# Linear Algebra for Team-Based Inquiry Learning

PREVIEW Edition — Instructor Version

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Steven Clontz
University of South Alabama

Drew Lewis

Contributing Authors

Jessalyn Bolkema
California State University, Dominguez Hills

Jeff Ford
Gustavus Adolphus College

Jordan Kostiuk
Brown University

Sharona Krinsky
California State University, Los Angeles

Jennifer Nordstrom
Linfield University

Kate Owens
College of Charleston

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[1] `tbil.org`

[2] `github.com/TeamBasedInquiryLearning/library/blob/main/LICENSE.md`

# TBIL Resource Library

This book is a part of the TBIL Resource Library[3], which includes a growing collection of course materials for courses aligned with the TBIL pedagogy.

This material is based upon work supported by the National Science Foundation under Award #2011807[4]. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

[3] tbil.org

[4] nsf.gov/awardsearch/showAward?AWD_ID=2011807

# For Instructors

**Features of the text.** The book is organized into Modules. You can think of a module as a chapter in a textbook.

In a TBIL course, each module starts with a **Readiness Assurance Process** to check understanding of and and solidify prerequisite knowledge (from other courses or previous modules) that will be needed in the upcoming module. The front page for each module provides a list of these Readiness Assurance outcomes, along with study resources and exercise students can use to refresh themselve on this prerequisite knowledge. Readiness Assurance Tests that assess this knowledge are available as part of the TBIL Resource Libary (see TBIL Resource Library). Join the TBIL community (see Community and Support) to gain access to these.

Within each module there is a varying amount of sections, one per learning outcome. Each section is designed to guide students into being able to demonstrate their understanding of that specific outcome. The learning outcome can be tested with a CheckIt Exercise, which are linked at the end of each section.

Within each section, students engage in Activities, which are interpersed with definitions, theorems, remarks, examples, etc. as needed to guide the learning process. The activities start in an exploratory way then building the concepts on the results of this exploration. The concepts are then practiced with less scaffolding to build fluency. Finally, towards the end of the section we connect the concepts and extend them to new settings.

Each section/learning outcome is designed with an exercise, a specific set of tasks, that the students need to be able to solve to demonstrate their competency. Virtually limitless randomized versions of the exercise can be generated via CheckIt to build a problem bank and allow for reassessment.

**Community and Support.** If you are adopting this text in your class, please fill out this short form[5] so we can track usage, let you know about updates, etc.

Implementation of these materials is supported by a TBIL community of practice which offers both formal and informal professional devlopment opportunities. The best way to connect with us is on our Slack workspace[6].

These materials are a product of our TBIL community. Feedback and suggestions

[5] forms.gle/Ktfbma6iBn2gN1W78

[6] chat.tbil.org/

for improvements are most welcome, either through our GitHub Repository[7] or the Slack workspace[8].

[7] github.com/TeamBasedInquiryLearning/library
[8] chat.tbil.org/

# Video Resources

Videos are available at the end of each section. A complete playlist of videos aligned with this text is available on YouTube[9].

[9] www.youtube.com/watch?v=kpOK7RhFEiQ&list=PLwXCBkIf7xBMo3zMnD7WVt39rANLlSdmj

# Contents

# Chapter 1

# Systems of Linear Equations (LE)

## Learning Outcomes

How can we solve systems of linear equations?
By the end of this chapter, you should be able to...

1. Translate back and forth between a system of linear equations, a vector equation, and the corresponding augmented matrix.
2. Explain why a matrix isn't in reduced row echelon form, and put a matrix in reduced row echelon form.
3. Determine the number of solutions for a system of linear equations or a vector equation.
4. Compute the solution set for a system of linear equations or a vector equation with infinitely many solutions.

**Readiness Assurance.** Before beginning this chapter, you should be able to...

1. Determine if a system to a two-variable system of linear equations will have zero, one, or infinitely-many solutions by graphing.
   - Review: Khan Academy[1]
2. Find the unique solution to a two-variable system of linear equations by back-substitution.
   - Review: Khan Academy[2]
3. Describe sets using set-builder notation, and check if an element is a member of a set described by set-builder notation.
   - Review: YouTube[3]

[1] `bit.ly/2l21etm`
[2] `www.khanacademy.org/math/algebra-basics/alg-basics-systems-of-equations/alg-basics-solving-systems-with-substitution/v/practice-using-substitution-for-systems`
[3] `youtu.be/xnfUZ-NTsCE`

## 1.1 Linear Systems, Vector Equations, and Augmented Matrices (LE1)

### Learning Outcomes

- Translate back and forth between a system of linear equations, a vector equation, and the corresponding augmented matrix.

### 1.1.1 Warm Up

**Activity 1.1.1** Consider the pairs of lines described by the equations below. Decide which of these are parallel, identical, or transverse (i.e., intersect in a single point).

**(a)**

$$\begin{aligned} -x_1 + 3x_2 &= 1 \\ 2x_1 - 5x_2 &= 2 \end{aligned}$$

**(b)**

$$\begin{aligned} -x_1 + 3x_2 &= 1 \\ 2x_1 - 6x_2 &= -2 \end{aligned}$$

**(c)**

$$\begin{aligned} -x_1 + 3x_2 &= 1 \\ 2x_1 - 6x_2 &= 3 \end{aligned}$$

### 1.1.2 Class Activities

**Definition 1.1.2** A **matrix** is an $m \times n$ array of real numbers with $m$ rows and $n$ columns:

$$\left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array}\right] = \left[\begin{array}{cccc} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{array}\right].$$

Frequently we will use matrices to describe an ordered list of its **column vectors**:

$$\left[\begin{array}{c} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{array}\right], \left[\begin{array}{c} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{array}\right], \cdots, \left[\begin{array}{c} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{array}\right] = \vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n.$$

When order is irrelevant, we will use set notation:

$$\left\{ \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \cdots, \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \right\} = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}.$$

$\Diamond$

**Definition 1.1.3** A **Euclidean vector** is an ordered list of real numbers

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

We will find it useful to almost always typeset Euclidean vectors vertically, but the notation $\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}^T$ is also valid when vertical typesetting is inconvenient. The set of all Euclidean vectors with $n$ components is denoted as $\mathbb{R}^n$, and vectors are often described using the notation $\vec{v}$.

Each number in the list is called a **component**, and we use the following definitions for the sum of two vectors, and the product of a real number and a vector:

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix} \qquad c \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{bmatrix}$$

$\Diamond$

**Example 1.1.4** Following are some examples of addition and scalar multiplication in $\mathbb{R}^4$.

$$\begin{bmatrix} 3 \\ -3 \\ 0 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 3+0 \\ -3+2 \\ 0+7 \\ 4+1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 7 \\ 5 \end{bmatrix}$$

$$-4 \begin{bmatrix} 0 \\ 2 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -4(0) \\ -4(2) \\ -4(-2) \\ -4(3) \end{bmatrix} = \begin{bmatrix} 0 \\ -8 \\ 8 \\ -12 \end{bmatrix}$$

$\square$

**Definition 1.1.5** A **linear equation** is an equation of the variables $x_i$ of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

A **solution** for a linear equation is a Euclidean vector

$$\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

that satisfies

$$a_1 s_1 + a_2 s_2 + \cdots + a_n s_n = b$$

(that is, a Euclidean vector whose components can be plugged into the equation). ◊

**Remark 1.1.6** In previous classes you likely used the variables $x, y, z$ in equations. However, since this course often deals with equations of four or more variables, we will often write our variables as $x_i$, and assume $x = x_1, y = x_2, z = x_3, w = x_4$ when convenient.

**Definition 1.1.7** A **system of linear equations** (or a **linear system** for short) is a collection of one or more linear equations.

$$\begin{array}{rrrrl} a_{11}x_1 + & a_{12}x_2 + & \ldots + & a_{1n}x_n & = b_1 \\ a_{21}x_1 + & a_{22}x_2 + & \ldots + & a_{2n}x_n & = b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + & \ldots + & a_{mn}x_n & = b_m \end{array}$$

Its **solution set** is given by

$$\left\{ \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \middle| \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \text{is a solution to all equations in the system} \right\}.$$

◊

**Remark 1.1.8** When variables in a large linear system are missing, we prefer to write the system in one of the following standard forms:

| Original linear system: | Verbose standard form: | Concise standard form: |
|---|---|---|
| $x_1 + 3x_3 = 3$ | $1x_1 + 0x_2 + 3x_3 = 3$ | $x_1 \quad\quad + 3x_3 = 3$ |
| $3x_1 - 2x_2 + 4x_3 = 0$ | $3x_1 - 2x_2 + 4x_3 = 0$ | $3x_1 - 2x_2 + 4x_3 = 0$ |
| $-x_2 + x_3 = -2$ | $0x_1 - 1x_2 + 1x_3 = -2$ | $- x_2 + x_3 = -2$ |

**Remark 1.1.9** It will often be convenient to think of a system of equations as a vector equation.

By applying vector operations and equating components, it is straightforward to see that the vector equation

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

is equivalent to the system of equations

$$\begin{alignedat}{4} x_1 & & & + 3x_3 &&= 3 \\ 3x_1 &-2x_2 & & + 4x_3 &&= 0 \\ & -\ x_2 & & +\ x_3 &&= -2 \end{alignedat}$$

**Definition 1.1.10** A linear system is **consistent** if its solution set is non-empty (that is, there exists a solution for the system). Otherwise it is **inconsistent**. $\Diamond$

**Fact 1.1.11** *All linear systems are one of the following:*

1. Consistent with one solution: *its solution set contains a single vector, e.g.* $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$
2. Consistent with infinitely-many solutions*: its solution set contains infinitely many vectors, e.g.* $\left\{ \begin{bmatrix} 1 \\ 2-3a \\ a \end{bmatrix} \middle| a \in \mathbb{R} \right\}$
3. Inconsistent*: its solution set is the empty set, denoted by either* $\{\}$ *or* $\emptyset$.

**Activity 1.1.12** All inconsistent linear systems contain a logical **contradiction**. Find a contradiction in this system to show that its solution set is the empty set.

$$\begin{aligned} -x_1 + 2x_2 &= 5 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

**Activity 1.1.13** Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

**(a)** Find several different solutions for this system:

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \begin{bmatrix} ? \\ 2 \end{bmatrix} \quad \begin{bmatrix} 0 \\ ? \end{bmatrix} \quad \begin{bmatrix} ? \\ ? \end{bmatrix} \quad \begin{bmatrix} ? \\ ? \end{bmatrix}$$

**(b)** Suppose we let $x_2 = a$ where $a$ is an arbitrary real number. Which of these expressions for $x_1$ in terms of $a$ satisfies both equations of the linear system?

A. $x_1 = -3a$

B. $x_1 = 3$

C. $x_1 = 2a + 3$

D. $x_1 = -4a + 6$

**Answer**. C. $x_1 = 2a + 3$

**(c)** Given $x_2 = a$ and the expression you found in the previous task, which of these describes the solution set for this system?

A. $\left\{ \left[ \begin{matrix} 2a+3 \\ a \end{matrix} \right] \middle| a \in \mathbb{R} \right\}$

B. $\left\{ \left[ \begin{matrix} a \\ 2a+3 \end{matrix} \right] \middle| a \in \mathbb{R} \right\}$

C. $\left\{ \left[ \begin{matrix} a \\ b \end{matrix} \right] \middle| a \in \mathbb{R} \right\}$

D. $\left\{ \left[ \begin{matrix} 2a+3 \\ 2b-3 \end{matrix} \right] \middle| a \in \mathbb{R} \right\}$

**Answer**. A. $\left\{ \left[ \begin{matrix} 2a-3 \\ a \end{matrix} \right] \middle| a \in \mathbb{R} \right\}$

**Activity 1.1.14** Consider the following linear system.

$$\begin{aligned} x_1 + 2x_2 \phantom{+ x_3} - x_4 &= 3 \\ x_3 + 4x_4 &= -2 \end{aligned}$$

Substitute $x_2 = a$ and $x_4 = b$, and then solve for $x_1$ and $x_3$:

$$x_1 = ? \qquad\qquad x_3 = ?$$

Then use these to describe the solution set

$$\left\{ \left[ \begin{matrix} ? \\ a \\ ? \\ b \end{matrix} \right] \middle| a, b \in \mathbb{R} \right\}$$

to the linear system.

**Observation 1.1.15** Solving linear systems of two variables by graphing or substitution is reasonable for two-variable systems, but these simple techniques won't usually cut it for equations with more than two variables or more than two equations. For example,

$$\begin{aligned} -2x_1 - 4x_2 + x_3 - 4x_4 &= -8 \\ x_1 + 2x_2 + 2x_3 + 12x_4 &= -1 \\ x_1 + 2x_2 + x_3 + 8x_4 &= 1 \end{aligned}$$

has the exact same solution set as the system in the previous activity, but we'll want to learn new techniques to compute these solutions efficiently.

**Remark 1.1.16** The only important information in a linear system are its coefficients and constants.

| Original linear system: | Verbose standard form: | Coefficients/constants: |
| --- | --- | --- |
| $x_1 + 3x_3 = 3$ | $1x_1 + 0x_2 + 3x_3 = 3$ | $1 \quad 0 \quad 3 \mid 3$ |
| $3x_1 - 2x_2 + 4x_3 = 0$ | $3x_1 - 2x_2 + 4x_3 = 0$ | $3 \ {-2} \ 4 \mid 0$ |
| $-x_2 + x_3 = -2$ | $0x_1 - 1x_2 + 1x_3 = -2$ | $0 \ {-1} \ 1 \mid -2$ |

**Definition 1.1.17** A system of $m$ linear equations with $n$ variables is often represented by writing its coefficients and constants in an **augmented matrix**: the $m \times n$ matrix of its coefficients augmented with the $m$ constant values as a final column.

$$\begin{aligned}
a_{11}x_1 + \ a_{12}x_2 + \ldots + \ a_{1n}x_n &= b_1 \\
a_{21}x_1 + \ a_{22}x_2 + \ldots + \ a_{2n}x_n &= b_2 \\
\vdots \qquad \quad \vdots \qquad \qquad \quad \vdots \quad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}$$

$$\left[\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{array}\right]$$

Sometimes, we will find it useful to refer only to the coefficients of the linear system (and ignore its constant terms). We call the $m \times n$ array consisting of these coefficients a **coefficient matrix**.

$$\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}$$

$\Diamond$

**Example 1.1.18** The corresponding augmented matrix for this system is obtained by simply writing the coefficients and constants in matrix form.

Linear system:

$$\begin{aligned}
x_1 \qquad \ + 3x_3 &= 3 \\
3x_1 - 2x_2 + 4x_3 &= 0 \\
- \ x_2 + \ x_3 &= -2
\end{aligned}$$

Augmented matrix:

$$\left[\begin{array}{ccc|c}
1 & 0 & 3 & 3 \\
3 & -2 & 4 & 0 \\
0 & -1 & 1 & -2
\end{array}\right]$$

Vector equation:

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

$\square$

### 1.1.3 Individual Practice

**Activity 1.1.19** Consider the following augmented matrices. For each of them, decide how many variables and how many equations the corresponding linear system has.

**(a)**

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 1 & -2 & 4 & 3 \\ 3 & -1 & 7 & -1 \end{array}\right]$$

**(b)**

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 1 & -2 & 4 & 3 \\ 3 & -1 & 7 & -1 \\ 3 & -1 & 7 & -1 \end{array}\right]$$

**(c)**

$$\left[\begin{array}{ccc|c} 2 & 0 & 3 & 3 \\ 1 & 0 & 4 & 3 \\ 3 & 0 & 7 & -1 \\ 3 & 0 & 7 & -1 \end{array}\right]$$

**(d)**

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 1 & -2 & 4 & 3 \\ 0 & 0 & 0 & 0 \\ 3 & -1 & 7 & -1 \end{array}\right]$$

### 1.1.4 Videos

Standalone

**Figure 1** Video: Converting between systems, vector equations, and augmented matrices

### 1.1.5 Exercises

Exercises available at https://tbil.org/preview/linear-algebra/exercises/#/bank/LE1/.

### 1.1.6 Mathematical Writing Explorations

**Exploration 1.1.20** Choose a value for the real constant $k$ such that the following system has one, many, or no solutions. In each case, write the solution set.

Consider the linear system:

$$\begin{aligned} x_1 - x_2 &= 1 \\ 3x_1 - 3x_2 &= k \end{aligned}$$

**Exploration 1.1.21** Consider the linear system:

$$\begin{aligned} ax_1 + bx_2 &= j \\ cx_1 + dx_2 &= k \end{aligned}$$

Assume $j$ and $k$ are arbitrary real numbers.

- Choose values for $a, b, c$, and $d$, such that $ad - bc = 0$. Show that this system is inconsistent.
- Prove that, if $ad - bc \neq 0$, the system is consistent with exactly one solution.

**Exploration 1.1.22** Given a set $S$, we can define a relation between two arbitrary elements $a, b \in S$. If the two elements are related, we denote this $a \sim b$.

Any relation on a set $S$ that satisfies the properties below is an **equivalence relation**.

- **Reflexive**: For any $a \in S, a \sim a$
- **Symmetric**: For $a, b \in S$, if $a \sim b$, then $b \sim a$
- **Transitive:** for any $a, b, c \in S, a \sim b$ and $b \sim c$ implies $a \sim c$

For each of the following relations, show that it is or is not an equivalence relation.

- For $a, b, \in \mathbb{R}$, $a \sim b$ if an only if $a \leq b$.
- For $a, b, \in \mathbb{R}$, $a \sim b$ if an only if $|a| = |b|$.

### 1.1.7 Sample Problem and Solution

Sample problem Example **??**.

# 1.2 Row Reduction of Matrices (LE2)

## Learning Outcomes

- Explain why a matrix isn't in reduced row echelon form, and put a matrix in reduced row echelon form.

### 1.2.1 Warm Up

**Activity 1.2.1** Consider the following matrices:

$$A = \left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 3 & -2 & 4 & 0 \\ 0 & -1 & 1 & -2 \end{array}\right], \quad B = \left[\begin{array}{ccc} 2 & 5 & 3 \\ 1 & -2 & 4 \\ 3 & -1 & 7 \end{array}\right]$$

**(a)** Write down a linear system whose augmented matrix is $A$. Can you write down another?

**(b)** Write down a linear system whose coefficient matrix is $B$. Can you write down another?

### 1.2.2 Class Activities

**Definition 1.2.2** Two systems of linear equations (and their corresponding augmented matrices) are said to be **equivalent** if they have the same solution set.

For example, both of these systems share the same solution set $\left\{\left[\begin{array}{c} 1 \\ 1 \end{array}\right]\right\}$.

$$\begin{aligned} 3x_1 - 2x_2 &= 1 \\ x_1 + 4x_2 &= 5 \end{aligned} \qquad\qquad \begin{aligned} 3x_1 - 2x_2 &= 1 \\ 4x_1 + 2x_2 &= 6 \end{aligned}$$

Therefore these augmented matrices are equivalent (even though they're not *equal*), which we denote with $\sim$:

$$\left[\begin{array}{cc|c} 3 & -2 & 1 \\ 1 & 4 & 5 \end{array}\right] \neq \left[\begin{array}{cc|c} 3 & -2 & 1 \\ 4 & 2 & 6 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 3 & -2 & 1 \\ 1 & 4 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & -2 & 1 \\ 4 & 2 & 6 \end{array}\right]$$

$\Diamond$

**Activity 1.2.3** Consider whether these matrix manipulations (A) must keep the *same* solution set, or (B) might result in a *different* solution set for the corresponding linear system.

**(a)** Swapping two rows, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & 3 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 1 & 2 & 4 \end{array}\right] \qquad \begin{aligned} x + 2y &= 4 \\ x + 3y &= 5 \end{aligned} \qquad \begin{aligned} x + 3y &= 5 \\ x + 2y &= 4 \end{aligned}$$

A. Solutions must be the *same*. B. Solutions might be *different*.

**Answer**. A. Same

**(b)** Swapping two columns, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & 3 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 1 & 4 \\ 3 & 1 & 5 \end{array}\right] \qquad \begin{aligned} x + 2y &= 4 \\ x + 3y &= 5 \end{aligned} \qquad \begin{aligned} 2x + y &= 4 \\ 3x + y &= 5 \end{aligned}$$

A. Solutions must be the *same*. B. Solutions might be *different*.

**Answer**. B. Different

**(c)** Add a constant to every term of a row, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & 3 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1+3 & 2+3 & 4+3 \\ 1 & 3 & 5 \end{array}\right] \qquad \begin{aligned} x + 2y &= 4 \\ x + 3y &= 5 \end{aligned} \qquad \begin{aligned} 4x + 5y &= 7 \\ x + 3y &= 5 \end{aligned}$$

A. Solutions must be the *same*. B. Solutions might be *different*.

**Answer**. B. Different

**(d)** Multiply a row by a nonzero constant, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & 3 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 3(1) & 3(2) & 3(4) \\ 1 & 3 & 5 \end{array}\right] \qquad \begin{aligned} x + 2y &= 3 \\ x + 3y &= 5 \end{aligned} \qquad \begin{aligned} 3x + 6y &= 12 \\ x + 3y &= 5 \end{aligned}$$

A. Solutions must be the *same*. B. Solutions might be *different*.

**Answer**. A. Same

**(e)** Add one row to another row, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & 3 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 1+1 & 3+2 & 5+4 \end{array}\right] \qquad \begin{aligned} x + 2y &= 4 \\ x + 3y &= 5 \end{aligned} \qquad \begin{aligned} ?\,x + \,?\,y &= \,? \\ ?\,x + \,?\,y &= \,? \end{aligned}$$

A. Solutions must be the *same*. B. Solutions might be *different*.

**Answer**. A. Same

**(f)** Replace a column with zeros, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & 3 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 1 & 0 & 5 \end{array}\right] \qquad \begin{aligned} x + 2y &= 4 \\ x + 3y &= 5 \end{aligned} \qquad \begin{aligned} ?\,x + \,?\,y &= \,? \\ ?\,x + \,?\,y &= \,? \end{aligned}$$

A. Solutions must be the *same*. B. Solutions might be *different*.

**Answer**. B. Different

**(g)** Replace a row with zeros, for example:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & 3 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} x + 2y &= 4 \\ x + 3y &= 5 \end{aligned} \qquad \begin{aligned} ?\,x + \,?\,y &= \,? \\ ?\,x + \,?\,y &= \,? \end{aligned}$$

A. Solutions must be the *same*. B. Solutions might be *different*.

**Answer**. B. Different

**Activity 1.2.4**

Initializing....

Standalone
Embed

How does adding row multiples to other rows affect a linear system's solution set?

A. Solutions must be the *same*. B. Solutions might be *different*.

**Answer**. A. Same

**Definition 1.2.5** The following three **row operations** produce equivalent augmented matrices.

1. Swap two rows, for example, $R_1 \leftrightarrow R_2$:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & 5 & 6 \\ 1 & 2 & 3 \end{array}\right]$$

2. Multiply a row by a nonzero constant, for example, $2R_1 \to R_1$:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 2(1) & 2(2) & 2(3) \\ 4 & 5 & 6 \end{array}\right]$$

3. Add a constant multiple of one row to another row, for example, $R_2 - 4R_1 \to R_2$:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4-4(1) & 5-4(2) & 6-4(3) \end{array}\right]$$

Observe that we will use the following notation: (Combination of old rows) $\to$ (New row). $\Diamond$

**Activity 1.2.6** Each of the following linear systems has the same solution set.

A)

$$\begin{aligned} x+2y+\phantom{3}z &= 3 \\ -x-\phantom{2}y+\phantom{3}z &= 1 \\ 2x+5y+3z &= 7 \end{aligned}$$

B)

$$\begin{aligned} 2x+5y+3z &= 7 \\ -x-\phantom{2}y+\phantom{3}z &= 1 \\ x+2y+\phantom{3}z &= 3 \end{aligned}$$

C)

$$\begin{aligned} x\phantom{+2y}-\phantom{2}z &= 1 \\ y+2z &= 4 \\ y+\phantom{2}z &= 1 \end{aligned}$$

D)

$$\begin{aligned} x+2y+\phantom{3}z &= 3 \\ y+2z &= 4 \\ 2x+5y+3z &= 7 \end{aligned}$$

E)

$$\begin{aligned} x\phantom{+2y}-\phantom{2}z &= 1 \\ y+2z &= 4 \\ z &= 3 \end{aligned}$$

F)

$$\begin{aligned} x+2y+\phantom{2}z &= 3 \\ y+2z &= 4 \\ y+\phantom{2}z &= 1 \end{aligned}$$

Sort these six equivalent linear systems from most complicated to simplest (in your opinion).

**Activity 1.2.7** Here we've written the sorted linear systems from Activity 1.2.6 as augmented matrices.

$$\left[\begin{array}{ccc|c} 2 & 5 & 3 & 7 \\ -1 & -1 & 1 & 1 \\ 1 & 2 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 3 \\ -1 & -1 & 1 & 1 \\ 2 & 5 & 3 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 3 \\ 0 & 1 & 2 & 4 \\ 2 & 5 & 3 & 7 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 3 \\ 0 & \boxed{1} & 2 & 4 \\ 0 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 1 \\ 0 & \boxed{1} & 2 & 4 \\ 0 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 1 \\ 0 & \boxed{1} & 2 & 4 \\ 0 & 0 & -1 & -3 \end{array}\right]$$

Assign the following row operations to each step used to manipulate each matrix to the next:

$R_3 - 1R_2 \to R_3$ $\qquad$ $R_2 + 1R_1 \to R_2$ $\qquad$ $R_1 \leftrightarrow R_3$

$R_3 - 2R_1 \to R_3$ $\qquad$ $R_1 - 2R_3 \to R_1$

**Definition 1.2.8** A matrix is in **reduced row echelon form** (**RREF**) if

1. The leftmost nonzero term of each row is 1. We call these terms **pivots**.
2. Each pivot is to the right of every higher pivot.
3. Each term that is either above or below a pivot is 0.
4. All zero rows (rows whose terms are all 0) are at the bottom of the matrix.

Every matrix has a unique reduced row echelon form. If $A$ is a matrix, we write RREF($A$) for the reduced row echelon form of that matrix. $\Diamond$

**Activity 1.2.9** Recall that a matrix is in **reduced row echelon form** (**RREF**) if

1. The leftmost nonzero term of each row is 1. We call these terms **pivots**.
2. Each pivot is to the right of every higher pivot.
3. Each term that is either above or below a pivot is 0.
4. All zero rows (rows whose terms are all 0) are at the bottom of the matrix.

For each matrix, mark the leading terms, and label it as RREF or not RREF. For the ones not in RREF, determine which rule is violated and how it might be fixed.

$$A = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad B = \left[\begin{array}{ccc|c} 1 & 0 & 4 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad C = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

**Activity 1.2.10** Recall that a matrix is in **reduced row echelon form** (**RREF**) if

1. The leftmost nonzero term of each row is 1. We call these terms **pivots**.
2. Each pivot is to the right of every higher pivot.
3. Each term that is either above or below a pivot is 0.
4. All zero rows (rows whose terms are all 0) are at the bottom of the matrix.

For each matrix, mark the leading terms, and label it as RREF or not RREF. For the ones not in RREF, determine which rule is violated and how it might be fixed.

$$D = \left[\begin{array}{ccc|c} 1 & 0 & 2 & -3 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad E = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 7 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad F = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

**Remark 1.2.11** In practice, if we simply need to convert a matrix into reduced row echelon form, we use technology to do so.

However, it is also important to understand the **Gauss-Jordan elimination** algorithm that a computer or calculator uses to convert a matrix (augmented or not) into reduced row echelon form. Understanding this algorithm will help us better understand how to interpret the results in many applications we use it for in Chapter 2.

**Activity 1.2.12** Consider the matrix

$$\begin{bmatrix} 2 & 6 & -1 & 6 \\ 1 & 3 & -1 & 2 \\ -1 & -3 & 2 & 0 \end{bmatrix}.$$

Which row operation is the best choice for the first move in converting to RREF?

A. Add row 3 to row 2 ($R_2 + R_3 \to R_2$)

B. Add row 2 to row 3 ($R_3 + R_2 \to R_3$)

C. Swap row 1 to row 2 ($R_1 \leftrightarrow R_2$)

D. Add -2 row 2 to row 1 ($R_1 - 2R_2 \to R_1$)

**Activity 1.2.13** Consider the matrix

$$\begin{bmatrix} \boxed{1} & 3 & -1 & 2 \\ 2 & 6 & -1 & 6 \\ -1 & -3 & 2 & 0 \end{bmatrix}.$$

Which row operation is the best choice for the next move in converting to RREF?

A. Add row 1 to row 3 ($R_3 + R_1 \to R_3$)

B. Add -2 row 1 to row 2 ($R_2 - 2R_1 \to R_2$)

C. Add 2 row 2 to row 3 ($R_3 + 2R_2 \to R_3$)

D. Add 2 row 3 to row 2 ($R_2 + 2R_3 \to R_2$)

**Activity 1.2.14** Consider the matrix

$$\begin{bmatrix} \boxed{1} & 3 & -1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Which row operation is the best choice for the next move in converting to RREF?

A. Add row 1 to row 2 ($R_2 + R_1 \to R_2$)

B. Add -1 row 3 to row 2 ($R_2 - R_3 \to R_2$)

C. Add -1 row 2 to row 3 ($R_3 - R_2 \to R_3$)

D. Add row 2 to row 1 ($R_1 + R_2 \to R_1$)

**Observation 1.2.15** The steps for the Gauss-Jordan elimination algorithm may be summarized as follows:

1. Ignoring any rows that already have marked pivots, identify the leftmost column with a nonzero entry.
2. Use row operations to obtain a pivot of value 1 in the topmost row that does not already have a marked pivot.
3. Mark this pivot, then use row operations to change all values above and below the marked pivot to 0.
4. Repeat these steps until the matrix is in RREF.

In particular, *once a pivot is marked, it should remain in the same position.* This will keep you from undoing your progress towards an RREF matrix.

**Activity 1.2.16** Complete the following RREF calculation (multiple row operations may be needed for certain steps):

$$A = \begin{bmatrix} 2 & 3 & 2 & 3 \\ -2 & 1 & 6 & 1 \\ -1 & -3 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & ? & ? & ? \\ -2 & 1 & 6 & 1 \\ -1 & -3 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & ? & ? & ? \\ 0 & ? & ? & ? \\ 0 & ? & ? & ? \end{bmatrix}$$

$$\sim \begin{bmatrix} \boxed{1} & ? & ? & ? \\ 0 & \boxed{1} & ? & ? \\ 0 & ? & ? & ? \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & ? & ? \\ 0 & \boxed{1} & ? & ? \\ 0 & 0 & ? & ? \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \boxed{1} & 0 & -2 & 0 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**Activity 1.2.17** Consider the matrix

$$A = \begin{bmatrix} 2 & 4 & 2 & -4 \\ -2 & -4 & 1 & 1 \\ 3 & 6 & -1 & -4 \end{bmatrix}.$$

Compute RREF($A$).

**Activity 1.2.18** Consider the non-augmented and augmented matrices

$$A = \begin{bmatrix} 2 & 4 & 2 & -4 \\ -2 & -4 & 1 & 1 \\ 3 & 6 & -1 & -4 \end{bmatrix} \qquad B = \left[\begin{array}{ccc|c} 2 & 4 & 2 & -4 \\ -2 & -4 & 1 & 1 \\ 3 & 6 & -1 & -4 \end{array}\right].$$

Can RREF($A$) be used to find RREF($B$)?

A. Yes, RREF($A$) and RREF($B$) are exactly the same.

B. Yes, RREF($A$) may be slightly modified to find RREF($B$).

C. No, a new calculation is required.

**Activity 1.2.19** Free browser-based technologies for mathematical computation are available online.

- Go to https://sagecell.sagemath.org/.
- In the dropdown on the right, you can select a number of different languages. Select "Octave" for the Matlab-compatible syntax used by this text.
- Type `rref([1,3,2;2,5,7])` and then press the Evaluate button to compute the RREF of $\left[\begin{array}{ccc} 1 & 3 & 2 \\ 2 & 5 & 7 \end{array}\right]$.
- Now try using whitespace to write out the matrix and compute RREF instead:

```
A = [1 3 2
     2 5 7]

rref(A)
```

**Activity 1.2.20** In the HTML version of this text, code cells are often embedded for your convenience when RREFs need to be computed.

Try this out to compute RREF $\left[\begin{array}{cc|c} 2 & -3 & 1 \\ 3 & 0 & 6 \end{array}\right]$.

```
rref([2,-3,1;3,0,6])
```

```
A = [2 -3 1
     3  0 6]

rref(A)
```

## 1.2.3 Individual Practice

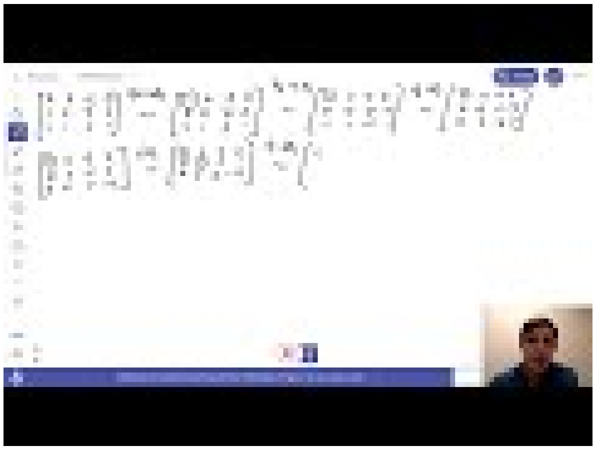
**Activity 1.2.21** Find three examples of linear systems for which the RREF of their augmented matrices is equal to

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

**Activity 1.2.22** Which of the following matrices are not in RREF?

$$A = \left[\begin{array}{ccc|c} 1 & 0 & 2 & -3 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad B = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad C = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

### 1.2.4 Videos



Standalone

**Figure 2** Video: Row reduction

### 1.2.5 Exercises

Exercises available at `https://tbil.org/preview/linear-algebra/exercises/#/bank/LE2/`.

### 1.2.6 Mathematical Writing Explorations

**Exploration 1.2.23** Prove that Gauss-Jordan Elimination preserves the solution set of a system of linear equations in $n$ variables. Make sure your proof includes each of the following. Just because I've used bullet points here does not mean you should use bullet points in your proof.

- Write an arbitrary system of linear equations in $n$ variables. Your notation should be unambiguous.
- Label an element of your solution set. You won't know what it is exactly, so you'll have to use a variable. Remember what it means (by definition!) to be in the solution set.
- Describe the three operations used in Gauss-Jordan Elimination.
- Consider all three operations in Gauss-Jordan Elimination. After each one is used, show that the element of the solution set you picked still satisfies the definition.

**Exploration 1.2.24** Let $M_{2,2}$ indicate the set of all $2\times 2$ matrices with real entries. Show that equivalence of matrices as defined in this section is an equivalence relation, as in exploration Exploration 1.1.22

### 1.2.7 Sample Problem and Solution

Sample problem Example **??**.

## 1.3 Counting Solutions for Linear Systems (LE3)

### Learning Outcomes

- Determine the number of solutions for a system of linear equations or a vector equation.

### 1.3.1 Warm Up

**Activity 1.3.1**

**(a)** Without referring to your Activity Book, which of the four criteria for a matrix to be in Reduced Row Echelon Form (RREF) can you recall?

**(b)** Which, if any, of the following matrices are in RREF? You may refer to the Activity Book now for criteria that you may have forgotten.

$$P = \left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & -3 \\ 0 & 3 & 3 & -\frac{3}{5} \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad Q = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 7 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad R = \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

### 1.3.2 Class Activities

**Remark 1.3.2** We will frequently need to know the reduced row echelon form of matrices during the remainder of this course, so unless you're told otherwise, feel free to use technology (see Activity 1.2.19) to compute RREFs efficiently.

**Activity 1.3.3** Consider the following system of equations.

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -x_1 + 3x_2 - 6x_3 &= 11 \\ 4x_1 + x_2 + x_3 &= 1. \end{aligned}$$

**(a)** Convert this to an augmented matrix and use technology to compute its reduced row echelon form:

$$\text{RREF}\left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right] = \left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right]$$

**(b)** Use the RREF matrix to write a linear system equivalent to the original system.

**(c)** How many solutions must this system have?

A. Zero

B. Only one

C. Infinitely-many

```
A = [ 3 -2 13  6
      2 -2 10  2
     -1  3 -6 11
      4  1  1  1 ]

rref(A)
```

**Activity 1.3.4** Consider the vector equation

$$x_1 \begin{bmatrix} 3 \\ 2 \\ -1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -2 \\ 0 \\ 7 \end{bmatrix} + x_3 \begin{bmatrix} 13 \\ 10 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 1 \\ -2 \end{bmatrix}$$

**(a)** Convert this to an augmented matrix and use technology to compute its reduced row echelon form:

$$\text{RREF} \left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right] = \left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right]$$

**(b)** Use the RREF matrix to write a linear system equivalent to the original system.

**(c)** How many solutions must this system have?

A. Zero B. Only one C. Infinitely-many

```
rref([3,-2,13,6;2,-2,10,2;-1,0,-3,1;3,7,0,-2])
```

**Activity 1.3.5** What contradictory equations besides $0 = 1$ may be obtained from the RREF of an augmented matrix?

A. $x = 0$ is an obtainable contradiction

B. $x = y$ is an obtainable contradiction

C. $0 = 17$ is an obtainable contradiction

D. $0 = 1$ is the only obtainable contradiction

**Activity 1.3.6** Consider the following linear system.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 4x_2 + 8x_3 &= 0 \\ 3x_1 + 6x_2 + 11x_3 &= 1 \\ x_1 + 2x_2 + 5x_3 &= -1 \end{aligned}$$

**(a)** Find its corresponding augmented matrix $A$ and find RREF$(A)$.

**(b)** Use the RREF matrix to write a linear system equivalent to the original system.

**(c)** How many solutions must this system have?

A. Zero B. One C. Infinitely-many

**Fact 1.3.7** *By finding* RREF($A$) *from a linear system's corresponding augmented matrix $A$, we can immediately tell how many solutions the system has.*

- *If the linear system given by* RREF($A$) *includes the contradiction*

$$0 = 1,$$

*that is, the* RREF *matrix includes the row*

$$\left[\begin{array}{ccc|c} 0 & \cdots & 0 & 1 \end{array}\right],$$

*then the system is* inconsistent*, which means it has* zero *solutions and we may write*

$$\textit{Solution set } = \{\} \qquad \textit{or} \qquad \textit{Solution set } = \emptyset.$$

- *If the linear system given by* RREF($A$) *sets each variable of the system to a single value; that is we have:*

$$\begin{aligned} x_1 &= s_1 \\ x_2 &= s_2 \\ &\vdots \\ x_n &= s_n \end{aligned}$$

*(with some possible extra $0 = 0$ equations), then the system is* consistent *with exactly* one *solution, and we may write*

$$\textit{Solution } = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \qquad \textit{but} \qquad \textit{Solution set } = \left\{ \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \right\}.$$

- *Otherwise, the system given by the* RREF *matrix must* not include a $0 = 1$ contradiction *while* including at least one equation with multiple variables. *This means it is* consistent *with* infinitely-many *different solutions. We'll learn how to find such solution sets in* *Section 1.4.*

**Activity 1.3.8** Consider each of the following systems of linear equations or vector equations.

**(a)**

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & - & 3\,x_3 & = & 8 \\ 3\,x_1 & - & 2\,x_2 & - & 5\,x_3 & = & 17 \\ x_1 & - & x_2 & - & 2\,x_3 & = & 7 \\ 10\,x_1 & - & 8\,x_2 & - & 21\,x_3 & = & 65 \end{array}$$

**(i)** Explain and demonstrate how to find a simpler linear system that has the same solution set.

**Answer**.

$$\begin{array}{rrrcr} x_1 & & & = & 2 \\ & x_2 & & = & -3 \\ & & x_3 & = & -1 \\ & & 0 & = & 0 \end{array}$$

**(ii)** Explain whether this solution set has no solutions, one solution, or infinitely-many solutions. If the set is finite, describe it using set notation.

**Answer**. The solution set has one solution. The solution set is $\left\{ \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} \right\}$.

**(b)**

$$\begin{array}{rcrcrcr} x_1 & - & 5\,x_2 & - & 15\,x_3 & = & -8 \\ & & x_2 & + & 3\,x_3 & = & 1 \\ x_1 & & & & & = & 2 \\ 5\,x_1 & - & 7\,x_2 & - & 21\,x_3 & = & -10 \end{array}$$

**(i)** Explain and demonstrate how to find a simpler linear system that has the same solution set.

**Answer**.

$$\begin{array}{rrcrcr} x_1 & & & & = & 0 \\ & x_2 & + & 3\,x_3 & = & 0 \\ & & & 0 & = & 1 \\ & & & 0 & = & 0 \end{array}$$

**(ii)** Explain whether this solution set has no solutions, one solution, or infinitely-many solutions. If the set is finite, describe it using set notation.

**Answer**. The solution set has no solutions. The solution set is $\emptyset$.

**(c)**

$$\begin{array}{rcrcrcr} -2\,x_1 & + & 2\,x_2 & + & 5\,x_3 & = & 1 \\ -x_1 & + & x_2 & + & 2\,x_3 & = & 1 \\ 2\,x_1 & - & 2\,x_2 & + & x_3 & = & -7 \\ -2\,x_1 & + & 2\,x_2 & + & 16\,x_3 & = & -10 \end{array}$$

**(i)** Explain and demonstrate how to find a simpler linear system that has the same solution set.

**Answer**.

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & & & = & -3 \\ & & & & x_3 & = & -1 \\ & & & & 0 & = & 0 \\ & & & & 0 & = & 0 \end{array}$$

**(ii)** Explain whether this solution set has no solutions, one solution, or infinitely-many solutions. If the set is finite, describe it using set notation.

**Answer**. The solution set has infinitely-many solutions.

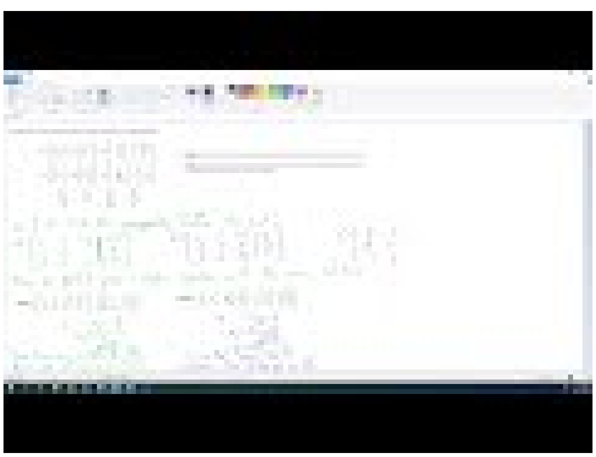
### 1.3.3 Individual Practice

**Activity 1.3.9** In Fact 1.1.11, we stated, but did not prove the assertion that all linear systems are one of the following:

1. *Consistent with one solution:* its solution set contains a single vector, e.g. $\left\{\left[\begin{array}{c}1\\2\\3\end{array}\right]\right\}$
2. *Consistent with infinitely-many solutions*: its solution set contains infinitely many vectors, e.g. $\left\{\left[\begin{array}{c}1\\2-3a\\a\end{array}\right] \middle| a\in\mathbb{R}\right\}$
3. *Inconsistent*: its solution set is the empty set, denoted by either $\{\}$ or $\emptyset$.

Explain why this fact is a consequence of Fact 1.3.7 above.

### 1.3.4 Videos



Standalone

**Figure 3** Video: Finding the number of solutions for a system

### 1.3.5 Exercises

Exercises available at `https://tbil.org/preview/linear-algebra/exercises/#/bank/LE3/`.

### 1.3.6 Mathematical Writing Explorations

**Exploration 1.3.10** A system of equations with all constants equal to 0 is called **homogeneous**. These are addressed in detail in section Section 2.7

- Choose three systems of equations from this chapter that you have already solved. Replace the constants with 0 to make the systems homogeneous. Solve the homogeneous systems and make a conjecture about the relationship between the earlier solutions you found and the associated homogeneous systems.
- Prove or disprove. A system of linear equations is homogeneous if an only if it has the the zero vector as a solution.

### 1.3.7 Sample Problem and Solution

Sample problem Example **??**.

## 1.4 Linear Systems with Infinitely-Many Solutions (LE4)

### Learning Outcomes

- Compute the solution set for a system of linear equations or a vector equation with infinitely many solutions.

### 1.4.1 Warm Up

**Activity 1.4.1** Write down any three linear systems and determine if they are consistent, have a single solution, or have infinitely many solutions.

### 1.4.2 Class Activities

**Activity 1.4.2** Consider this simplified linear system found to be equivalent to the system from Activity 1.3.6:

$$\begin{aligned} x_1 + 2x_2 \phantom{+x_3} &= 4 \\ x_3 &= -1 \end{aligned}$$

Earlier, we determined this system has infinitely-many solutions.

**(a)** Let $x_1 = a$ and write the solution set in the form $\left\{ \begin{bmatrix} a \\ ? \\ ? \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$.

**(b)** Let $x_2 = b$ and write the solution set in the form $\left\{ \begin{bmatrix} ? \\ b \\ ? \end{bmatrix} \middle| \, b \in \mathbb{R} \right\}$.

**(c)** Which of these was easier? What features of the RREF matrix $\left[\begin{array}{ccc|c} \boxed{1} & 2 & 0 & 4 \\ 0 & 0 & \boxed{1} & -1 \end{array}\right]$ caused this?

**Definition 1.4.3** Recall that the pivots of a matrix in RREF form are the leading 1s in each non-zero row.

The pivot columns in an augmented matrix correspond to the **bound variables** in the system of equations ($x_1, x_3$ below). The remaining variables are called **free variables** ($x_2$ below).

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 0 & 4 \\ 0 & 0 & \boxed{1} & -1 \end{array}\right]$$

To efficiently solve a system in RREF form, assign letters to the free variables, and then solve for the bound variables. ◊

**Activity 1.4.4** Find the solution set for the system

$$\begin{aligned} 2x_1 - 2x_2 - 6x_3 + x_4 - x_5 &= 3 \\ -x_1 + x_2 + 3x_3 - x_4 + 2x_5 &= -3 \\ x_1 - 2x_2 - x_3 + x_4 + x_5 &= 2 \end{aligned}$$

by doing the following.

**(a)** Row-reduce its augmented matrix.

**(b)** Assign letters to the free variables (given by the non-pivot columns):

$$? = a$$

$$? = b$$

**(c)** Solve for the bound variables (given by the pivot columns) to show that

$$? = 1 + 5a + 2b$$

$$? = 1 + 2a + 3b$$

$$? = 3 + 3b$$

**(d)** Replace $x_1$ through $x_5$ with the appropriate expressions of $a, b$ in the following set-builder notation.

$$\left\{ \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array}\right] \middle| \, a, b \in \mathbb{R} \right\}$$

**Remark 1.4.5** Don't forget to correctly express the solution set of a linear system. Systems with zero or one solutions may be written by listing their elements, while systems with infinitely-many solutions may be written using set-builder notation.

- *Inconsistent*: $\emptyset$ or $\{\}$
  - (not $0$ or $\left[\begin{array}{c}0\\0\\0\end{array}\right]$)
- *Consistent* with *one solution*: e.g. $\left\{\left[\begin{array}{c}1\\2\\3\end{array}\right]\right\}$
  - (not just $\left[\begin{array}{c}1\\2\\3\end{array}\right]$)
- *Consistent* with *infinitely-many solutions*: e.g. $\left\{\left[\begin{array}{c}1\\2-3a\\a\end{array}\right] \,\middle|\, a\in\mathbb{R}\right\}$
  - (not just $\left[\begin{array}{c}1\\2-3a\\a\end{array}\right]$)

**Activity 1.4.6** Consider the following system of linear equations.

$$x_1\left[\begin{array}{c}1\\0\\1\end{array}\right]+x_2\left[\begin{array}{c}0\\1\\-1\end{array}\right]+x_3\left[\begin{array}{c}-1\\5\\-5\end{array}\right]+x_4\left[\begin{array}{c}-3\\13\\-13\end{array}\right]=\left[\begin{array}{c}-3\\12\\-12\end{array}\right].$$

**(a)** Explain how to find a simpler system or vector equation that has the same solution set.

**(b)** Explain how to describe this solution set using set notation.

**Activity 1.4.7** Consider the following system of linear equations.

$$\begin{array}{ccccccc}
x_1 & & & - & 2\,x_3 & = & -3\\
5\,x_1 & + & x_2 & - & 7\,x_3 & = & -18\\
5\,x_1 & - & x_2 & - & 13\,x_3 & = & -12\\
x_1 & + & 3\,x_2 & + & 7\,x_3 & = & -12
\end{array}$$

**(a)** Explain how to find a simpler system or vector equation that has the same solution set.

**(b)** Explain how to describe this solution set using set notation.

### 1.4.3 Individual Practice

**Activity 1.4.8** Consider the following linear system, its augmented matrix $A$, and RREF($A$):

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & 4 \\ & & x_2 & - & 2\,x_3 & = & -1 \\ & & x_2 & - & 2\,x_3 & = & -3 \\ x_1 & + & 2\,x_2 & - & 5\,x_3 & = & 0 \end{array}$$

$$A = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & 1 & -2 & -3 \\ 1 & 2 & -5 & 0 \end{array}\right], \quad \text{RREF}(A) = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

All of the following statements are not accurate or otherwise incorrect; identify what is problematic about the statements and correct them.

**(a)** The matrix $A$ is inconsistent.

**(b)** The linear system has two bound variables and one free variable.

**(c)** The solution set to the given linear system is $\{\emptyset\}$.

**Activity 1.4.9** Consider the following linear system, its augmented matrix $B$, and RREF($B$):

$$\begin{array}{rcrcrcrcrcr} 2\,x_1 & - & 2\,x_2 & - & 8\,x_3 & + & 3\,x_4 & - & 9\,x_5 & = & -17 \\ -x_1 & & & + & x_3 & - & x_4 & + & 2\,x_5 & = & 6 \\ 2\,x_1 & - & x_2 & - & 5\,x_3 & + & x_4 & - & 5\,x_5 & = & -10 \\ -x_1 & + & 3\,x_2 & + & 10\,x_3 & & & + & 7\,x_5 & = & 6 \end{array}$$

$$B = \left[\begin{array}{ccccc|c} 2 & -2 & -8 & 3 & -9 & -17 \\ -1 & 0 & 1 & -1 & 2 & 6 \\ 2 & -1 & -5 & 1 & -5 & -10 \\ -1 & 3 & 10 & 0 & 7 & 6 \end{array}\right]$$

$$\text{RREF}(B) = \left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & -1 & -3 \\ 0 & 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

All of the following statements are not accurate or otherwise incorrect; identify what is problematic about the statements and correct them.

**(a)** The matrix $B$ is consistent with infinitely many solutions.

**(b)** The solution set is given by $\left[\begin{array}{c} a+b-3 \\ -3\,a-2\,b+1 \\ a \\ b-3 \\ b \end{array}\right]$.

**(c)** The variables $x_3, x_5$ are free. Setting them equal to $a, b$ respectively and solving for the bound variables, the solution set to the linear system is given by $\left\{ \left[ \begin{array}{c} a+b-3 \\ -3\,a-2\,b+1 \\ b-3 \end{array} \right] \middle| \, a, b \in \mathbb{R} \right\}$.

### 1.4.4 Videos

Standalone

**Figure 4** Video: Solving a system of linear equations with infinitely-many solutions

### 1.4.5 Exercises

Exercises available at `https://tbil.org/preview/linear-algebra/exercises/#/bank/LE4/`.

### 1.4.6 Mathematical Writing Explorations

**Exploration 1.4.10** Construct a system of 3 equations in 3 variables having:

- 0 free variables
- 1 free variable
- 2 free variables

In each case, solve the system you have created. Conjecture a relationship between the number of free variables and the type of solution set that can be obtained from a given system.

**Exploration 1.4.11** For each of the following, decide if it's true or false. If you think it's true, can we construct a proof? If you think it's false, can we find a counterexample?

- If the coefficient matrix of a system of linear equations has a pivot in the rightmost column, then the system is inconsistent.
- If a system of equations has two equations and four unknowns, then it must be consistent.

- If a system of equations having four equations and three unknowns is consistent, then the solution is unique.
- Suppose that a linear system has four equations and four unknowns and that the coefficient matrix has four pivots. Then the linear system is consistent and has a unique solution.
- Suppose that a linear system has five equations and three unknowns and that the coefficient matrix has a pivot in every column. Then the linear system is consistent and has a unique solution.

### 1.4.7 Sample Problem and Solution

Sample problem Example **??**.

# Chapter 2

# Euclidean Vectors (EV)

## Learning Outcomes

What is a space of Euclidean vectors?
By the end of this chapter, you should be able to...

1. Determine if a Euclidean vector can be written as a linear combination of a given set of Euclidean vectors by solving an appropriate vector equation.
2. Determine if a set of Euclidean vectors spans $\mathbb{R}^n$ by solving appropriate vector equations.
3. Determine if a subset of $\mathbb{R}^n$ is a subspace or not.
4. Determine if a set of Euclidean vectors is linearly dependent or independent by solving an appropriate vector equation.
5. Explain why a set of Euclidean vectors is or is not a basis of $\mathbb{R}^n$.
6. Compute a basis for the subspace spanned by a given set of Euclidean vectors, and determine the dimension of the subspace.
7. Find a basis for the solution set of a homogeneous system of equations.

**Readiness Assurance.** Before beginning this chapter, you should be able to...

1. Use set builder notation to describe sets of vectors.
   - Review: YouTube[1]
2. Add Euclidean vectors and multiply Euclidean vectors by scalars.
   - Review: Khan Academy (1)[2] (2)[3]

[1] `youtu.be/xnfUZ-NTsCE`
[2] `www.khanacademy.org/math/linear-algebra/vectors-and-spaces/vectors/v/adding-vectors`
[3] `www.khanacademy.org/math/linear-algebra/vectors-and-spaces/vectors/v/`

3. Perform basic manipulations of augmented matrices and linear systems.
   - Review: Section 1.1, Section 1.2, Section 1.3

# 2.1 Linear Combinations (EV1)

## Learning Outcomes

- Determine if a Euclidean vector can be written as a linear combination of a given set of Euclidean vectors by solving an appropriate vector equation.

## 2.1.1 Warm Up

**Activity 2.1.1** Discuss which of the vectors $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$ is a solution to the given vector equation:

$$x_1 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$$

## 2.1.2 Class Activities

**Note 2.1.2** We've been working with **Euclidean vector spaces** of the form

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \middle| x_1, x_2, \ldots, x_n \in \mathbb{R} \right\}.$$

There are other kinds of **vector spaces** as well (e.g. polynomials, matrices), which we will investigate in Section **??**. But understanding the structure of *Euclidean* vectors on their own will be beneficial, even when we turn our attention to other kinds of vectors.

We will use the phrase **vector space** freely from this point on, even while delaying a formal definition. Readers can choose to interpret this to mean *Euclidean vector space*, i.e $\mathbb{R}^n$ for some $n$, if they wish; we do this as all of the statements we make using the term **vector space** are also true for all vector spaces as defined in Definition **??**.

Likewise, when we multiply a vector by a real number, as in $-3 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ -6 \end{bmatrix}$, we refer to this real number as a **scalar**.

---
`multiplying-vector-by-scalar`

We often use letters like $V$ and $W$ to refer to vector spaces (Euclidean or otherwise)

**Definition 2.1.3** A **linear combination** of a set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is given by $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$ for any choice of scalar multiples $c_1, c_2, \dots, c_n$.

For example, we can say $\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ since

$$\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

◊

**Definition 2.1.4** The **span** of a set of vectors is the collection of all linear combinations of that set:

$$\operatorname{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \,|\, c_i \in \mathbb{R}\}.$$

For example:

$$\operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\} = \left\{a\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \,\middle|\, a, b \in \mathbb{R}\right\}.$$

◊

**Activity 2.1.5** Consider $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$.

**(a)** Sketch the four Euclidean vectors

$$1\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \quad 0\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad -2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$$

in the $xy$ plane by placing a dot at the $(x, y)$ coordinate associated with each vector.

**(b)** Sketch a representation of all the vectors belonging to

$$\operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\} = \left\{a\begin{bmatrix} 1 \\ 2 \end{bmatrix} \,\middle|\, a \in \mathbb{R}\right\}$$

in the $xy$ plane by plotting their $(x, y)$ coordinates as dots. What best describes this sketch?

A. A line B. A plane C. A parabola D. A circle

**Remark 2.1.6** It is important to remember that

$$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \neq \operatorname{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}.$$

For example,

$$\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix},\begin{bmatrix}1\\2\\1\end{bmatrix}\right\}$$

is a set containing exactly two vectors, while

$$\operatorname{span}\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix},\begin{bmatrix}1\\2\\1\end{bmatrix}\right\}=\left\{a\begin{bmatrix}1\\-1\\2\end{bmatrix}+b\begin{bmatrix}1\\2\\1\end{bmatrix}\middle|\, a,b\in\mathbb{R}\right\}$$

is a set containing infinitely-many vectors.

**Activity 2.1.7** Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

**(a)** Sketch the following five Euclidean vectors in the $xy$ plane.

$$1\begin{bmatrix}1\\2\end{bmatrix}+0\begin{bmatrix}-1\\1\end{bmatrix}=?\qquad 0\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}=?\qquad 1\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}=?$$

$$-2\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}=?\qquad -1\begin{bmatrix}1\\2\end{bmatrix}+-2\begin{bmatrix}-1\\1\end{bmatrix}=?$$

**(b)** Sketch a representation of all the vectors belonging to

$$\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}=\left\{a\begin{bmatrix}1\\2\end{bmatrix}+b\begin{bmatrix}-1\\1\end{bmatrix}\middle|\, a,b\in\mathbb{R}\right\}$$

in the $xy$ plane. What best describes this sketch?

A. A line B. A plane C. A parabola D. A circle

**Activity 2.1.8** Sketch a representation of all the vectors belonging to $\operatorname{span}\left\{\begin{bmatrix}6\\-4\end{bmatrix},\begin{bmatrix}-3\\2\end{bmatrix}\right\}$ in the $xy$ plane. What best describes this sketch?

A. A line

B. A plane

C. A parabola

D. A cube

**Activity 2.1.9** Consider the following questions to discover whether a Euclidean vector belongs to a span.

**(a)** The Euclidean vector $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belongs to $\operatorname{span}\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix},\begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ exactly when there exists a solution to which of these vector equations?

A. $x_1 \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$

B. $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$

C. $x_1 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} = 0$

**(b)** Use technology to find RREF of the corresponding augmented matrix, and then use that matrix to find the solution set of the vector equation.

**(c)** Given this solution set, does $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belong to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$?

**Observation 2.1.10** The following are all equivalent statements:

- The vector $\vec{b}$ belongs to span$\{\vec{v}_1, \ldots, \vec{v}_n\}$.
- The vector $\vec{b}$ is a linear combination of the vectors $\vec{v}_1, \ldots, \vec{v}_n$.
- The vector equation $x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{b}$ is consistent.
- The linear system corresponding to $\left[\vec{v}_1 \, \ldots \, \vec{v}_n \,|\, \vec{b}\right]$ is consistent.
- RREF $\left[\vec{v}_1 \, \ldots \, \vec{v}_n \,|\, \vec{b}\right]$ doesn't have a row $[0 \cdots 0 \,|\, 1]$ representing the contradiction $0 = 1$.

**Activity 2.1.11** Consider the following claim:

$\begin{bmatrix} -6 \\ 2 \\ -6 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$, and $\begin{bmatrix} -4 \\ 1 \\ -5 \end{bmatrix}$.

**(a)** Write a statement involving the solutions of a vector equation that's equivalent to this claim.

**(b)** Explain why the statement you wrote is true.

**(c)** Since your statement was true, use the solution set to describe a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$, and $\begin{bmatrix} -4 \\ 1 \\ -5 \end{bmatrix}$ that equals $\begin{bmatrix} -6 \\ 2 \\ -6 \end{bmatrix}$.

**Activity 2.1.12** Consider the following claim:

$$\begin{bmatrix} -5 \\ -1 \\ -7 \end{bmatrix} \text{ belongs to span } \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ -5 \end{bmatrix} \right\}.$$

**(a)** Write a statement involving the solutions of a vector equation that's equivalent to this claim.

**(b)** Explain why the statement you wrote is false, to conclude that the vector does not belong to the span.

## 2.1.3 Individual Practice

**Activity 2.1.13** Before next class, find some time to do the following:

**(a)** Without referring to your activity book, write down the definition of a linear combination of vectors.

**(b)** Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$. Write down an example $\vec{w}_1 = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ of a linear combination of $\vec{u}, \vec{v}$. Then write down an example $\vec{w}_2 = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ that is *not* a linear combination of $\vec{u}, \vec{v}$.

**(c)** Draw a rough sketch of the vectors $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$, $\vec{w}_1 = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$, and $\vec{w}_2 = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ in $\mathbb{R}^3$.

### 2.1.4 Videos

Standalone

**Figure 5** Video: Linear combinations

### 2.1.5 Exercises

Exercises available at https://tbil.org/preview/linear-algebra/exercises/#/bank/EV1/.

### 2.1.6 Mathematical Writing Explorations

**Exploration 2.1.14** Suppose $S = \{\vec{v_1}, \ldots, \vec{v_n}\}$ is a set of vectors. Show that $\vec{v_0}$ is a linear combination of members of $S$, if an only if there are a set of scalars $\{c_0, c_1, \ldots, c_n\}$ such that $\vec{z} = c_0\vec{v_0} + \cdots + c_n\vec{v_n}$. We can do this in a few parts. I've used bullets here to indicate all that needs to be done. This is an "if and only if" proof, so it needs two parts.

- First, assume that $\vec{0} = c_0\vec{v_0} + \cdots + c_n\vec{v_n}$ has a solution, with $c_0 \neq 0$. Show that $\vec{v_0}$ is a linear combination of elements of $S$.
- Next, assume that $\vec{v_0}$ is a linear combination of elements of $S$. Can you find the appropriate $\{c_0, c_1, \ldots, c_n\}$ to make the equation $\vec{z} = c_0\vec{v_0} + \cdots + c_n\vec{v_n}$ true?
- In either of your proofs above, does the case when $\vec{v_0} = \vec{z}$ change your thinking? Explain why or why not.

### 2.1.7 Sample Problem and Solution

Sample problem Example **??**.

## 2.2 Spanning Sets (EV2)

### Learning Outcomes

- Determine if a set of Euclidean vectors spans $\mathbb{R}^n$ by solving appropriate vector equations.

### 2.2.1 Warm Up

**Activity 2.2.1** Given a set of ingredients and a meal, a recipe is a list of amounts of each ingredient required to prepare the given meal.

**(a)** Use the words *vector* and *linear combination* to create a new statement that is analogous to one above.

**(b)** Building on your analogy, what role might the word *span* play?

### 2.2.2 Class Activities

**Observation 2.2.2** Any single non-zero vector/number $x$ in $\mathbb{R}^1$ spans $\mathbb{R}^1$, since $\mathbb{R}^1 = \{cx \,|\, c \in \mathbb{R}\}$.

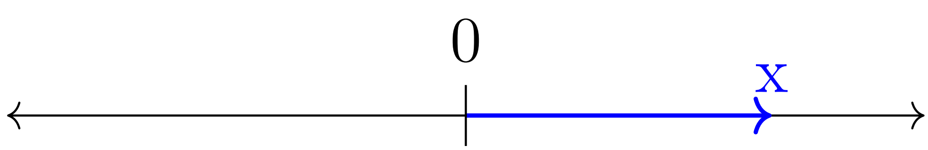



**Figure 6** An $\mathbb{R}^1$ vector

**Activity 2.2.3** How many vectors are required to span $\mathbb{R}^2$? Sketch a drawing in the $xy$ plane to support your answer.

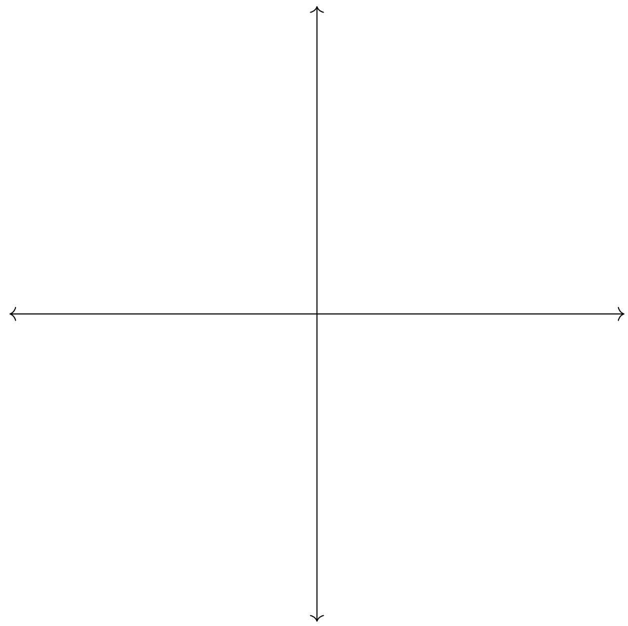


**Figure 7** The $xy$ plane $\mathbb{R}^2$

A. 1

B. 2

C. 3

D. 4

E. Infinitely Many

**Activity 2.2.4** How many vectors are required to span $\mathbb{R}^3$?

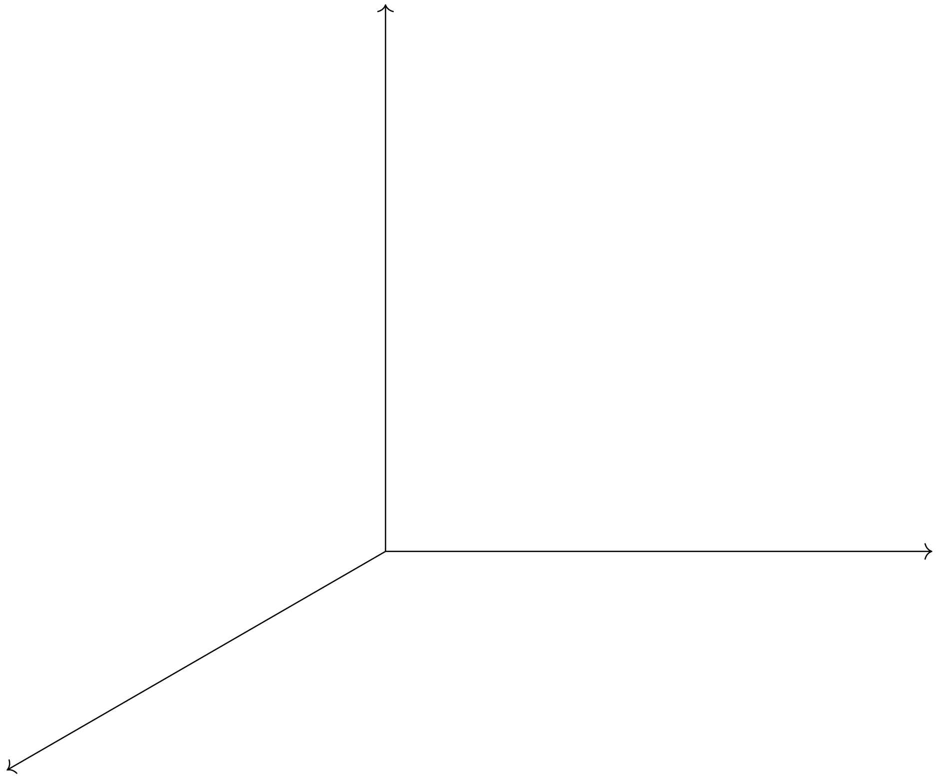


**Figure 8** $\mathbb{R}^3$ space

A. 1

B. 2

C. 3

D. 4

E. Infinitely Many

**Fact 2.2.5** *At least $n$ vectors are required to span $\mathbb{R}^n$.*

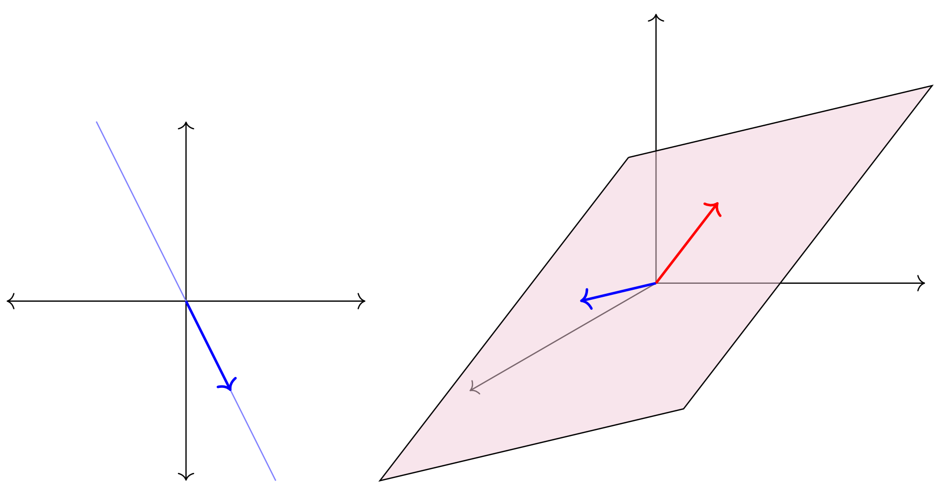


***Figure 9*** *Failed attempts to span $\mathbb{R}^n$ by $< n$ vectors*

**Activity 2.2.6** Consider the question: Does every vector in $\mathbb{R}^3$ belong to

$$\operatorname{span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}-2\\0\\1\end{bmatrix},\begin{bmatrix}-2\\-2\\2\end{bmatrix}\right\}?$$

**(a)** Determine if $\begin{bmatrix}7\\-3\\-2\end{bmatrix}$ belongs to $\operatorname{span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}-2\\0\\1\end{bmatrix},\begin{bmatrix}-2\\-2\\2\end{bmatrix}\right\}$.

**Answer**. The vector belongs to the span.

**(b)** Determine if $\begin{bmatrix}0\\-4\\3\end{bmatrix}$ belongs to $\operatorname{span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}-2\\0\\1\end{bmatrix},\begin{bmatrix}-2\\-2\\2\end{bmatrix}\right\}$.

**Answer**. The vector belongs to the span.

**(c)** Determine if $\begin{bmatrix}2\\5\\7\end{bmatrix}$ belongs to $\operatorname{span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}-2\\0\\1\end{bmatrix},\begin{bmatrix}-2\\-2\\2\end{bmatrix}\right\}$.

**Answer**. The vector does not belong to the span.

**Activity 2.2.7** We'd prefer a more methodical method to decide if every vector in $\mathbb{R}^n$ belongs to some spanning set, compared to the guess-and-check method we used in Activity 2.2.6.

**(a)** An arbitrary vector $\begin{bmatrix}?\\?\\?\end{bmatrix}$ belongs to $\operatorname{span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}-2\\0\\1\end{bmatrix},\begin{bmatrix}-2\\-2\\2\end{bmatrix}\right\}$ provided the equation

$$x_1\begin{bmatrix}1\\-1\\0\end{bmatrix}+x_2\begin{bmatrix}-2\\0\\1\end{bmatrix}+x_3\begin{bmatrix}-2\\-2\\2\end{bmatrix}=\begin{bmatrix}?\\?\\?\end{bmatrix}$$

has...

A. no solutions.

B. exactly one solution.

C. at least one solution.

D. infinitely-many solutions.

**Answer**. A.

**(b)** We're guaranteed at least one solution if the RREF of the corresponding augmented matrix has no contradictions; likewise, we have no solutions if the RREF corresponds to the contradiction $0=1$. Given

$$\left[\begin{array}{ccc|c}1&-2&-2&?\\-1&0&-2&?\\0&1&2&?\end{array}\right]\sim\left[\begin{array}{ccc|c}1&0&2&?\\0&1&2&?\\0&0&0&?\end{array}\right]$$

we may conclude that the set does not span all of $\mathbb{R}^3$ because...

A. the row $[0\,1\,2\,|\,?\,]$ prevents a contradiction.

B. the row $[0\,1\,2\,|\,?\,]$ allows a contradiction.

C. the row $[0\,0\,0\,|\,?\,]$ prevents a contradiction.

D. the row $[0\,0\,0\,|\,?\,]$ allows a contradiction.

**Answer**. D.

**Fact 2.2.8** *The set $\{\vec{v}_1, \ldots, \vec{v}_n\}$ spans all of $\mathbb{R}^n$ exactly when the vector equation*

$$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{w}$$

*is consistent for* every *vector $\vec{w}$.*

*Likewise, the set $\{\vec{v}_1, \ldots, \vec{v}_n\}$ fails to span all of $\mathbb{R}^n$ exactly when the vector equation*

$$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{w}$$

*is inconsistent for* some *vector $\vec{w}$.*

*Note these two possibilities are decided based on whether or not the RREF of the vector equation's coefficient matrix (that is,* $\mathrm{RREF}[\vec{v}_1\ \ldots\ \vec{v}_n]$*) has either all pivot rows, or at least one non-pivot row (a row of zeroes):*

$$\left[\begin{array}{ccc|} 1 & -2 & -2 \\ -1 & 0 & -2 \\ 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right].$$

**Activity 2.2.9** Consider the set of vectors $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix} \right\}$ and the question "Does $\mathbb{R}^4 = \operatorname{span} S$?"

**(a)** Rewrite this question in terms of the solutions to a vector equation.

**(b)** Answer your new question, and use this to answer the original question.

**Activity 2.2.10** Let $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^7$ be three Euclidean vectors, and suppose $\vec{w}$ is another vector with $\vec{w} \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. What can you conclude about $\operatorname{span}\{\vec{w}, \vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

A. $\operatorname{span}\{\vec{w}, \vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is larger than $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

B. $\operatorname{span}\{\vec{w}, \vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is the same as $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

C. $\operatorname{span}\{\vec{w}, \vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is smaller than $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

### 2.2.3 Individual Practice

**Activity 2.2.11** One of our important results in this lesson is Fact 2.2.5, which states that a set of $n$ vectors is required to span $\mathbb{R}^n$. While we developed some geometric intuition for why this true, we did not prove it in class. Before coming to class next time, follow the steps outlined below to convince yourself of this fact using the concepts we learned in this lesson.

**(a)** Let $\{\vec{v}_1, \ldots, \vec{v}_m\}$ be a set of vectors living in $\mathbb{R}^n$ and assume that $m < n$. How many rows and how many columns will the matrix $[\vec{v}_1 \cdots \vec{v}_m]$ have?

**(b)** Given no additional information about the vectors $\vec{v}_1, \ldots, \vec{v}_m$, what is the maximum possible number of pivots in $\text{RREF}[\vec{v}_1 \ \ldots \ \vec{v}_m]$?

**(c)** Conclude that our given set of vector cannot span all of $\mathbb{R}^n$.

### 2.2.4 Videos

Standalone

**Figure 10** Video: Determining if a set spans a Euclidean space

### 2.2.5 Exercises

Exercises available at `https://tbil.org/preview/linear-algebra/exercises/#/bank/EV2/`.

### 2.2.6 Mathematical Writing Explorations

**Exploration 2.2.12** Construct each of the following, or show that it is impossible:

- A set of 2 vectors that spans $\mathbb{R}^3$
- A set of 3 vectors that spans $\mathbb{R}^3$
- A set of 3 vectors that does not span $\mathbb{R}^3$
- A set of 4 vectors that spans $\mathbb{R}^3$

For any of the sets you constructed that did span the required vector space, are any of the vectors a linear combination of the others in your set?

**Exploration 2.2.13** Based on these results, generalize this a conjecture about how a set of $n-1, n$ and $n+1$ vectors would or would not span $\mathbb{R}^n$.

### 2.2.7 Sample Problem and Solution

Sample problem Example **??**.

## 2.3 Subspaces (EV3)

### Learning Outcomes

- Determine if a subset of $\mathbb{R}^n$ is a subspace or not.

### 2.3.1 Warm Up

**Activity 2.3.1** Consider the linear equation

$$x+2y+z=0.$$

**(a)** Verify that both $\vec{v}=\begin{bmatrix}1\\-1\\1\end{bmatrix}$ and $\vec{w}=\begin{bmatrix}1\\0\\-1\end{bmatrix}$ are solutions.

**(b)** Is the vector $2\vec{v}-3\vec{w}$ also a solution?

### 2.3.2 Class Activities

**Observation 2.3.2** Recall that if $S=\{\vec{v}_1,\dots,\vec{v}_n\}$ is subset of vectors in $\mathbb{R}^n$, then $\operatorname{span}(S)$ is the set of all linear combinations of vectors in $S$. In EV2 (Section 2.2), we learned how to decide whether $\operatorname{span}(S)$ was equal to all of $\mathbb{R}^n$ or something strictly smaller.

**Activity 2.3.3** Let $S$ denote a set of vectors in $\mathbb{R}^n$ and suppose that $\vec{u},\vec{v}\in\operatorname{span}(S)$, $c\in\mathbb{R}$ and that $\vec{w}\in\mathbb{R}^n$. Which of the following vectors might *not* belong to $\operatorname{span}(S)$?

A. $\vec{0}$

B. $\vec{u}+\vec{w}$

C. $\vec{u}+\vec{v}$

D. $c\vec{u}$

**Definition 2.3.4** A **homogeneous** system of linear equations is one of the form:

$$\begin{alignedat}{4} a_{11}x_1 &+ a_{12}x_2 &+ \ldots + &a_{1n}x_n &= 0 \\ a_{21}x_1 &+ a_{22}x_2 &+ \ldots + &a_{2n}x_n &= 0 \\ \vdots & \quad\ \vdots & & \ \vdots & \vdots \\ a_{m1}x_1 &+ a_{m2}x_2 &+ \ldots + &a_{mn}x_n &= 0 \end{alignedat}$$

This system is equivalent to the vector equation:

$$x_1\vec{v}_1+\cdots+x_n\vec{v}_n=\vec{0}$$

and the augmented matrix:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{array}\right]$$

◊

**Activity 2.3.5** Consider the homogeneous vector equation $x_1\vec{v}_1+\cdots+x_n\vec{v}_n=\vec{0}$.

**(a)** Is this equation consistent?

A. no.

B. yes.

C. more information is required.

**(b)** Note that if $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ are both solutions, we know that

$$a_1\vec{v}_1+\cdots+a_n\vec{v}_n=\vec{0} \text{ and } b_1\vec{v}_1+\cdots+b_n\vec{v}_n=\vec{0}.$$

Therefore by adding these equations:

$$(a_1+b_1)\vec{v}_1+\cdots+(a_n+b_n)\vec{v}_n=\vec{0},$$

we may conclude that the vector $\begin{bmatrix} a_1+b_1 \\ \vdots \\ a_n+b_n \end{bmatrix}$ is...

A. another solution.

B. not a solution.

C. is equal to $\vec{0}$.

**(c)** Similarly, if $c \in \mathbb{R}$, then since multiplying by $c$ yields:

$$(ca_1)\vec{v}_1 + \cdots + (ca_n)\vec{v}_n = \vec{0},$$

we may conclude that the vector $\begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix}$ is...

A. another solution.

B. not a solution.

C. is equal to $\vec{0}$.

D. The empty set.

**Observation 2.3.6** If $S$ is any set of vectors in $\mathbb{R}^n$, then the set span$(S)$ has the following properties:

- the set span$(S)$ is non-empty.
- the set span$(S)$ is closed under addition: for any $\vec{u}, \vec{v} \in \text{span}(S)$, the sum $\vec{u} + \vec{v}$ is also in span$(S)$.
- the set span$(S)$ is closed under scalar multiplication: for any $\vec{u} \in \text{span}(S)$ and scalar $c \in \mathbb{R}$, the product $c\vec{u}$ is also in span$(S)$.

Likewise, if $W$ is the solution set to a homogenous vector equation, it too satisfies:

- the set $W$ is non-empty.
- the set $W$ is closed under addition: for any $\vec{u}, \vec{v} \in W$, the sum $\vec{u} + \vec{v}$ is also in $W$.
- the set span$(S)$ is closed under scalar multiplication: for any $\vec{u} \in W$ and scalar $c \in \mathbb{R}$, the product $c\vec{u}$ is also in $W$.

**Definition 2.3.7** A subset $W$ of a vector space is called a **subspace** provided that it satisfies the following properties:

- the subset is non-empty.
- the subset is **closed under addition**: for any $\vec{u}, \vec{v} \in W$, the sum $\vec{u} + \vec{v}$ is also in $W$.
- the subset is **closed under scalar multiplication**: for any $\vec{u} \in W$ and scalar $c \in \mathbb{R}$, the product $c\vec{u}$ is also in $W$.

$\Diamond$

**Observation 2.3.8** Note the similarities between a planar subspace spanned by two non-colinear vectors in $\mathbb{R}^3$, and the Euclidean plane $\mathbb{R}^2$. While they are not the same thing (and shouldn't be referred to interchangably), algebraists call such similar spaces **isomorphic**; we'll learn what this means more carefully in a later chapter.

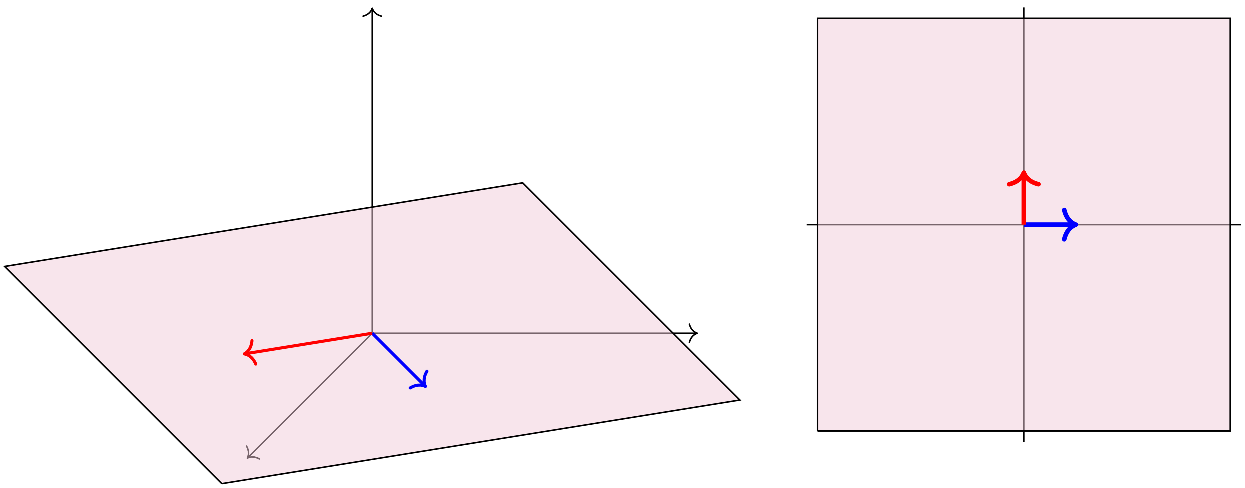


**Figure 11** A planar subset of $\mathbb{R}^3$ compared with the plane $\mathbb{R}^2$.

**Activity 2.3.9** Let $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| x + 2y + z = 0 \right\}$.

**(a)** Is $W$ the empty set?

**(b)** Let's assume that $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ are in $W$. What are we allowed to assume?

A. $x + 2y + z = 0$.

B. $a + 2b + c = 0$.

C. Both of these.

D. Neither of these.

**(c)** Which equation must be verified to show that $\vec{v} + \vec{w} = \begin{bmatrix} x+a \\ y+b \\ z+c \end{bmatrix}$ also belongs to $W$?

A. $(x + a) + 2(y + b) + (z + c) = 0$.

B. $x + a + 2y + b + z + c = 0$.

C. $x + 2y + z = a + 2b + c$.

**(d)** Use the assumptions from (a) to verify the equation from (b).

**(e)** Is $W$ is a subspace of $\mathbb{R}^3$?

A. Yes

B. No

C. Not enough information

**(f)** Show that $k\vec{v} = \begin{bmatrix} kx \\ ky \\ kz \end{bmatrix}$ also belongs to $W$ for any $k \in \mathbb{R}$ by verifying $(kx) + 2(ky) + (kz) = 0$ under these assumptions.

**(g)** Is $W$ is a subspace of $\mathbb{R}^3$?

A. Yes

B. No

C. Not enough information

**Activity 2.3.10** Let $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| x + 2y + z = 4 \right\}$.

**(a)** Is $W$ the empty set?

**(b)** Which of these statements is valid?

A. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *and* $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \in W$, so $W$ *is* a subspace.

B. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *and* $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \in W$, so $W$ *is not* a subspace.

C. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *but* $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \notin W$, so $W$ *is* a subspace.

D. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *but* $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \notin W$, so $W$ *is not* a subspace.

**(c)** Which of these statements is valid?

(a) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *and* $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in W$, so $W$ *is* a subspace.

(b) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *and* $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in W$, so $W$ *is not* a subspace.

(c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *but* $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin W$, so $W$ *is* a subspace.

(d) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in W$, *but* $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin W$, so $W$ *is not* a subspace.

**Remark 2.3.11** In summary, *any one* of the following is enough to prove that a nonempty subset $W$ is *not* a subspace:

- Find specific values for $\vec{u}, \vec{v} \in W$ such that $\vec{u} + \vec{v} \notin W$.
- Find specific values for $c \in \mathbb{R}, \vec{v} \in W$ such that $c\vec{v} \notin W$.
- Show that $\vec{0} \notin W$.

If you cannot do any of these, then $W$ can be proven to be a subspace by doing *all* of the following:

1. Show that $W$ is non-empty.
2. For all $\vec{v}, \vec{w} \in W$ (not just specific values), $\vec{u} + \vec{v} \in W$.
3. For all $\vec{v} \in W$ and $c \in \mathbb{R}$ (not just specific values), $c\vec{v} \in W$.

**Activity 2.3.12** Consider these subsets of $\mathbb{R}^3$:

$$R = \left\{ \left[ \begin{array}{c} x \\ y \\ z \end{array} \right] \middle| \, y = z + 1 \right\} \qquad S = \left\{ \left[ \begin{array}{c} x \\ y \\ z \end{array} \right] \middle| \, y = |z| \right\} \qquad T = \left\{ \left[ \begin{array}{c} x \\ y \\ z \end{array} \right] \middle| \, z = xy \right\}.$$

**(a)** Show $R$ isn't a subspace by showing that $\vec{0} \notin R$.

**(b)** Show $S$ isn't a subspace by finding two vectors $\vec{u}, \vec{v} \in S$ such that $\vec{u} + \vec{v} \notin S$.

**(c)** Show $T$ isn't a subspace by finding a vector $\vec{v} \in T$ such that $2\vec{v} \notin T$.

**Activity 2.3.13** Consider the following two sets of Euclidean vectors:

$$U = \left\{ \left[ \begin{array}{c} x \\ y \end{array} \right] \middle| \, 7\,x + 4\,y = 0 \right\} \qquad W = \left\{ \left[ \begin{array}{c} x \\ y \end{array} \right] \middle| \, 3\,xy^2 = 0 \right\}$$

Explain why one of these sets is a subspace of $\mathbb{R}^2$ and one is not.

**Activity 2.3.14** Consider the following attempted proof that

$$U = \left\{ \left[ \begin{array}{c} x \\ y \end{array} \right] \middle| \, x + y = xy \right\}$$

is closed under scalar multiplication.

> Let $\left[ \begin{array}{c} x \\ y \end{array} \right] \in U$, so we know that $x + y = xy$. We want to show $k \left[ \begin{array}{c} x \\ y \end{array} \right] = \left[ \begin{array}{c} kx \\ ky \end{array} \right] \in U$, that is, $(kx) + (ky) = (kx)(ky)$. This is verified by the following calculation:
>
> $$(kx) + (ky) = (kx)(ky)$$

$$\begin{aligned} k(x+y) &= k^2xy \\ 0[k(x+y)] &= 0[k^2xy] \\ 0 &= 0 \end{aligned}$$

Is this reasoning valid?

A. Yes

B. No

**Remark 2.3.15** Proofs of an equality LEFT = RIGHT should generally be of one of these forms:

1. Using a chain of equalities:

$$\begin{aligned} \text{LEFT} &= \cdots \\ &= \cdots \\ &= \cdots \\ &= \text{RIGHT} \end{aligned}$$

   Alternatively:

$$\begin{aligned} \text{LEFT} &= \cdots & \text{RIGHT} &= \cdots \\ &= \cdots & &= \cdots \\ &= \cdots & &= \cdots \\ &= \text{SAME} & &= \text{SAME} \end{aligned}$$

2. When the assumption THIS = THAT is already known or assumed to be true :

$$\begin{aligned} & \text{THIS} = \text{THAT} \\ \Rightarrow \quad & \cdots = \cdots \\ \Rightarrow \quad & \cdots = \cdots \\ \Rightarrow \quad & \text{LEFT} = \text{RIGHT} \end{aligned}$$

**Warning 2.3.16** The following proof is *invalid*.

$$\begin{aligned} & \text{LEFT} = \text{RIGHT} \\ \Rightarrow \quad & \cdots = \cdots \\ \Rightarrow \quad & \cdots = \cdots \\ \Rightarrow \quad & 0 = 0 \\ \Rightarrow \quad & \text{ANYTHING} = \text{ANYTHING} \end{aligned}$$

Basically, you cannot prove something is true by assuming it's true, and it's not helpful to prove to someone that zero equals itself (they probably already know that).

### 2.3.3 Individual Practice

**Remark 2.3.17** Recall that in Activity 2.2.1 we used the words *vector*, *linear combination*, and *span* to make an analogy with recipes, ingredients, and meals. In this analogy, a *recipe* was defined to be a list of amounts of each ingredient to build a particular meal.

**Activity 2.3.18**

**(a)** Given the set of ingredients $S = \{\text{flour}, \text{yeast}, \text{salt}, \text{water}, \text{sugar}, \text{milk}\}$, how should we think of the subspace $\text{span}(S)$?

**(b)** What is one meal that lives in the subspace $\text{span}(S)$?

**(c)** What is one meal that does not live in the subspace $\text{span}(S)$?

**Activity 2.3.19** Let

$$W = \left\{ \left[\begin{array}{c} x \\ y \\ z \\ w \end{array}\right] \middle| x + y = 3z + 2w \right\}.$$

The set $W$ is a subspace. Below are two attempted proofs of the fact that $W$ is closed under vector addition. Both of them are invalid; explain why.

**(a)** Let $\vec{u} = \left[\begin{array}{c} 1 \\ 4 \\ 1 \\ 1 \end{array}\right], \vec{v} = \left[\begin{array}{c} 2 \\ -1 \\ 1 \\ -1 \end{array}\right]$. Then both $\vec{u}, \vec{v}$ are elements of $W$. Their sum is

$$\vec{w} = \left[\begin{array}{c} 3 \\ 3 \\ 2 \\ 0 \end{array}\right]$$

and since

$$3 + 3 = 3 \cdot (2) + 2 \cdot (0),$$

it follows that $\vec{w}$ is also in $W$ and so $W$ is closed under vector addition.

**(b)** If $\left[\begin{array}{c} x \\ y \\ z \\ w \end{array}\right], \left[\begin{array}{c} a \\ b \\ c \\ d \end{array}\right]$ are in $W$, we need to show that $\left[\begin{array}{c} x+a \\ y+b \\ z+c \\ w+d \end{array}\right]$ is also in W. To be in $W$, we need

$$(x + a) + (y + b) = 3(z + c) + 2(w + d).$$

Well, if

$$(x + a) + (y + b) = 3(z + c) + 2(w + d),$$

then we know that

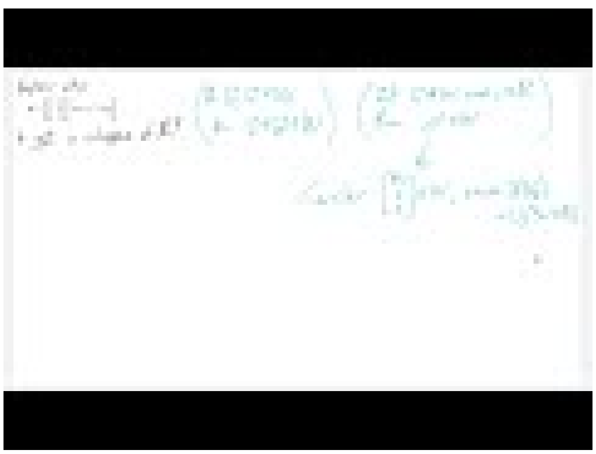
$$x + y - 3z - 2w + a + b - 3c - 2d = 0$$

by moving everything over to the left hand side. Since we are assumming that $x+y-3z-2w=0$ and $a+b-3c-2d=0$, it follows that $0=0$, which is true, which proves that vector addition is closed.

### 2.3.4 Videos

Standalone

**Figure 12** Video: Showing that a subset of a vector space is a subspace



Standalone

**Figure 13** Video: Showing that a subset of a vector space is not a subspace

### 2.3.5 Exercises

Exercises available at `https://tbil.org/preview/linear-algebra/exercises/#/bank/EV3/`.

### 2.3.6 Mathematical Writing Explorations

**Exploration 2.3.20** A square matrix $M$ is **symmetric**if, for each index $i, j$, the entries $m_{ij} = m_{ji}$. That is, the matrix is itself when reflected over the diagonal from upper left to lower right. Prove that the set of $n \times n$ symmetric matrices is a subspace of $M_{n \times n}$.

**Exploration 2.3.21** The space of all real-valued function of one real variable is a vector space. First, define $\oplus$ and $\odot$ for this vector space. Check that you have closure (both kinds!)

and show what the zero vector is under your chosen addition. Decide if each of the following is a subspace. If so, prove it. If not, provide the counterexample.

- The set of even functions, $\{f : \mathbb{R} \to \mathbb{R} : f(-x) = f(x) \text{ for all } x\}$.
- The set of odd functions, $\{f : \mathbb{R} \to \mathbb{R} : f(-x) = -f(x) \text{ for all } x\}$.

**Exploration 2.3.22** Give an example of each of these, or explain why it's not possible that such a thing would exist.

- A nonempty subset of $M_{2\times 2}$ that is not a subspace.
- A set of two vectors in $\mathbb{R}^2$ that is not a spanning set.

**Exploration 2.3.23** Let $V$ be a vector space and $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ a subset of $V$. Show that the span of $S$ is a subspace. Is it possible that there is a subset of $V$ containing fewer vectors than $S$, but whose span contains all of the vectors in the span of $S$?

### 2.3.7 Sample Problem and Solution

Sample problem Example **??**.

## 2.4 Linear Independence (EV4)

### Learning Outcomes

- Determine if a set of Euclidean vectors is linearly dependent or independent by solving an appropriate vector equation.

### 2.4.1 Warm Up

**Activity 2.4.1** Consider the vector equation

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ 4 \end{bmatrix}.$$

**(a)** Decide which of $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a solution vector.

**(b)** Consider now the following vector equation:

$$y_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y_2 \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + y_3 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + y_4 \begin{bmatrix} -1 \\ 7 \\ 4 \end{bmatrix} = \vec{0}.$$

How is this vector equation related to the original one?

**(c)** Use the solution vector you found in part (a) to construct a solution vector to this new equation.

### 2.4.2 Class Activities

**Activity 2.4.2** Consider the two sets

$$S=\left\{\begin{bmatrix}2\\3\\1\end{bmatrix},\begin{bmatrix}1\\1\\4\end{bmatrix}\right\}\qquad T=\left\{\begin{bmatrix}2\\3\\1\end{bmatrix},\begin{bmatrix}1\\1\\4\end{bmatrix},\begin{bmatrix}-1\\0\\-11\end{bmatrix}\right\}.$$

Which of the following is true?

A. $\operatorname{span} S$ is bigger than $\operatorname{span} T$.

B. $\operatorname{span} S$ and $\operatorname{span} T$ are the same size.

C. $\operatorname{span} S$ is smaller than $\operatorname{span} T$.

**Definition 2.4.3** We say that a set of vectors is **linearly dependent** if one vector in the set belongs to the span of the others. Otherwise, we say the set is **linearly independent**.

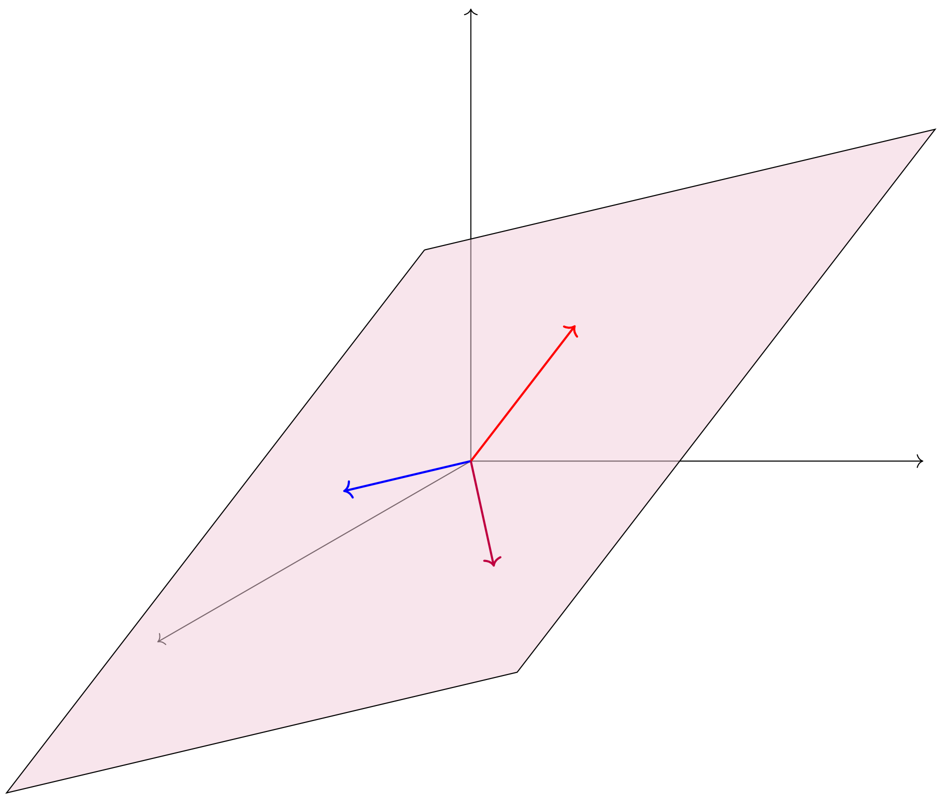


**Figure 14** A linearly dependent set of three vectors

You can think of linearly dependent sets as containing a redundant vector, in the sense that you can drop a vector out without reducing the span of the set. In the above image, all three vectors lay in the same planar subspace, but only two vectors are needed to span the plane, so the set is linearly dependent. $\Diamond$

**Activity 2.4.4** Consider the following three vectors in $\mathbb{R}^3$:

$$\vec{v}_1 = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \text{ and } \vec{v}_3 = \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix}.$$

**(a)** Let $\vec{w} = 3\vec{v}_1 - \vec{v}_2 - 5\vec{v}_3 = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$. The set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w}\}$ is...

A. linearly dependent: at least one vector is a linear combination of others

B. linearly independent: no vector is a linear combination of others

**(b)** Find

$$\text{RREF}\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{w} \end{bmatrix} = \text{RREF}\begin{bmatrix} -2 & 1 & -2 & ? \\ 0 & 3 & 5 & ? \\ 0 & 0 & 4 & ? \end{bmatrix} = ?.$$

What does this tell you about solution set for the vector equation $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{w} = \vec{0}$?

A. It is inconsistent.

B. It is consistent with one solution.

C. It is consistent with infinitely many solutions.

**(c)** Which of these might explain the connection?

A. A pivot column establishes linear independence and creates a contradiction.

B. A non-pivot column both describes a linear combination and reveals the number of solutions.

C. A pivot row describes the bound variables and prevents a contradiction.

D. A non-pivot row prevents contradictions and makes the vector equation solvable.

**Fact 2.4.5** *For any vector space, the set $\{\vec{v}_1, \dots \vec{v}_n\}$ is linearly dependent if and only if the vector equation $x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}$ is consistent with infinitely many solutions.*

*Likewise, the set of vectors $\{\vec{v}_1, \dots \vec{v}_n\}$ is linearly independent if and only the vector equation*

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}$$

*has exactly one solution:* $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$.

**Activity 2.4.6** Find

$$\mathrm{RREF}\left[\begin{array}{ccccc|c} 2 & 2 & 3 & -1 & 4 & 0 \\ 3 & 0 & 13 & 10 & 3 & 0 \\ 0 & 0 & 7 & 7 & 0 & 0 \\ -1 & 3 & 16 & 14 & 1 & 0 \end{array}\right]$$

and mark the part of the matrix that demonstrates that

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is linearly dependent (the part that shows its linear system has infinitely many solutions).

**Activity 2.4.7**

**(a)** Write a statement involving the solutions of a vector equation that's equivalent to each claim:

**(i)** "The set of vectors $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 11 \\ 6 \\ 2 \end{bmatrix} \right\}$ is linearly *independent*."

**(ii)** "The set of vectors $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 11 \\ 6 \\ 2 \end{bmatrix} \right\}$ is linearly *dependent*."

**(b)** Explain how to determine which of these statements is true.

**Observation 2.4.8** Compare the following results:

- A set of $\mathbb{R}^m$ vectors $\{\vec{v}_1, \ldots \vec{v}_n\}$ is linearly independent if and only if $\mathrm{RREF}\left[\begin{array}{ccc} \vec{v}_1 & \ldots & \vec{v}_n \end{array}\right]$ has all pivot *columns*.
- A set of $\mathbb{R}^m$ vectors $\{\vec{v}_1, \ldots \vec{v}_n\}$ is linearly dependent if and only if $\mathrm{RREF}\left[\begin{array}{ccc} \vec{v}_1 & \ldots & \vec{v}_n \end{array}\right]$ has at least one non-pivot *column*.
- A set of $\mathbb{R}^m$ vectors $\{\vec{v}_1, \ldots \vec{v}_n\}$ spans $\mathbb{R}^m$ if and only if $\mathrm{RREF}\left[\begin{array}{ccc} \vec{v}_1 & \ldots & \vec{v}_n \end{array}\right]$ has all pivot *rows*.
- A set of $\mathbb{R}^m$ vectors $\{\vec{v}_1, \ldots \vec{v}_n\}$ fails to span $\mathbb{R}^m$ if and only if $\mathrm{RREF}\left[\begin{array}{ccc} \vec{v}_1 & \ldots & \vec{v}_n \end{array}\right]$ has at least one non-pivot *row*.

**Activity 2.4.9** What is the largest number of $\mathbb{R}^4$ vectors that can form a linearly independent set?

A. 3

B. 4

C. 5

D. You can have infinitely many vectors and still be linearly independent.

**Activity 2.4.10** Is it possible for the set of Euclidean vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n, \vec{0}\}$ to be linearly independent?

A. Yes

B. No

## 2.4.3 Individual Practice

**Remark 2.4.11** Recall that in Activity 2.2.1 we used the words *vector*, *linear combination*, and *span* to make an analogy with recipes, ingredients, and meals. In this analogy, a *recipe* was defined to be a list of amounts of each ingredient to build a particular meal.

**Activity 2.4.12** Consider the statement: The set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent because the vector $\vec{v}_3$ is a linear combination of $\vec{v_1}$ and $\vec{v}_2$. Construct an analogous statement involving ingredients, meals, and recipes, using the terms *linearly (in)dependent* and *linear combination*.

**Activity 2.4.13** The following exercises are designed to help develop your geometric intution around linear dependence.

**(a)** Draw sketches that depict the following:

- Three linearly independent vectors in $\mathbb{R}^3$.
- Three linearly dependent vectors in $\mathbb{R}^3$.

**(b)** If you have three linearly dependent vectors, is it necessarily the case that one of the vectors is a multiple of the other?

## 2.4.4 Videos

Standalone

**Figure 15** Video: Linear independence

### 2.4.5 Exercises

Exercises available at `https://tbil.org/preview/linear-algebra/exercises/#/bank/EV4/`.

### 2.4.6 Mathematical Writing Explorations

**Exploration 2.4.14** Prove the result of Observation 2.4.8, by showing that, given a set $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ of vectors, $S$ is linearly independent iff the equation $x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n = \vec{0}$ is only true when $x_1 = x_2 = \cdots = x_n = 0$.

### 2.4.7 Sample Problem and Solution

Sample problem Example **??**.

## 2.5 Identifying a Basis (EV5)

### Learning Outcomes

- Explain why a set of Euclidean vectors is or is not a basis of $\mathbb{R}^n$.

### 2.5.1 Warm Up

**Remark 2.5.1** Recall that in Activity 2.2.1 we used the words *vector*, *linear combination*, and *span* to make an analogy with recipes, ingredients, and meals. In this analogy, a *recipe* was defined to be a list of amounts of each ingredient to build a particular meal.

**Activity 2.5.2** Consider the following set of ingredients:

$$S = \{\text{tomato}, \text{olive oil}, \text{dough}, \text{cheese}, \text{pizza sauce}, \text{garlic}\}$$

**(a)** Does "pizza" live inside of $\operatorname{span}(S)$?

**(b)** Identify which ingredients in $S$ make the set linearly dependent.

**(c)** Can you think of a subset $S'$ of $S$ that is linearly independent and for which "pizza" is still in $\operatorname{span} S'$?

### 2.5.2 Class Activities

**Activity 2.5.3** Consider the set of vectors

$$S = \left\{ \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -16 \\ -5 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

**(a)** Express the vector $\begin{bmatrix} 5 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ as a linear combination of the vectors in $S$, i.e. find scalars such that

$$\begin{bmatrix} 5 \\ 2 \\ 0 \\ 1 \end{bmatrix} = ? \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} + ? \begin{bmatrix} 2 \\ 4 \\ 1 \\ 1 \end{bmatrix} + ? \begin{bmatrix} 0 \\ -16 \\ -5 \\ -3 \end{bmatrix} + ? \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + ? \begin{bmatrix} 3 \\ 3 \\ 0 \\ 1 \end{bmatrix}.$$

**(b)** Find a *different* way to express the vector $\begin{bmatrix} 5 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ as a linear combination of the vectors in $S$.

**(c)** Consider another vector $\begin{bmatrix} 8 \\ 6 \\ 7 \\ 5 \end{bmatrix}$. Without computing the RREF of another matrix, how many ways can this vector be written as a linear combination of the vectors in $S$?

A. Zero.

B. One.

C. Infinitely-many.

D. Computing a new matrix RREF is necessary.

**Activity 2.5.4** Let's review some of the terminology we've been dealing with...

**(a)** If every vector in a vector space can be constructed as one or more linear combinations of vectors in a set $S$, we can say...

A. the set $S$ spans the vector space.

B. the set $S$ fails to span the vector space.

C. the set $S$ is linearly independent.

D. the set $S$ is linearly dependent.

**(b)** If the zero vector $\vec{0}$ can be constructed as a *unique* linear combination of vectors in a set $S$ (the combination multiplying every vector by the scalar value 0), we can say...

A. the set $S$ spans the vector space.

B. the set $S$ fails to span the vector space.

C. the set $S$ is linearly independent.

D. the set $S$ is linearly dependent.

**(c)** If every vector of a vector space can either be constructed as a *unique* linear combination of vectors in a set $S$, or not at all, we can say...

A. the set $S$ spans the vector space.

B. the set $S$ fails to span the vector space.

C. the set $S$ is linearly independent.

D. the set $S$ is linearly dependent.

**Definition 2.5.5** A **basis** of a vector space $V$ is a set of vectors $S$ contained in $V$ for which

1. *Every* vector in the vector space can be expressed as a linear combination of the vectors in $S$.

2. For each vector $\vec{v}$ in the vector space, there is only *one* way to write it as a linear combination of the vectors in $S$.

These two properties may be expressed more succintly as the statement ”Every vector in $V$ can be expressed *uniquely* as a linear combination of the vectors in $S$”. $\Diamond$

**Observation 2.5.6** In terms of a vector equation, a set $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis of a vector space if the vector equation

$$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{w}$$

has a *unique* solution for *every* vector $\vec{w}$ in the vector space.

Put another way, a basis may be thought of as a minimal set of “building blocks” that can be used to construct any other vector of the vector space.

**Activity 2.5.7** Let $S$ be a basis (Definition 2.5.5) for a vector space. Then...

A. the set $S$ must both span the vector space and be linearly independent.

B. the set $S$ must span the vector space but could be linearly dependent.

C. the set $S$ must be linearly independent but could fail to span the vector space.

D. the set $S$ could fail to span the vector space and could be linearly dependent.

**Activity 2.5.8** The vectors

$$\hat{\imath} = (1,0,0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \hat{\jmath} = (0,1,0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \hat{k} = (0,0,1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

form a basis $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ used frequently in multivariable calculus.

Find the unique linear combination of these vectors

$$?\,\hat{\imath} + \,?\,\hat{\jmath} + \,?\,\hat{k}$$

that equals the vector

$$(3,-2,4)=\begin{bmatrix}3\\-2\\4\end{bmatrix}$$

in $xyz$ space.

**Definition 2.5.9** The **standard basis** of $\mathbb{R}^n$ is the set $\{\vec{e}_1,\dots,\vec{e}_n\}$ where

$$\vec{e}_1=\begin{bmatrix}1\\0\\0\\\vdots\\0\\0\end{bmatrix}\qquad \vec{e}_2=\begin{bmatrix}0\\1\\0\\\vdots\\0\\0\end{bmatrix}\qquad \cdots \qquad \vec{e}_n=\begin{bmatrix}0\\0\\0\\\vdots\\0\\1\end{bmatrix}.$$

In particular, the standard basis for $\mathbb{R}^3$ is $\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}=\{\hat{\imath},\hat{\jmath},\hat{k}\}$. $\Diamond$

**Instructor Note.** In Octave, using ”format rat” aligns the columns nicely and actually converts decimals to fractions, when applicable.

**Activity 2.5.10** Take the RREF of an appropriate matrix to determine if each of the following sets is a basis for $\mathbb{R}^4$. When using Octave on these large matrices, try first typing ”format rat” at the top before you enter the matrices. What do you notice about the formatting? Is this helpful?

**(a)**

$$\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$$

A. A basis, because it both spans $\mathbb{R}^4$ and is linearly independent.

B. Not a basis, because while it spans $\mathbb{R}^4$, it is linearly dependent.

C. Not a basis, because while it is linearly independent, it fails to span $\mathbb{R}^4$.

D. Not a basis, because not only does it fail to span $\mathbb{R}^4$, it's also linearly dependent.

**(b)**

$$\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}2\\0\\0\\3\end{bmatrix},\begin{bmatrix}4\\3\\0\\2\end{bmatrix},\begin{bmatrix}-3\\0\\1\\3\end{bmatrix}\right\}$$

A. A basis, because it both spans $\mathbb{R}^4$ and is linearly independent.

B. Not a basis, because while it spans $\mathbb{R}^4$, it is linearly dependent.

C. Not a basis, because while it is linearly independent, it fails to span $\mathbb{R}^4$.

D. Not a basis, because not only does it fail to span $\mathbb{R}^4$, it's also linearly dependent.

**(c)**

$$\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}2\\0\\0\\3\end{bmatrix},\begin{bmatrix}3\\13\\7\\16\end{bmatrix},\begin{bmatrix}-1\\10\\7\\14\end{bmatrix},\begin{bmatrix}4\\3\\0\\2\end{bmatrix}\right\}$$

A. A basis, because it both spans $\mathbb{R}^4$ and is linearly independent.

B. Not a basis, because while it spans $\mathbb{R}^4$, it is linearly dependent.

C. Not a basis, because while it is linearly independent, it fails to span $\mathbb{R}^4$.

D. Not a basis, because not only does it fail to span $\mathbb{R}^4$, it's also linearly dependent.

**(d)**

$$\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}4\\3\\0\\2\end{bmatrix},\begin{bmatrix}-3\\0\\1\\3\end{bmatrix},\begin{bmatrix}3\\6\\1\\5\end{bmatrix}\right\}$$

A. A basis, because it both spans $\mathbb{R}^4$ and is linearly independent.

B. Not a basis, because while it spans $\mathbb{R}^4$, it is linearly dependent.

C. Not a basis, because while it is linearly independent, it fails to span $\mathbb{R}^4$.

D. Not a basis, because not only does it fail to span $\mathbb{R}^4$, it's also linearly dependent.

**(e)**

$$\left\{\begin{bmatrix}5\\3\\0\\-1\end{bmatrix},\begin{bmatrix}-2\\1\\0\\3\end{bmatrix},\begin{bmatrix}4\\5\\1\\3\end{bmatrix}\right\}$$

A. A basis, because it both spans $\mathbb{R}^4$ and is linearly independent.

B. Not a basis, because while it spans $\mathbb{R}^4$, it is linearly dependent.

C. Not a basis, because while it is linearly independent, it fails to span $\mathbb{R}^4$.

D. Not a basis, because not only does it fail to span $\mathbb{R}^4$, it's also linearly dependent.

**Activity 2.5.11** If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis for $\mathbb{R}^4$, that means $\text{RREF}[\vec{v}_1\,\vec{v}_2\,\vec{v}_3\,\vec{v}_4]$ has a pivot in every row (because it spans), and has a pivot in every column (because it's linearly independent).

What is $\text{RREF}[\vec{v}_1\,\vec{v}_2\,\vec{v}_3\,\vec{v}_4]$?

$$\text{RREF}[\vec{v}_1\,\vec{v}_2\,\vec{v}_3\,\vec{v}_4] = \begin{bmatrix}? & ? & ? & ?\\ ? & ? & ? & ?\\ ? & ? & ? & ?\\ ? & ? & ? & ?\end{bmatrix}$$

**Fact 2.5.12** *The set* $\{\vec{v}_1, \ldots, \vec{v}_m\}$ *is a basis for* $\mathbb{R}^n$ *if and only if* $m = n$ *and* $\mathrm{RREF}[\vec{v}_1 \, \ldots \, \vec{v}_n] = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}$.

*That is, a basis for* $\mathbb{R}^n$ *must have exactly* $n$ *vectors and its square matrix must row-reduce to the so-called* ***identity matrix*** *containing all zeros except for a downward diagonal of ones. (We will learn where the identity matrix gets its name in a later module.)*

### 2.5.3 Individual Practice

**Activity 2.5.13** Let $S$ denote a set of vectors in $\mathbb{R}^n$. Without referring to your Activity Book, write down:

**(a)** The definition of what it means for $S$ to be linearly independent.

**(b)** The definition of what it means for $S$ to span $\mathbb{R}^n$.

**(c)** The definition of what it means for $S$ to be a basis for $\mathbb{R}^n$.

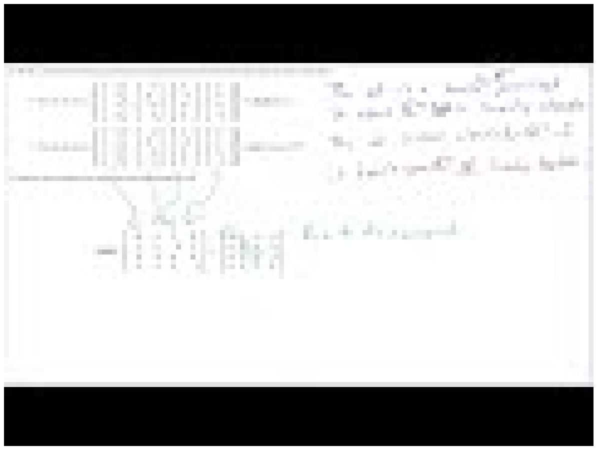
**Activity 2.5.14** You are going on a trip and need to pack. Let $S$ denote the set of items that you are packing in your suitcase.

**(a)** Give an example of such a set of items $S$ that you would say "spans" everything you need, but is linearly dependent.

**(b)** Give an example of such a set of items $S$ that is linearly independent, but does not "span" everything you need.

**(c)** Give an example of such a set $S$ that you might reasonably consider to be a "basis" for what you need?

### 2.5.4 Videos



Standalone

**Figure 16** Video: Verifying that a set of vectors is a basis of a vector space

### 2.5.5 Exercises

Exercises available at https://tbil.org/preview/linear-algebra/exercises/#/bank/EV5/.

### 2.5.6 Mathematical Writing Explorations

**Exploration 2.5.15**

- What is a basis for $M_{2,2}$?
- What about $M_{3,3}$?
- Could we write each of these in a way that looks like the standard basis vectors in $\mathbb{R}^m$ for some $m$? Make a conjecture about the relationship between these spaces of matrices and standard Eulidean space.

**Exploration 2.5.16** Recall our earlier definition of symmetric matrices. Find a basis for each of the following:

- The space of $2 \times 2$ symmetric matrices.
- The space of $3 \times 3$ symmetric matrices.
- The space of $n \times n$ symmetric matrices.

**Exploration 2.5.17** Must a basis for the space $P_2$, the space of all quadratic polynomials, contain a polynomial of each degree less than or equal to 2? Generalize your result to polynomials of arbitrary degree.

### 2.5.7 Sample Problem and Solution

Sample problem Example **??**.

## 2.6 Subspace Basis and Dimension (EV6)

### Learning Outcomes

- Compute a basis for the subspace spanned by a given set of Euclidean vectors, and determine the dimension of the subspace.

### 2.6.1 Warm Up

**Activity 2.6.1** Consider the set $S$ of vectors in $\mathbb{R}^4$ given by

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

**(a)** Is the set $S$ linearly independent or linearly dependent?

**(b)** How would you describe the subspace $\operatorname{span} S$ geometrically?

**(c)** What do the spaces $\operatorname{span} S$ and $\mathbb{R}^2$ have in common? In what ways do they differ?

### 2.6.2 Class Activities

**Observation 2.6.2** Recall from section Section 2.3 that a **subspace** of a vector space is the result of spanning a set of vectors from that vector space.

Recall also that a linearly dependent set contains “redundant” vectors. For example, only two of the three vectors in Figure 14 are needed to span the planar subspace.

**Activity 2.6.3** Consider the subspace of $\mathbb{R}^4$ given by $W = \operatorname{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}$.

**(a)** Mark the column of $\operatorname{RREF} \begin{bmatrix} 2 & 2 & 2 & 1 \\ 3 & 0 & -3 & 5 \\ 0 & 1 & 2 & -1 \\ 1 & -1 & -3 & 0 \end{bmatrix}$ that shows that $W$’s spanning set is linearly dependent.

**(b)** What would be the result of removing the vector that gave us this column?

A. The set still spans $W$, and remains linearly dependent.

B. The set still spans $W$, but is now also linearly independent.

C. The set no longer spans $W$, and remains linearly dependent.

D. The set no longer spans $W$, but is now linearly independent.

```
rref([2,2,2,1; 3,0,-3,5; 0,1,2,-1; 1,-1,-3,0])
```

**Definition 2.6.4** Let $W$ be a subspace of a vector space. A **basis** for $W$ is a linearly independent set of vectors that spans $W$ (but not necessarily the entire vector space). $\Diamond$

**Observation 2.6.5** So given a set $S = \{\vec{v}_1, \dots, \vec{v}_m\}$, to compute a basis for the subspace $\operatorname{span} S$, simply remove the vectors corresponding to the non-pivot columns of $\text{RREF}[\vec{v}_1 \, \dots \, \vec{v}_m]$. For example, since

$$\text{RREF}\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & -2 & 0 \\ 3 & 6 & -2 & 1 \end{bmatrix} = \begin{bmatrix} \boxed{1} & 2 & 0 & 1 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

the subspace $W = \operatorname{span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}, \begin{bmatrix}0\\-2\\-2\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$ has $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\-2\\-2\end{bmatrix}\right\}$ as a basis.

**Activity 2.6.6**

**(a)** Find a basis for $\operatorname{span} S$ where

$$S = \left\{\begin{bmatrix}2\\3\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\5\\-1\\0\end{bmatrix}\right\}.$$

**(b)** Find a basis for $\operatorname{span} T$ where

$$T = \left\{\begin{bmatrix}2\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\5\\-1\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\\1\end{bmatrix}\right\}.$$

**Observation 2.6.7** Even though we found different bases for them, $\operatorname{span} S$ and $\operatorname{span} T$ are exactly the same subspace of $\mathbb{R}^4$, since

$$S = \left\{\begin{bmatrix}2\\3\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\5\\-1\\0\end{bmatrix}\right\} = \left\{\begin{bmatrix}2\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\5\\-1\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\\1\end{bmatrix}\right\} = T.$$

Thus the basis for a subspace is not unique in general.

**Fact 2.6.8** *Any non-trivial real vector space has infinitely-many different bases, but all the bases for a given vector space are exactly the same size.*

*For example,*

$$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \right\}$$

*are all valid bases for $\mathbb{R}^3$, and they all contain three vectors.*

**Definition 2.6.9** The **dimension** of a vector space or subspace is equal to the size of any basis for the vector space.

As you'd expect, $\mathbb{R}^n$ has dimension $n$. For example, $\mathbb{R}^3$ has dimension 3 because any basis for $\mathbb{R}^3$ such as

$$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \right\}$$

contains exactly three vectors. ◊

**Activity 2.6.10** Consider the following subspace $W$ of $\mathbb{R}^4$:

$$W = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -5 \\ 5 \end{bmatrix}, \begin{bmatrix} 12 \\ -3 \\ 15 \\ -18 \end{bmatrix} \right\}.$$

**(a)** Explain and demonstrate how to find a basis of $W$.

**(b)** Explain and demonstrate how to find the dimension of $W$.

**Activity 2.6.11** The dimension of a subspace may be found by doing what with an appropriate RREF matrix?

A. Count the rows.

B. Count the non-pivot columns.

C. Count the pivots.

D. Add the number of pivot rows and pivot columns.

## 2.6.3 Individual Practice

**Activity 2.6.12** In Observation 2.6.5, we found a basis for the subspace

$$W = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

To do so, we use the results of the calculation:

$$\text{RREF}\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & -2 & 0 \\ 3 & 6 & -2 & 1 \end{bmatrix} = \begin{bmatrix} \boxed{1} & 2 & 0 & 1 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

to conclude that the set $\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}\right\}$, the set of vectors *corresponding* to the pivot columns of the RREF, is a basis for $W$.

**(a)** Explain why neither of the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are elements of $W$.

**(b)** Explain why this shows that, in general, when we calculate a basis for $W = \text{span}\{\vec{v}_1, \ldots, \vec{v}_n\}$, the pivot columns of $\text{RREF}[\vec{v}_1 \ldots \vec{v}_n]$ themselves do not form a basis for $W$.

### 2.6.4 Videos

Standalone

**Figure 17** Video: Finding a basis of a subspace and computing the dimension of a subspace

### 2.6.5 Exercises

Exercises available at `https://tbil.org/preview/linear-algebra/exercises/#/bank/EV6/`.

### 2.6.6 Mathematical Writing Explorations

**Exploration 2.6.13** Prove each of the following statements is true.

- If $\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_m\}$ and $\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}$ are each a basis for a vector space $V$, then $m = n$.
- If $\{\vec{v}_1, \vec{v}_2 \ldots, \vec{v}_n\}$ is linearly independent, then so is $\{\vec{v}_1, \vec{v}_1 + \vec{v}_2, \ldots, \vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_n\}$.
- Let $V$ be a vector space of dimension $n$, and $\vec{v} \in V$. Then there exists a basis for $V$

which contains $\vec{v}$.

**Exploration 2.6.14** Suppose we have the set of all function $f : S \to \mathbb{R}$. We claim that this is a vector space under the usual operation of function addition and scalar multiplication. What is the dimension of this space for each choice of $S$ below:

- $S = \{1\}$
- $S = \{1, 2\}$
- $S = \{1, 2, \ldots, n\}$
- $S = \mathbb{R}$

**Exploration 2.6.15** Suppose you have the vector space $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x + y + z = 1 \right\}$ with the operations $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \oplus \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 - 1 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$ and $\alpha \odot \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} \alpha x_1 - \alpha + 1 \\ \alpha y_1 \\ \alpha z_1 \end{pmatrix}$. Find a basis for $V$ and determine it's dimension.

### 2.6.7 Sample Problem and Solution

Sample problem Example **??**.

## 2.7 Homogeneous Linear Systems (EV7)

### Learning Outcomes

- Find a basis for the solution set of a homogeneous system of equations.

### 2.7.1 Warmup

**Remark 2.7.1** Recall from Section 2.3 that a **homogeneous** system of linear equations is one of the form:

$$\begin{array}{rl} a_{11}x_1 + \ a_{12}x_2 + \ldots + \ a_{1n}x_n & = 0 \\ a_{21}x_1 + \ a_{22}x_2 + \ldots + \ a_{2n}x_n & = 0 \\ \vdots \qquad \vdots \qquad\qquad \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & = 0 \end{array}$$

This system is equivalent to the vector equation:

$$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}$$

and the augmented matrix:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{array}\right].$$

**Activity 2.7.2** In Section 2.3, we observed that if

$$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}$$

is a homogeneous vector equation, then:

- The zero vector $\vec{0}$ is a solution;
- The sum of any two solutions is again a solution;
- Multiplying a solution by a scalar produces another solution.

Based on this recollection, which of the following best describes the solution set to the homogeneous equation?

A. A basis for $\mathbb{R}^n$.

B. A subspace of $\mathbb{R}^n$.

C. All of $\mathbb{R}^n$.

D. The empty set.

### 2.7.2 Class Activities

**Activity 2.7.3** Consider the homogeneous system of equations

$$\begin{aligned} x_1 + 2x_2 \phantom{{}-x_3} + \phantom{2}x_4 &= 0 \\ 2x_1 + 4x_2 - x_3 - 2x_4 &= 0 \\ 3x_1 + 6x_2 - x_3 - \phantom{2}x_4 &= 0 \end{aligned}$$

**(a)** Find its solution set (a subspace of $\mathbb{R}^4$).

**(b)** Rewrite this solution space in the form

$$\left\{ a\begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} + b\begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \middle|\, a, b \in \mathbb{R} \right\}.$$