Steeb

Chapter 1

Qubits

1.1 Introduction

A single *qubit* is a two-state system, such as a two-level atom. The states (kets) $|h\rangle$ and $|v\rangle$ of the horizontal and vertical polarization of a photon can also be considered as a two-state system. Another example is the relative phase and intensity of a single photon in two arms of an interferometer. The underlying Hilbert space for the qubit is $\mathbb{C}^2$. An arbitrary orthonormal basis for $\mathbb{C}^2$ is denoted by $\{ |0\rangle, |1\rangle \}$, where (scalar product)

$$\langle 0|0\rangle = \langle 1|1\rangle = 1, \qquad \langle 0|1\rangle = \langle 1|0\rangle = 0.$$

Any pure quantum state $|\psi\rangle$ (qubit) of this system can be written, up to a phase, as a *superposition* (linear combination of the states)

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle\,, \quad |\alpha|^2 + |\beta|^2 = 1\,, \quad \alpha, \beta \in \mathbb{C}.$$

The classical boolean states, 0 and 1, can be represented by a fixed pair of orthonormal states of the qubit. The *standard basis* in $\mathbb{C}^2$ is given by

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and the *Hadamard basis* in $\mathbb{C}^2$ is given by

$$|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Up to an overall phase an arbitrary normalized state in $\mathbb{C}^2$ can be written as

$$|\psi\rangle = \begin{pmatrix} e^{i\phi}\cos(\theta) \\ \sin(\theta) \end{pmatrix}.$$

For any orthonormal basis $\{|0\rangle, |1\rangle\}$ in $\mathbb{C}^2$ we have

$$|0\rangle\langle 0| + |1\rangle\langle 1| = I_2$$

where I_2 is the 2×2 identity matrix. The 2×2 matrices

$$|0\rangle\langle 0|, \qquad |1\rangle\langle 1|$$

are projection matrices with

$$|0\rangle\langle 0|1\rangle\langle 1| = 0_2.$$

Furthermore

$$(|0\rangle\langle 1| + |1\rangle\langle 0|)^2 = |1\rangle\langle 1| + |0\rangle\langle 0| = I_2.$$

Given two normalized states $|\psi\rangle$, $|\phi\rangle$ in $\mathbb{C}^2$, then $0 \leq |\langle\psi|\phi\rangle|^2 \leq 1$ provides a probability. Let $|\psi\rangle \in \mathbb{C}^2$ and normalized. Then

$$\rho = |\psi\rangle\langle\psi|$$

is a density matrix (*pure state*). We have

$$\rho^2 = |\psi\rangle\langle\psi|\psi\rangle\langle\psi| = |\psi\rangle\langle\psi| = \rho.$$

If the qubit represents a *mixed state* one uses a two-dimensional *density matrix* ρ for its representation. We therefore express one qubit as

$$\rho = \frac{1}{2}(I_2 + \mathbf{n}\cdot\boldsymbol{\sigma}) \equiv \frac{1}{2}(I_2 + n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3)$$

where $\mathbf{n} \in \mathbb{R}^3$,

$$\mathbf{n}\cdot\mathbf{n} \equiv n_1^2 + n_2^2 + n_3^2 \leq 1$$

and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ denote the *Pauli spin matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For pure states we have $\mathbf{n}\cdot\mathbf{n} = 1$ and $\rho = |\psi\rangle\langle\psi|$. The Pauli spin matrices are hermitian and unitary and admit the eigenvalues $+1$ and -1.