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SELF-TUNING CONSENSUS ON DIRECTED GRAPHS IN THE CASE OF UNKNOWN CONTROL DIRECTIONS*

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Abstract. This paper considers the problem of leaderless self-tuning consensus in multiagent discrete time systems with unknown control directions. We introduce novel algorithms for adaptive tuning of interagent coupling parameters. The control algorithm is fully distributed and the agent control signals are continuous functions in their arguments. Networked systems on directed graphs are analyzed. Assuming that the graph is strongly connected it is proved that all agent states converge toward the same value, and the interagent coupling parameters are convergent sequences.

Key words. multiagent systems, self-tuning consensus, directed graphs, adaptive synchronization

AMS subject classifications. 93A, 93D, 93E

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1. Introduction.

During the past several years a number of papers have appeared that consider the problem of cooperative control, synchronization, and consensus in networked dynamical systems. Many of the results in this area are focused on cases with known system dynamics. Extensive lists of references on distributed coordination, synchronization, and consensus in networked control systems can be found in the recent survey paper [1] and recently published research monographs [2, 3, 4]. Since the topic of our work is adaptive consensus of multiagent dynamical systems, we only review and comment on the related results concerning systems with unknown parameters and uncertainties in agent dynamics. One of the first works on adaptive synchronization of networked control systems is [5]. The authors assume that the synchronous solutions of the autonomous network are global information available to all agents. The local control input is proportional to the error between the individual agent states and the known synchronous solution. It is shown that this error converges to zero. Similar analysis is presented in [6]. The assumption that the synchronous solution is known in advance and that it is a bounded signal, is a restrictive requirement. Distributed synchronization in complex networks is considered in [7] where authors propose a novel adaptive algorithm for tuning the interagent coupling parameters. The presented analysis requires that a certain matrix that depends on the coupling parameters be negative semidefinite at all times $t > 0$. The problem of steering a group of agents to a desired reference velocity is analyzed in [8] by incorporating relative position and relative velocity feedbacks. Interesting results on a leader-following problem in multiagent systems are presented in [9, 10, 11] where it is assumed that the leader trajectory is known to all agents, and the system dynamics is linear with respect to unknown parameters and known basis functions. The agents are coupled via fixed nonadaptive gain whose value is a global piece of information. The algorithm presented in [11] achieves the desired leader following in a finite time.

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Important results on adaptive dynamics protocols in multiagent systems are reported in [12] where it is shown that each agent state converges towards the average of its neighbor's states. Extension of the above results to systems with general linear and Lipschitz nonlinear dynamics is given in [13]. In [12, 13] it is assumed that the linear dynamics is known, and the network topology is characterized by an undirected graph. Recently interesting results were reported in [14] where multiagent systems with general linear dynamics and directed graph topology are considered. There, the authors propose a fully distributed adaptive protocol for the leader-following problem. As is remarked in [15] the existing approaches to adaptive multiagent consensus including the above-mentioned, are not suited for networked systems with unknown control directions. For the first time this problem has been considered in [15, 16] where consensus is achieved by using a Nussbaum-type controller. In [16] continuous-time integrator agent dynamics is analyzed, and [15] considers a more general continuous-time model linear with respect to unknown parameters and known basis functions. Both [15] and [16] treat the case of undirected graph topology.

Motivated by [15, 16], in this paper we propose distributed algorithms for achieving consensus among agents with unknown control directions in the case of directed network graphs. As stated in the conclusion section of [15], this is still an open problem. We analyze discrete-time agent dynamics and show that (i) the difference between any two agent states converges to zero, and (ii) agent states and coupling parameters are convergent sequences. Statement (i) is demonstrated in [15, 16] as well, for undirected graphs, and instead of (ii) they show that agent states and coupling gains are bounded functions. The consensus solution proposed in this paper utilizes a novel discrete-time control protocol reminiscent of a continuous-time Naussbaum-type controller. In 1983 Naussbaum [17] presented a method for adaptive stabilization of a first order continuous-time system without requiring the knowledge of the sign of the high-frequency gain. This approach came to be known as "universal stabilization" and was further advanced in follow-up papers (see, for example, [18, 19, 20, 21]). An overview of major contributions on universal stabilization with a large list of references can be found in [22, 23]. Most of the work on this topic is confined to continuous-time systems. Extension to the discrete-time case is not straightforward. In [24] a discrete-time version of the Nussbaum gain is generated by means of switching curves. Discrete-time stabilizers presented in [25, 26] do not rely on a Nussbaum-type regulation, and both controllers require the knowledge of an upper bound of the feedback gain. The consensus problem in our paper is solved by a smooth controller (control law is a continuous function of its arguments) without involving switching functions or switching curves. There is no need for a priori knowledge of an upper bound on the high-frequency gain.

In this paper we use the following notation: $\Re$ denotes the set of real numbers; $\Re^N$ is the set of N-dimensional vectors with real entries; $\Re^{N\times N}$ denotes the space of $N \times N$ real matrices; $\mathbb{N}$ is the set of natural numbers; the superscript T denotes the transpose of a matrix; $\rho(A)$ denotes the spectral radius of the matrix A; $Span(x)$ is the vector space spanned by the vector x; $\| x \|$ denotes the Euclidean norm of vector x; and $sgn(b)$ is the sign function of a real number b. In this paper ℓ is used to denote a vector with all entries equal to one. The dimension of ℓ will be clear from the context. For two functions $f(s)$ and $g(s)$ defined on some subset of real numbers we write $f(s) = O\left(g(s)\right)$ as $s \to \infty$ iff there exist constants $c > 0$ and s_0 such that $|f(s)| \leq c|g(s)|$ $\forall s \geq s_0$. When performing majorizations and in certain upper bounds, we use c_i, $i = 1, 2, \ldots$ to denote nonnegative constants whose values are unimportant.

2. Problem formulation. The aim of this paper is to propose a distributed algorithm for leaderless coordination in networked systems with unknown control directions so that certain states of all systems converge towards a common (consensus) value. Such consensus value may represent a destination point of a rendezvous task for a group of robots, or a sailing velocity of a group of unmanned vehicles with a desired formation. Inspired by the emergent behavior in flocks and robot formation control, we consider a system of N identical agents whose dynamics is given by

$$x_i(t+1) = x_i(t) + bu_i(t), \quad b \neq 0, \quad i = 1, \dots, N, \tag{2.1}$$

where $t \in \{0, 1, 2, \dots\}$ is the discrete time index, $x_i(t) \in \Re$ and $u_i(t) \in \Re$ are the ith agent state and input signals, respectively. Initial values $x_i(0), i = 1, \dots, N$, are finite numbers. Without loss of generality we assume that $x_i(k) = 0, u_i(k) = 0$ $\forall k < 0$. It is assumed that the high-frequency gain b is an unknown parameter. The communication topology of the considered network is described by a directed graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$, where $\mathbf{V} = \{1, 2, \dots, N\}$ is the set of nodes and $\mathbf{E} \subseteq \mathbf{V} \times \mathbf{V}$ is the set of edges (communication links). The node i represents the agent i, and the ordered pairs (i, j), $i \neq j$, denote edges. If the ith agent can receive information from the jth agent then $(j, i) \in \mathbf{E}$. The set of neighbors of node i is denoted by $\mathcal{N}_i = \{j \in \mathbf{V} | (i, j) \in \mathbf{E}\}$. The above described scenario can be encountered in problems of velocity coordination in robot formation control. As is described in [15], for robots produced in the same batch it is reasonable to assume that all units have the same dynamics and the same control directions. In order to simplify the underlying algebra, and for the sake of clarity, we are considering the case of a first order one dimensional agent dynamics (2.1). This model can be viewed as a discrete-time version of the following continuous-time rigid body dynamics

$$m\frac{dx_i(\tau)}{d\tau} = u_i(\tau), \quad \tau \geq 0,$$

where m is the mass, $x_i(\tau)$ is the velocity, and $u_i(\tau)$ is the external force driving the motion of the ith agent. The parameter b in (2.1) can be interpreted as the inverse of mass m.

In this paper we make reference to the following definition of leaderless consensus.

DEFINITION 1. *Multiagent system defined by* (2.1) *achieves wide-sense consensus if*

$$\lim_{t\to\infty} (x_i(t) - x_j(t)) = 0 \quad \forall i, j \in \mathbf{V}. \tag{2.2}$$

If in addition to (2.2) $\lim_{t\to\infty} x_i(t) = x_c$ $\forall i \in \mathbf{V}$ *for some finite* x_c*, we say that the multiagent system achieves strict-sense consensus.*

Obviously strict-sense consensus implies wide-sense consensus. The opposite is not true. Take for example $x_i(t) = cos(log(t + i))$ $\forall i \in \mathbf{V}$. Then (2.2) holds, while $\{x_i(t)\}$, $t \geq 0$, does not have a limit.

The objective is to design $u_i(t)$, $i \in \mathbf{V}$, so that the multiagent system (2.1) achieves strict-sense consensus. The stability analysis in case of the unknown control directions can be a fairly complex task algebraically. For the sake of presenting clear and easy to follow proofs, we gradually increase the complexity of our analysis by first considering the case of known sign of the parameter b in (2.1). The case of the unknown sign of the control direction is considered in section 4.

3. The case of known sign of b. In this section we assume that $sgn(b)$ is known. Let $\varphi_i(t)$ represent the missmatch between the ith agent state and the one step delayed average of neighboring states, i.e.,

$$\varphi_i(t) := x_i(t) - \frac{1}{1+N_i}\sum_{j\in\overline{\mathcal{N}_i}} x_j(t-1), \quad i \in \mathbf{V}, \tag{3.1}$$

where, $\overline{\mathcal{N}_i} = \mathcal{N}_i \cup \{i\}$, and N_i is the cardinality of the set $\mathcal{N}_i$. Since the sign of b is known, we can use the protocol $u_i(t) = \hat{\theta}_i(t)\varphi_i(t)$, where the adaptive gain $\hat{\theta}_i(t)$ can be updated by the gradient algorithm similar to [27]. The gradient-based protocol presented in [27] cannot handle the case of the unknown sign of parameter b. In this paper we propose the following distributed controller $\forall i \in \mathbf{V}$,

$$u_i(t) = -sgn(b)f_i(t)\frac{\varphi_i(t)}{r_i(t)} + \frac{1}{1+N_i}\sum_{j\in\overline{\mathcal{N}_i}} u_j(t-1), \tag{3.2}$$

with $f_i(t)$ defined by

$$f_i(t) = \theta_i(t)^{m_i-1}\sum_{n=0}^{m_i-1}\left(\frac{\theta_i(t-1)}{\theta_i(t)}\right)^n, \quad m_i > 1, \tag{3.3}$$

where $\theta_i(t)$ is generated by the following recursion,

$$\theta_i(t) = \theta_i(t-1) + \mu_i\frac{\varphi_i(t)^2}{r_i(t)}, \quad \theta_i(0) > 0, \quad \mu_i > 0, \tag{3.4}$$

with $r_i(t)$ given by

$$r_i(t) = r_i(t-1) + \varphi_i(t)^2, \quad r_i(0) \geq 1, \tag{3.5}$$

and $0 < \mu_i < \infty$ is the designer selected step size. In (3.3), $m_i > 1$ is an integer selected by the designer. In its simplest form $f_i(t) = \theta_i(t) + \theta_i(t-1)$ for $m_i = 2$. Note that at time index t the control protocol $u_i(t)$ (see (3.2)) presumes the knowledge of the values of neighboring control signals $u_j(t-1)$ and the states $x_j(t-1)$, $j \in \mathcal{N}_i$, at the time index $t-1$. Since both $u_j(t-1)$ and $x_j(t-1)$, $j \in \mathcal{N}_i$, $i \in \mathbf{V}$, are available at time t, they are transmitted to the ith agent for use in (3.2). We remark that in section 4 it is demonstrated that the protocol (3.2)–(3.5) can be used to solve the case of unknown sign of parameter b by replacing the time-varying gain $sgn(b)f_i(t)$ in (3.2) with a suitably chosen function of $\theta_i(t)$ generated by (3.4). By using (2.1) and (3.1) we can derive

$$\varphi_i(t+1) = \varphi_i(t) + b\left(u_i(t) - \frac{1}{1+N_i}\sum_{j\in\overline{\mathcal{N}_i}} u_j(t-1)\right), \quad i \in \mathbf{V}. \tag{3.6}$$

Substituting the control signal (3.2) in (3.6) yields

$$\varphi_i(t+1) = \varphi_i(t) - |b|f_i(t)\frac{\varphi_i(t)}{r_i(t)}, \quad i \in \mathbf{V}. \tag{3.7}$$

In order to examine (3.7) we need the following technical result.

LEMMA 3.1.

$$\theta_i(t) \le \theta_i(0) + \mu_i log\ r_i(t), \tag{3.8}$$

$$I_{1i}(t) := \sum_{k=1}^{t} (log\ r_i(k))^n \frac{\varphi_i(k)^2}{r_i(k)^2} \le c_1 < \infty \tag{3.9}$$

$\forall t \ge 1$, $0 \le n < \infty$, $i \in \mathbf{V}$, and some constant $c_1 > 0$.

Proof. The first statement follows from (3.4) and (3.5), i.e.,

$$\begin{aligned}\theta_i(t) &= \theta_i(0) + \mu_i \sum_{k=1}^{t} \frac{1}{r_i(k)} \int_{r_i(k-1)}^{r_i(k)} dy \le \theta_i(0) + \mu_i \sum_{k=1}^{t} \int_{r_i(k-1)}^{r_i(k)} \frac{dy}{y} \\ &= \theta_i(0) + \mu_i \left(log\ r_i(t) - log\ r_i(0)\right).\end{aligned} \tag{3.10}$$

We next prove (3.9). If $\{r_i(k)\}$, $k \ge 0$, is a bounded sequence, then

$$I_{1i}(t) \le c_2 \sum_{k=1}^{t} \frac{\varphi_i(k)^2}{r_i(k)^2} \tag{3.11}$$

for some finite $c_2 > 0$. If $r_i(k) \to \infty$ as $k \to \infty$, then $(log\ r_i(k))^n / r_i(k)^{\varepsilon_1} \to 0$ as $k \to \infty$ for $\varepsilon_1 > 0$. Hence, $I_{1i}(t)$ satisfies $I_{1i}(t) \le c_3 \sum_{k=1}^{t} \varphi_i(k)^2 / r_i(k)^{2-\varepsilon_1}$ for some positive constant c_3. Then from (3.5) for any $0 \le \varepsilon_2 < 1$

$$\begin{aligned}\sum_{k=1}^{t} \frac{\varphi_i(k)^2}{r_i(k)^{2-\varepsilon_2}} &= \sum_{k=1}^{t} \frac{1}{r_i(k)^{2-\varepsilon_2}} \int_{r_i(k-1)}^{r_i(k)} dy \le \sum_{k=1}^{t} \int_{r_i(k-1)}^{r_i(k)} \frac{dy}{y^{2-\varepsilon_2}} \\ &\le \frac{1}{(1-\varepsilon_2) r_i(0)^{1-\varepsilon_2}},\end{aligned}$$

i.e., statement (3.9) holds. ◻

In the following we show that the equilibrium state $\varphi_i(t) = 0$, $i \in \mathbf{V}$, of the system dynamics described by (3.3)–(3.5) and (3.7) is globally asymptotically stable in the sense that $\varphi_i(t) \to 0$ as $t \to \infty$ for all initial conditions $\varphi_i(0)$, and all $\theta_i(0) > 0$ and $r_i(0) \ge 1$. We actually prove that $\varphi_i(t)$ converges to zero exponentially fast. In addition we demonstrate that $\{\theta_i(t)\}$ and $\{r_i(t)\}$, $t \ge 0$, are convergent sequences.

THEOREM 3.2. *Consider the nonlinear dynamics described by* (3.7) *(or, equivalently, by* (2.1) *and* (3.2)*). Then for all $\varphi_i(0)$ (or $x_i(0)$),*

$$\bar{\theta}_i := \lim_{t\to\infty} \theta_i(t) \quad \textit{exists}, \tag{3.12}$$

$$\bar{r}_i := \lim_{t\to\infty} r_i(t) \quad \textit{exists}, \tag{3.13}$$

$$|\varphi_i(t)| \le c_4 \rho_1^t, \quad 0 < \rho_1 < 1, \quad 0 < c_4 < \infty. \tag{3.14}$$

Proof. After squaring both sides of (3.7) we obtain

$$\varphi_i(t+1)^2 = \varphi_i(t)^2 - 2|b| f_i(t) \frac{\varphi_i(t)^2}{r_i(t)} + b^2 f_i(t)^2 \frac{\varphi_i(t)^2}{r_i(t)^2}, \quad i \in \mathbf{V}. \tag{3.15}$$

By substituting $\left(\mu_i \varphi_i(t)^2 / r_i(t)\right) = \theta_i(t) - \theta_i(t-1)$ (see (3.4)) in the second term on the right-hand side (RHS) of (3.15) one obtains $\forall i \in \mathbf{V}$

$$\varphi_i(t+1)^2 = \varphi_i(t)^2 - \frac{2|b|}{\mu_i} \left(\theta_i(t)^{m_i} - \theta_i(t-1)^{m_i}\right) + b^2 f_i(t)^2 \frac{\varphi_i(t)^2}{r_i(t)^2}, \tag{3.16}$$

where we used the fact that by (3.3),

$$f_i(t)(\theta_i(t) - \theta_i(t-1)) = \theta_i(t)^{m_i} - \theta_i(t-1)^{m_i}. \tag{3.17}$$

After "telescoping" (3.16) we arrive at

$$\varphi_i(t+1)^2 = \varphi_i(0)^2 - \frac{2|b|}{\mu_i}\left(\theta_i(t)^{m_i} - \theta_i(0)^{m_i}\right) + b^2 I_{2i}(t), \tag{3.18}$$

where

$$I_{2i}(t) := \sum_{k=1}^{t} f_i(k)^2 \frac{\varphi_i(k)^2}{r_i(k)^2}, \quad i \in \mathbf{V}. \tag{3.19}$$

Observe that by (3.4) $(\theta_i(t-1)/\theta_i(t)) \leq 1$. Then (3.3) and (3.8) imply $f_i(t) \leq c_5 + c_6 \left(log\ r_i(t)\right)^{m_i - 1}, i \in \mathbf{V}$, for some positive constants c_5 and c_6. Hence, application of (3.9) to (3.19) gives $I_{2i}(t) \leq c_7 < \infty\ \forall i \in \mathbf{V}$ for some constant $c_7 > 0$. Consequently (3.18) implies that $\{\theta_i(t)\}$, $t \geq 0$, $i \in \mathbf{V}$, is a bounded sequence. Since $\{\theta_i(t)\}$, $t \geq 0$, is a nondecreasing positive sequence, it must have a limit. Thus, statement (3.12) is proved. From (3.4) and (3.12) we obtain

$$\lim_{t\to\infty}\left(\theta_i(0) + \mu_i \sum_{k=1}^{t} \frac{\varphi_i(k)^2}{r_i(k)}\right) = \overline{\theta}_i, \quad i \in \mathbf{V}. \tag{3.20}$$

If $r_i(t) \to \infty$ as $t \to \infty$, application of Kronecker's lemma [28, p. 503] to (3.20) yields

$$\lim_{t\to\infty} \frac{1}{r_i(t)} \sum_{k=1}^{t} \varphi_i(k)^2 = 0, \quad i \in \mathbf{V}. \tag{3.21}$$

On the other hand (3.4) implies $r_i(t) = r_i(0) + \sum_{k=1}^{t} \varphi_i(k)^2$, or $\lim_{t\to\infty}\left(\sum_{k=1}^{t} \varphi_i(k)^2\right)/ r_i(t) = 1$, which contradicts (3.21). Thus $r_i(t) < \infty\ \forall t \geq 0$, $i \in \mathbf{V}$. Since by (3.5) $r_i(t)$ is nondecreasing, it must have a limit, i.e., statement (3.13) holds. We now prove (3.14) by analyzing (3.7). Note that (3.7) implies

$$\varphi_i(t+1) = (1 - a_i(t))\varphi_i(t), \quad i \in \mathbf{V}, \tag{3.22}$$

where

$$a_i(t) = |b| f_i(t)/r_i(t). \tag{3.23}$$

The following analysis of (3.22) considers two cases for $\varphi_i(0)$.

Case 1: *In* (3.22), $\varphi_i(0) \neq 0$. Since $\theta_i(t) \to \overline{\theta}_i$ and $r_i(t) \to \overline{r}_i$ as $t \to \infty$, we can conclude that $\overline{a}_i := \lim_{t\to\infty} a_i(t)$ exists. Since by (3.4) and (3.5), $\overline{\theta}_i > 0$ and $\overline{r}_i > 0$, from (3.3) and (3.22) it follows that $\overline{a}_i > 0$. Therefore (3.22) can be written as

$$\varphi_i(t+1) = (1 - \overline{a}_i)\varphi_i(t) + \tilde{a}_i(t)\varphi_i(t), \quad i \in \mathbf{V}, \tag{3.24}$$

with $\tilde{a}_i = \overline{a}_i - a_i(t) \to 0$ as $t \to \infty$. Then

$$\varphi_i(t+1) = (1 - \overline{a}_i)^t \varphi_i(0) + \sum_{k=0}^{t} (1 - \overline{a}_i)^{t-k} \tilde{a}_i(k)\varphi_i(k) \tag{3.25}$$

$\forall i \in \mathbf{V}$. We now show that $|1-\overline{a}_i| \le 1-\varepsilon_3$ for some $0<\varepsilon_3<1$ whenever $\varphi_i(0) \ne 0$. Observe that (3.5) and (3.13) imply

$$\lim_{t\to\infty} \varphi_i(t) = 0 \quad \forall i \in \mathbf{V}. \tag{3.26}$$

On the other hand by Stoltz's theorem [29, p. 85], the second term on the RHS of (3.25) tends to zero, i.e.,

$$\begin{aligned}\lim_{t\to\infty} \frac{\sum_{k=0}^{t}(1-\overline{a}_i)^{-k}\tilde{a}_i(k)\varphi_i(k)}{(1-\overline{a}_i)^{-t}} &= \lim_{t\to\infty} \frac{(1-\overline{a}_i)^{-t}\tilde{a}_i(t)\varphi_i(t)}{(1-\overline{a}_i)^{-t}-(1-\overline{a}_i)^{-t+1}} \\ &= \lim_{t\to\infty} \frac{\tilde{a}_i(t)\varphi_i(t)}{\overline{a}_i} = 0,\end{aligned} \tag{3.27}$$

by virtue of the fact that $\overline{a}_i > 0$, and $\tilde{a}_i(t) \to 0$ and $\varphi_i(t) \to 0$ as $t\to\infty$. Hence from (3.25), (3.26), and (3.27) we conclude that $(1-\overline{a}_i)^t \to 0$ as $t\to\infty$ for all $\varphi_i(0) \ne 0$. Therefore there exists ε_3 $(0<\varepsilon_3<1)$ so that

$$|1-\overline{a}_i| \le 1-\varepsilon_3 \quad \forall i \in \mathbf{V}. \tag{3.28}$$

On the other hand since $a_i(t) \to \overline{a}_i$ as $t\to\infty$, it follows that $\exists t_0^{(i)}$ so that $\forall t \ge t_0^{(i)}$ we have $|a_i(t)-\overline{a}_i| \le \varepsilon_3/2$, where ε_3 is the same as in (3.28). Then $|1-a_i(t)| \le |1-\overline{a}_i| + |\overline{a}_i - a_i(t)| \le 1-\varepsilon_3/2$ for all $t \ge t_0^{(i)}$, $i \in \mathbf{V}$. Thus from (3.22) one obtains

$$|\varphi_i(t+1)| = \left| \prod_{k=t_0^{(i)}}^{t} (1-a_i(k))\varphi_i(t_0^{(i)}) \right| \le \left(1-\frac{\varepsilon_3}{2}\right)^{t-t_0^{(i)}} \cdot |\varphi_i(t_0^{(i)})| \tag{3.29}$$

for some $\varepsilon_3, 0<\varepsilon_3<1$, and for all $i \in \mathbf{V}$. Statement (3.14) follows from (3.29) by virtue of the fact that $\varphi_i(t_0^{(i)})$ is finite (see (3.26)).

Case 2: *In* (3.22), $\varphi_i(0)=0$. Then from (3.22) we have $\varphi_i(t)=0$ $\forall t \ge 0$, i.e., (3.14) holds. Thus the theorem is proved. ∎

In order to show that the proposed algorithm guarantees strict-sense consensus (see Definition 1) we introduce the following assumption.

Assumption A1. The underlying network graph is strongly connected.

THEOREM 3.3. *Let Assumption* A1 *hold for the multiagent system* (2.1). *Then the control Algorithm* (3.2)–(3.5) *provides strict-sense consensus.*

Proof. We first demonstrate that the considered algorithm guarantees wide-sense consensus (see Definition 1). Define

$$x(t)^T := [x_1(t), \dots, x_N(t)], \tag{3.30}$$

$$\varphi(t)^T := [\varphi_1(t), \dots, \varphi_N(t)]. \tag{3.31}$$

Then (3.1) can be written in the form

$$\varphi(t+1) = x(t+1) - Qx(t), \tag{3.32}$$

where $Q \in R^{N\times N}$ is given by $Q = [q_{ij}]$ with

$$q_{ij} = \begin{cases} 1/(1+N_i), & j \in \mathcal{N}_i, \quad j = i, \\ 0, & \text{otherwise}. \end{cases} \tag{3.33}$$

Since by Assumption A1 the considered network graph is strongly connected, stochastic matrix Q is irreducible, implying that $\lambda_1 = 1$ is an algebraically simple eigenvalue of Q, and it is its maximal eigenvalue [30, Thm. 8.4.4., p. 508]. Let ℓ and v denote the right and left eigenvectors of Q corresponding to $\lambda_1 = 1$, i.e., $Q\ell = \ell$ and $v^T Q = v^T$. From (3.33) it is not difficult to see that $\ell \in \Re^N$ is equal to $\ell^T = [1, \dots, 1]$. Note that according to the Peron–Frobenius theorem [30, Thm. 8.4.4., p. 508] ℓ and v are positive vectors. Since $q_{ii} > 0$, $i = 1, \dots, N$, the matrix Q is primitive implying that it has only one eigenvalue of maximal modulus. Based on the above discussion, the matrix Q can be decomposed as follows:

$$Q = Q_1 + \ell v^T, \quad \ell^T v = 1, \tag{3.34}$$

where v is normalized so that $\ell^T v = 1$ and Q_1 is a matrix with the spectral radius $\rho(Q_1) < 1$. In addition Q_1, ℓ, and v satisfy $Q_1 \ell = 0$ and $v^T Q_1 = 0$. We are now equipped to show that the process

$$\delta(t+1) := x(t+1) - \ell v^T x(t) \tag{3.35}$$

converges to zero, implying that all elements of $x(t+1)$ asymptotically behave as $v^T x(t)$, thereby yielding (2.2). Substituting (3.34) into (3.32) yields

$$\varphi(t+1) = x(t+1) - \ell v^T x(t) - Q_1 \left(x(t) - \ell v^T x(t-1) \right),$$

where we used the fact that in (3.34), $Q_1 \ell = 0$. By using the definition of $\delta(t)$ (see (3.35)) in the previous equation, one can derive $\varphi(t+1) = \delta(t+1) - Q_1 \delta(t)$ from where it follows that

$$\left(I - q^{-1} Q_1 \right) \delta(t+1) = \varphi(t+1), \tag{3.36}$$

where q^{-1} is the unit delay operator ($q^{-1}\delta(t+1) := \delta(t)$). Since all eigenvalues of Q_1 are strictly inside the unit circle, $(I - z^{-1}Q_1)^{-1}$ is a stable transfer function (here z is a complex variable) which together with (3.14) and (3.36) gives

$$|\delta(t+1)| \le c_8 \rho_2^t, \quad 0 < \rho_2 < 1, \forall t \ge 0. \tag{3.37}$$

Observe that (3.35) implies $x_i(t+1) = \delta_i(t+1) + v^T x(t)$ $\forall i \in \mathbf{V}$ where $\delta_i(t)$ is the ith component of the vector $\delta(t)$. Then from (3.35) and (3.37) it follows that (2.2) holds. Note that (2.2) does not guarantee convergence of $x_i(t)$, $i \in \mathbf{V}$. Take, for example, $x_i(t) = (-1)^t$ or $x_i(t) = cos(log(t+i))$ $\forall i \in \mathbf{V}$. Next we prove that $x_i(t) \to x_c$ $\forall i \in \mathbf{V}$ for some finite x_c. From (3.35) one obtains

$$x(t+1) = (\ell v^T)^t x(0) + \sum_{k=0}^{t} (\ell v^T)^{t-k} \delta(k+1). \tag{3.38}$$

By virtue of the fact that ℓv^T is an idempotent matrix (remember that $\ell^T v = 1$), (3.38) implies

$$x(t+1) = \ell v^T x(0) + \ell v^T \sum_{k=0}^{t-1} \delta(k+1) + \delta(t+1). \tag{3.39}$$

On the other hand, by (3.37) we can conclude that $\sum_{k=0}^{\infty} \delta(k+1)$ is an absolutely convergent series. Let $\overline{\delta} := \lim_{t\to\infty} \sum_{k=0}^{t-1} \delta(k+1)$. Then from (3.39) and (3.37) it follows that $\overline{x} := \lim_{t\to\infty} x(t) = \ell v^T \left(x(0) + \overline{\delta} \right)$. Hence $\overline{x} \in Span(\ell)$, where $\ell \in \Re^N$ is the vector with all entries equal to one. Thus the theorem is proved. ∎

4. The unknown sign case.

In this section we assume that the sign of parameter b in (2.1) is unknown. Instead of (3.2) we propose the following control signal $\forall i \in \mathbf{V}$,

$$u_i(t) = M_i(t)\frac{\varphi_i(t)}{r_i(t)} + \frac{1}{1+N_i}\sum_{j\in\overline{\mathcal{N}}_i} u_j(t-1), \quad i \in \mathbf{V}, \tag{4.1}$$

where

$$M_i(t) = \frac{L_i(t) - L_i(t-1)}{\theta_i(t) - \theta_i(t-1)} \tag{4.2}$$

with

$$L_i(t) = \theta_i(t)^{m_i} cos\, \theta_i(t), \tag{4.3}$$

where $m_i \geq 1$ is a finite integer, while $\theta_i(t)$ and $r_i(t)$ are generated by (3.4) and (3.5). The following lemma states that $M_i(t)$ is finite for every finite $\theta_i(t)$.

LEMMA 4.1. *If $\{\theta_i(t)\}$, $t \geq 0$, is a bounded sequence, then $\{M_i(t)\}$, $t \geq 0$, $i \in \mathbf{V}$, is bounded as well.*

Proof. Note that from (4.3) we can obtain

$$\begin{aligned} L_i(t) - L_i(t-1) &= (\theta_i(t)^{m_i} - \theta_i(t-1)^{m_i})\, cos\, \theta_i(t) \\ &\quad -2\theta_i(t-1)^{m_i} sin\left(\frac{\theta_i(t)+\theta_i(t-1)}{2}\right) sin\left(\frac{\theta_i(t)-\theta_i(t-1)}{2}\right). \end{aligned} \tag{4.4}$$

Since

$$\frac{\theta_i(t)^{m_i} - \theta_i(t-1)^{m_i}}{\theta_i(t) - \theta_i(t-1)} = \theta_i(t)^{m_i-1}\sum_{n=0}^{m_i-1}\left(\frac{\theta_i(t-1)}{\theta_i(t)}\right)^n, \tag{4.5}$$

from (4.4) one can conclude that $(L_i(t) - L_i(t-1))\,/\,(\theta_i(t) - \theta_i(t-1))$ is bounded for all bounded $\theta_i(t)$, where we used the fact that by (3.4) $\theta_i(t) > 0$ $\forall t \geq 0$, and $|(1/y)sin(y)| \leq 1$ $\forall y$. Thus the lemma is proved. ∎

The following proposition examines the properties of the nonlinear dynamics defined by (2.1) and (4.1).

THEOREM 4.2. *The multiagent system defined by* (2.1) *and* (4.1) *provides* $\forall i \in \mathbf{V}$,

$$\overline{\theta}_i := \lim_{t\to\infty} \theta_i(t) \quad \textit{exists}, \tag{4.6}$$

$$\overline{r}_i := \lim_{t\to\infty} r_i(t) \quad \textit{exists}, \tag{4.7}$$

$$\lim_{t\to\infty} \varphi_i(t) = 0 \tag{4.8}$$

for all $x_i(0)$ and all $\theta_i(0) > 0$, $r_i(0) \geq 1$.

Proof. After substituting (4.1) into (2.1) we obtain

$$\varphi_i(t+1) = \varphi_i(t) + bM_i(t)\frac{\varphi_i(t)}{r_i(t)} \quad \forall i \in \mathbf{V}. \tag{4.9}$$

Squaring both sides of the above equation gives

$$\varphi_i(t+1)^2 = \varphi_i(t)^2 + 2bM_i(t)\frac{\varphi_i(t)^2}{r_i(t)} + b^2 M_i(t)^2 \frac{\varphi_i(t)^2}{r_i(t)^2} \tag{4.10}$$

$\forall i \in \mathbf{V}$. By substituting $\mu_i \varphi_i(t)^2/r_i(t) = \theta_i(t) - \theta_i(t-1)$ (see (3.4)), and (4.2) into the second term on the RHS of (4.10) if follows that $\forall i \in \mathbf{V}$

$$\varphi_i(t+1)^2 = \varphi_i(t)^2 + \frac{2b}{\mu_i}\left(L_i(t) - L_i(t-1)\right) + b^2 M_i(t)^2 \frac{\varphi_i(t)^2}{r_i(t)^2}. \tag{4.11}$$

Backward iteration of the last equation with respect to $\varphi_i(t)^2$ yields

$$\varphi_i(t+1)^2 = \varphi_i(1)^2 + \frac{2b}{\mu_i}\left(L_i(t) - L_i(0)\right) + I_{3i}(t), \tag{4.12}$$

where

$$I_{3i}(t) := b^2 \sum_{k=1}^{t} M_i(k)^2 \frac{\varphi_i(k)^2}{r_i(k)^2}, \quad i \in \mathbf{V}. \tag{4.13}$$

We now show that $I_{3i}(t)$ is bounded $\forall t \geq 0$. From (4.2), (4.4), and (4.5) it is not difficult to derive $|M_i(t)| \leq m_i \theta_i(t)^{m_i - 1} + \theta_i(t-1)^{m_i}$ where we used the fact that by (3.4) $(\theta_i(t-1)/\theta_i(t)) \leq 1$ and $|sin(y)/y| \leq 1$ $\forall y$. Then by using (3.8) one can obtain $|M_i(t)| \leq c_9 \left(log\, r_i(t)\right)^{m_i} + c_{10}$, $0 < c_9, c_{10} < \infty$. Hence, based on the second statement of Lemma 3.1, we conclude that

$$\overline{I}_{3i}(t) := \lim_{t\to\infty} I_{3i}(t) \quad \text{exists } \forall i \in \mathbf{V}. \tag{4.14}$$

We can now demonstrate that $\theta_i(t) < \infty$ $\forall t \geq 1$. Assume that $\theta_i(t) \to \infty$ as $t \to \infty$. Then (4.3), (4.12), and (4.14) yield

$$\liminf_{t\to\infty} \frac{\varphi_i(t+1)^2}{\theta_i(t)^{1/2}} = -\infty \tag{4.15}$$

which contradicts the fact that $\varphi_i(t)^2/\theta_i(t)^{1/2} \geq 0$ $\forall t \geq 0$. Thus $\{\theta_i(t)\}$ is bounded, and by (3.4) is a nondecreasing sequence. Therefore it must have a limit, which proves (4.6). The proof of (4.7) is the same as the proof of (3.13). The third statement of this theorem is a direct consequence of (3.5) and (4.7). Thus the theorem is proved. ∎

Let $p_i(t)$ denote the control gain in (4.1),

$$p_i(t) := -b\frac{M_i(t)}{r_i(t)}, \quad i \in \mathbf{V}. \tag{4.16}$$

Since by (4.6), $\theta_i(t) - \theta_i(t-1) \to 0$ as $t \to \infty$, we have

$$\lim_{t\to\infty} \frac{sin\left(\theta_i(t) - \theta_i(t-1)\right)/2}{\theta_i(t) - \theta_i(t-1)} = \frac{1}{2}. \tag{4.17}$$

Then from (4.2)–(4.7) one can obtain that $p_i(t) \to \overline{p}_i$ as $t \to \infty$, where the limit gain $\overline{p}_i$ is given by

$$\overline{p}_i = -b\left(m_i \overline{\theta}_i^{m_i - 1} cos\, \overline{\theta}_i - \overline{\theta}_i^{m_i} sin\, \overline{\theta}_i\right)/\overline{r}_i, \quad i \in \mathbf{V}. \tag{4.18}$$

Equation (4.9) can now be written in the form

$$\varphi_i(t+1) = (1-\overline{p}_i)\varphi_i(t) + \tilde{p}_i(t)\varphi_i(t), \quad i \in \mathbf{V}, \tag{4.19}$$

where $\tilde{p}_i(t) := \overline{p}_i - p_i(t) \to 0$ as $t \to \infty$. We are now ready to formulate the following proposition.

THEOREM 4.3. *Let Assumption* A1 *hold. Then the control protocol* (4.1) *provides*

$$\lim_{t\to\infty} (x_i(t) - x_j(t)) = 0, \quad i, j \in \mathbf{V}, \tag{4.20}$$

and if the limit gain $\overline{p}_i \neq 0$ *then*

$$|\varphi_i(t)| \leq c_{11}\rho_2^t, \quad 0 < \rho_2 < 1, \quad 0 < c_{11} < \infty, \tag{4.21}$$

$$\lim_{t\to\infty} x_i(t) = x_c \tag{4.22}$$

for some finite x_c *and* $\forall i \in \mathbf{V}$.

Proof. The proof of the first statement follows along exactly the same lines as the proof of Theorem 3.3 up untill (3.36). Then from (3.36) and (4.8) we can conclude that $\delta(t+1) \to 0$ as $t \to \infty$ by virtue of the fact that $(I - z^{-1}Q_1)^{-1}$ is a stable transfer function (remember $\rho(Q_1) < 1$). Consequently, definition (3.35) implies that $x_i(t+1) - v^T x(t) \to 0$ as $t \to \infty$ $\forall i \in \mathbf{V}$. Hence, (4.20) holds. Similarly, the proof of the second statement of this theorem (relation (4.21)) is the same as the proof of (3.14) (see the analysis starting with (3.24) until the end of the proof of Theorem 3.2, and apply it to (4.19)). The proof of (4.22) is identical to the proof of Theorem 3.3. ∎

Remark 1. Observe that statement (4.21) requires the limit gain $\overline{p}_i$ to be $\overline{p}_i \neq 0$. In the case of known $sgn(b)$, the gain $a_i(t)$ (3.23) by construction satisfies $a_i(t) \neq 0$ $\forall t \geq 0$. This follows from the fact that $\theta_i(t)$ and $r_i(t)$ are positive sequences (see (3.4) and (3.5)). In case the sign of the parameter b is unknown, the control gain $p_i(t)$ defined by (4.16) must be allowed to change its sign (cross zero value), as a consequence of which there is no guarantee that $\overline{p}_i \neq 0$. Of course if $\overline{p}_i \neq 0$, then it must be $|1 - \overline{p}_i| \leq 1 - \varepsilon_4$ for some $0 < \varepsilon_4 < 1$. The proof of this claim is the same as the proof of (3.14) or (4.21). Therefore the limit gain $\overline{p}_i$ is guaranteed to at worst provide marginal stability of (4.9) or (4.19). This worst case scenario is characterized by $\overline{p}_i = 0$. Note that by using (4.3), (4.6), and (4.8) in (4.12) we can derive

$$\varphi_i(1)^2 + \frac{2b}{\mu_i}\left(\overline{\theta}_i^{m_i} cos\, \overline{\theta}_i - \theta_i(0)^{m_i} cos\, \theta_i(0)\right) + \overline{I}_{3i} = 0 \tag{4.23}$$

$\forall i \in \mathbf{V}$, where $\overline{I}_{3i}$ is given by (4.14). Equation (4.23) together with $\overline{p}_i = 0$ (see (4.18)) defines conditions on various parameters determining the case of marginal stability of (4.19).

Remark 2. Instead of (4.3), in (4.1) we can use

$$L_i(t) = (log\, \theta_i(t))\, cos\, \theta_i(t), \quad i \in \mathbf{V}, \tag{4.24}$$

provided that the recursion (3.4) is initialized with $\theta_i(0) > 1$. It is not difficult to verify that for this $L_i(t)$, function $M_i(t)$ defined by (4.2) is finite for every finite $\theta_i(t)$. Let

$$\Delta y(t) := y(t) - y(t-1) \quad \forall t, \quad y \in \Re. \tag{4.25}$$

Then (4.2) can be written as follows:

$$M_i(t) = \frac{\Delta L_i(t)}{\Delta\theta_i(t)} = \frac{\Delta log\ \theta_i(t)}{\Delta\theta_i(t)} cos\ \theta_i(t) + \frac{\Delta cos\ \theta_i(t)}{\Delta\theta_i(t)} log\ \theta_i(t-1) \tag{4.26}$$
$$= \frac{1}{\Delta\theta_i(t)}\left[log\left(1+\frac{\Delta\theta_i(t)}{\theta_i(t-1)}\right) - 2sin\frac{\theta_i(t)+\theta_i(t-1)}{2}\cdot sin\frac{\Delta\theta_i(t)}{2} log\ \theta_i(t-1)\right].$$

Since by (3.4) $\Delta\theta_i(t) \geq 0\ \forall t \geq 0$, and $log(1+w) \leq w\ \forall w \geq 0$, we have

$$\frac{1}{\Delta\theta_i(t)} log\left(1+\frac{\Delta\theta_i(t)}{\theta_i(t-1)}\right) \leq \frac{1}{\theta(t-1)}. \tag{4.27}$$

Therefore by using the well-known fact that $|\left(sin(\Delta\theta_i(t)/2)\right)/\Delta\theta_i(t)| \leq 1/2$, from (4.26) and (4.27) one obtains

$$|M_i(t)| \leq \frac{1}{\theta_i(t-1)} + log\ \theta_i(t-1), \tag{4.28}$$

implying that if $\{\theta_i(t)\}$, $t \geq 0$, is a bounded sequence, so is $\{M_i(t)\}$, $t \geq 0$. In this case Theorems 4.2 and 4.3 still hold provided that in (3.4) $\theta_i(0) > 1$. The minor technical difference occurs in (4.15) which is derived from (4.12) by assuming that $\theta_i(t) \to \infty$ as $t \to \infty$. When $L_i(t)$ is given by (4.24), from (4.12) one can obtain

$$\liminf_{t\to\infty} \frac{\varphi_i(t+1)^2}{\left(log\ \theta_i(t)\right)^{1/2}} = -\infty, \tag{4.29}$$

which is in contradiction to $\varphi(t+1)^2/\left(log\ \theta_i(t)\right)^{1/2} \geq 0\ \forall t \geq 0$. Therefore $\{\theta_i(t)\}$, $t \geq 0$, must be a bounded sequence. The rest of the analysis is the same as the proof of Theorems 4.2 and 4.3. In this case the limit value of the gain $p_i(t)$ defined by (4.16) can be calculated as follows. Observe that by (4.6) $\Delta\theta_i(t) := \theta_i(t) - \theta_i(t-1) \to 0$ as $t \to \infty$. Then again by (4.6)

$$\lim_{t\to\infty} \frac{1}{\Delta\theta_i(t)} log\left(1+\frac{\Delta\theta_i(t)}{\theta_i(t-1)}\right) = \frac{1}{\overline{\theta}_i}. \tag{4.30}$$

Hence by using (4.6), (4.7), and (4.30) in (4.26) and (4.16) we derive

$$\overline{p}_{1i} := -\lim \frac{\Delta L_i(t)}{r_i(t)\Delta\theta_i(t)} = -\frac{1}{\overline{r}_i}\left[\frac{cos\ \overline{\theta}_i}{\overline{\theta}_i} - \left(sin\ \overline{\theta}_i\right) log\ \overline{\theta}_i\right] \tag{4.31}$$

$\forall i \in \mathbf{V}$.

Remark 3. It is important to observe that the limit gains (4.18) and (4.31) do not take on large values. Since by (3.8), (4.6), and (4.7), $\overline{\theta}_i \leq \theta_i(0) + \mu_i log\ \overline{r}_i$, (4.18) implies $|\overline{p}_i| = O\left[(log\overline{r}_i)^{m_i}/\overline{r}_i\right]$, where $m_i \geq 1$ is a finite integer. Similarly from (4.31) we can obtain $|\overline{p}_{1i}| = O\left[(log\overline{r}_i)/\overline{r}_i\right]$. Obviously even in the case of large $\overline{r}_i$ (see (3.5) and (4.7)), the limit gain $\overline{p}_i$ or $(\overline{p}_{1i})$ is relatively small, which is confirmed by the simulation experiment in the next section.

5. Simulation example:.

In this example we consider the algorithm defined by (4.1)–(4.3), (3.1), (3.4), and (3.5), and a network of six agents whose directed graph topology is characterized by the following adjacency matrix

$$A_d = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

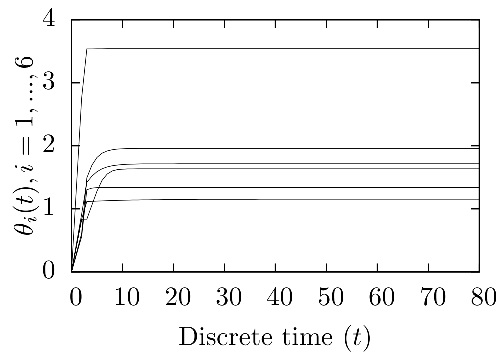

FIG. 1. *Convergence of* $\theta_i(t)$.

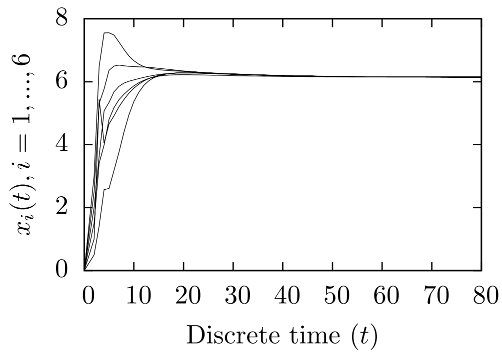

FIG. 2. *Convergence of agent states.*

Let $A_d(i,j)$ be the ijth element of A_d. If agent i can directly receive information from the jth agent $A_d(i,j) = 1$. Otherwise $A_d(i,j) = 0$. In (2.1) we set $b = 2$ and $x_i(0) = i(-1)^i$ $\forall i \in \mathbf{V}$. In (4.3) we select $m_i = 2$, and in (3.4), $\mu_i = 1$ $\forall i \in \mathbf{V}$. Figure 1 presents sequences $\{\theta_i(t)\}$, $t \geq 0$ $\forall i \in \mathbf{V}$ given by (3.4), and Figure 2 shows that all states $x_i(t)$ $\forall i \in \mathbf{V}$, converge to the same value. Time evolution of the input signals $u_i(t)$ $\forall i \in \mathbf{V}$ is depicted in Figure 3. The interagent coupling parameters (or control gains) $\{p_i(t)\}$, $t \geq 0$, $\forall i \in \mathbf{V}$ defined by (4.16) (see also (4.1)) are shown in Figure 4. The above figures confirm theoretically derived conclusions.

6. Conclusion. In this paper we have shown that the multiagent leaderless consensus on directed graphs in case of discrete-time systems with unknown control directions is possible. We have introduced "smooth" control protocol reminiscent of Naussbaum-type controllers for continuous-time systems. Provided that the underlying graph is strongly connected, it is shown that the networked system achieves strict sense consensus, and the coupling parameters are convergent sequences. As future research topics it is important to analyze performance of the above algorithm in the case of time-varying graph topology, presence of noise in interagent communications, and robustness with respect to modeling errors in agent dynamics. At this time it is unclear to us how to extend the above results to systems of general order. In this case $u_i(t)$ and $M_i(t)$ given by (4.1) and (4.2) have to be redefined. This remains an open problem. Another important topic is to extend the above results to distributed event-trigered control protocols along the concepts proposed in [31, 32].

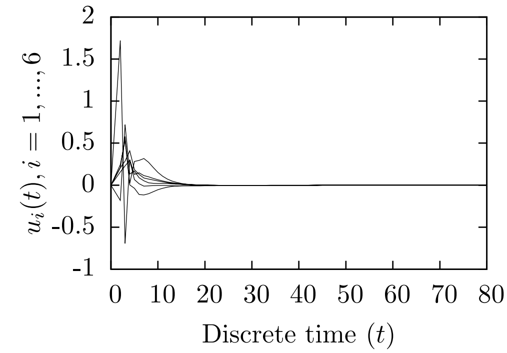

Fig. 3. *Evolution of agent input signals.*

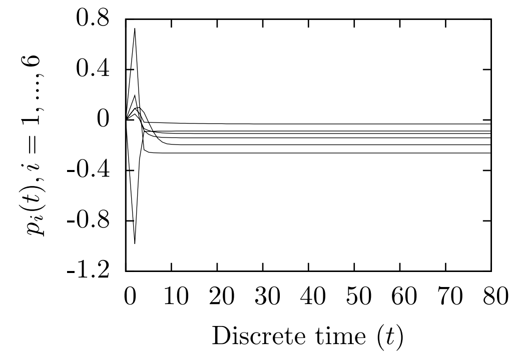

Fig. 4. *Agent control gains.*

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