cases} \tag{9.28}$$

a) Identify the critical points of the nonlinear system.
b) Write down the linearization of the nonlinear system around the constant/equilibrium solutions.
c) Classify the critical points in terms of their stability type.

Solution.

a) We look for $(x, y) \in \mathbb{R}^2$ satisfying

$$\begin{cases} x^2 + y^2 - 6 = 0 \\ x^2 - y = 0. \end{cases} \tag{9.29}$$

The second equation implies $y = x^2$, therefore $y^2 + y - 6 = (y-2)(y+3) = 0 \implies y = 2$ and $y = -3$. Since $x^2 = y$, if $y < 0$ then x is complex-valued, so we only get real critical points when $y = 2$. Here we have two critical points $(\sqrt{2}, 2)$ and $(-\sqrt{2}, 2)$.

b) The linearization is in the form of $\boldsymbol{X}' = A(\boldsymbol{X} - \boldsymbol{X}_0)$ where

$$A(x_0, y_0) = \begin{pmatrix} 2x|_{x=x_0} & 2y|_{y=y_0} \\ 2x|_{x=x_0} & -1 \end{pmatrix}. \tag{9.30}$$

Therefore at the two critcal points we have

$$A(\sqrt{2}, 2) = \begin{pmatrix} 2\sqrt{2} & 4 \\ 2\sqrt{2} & -1 \end{pmatrix},\ A(-\sqrt{2}, 2) = \begin{pmatrix} -2\sqrt{2} & 4 \\ -2\sqrt{2} & -1 \end{pmatrix}. \tag{9.31}$$

c) To classify the critical points, one option here is to look for the eigenvalues directly and examine the signs of the real parts of the eigenvalues, but since we are only interested in their signs and not their magnitudes, we can also infer this information by remembering that the determinant of A is the product of the two eigenvalues and the trace of A is the sum of the eigenvalues.

If λ_1, λ_2 are two eigenvalues, then $\lambda_1\lambda_2 > 0$ if they have the same sign and $\lambda_1, \lambda_2 < 0$ if they are of opposite signs. If $\det A > 0$ and $\operatorname{tr} A > 0$, then the two roots must be positive and likewise if $\det A > 0$ and $\operatorname{tr} A < 0$, the two roots must be negative. So in the case that A admits two real eigenvalues, we can classify their signs by computing its determinant and trace.

If $\lambda_1 = \alpha - i\beta, \lambda_2 = \alpha + i\beta$ are two complex roots, then

$$\lambda_1\lambda_2 = \alpha^2 + \beta^2 > 0 \text{ assuming } \alpha, \beta \neq 0. \tag{9.32}$$

Since $\operatorname{tr} A = 2\alpha$, if $\operatorname{tr} A > 0$ then the real parts of the eigenvalues are positive, and likewise if $\operatorname{tr} A < 0$ then the real parts of the eigenvalues are negative. In summary,

- If $\det A < 0$ then we have a saddle point.
- If $\det A > 0$ and $\operatorname{tr} A > 0$, the real parts of the eigenvalues are positive, so we have an unstable node.
- If $\det A > 0$ and $\operatorname{tr} A < 0$, the real parts of the eigenvalues are negative, so we have a stable node.

With this in mind we then calculate

$$\det A(\sqrt{2}, 2) < 0 \implies (\sqrt{2}, 2) \text{ is an unstable saddle point} \tag{9.33}$$

$$\det A(-\sqrt{2}, 2) > 0, \operatorname{tr} A(-\sqrt{2}, 2) < 0 \implies (\sqrt{2}, 2) \text{ is a stable node.} \tag{9.34}$$

□