Supplementary information for "Back to Einstein: How to include sediment burial in bedload diffusion models?"

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1 Calculation of the distribution function

Owing to the convolution structure of manuscript equations (1-3), their solution is a formidable problem. Luckily, we have the device of Laplace transforms. These project integro-differential equations into an alternate space in which convolutions are unraveled [e.g., *Arfken*, 1985]. The double Laplace transform of a joint probability distribution $p(x,t)$ is defined by

$$\tilde{p}(\eta, s) = \int_0^\infty dx e^{-\eta x} \int_0^\infty dt e^{-st} p(x,t). \tag{1}$$

The Laplace-transformed moments of x are linked to derivatives of the double transformed distribution (1) [cf., *Berezhkovskii and Weiss*, 2002]. Equation (1) implies

$$\langle \tilde{x}(s)^k \rangle = (-)^k \partial_\eta^k \tilde{p}(\eta, s)\Big|_{\eta=0}. \tag{2}$$

The operator $\langle \circ \rangle$ denotes the ensemble average [e.g., *Kittel*, 1958]. This means we can compute the variance of position as $\sigma_x^2(t) = \langle x^2 \rangle - \langle x \rangle^2 = \mathcal{L}^{-1}\{\langle \tilde{x}^2 \rangle; t\} - \mathcal{L}^{-1}\{\langle \tilde{x} \rangle; t\}^2$, where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform [e.g., *Arfken*, 1985]. This is a powerful tool, since we can use it to derive the positional variance without integrating the distribution in equation (7) of the manuscript.

Double transforming manuscript equations (1-3) using the definition (1) gives

$$\tilde{\omega}_{1T}(\eta, s) = \theta_1 \tilde{g}_1(\eta, s) + \tilde{\omega}_2(\eta, s)\tilde{g}_1(\eta, s) - \tilde{\omega}_{1F}(\eta, s), \tag{3}$$

$$\tilde{\omega}_{1F}(\eta, s) = \theta_1 \tilde{g}_1(\eta, s+\kappa) + \tilde{\omega}_2(\eta, s)\tilde{g}_1(\eta, s+\kappa), \tag{4}$$

$$\tilde{\omega}_2(\eta, s) = \theta_2 \tilde{g}_2(\eta, s) + \tilde{\omega}_{1F}(\eta, s)\tilde{g}_2(\eta, s). \tag{5}$$

This algebraic system solves for

$$\tilde{\omega}_{1T}(\eta, s) = \frac{\theta_1 + \theta_2 \tilde{g}_2(\eta, s)}{1 - \tilde{g}_1(\eta, s+\kappa)\tilde{g}_2(\eta, s)} \big\{ \tilde{g}_1(\eta, s) - \tilde{g}_1(\eta, s+\kappa) \big\}, \tag{6}$$

$$\tilde{\omega}_{1F}(\eta, s) = \frac{\theta_1 + \theta_2 \tilde{g}_2(\eta, s)}{1 - \tilde{g}_1(\eta, s+\kappa)\tilde{g}_2(\eta, s)} \tilde{g}_1(\eta, s+\kappa), \tag{7}$$

$$\tilde{\omega}_2(\eta, s) = \frac{\theta_2 + \theta_1 \tilde{g}_1(\eta, s+\kappa)}{1 - \tilde{g}_1(\eta, s+\kappa)\tilde{g}_2(\eta, s)} \tilde{g}_2(\eta, s). \tag{8}$$

Double transforming manuscript equations (4-6) gives

$$\tilde{p}_0(\eta, s) = \frac{1}{s}\tilde{\omega}_{1T}(\eta, s), \tag{9}$$

$$\tilde{p}_1(\eta, s) = \theta_1 \tilde{G}_1(\eta, s) + \tilde{\omega}_2(\eta, s)\tilde{G}_1(\eta, s), \tag{10}$$

$$\tilde{p}_2(\eta, s) = \theta_2 \tilde{G}_2(\eta, s) + \tilde{\omega}_{1F}(\eta, s)\tilde{G}_2(\eta, s). \tag{11}$$

The total probability is $p(x,t) = p_0(x,t) + p_1(x,t) + p_2(x,t)$. Using equations (6-11) this becomes, in the double Laplace representation,

$$\tilde{p}(\eta,s) = \frac{1}{s}\frac{\theta_1 + \theta_2\tilde{g}_2(\eta,s)}{1-\tilde{g}_1(\eta,s+\kappa)\tilde{g}_2(\eta,s)}\big\{\tilde{g}_1(\eta,s) - \tilde{g}_1(\eta,s+\kappa)\big\} + \frac{\theta_1\big[\tilde{G}_1(\eta,s) + \tilde{g}_1(\eta,s+\kappa)\tilde{G}_2(\eta,s)\big] + \theta_2\big[\tilde{G}_2(\eta,s) + \tilde{g}_2(\eta,s)\tilde{G}_1(\eta,s)\big]}{1-\tilde{g}_1(\eta,s+\kappa)\tilde{g}_2(\eta,s)}. \tag{12}$$

Plugging the propagators outlined in manuscript equations (8-9) into equation (12) gives

$$\tilde{p}(\eta,s) = \frac{1}{s}\frac{(s+\kappa+k')s + \theta_1(s+\kappa)\eta v + \kappa k_2}{(s+\kappa+k_1)\eta v + (s+\kappa+k')s + \kappa k_2}. \tag{13}$$

In this equation, $k' = k_1 + k_2$, and we have used the normalization requirement of the initial probabilities: $\theta_1 + \theta_2 = 1$. The double inverse transform of this equation provides the distribution $p(x,t)$. We invert the transform over η first. Using the results 15.103 (transform of exponential), 15.123 (transform of derivative), and 15.141 (transform of Dirac delta function) from *Arfken* [1985] provides

$$\tilde{p}(x,s) = \theta_1\frac{s+\kappa}{s(s+\kappa+k_1)}\delta(x) + \frac{1}{v}\Big(\frac{(s+\kappa+k')s+\kappa k_2}{s(s+\kappa+k_1)} - \frac{\theta_1(s+\kappa)[s(s+\kappa+k_1)+\kappa k_2]}{s(s+\kappa+k_1)^2}\Big)\exp\Big[-\frac{(s+\kappa+k')s+\kappa k_2}{s+\kappa+k_1}\frac{x}{v}\Big]. \tag{14}$$

Inverting the remaining transform over s, applying results 15.152 (substitution), 15.164 (translation), and 15.175 (transform of te^{kt}) from *Arfken* [1985], and defining the shorthand notations $\tau = k_1(t - x/v)$, $\xi = k_2 x/v$, and $\Omega = (\kappa + k_1)/k_1$, gives the simpler form

$$p(x,t) = \theta_1\Big[1 - \frac{k_1}{\kappa+k_1}\big(1-e^{-(\kappa+k_1)t}\big)\Big]\delta(x) + \frac{1}{v}\exp[\Omega\tau - \xi] \times \mathcal{L}^{-1}\Big\{\Big(\theta_2 + \frac{\theta_1 k_1 + \theta_2 k_2}{s} + \frac{\theta_1 k_1 k_2}{s^2} + \frac{\theta_2\kappa k_2}{s(s-\kappa-k_1)} + \frac{\theta_1\kappa k_1 k_2}{s^2(s-\kappa-k_1)}\Big) \times \exp\Big[\frac{k_1\xi}{s}\Big]; \tau/k_1\Big\}. \tag{15}$$

Using entries 2.2.2.1, 2.2.2.8, and 1.1.1.13 from *Prudnikov et al.* [1992] in conjunction with the definition of the Marcum Q-function $\mathcal{P}_\mu(x,t)$ [e.g., *Temme*, 1996], and inserting the Heaviside functions to account for the fact that grains can neither travel backwards nor at speeds exceeding v, we finally arrive at manuscript equation (10) for the joint distribution $p(x,t)$.

2 Calculation of the moments

We compute the first two moments of position x and ultimately its variance using equation (2). The first two derivatives of the double Laplace transformed distribution (13) are

$$\partial_\eta\tilde{p}(\eta,s) = -v\frac{1}{s}\frac{[(s+\kappa+k')s+\kappa k_2][\theta_2(s+\kappa)+k_1]}{[\eta v(s+\kappa+k_1) + (s+\kappa+k')s+\kappa k_2]^2}, \tag{16}$$

$$\partial_\eta^2\tilde{p}(\eta,s) = 2v^2\frac{1}{s}\frac{(s+\kappa+k_1)[(s+\kappa+k')s+\kappa k_2][\theta_2(s+\kappa)+k_1]}{[\eta v(s+\kappa+k_1) + (s+\kappa+k')s+\kappa k_2]^3}. \tag{17}$$

Evaluating these at $\eta = 0$ and applying equation (2) provides the Laplace transformed moments

$$\frac{\langle \tilde{x}(s) \rangle}{v} = \frac{1}{s} \frac{\theta_2(s+\kappa) + k_1}{(s+\kappa+k')s + \kappa k_2} = \frac{1}{s} \frac{\theta_2(s+\kappa) + k_1}{(s+a+b)(s+a-b)}, \tag{18}$$

$$\frac{\langle \tilde{x}^2(s) \rangle}{2v^2} = \frac{1}{s} \frac{(s+\kappa+k_1)(\theta_2(s+\kappa)+k_1)}{[(s+\kappa+k')s+\kappa k_2]^2} = \frac{1}{s} \frac{(s+\kappa+k_1)(\theta_2(s+\kappa)+k_1)}{(s+a+b)^2(s+a-b)^2}. \tag{19}$$

The parameters $a = (\kappa + k')/2$ and $b^2 = a^2 - \kappa k_2$ were introduced to factorize the denominators. These equations can be inverted using the properties 15.164 (translation), 15.11.1 (integration), and 15.123 (differentiation) from *Arfken* [1985] after expansion in partial fractions. For the mean, the calculation is

$$\frac{2b}{v} \langle x \rangle = \big[\theta_2 + (k_1 + \theta_2 \kappa) \int_0^t dt \big] \mathcal{L}^{-1} \Big\{ \frac{1}{s+a-b} - \frac{1}{s+a+b}; t \Big\} \tag{20}$$

$$= \Big[\theta_2 + \frac{k_1 + \theta_2\kappa}{b-a}\Big] e^{(b-a)t} - \Big[\theta_2 - \frac{k_1+\theta_2\kappa}{a+b}\Big] e^{-(a+b)t} - \Big[\frac{k_1+\theta_2\kappa}{b-a} + \frac{k_1+\theta_2\kappa}{a+b}\Big]. \tag{21}$$

This equation rearranges to manuscript equation (11). The second moment (19) is

$$\frac{2b^2}{v^2}\langle x^2 \rangle = \Big[\theta_2(\delta(t) + \partial_t) + (\theta_2(2\kappa + k_1) + k_1) + (\kappa + k_1)(\theta_2\kappa + k_1) \int_0^t dt\Big]$$
$$\times \mathcal{L}^{-1}\Big\{ \frac{1}{(s+a-b)^2} + \frac{1}{(s+a+b)^2} - \frac{1}{b(s+a-b)} + \frac{1}{b(s+a+b)}; t\Big\}. \tag{22}$$

This becomes

$$\frac{2b^3}{v^2}\langle x^2 \rangle = \Big[\theta_2\partial_t + [\theta_2(2\kappa + k_1) + k_1] + (\kappa + k_1)(\theta_2\kappa + k_1) \int_0^t dt\Big]$$
$$\times \Big((bt-1)e^{(b-a)t} + (bt+1)e^{-(a+b)t}\Big) \tag{23}$$

which evaluates to manuscript equation (12). Finally, $\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2$ derives the variance in manuscript equation (13).

3 Limiting behavior of the moments

We determine the diffusion exponents γ in the local, intermediate, and global ranges using the two limiting cases described in the discussion of the manuscript. Limit (1) is $\kappa \to 0$. We take this limit in equations (18) and (19) with initial condition $\theta_1 = 1$ to obtain

$$\langle \tilde{x} \rangle = v k_1 \frac{1}{s^2(s+k')}, \tag{24}$$

$$\langle \tilde{x}^2 \rangle = 2v^2 k_1 \frac{s+k_1}{s^3(s+k')^2}. \tag{25}$$

Inverting these equations provides the variance

$$\sigma_x^2 = 2v^2 \frac{k_1}{k'^4}\Big(k_1\big[\frac{1}{2} - k'te^{-k't} - \frac{1}{2}e^{-2k't}\big] + k_2\big[-2 + k't + (2+k't)e^{-k't}\big]\Big). \tag{26}$$

This result encodes two ranges of diffusion and can also be derived from the governing equations of the *Lisle et al.* [1998] and *Lajeunesse et al.* [2018] models. Expanding for small t provides $\sigma_x^2(t) = v^2 k_1 t^3/3$ – local range super-diffusion. Expanding for large t provides $\sigma_x^2(t) = 2v^2 k_1 k_2 t/k'^3$ – intermediate range normal diffusion.

We further investigate limit (1) for arbitrary initial conditions. By applying Tauberian theorems, we assert the $t \to 0$ variance is determined by the $s \to \infty$ limits of (18) and (19) [e.g., *Weiss*, 1994; *Weeks and Swinney*, 1998]. Expanding these equations in powers of $1/s$ and inverting the resulting transforms gives

$$\langle x \rangle = v\theta_2 t + \frac{1}{2} v(\theta_1 k_1 - \theta_2 k_2)t^2 + O(t^3), \tag{27}$$

$$\langle x^2 \rangle = v^2\theta_2 t^2 + \frac{1}{3} v^2(\theta_1 k_1 - 2\theta_2 k_2)t^3 + O(t^4). \tag{28}$$

This equation highlights the effect of initial conditions on the diffusion characteristics of the local range:

$$\sigma_x^2(t) \sim v^2\theta_1\theta_2 t^2 + \frac{1}{3} v^2(\theta_1 k_1 + \theta_2 k_2)t^3. \tag{29}$$

We have taken only leading order terms for any option of θ_1 and θ_2. Equation (29) shows local range exponent $\gamma = 2$ when initial conditions are mixed (both are non-zero) and $\gamma = 3$ when initial conditions are pure (one is zero).

Limit (2) is $1/k_2 \to 0$ and $v \to \infty$ while $v/k_2 = l$. Under this limit, equations (18) and (19) provide

$$\langle \tilde{x} \rangle = k_1 l \frac{1}{s(s+\kappa)}, \tag{30}$$

$$\langle \tilde{x}^2 \rangle = 2l^2 k_1 \frac{s+\kappa+k_1}{s(s+\kappa)^2}. \tag{31}$$

Inverting these equations and introducing the variables $c = lk_1$ (an effective velocity) and $D_d = l^2 k_1$ (a diffusivity) provides positional variance

$$\sigma_x^2(t) = \frac{2D_d(1-e^{-\kappa t})}{\kappa} + \frac{(1 - e^{-2\kappa t} - 2e^{-\kappa t}\kappa t)c^2}{\kappa^2}. \tag{32}$$

This is mathematically identical to the key result of *Wu et al.* [2019]. Expanding for small t provides $\sigma_x^2(t) = 2D_d t$ – intermediate range normal diffusion, while sending $t \to \infty$ provides $\sigma_x^2 = (2D_d\kappa + c^2)/\kappa^2$ – a constant variance in the geomorphic range. The global range is characterized by competition between terms in equation (32), and shows $2 \leq \gamma \leq 3$ depending on the ratio k_1/κ [cf., *Wu et al.*, 2019]. Finally, both equations (26) and (32) reduce to the Einstein result $\sigma_x^2(t) = 2D_d t$ in further simplified limits.

References

Arfken, G., *Mathematical Methods for Physicists*, Academic Press, Inc., doi:10.1063/1.3062258, 1985.

Berezhkovskii, A. M., and G. H. Weiss, Detailed description of a two-state non-Markov system, *Physica A: Statistical Mechanics and its Applications*, *303*(1-2), 1–12, doi:10.1016/S0378-4371(01)00431-9, 2002.

Kittel, C., *Elementary Statistical Physics*, R.E. Krieger Pub. Co., 1958.

Lajeunesse, E., O. Devauchelle, and F. James, Advection and dispersion of bed load tracers, *Earth Surface Dynamics*, *6*(November), 389–399, 2018.

Lisle, I. G., C. W. Rose, W. L. Hogarth, P. B. Hairsine, G. C. Sander, and J.-Y. Parlange, Stochastic sediment transport in soil erosion, *Journal of Hydrology*, *204*, 217–230, 1998.

Prudnikov, A., Y. A. Brychkov, and O. Marichev, *Integrals and Series: Volume 5: Inverse Laplace Transforms*, Gordon and Breach Science Publishers, 1992.

Temme, N. M., *Special functions: An introduction to the classical functions of mathematical physics*, John Wiley & Sons Ltd., 1996.

Weeks, E. R., and H. L. Swinney, Anomalous diffusion resulting from strongly asymmetric random walks, *Physical Review E - Statistical Physics, Plasmas, Fluids, and Related Interdisciplinary Topics*, *57*(5), 4915–4920, doi:10.1103/PhysRevE.57.4915, 1998.

Weiss, G. H., *Aspects and applications of the random walk*, North Holland, Amsterdam, 1994.

Wu, Z., E. Foufoula-Georgiou, G. Parker, A. Singh, X. Fu, and G. Wang, Analytical solution for anomalous diffusion of bedload tracers gradually undergoing burial, *Journal of Geophysical Research: Earth Surface*, *124*(1), 21–37, doi:10.1029/2018JF004654, 2019.