BIRTH-DEATH MODELS OF BEDLOAD TRANSPORT
A SYNTHETIC LITERATURE REVIEW

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ABSTRACT

A foundational problem in river science is the calculation of the bedload flux, or the rate of movement of bed material. This is a significant problem because bed material movement shapes river channels and determines habitat suitability to a large extent. Conceptualizing bedload transport as a random switching between motion and rest states leads naturally to an analogy with ecological population modeling. Erosion, movement, and deposition are analogous to birth, migration, and death. As in ecology, details of these processes are considered difficult to predict, but their rates can be measured. These process analogies have opened up new ways to model bedload transport as a stochastic process. I review approaches which model the bedload flux as a birth-death process. These approaches are exciting because they provide a new theoretical framework for understanding the statistical properties of bedload fluxes – fluctuations, intermittency, spatial and temporal correlations. However, existing approaches contain many simplifying assumptions, so the literature review reveals a handful of obvious opportunities for extensions to these birth-death models.

The review is used to define a research trajectory, which I extrapolate from in order to discern a set of possible extensions. I develop some of these extensions myself, and present (1) a new birth-death model incorporating bed surface affects in sediment entrainment and deposition for multiple size fractions, (2) a joint probabilistic modeling of bed elevation and the bedload flux, and (3) a two-dimensional birth-death theory of bedload motion, where particles are free to move in longitudinal and transverse directions. After these extensions are completed, I conclude with a discussion of the scope and limitations of birth-death approaches, and I frame some possible extensions in terms of interdisciplinary literature in physics, chemistry, and ecology. The wider directive of this synthetic review is to foster future research in birth-death modelling of bedload transport, because I believe this theoretical framework provides a foundation for deeper understanding of the relationship between channel morphology and bedload fluctuations, with potential to clarify both of these subjects, each of contemporary relevance to ecology and engineering.

1 Introduction: birth-death models for the probability distribution of bedload flux

In river science, a fundamental problem is the determination of the bedload flux, or the rate of downstream movement of bedload grains [*Ballio et al.*, 2014]. Since bedload transport has strong feedback with stream morphology [*Church*, 2006; *Recking et al.*, 2016], its prediction is useful to a wide range of environmental considerations. These considerations span a wide range, from aquatic habitat restoration to energy production [*Kondolf et al.*, 2014; *Wohl et al.*, 2015]. Unfortunately, existing approaches to compute the bedload flux are inadequate. Predictions regularly deviate by two orders of magnitude from measured values [*Gomez and Church*, 1989; *Barry et al.*, 2004; *Bathurst*, 2007; *Recking et al.*, 2012].

Predicting bedload fluxes is challenging because transport is not always well correlated to average characterizations of flow and bed material. Local fluxes can range through orders of magnitude as details of turbulent fluctuations and bed organization vary, while average characterizations of flow and sediment remain constant [*Sumer et al.*, 2003; *Charru et al.*, 2004; *Hassan et al.*, 2008; *Venditti et al.*, 2017]. The same turbulence and sediment organization details which

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correlate with the bedload flux also interact with it. Turbulent impulses induce sediment motion [*Valyrakis et al.*, 2010; *Celik et al.*, 2014; *Amir et al.*, 2014; *Shih et al.*, 2017], moving sediment affects turbulent characteristics [*Singh et al.*, 2010; *Santos et al.*, 2014; *Liu et al.*, 2016], and bedload fluxes modify the stability and arrangement of bed surface grains [*Kirchner et al.*, 1990; *Charru et al.*, 2004; *Hassan et al.*, 2008]. The bedload flux is in a cyclical feedback with its controls: these controls are the details of turbulence and bed organization [*Jerolmack and Mohrig*, 2005].

As a result, bedload fluxes exhibit wide fluctuations, even as average characterizations of flow and bed organization remain steady [*Ancey et al.*, 2014]. In the most controlled laboratory experiments, with steady flows driving uniform glass beads under conditions which do not favor bedform development, instantaneous fluxes are often as much as 200% mean values [*Böhm et al.*, 2004; *Ancey et al.*, 2008; *Heyman et al.*, 2014, 2016]. In natural streams, with temporally or spatially varying grain size distributions [*Lisle and Madej*, 1992; *Chen and Stone*, 2008], unsteady flows [*Mao*, 2012; *Ferrer-Boix and Hassan*, 2015], a variety of dynamic bed surface structures [*Hassan et al.*, 2008; *Venditti et al.*, 2017], variable sediment supply [*Madej et al.*, 2009; *Elgueta*, 2018], lateral adjustment [*Pitlick et al.*, 2013; *Redolfi et al.*, 2018], and migrating bedforms [*Gomez and Church*, 1989; *Dhont and Ancey*, 2018] all imparting additional sources of variabilility through their feedbacks with the flux, instantaneous fluxes can be even larger.

Apparently, bedload fluxes have statistical characteristics. Einstein was the first to recognize the statistical character of bedload transport [*Einstein*, 1937]. He understood transport as a random switching between states of motion and rest [*Einstein*, 1942, 1950]. He called the transition from rest to motion entrainment, and he characterized it with a probability which was linked to extreme events in fluid turbulence [*Einstein and El-Samni*, 1949; *Einstein*, 1950]. He called the transition from motion to rest deposition, and he considered it an implicit function of the bedload flux [*Einstein*, 1942, 1950]. By convolving over sequences of entrainment and deposition, employing some semi-emprical arguments, Einstein developed a formula for the average bedload flux [*Einstein*, 1950].

Many investigators have criticized and developed Einstein's ideas [*Paintal*, 1971; *Yalin*, 1972; *Shen*, 1980; *Lisle et al.*, 1998; *Papanicolaou et al.*, 2002; *Sun and Donahue*, 2000; *Ancey et al.*, 2006a; *Armanini et al.*, 2015]. These revisions focus on the more ad hoc elements of Einstein's derivations, and they lend a more mechanistic basis, a firmer mathematical foundation, and more generality to the approach. Although all of these authors accepted the statistical character of bedload motion, none developed theories for a probability distribution for the bedload flux. This is desirable since its higher moments give unambiguous measurements of the magnitude of bedload fluctuations.

The extension of the Einstein approach to obtain a probability distribution of the bedload flux, with fluctuations stemming from the random character of entrainment and deposition, was first explored by *Lisle et al.* [1998]; *Sun and Donahue* [2000], and it was further developed by *Ancey et al.* [2006a], and the more recent work I will discuss extensively in this review. All of these authors revisited Einstein's ideas from a foundation in the stochastic mathematics which became formalized somewhat after Einstein's work [e.g. *Cox and Miller*, 1965], treating the transitions between motion and rest states as random events characterized by probabilities. This enabled them to apply the formalism of continuous time Markov processes to derive a probability distribution of the bedload flux, with a mean value which was an improved version of the *Einstein* [1950] formula, very similar to the revised formula of *Yalin* [1972], and a variance which provided an unambiguous prediction of the magnitude of bedload fluctuations.

Deterministic processes trace the evolution of some set of variables $\{x_1(t), x_2(t), \dots\}$ through time. Stochastic processes, in contrast, trace the evolution of a probability distribution for random variables $P_{X_1,X_2,\dots}(x_1, x_2, \dots; t)$ through time. The Markov moniker refers to the amount of memory in the process. If the probability distribution of the variables of interest can be predicted in the future using only distribution functions from the present, the process is Markovian. Otherwise, if predicting the future requires knowledge of the entire history, the process is non-Markovian [*Cox and Miller*, 1965; *van Kampen*, 1992]. Markov birth-death processes consider the probabilistic creation and annhilation of members of a population [*Cox and Miller*, 1965; *van Kampen*, 1992].

To model the bedload flux with such a process, the population considered is the number of moving particles within a control volume over the bed. This population is subject to creation by entrainment and annhilation by deposition [*Ancey et al.*, 2008; *Turowski*, 2009; *Heyman et al.*, 2013; *Ma et al.*, 2014; *Ancey and Heyman*, 2014; *Ancey et al.*, 2015]. This approach is exciting because Markov birth-death processes are highly studied in physics, chemistry, and population ecology, so there are many extensions readily available [*Bailey*, 1968; *Cox and Miller*, 1965; *Pielou*, 1977; *van den Broek*, 2012; *van Kampen*, 1992; *Gillespie*, 1992; *Field and Tough*, 2010; *Mendez et al.*, 2015]. At the same time, birth-death modeling of the bedload flux remains relatively undeveloped: birth-death models are all one-dimensional and designed for a single grain size.

In this article, I review the literature on Markov birth-death models of the bedload flux. In the review I assume some familiarity with discrete state continuous time Markov processes, which could be gleaned from reading the appropriate chapter in *Cox and Miller* [1965]. These theories are exciting because they describe the mean bedload flux and its fluctuations within a volume [*Ancey et al.*, 2006a, 2008; *Turowski*, 2009], statistical properties of the flux at a point in space [*Heyman et al.*, 2013; *Ma et al.*, 2014], and the spatial and temporal correlations in the bedload flux within a reach of channel [*Ancey et al.*, 2014, 2015]. All of these considerations are new, and they are unique within the river science literature in that they do not ignore the statistical character of bedload transport. They initiate a theoretical framework from which the feedbacks between the arrangement of bed particles, turbulence, and channel morphology which lead to bedload fluctuations can be more carefully studied and understood. In reverse, the capacity of bedload models to describe fluctuations provides an additional benchmark against which to judge models, a task which is notably difficult [*Iverson*, 2013].

One of the first birth-death models of the bedload flux I will review is *Ancey et al.* [2006a]: these authors generalized *Einstein* [1950] using a birth-death framework to obtain a statistical distribution of the bedload flux within a control volume. They found the fluctuations predicted by the Einstein-like model were not large enough to describe experimental data. In order to match the wide fluctuations of experimental data, *Ancey et al.* [2008] reworked their earlier control volume model to include a collective entrainment effect. Collective entrainment is a mathematical term charaterizing the correlations between moving grains due to turbulence, collisions, or granular avalanche effects.

The *Ancey et al.* [2008] model has been generalized and applied in a handful of followup works. *Turowski* [2009] generalized it to include limited sediment availability. *Heyman et al.* [2013] and *Ma et al.* [2014] studied how the *Ancey et al.* [2008] model could be interpreted to describe the statistics of the bedload flux at a fixed point in space, rather than just a control volume like that considered by *Einstein* [1950]. Control volumes are less desirable because transport is practically easier to measure at a point in space than it is within a control volume.

Ancey et al. [2014, 2015] generalized the *Ancey et al.* [2008] model in order to understand spatial characteristics of bedload transport. They considered a reach of channel as an array of adjacent control volumes (cells), and they coupled the *Ancey et al.* [2008] model between adjacent cells. They mapped this generalized birth-death model for the bedload flux distributed over an array of control volumes onto an advection diffusion equation for the bedload flux. Some reprecussions of this model were further explored in context of experimental data by *Heyman et al.* [2014].

All of these works contain many simplifying assumptions, and many of these were carefully highlighted by their authors. These studies were selected for this review because, taken together, they define a research trajectory which can be extrapolated to suggest future research topics. To be sure, there is a long road ahead before Markov models can accomodate all of the complexities of natural river channels, some of which I've mentioned. The goal of this review is to indicate this trajectory in order to motivate future research: this is pursued in sections 2 and 3. I discuss some of the future research topics I believe the trajectory points out in section 4, and I develop a few of these ideas myself in 5. Along the way, I have somewhat changed the mathematical treatment and the notation from some of the original papers in order to highlight the historical progression between each work, and to illustrate that these works all share the same theme.

My own extensions concern: (1) birth-death modeling of the fractional transport rates of a sediment mixture – earlier works concentrate on a single grain size [*Ancey et al.*, 2006a, 2008; *Turowski*, 2009]; (2) the bedload flux of a single grain size when bed elevations vary, modifying entrainment and deposition characteristics – earlier works concentrate on a flat bed with no bedforms [*Ancey et al.*, 2006a, 2008; *Turowski*, 2009]; and (3) a model for bedload flux including two-dimensional diffusion – earlier works concetrate on one dimensional diffusion [*Ancey et al.*, 2014, 2015; *Heyman et al.*, 2014]. Once these extensions are presented, in section 6 I recap the literature review, my identifications of underlying assumptions and future research directions, and my own extended Markov birth-death models of the bedload flux.

2 Background: the foundational stochastic model of bedload transport

Einstein [1950] can be credited with the first attempt to understand the bedload flux as random quantity. Einstein understood the movement of individual bedload grains as a random succession of motion and rest intervals. He termed the transition from rest to motion entrainment, and characterized it by a probability related to fluid turbulence. When particles are set in motion, Einstein considered they moved in discrete jumps of the same average distance. We define the bedload flux as the volume of grains passing a stream cross section per unit time. It is a volume of grains per time per unit width of stream, so its units are area/time. Therefore, Einstein's mean flux formula is derived from summing the alternate start-stop motions of all bed particles which cross a surface in a time interval.

The conceptual picture Einstein considered is depicted in figure 1. All bed particles are identical and are considered to have the same average geometry when at rest on the bed, meaning their entrainment characteristics are identical. For continuity with the rest of the review, we consider sediment particles are spheres of radius a, although *Einstein* [1950] was more general. We can follow Einstein to compute the mean number of these particles crossing a flow cross-sectional surface S in an interval of time. This will construct the mean bedload flux $\langle q_s \rangle$. In subsequent sections,

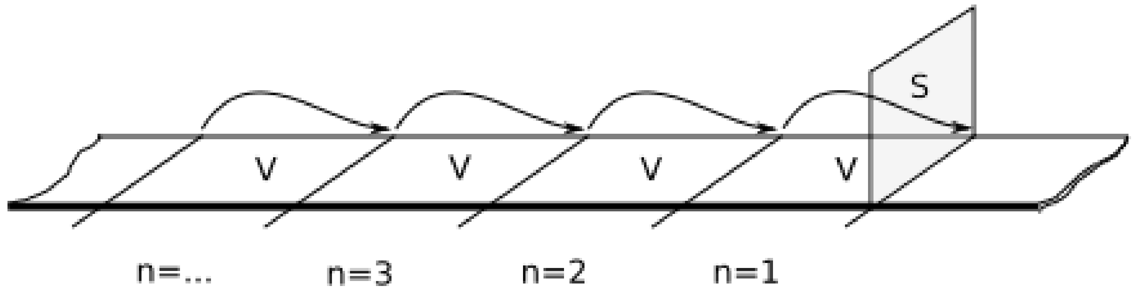

Figure 1: Einstein's conceptual picture (adapted from *Yalin* [1972]). Particles move in discrete jumps of length L from left to right through an array of adjacent control volumes. The bedload flux is the volume of bedload particles crossing the surface S per unit width and time.

we consider q_s a random variable, and we will review approaches to derive its full probability distribution. From these contemporary models, an Einstein-like mean flux formula emerges from taking the mean of the probability distribution for q_s. Hence we use averaging brackets $\langle \cdot \rangle$ for the sake of continuity.

Einstein's derivation goes as follows: we partition the channel into a sequence of identical control volumes V. Each volume has dimensions L, w, and h ($V = lwh$). L is the downstream length of each control volume, and it is also the average jump distance of particles. w and h are the width and height of each control volume. Let P_n be the probability that any individual grain on the bed is entrained *at least* n times during the time interval T. That is, let P_n be the probability that an individual grain undergoes *at least* n jumps of length L in an interval T, meaning it travels a distance nL or more in T. Considering that each control volume contains N_V grains at rest within it, it follows that on average

$$N_V P_n. \tag{1}$$

grains are displaced *at least* a distance nL from a control volume within the time interval T.

Now we derive the mean flux. Note that the grains passing through a cross-sectional surface S (of area lw) in the time T could have come from any upstream location. Therefore, the number of grains crossing S in T is

$$\sum_{n=1}^{\infty} N_V P_n. \tag{2}$$

This is the net arrival of grains in the interval T from all upstream control volumes. Multiplying by the particle volume ν_p and dividing by the timescale T and channel width w gives the mean bedload flux (volume per width per time):

$$\langle q_s \rangle = \frac{\nu_p N_V}{wT} \sum_{n=1}^{\infty} P_n = A \frac{a^2}{T} \sum_{n=1}^{\infty} P_n. \tag{3}$$

In the second equality, we have introduced $\nu_p = 4\pi a^3/3$, and two assumptions of Einstein. First, is the reasonable assumption that the number N_V of particles on the surface within V is proportional to the ratio of bed surface and particle areas: $N_V \propto Lw/a^2$, and second is the assumption that the travel distance L is proportional to particle size: $L \propto a$. The latter assumption is more difficult to justify [e.g. *Yalin*, 1972]. A is a constant of proportionality.

Einstein implies the probability that an individual grain travels a distance nL or more in T is equivalent to the probability that an individual grain travels at least a distance L successively n times in T. Hence $P_n = P_1^n$ by the law of multiplication of probabilities, so that $\sum_{n=1}^{\infty} P_n = \sum_{n=1}^{\infty} P_1^n = P_1/(1-P_1)$, using the geometric series (noting $P_1 < 1$ because it is a probability). Owing to this substitution, Einstein's mean bedload formula takes the form

$$\langle q_s \rangle = A \frac{a^2}{T} \frac{P_1}{1-P_1}, \tag{4}$$

where A is a constant of proprotionality. The $P_1/(1-P_1)$ structure of Einstein's formula is a distinctive characteristic. In order to use the apply the Einstein formula to predict bedload transport in natural streams or experiments, the timescale T and the probability P_1 that at least one grain entrains in T must be determined. Then the formula can be calibrated to a given setting by adjusting the constant A.

In the original work, Einstein considered the timescale T proportional to the "time of the grain" formed by the grain's size a and its settling velocity w in still fluid: $T \propto a/w$. We will revisit this assumption shortly. Regarding the probability P_1, one of Einstein's most influential and enduring ideas within hydraulic engineering is his conception of the probability P_1 that at least one particle entrains in T. He considered that entrainment is driven by the fluctuating lift force imparted by the turbulent fluid, and it is resisted by the submerged weight of grains. Therefore, he formulated P_1 as an exceedance probability of the lift force over weight, so he wrote P_1 as an integral over the probability distribution of fluid shear stress. Einstein developed an alternative perspective on the incipient motion of particles that embraced turbulence and did not rely on the critical shear stress concept [e.g. *Shields*, 1936].

This formulation of entrainment probability in terms of the exceedance of random driving quantities over (possibly random) resisting quantities has been generalized and extended a great deal. *Paintal* [1971] made a significant extension by including random granular geometry of bed particles into the force balance. More recently, *Tregnaghi et al.* [2012] ammended the theory to include the impulse concept: the experimentally verified idea that force magnitude and duration are of equal importance for entrainment [e.g. *Diplas et al.*, 2008; *Celik et al.*, 2014]. Refined theories of the entrainment probability, all fundamentally similar to the original ideas of *Einstein* [1950], have been carefully reviewed by *Dey and Papanicolaou* [2008]; *Dey et al.* [2018], and they are a subject of intensive ongoing research.

As noted by *Yalin* [1972], who wrote a careful review and revision of *Einstein* [1950], the Einstein theory for $\langle q_s \rangle$ is theoretically sound, at least to equation 4, and provided Einstein's probabilities P_n are interpreted in a careful way. I have mimiced Yalin to say P_n is the probability of *at least* n jumps in T. Had I said P_n was the probability of (only) n jumps in T, the relationship $P_n = P_1^n$ would not follow, so the Einstein flux would not develop the notable $P_1/(1-P_1)$ form as in 4. Einstein was not as clear as he could have been on this distinction. Notably, the more general birth-death theories which reproduce Einstein as a limiting case [e.g. *Ancey et al.*, 2006a] use probability concepts relating to (only) one entrainment in a time interval. Therefore, to link back to Einstein from these contemporary theories, we need to follow *Yalin* [1972] to express the mean flux $\langle q_s \rangle$ in terms of the probabilities of (only) n entrainments in a time interval, which we can denote by a lower case letter p_n. These two probabilties, P_n (at least) and p_n (only), have probably been conflated often within the literature.

Yalin appealed to scaling arguments in order to refine Einstein's theory. He highlighted that Einstein's prescription of $T \propto a/w$, where w is the settling velocity, was not supported by any experimental evidence or theoretical argument. Yalin noted that within Einstein's theory, "T appears in connection to the study of detachment of individual grains due to turbulent fluctuations", from which he concluded that T should be actually be measured in terms of a period τ of turbulent fluctuations:

$$T = N_\tau \tau, \tag{5}$$

where N_τ is the dimensionless number of turbulent fluctuations in T (which is considered unknown but very large).

Yalin introduces a probability p of (only) one entrainment during an individual turbulent fluctuation. Since the number N_τ of turbulent fluctuations in T is very large, he considers this probability is very small, so that the probability of (only) n detachments in T can be computed with the Poisson distribution:

$$p_n = \frac{(N_\tau p)^n}{n!} e^{-N_\tau p}. \tag{6}$$

The probability P_n of *at least* n detachments in T can be expressed in terms of the probability p_n of (only) n detachments in T as

$$P_n = \sum_{i=n}^{\infty} p_i. \tag{7}$$

Owing to equations 5, 6, and 7, the mean bedload flux 3 becomes

$$\langle q_s \rangle = A \frac{a^2}{N_\tau \tau} \sum_{n=1}^{\infty} \sum_{i=n}^{\infty} p_i = A \frac{a^2}{N_\tau \tau} \sum_{n=1}^{\infty} n p_n = A \frac{aL}{\tau} p. \tag{8}$$

The unknown number of turbulent fluctuations N_τ cancels. Hence, Yalin's ammended Einstein formula is proportional to p, rather than $P_1/(1-P_1)$ as in *Einstein* [1950]. In Yalin's mind, the probability p is the probability of entrainment of only one grain in a small interval τ related to the period of turbulent fluctuations.

When Einstein's conceptual picture of bedload transport is revisited from a Markov process framework, the probability distribution $P(q_s)$ of the bedload rate can be derived from his assumptions [*Ancey et al.*, 2006a]. This distribution implies a mean bedload formula $\langle q_s \rangle = \sum q_s P(q_s)$ formally similar to Yalin's Einstein-like formula 8, and it provides additional information about the magnitude of bedload fluctuations, which are characterized without ambiguity by the variance $\langle \delta q_s^2 \rangle$, where $\delta q_s = q_s - \langle q_s \rangle$ is the deviation of the bedload flux from the mean. Jumping ahead a little, the fluctuations $\langle(\delta q_s)^2\rangle$ derived in this way are not large enough to match experimental data [*Ancey et al.*, 2006a], and this is because *Einstein* [1950], *Yalin* [1972], and *Ancey et al.* [2006a] each assumed the transitions between rest and motion were independent between all particles.

Ancey et al. [2008] fixed this problem by describing bedload transport within a more general birth-death process framework [*Cox and Miller*, 1965]. They included a collective interaction into the entrainment rates of bed particles in their model, and with this inclusion they derived bedload fluctuations of realistic magnitude. Their collective interaction biases entrainment, making it more likely when particles are already moving, a point we will discuss. It leads to clouds of active particles, and a wide tail on the bedload flux probability distribution: both of these features are in accord with experimental observations [*Drake et al.*, 1988; *Ancey et al.*, 2006a, 2008]. I'll start the review with *Ancey et al.* [2006a], which is essentially *Einstein* [1950] revisited from the birth-death framework, lending it more generality by treating the bedload flux as a probability distribution. The mean of this probability distribution reproduces equation 8, which is a satisfying historical coherence. Followups and refinements of the *Ancey et al.* [2006a] work constitute the bulk of the review in section 3.

3 Review: birth-death theories of bedload transport

3.1 Formalizing Einstein: a foundation in Markov process theory

A significant theoretical development was made in stochastic modeling of the bedload flux by *Ancey et al.* [2006a]. They reinterpreted the concepts of *Einstein* [1950] to develop a new model of the bedload flux expressed as a Markov birth-death processes. The motivation for the *Ancey et al.* [2006a] work was a series of experiments: *Böhm et al.* [2004] measured the bedload flux of glass beads at the outlet of a narrow flume using a percussion sensor, while *Ancey et al.* [2006a] gathered a higher resolution dataset by filming particle motion with a video camera through the plexiglass sidewall of a narrow flume. They obtained bedload flux and motion characteristics using image processing. Conditions were highly controlled: uniform 6mm diameter glass beads were used as sediment. The flume was 6.5mm wide, so transport was quasi 1D. It only occurred in the downstream direction, and experiments were conducted under steady state conditions when the rate of sediment input was matched by the yield at the outlet. Despite this idealized level of control, measured fluxes had large fluctuations. Instantaneous values were as much as four times mean values.

The prevalence of large bedload fluctuations even under these idealized laboratory conditions motivated *Ancey et al.* [2006a] to revisit Einstein's assumptions to develop a model of the bedload flux as a random variable. The assumptions underlying their model were:

1. The sediment system can be characterized using a control volume with a size much larger than the scale of a single grain, similar to *Einstein* [1950]. A series of volumes as depicted in 1 is not necessary.
2. The motion of each grain is independent of every other, and can be characterized as a random succession of motion and rest intervals.
3. Transitions between motion and rest states are characterized by probabilities per unit time, or rates.
4. Choosing a particle at random from the system, the possible transitions of that particle are sufficiently determined by its present state, i.e., motion or rest, and knowledge of its complete history of motion/rest transitions does not improve knowledge of its future state. This is the Markov property.

These assumptions are enough to develop a birth-death model for the bedload flux. Within this model, bedload fluctuations are due to variations in the number of moving particles within the control volume. As we will see, *Ancey et al.* [2006a] obtained a probability distribution for the bedload flux. The mean of this probability distribution is the Einstein-like formula of Yalin 8. The variance of this distribution is an unambiguous determination of the expected magnitude of bedload fluctuations.

3.1.1 Theoretical development: the number of moving particles within a control volume

Following *Ancey et al.* [2006a] we consider a control volume as in figure 2, and we assume that a particle's dynamics are independent of its history and of the dynamics of all other particles. The probability of entrainment or deposition of a particle at each instant only depends on whether a particle is in motion or at rest at that instant. Hence, the stochastic dynamics of each particle satisfy the Markov property: it is independent of the past dynamics and only contingent on the current state of the particle. Since there are no interactions between particles to take account of, their dynamics, which are random successions between motion and rest states, can be considered separately. At the end, bedload transport within the control volume can be understood as the net contributions from all individual particles within it.

Therefore, we consider an individual particle within the control volume. We denote the probability that this particle is in the rest state at some time t by $\pi_0(t)$ and the probability that it is in the motion state by $\pi_1(t)$. We consider that within a small time interval δt, if the particle is at rest, the probability for the particle to transition into motion (entrainment) is $\sigma^{-1}\delta t$. Similarly, if the particle is in motion, the probability that it transitions into rest (deposition) is $\tau^{-1}\delta t$. Hence σ^{-1} is the rate of entrainment of an individual grain, and τ^{-1} is the rate of deposition. Since these rates have dimensions of inverse time, σ can be interpreted as a timescale of rest, and τ can be interpreted as a timescale of motion.

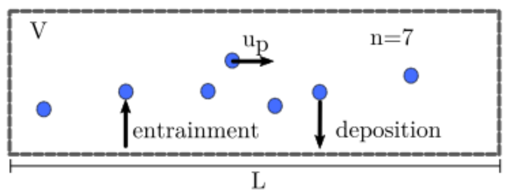

Figure 2: A control volume V of downstream length L is defined over a region of mobile bed. The number n of particles in motion is considered a random variable due to entrainment and deposition processes. Particles in motion move with velocity u_p.

Using these transition rates and the standard method of setting up a master equation for a birth-death Markov process [*Cox and Miller*, 1965; *Pielou*, 1977; *Gillespie*, 1992, e.g.] develops the following equations describing the flow of probability through time:

$$\dot{\pi}_0(t) = \tau^{-1}\pi_1(t) - \sigma^{-1}\pi_0(t) \tag{9}$$

$$\dot{\pi}_1(t) = \sigma^{-1}\pi_0(t) - \tau^{-1}\pi_1(t). \tag{10}$$

This two state stochastic process, illustrated in figure 3 is known as the random telegraph process [*Gillespie*, 1992; *Gardiner*, 1983], and it is often invoked as a pedagogical example in textbooks on stochastic processes.

These equations can be solved for the probability that a particle is at rest – $\pi_0(t)$ – or in motion – $\pi_1(t)$ – by elementary techniques. If the particle is initially at rest, so that $\pi_0(0) = 1$ and $\pi_1(0) = 0$, the solution is

$$\pi_0(t) = e^{-t/t_c} + \frac{\tau}{\sigma+\tau}(1 - e^{-t/t_c}), \tag{11}$$

$$\pi_1(t) = \frac{\sigma}{\sigma+\tau}(1 - e^{-t/t_c}). \tag{12}$$

Here $t_c = \sigma\tau/(\sigma+\tau)$ is a correlation timescale which is the geometric mean of the motion and rest timescales. We can see that, after a time $t \gg t_c$, when $e^{-t/t_c} \approx 0$, the trajectory of the particle is independent of (forgets) the initial condition and settles into a stationary state with no time dependence. In this steady state, the probability that the particle is in motion is

$$\pi_1 = \frac{\tau}{\sigma+\tau} = \xi, \tag{13}$$

while the probability that the particle is at rest is

$$\pi_0 = \frac{\sigma}{\sigma+\tau} = 1 - \xi. \tag{14}$$

We can interpret ξ as the fraction of time spent in the motion state by the particle on average. To see this, we consider the distribution of residence times in the motion state. Let $R(T_m)$ be the probability the particle has been in motion for time T_m. Then $R(T_m + \delta t)$ is the probability the particle remains in motion for another period δt. This is just the probability the particle was in motion at T_m, times the probability that the particle did not deposit in δt. In symbols, $R(T_m + \delta t) = R(T_m)(1 - \tau^{-1}\delta t)$. As $\delta t \to 0$, this develops the differential equation

$$\frac{d}{dT_m}R(T_m) = -\frac{1}{\tau}R(T_m), \tag{15}$$

and the solution is the exponential distribution:

$$R(T_m) = \tau e^{-T_m/\tau}. \tag{16}$$

Hence the mean time spent in motion is τ. By a similar argument, the distribution of resting times is exponential with parameter σ, so that σ is the mean time spent at rest. Accordingly $\xi = \tau/(\sigma+\tau)$ represents a fraction of time spent in motion by the particle.

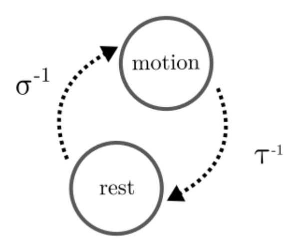

Figure 3: Each particle in the control volume switches between motion and rest states with transition rates σ^{-1} and τ^{-1}.

This is an important link: the parameters τ and σ are observable quantities. The rate paramters entering the *Ancey et al.* [2006a] model reviewed in this section are summarized in table **??**.

We have followed *Ancey et al.* [2006a] to a description of the random start-stop motions of an individual particle. The probability that the particle is found in motion is ξ and the probability that it is found at rest is $1 - \xi$. Hence an observation of whether or not the particle is in motion is a Bernoulli trial with known probabilities, like flipping a coin. These two probabilities are ratios of observable quantities.

Now in order to get at the bedload flux, consider that on the bed surface within the control volume there are N particles at any time, each switching randomly from motion to rest as an equilibrium telegraph process. We ask, of the N particles, what is the probability that n of them are in motion, $P(n)$? The sum of N Bernoulli trials is a binomial distribution [*Feller*, 1968], so the probability is

$$P(n) = \binom{N}{n}\xi^{N-n}(1-\xi)^n \sim \mathrm{Bi}(\xi N, \xi(1-\xi)N). \tag{17}$$

This is just the probability that $N-n$ particles are at rest with probability $1-\xi$ while n are in motion with probability ξ. The factor $\binom{N}{n}$ represents the number of ways to select the n moving particles from the N total particles. $\mathrm{Bi}(a,b)$ is a shorthand for a binomial distribution of mean a and variance b. Hence *Ancey et al.* [2006a] developed a probability distribution for the number of particles in motion in a control volume using the conceptual picture of *Einstein* [1950].

Now we outline two important limits of the binomial particle activity distribution as the number N of particles available for motion in the control volume becomes very large. The first limit is to consider that as N becomes large, the probability ξ that any of these N particles is observed in motion remains fixed. In this case, the binomial distribution, equation 17, tends to a Gaussian distribution:

$$P(n) \sim \mathcal{N}[N\xi, N\xi(1-\xi)]. \tag{18}$$

This follows from applying Stirling's formula $\ln(N!) \approx N\ln(N) - N$. The second way is to consider that as N becomes large, the probability ξ that any one of these N particles is observed in motion decreases accordingly to leave the product $N\xi := \lambda$ fixed. Under this limit, the binomial distribution tends to a Poisson distribution:

$$P(n) \sim \frac{\lambda^n}{n!}e^{-\lambda}. \tag{19}$$

This follows from taking only the highest order of N from the binomial coefficients, and using one of the definitions of the exponential function: $\lim_{N\to\infty}(1-\frac{\lambda}{N})^N = e^{-\lambda}$. We will use these limits to evaluate the bedload flux from the particle activity distribution derived in the *Ancey et al.* [2006a] model, and to make contact between this early birth-death model of the bedload flux, and the more sophisticated models which follow it.

3.1.2 From the number of moving particles to the bedload flux

Usually, the bedload flux is defined as the number of moving particles crossing a surface S perpendicular to the flow direction: $q_s = \int_S \mathbf{u}_p \cdot d\mathbf{S}$, where $d\mathbf{S}$ is an area increment with direction perpendicular to surface S [*Ballio et al.*, 2014]. The surface S used within the definition of the flux does not have an obvious relationship to the control volume in which we considered the random number of particles in motion. We need to connect the properties of bedload motion on a surface to the number of active particles within a volume.

The surface integral definition of flux just stated is often used in continuum field theories such as classical electrodynamics and kinetic theory, but it is not well suited to discrete particles. *Ancey et al.* [2006a] and *Furbish et al.* [2012] have advocated an alternative formulation based on ensemble averaging. This alternative definition hinges on the probability $P[\mathbf{u}_p|\mathbf{x},t]$ that a particle contacts the control surface S at position $\mathbf{x}$ and time t with velocity $\mathbf{u}_p$. This conditional probability is considered to result from a very large collection of identical systems selected at random moments in their evolution: that is, the conditional probability is an ensemble quantity [*Kittel*, 1958].

In terms of this ensemble probability, the flux (per unit width) is [*Ancey et al.*, 2006a]:

$$q_s = \frac{1}{w}\int_{\text{all } \mathbf{u}_p}\int_S P[\mathbf{u}_p|\mathbf{x},t]\mathbf{u}_p \cdot d\mathbf{S}d\mathbf{u}_p. \tag{20}$$

I have slightly modified this equation with the factor of w^{-1}. In steady conditions, when the probability is independent of time – $\partial P/\partial t = 0$, the ensemble definition of the flux 20 can be related to the number of moving particles in the control volume we derived in the previous section. This link is only approximate, which I believe an under-emphasized point in the literature.

The approximation is developed by swapping the ensemble interpretation of P by a frequentist interpretation. This is done by integrating along the control volume. If the control volume is sufficiently long, $L \to \infty$, it can be seen as a stack of very many independent cross sectional surfaces. These comprise a stack of replicas of S, and at an instant, each surface constitutes one realization (particles intersecting S with some set of positions and velocities) contributing to P. We can define P by counting occurences along this stack of replica surfaces. This becomes an integral across the control volume.

With this concept of replica surfaces in mind, we can formalize the link with symbols. The n particles of radius a will be distributed throughout the volume at some set of positions $\mathbf{x}_i$ with some set of velocities $\mathbf{u}_i$, where $i = 0, 1, \ldots, n$. Thus the cloud of particles within the control volume at some instant can be represented by its density in a position-velocity phase space:

$$\rho(\mathbf{x}, \mathbf{u}_p) = \sum_{i=1}^{n} M_a(\mathbf{x} - \mathbf{x}_i)\delta^3(\mathbf{u}_p - \mathbf{u}_i). \tag{21}$$

Here $M_a(\mathbf{z})$ is a marker function with definition

$$M_a(\mathbf{z}) = \begin{cases} 1 & \text{if } |\mathbf{z}| < a \\ 0 & \text{otherwise} \end{cases}. \tag{22}$$

It is 1 within a sphere of radius a around its argument, and zero otherwise. Its volume integral is $\int M_a(\mathbf{x})dV = 4\pi a^3/3 = \nu_p$, the particle volume. There are other applications of such functions in *Furbish et al.* [2012] and *Ballio et al.* [2014].

Using this density, for sufficiently large L, we can write the conditional probability $P[\mathbf{u}_p|\mathbf{x}]$ approximately as an integral over the stack of replica surfaces. This is an exchange of the ensemble interpretation of P for a frequentist interpretation:

$$P[\mathbf{u}_p|\mathbf{x}] \approx \frac{1}{L}\int_0^L dx \rho(\mathbf{x}, \mathbf{u}_p). \tag{23}$$

The integral runs along the downstream coordinate. This relationship is only exact as $L \to \infty$. Presumably, for fixed L, the quality of this approximation would scale with spatial correlations among bedload positions and velocities. The more correlated bedload positions and velocities are, the worse the approximation will become.

Using this swap, the bedload flux becomes

$$q_s \approx \frac{1}{w}\int_{\text{all } \mathbf{u}_p}\int_S \frac{1}{L}\int_0^L dx \sum_{i=1}^{n} M_a(\mathbf{x} - \mathbf{x}_i)\delta^3(\mathbf{u}_p - \mathbf{u}_i)\mathbf{u}_p \cdot \mathbf{k} dS d\mathbf{u}_p = \frac{\nu_p}{Lw}\sum_{i=1}^{n} \mathbf{u}_i \cdot \mathbf{k} \tag{24}$$

It is expressed in terms of the number of particles n within the control volume. n is the random variable we considered earlier as the result of N telegraph processes. If we write the downstream component of each particle's velocity as $\mathbf{u}_i \cdot \mathbf{k} = u_i$, the bedload flux takes the simple looking form

$$q_s \approx \frac{\nu_p}{Lw}\sum_{i=1}^{n} u_i \tag{25}$$

in terms of the number of particles n within the (necessarily large) control volume and their downstream velocities u_i, where $i = 1, 2, \ldots, n$. This equation becomes exact as $L \to \infty$.

There are options for further analysis of the flux in this way:

1. We can consider the downstream velocities of particles as random variables governed by a probability distribution [*Roseberry et al.*, 2012; *Furbish et al.*, 2016], in which case the bedload flux is a sum of a random number n of random variables u_i. In stochastic processes parlance, a sum of a random number of random variables is called a compound stochastic process [*Feller*, 1968]. Treating q_s as a compound process has not been fully explored, to my knowledge.
2. We can consider that bedload particles, once set in motion, will all move with approximately the same average velocity, so that the sum 25 simplifies. This idea has some theoretical support [e.g. *Ancey et al.*, 2003], although it conflicts with experimental findings [*Heyman et al.*, 2016], so it should be considered critically. This is the approach taken by *Ancey et al.* [2006a].

For now, we will follow *Ancey et al.* [2006a] to consider that all particles move at approximately the same downstream particle velocity u_s, so that the bedload flux becomes

$$q_s(n) \approx \frac{\nu_p u_s}{Lw} n. \tag{26}$$

The flux is directly proportional to the number n of moving particles within the control volume. The bedload flux is a random quantity which inherits its statistical properties from n.

3.1.3 The bedload flux as a random variable: Einstein revisited, Yalin an intermediary

Now we develop the probability distribution of the bedload flux by combining the results we have obtained so far: (1) the probability distribution of the number of moving particles within a control volume equation 17, and (2) the relationship between the number of moving particles and the bedload flux equation 26. Using the law for transformation of discrete random variables [e.g. *Feller*, 1968],

$$P(q_s) \approx \sum_{n=0}^{N} P(n)\delta_{n,Lwq_s/(\nu_p u_s)}, \tag{27}$$

but this is not valid. Since the bedload flux is only approximately given by 26 there is no guarantee that $q_s Lw/(\nu_p u_s)$ will be an integer, and this transform cannot be carried through.

Instead, we can take advantage of the large L assumption embedded into the equation 26. Because L is necessarily large and $N \propto L$, it follows that N is also large when 26 is valid. Therefore, we can also take a limit of the binomial distribution $P(n)$ as the number N of particles in the control volume becomes very large, while the probability that any individual particle is observed in motion remains fixed. We can use the resulting distribution, equation 18, with 26, to derive the bedload flux distribution without the mathematical ambiguity of 27. The probability distribution for the bedload flux becomes, using the law for transformations of continuous random variables [e.g. *Feller*, 1968] with 26 and 18:

$$P(q_s) \approx \int_0^\infty dn\delta(n - Lwq_s/(\nu_p u_s))P(n) \approx \mathcal{N}\big(N\xi[\nu_p u_s/(Lw)], N\xi(1-\xi)[\nu_p u_s/(Lw)]^2\big) \tag{28}$$

Ancey et al. [2006a] obtain a normal distribution for the bedload flux.

Now we discuss the connection of the *Ancey et al.* [2006a] formalism to the earlier work of *Einstein* [1950]. From equation 28, the mean flux is

$$\langle q_s \rangle = \nu_p u_s \xi \frac{N}{Lw}. \tag{29}$$

According to *Einstein* [1950], the ratio $N/(Lw)$ is related to the particle size via $N/(Lw) = 1/a^2$. Since the particle volume is $\nu_p = 4\pi a^3/3$, the mean flux becomes

$$\langle q_s \rangle = \frac{4}{3}\pi a u_s \xi. \tag{30}$$

Einstein famously considered that the average jump distance of a particle within a single motion interval was proportional to the particle size [*Einstein*, 1950; *Yalin*, 1972]. If we denote the jump distance as l to distinguish it from the control volume size L, this means $a \propto l$. Similarly, Einstein held that the mean transport rate scaled with an inverse characteristic timescale. Originally, it was expressed in terms of the particle size and the settling velocity [*Einstein*, 1950]. Later on, *Yalin* [1972] concluded it was a timescale of turbulent fluctuations, and there have been similar interpretations by other investigators [*Paintal*, 1971; *Cheng*, 2004; *Armanini et al.*, 2015]. Apparently, in this case, to make the link from *Ancey et al.* [2006a] to *Einstein* [1950], the timescale should be $\tau = ka/u_s$, where k is a constant of proportionality. With these substitutions, the bedload flux becomes

$$\langle q_s \rangle = A\frac{al}{\tau}\xi, \tag{31}$$

where A is a constant of proportionality. This is similar to Yalin's reinterpretation of Einstein, equation 8. In the *Ancey et al.* [2006a] model, ξ, the fraction of time spent in motion by a single particle, plays the role of p, which was the probability of (only) one entrainment in a unit of time. In contrast to the p and τ of *Einstein* [1950], the parameters of this model, ξ and τ, are measurable quantities with a clear physical basis. ξ is the fraction of time spent in motion by an individual particle. τ is the ratio of particle size and the average downstream velocity of sediment in motion. We can conclude that *Ancey et al.* [2006a] has reframed and extended Einstein's theory using stochastic mathematics, treating the bedload flux as a statistical quantity. The parameters of the refined model have a sound physical interpretation.

Now we will examine the magnitude of bedload fluctuations expected from the model. We can examine the variance of the bedload flux, and compare this to the mean of the flux, to measure the strength of fluctuations. However, a clearer conclusion emerges from comparing the mean and variance of the particle activity, derived from the distributions 17 or either of its large N limits. The ratio of the variance to the mean is

$$\frac{\langle \delta n^2 \rangle}{\langle n \rangle} = 1 - \xi. \tag{32}$$

We conclude that according to the *Ancey et al.* [2006a] model, because $\xi > 0$, the variance in the number of active particles within the observation window should never exceed the mean.

All of the experiments of *Ancey et al.* [2006a] violated this condition, undermining the model, and leading us to examine its assumptions. *Ancey et al.* [2006a] hypothesized that the inability of the birth-death model they developed to describe experimental fluctuations was due to the lack of interaction between particles as they transition between motion and rest states. They noted that particles preferentially moved in groups, and that the entrainment of one particle led to subsequent entrainment of neighboring particles which were contingent on it for stability. Therefore they extended their 2006 model to include a collective motion effect in *Ancey et al.* [2008]. As it turns out, this inclusion leads to a negative binomial distribution of the bedload flux. This distribution has a sufficiently heavy tail to describe experimental bedload fluctuations. We focus on the *Ancey et al.* [2008] model and the collective entrainment process it introduces in the next section.

3.2 Describing bedload fluctuations: Including correlations between particle motions

Ancey et al. [2006b, 2008] developed a more general birth-death model in order to describe the relatively heavy tailed bedload probability distributions seen in experiments. In the last ten years, this model has been restated and more deeply studied by several authors [*Turowski*, 2010; *Heyman et al.*, 2013; *Heyman*, 2014; *Heyman et al.*, 2014; *Ma et al.*, 2014], and has led into exciting generalizations [*Turowski*, 2010; *Ancey et al.*, 2014, 2015]. These are the real focus of this review.

Instead of considering the motion of every bedload particle as an independent process involving transitions between motion and rest states, and summing a collection of these to get the total number of moving particles within the control volume, *Ancey et al.* [2008] consider the total number of moving particles within the control volume as a state. In this case, instead of just two states (motion and rest), there is an infinite ladder of states ($n = 0, 1, 2 \dots$), each representing the number of active particles within the control volume. The ladder of states is illustrated diagramatically in figure 4.

The advantage of this approach is when all particles are treated simulatenously, the transition rates can be made to depend on the number of active particles. The state is a collective property of the moving particles within the control volume. This allows interactions between particles to be introduced. *Ancey et al.* [2008] considers the population of moving particles changes due to entrainment (birth) and deposition (death), as before, but they also introduce migration processes for particles to move into (emigration) and out of (immigration) the control volume. We note that these migration processes do not fundamentally modify the structure of the equations which result. However, they are physically expected, so their inclusion is rational.

The key result of *Ancey et al.* [2008] is a successful description of wide bedload fluctuations. To obtain this, they made a component of the entrainment rate increase with the number of active particles. In effect, they introduced a positive feedback on entrainment: when more particles are in motion, entrainment becomes more likely. They termed this feedback term "collective entrainment". The collective entrainment term fundamentally modifies the probability distribution of the bedload flux and lends it a wide tail, so the model is capable of describing realistically large fluctuations. Rather than a binomial distribution of the number of moving particles, as in 17, *Ancey et al.* [2008] derive a negative binomial distribution. Within their model, collective entrainment is the source of the difference between these two distributions. When it is turned off, *Ancey et al.* [2008] reduces to *Ancey et al.* [2006a].

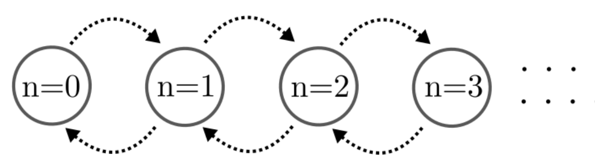

Figure 4: Birth, death, immigration, and emigration rates characterize the flow of probability up and down an infinite ladder of states. Each state represents n particles in motion, and n runs across all positive integers including zero.

Collective entrainment has been attributed to several physical mechanisms. Usually, the feedback is considered due to collisions of moving particles with the bed [*Ancey et al.*, 2008; *Heyman et al.*, 2013; *Heyman*, 2014] and the advection of turbulent structures implying waves of entrainment [*Ancey et al.*, 2008; *Heyman et al.*, 2014]. Another consideration is that collective entrainment is the result of small avalanches, or local rearrangements of the bed surface triggered by the entrainment of an individual particle [*Heyman et al.*, 2014; *Heyman*, 2014]. Entraining one grain could destabilize all of those contingent on it, so local microstructure [**?**] may imply collective entrainment. These mechanisms for the collective entrainment term are reasonable, but they only hypotheses at this stage, and research is needed.

The layout of this section is as follows: First, I will follow *Ancey et al.* [2008] to develop a master equation for the number n of active particles within the control volume, obtaining a probability distribution $P(n)$. Then we can analyze the mean and variance of n and develop the linkage to the probability distribution of the bedload flux, $P(q_s)$ using the relationship between volume and surface definitions of the flux, equation 26. This flux will be shown to link back to *Einstein* [1950] as a limiting case. The magnitude of fluctuations will be considered, and we will conclude by a discussion of the model's assumptions and successes. In subsequent sections, we will examine generalizations of and other ways to study this model.

3.2.1 Master equation of the collective entrainment model

Consider the probability that there are n particles in motion at time t. This can be denoted $P(n,t)$. The random variable n takes on values $0, 1, 2, \ldots$ – non-negative integers of arbitrary magnitude. Each of these states has an associated probability $p_n(t)$. The population is subject to four transitions within a small time increment δt:

1. Entrainment of a single bed particle (birth) can happen in time δt with probability $\lambda_0 + \mu n$. The second term μn is the collective entrainment effect. As the number of active particles increases, so does the rate of entrainment. This interaction term induces realistically wide bedload fluctuations.
2. Deposition of a single bed particle can happen in δt with probability σn. This is similar to the telegrapher's process considered earlier. The rate of deposition increases with the number of active particles, just as it does implicitly in *Ancey et al.* [2006a]. The deposition of each particle is independent of every other.
3. Immigration of a particle from upstream into the control volume can happen in δt with probability λ_1. This term does not fundamentally modify the structure of the equations we will derive for the probability distribution of the bedload rate.
4. Emigration of a particle from the control volume to downstream can happen with probability ν. This term is significant because it defines a bedload flux across a surface as in $q_s = \int dS\mathbf{k}\cdot\mathbf{u}_s$. Counting emigration events provides an alternate definition of the flux. This counting problem was solved by *Ma et al.* [2014]. We will consider this later. In this section, we will continue to interpret the flux as in section 3.1.2, by converting an ensemble average to a volume average.

These transitions are depicted in figure 5. The symbols of 3.2 are summarized in table 1.

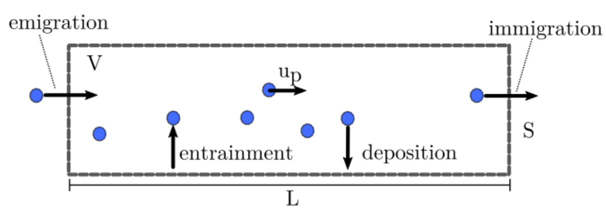

Figure 5: As before, the model is based on analyzing the number of particles within a control volume V which has downstream length L. Particles appear into the control volume due to entrainment and emigration. They disappear due to deposition and immigration. The downstream boundary is a stream cross section S.

We can consider the flow of probability up and down the ladder of states ($n = 0, 1, 2, \ldots$) which is induced by these transitions, entrainment (including the collective entrainment term), deposition, immigration, and emigration, using the formalism of more general Markov birth-death processes [*Cox and Miller*, 1965; *Pielou*, 1977]. By the standard way of developing master equations [e.g. *Cox and Miller*, 1965], we find an infinite system of equations for the probabilities of finding n moving particles within the control volume:

$$\begin{aligned}P(n,t+\delta t) &= \alpha\delta t(n+1)P(n+1,t)\\ &\quad + [\lambda+(n-1)\mu]\delta t P(n-1,t)\\ &\quad + [1-\delta t[\lambda + n\alpha + n\mu]]P(n,t). \end{aligned} \tag{33}$$

Here $\alpha = \nu + \sigma$ summarizes the contributions of emigration and deposition, which both act to decrease the number of particles n, and $\lambda = \lambda_0 + \lambda_1$ summarizes the contributions of immigration and (individual) entrainment, which both act to increase the number of particles n.

As $\delta t \to 0$ these equations become a hirearchy of differential-difference equations for the probabilities $P(n,t)$ of finding n particles in the control volume at time t:

$$\frac{d}{dt}P(n,t) = (n+1)\alpha P(n+1,t) + [\lambda+(n-1)\mu]P(n-1,t) - [\lambda + n(\alpha+\mu)]P(n,t). \tag{34}$$

These master equations define a birth-death immigration emigration model [e.g. *Cox and Miller*, 1965; *Gardiner*, 1983]. It can be solved for the probabilities $P(n,t)$ by introducing the probability generating function $G(z,t) = \sum_{n=0}^{\infty} z^n P(n,t)$ [*Cox and Miller*, 1965; *Gardiner*, 1983; *Ancey et al.*, 2008]. Multiplying 34 by z^n and summing over all n gives, after a careful manipulation of the sums in order to form some function of $G(z,t)$ in every term,

$$\frac{\partial}{\partial t}G(z,t) = \lambda(z-1)G(z,t) + \{\sigma + \mu z^2 + \nu - (\mu+\sigma+\nu)z\}\frac{\partial}{\partial z}G(z,t). \tag{35}$$

In effect, the probability generating function exchanges the discrete variable n for a continous variable z. This first order partial differential equation for G can be solved by the method of characteristics [*Cox and Miller*, 1965; *Garabedian*, 1964]. If there are initially N_0 moving particles within the control volume, the solution is

$$G(z,t) = \Big(\frac{\alpha-\mu}{(K-1)\mu z+\alpha-K\mu}\Big)^{n+\lambda/\mu}\Big(\frac{(K\alpha-\mu)z+\alpha(1-K)}{\alpha-\mu}\Big)^{n}. \tag{36}$$

The factor K is the autocorrelation function $K(t)=\exp(-t(\alpha-\mu))$.

A useful feature of the probability generating function is that it generates the hirearchy of probabilities $P(n,t)$ via the Taylor expansion around $z=0$: $G(z,t)=\sum_{n=0}^{\infty}\frac{z^n}{n!}\big(\frac{\partial}{\partial z}\big)^n G(z,t)|_{z=0}$ [*Cox and Miller*, 1965]. Comparing this formula with the definition of G, $G(z,t)=\sum_{n=0}^{\infty}z^nP(n,t)$, the coefficient of z^n in the power series expansion of $G(z,t)$ is apparently $P(n,t)$. The generating function provides the probability of finding n particles in the control volume.

Sufficiently far from the intial time, when $K(t)\approx 0$ in the expression of G, meaning t is much larger than the autocorrelation time $(\alpha-\mu)^{-1}$ so the system has forgotten its initial condition, the power series expansion of $G(z,t)\to G(z)$ generates a set of stationary probabilities for the number of bedload particles in motion within the control volume:

$$P(n)=\frac{\Gamma(r+n)}{\Gamma(r)\Gamma(1+n)}p^r(1-p)^n,\, n=0,1,\ldots. \tag{37}$$

Here $r=\lambda/\mu$ and $p=1-\mu/\alpha$ characterize the strength of the collective entrainment factor μ relative to the other transitions which act to increase (λ) or decrease (α) the population. $\Gamma(x)=\int_0^\infty z^{x-1}e^{-z}dz$ is the well-known Γ function of mathematical physics [*Boas*, 2005; *Mathews and Walker*, 1971]. It is a generalization of $x!$ to non-integer values of x. This stationary distribution only exists when $\alpha>\mu$: the rate at which sediment particles appear in the control volume is lower than their disappearance rate. The set of symbols introduced in the *Ancey et al.* [2008] formulation is summarized in table 1.

Table 1: The parameters used in section 3.2

Symbol	Meaning
ν	emigration
σ	deposition
λ_0	immigration
μ	collective entrainment
λ_1	entrainment
λ	$\lambda_0+\lambda_1$
α	$\sigma+\nu$
r	λ/μ
p	$1-\mu/\alpha$

This stationary distribution equation 37 is the negative binomial distribution:

$$P(n)\sim \text{NegBin}(r,p). \tag{38}$$

It is a generalization of the binomial distribution for the number of active particles, equation 17, which was derived in the *Ancey et al.* [2006a] paper. The negative binomial distribution has a relatively heavy tail, so that large deviations in the number of active particles n are possible. These large fluctuations are the result of the collective entrainment rate μ, and they allow the birth-death immigration-emigration model to express bedload fluctuations of realistic magnitude. As we will show, they become relatively small as collective entrainment is turned off, $\mu\to 0$.

3.2.2 The bedload flux: wide fluctuations and the Einstein limit

As discussed, the negative binomial distribution supports wide fluctuations, controlled by the collective entrainment parameter μ. The mean number of particles in the control volume, taken over the distribution 38, is

$$\langle n\rangle=\sum_{n=0}^{\infty}nP(n)=\frac{\lambda\alpha}{\alpha-\mu}. \tag{39}$$

Likewise the variance is

$$\langle\delta n^2\rangle=\sum_{n=0}^{\infty}(n-\langle n\rangle)^2P(n)=\frac{\lambda\alpha}{\alpha-\mu^2}, \tag{40}$$

which means their ratio is

$$\frac{\langle\delta n^2\rangle}{\langle n\rangle}=\frac{1}{1-\mu/\alpha}. \tag{41}$$

It is interesting to compare this to equation 32 bounding the magnitude of fluctuations in the *Ancey et al.* [2006a] model. In contrast, within the *Ancey et al.* [2008] model, the variance must exceed the mean, because as discussed, in steady state $\alpha > \mu$. The magnitude of fluctuations can grow arbitrarily with the collective entrainment parameter μ. This is a coherent conclusion, since collective entrainment was introduced in order to represent the collective effects of turbulent fluctuations, collision-induced entrainment, and granular avalanches; and with the expressed intent of generating realistically wide bedload fluctuations, exceeding those of the *Ancey et al.* [2006a] model.

Armed with the probability distribution for the number of particles in the control volume, equation 38, and the link between control volume and surface statistics derived in section 3.1.2, equation 26, we can derive the probability distribution of the bedload flux in a similar way as equation 28. Using the law for transformation of probabilities gives

$$P(q_s) \approx \frac{\Gamma(r + \frac{Lw}{\nu_p u_s} q_s)}{\Gamma(r)\Gamma(1 + \frac{Lw}{\nu_p u_s} q_s)} p^r (1-p)^{\frac{Lw}{\nu_p u_s} q_s} \tag{42}$$

as the probability distribution of the bedload flux q_s. The bedload flux, as derived in *Ancey et al.* [2008], is a random variable with a negative binomial distribution.

Now we link back to the *Einstein* [1950] theory. Einstein considered neither immigration (λ_0), emigration (ν), nor collective entrainment (μ). When all of these parameters are set to zero, $\mu = \lambda_0 = \nu = 0$, the parameters r and p of the distribution 37 become $r \to \infty$ and $p = 0$. With these limits, the distribution 37 tends to a Poisson distribution. There are several ways to show this. The simplest way is to take the appropriate limits ($t \to \infty$, $\mu = \lambda_1 = \nu = 0$) into the generating function 36, giving

$$G(z) = e^{-\lambda_0 (z-1)/\sigma}. \tag{43}$$

This inverts to ($P(n) = \frac{1}{n!}(\partial/\partial z)^n G(z)|_{z=1}$)

$$P(n) = \frac{(\lambda_1/\sigma)^n}{n!} e^{-\lambda_1/\sigma}, \tag{44}$$

the Poisson distribution with rate λ_0/σ. The *Ancey et al.* [2008] model links to the limiting result of *Ancey et al.* [2006a], equation **??**.

Using equation 26 with this Poisson distribution, equation 44, we compute the number of active paricles n in the absence of collective entrainment:

$$\langle q_s \rangle = \frac{\nu_p u_s}{Lw} \sum_{n=0}^{\infty} n \frac{(\lambda_1/\sigma)^n}{n!} e^{-\lambda_1/\sigma} = \frac{\nu_p u_s}{Lw} \frac{\lambda_1}{\sigma}. \tag{45}$$

To connect equation 45 with the Einstein-like formula of Yalin, equation 8, requires a delicate interpretation of the parameters λ_0 and σ. The rate of entrainment λ_1 pertains to all particles within the control volume, of which there is a very large number $N \to \infty$. Meanwhile, the rate of deposition σ pertains to an individual particle. These interpretations are consistent with the master equation 33. Hence if we define the probability that (only) one particle entrains in a small interval δt as λ', then $\lambda = N\lambda'$, and the mean flux is

$$\langle q_s \rangle = \frac{\nu_p u_s}{Lw} \frac{N\lambda'}{\sigma}. \tag{46}$$

If we can again interpret the ratio $N/(Lw)$ as proportional to the area of an individual particle, using $\nu_p \propto a^3$, and the travel distance $l \propto a$, we get

$$\langle q_s \rangle = A \frac{al}{a/u_s} \frac{\lambda'}{\sigma}, \tag{47}$$

again in accord with the Yalin result 8.

We have shown that the assumptions of *Einstein* [1950] can be mixed with stochastic mathematics to develop a more coherent and physically-based theory of the bedload flux, as performed by *Ancey et al.* [2006a]. Since the fluctuations predicted by the Gaussian bedload flux of *Ancey et al.* [2006a] are not sufficiently large to describe experiments, we have mimiced *Ancey et al.* [2008] to include collective motion effects in order to express bedload fluctuations of sufficient magnitude from a birth-death model. Turning off the extra processes in the *Ancey et al.* [2008] model leads to a coherent connection back to a limiting result of *Ancey et al.* [2006a], and further to back to Yalin's revision of Einstein [*Yalin*, 1972].

However, there is obviously some ambiguity around the number of particles available for motion, which was denoted N in previous sections in this review. Apparently, the *Ancey et al.* [2008] model considers the $N \to \infty$ limit, while the probability that any one particle is observed in motion goes to zero, similar to the Poisson limit of the *Ancey et al.* [2006a] model. Of course, the number of particles available for motion is not unlimited. Over a purely alluvial bed composed to gravel, it must be limited by the number of particles on the bed surface. Within the idealization of *Einstein* [1950], this constraint was expressed by the equation $Na^2 \propto Lw$.

In the collective model of *Ancey et al.* [2008], limiting the number of particles available for motion is more difficult. The most obvious way would be to consider that the entrainment rate of all particles is proportional to the entrainment rate of a single particle, times the number of particles at rest, such as $\lambda'(N-n)$. However, this does not work because particles can also arrive from upstream, meaning n can exceed N. In this case, to consider the rate of entrainment decreases with the number of active particles, one would need to disciminate how the particles got into the control volume in the first place. Did they entrain, or did they emigrate? This is non-Markovian. A second way to impose that a finite number of bed particles are available for entrainment is to consider the Markovian dynamics of two species at once: the bed surface particles, and the particles in motion. This path was taken by *Turowski* [2009], and this is the next section of the review.

3.3 What about sediment availability?: Turowski's two species model

Semi-alluvial channels, where alluvium sits on top of bedrock, will violate one of the key assumptions of the *Ancey et al.* [2008] framework. This model implicitly assumes that the bed surface has an infinite number of particles available for entrainment. In reality, this situation will never be satisfied, even in alluvial channels, because there are a wide variety of stabilizing processes acting on bed surfaces [*Hassan et al.*, 2008; **?**].

Turowski was concerned with bedrock scour due to alluvium. Therefore he extended the Ancey framework. The Ancey framework takes account of one population: the number of moving particles. Turowski's extension included a second population into the framework: the number of particles available for entrainment. He considered that as a particle entrains into a motion state, the number of particles available for entrainment decreases, and likewise as a particle deposits from the motion state, the number of particles available for entrainment increases. Therefore, he considered a probabilisitic population model of two coupled populations: such an inquiry is entirely new in stochastic theory of the bedload flux, although it is commonplace in ecological modeling [e.g. *Pielou*, 1977]

Turowski considered a joint probability distribution for the random number N of moving particles and M of particles available for entrainment. This can be written $\pi_{n,m}(t)$. Within a small time interval dt he considered birth, death, immigration, and emigration transitions governing the probabilities $\pi_{n,m}(t)$ similar to *Ancey et al.* [2008]. Like *Ancey et al.* [2008], he also included a collective entrainment contribution, arranging that the model expressed wide Negative-Binomial-like fluctuations in the bedload rate.

Turowski's transition probabilities in dt varied by including the number of available particles m into the entrainment and deposition probabilities. He considered a rate of entrainment $(\lambda_0 + \mu n)m$: when the population of particles available for entrainment is zero ($m = 0$), the rate of entrainment is zero. The entrainment process, when it occurs, enacts a change of state $(n, m) \to (n+1, m-1)$: the number of active particles increases by one, while the number of particles available for entrainment decreases by one.

Similarly, Turowski's deposition process was generalized from *Ancey et al.* [2008]. Deposition occurs with rate σn in time dt, just as in Ancey et al, but when this deposition process occurs it enacts the change $(n, m) \to (n-1, m+1)$ in the coupled populations. The Kolmogorov equation is

$$\begin{aligned} P(m,n;t+dt) &= P(m,n-1,t)\lambda_0 dt + P(m+1,n-1,t)((m+1)\lambda_1 \\ &\quad + (m+1)(n-1)\mu)dt + P(m-1,n+1,t)(n+1)\sigma dt + P(m,n+1,t)(n+1)\nu dt \\ &\quad + P(m,n,t) * [1 - dt(\lambda_0 + m\lambda_1 + n\nu + n\sigma + mn\mu). \end{aligned} \tag{48}$$

which in the limit $dt \to 0$ becomes a Master equation for the joint probabilities $\pi_{n,m}(t)$:

$$\begin{aligned} \frac{d}{dt}P(m,n,t) &= \lambda_0 P(m,n-1,t) + \\ &\quad ((m+1)\lambda_1 + (m+1)(n-1)\mu)P(m+1,n-1,t) + (n+1)\sigma P(m-1,n+1,t) \\ &\quad + (n+1)\nu P(m,n+1,t) - (\lambda_0 + m\lambda_1 + n\nu + n\sigma + mn\mu)P(m,n,t). \end{aligned} \tag{49}$$

Again, as a reminder, the number of active particles is n and the number of particles available for entrainment is m: this master equation describes a hirearchy of joint probability distributions for these discrete random varaibles. Exact solutions via counting arguments [e.g. *Ancey et al.*, 2006a] or probability generating functions [e.g. *Ancey et al.*, 2008] are evidently no longer tractable. *Turowski* [2009] resorted to a numerical solution. Numerical algorithms for stochastic birth-death models are relatively easy to implement and are well described in a number of references [e.g. *Gillespie*, 1977, 1992]. Turowski pursued one of these algorithms and related his transport rate computations to bedrock erosion and alluvial cover within a bedrock channel.

Now we briefly describe the stochastic simulation algorithm of Gillespie 1977 ... Then we reproduce Turowski's results using it ...

We also explore the possibility of an exact solution for the Turowski model.. go harder than he did.

Turowski's master equation reduces to the Ancey et al result as $m \to \infty$. As m gets very large, $m - 1 \approx m \approx m + 1$: effectively, large m makes m constant as far as the master equation perceives. In this limit the probability $P(m, n, t) \approx P(n, t)$, and since m is effectively constant, one can take $m\lambda_1 \to \lambda_1'$ and $m\mu \to \mu'$. With these substitutions the master equation reproduces the *Ancey et al.* [2008] result. For our purposes, given our concern with alluvial channels as outlined in the introduction, the main takeaway of Turowski's work is that the birth-death modelling approach can be applied to study the stochastic dynamics of multiple populations whose variations are correlated. This is similar to competing species in ecology [*Pielou*, 1977] or to multiple reacting chemical species [*Gardiner*, 1983].

3.4 The waiting time between emigration events: Bedload flux from waiting time statistics

These studies are concerned with linking the birth-death formulation, which in the original interpretation deals with transport characteristics within an observation window [*Ancey et al.*, 2008], to the transport characteristics at a point – the number of particles crossing a plane perpendicular to the flow direction.

Heyman et al. [2013] was concerned with the statistics of the time interval between successive emigration events, while **?** focused on actually counting the number of emigration events in a unit time: that is, they computed the bedload flux over the downstream boundary of the control volume *Ancey et al.* [2008].

The waiting time between successive transport events is significant in a broad range of transport studies. First, there are probabilistic transport rate formulations which use the waiting time distribution as their input [*Turowski*, 2010]. Second, the waiting time between successive transport events, as we shall show, reflects back on the underlying entrainment and deposition characteristics, so it provides fundamental information into the somewhat mysterious physics of bedload transport.

The original *Einstein* [1937, 1950] approach reveals that the waiting time between successive transport events should be an exponentially distributed random variable. However, Heyman et al (2013) demonstrates that the inclusion of collective entrainment due to *Ancey et al.* [2008] modifies the waiting time distribution in a non-trivial way. The waiting time between successive entrainment events has contributions at two different scales: first, there is a shorter timescale of individual entrainment events, controlled by fluid turbulence and essentially recognized within the *Einstein* [1937] formalism, and second, there is a longer timescale due to collective entrainment. These authors showed that within the *Ancey et al.* [2008] birth-death model, at low transport rates, when entrainment and motion are highly intermittent, the short timescale and the long timescale are well separated. While at higher transport rates, these two timescales blend together as individual and collective entrainment contributions become indistinguishable.

I'll now sketch the *Heyman et al.* [2013] derivation of the statistics of the waiting time between successive emigration events, and show how the two different timescales due to individual and collective entrainment arise. For *Heyman et al.* [2013] the variable of interest is $S_k = \sum_{i=0}^{k} T - i$: S_k is the time when the kth emigration event occurs. $T_1, T_2, \ldots$ are the waiting times: the time from $t = 0$ to the first emigration event is T_1. The time from T_1 to the second emigration event is T_2, and so on. For a Markov process such as the *Ancey et al.* [2008] model, the waiting times are independant and identically distributed. Within the Einstein model we have already seen that the waiting time probability distribution is expoential. However, in the *Ancey et al.* [2008] formalism there is no reason to expect an exponential distribution.

Heyman et al define $F_n(t)$ as the probability that there are n particles in motion and no emigration event occurred in time t:

$$F_n(t) = Pr(T > t, N(t) = n). \tag{50}$$

By extension, the probability that there are any number of particles and no emigration event occurred in time t is

$$F(t) = Pr(T > t) = \sum_{n=0}^{\infty} F_n(t). \tag{51}$$

Now $F(t + \Delta t)$ is equivalent to the probability that any other event but emigration occurs in Δt. Therefore we can write a master equation for $F(t)$:

$$F_n(t + \Delta t) = F_n(t)[1 - (\lambda + n(\sigma + \mu + \gamma))\Delta t] + F_{n+1}(t)\sigma(n + 1)\Delta t + F_{n-1}(t)\lambda + \mu(n - 1)\Delta t + o(\Delta t). \tag{52}$$

This equation holds for $n \geq 1$. At $n = 0$ deposition and collective entrainment processes are not possible, therefore

$$F_0(t + \Delta t) = F_0(t)[1 - \lambda\Delta t] + F_1(t)\sigma\Delta t + o(\Delta t) \tag{53}$$

Again dividing by Δt and taking $\Delta t \to 0$ gives the differential difference equations

$$F_0'(t) = \lambda F_0(t) + \sigma F_1(t) \tag{54}$$

$$F_n'(t) = -(\lambda + n(\sigma + \mu + \gamma))F_n(t) + \sigma(n+1)F_{n+1}(t) + [\lambda + \mu(n-1)]F_{n-1}(t) \text{ if } n \geq 1 \tag{55}$$

for the waiting time distribution.

Summing all of the terms gives the simple equation

$$\sum_{n=0}^{\infty} F_n'(t) = -\gamma \sum_{n=0}^{\infty} nF_n(t), \tag{56}$$

which, denoting the pdf of waiting times T as $f_T(t)$ gives the simple relationship:

$$f_T(t) = -F'(t) = \gamma\langle F_n(t)\rangle. \tag{57}$$

Therefore the probability distribution function of waiting times between emigration events, $f_T(t)$, is an average of the probability that there are n particles and no emigration event occurred in t over all n.

The general solution of $F_n(t)$ can be obtained from the differential equations with a generating function:

$$G(z,t) = \sum_{n=0}^{\infty} F_n(t)z^n. \tag{58}$$

Multipling the differential equation by z^n and summing over all n gives

$$\frac{\partial G}{\partial t} = (\sigma + \mu z^2 - z(\alpha + \mu))\frac{\partial G}{\partial t} + (z-1)\lambda G, \tag{59}$$

where the shorthand $\alpha = \gamma + \sigma$ has been introduced.

This equation can be solved with the method of characteristics, giving a closed form solution for $G(z,t)$. A useful property of generating functions (and the source of their name) is $\langle F_n\rangle = \partial G(z,t)/\partial t|_{z=1}$, therefore the probability distribution of waiting times between emigration events in the *Ancey et al.* [2008] birth-death model of bedload flux is

$$f_T(t) = \gamma(z_1 - z_2)^{\lambda/\mu}\big(\frac{\alpha - \mu}{A(t) - B(t)}\big)^{\lambda/\mu+1} e^{-\lambda(1-z_2)t}\big(\frac{\lambda/\mu + 1)B(t)}{A(t) - B(t)}[(1-z_2)e^{-(z_1-z_2)t} + z_1 - 1] + e^{-\mu(z_1-z_2)t} - 1\big) \tag{60}$$

When collective entrainment is turned off ($\mu = 0$), the probability distribution becomes $f_T(t) = \frac{\alpha}{\lambda\gamma}e^{-\alpha t/(\lambda\gamma)}$: the exponential waiting time between emigration events is recovered.

This probability distribution for waiting times between emigration events in the *Ancey et al.* [2008] model has, in general, contributions from two timescales. The relatively common individual entrainment process imparts a 'fast' timescale to the probability distribution of emigration waiting time, while the relatively rare collective entrainment process imparts a 'slower' timescale. When collective entrainment is turned off, only the fast timescale is present in the pdf. When the entrainment rate is relatively low, the fast timescale is much shorter than the slow timescale, so that the timescales are well separated and the waiting time distribution is not well approximated by any common distribution in the literature (Gamma, exponential): the behavior of the *Ancey et al.* [2008] model at low transport rates has defied prediction by any other framework.

The most significant point of the *Heyman et al.* [2013] paper is that the probability distribution function of waiting times between emigration events can be used to calibrate the *Ancey et al.* [2008] birth death model to flume experiments. This is because short emigration times control $\alpha - \mu$ and long emigration times control λ. The evaluation of waiting times between emigration events supports a calibration of the stochastic birth-death model of *Ancey et al.* [2008] on flume studies where the transport rate is measured at the outlet of the flume, but the erosion, deposition, or emigration rates are not necessarily measured. In contrast to **?** the mathematics are not much beyond the original formulation of the *Ancey et al.* [2008] model, so this work may be more readily accepted by the research community.

3.5 The bedload flux at a point: the counting statistics of emigration events

Up to this point, the Markovian approaches to understand a fluctuating bedload rate were based upon analyzing the stochastic number of active particles within a control volume. The bedload flux definition considered for these studies was somewhat unconventional. Typically it is defined as the rate of bedload crossing a vertical plane perpendicular to the flow direction [*Ballio et al.*, 2014].

The studies of *Heyman et al.* [2013] and **?** demonstrated that the birth-death model of *Ancey et al.* [2008] can also be used to define a flux at a position in space. Although the birth-death model of *Ancey et al.* [2008] counts the number of moving particles at each instant of time, it does not count the number of emigration events which contribute to this number. Counting the number of emigration events in an interval of time is a much more difficult problem. Ma et al (2014) approached this counting problem using a path integral approach: this approach is essentially outlined in **?**. The path integral approach draws heavily from the stochastic simulation algorithm of *Gillespie* [1977].

The conventional definition of the bedload flux is an integral of particles through a surface perpendicular to the flow:

$$Q_s(t) = \int\int_A dA\mathbf{u}_s \cdot \mathbf{n}, \tag{61}$$

where $\mathbf{n}$ is a unit vector perpendicular to the average flow direction and $\mathbf{u}_s$ is the velocity of bed load. This instantaneous flux is very difficult to measure, so usually it is averaged over some time interval δt:

$$Q_s(t;\delta t) = \frac{1}{\delta t}\int\int_A \int_t^{t+\delta t} \mathbf{u}_s \cdot \mathbf{n} d\tau dA = \frac{\delta V_s}{\delta t} = v_s \frac{\delta S(t;\delta t)}{\delta t}. \tag{62}$$

$Q_s(t;\delta t)$ is the volume flux averaged over δt; δV_s is the volume of bed load particles passing through A within δt; $S(t)$ is the cumulative number of particles passing through the cross section since t=0, and $\delta S(t;dt)$ is the number of particles passing through the surface between t and $t+\delta t$, i.e. $\delta S(t;\delta t) = S(t+\delta t) - S(t)$. This equation means the bedload flux at t, when viewed in a time-averaged sense, is the number of particles crossing the surface between t and $t+\delta t$. Because volume units can always be multiplied in later, the bedload flux is written

$$q_s(t) = \frac{\delta S(t;\delta t}{\delta t}. \tag{63}$$

This bedload flux is a random variable. A fluctuation is defined strictly as $q_s' = q_s - \langle q_s \rangle$, and the magnitude of fluctuations is defined without ambiguity by the variance $\langle q_s'^2 \rangle$. This $\delta S(t;dt)/\delta t$ is the number of emigration events between t and $t+\delta t$, so if this number could be counted, the bedload flux would be known at the downstream boundary of the control volume: the bedload flux would be known at a point in space.

The path integral formulation performs this counting problem. For the sake of computation, the time interval t is discretized as $i\Delta t = t$, $i = 0, 1, 2, \ldots$. In the end, the limit $\Delta t \to 0$ will be taken. If the increments Δt are sufficiently small, Ma et al argued that emigration events during one of these intervals will be approximately Poissonian so that

$$P[\Delta S_i = n | N_i(t) = N_i] = \exp[-f_3(N_i)\Delta t]\frac{(f_3(N_i)\Delta t)^N}{n!}. \tag{64}$$

This equation expresses the probability that n particles emigrate between t and $t+\delta t$ is a Poisson distribution with rate parameter $f_3(N_i)\Delta t$, where $f_3(N_i) = \gamma N_i$ is the emigration rate.

Now Ma et al argued that because the entrainment rate $f_1 = \lambda + \mu N(L)$, the deposition rate $f_2 = \sigma N(L)$, and the emigration rate $f_3 = \gamma N(1)$, if the observation window is sufficiently large, entrainment and deposition processes dominate, so that the number of moving particles N is perturbed only weakly by emigration. In this limit of sufficiently large observation window size, it then holds that

$$P(\sum \Delta S_i = n | N_1, N_2, \ldots) = \exp[-\sum f_3(N_i)\Delta t]\frac{(f_3(N_i)\Delta t)^n}{n!}, \tag{65}$$

which in the continuum limit becomes a path integral counting emigration events:

$$P[\delta S = n | N(t) : t_0 < t < t_0 + \delta t] = \exp[-\int f_3(N(\tau))d\tau]\frac{[\int f_3[N(\tau)]d\tau]^n}{n!}. \tag{66}$$

Denoting the integral over the process $f_3(N(\tau))$ as $\alpha = \int_{t_0}^{t_0+\delta t} d\tau f_3[N(\tau)]$, the probability for n emigration events in δt is written

$$P(\delta S = n) = \int d\alpha \frac{e^{-\alpha}\alpha^n}{n!}P(\alpha, t). \tag{67}$$

From this difficult derivation a simple formula emerges: the probability of n emigration events in δt is an integral over all possible realizations of the process $N(t)$, where each process is given a weight $e^{-\alpha}\alpha^n/n!P(\alpha,t)d\alpha$. This is called a Cox process. It's a generalized Poisson process where the parameter α is also a random variable. It represents a superposition of a series of Poisson distributions taking every possible path of $N(t)$ into account. This is a stochastic description of the bedload transport flux based on the counting statistics of emigration events.

Notably, because of the time integral of $N(t)$ involved in this relationship, it has a memory. It is non-Markovian. This character reveals a very interesting fact: the fluctuations of the transport rate depend on the time scale of observation. In fact, Ma et al derived the expression for the variance:

$$\delta t \mathrm{var}[q_s(\delta t)] = \gamma^2 \frac{\lambda(\sigma+\gamma)}{(\sigma+\gamma-\mu)^2} 2t_c[\delta t - t_c(1-e^{-\delta t/t_c}] + \frac{\gamma\lambda}{\sigma+\gamma-\mu}\delta t \tag{68}$$

where $t_c = 1/(\sigma+\gamma-\mu)$ is the autocorrelation time of $N(t)$, representing memory in the system. Apparently, the variance of the bedload flux at a point depends on the observation or averaging time-scale δt.

In fact, Ma et al discriminated three regimes of behavior from 68:

$$\delta t \mathrm{var}[q_s(\delta t)] \approx \begin{cases} \frac{\gamma\lambda}{\sigma+\gamma-\mu}, & 0 < \delta t \ll t_l \\ \delta t \frac{\gamma^2\lambda(\sigma+\gamma)}{(\sigma+\gamma-\mu)^2} & t_l < \delta t < t_c \\ 2\gamma^2 \frac{\lambda(\sigma+\gamma)}{(\sigma+\gamma-\mu)^2} t_c + \frac{\gamma\lambda}{\sigma+\gamma-\mu} & t_c \ll \delta t < \infty \end{cases} \tag{69}$$

When this three-regime equation is divided by the mean bedload rate mean$[q_s(\delta t)]$, the behavior is discriminated by a dimensionless number Ma et al called Ra:

$$Ra(\delta t) \approx \begin{cases} 1 & 0 < deltat \ll t_l \\ \frac{deltat}{t_l} & t_l < \delta t < t_c \\ \frac{2t_c}{t_l} + 1 & t_c \ll \delta t < \infty \end{cases} \tag{70}$$

The essential point is that Ma et al derived three regimes of fluctuations from the *Ancey et al.* [2008] model. Depending on the sampling interval δt, three different scaling relations are possible for the magnitude of fluctuations in the bedload flux. These intervals are discriminated by two timescales t_c and t_l. There are probably deep consequences of this research which remain to be fully understood. At this stage, it's sufficient to note that fluctuations differ with observation timescale, and that this effect can be discriminated into stages with a dimensionless number Ra.

The origin of the two timescales still needs to be discussed.

There is another way to get information about the bedload flux at the downstream boundary of the control volume defined by *Ancey et al.* [2008], and this was examined by *Heyman et al.* [2013, 2016]. Rather than actually counting the number of emigration events with the complicated path integral approach, Heyman et al. took a somewhat simpler angle: they examined the waiting time between successive emigration events.

3.6 The diffusion of bedload from a microstructural basis

Ancey et al. [2008] developed a generalized version of *Einstein* [1950] which took account of a new process to describe the large fluctuations seen in experimental bedload flux signals: collective entrainment. *Turowski* [2009] generalized this model to finite sediment availability, while *Heyman et al.* [2013] and *Ma et al.* [2014] used the *Ancey et al.* [2008] model in its original form to understand the distribution of waiting times between emigration events and the bedload flux probability distribution at a point in space, respectively. These two works developed new understanding of spatial and temporal aspects of bedload flux: *Heyman et al.* [2013] demonstrated that the waiting time between successive emigration events expressed contributions from slow and fast timescales related to individual and collective entrainment, respectively. *Ma et al.* [2014] highlighted the role of observation timescale on the magnitude of flutuations, and discriminated three distinct scaling regimes for bedload fluctuations which are contingent on a dimensionless number $Ra(\delta t)$, where δt is the timescale of observation.

However, up to this point in early 2014, stochastic birth-death models had focused on a single region of space – a control volume [*Einstein*, 1950; *Ancey et al.*, 2006a, 2008], or a single point in space – the downstream end of the control volume, or the plane across which emigration events happen [*Heyman et al.*, 2013; **?**]. This approach is obviously oversimplified. Variability of sediment transport rates is almost a defining feature of them [*Hassan et al.*, 2008; **?**; *Nelson et al.*, 2014]. There is a need to take account of spatial differences in entrainment and deposition characteristics in calculating bedload fluxes. Additionally, there are a wide set of issues centered around bedload diffusion– or the spreading of bedload particles as they are tracked through time and subjected to random entrainment and deposition events [**?**]. Indeed, this study of bedload diffusion was the original research directive of *Einstein* [1937].

Often, bedload diffusion has been understood through deterministic partial differential equation models. These advection diffusion equations have been derived by considering conservation of mass [], but they were not very well connected to the underlying stochastic dynamics of bedload transport. Generating this connection between a microstructural stochastic model like *Ancey et al.* [2008] and the continuum diffusion equation treatment of bedload diffusion [e.g. *Parker and Toro-Escobar*, 2002] was the directive of the next papers on birth-death modeling [*Ancey et al.*, 2014, 2015]. These papers took a conceptually straightforward extension of the *Ancey et al.* [2008] model which leads to difficult mathematics.

Rather than considering a single control volume, as in *Ancey et al.* [2008], *Ancey et al.* [2014, 2015] considered an infinite array of adjacent control volumes (cells), indexed by $i = 1, 2, \dots, M$. Bedload particles can entrain and deposit within each cell, and they can also migrate between cells. Each of these transitions are characterized by probabilities per unit time in generalization of *Ancey et al.* [2008] across spatial extent. The ith cell has a random number of particles N_i in motion within it. Therefore the state of the system at an instant of time is fully characterized by the set of these numbers: $\mathbf{n} = (n_1(t), n_2(t), \dots n_M(t))$.

One target is the grand probability distribution of the number of active particles within each cell at an arbitrary time: $P(\mathbf{n}; t)$. Another target is the continuum limit: letting the length Δx of each cell shrink to zero develops an advection-diffusion equation for bedload diffusion. Ancey et al termed this a "Stochastic interpretation of bedload diffusion": and the latter target has much more fundamental scope. It explores the link between microscale stochastic models of bedload transport and mesoscale deterministic models based upon advection-diffusion equations.

Now I'll sketch the *Ancey et al.* [2014, 2015] derivation of a birth-death-migration model across an array of cells. There are essentially three transitions to take account of:

1. Entrainment can occur within each cell (i) at probability per unit time $(\lambda_i + \mu_i n_i)\delta t$
2. Deposition can occur within each cell (i) at probability per unit time $\sigma_i n_i \delta t$
3. Migration can occur from cell i to cell $i+1$ (downstream) at probability per unit time ν_i

Now some notation is introduced to write the effect of these transitions on $\mathbf{n}$ in shorthand. If the change of state due to one of these transitions is written as $\Delta\mathbf{n}$, then the effect of these transitions can be characterized using vectors $\mathbf{r}_i^j$ and $\mathbf{r}_i^\pm$. The vectors $\mathbf{r}_i^j$ are of the same dimension as $\mathbf{n}$ and all but two of their entries are zero: $r_i = 1$, $r_j = -1$, $r_k = 0$ for $k \neq i, j$. The vectors $\mathbf{r}_i^\pm$ are the same dimension as $\mathbf{n}$ and all but their ith entry is zero: $r_i = \pm 1$, $r_j = 0$, $j \neq i$.

Thus the transition probabilities of each process can be written:

1. $p_i^3 = \text{Prob}(\mathbf{N} = \mathbf{n} + \mathbf{r}_i^+; t + \delta t) = (\lambda_i + \mu_i N_i)\delta t$ – entrainment
2. $p_i^2 = \text{Prob}(\mathbf{N} = \mathbf{n} + \mathbf{r}_i^-; t + \delta t) = \sigma_i N_i \delta t$ – deposition
3. $p_i^1 = \text{Prob}(\mathbf{N} = \mathbf{nn} + \mathbf{r}_i^{j-1}; t + \delta t) = \nu_{i-1} N_{i-1} \delta t$ – migration

These transition probabilities develop a master equation analogous to the one from *Ancey et al.* [2008], but generalized to a collection of M cells:

$$\begin{aligned}\frac{\partial}{\partial t} P(\mathbf{n}; t) = \sum_{i=1}^{M} (n_i + 1)[P(\mathbf{n} + \mathbf{r}_{i+1}^j, t)\nu_i + P(\mathbf{n} + \mathbf{r}_i^+, t)\sigma_i] \\ + P(\mathbf{n} + \mathbf{r}_i^-, t)(\lambda_i + \mu_i(n_i - 1)) \\ + P(\mathbf{n} + \mathbf{r}_i^{j-1}, t)\nu_{i-1} n_{i-1} \\ - P(\mathbf{n}, t)(\nu_{i-1} n_{i-1} + \lambda_i + \mu_i n_{i+1} + \nu_i n_i + \sigma_i n_i)\end{aligned} \quad (71)$$

Information about solving this equation is sparse in the literature. The closest approach has been in ecology: these birth/death/migration-type stochastic models have been considered in context of ecological population dynamics since at least the sixties [*Bailey*, 1968]. One exact solution of a very similar stochastic model has been attained using path integral approaches borrowed from quantum field theory [*Field and Tough*, 2010]. By any metric, the analytical barrier this multidimensional master equation presents is not trivial, although definitely numerical solutions are possible [e.g. *Gillespie*, 1992].

In order to link this equation to spatial diffusion, *Ancey et al.* [2014, 2015] resorted to a Poisson space representation [*Gardiner*, 1983]. In effect, the Poisson transformation converts the discrete random variable $\mathbf{N}$ to a continuous random variable $\mathbf{a}$ (the Poisson rate). The Poisson transform is

$$P(\mathbf{n}, t) = \prod_i \int \frac{e^{-a_i} a_i^n}{n!} f(\mathbf{a}, t) d\mathbf{a}, \quad (72)$$

where $\mathbf{a} = (a_i) \geq 0$ for $i = 1, 2, \dots$ and $f(\mathbf{a}, t)$ is the multivariable probability density of the continuous vector $\mathbf{a}$.

Ancey et al. [2014, 2015] used this transformation to obtain a Langevin equation representation of the dynamics of the random variable **a**. That is, they obtained a stochastic differential equation representation of **a**:

$$da_i(t) = (\lambda_i - a_i(\sigma_i - \mu_i) + \nu_{i-1}a_{i-1} - \nu_i a_i)dt + \sqrt{2\mu_i a_i}dW_i(t), \tag{73}$$

and the $W_i(t)$ is a white noise term (Wiener Process) on cell i. The mean steady state solution of this Langevin equation is

$$\langle a \rangle_s s = \frac{\lambda}{\sigma - \mu} \tag{74}$$

for all i.

Now *Ancey et al.* [2014, 2015] sought out to find an advection diffusion equation as the limit of cell size $\Delta x \to 0$, so they went on to impose this limit on the Langevin equation, thereby obtained an advection diffusion equation. Unfortunately, the limit is more subtle than expected. This did not really work for reasons I need to understand better, and it did not work in ways I need to learn to explain better. More study is needed on spatially varying birth-death models, from me, and from others especially.

3.7 Spatial correlations in bedload transport:

Heyman et al. [2014] used the *Ancey et al.* [2014] model to examine a set of experiments resolving spatial correlations of bedload transport. They found a stochastic differential equation for the mean of the poisson rate η: They related this to a point process framework which allowed them to analyze spatial correlations in the bedload rate and relate this to their birth-death model. Somehow this whole thing is contingent on their two-point spatial correlation process. They state that the whole problem is linear so that it'd be straightforward to extend it to 2d. slowing down ...

4 Scope: identifying future research directions

4.1 recap

Einstein developed first stochastic theories describing the diffusion and flux of bedload. Ancey 2006, building on earlier work from Lisle 1998 and the many investigators who criticized and revised Einstein, formalized Einstein's assumptions over a foundation in Markov process theory, and extended his work by treating the bedload flux within a control volume as the cooperation of many independent two-state Markov processes: each bed particle within the control volume makes random transitions between motion and rest. Thus he derived a statistical distribution of the bedload rate which reproduces Einstein's result when the mean is taken. However, the magnitude of fluctuations predicted by this model are too small.

Ancey 2008 extended the Einstein theory by linking the transition rates between particles within the observation window with a collective entrainment term: they prescribed that the transition rate from rest to motion depends on the number of active particles. This is a simple way to include the effects of coherent turbulence and impact-based entrainment: it is an experimental observation that moving particles tend to come in waves [*Drake et al.*, 1988]. The Ancey 2008 model was demonstrated capable of describing fluctuations, and all of its parameters have physical meanings, but notably there is no clear suggestion as to how collective and individual entrainment processes can be separated within experiments. Clearly, approaches to derive entrainment probabilities are focused on the individual entrainment rate [*Dey et al.*, 2018], and there have been no developments with regard to computing the collective entrainment rate: there is no clear physical model of collective entrainment yet.

Turowski 2009 extended the Ancey work to take account of limited supply, in order to describe semi-alluvial channels where alluvial deposits lie on top of bedrock. He continued to work within the Einstein paradigm by describing the transport rate within a control volume, rather than by counting the number of particles leaving the control volume: although the Ancey 2008 work essentially outlines this possibiilty. The transport rate at a point should be considered as the number of emigration events in a unit time.

Heyman et al 2013 and Ma et al 2014 went on to consider the statistics of emigration: they tried to discern what could be learned about the transport rate at a fixed point from the control volume formalism of Ancey et al and Einstein before that. Heyman et al was concerned with the statistics of the waiting time between successive emigration events. The original Einstein (1937, 1950) assumptions generate an exponential waiting time ditsribution with a timescale related to the entrainment rate. However, Heyman et al (2013) showed that when collective entrainment effects were included there are two timescales: one fast timescale related to the individual entrainment rate, and one slower timescale related to collective entrainment effects. Thus, the distribution of waiting times between successive emigration events, which is a useful concept for alternative stochastic models of the bedload flux distribution [e.g. *Turowski*, 2010], is a more subtle object if collective entrainment occurs. These effects are accentuated at low transport rates, where the fast and slow timescales are more disparate. At high rates, the timescales become comparable and the waiting time between successive emigration events blends into an Einstein-like exponential.

All of these approaches considered bedload transport within a control volume or at a point, but indeed, a very large set of contemporary river science studies are concerned with the diffusion or spreading of bed material through a downstream reach of river [**?**]. Bedload diffusion is not captured by a model at a single location: often, it has been described using an advection diffusion equation. These advection diffusion equations follow from considerations of mass balance within river channels, but they were not derived from any underlying microstructural model until [*Ancey et al.*, 2014]. *Ancey et al.* [2014] extended the previous control volume model [*Ancey et al.*, 2008] to an array of control volumes, where emigration from the ith volume is immigration to the $i+1$th volume.

In an approach widely used in chemical physics [*Gardiner*, 1983] and ecology [*Bailey*, 1968], *Ancey et al.* [2015, ?] derived an advection diffusion equation from their microstructural model in a limit as the size of each control volume goes to zero while the number becomes infinite. Thus, in their interpretation, bedload diffusion emerges in the continuum limit of coupled birth-death immigration-emigration models. They rederived the famous Exner equation of bedload diffusion.

Taken together, this research exhibits a coherent progression from the original work of Einstein to a more mathematically and physically based set of more general approaches. Within these approaches the bedload flux is understood as a random quantity. The mean bedload flux and the magnitude of its fluctuations are modelled in an unambiguous way [*Ancey et al.*, 2006a, 2008] considering no limitations in sediment availability, and it was shown that collective effects are a necessary inclusion to properly describe fluctuations in bedload transport [*Ancey et al.*, 2008]. An extension to a more realistic situation of finite supply was developed [*Turowski*, 2009], and the relationships between these control volume based models and the definition of bedload flux as particles crossing a plane perpendicular to the flow were developed [*Heyman et al.*, 2013; **?**; *Ballio et al.*, 2014].

These approaches did not address spatial correlations or variations in bedload fluxes, although the problem of bedload diffusing through a reach of channel is of contemporary significance [**?**]. This extension was the most recent development in birth-death modelling of bedload [*Ancey et al.*, 2014, 2015]. By extending the *Ancey et al.* [2008] model to an array of adjacent control volumes, an advection-diffusion equation describing the concentration of bedload in motion was developed, providing the first connection between a microscale stochastic model of bedload transport with a macroscopic advection-diffusion formulation. *Heyman et al.* [2014] examined the spatial correlations between local bedload fluxes expressed by the *Ancey et al.* [2014] model. This is the state of the art of bedload flux models.

4.2 Assumptions, calibraiton problems, and unclear aspects: A criticism of the birth-death approach

First, the various glaring assumptions of the birth-death approaches are highlighted.

1. Models only consider one particle size, while natural streams exhibit wide distributions of sizes and bedload transport expresses a wide set of effects related to particle size segregation [*Wilcock and Crowe*, 2003; *Parker and Klingeman*, 1982; *Chen and Stone*, 2008].
2. There is no clear means to discriminate collective and individual entrainment processes within experiments.
3. There is only a one-way feedback in these models: bedload is considered subordinate to the fluid flow, and the effect of bedload transport back onto the fluid flow is considered negligible. In fact, there are measurable influences of bedload transport on the fluid phase []; it's not yet clear whether a one-way coupled scheme such as this can actually describe natural streams across a realistic range of conditions.
4. Collective entrainment is somewhat of a catch-all term with the physical mechanisms which may contribute to it unresolved. These may include collective entrianment, the formation and disintegration of particle clusters, interactions between the motion of grains of idfferent size, turbulent structure, local avalanche behavior, and bed form migration.
5. there have been no theories developed to compute the collective entrainment rate from any simplified mechanical model, although there is a very large set of work concerned with calculating the individual entrainment rate from considerations of fluid turbulence and random granular arrangement [*Einstein and El-Samni*, 1949; *Einstein*, 1950; *Grass*, 1970; *Paintal*, 1971; *Cheng et al.*, 1998; *Wu and Yang*, 2004; *Dey and Papanicolaou*, 2008; *Tregnaghi et al.*, 2012; *Dey et al.*, 2018]. Calculating the collective entrainment rate from underlying principles appears on the surface very difficult. If it is considered to stem from collisions of moving grains with stationary grains, the collective entrainment probability will be related to the probability of collisions with the granular bed. If it is considered to stem from granular avalanches, where coherent turbulent structures initiate collections or clusters of grains into motion simultaneously, the collective entrainment probability will

depend on the collective dynamics of the granular assembly – leading immediately into the murky physics of force balance within granular assemblies.

6. Einstein-like assume a clean division between rest and motion states, which is no doubt an idealization. Bedload transport makes a continuous transition from the idealized start-stop motions of Einstein-like models to a less idealized granular flow or creep, where all particles move together in a coorelated fluid-like flow [], meaning bedload models should only hold at relatively low mobility stages.
7. At the same time, the divison of motion into only two categories may be flawed. A wide collection of studies have highlighted different modes of motion. *Einstein* [1950] considered that bedload was particles moving "in a rolling, sliding, or saltating mode". What if rolling, sliding, and saltating modes of motion were treated independently? In an *Ancey et al.* [2006a] model of independent particles switching between states, this would require a four-state model. There are too many transition probabilities within such a model probably to calibrate the model from experiments. Again, in order to consider such a scenario we would need more knowledge of underlying mechanics.

Incorporating multiple grain sizes in birth-death models is possible, in principle. As a first approximation, it is easy to introduce multiple grain sizes without including interactions between grain sizes in the entrainment and deposition probabilities for each size fraction. However, spatial and temporal heterogeneity in the bed surface characteristics are a hallmark of bedload transport of gravel mixtures [*Hassan et al.*, 2008]. This leaves models such as *Ancey et al.* [2008] somewhat without basis when multiple grain sizes are considered. The concept of **?** is essential to include when multiple grain sizes are considered: the bed surface state determines which grains can entrain [e.g. *Wilcock and Crowe*, 2003; *Parker and Klingeman*, 1982], and presumably which grains can deposit, as well. The entrainment and deposition rates of each size fraction will need to depend on the bed surface state, and fractional transport will set up spatial hetereogenieties in the bed surface state, so that an *Ancey et al.* [2014] type model, incorporating the possiblity of spatial hetereogeneity, should be mixed with a *Turowski* [2009] type model to incorporate the effect of the bed surface state on the entrainment and deposition rates of each size fraction. This extension will not be easy, and it will introduce many undetermined parameters: to pin down this transport model of multiple interacting size fractions will take a lot of work.

These stochastic models are admittedly difficult to calibrate. Their calibration requires a large dataset which is prohibitive to measure within natural streams. Therefore, their applicability in real streams remains limited until the inputs of stochastic models can be computed from physical theories based upon practically measurable quantities. This quest for relationships to compute the inputs of stochastic models from measurable quantities has been called "stochastic closure" for the analogous problem in turbulence [*Heyman et al.*, 2016]. The difficult issues precluding the application of stochastic models to natural streams should not be downplayed: much more work is needed.

5 synthesis: three new birth death models

5.1 Two-dimensional bedload diffusion

5.2 A stochastic model of transport with bed elevation variations

5.3 Sediment doesn't deposit just anywhere

6 Conclusion: progress mining from other fields

eh

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