1a) Prove that $n(n+7)/4$, or $\frac{n^2}{4} + \frac{7n}{4} \in O(n^3)$ is true.

Proof:

by the Big-Oh definition, ~~given~~ runtime $T(n) = \frac{n^2}{4} + \frac{7n}{4}$ is $O(n^3)$ if $T(n) \leq c \cdot n^3$ for some $n \geq n_0$. Let's check this condition.

If $\frac{n^2}{4} + \frac{7n}{4} \leq c \cdot n^3$ then $\frac{1}{4} + \frac{7}{4n^2} \leq c$. Therefore, the Big-Oh condition holds for $n \geq n_0 = 1$ and $c \geq 2$ $(= 1/4 + 7/4)$. Larger values of n_0 result in smaller factors c (e.g. for $n_0 = 1$, $c \geq 2$ and for $n_0 = 2$, $c \geq 0.6875$, etc.). In any case, the above statement is valid.

1b Prove that $n(n+6)/5$, or $\frac{n^2}{5} + \frac{6n}{5} \in O(n^2)$ is true.

Proof:

by the Big-Oh definition, runtime $T(n) = \frac{n^2}{5} + \frac{6n}{5}$ is $O(n^2)$ if $T(n) \leq c \cdot n^3$ for some $n \geq n_0$. Let's check this condition. If $\frac{n^2}{5} + \frac{6n}{5} \leq c \cdot n^2$ then $1/5 + 6/5 \leq c$. Therefore, the Big-Oh condition holds for $n \geq n_0 = 1$ and $c \geq 1.4$ $(= 1/5 + 6/5)$. Larger values of n_0 result in smaller factors c (e.g. for $n_0 = 1$, $c \geq 1$ and for $n_0 = 2$, $c \geq .8$, etc.). In any case, the above statement is valid.

2a)

Sidework 1: for $(n^2)^{18} \Rightarrow n^{36}$: say $n^{36} \in \Theta(n^{36})$

(a) Test $\Omega(n^{36}) \ni n^{36}$

$n^{36} \geq c \cdot n^{36}$ say $c=1, n=2$

$n^{36} \geq n^{36}$

$2^{36} \geq 2^{36}$

(b) Test $O(n^{36}) \ni n^{36}$

$n^{36} \leq c \cdot n^{36}$ say $c=1, n=2$

$n^{36} \leq n^{36}$

$2^{36} \leq 2^{36}$

Sidework 2: does this work with $\Theta(n^{35})$?

(a) Test $\Omega(n^{35}) \ni n^{36}$

$n^{36} \geq c \cdot n^{35}$ say $c=1, n=2$

$n^{36} \geq n^{35}$

$2^{36} \geq 2^{35}$

(b) Test $O(n^{35}) \ni n^{36}$ say $c=1$,

$n^{36} \leq 1 \cdot n^{35}$

$n \leq 1$ as n increases, c must increase

$\Downarrow$

$n \leq c$

Proof: I will prove that the tightest-bound notation for big theta, Θ for a function $f(n) = (n^2)^{18}$ is n^{36}. To simplify, let's refer to $f(n)$ as n^{36} since $(n^2)^{18} = n^{36}$. Next, we let positive, real constants c and n_0 and $n \geq n_0 \geq 1$ according to examine the two required notations for $\Theta(n^{36})$: (a) $\Omega(n^{36})$ and (b) $O(n^{36})$:

(a) $f(n) = n^{36} \in \Omega(n^{36})$ is evaluated such that $n^{36} \geq c \cdot n^{36}$. Again, as $c=1$, the inequality holds. Conversely $O(n^{36})$ also holds as it is evaluated by the expression $n^{36} \leq c \cdot n^{36}$. Testing the next tightest bound, $\Theta(n^{35})$ fails along the evaluation of $n^{36} \in \Theta(n^{35})$: $n^{36} \leq c \cdot n^{35}$ sets up a relation such that c must increase as n increases, disproving $f(n) \in \Theta(n^{35})$

[c] claim: $n(n+9)/8 \in \Theta(n^3)$ is false

→ <u>Sidework</u> ~~Fam~~ first, for $\Omega(n^3)$: reminder that O will be

$$n(n+9)/8 \Longrightarrow \frac{n^2}{8}+\frac{9n}{8} \geq c \cdot n^3 \qquad \frac{n^2}{8}+\frac{9n}{8} \leq c \cdot n^3$$

let's say $c=1$

$$\frac{n^2}{8}+\frac{9n}{8} \geq n^3 \Longrightarrow n^2+9n \geq 8n^3$$

(side tracked...)

rule: if $A+B+C+D=0$, $x=1$

$Ax^3+Bx^2+Cx+D=0$, ~~then one value of x is 1~~

$$\underbrace{(-8n^3+n^2+9n)}_{-1} \geq \underbrace{0}_{-1}$$

$$8n^3-n^2-9n \leq 0$$

$8-1-9=-2$ ✗ nope

rule: if alternating coefficients summed are the same, then $x=-1$

$$8n^2-n-9 \leq 0$$

$8-9 \quad -1+0$ ✓ one $n \leq -1$

$$n \leq \frac{1 \pm \sqrt{(1)-4(8)(-9)}}{2(8)}$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$n \leq \frac{1 \pm \sqrt{289}}{16} \Longrightarrow n \leq \frac{1 \pm 17}{16}$$

$$n \leq \frac{18}{16} \quad n \leq -1$$

$$n \leq 1.125 \quad n \leq -1$$

Proof:

Let $c=1$ and it will be shown that n^3 fails to account for the Big Ω of the function $f(n)=n(n+9)/8$ according to the definition. If $c=1$, $\frac{n^2}{8}+\frac{9n}{8} \geq k^3$ must be true but this is impossible, as increasing values of n result in smaller values of c in order for this to hold up. This contradicts the big omega definition which contradicts the big theta definition as well.

1d

~~Example 1: Prove that running time $T(n) = n^3 + 20n + 1$ is $O(n$~~

Prove that $n(n+2)/9 \in \Omega(n)$

Side work

$$n(n+2)/9 \Longrightarrow \frac{n^2}{9} + \frac{2n}{9} \geq c \cdot n \quad \forall n \geq n_0 \geq 1$$

let's try $c = 1$

$$\frac{n^2}{9} + \frac{2n}{9} \geq n \Longrightarrow n^2 + 2n \geq 9n$$

$ax^2 + bx + c = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Solve for expression in relation to 0

$$n^2 - 7n \geq 0 \qquad n \geq \frac{7 \pm \sqrt{49}}{2}$$

$$\begin{array}{r} 64 \\ +16 \\ \hline 80 \end{array} \qquad \begin{array}{r} 49 \\ +14 \\ \hline 63 \end{array}$$

$\boxed{n \geq 7}$, $n \geq 0$

relevant

examine factors of c vs n_0

$n_0 = 7$: $7(7+2)/9 \geq c \Longrightarrow 7 \geq c$

$n_0 = 8$: $8(8+2)/9 \geq c \Longrightarrow 8.\bar{8} \geq c$

Proof

Let $c = 1$, $n_0 = 7$ and they are constants such that $n \geq n_0$.
If $n \geq n_0 = 7$, then, in accordance with the quadratic formula,

$$n \geq \frac{7 \pm \sqrt{49}}{2} \Longrightarrow n^2 - 7n \geq 0 \Longrightarrow \frac{n^2}{9} + \frac{2n}{9} \geq c \cdot n \quad \forall n \geq 7.$$

Larger values of n_0 result in larger factors c (e.g. for $n_0 = 7$, $c \geq 7$; for $n_0 = 8$, $c \geq 8.\bar{8}$). In any case $n(n+2)/9 \in \Omega(n)$ is true.

2a

Sidework 1: for $(n^2)^{18} \Rightarrow n^{36}$: say $n^{36} \in \Theta(n^{36})$

(a) Test $\Omega(n^{36}) \ni n^{36}$

$n^{36} \geq c \cdot n^{36}$ say $c=1, n=2$

$n^{36} \geq n^{36}$

$2^{36} \geq 2^{36}$

(b) Test $O(n^3) \ni n^{36}$

$n^{36} \leq c \cdot n^{36}$ say $c=1, n=2$

$n^{36} \leq n^{36}$

$2^{36} \leq 2^{36}$

Sidework 2: does this work with $\Theta(n^{35})$?

(a) Test $\Omega(n^{35}) \ni n^{36}$

$n^{36} \geq c \cdot n^{35}$ say $c=1, n=2$

$n^{36} \geq n^{35}$

$2^{36} \geq 2^{35}$

(b) Test $O(n^{35}) \ni n^{36}$ say $c=1$,

$n^{36} \leq 1 \cdot n^{35}$

$n \leq 1$ as n increases, c must increase

$\Downarrow$

$n \leq c$

Proof: I will prove that the tightest-bound notation for big theta, Θ for a function $f(n) = (n^2)^{18}$ is n^{36}. To simplify, let's refer to $f(n)$ as n^{36} since $(n^2)^{18} = n^{36}$. Next, we let positive, real constants c and n_0 accordingly: $c=1$ and $n \geq n_0 \geq 1$ to examine the two required notations for $\Theta(n^{36})$: (a) $\Omega(n^{36})$ and (b) $O(n^{36})$:

(a) $f(n) = n^{36} \in \Omega(n^{36})$ is evaluated such that $n^{36} \geq c \cdot n^{36}$. Again, as $c=1$, the inequality holds. Conversely $O(n^{36})$ also holds as it is evaluated by the expression $n^{36} \leq c \cdot n^{36}$. Testing the next tightest bound, $\Theta(n^{35})$ fails along the evaluation of $n^{36} \in \Theta(n^{35})$: $n^{36} \leq c \cdot n^{35}$ sets up a relation such that c must increase as n increases, disproving $f(n) \in \Theta(n^{35})$.

~~Let's Prove that $n(n+2) \in \Omega(n^2)$ is true~~

~~Proof~~:

2-b) $\text{sqrt}(n^{16}) \Rightarrow n^8$

sidework 1: $n^8 \in \Theta(n^8)$

(a) Test $\Omega(n^8) \ni n^8$: say $c=1, n=2$

$$n^8 \geq c \cdot n^8 \Longrightarrow 2^8 \geq 1 \cdot 2^8$$

$$256 \geq 256$$

(b) Test $O(n^8) \ni n^8$: say $c=1, n=2$

$$n^8 \leq c \cdot n^8 \Longrightarrow 2^8 \leq 1 \cdot 2^8$$

$$256 \leq 256$$

sidework 2 just test $O(n^7) \ni n^8$

$$n^8 \leq c \cdot n^7 \quad \text{say } c=1$$

$$n^8 \leq 1 \cdot n^7 \Longrightarrow \left.\begin{array}{l} n \leq 1 \\ n \leq c \end{array}\right\} \therefore \text{ as } n \text{ increases, } c \text{ must increase}$$

Proof: I will prove that the tightest-bound notation for big theta, Θ for a function $f(n) = \sqrt{n^{16}}$ is n^8. $f(n)$, in simpler terms, is n^8. Let positive, real constants $c=1$ and n_0 such that $n \geq n_0 \geq 1$ evaluate all relevant notations of Ω and O, both of which $f(n)$ must be a subset of to satisfy ~~any notation~~ $f(n) \in \Theta(n^8)$. ~~Again, as test~~ ~~In evaluating~~ ~~the terms used to evaluate~~ ~~In evaluating~~

In evaluating both (a) $f(n) \in \Omega(n^8)$ and (b) $f(n) \in O(n^8)$, the following terms can be used to evaluate them: (a) $n^8 \geq c \cdot n^8$ and (b) $n^8 \leq c \cdot n^8$. Under the condition that $c=1$, these inequalities clearly are satisfied, proving $f(n) = \sqrt{n^{16}} \in \Theta(n^8)$. To test the next tightest bound, $f(n) \in \Theta(n^7)$, it's only necessary to evaluate $f(n) \in O(n^7)$. Here again, the next implied expression $n^8 \leq c \cdot n^7$ requires that c grows as n's value increases, disproving $f(n) \in \Theta(n^7)$.

real proof!

2c Proof: $f(n) = n^2 \cdot \log n/2 \in \Theta(n^2 \cdot \log n/2)$ as the tightest bound.

Let positive constant $c = 1$ and we shall examine:

ⓐ $f(n) \in \Omega(n^2 \cdot \log n/2)\ \forall n \geq n_0 \geq 1$

ⓑ $f(n) \in O(n^2 \cdot \log n/2)\ \forall n \geq n_0 \geq 1$

First, condition ⓐ: $n^2 \cdot \log n/2 \geq 1 \cdot n^2 \cdot \log n/2$ is obviously true.

Second, condition ⓑ: $n^2 \cdot \log n/2 \leq 1 \cdot n^2 \cdot \log n/2$ is obviously true.

Thus the hypothesis is true.

- ~~It is also the tightest bound as the dominant exponent on the right-hand side, n^2, cannot be altered to be linear.~~ This hypothetical Θ notation, $n \cdot \log n/2$, would break along the aforementioned condition ⓑ as follows:

$(f(n) =) n^2 \cdot \log n/2 \leq 1 \cdot n \cdot \log n/2$. This assignment forces c to increase as n increases, breaking the Big O condition, and breaking it as a tighter bound. $\Theta(n^2)$ also fails as, for $O(n^2) \ni f(n)$, c must increase even after the minimum value for n has been established:

Sidework:

~~Proof~~: $f(n) = n^2 \cdot \log n/2 \in \Theta(n^2)$ as the tightest bounds. Side Proof!

Side Proof: Let positive constants $c = 1$ and $n_0 = 20$. The following conditions hold for the hypothesis to hold:

ⓐ $f(n) \in \Omega(n^2)\ \forall n \geq n_0 \geq 20$:

$n^2 \cdot \log n/2 \geq n^2 \cdot c$

$20^2 \cdot \log 20/2 \geq 20^2 \cdot 1 \Rightarrow 400 \cdot 1 \geq 400 \cdot 1$, which is obviously true

ⓑ $f(n) \in O(n^2)\ \forall n \geq n_0 \geq 20$ cannot be proven, however:

$n^2 \cdot \log n/2 \leq c \cdot n^2$

~~$\log n/2 \leq c$~~ let $c = 1$ and $n = 20$

$21^2 \cdot \log 21/2 \leq 1 \cdot 21^2 \Rightarrow 441 \cdot (1.02) \leq 1 \cdot 441$ is obviously false

~~Given ⓐ showed the minimum of n to be 20,~~

~~$1 \leq c$~~

~~which is supported by the definition of Big O. Therefore, the above~~