3. Inductive Proof of Fibonacci Sequence

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The Fibonacci numbers read along the following pattern supposing positive, real constants i and k:

$$F_0 = 1, F_1 = 1, F_2 = 3, F_3 = 4, ...F_i = F_{i-1} + F_{i-2}$$

These must satisfy the term $F_i < (5/3)^i \quad \forall i \geq 1$. Our base case is satisfied accordingly: $F_2 = 2 < 25/9$, such that our hypothesis is that the theorem is true while $i < k$ or, $i = 1, 2, ...k$ or, $\lim_{i \to k}$. Formally, the theorem holds true such that $F_{k+1} = F_k + F_{k-1}$. Thus we shall compose the proof with a clever use of the number 1 to employ it to prove the hypothesis on the right-hand side. Here are some key expository predicates that help to explain the formal proof:

1. $1 = \frac{(5/3)}{(5/3)} = \frac{(5/3)}{(5/3)}^2$

2. $(5/3)^k + (5/3)^{k-1} = \frac{(5/3)}{(5/3)}(5/3)^k + \frac{(5/3)}{(5/3)}^2(5/3)^{k-1}$

3. $\frac{(5/3)}{(5/3)}(5/3)^k + \frac{(5/3)}{(5/3)}^2(5/3)^{k-1} = (3/5)(5/3)^{k+1} + (3/5)^2(5/3)^{k+1}$

   because $\frac{(5/3)^{k+1}}{(5/3)} = (3/5)(5/3)^{k+1}$

   and $\frac{(5/3)^{k+1}}{(5/3)^2} = (3/5)^2(5/3)^{k+1}$

Accordingly, the proof reads:

$$F_{k+1} < (5/3)^k + (5/3)^{k-1}$$

$$F_{k+1} < (3/5)(5/3)^{k+1} + (3/5)^2(5/3)^{k+1}$$
$$F_{k+1} < (3/5)(5/3)^{k+1} + (9/25)(5/3)^{k+1}$$
$$F_{k+1} < (3/5 + 9/25)(5/3)^{k+1}$$
$$F_{k+1} < (24/25)(5/3)^{k+1}$$
$$F_{k+1} < (5/3)^{k+1}$$

The proof hinges on converting the right-hand side to terms of F_{k+1}, which occurs between the first and second step of the illustration. These are explained in the predicates, especially in predicate 3. Using terms on the right hand side with aforementioned uses of the number 1, we conclude that $F_{k+1} < (5/3)^{k+1}$, which proves the theorem.