Question(s): Which combinations of observables $\mathcal{R}_{ijkl}$ do we have to use in order to achieve the maximum rank of the covariance matrix? Does a simple rule exist in order to pick out the non-redundant observables?

1 Repetition

The generalized shear-ratio test by Schneider (2015) is given by

$$\mathcal{R}_{ijkl} := R_{jkl} \left[R_{ijk} - M_{jk} \right] - R_{ikl} + M_{kl} = 0 \; , \tag{1}$$

$$\tag{2}$$

with (non-observable) redshift-dependent multiplicative shear bias ratio M and observable shear-ratio R,

$$R_{ijk} := B_{ijk} M_{jk} = \frac{\beta(z_i, z_k)}{\beta(z_i, z_j)} \frac{M(z_k)}{M(z_j)} \; , \tag{3}$$

which is given by the ratio of two particular lensing efficiencies β, which is the ratio of two angular diameter distances

$$\beta(z_i, z_j) := \frac{D_{\rm ang}(z_i, z_k)}{D_{\rm ang}(z_k)} \; . \tag{4}$$

The first indice of a shear-ratio corresponds to the redshift z of the lens galaxy (population) and the other two indices correspond to the redshifts of the different source galaxy populations and we assume (here: simply by selection), that

$$z_i < z_j < z_k < z_l \; . \tag{5}$$

Because the (potential) redshift-dependent multiplicative shear bias $M(z)$ and its ratios are not observable, we first have to define an observable quantity, given by

$$\mathcal{R}^{\rm obs}_{ijkl} := R_{jkl} \left[R_{ijk} - 1 \right] - R_{ijl} + 1 = R_{jkl} R_{ijk} - R_{jkl} - R_{ijl} + 1 \; . \tag{6}$$

For perfect observables, i.e., without shear biases ($M \to 1$) and redshift uncertainties ($\delta z \to 0$), Eq. (2) and Eq. (6) reduce to[1]

$$\mathcal{B}_{ijkl} := B_{jkl}B_{ijk} - B_{jkl} - B_{ijl} + 1 \; . \tag{7}$$

2 Proof

For shear-ratios R_{ijl} and multiplicative bias ratios M_{jl}, the following two relations hold in general

$$R_{ijl} := R_{ijk}R_{ikl} \; , \tag{8}$$
$$M_{jl} := M_{jk}M_{kl} \; . \tag{9}$$

Using Eq. (7), it follows that an observable $\mathcal{R}_{ijkl}$ can be rewritten as

$$\begin{aligned}
\mathcal{R}^{\mathrm{obs}}_{ijkl} &= R_{jkl}[R_{ijk} - 1] - R_{ijl} + 1 & (10)\\
&= B_{jkl}M_{kl}B_{ijk}M_{jk} - B_{jkl}M_{kl} - B_{ijl}M_{jl} + 1\\
&= B_{jkl}B_{ijk}M_{kl}M_{jk} - B_{jkl}M_{kl} - B_{ijl}M_{jl} + 1\\
&= (B_{ijl} - 1 + B_{jkl})M_{kl}M_{jk} - B_{jkl}M_{kl} - B_{ijl}M_{jl} + 1\\
&= B_{jkl}M_{jl} - B_{jkl}M_{kl} - M_{jl} + 1\\
&= B_{jkl}\left[M_{jl} - M_{kl}\right] - M_{jl} + 1 \; , & (11)
\end{aligned}$$

which is **independent** of the redshift of the first lens galaxy population, i.e., z_i, and it does only contain a single perfect shear-ratio B.

This suggests, that the achievable rank of the covariance matrix $\Sigma =< \mathcal{R}_{ijkl}\mathcal{R}_{mnop} >$ is smaller than the naive counting of

$$\mathrm{Rank}(\Sigma) = \binom{N}{4} = \frac{N!}{4!(N-4)!} \tag{12}$$

actually suggests.

[1] In our approach, we assume that lens and source galaxies have arbitrarily small redshift intervalls $\Delta z \to 0$.

Consider now the case of $N = 5$, obeying the rule given in Eq. (5). This yields in total five possible combinations of adoptable generalized shear-ratios. Using Eq. (11), it follows that the observables can be written as

$$\begin{aligned}
\mathcal{R}_{1234} &= B_{234}[M_{34}(M_{23} - 1)] + 1 - M_{24} \\
\mathcal{R}_{1235} &= B_{235}[M_{35}(M_{23} - 1)] + 1 - M_{25} \\
\mathcal{R}_{1245} &= B_{245}[M_{45}(M_{24} - 1)] + 1 - M_{25} \\
\mathcal{R}_{1345} &= B_{345}[M_{45}(M_{34} - 1)] + 1 - M_{35} \\
\mathcal{R}_{2345} &= B_{345}[M_{45}(M_{34} - 1)] + 1 - M_{35}
\end{aligned}$$

Obviously, $\mathcal{R}_{1345}$ and $\mathcal{R}_{2345}$ provide the same information. This is due to the fact, that Eq. (6) states, that a shear-ratio with a first index for the redshift larger than one can constructed from linear combinations of shear-ratios with an index of one. We now have to validate, whether the relations given in Eq. (8) and Eq. (9) hide a redundancy in the three observables $\mathcal{R}_{1234}$, $\mathcal{R}_{1235}$ and $\mathcal{R}_{1245}$.

Therfore, we consider the linear combination of

$$\begin{aligned}
\mathcal{R}_{1235} - \mathcal{R}_{1245} &= B_{235}[M_{35}(M_{23} - 1)] + 1 - M_{52} - [B_{245}[M_{45}(M_{24} - 1)] + 1 - M_{25}] \\
&= B_{235}[M_{25} - M_{35}] - B_{245}[M_{25} - M_{45}] \\
&= B_{234}B_{245}[M_{25} - M_{35}] - B_{245}[M_{25} - M_{45}] \\
&= B_{245}\left[B_{234}(M_{25} - M_{35}) + (M_{45} - M_{25})\right] \\
&= B_{245}\left[B_{234}(M_{24}M_{45} - M_{34}M_{45}) + (M_{45} - M_{24}M_{45})\right] \\
&= B_{245} \cdot M_{45} \cdot [B_{234}(M_{34} - M_{24}) + (1 - M_{24})] \\
&= [B_{245} \cdot M_{45}] \cdot \mathcal{R}_{1234} \ ,
\end{aligned} \tag{13}$$

which shows, that one of the remaining three observables is redundant. This means that for $N = 5$, only $N_{\rm ind} = 3$ observables are independent. In order to explore this relation for a larger number of observables, we consider the case of $N = 6$, which yields in total the following 15 observables:

$$\begin{aligned}\mathcal{R}_{1234}, \mathcal{R}_{1235}, \mathcal{R}_{1236}, \mathcal{R}_{1245}, \mathcal{R}_{1246}, \mathcal{R}_{1256}\\ \mathcal{R}_{1345}, \mathcal{R}_{1346}, \mathcal{R}_{1356},\\ \mathcal{R}_{1456},\\ \mathcal{R}_{2345}, \mathcal{R}_{2346}, \mathcal{R}_{2356}, \mathcal{R}_{2456},\\ \mathcal{R}_{3456}\end{aligned}$$

However, applying the findings from Eq. (11) and Eq. (13), and that

$$\begin{aligned}\mathcal{R}_{1246} - \mathcal{R}_{1256} &= B_{246}M_{26} - B_{246}M_{46} + 1 - M_{26} - [B_{256}M_{26} - B_{256}M_{56} + 1 - M_{26}]\\ &= B_{256}\,[B_{245}M_{26} - B_{245}M_{46} + M_{56} - M_{26}]\\ &= [B_{256} \cdot M_{56}] \cdot \mathcal{R}_{1245}\ ,\end{aligned} \tag{14}$$

$$\begin{aligned}\mathcal{R}_{1236} - \mathcal{R}_{1246} &= B_{236}M_{26} - B_{236}M_{36} + 1 - M_{26} - [B_{246}M_{26} - B_{246}M_{46} + 1 - M_{26}]\\ &= B_{246}\,[M_{46} - M_{26} + B_{234}M_{26} - B_{234}M_{36}]\\ &= B_{245}B_{256}\,[M_{45}M_{56} - M_{25}M_{56} + B_{234}M_{25}M_{56} - B_{234}M_{35}M_{56}]\\ &= [B_{245} \cdot M_{45}] \cdot [B_{256} \cdot M_{56}] \cdot \mathcal{R}_{1234}\ ,\end{aligned} \tag{15}$$

$$\begin{aligned}\mathcal{R}_{1346} - \mathcal{R}_{1356} &= B_{346}M_{36} - B_{346}M_{46} + 1 - M_{36} - [B_{356}M_{36} - B_{356}M_{56} + 1 - M_{36}]\\ &= B_{356}\,[M_{56} - M_{36} + B_{345}M_{36} - B_{345}M_{46}]\\ &= B_{356} \cdot M_{56}\,[1 - M_{35} + B_{345}M_{35} - B_{345}M_{45}]\\ &= [B_{356} \cdot M_{56}] \cdot \mathcal{R}_{1345}\ ,\end{aligned} \tag{16}$$

the number of independent observables reduces in this case to only $N_{\text{ind}} = 6$. For our purposes, it is sufficient to choose the following observables:

$$\mathcal{R}_{1234}, \mathcal{R}_{1245}, \mathcal{R}_{1256}, \mathcal{R}_{1345}, \mathcal{R}_{1356}, \mathcal{R}_{1456}\ ,$$

in order to achieve the maximum rank of the covariance matrix. We conclude, that the rank is related to the number of observables by

$$\text{Rank}(\Sigma) = \frac{1}{2}(N-2)\,(N-3)\ , \tag{17}$$

which is in agreement to the emperically found result in the work by Kotula (2015, master thesis). Furthermore, a simple rule, in order to choose the non-redundant observables can be summarized as follows:

$$\text{For } i = 1 \text{ (fixed), take the } \sum_{j=2}^{j<k} \sum_{k=3}^{l=N-1} \mathcal{R}_{i,j,k,k+1}.$$

However, we mention, that this result does certainly not hold in reality, because we here use an isolated approach for the multiplicative shear bias. In reality, the redshifts have uncertainties, the galaxies are distributed over redshift intervalls, the shear-ratio is estimated from true galaxy shears (not like we did here, where we estimated these ratios directly from angular diameter distance ratios) and therefore, depend on the quality of observations and cosmological parameters. In addition, the multiplicative shear bias can not be seperated from the true shear-ratio $\mathcal{B}$. Hence, one can not simply apply Eq. (7).