Quantum Computing, Rieffel & Polak Chapters 1 and 2

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The information presented here, although greatly condensed, comes almost entirely from the textbook

Chapter 1 – Introduction

- Quantum computing is a beautiful combination of quantum physics, CS, and information theory
- Includes quantum computing, quantum cryptography, quantum communications, and quantum games
  - Uses quantum mechanics rather than classical mechanics to model information and its processing
- The book consists of three parts
  - Part I Quantum Building Blocks
  - Part II Quantum Algorithms
  - Part III Entanglement and Robust Quantum Computation

Chap 2 Single-Qubit Quantum Systems

- Quantum bits are the fundamental units of information in quantum computing, just as bits are fundamental units in classical computing
- There are many ways to realize classical bits
- Similarly, there are many ways to realize qubits
- We begin by examining the behavior of polarized photons, a possible realization of quantum bits

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Quantum mechanics of photon polarization

- A simple experiment illustrates some of the non-intuitive behavior of quantum systems
- Minimal equipment required
  - Laser pointer and three polarized lenses (polaroids) available from any camera supply store
  - 3D movies project the same scene into both eyes, but from slightly different perspectives
    - The viewer wears low-cost eyeglasses which contain a pair of different **polarizing** filters

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Quantum mechanics of photon polarization

- Shine a beam of light on a projection screen
- Insert polaroid A between light source and screen

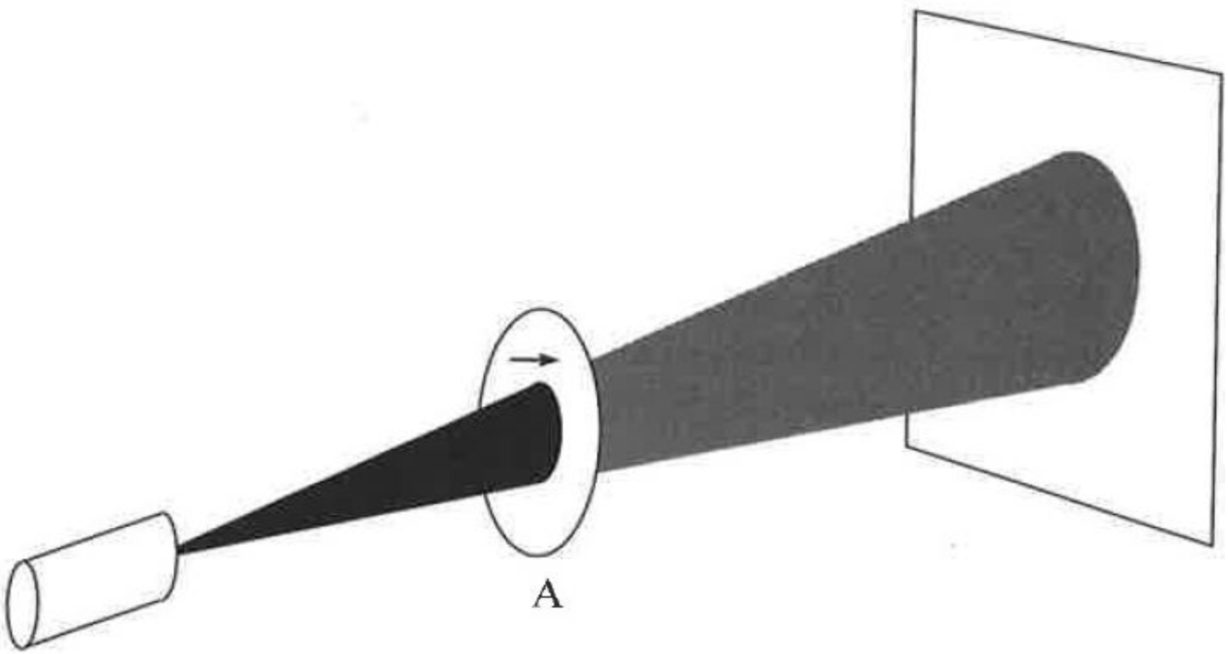

Figure 2.1
Single polaroid attenuates unpolarized light by 50 percent.

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Quantum mechanics of photon polarization

- Insert polaroid C between A and screen and rotate it

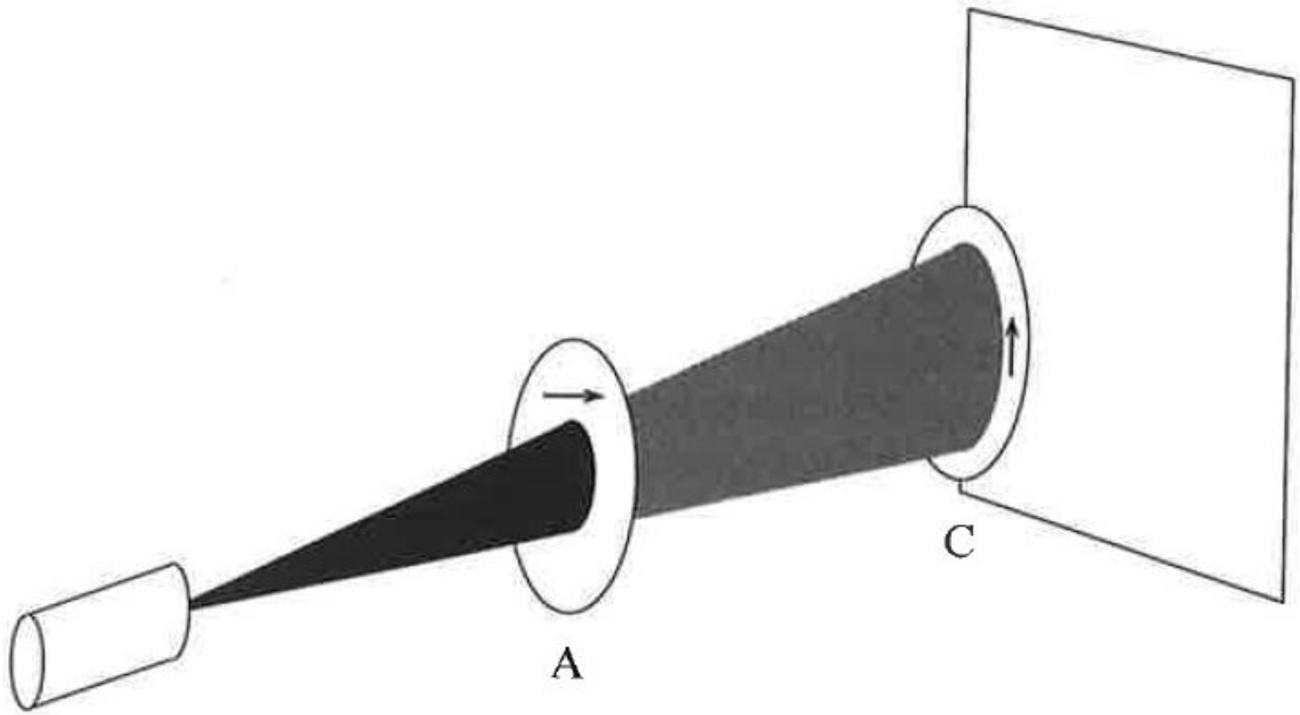

Figure 2.2
Two orthogonal polaroids block all photons.

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Quantum mechanics of photon polarization

- Finally, insert polaroid B between A and C

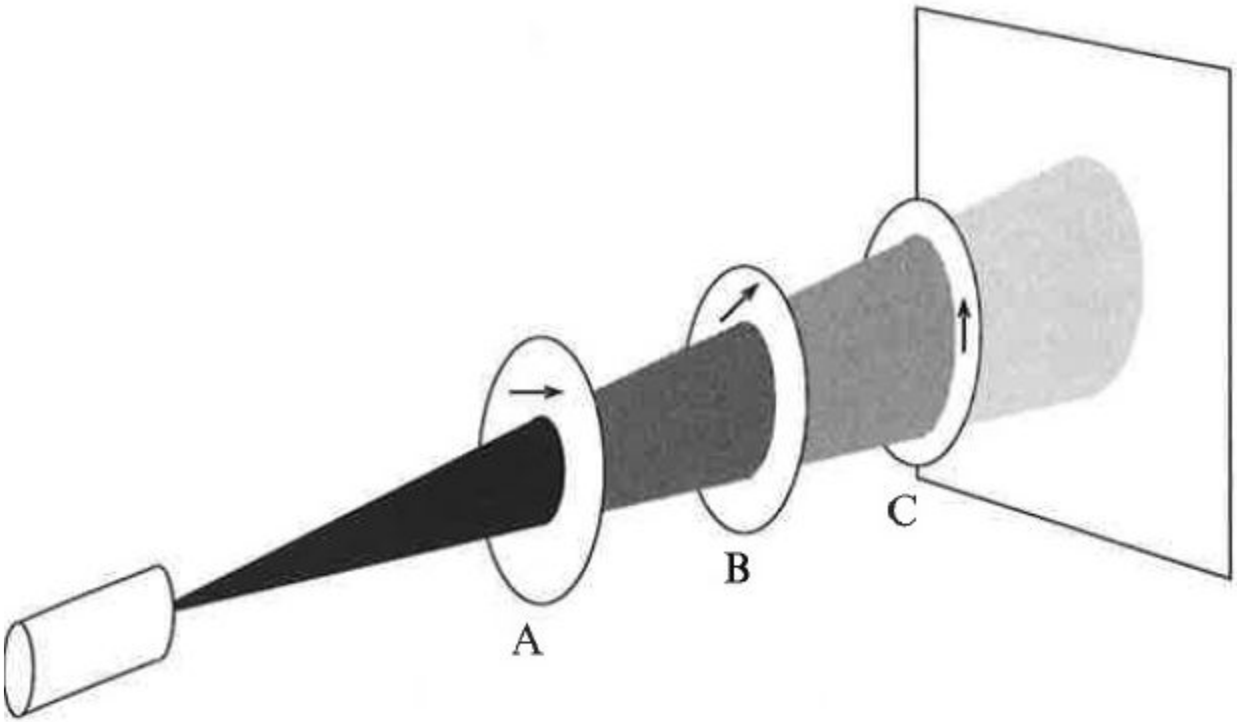

Figure 2.3
Inserting a third polaroid allows photons to pass.

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Quantum mechanics of photon polarization

- Turns out that the results of this experiment can be explained classically in terms of waves
- The same experiment can be performed with more sophisticated equipment using a single-photon emitter to yield the same results
  - Results explained only with quantum mechanics
- The explanation consists of two parts
  - Model of photon’s polarization state
  - Model of interaction between photon and polaroid

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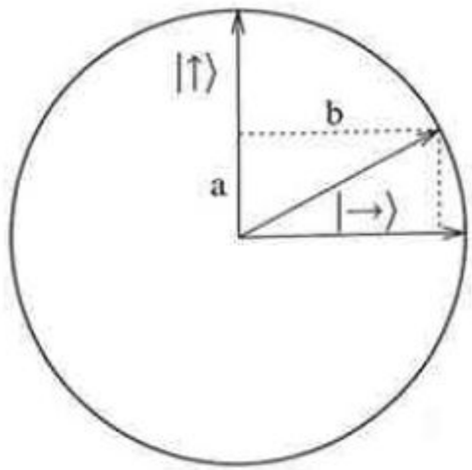
Quantum mechanics of photon polarization

- A photon's polarization state can be represented by a superposition of base vectors



$$|v\rangle = a|\uparrow\rangle + b|\rightarrow\rangle$$

- Now, when photon with polarization $|v\rangle$ meets polaroid $|\uparrow\rangle$, the photon gets thru with prob a^2
  - The probability is the square of the coefficient

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Quantum mechanics of photon polarization

- What happens when polaroid B with preferred axis $|\nearrow\rangle$ is inserted?
- We note that $$|\rightarrow\rangle = \frac{1}{\sqrt{2}}|\nearrow\rangle - \frac{1}{\sqrt{2}}|\nwarrow\rangle$$
- Therefore, ½ of the photons get thru B
- In summary, 1/8 of the photons get thru ABC
  - 1/2 get thru A, 1/2 get thru B, 1/2 get thru C

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Quantum mechanics of photon polarization

Exercise 2.1. Given the preferred axis of polaroid B as $|v\rangle = \cos\theta|\rightarrow\rangle + \sin\theta|\uparrow\rangle$, the fraction of photons reaching the screen

$= \frac{1}{2}$ (thru A) $\times \cos^2\theta$ (thru B) $\times \sin^2\theta$ (thru C)

For $\theta = 45^\circ$, fraction reaching screen $= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8}$

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Quantum mechanics of photon polarization

To determine θ yielding most light on screen, find θ for which the derivative is zero.

$$\frac{d}{d\theta}(\cos^2\theta \sin^2\theta) = \sin\theta\cos\theta(\cos^2\theta - \sin^2\theta) = 0$$

minima at $\theta = 0°$ and $90°$

maximum at $\cos^2\theta = \sin^2\theta \Rightarrow \theta = 45°$

Chap 2 Single-Qubit Quantum Systems

Single quantum bits

- The set of the infinite number of possible states of a physical quantum system is called the state space
- States of a two-state system can be represented in terms of two orthonormal basis states $|0\rangle$ and $|1\rangle$
  - The basis states can also be written as $\binom{1}{0}$ and $\binom{0}{1}$
  - Arbitrary state $|v\rangle = a|0\rangle + b|1\rangle$ can be written as $\binom{a}{b}$
  - This is a complex vector space with
  - Inner product $\langle v_2|v_1\rangle$ satisfying (bra = $\langle v|$, ket = $|v\rangle$)
    - $\langle v|v\rangle$ is non-negative real,
    - $\langle v_2|v_1\rangle = \overline{\langle v_1|v_2\rangle}$, and
    - $(a\langle v_2| + b\langle v_3|)|v_1\rangle = a\langle v_2|v_1\rangle + b\langle v_3|v_1\rangle$

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Single-qubit measurement

- Quantum theory postulates that any device that measures a two-state quantum system must have two preferred states whose representative vectors form an orthonormal basis for the associated vector space
- And measurement of a state transforms the state into one of the measuring device's associated basis vectors

$$|u\rangle \text{ or } |u^{\perp}\rangle$$

- And the probability the state is measured as $|u\rangle$ or $|u^{\perp}\rangle$ the square of the amplitude of that basis vector
  - For example, the state $|v\rangle = a|u\rangle + b|u^{\perp}\rangle$ is measured as $|u\rangle$ with probability a^2

Chap 2 Single-Qubit Quantum Systems

Quantum key distribution protocol

- Now we can describe the first application
  - Relies on quantum effects for security
- Quantum key distribution protocol establishes a symmetric key between 2 parties, Alice & Bob
- Alice & Bob connected by two public channels
  - Bidirectional classical channel
  - Unidirectional quantum channel for sending qubits
  - Channels observed by eavesdropper Eve

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Quantum key distribution protocol

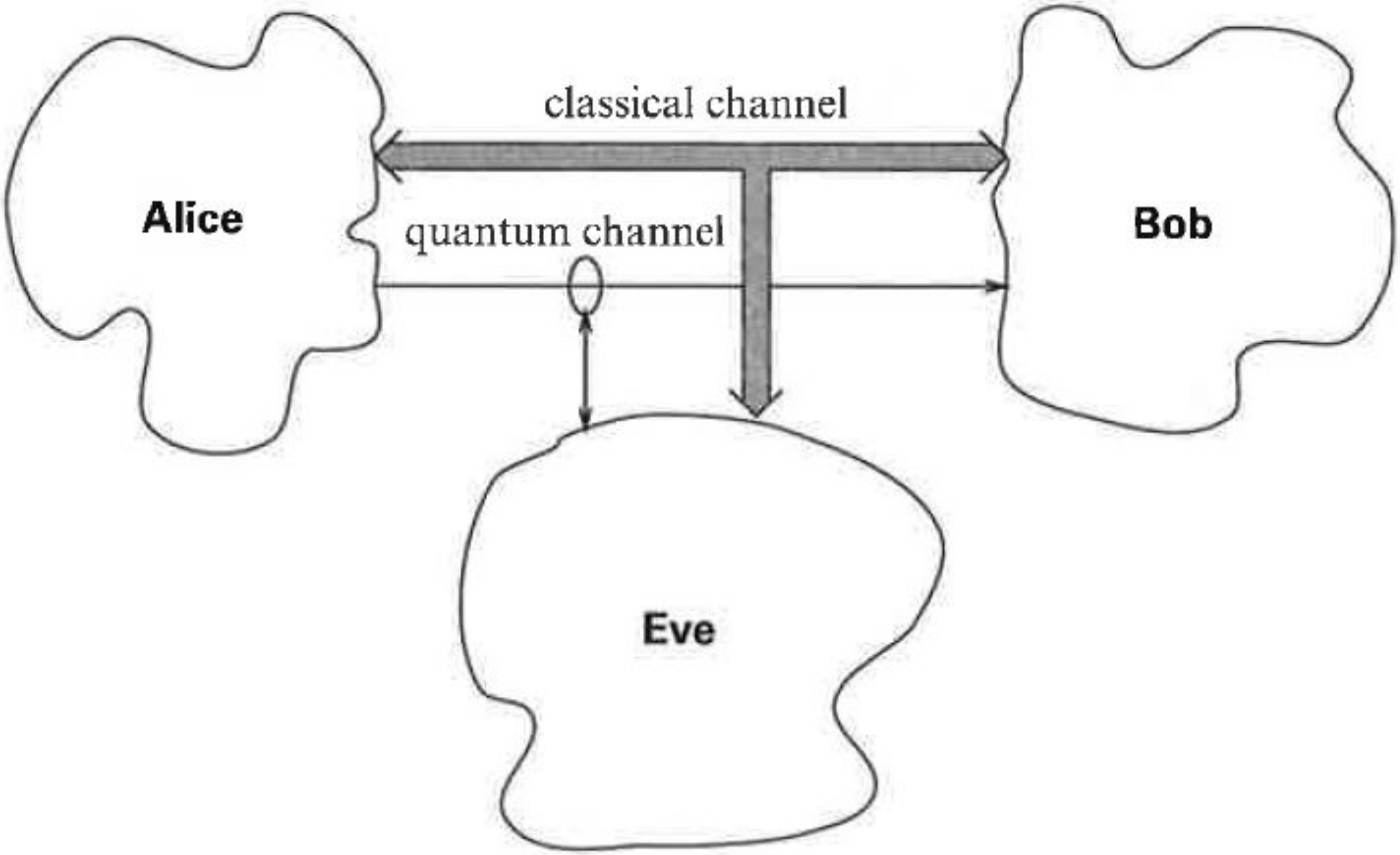

Figure 2.5
Alice and Bob wish to agree on a common key not known to Eve.

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Quantum key distribution protocol

- Alice generates a random sequence of bits
  - Random subset of sequence will be the private key
- Alice randomly encodes each bit of the sequence in the polarization state of a photon
  - Randomly choosing for each bit one of the bases

the standard basis,

$$0 \to |\uparrow\rangle$$

$$1 \to |\rightarrow\rangle$$

or the Hadamard basis,

$$0 \to |\nearrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\rightarrow\rangle)$$

$$1 \to |\nwarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\rightarrow\rangle)$$

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Quantum key distribution protocol

- Alice sends the sequence of photons to Bob
- Bob measures the state of each photon he receives by randomly picking either basis
- Over the classical channel, Alice and Bob tell each other the bases they used for each bit
- When choice of bases agree, Bob's measured bit values agree with bit values Alice sent
- Without revealing bit values, they discard all bits on which their bases differed (about 50%)

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Quantum key distribution protocol

- About 50% of bits transmitted remain
- Then Alice and Bob compare a certain number of bit values to check if eavesdropping occurred
- The checked bits are also discarded
- The remaining bits will now be used as the private key

Chap 2 Single-Qubit Quantum Systems

State space of single-qubit system

- The state space of a quantum system is the set of all possible states of the system
- The state space for a single qubit system, no matter how realized, is the set of qubit values

$$\{a|0\rangle + b|1\rangle\}$$

where $|a|^2 + |b|^2 = 1$ and $a|0\rangle + b|1\rangle$ and $a'|0\rangle + b'|1\rangle$ are considered the same qubit value if

$a|0\rangle + b|1\rangle = c(a'|0\rangle + b'|1\rangle)$ for some modulus one complex number c.

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State space of single-qubit system

- Relative phases versus global phases
- Global phase is
  - The multiple by which two vectors representing the same quantum state differ
  - It has no physical meaning
  - This is a common source of confusion for newcomers to the field

Chap 2 Single-Qubit Quantum Systems

State space of single-qubit system

- The relative phase of superposition $\{a|0\rangle + b|1\rangle\}$
  - Is a measure of the angle in the complex plane between the two complex numbers *a* and *b*
  - Specifically, it is the modulus one complex number $e^{i\phi}$ satisfying $a/b = e^{i\phi}|a|/|b|$
  - Two superpositions $a|0\rangle + b|1\rangle$ and $a'|0\rangle + b'|1\rangle$ whose amplitudes have the same magnitudes but differ in relative phase represent different states
  - The physically meaningful relative phase and the physically meaningless global phase should not be confused

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State space of single-qubit system

- Geometric views of single qubit state space
  - Two ways of looking at complex projective space
- 1. Extended complex plane
  - Complex plane C with additional point labeled ∞
- 2. Bloch sphere

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State space of single-qubit system

- Extended complex plane (with added ∞)
  - Correspondence between the set of all complex numbers and single-qubit states

$$a|0\rangle + b|1\rangle \mapsto b/a = \alpha$$

and its inverse

$$\alpha \mapsto \frac{1}{\sqrt{1+|\alpha|^2}}|0\rangle + \frac{\alpha}{\sqrt{1+|\alpha|^2}}|1\rangle$$

$$\begin{aligned}
|0\rangle &\mapsto 0\\
|1\rangle &\mapsto \infty\\
|+\rangle &\mapsto +1\\
|-\rangle &\mapsto -1\\
|\mathbf{i}\rangle &\mapsto \mathbf{i}\\
|-\mathbf{i}\rangle &\mapsto -\mathbf{i}.
\end{aligned}$$

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State space of single-qubit system

- Bloch sphere
  - Starting with the previous representation, map each state represented by the complex number $\alpha = s + \mathbf{i}t$ onto the unit sphere in 3D
  - The points $(x, y, z) \in \mathbf{C}$ satisfying $|x|^2 + |y|^2 + |z|^2 = 1$ via the standard stereographic projection map

$$(s, t) \mapsto \left(\frac{2s}{|\alpha|^2 + 1}, \frac{2t}{|\alpha|^2 + 1}, \frac{1 - |\alpha|^2}{|\alpha|^2 + 1} \right)$$

further requiring that $\infty \mapsto (0, 0, -1)$

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State space of single-qubit system

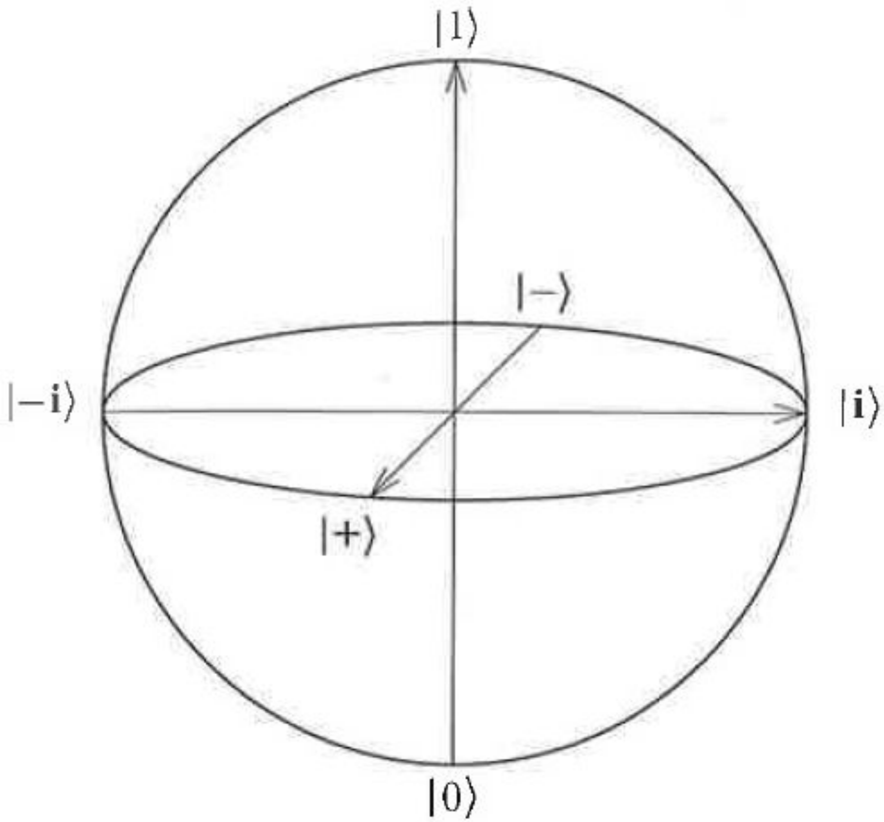

Figure 2.6
Location of certain single-qubit states on the surface of the Bloch sphere.

$$|0\rangle \mapsto (0, 0, 1)$$

$$|1\rangle \mapsto (0, 0, -1)$$

$$|+\rangle \mapsto (1, 0, 0)$$

$$|-\rangle \mapsto (-1, 0, 0)$$

$$|\mathbf{i}\rangle \mapsto (0, 1, 0)$$

$$|-\mathbf{i}\rangle \mapsto (0, -1, 0)$$

Chap 2 Single-Qubit Quantum Systems

State space of single-qubit system

- 3 representations of single-qubit state space
  - 1. Vectors in ket notation $a|0\rangle + b|1\rangle$ with complex coeficients a and b subject to $|a|^2 + |b|^2 = 1$
    - Where a and b are unique up to a unit complex factor
    - Representation not one-to-one
  - 2. Extended complex plane
    - One-to-one representation
  - 3. Bloch sphere
    - One-to-one representation