Joint stochastic theory of bedload transport and bed elevations: derivation of heavy-tailed resting times

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Key Points:

- We model fluvial bedload activity and local bed elevation as a two-species stochastic birth-death process.
- Computations show universal heavy-tailed power-law distributions of resting times for sediment undergoing burial with tail parameter $\alpha \approx 1.18$.
- We discuss implications for bedload diffusion and propose a new theoretical framework for fluvial morphodynamics.

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Abstract
A consensus has formed that fluvial bedload resting times lie on heavy-tailed statistical distributions which may result from sediment burial. However, due to observational difficulties, only a handful of experiments have resolved these distributions, and there have been few theoretical attempts to build understanding, leaving their generating mechanism and specific characteristics uncertain. In this work, we present a new theory describing bedload transport and bed elevation changes as a joint stochastic process and derive resting time distributions for sediment undergoing burial from the joint dynamics. Our theory implies heavy-tailed power-law distributions of resting times with tail behavior completely characterized by the mean erosion rate and its scaling with bed elevation changes. Obtained resting time distributions are remarkably independent of changes in bed elevation statistics linked to bedload fluctuations, and we hypothesize this may be a consequence of universal extremal properties of correlated random walks which being increasingly realized in physics.

1 Introduction

The majority of classic studies into fluvial sediment transport have attempted to relate the bulk downstream flux of bedload to characteristics of the hydraulic forcing (e.g. Yalin, 1972), yet the relevance of this approach to environmental problems is limited, as many contemporary issues require knowledge of the differences between motions of individual grains, and not just their average motion characteristics. For example, the export of contaminants from channels (e.g. Malmon, Reneau, Dunne, Katzman, & Drakos, 2005) and the morphological response of channels to ecological restoration efforts (e.g. Gaeuman, Stewart, Schmandt, & Pryor, 2017) or to changes in hydrology or sediment supply (e.g. Hassan & Bradley, 2017) is not determined by bulk bedload fluxes, highlighting individual motions of bedload as an important topic for geophysics research.

A significant complication is that individual grains transport within a noisy environment, with noise sources ranging across spatial and temporal scales from smaller scale fluid turbulence (Celik, Diplas, & Dancey, 2014) and variability in the arrangement of bed surface grains (Gordon, Carmichael, & Isackson, 1972), to larger scale channel morphology changes (Hassan & Bradley, 2017) and unsteady watershed hydrology (Phillips, Martin, & Jerolmack, 2013). As a result, the transport characteristics of individual grains are not deterministic (e.g. Einstein, 1937), even in the most controlled laboratory experiments (e.g. Böhm, Ancey, Frey, Reboud, & Ducottet, 2004; Charru, Mouilleron, & Eiff, 2004; Fathel, Furbish, & Schmeeckle, 2015; Heyman, Bohorquez, & Ancey, 2016).

In response to this, researchers have long considered probabilistic theories of individual motions based on random walk concepts, whereby bedload motions are approximated as alternating sequences of steps and rests, with step lengths and resting times treated as random variables drawn from statistical distributions (Bradley & Tucker, 2012; Einstein, 1937; Hassan, Church, & Schick, 1991; Nakagawa & Tsujimoto, 1976; Yano, 1969). In these theories, differences between the random motions of one grain and the next imply bedload diffusion, or a spreading apart of grains through time. Over long timescales, the diffusion characteristics predicted by these models critically differ depending on whether the step length and resting time distributions have light or heavy tails (e.g. Bradley, 2017).

Heavy-tailed distributions have exceedance functions $P(X > x) \sim x^{-\alpha}$ with tail parameters $\alpha < 2$, meaning large values of x are relatively common, while light-tailed distributions have $\alpha \geq 2$, meaning large values of x are relatively rare. If both resting time and step distance distributions have light tails, the diffusion is said to be normal or Fickian, with a variance of particle positions σ_x^2 scaling with time t as $\sigma_x^2 \propto t$. However, if either distribution has a heavy-tail, the diffusion is called anomalous, with a variance of particle position scaling as $\sigma_x^2 \propto t^\gamma$, where $\gamma \neq 1$. In this expression, $\gamma < 1$ is called sub-diffusion and $\gamma > 1$ is super-diffusion. In strongly asymmetric random walks such as bedload transport, heavy-tailed step lengths imply super-diffusion, while heavy-tailed resting times imply either super or sub-diffusion, depending on α (Weeks & Swinney, 1998; Weeks, Urbach, & Swinney, 1996).

Tracer experiments in gravel bed rivers show anomalous bedload diffusion (Bradley, 2017; Phillips et al., 2013), light-tailed step lengths (Bradley & Tucker, 2012; Hassan, Voepel, Schumer, Parker, & Fraccarollo, 2013), and heavy-tailed resting times (Bradley, 2017; Olinde & Johnson, 2015; Pretzlav, 2016; Voepel, Schumer, & Hassan, 2013), forming a coherent experimental picture of super-diffusive bedload transport, at least at long observation timescales (e.g. Martin, Jerolmack, & Schumer, 2012; Nikora, 2002). However, field studies have not resolved the mechanism generating observed heavy-tailed resting times (e.g. Bradley, 2017), and empirical distributions display clear differences in their form and characteristics, with different tail parameters (e.g. Olinde & Johnson, 2015) and sometimes truncation (e.g. Bradley, 2017) or tempering to light tails at large resting times (e.g. Voepel et al., 2013). These differences and the mechanism generating heavy-tailed resting times deserve further research attention.

A predominant hypothesis is that heavy-tailed resting times and anomalous diffusion originate from sediment burial (Martin, Purohit, & Jerolmack, 2014; Voepel et al., 2013; Wu et al., 2019). Conceptually, when grains rest on the bed surface, material transported from upstream can deposit on top of them, preventing entrainment until it's removed, driving up resting times and imparting a heavy tail to the distribution. Martin et al. (2014) have provided the only direct support for this hypothesis. They traced grains in a narrow flume with clear sidewalls, directly resolving burial as the generator of heavy-tailed resting times, and they described their results with a theoretical model which is formally similar to an earlier effort by Voepel et al. (2013).

The models of Voepel et al. (2013) and Martin et al. (2014) consider bed elevations as a random walk and interpret resting times as return periods from above in the bed elevation time-series (e.g. Redner, 2007). Both models are successful in describing different experimental resting time distributions. However, the assumptions and results of these models are inconsistent with one another, and their treatment of bed elevations as a process independent of sediment transport is questionable at first glance, since the erosion and deposition of individual grains are the source of bed elevation changes (e.g. Wong, Parker, DeVries, Brown, & Burges, 2007).

In this work, we approach the problem from a different angle, making an extension of the stochastic bedload transport theory of Ancey, Davison, Böhm, Jodeau, and Frey (2008) to link bed elevation changes to the erosion and deposition events of individual grains, and we derive resting times as a consequence of this theory. The key assumptions of our model are: (1) bedload erosion and deposition can be characterized by probabilities per unit time, or rates (e.g. Ancey et al., 2008; Einstein, 1950); and (2) these rates are contingent on the local bed elevation, encoding the property that erosion of sediment is emphasized from regions of exposure, while deposition is emphasized in regions of shelter (e.g. Sawai, 1987; Wong et al., 2007). Our theory generates heavy-tailed distributions with no tempering and a universal tail parameter $\alpha \approx 1.18$ for a particular non-dimensionalization of the resting time, showing close correspondence to the findings of Martin et al. (2014) and suggesting a correction to some imperfections in their results. We conclude the paper by framing our work in relation to earlier ideas and discussing the implications of this work on open problems in individual bedload motions and anomalous diffusion.

2 Stochastic theory

We define a volume of downstream length L which contains some number n of moving particles in the water flow and some number m of stationary particles composing the bed at time t. For simplicity, we consider all particles as approximately spherical with the same diameter $2a$, so their mobility and packing characteristics are similar. Following Ancey et al.

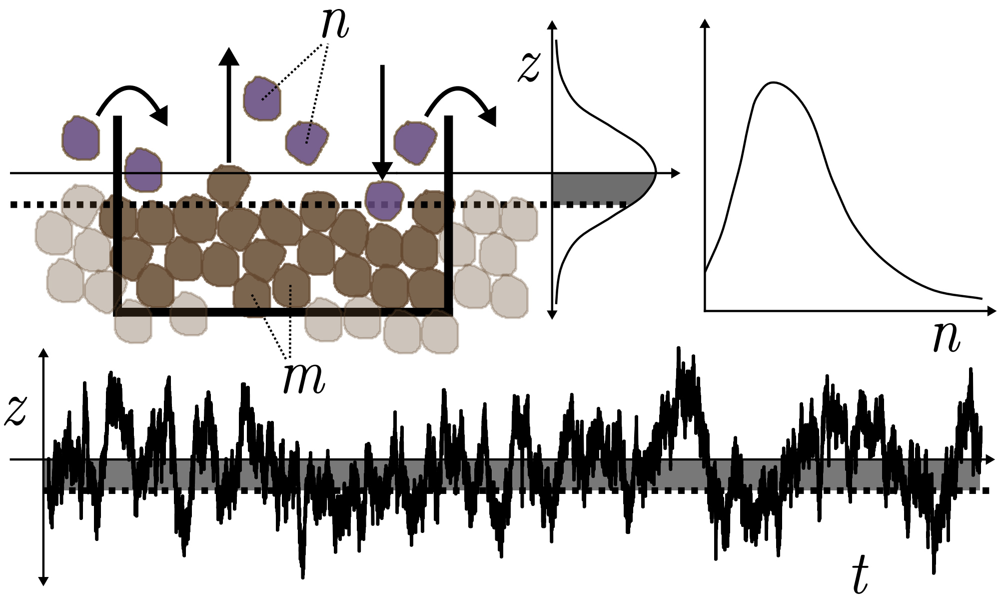

Figure 1. Definition sketch of a control volume containing n moving grains and m resting grains. Migration, entrainment, and deposition processes are represented by curved arrows, and the bed elevation at some instant is depicted by dotted line. The bed is presented in a degraded state, where $m < m_0$. The distributions of n and m are indicated in the upper right panel, while the bottom panel is a time-series of bed elevations (1).

(2008), we prescribe four events which can occur at any instant to modify the populations n and m, and we characterize these events using probabilities per unit time, or rates. These are: (1) migration of a moving particle into the volume from upstream ($n \to n+1$); (2) the entrainment (i.e., erosion) of a stationary particle into motion within the volume ($m \to m-1$ and $n \to n+1$); (3) the deposition of a moving particle to rest within the volume ($m \to m+1$ and $n \to n-1$); and (4) the migration of a moving particle out of the volume to downstream ($n \to n-1$). As the events occur at random intervals, they set up a joint stochastic evolution of the populations n and m characterized by a joint probability distribution $P(n, m, t)$ having marginals $P(n,t) = \sum_m P(n, m, t)$ and $P(m, t) = \sum_n P(n, m, t)$ for the number of particles in motion and at rest in the volume at t. These concepts are depicted in figure 1.

The populations n and m provide the bulk bedload flux q_s and the local bed elevation z. The mean bedload transport rate is given by $q_s \propto u_s \langle n \rangle$, where u_s is the characteristic velocity of moving bedload and $\langle n \rangle = \sum_{n,m} nP(n, m)$ is the mean number of grains in motion (e.g. Ancey et al., 2008; Charru et al., 2004; Furbish, Haff, Roseberry, & Schmeeckle, 2012). The bed elevation is related to m though the packing geometry of the bed. To derive this, we prescribe a mean number of grains at rest m_0 and introduce a packing fraction ϕ of grains in the bed (Torquato, 2018). Considering a two-dimensional bed (e.g. Einstein, 1950; Paintal, 1971), the deviation from the mean bed elevation is

$$z(m) = \frac{\pi a^2}{\phi L}(m - m_0) = z_1(m - m_0). \tag{1}$$

The constant $z_1 = \pi a^2/(\phi L)$ is an important scale of the problem. z_1 is the magnitude of bed elevation change (in an average sense across the control volume) associated with the addition or removal of a single grain. We write the rates of the four possible transitions as (e.g. Ancey et al., 2008):

$$R_{MI}(n+1,m|n,m) = \nu \qquad \text{migration in,} \tag{2}$$
$$R_E(n+1,m-1|n,m) = \lambda(m) + \mu(m)n \qquad \text{entrainment,} \tag{3}$$
$$R_D(n-1,m+1|n,m) = \sigma(m)n \qquad \text{deposition,} \tag{4}$$
$$R_{MO}(n-1,m|n,m) = \gamma n \qquad \text{migration out.} \tag{5}$$

These rates are independent of the past history of the populations and depend only on the current populations (n,m). As a result, the system is Markovian (e.g. Cox & Miller, 1965; van Kampen, 1992), meaning time intervals between subsequent transitions are exponentially distributed (e.g. Gillespie, 2007).

In (2-5), ν and γ are constants characterizing migration rates of individual grains into and out of the volume. They lack any dependence on the populations n and m. In contrast, $\lambda(m)$, $\mu(m)$, and $\sigma(m)$, characterizing the entrainment, collective entrainment (e.g. Ancey et al., 2008; Heyman, Ma, Mettra, & Ancey, 2014; Heyman, Mettra, Ma, & Ancey, 2013), and deposition rates of individual grains are considered to depend on m. As is well-known, bed elevation changes modify the likelihood of entrainment and deposition in a negative feedback (Sawai, 1987; Wong et al., 2007); that is, aggradation increases the likelihood of entrainment, while degradation increases the likelihood of deposition. Wong et al. (2007) concluded that bed elevation changes induce an exponential variation in entrainment and deposition probabilities, while Sawai (1987) concluded that the variation is linear. For simplicity, we incorporate the scaling of Sawai (1987) and note its equivalence to the Wong et al. (2007) scaling when bed elevation changes are small. Because experimental distributions of bed elevations are usually symmetrical, (Martin et al., 2014; Singh, Fienberg, Jerolmack, Marr, & Foufoula-Georgiou, 2009; Wong et al., 2007), we expect the erosion and deposition feedbacks to be anti-symmetrical. That is, as bed elevation changes drive up (down) erosion rates, so they drive down (up) deposition rates to the same degree.

Summarizing these ideas, the entrainment and deposition rates can be written $\chi(m) = \chi_0(1 \pm z_1 z(m)/(2l)^2)$, where $\chi = \lambda, \mu, \sigma$, and the entrainment parameters take the plus sign, while deposition takes the minus, and we have introduced a length scale l. As we'll see, the variance of bed elevation turns out to be given by $\text{var}(z) = (l/z_1)^2$. Accordingly, l characterizes the range of bed elevation variations, which could be interpreted as the active layer depth (e.g. Church, 2017). Another perspective is that l is the distance of bed elevation change at which the entrainment and deposition rates are significantly affected. With these substitutions, the local bed elevation-dependent entrainment and deposition rates (3-4) can be written:

$$R_E(n+1,m-1|n,m) = [\lambda_0 + \mu_0 n]\Big[1 + \frac{z_1 z(m)}{(2l)^2}\Big], \qquad \text{entrainment,} \tag{6}$$
$$R_D(n-1,m+1|n,m) = \sigma_0\Big[1 - \frac{z_1 z(m)}{(2l)^2}\Big]n, \qquad \text{deposition.} \tag{7}$$

At $z(m) = 0$, the rates reduce to those of the Ancey et al. (2008) theory. Away from this elevation, entrainment and deposition are alternatively suppressed and accentuated depending on the sign of $z(m)$.

In terms of the transition rates (2-7), we can obtain the Master equation for the probability flow using the forward Kolmogorov equation $\partial P(n,m;t)/\partial t = \sum_{n',m'} R(n,m)P(n',m';t)$ (e.g. Ancey et al., 2008; Cox & Miller, 1965; Gillespie, 1992) as

$$\begin{aligned}\frac{\partial P}{\partial t}(n,m;t) = \nu P(n-1,m;t) + \{\lambda(m+1) + [n-1]\mu(m+1)\}P(n-1,m+1;t)\\ + [n+1]\sigma(m-1)P(n+1,m-1;t) + [n+1]\gamma P(n+1,m;t)\\ - \{\nu + \lambda(m) + n\mu(m) + n\sigma(m) + n\gamma\}P(n,m;t).\end{aligned} \tag{8}$$

The joint probability distribution $P(n, m; t)$ solving this equation will fully characterize the statistics of n and m. We anticipate that solutions will adjust from the initial conditions to a steady-state distribution $P_s(n, m)$, independent of time, if the constant factors in the transition rates are representative of steady bedload transport conditions. This Master equation describes a two-species stochastic birth-death model (e.g. Cox & Miller, 1965) of a type well-known in the population ecology literature (e.g. Pielou, 1977; Swift, 2002). In our context, the two species are the moving and stationary grains in the volume.

3 Numerical simulations

Unfortunately, (8) does not appear to admit an analytical solution (but see Swift (2002) for a standard method which fails in this case). The difficulty stems from the product terms between n and m. In response, we numerically simulate (8) using the Gillespie algorithm (Gillespie, 1977, 1992, 2007). The Gillespie algorithm leverages the defining property of a Markov process: when transition rates do not depend on the past, time intervals between transitions are exponentially distributed (e.g. Cox & Miller, 1965).

Therefore, to step the Markov process through a single transition, it's enough to draw a random value from the exponential distribution of transition intervals to determine the time of the next transition. Then, drawing another random value to choose the type of transition which occurs, using the relative probabilities (2-7), the transition can be enacted by shifting t, n and m by the appropriate values (i.e., entrainment is $m \to m - 1$ and $n \to n + 1$, and so on). This procedure can be iterated to form an exact realization of the stochastic process (e.g. Gillespie, 2007).

Table 1. Parameters from Ancey et al. (2008) experiments describing the rates of migration in, entrainment, deposition, and migration out when $z(m) = 0$. All units are s^{-1} (probability/time). In our model, bed elevation changes modulate these rates in accord with (2-7).

Flow	ν	λ_0	μ_0	σ_0	γ
(a)	5.45	6.59	3.74	4.67	0.77
(g)	7.74	8.42	4.34	4.95	0.56
(i)	15.56	22.07	3.56	4.52	0.68
(l)	15.52	14.64	4.32	4.77	0.48
(n)	15.45	24.49	3.64	4.21	0.36

In this way, we simulated 5 flow conditions with 10 different values of l taken across a range from $l = a$ (1 grain radius) to $l = 10a$ (10 grain radii). These values lie in the range exhibited in the majority of available experimental data (Martin et al., 2014; Singh et al., 2009; Wong et al., 2007). For the migration, entrainment, and deposition parameters at each flow condition $(\nu, \lambda_0, \mu_0, \sigma_0, \gamma)$, we used values measured by Ancey et al. (2008) in a series of flume experiments. These are summarized in table 1. Flow conditions are labeled (a), (g), and so on, roughly in order of increasing bedload flux (see Ancey et al. (2008) for more details). However, our conclusions are not dependent on these parameters. In all simulations, we take the packing fraction $\phi = 0.6$, a typical value for a pile of spheres (e.g. Bennett, 1972), and set $L = 22.5$cm and $a = 0.3$cm, in accord with the Ancey et al. (2008) experiments. Each simulation was run for 1500hrs of virtual time, a period selected to ensure convergence of the resting time statistics.

4 Results

Our simulations show noisy time-series of bedload activities and bed elevations (as seen in the bottom panel of figure 1). From our chosen initial conditions, all simulations show a rapid attainment of steady state conditions followed by a steady-state stochastic dynamics of n and m, supporting a time-independent joint distribution $P_s(n, m)$. We compute this joint distribution by counting occurrences of the states (n, m) in the simulated time series.

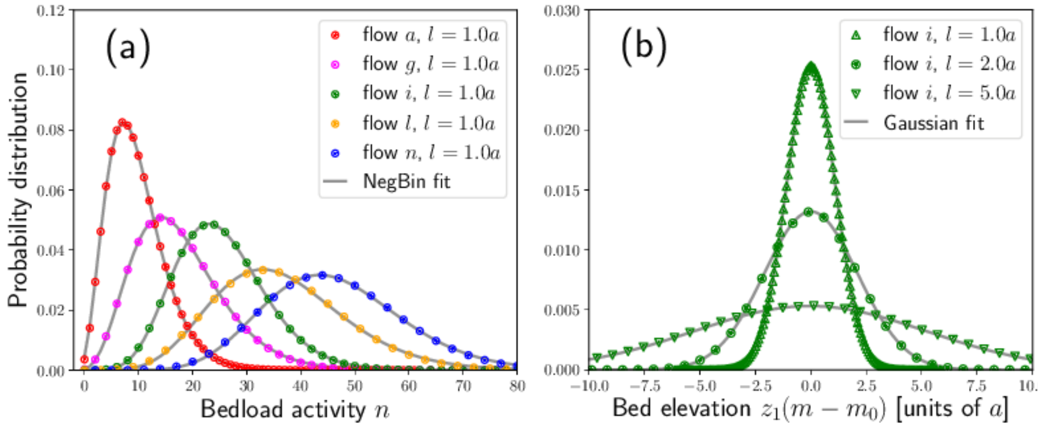

Figure 3. Marginal distributions of n and m for a subset of simulations. Some points have been omitted for clarity.

From this joint distribution we compute marginals $P(n)$ and $P(m)$ as explained in section 2.

Some of these marginal distributions are displayed in figure 3. Neglecting changes in bed elevation, Ancey et al. (2008) analytically derived negative binomial (NegBin) distributions for the bedload activity n, and this functional form seems preserved after including bed elevation changes, because all our computed distributions admit clean NegBin fits (figure 3a). For m, our computations show Gaussian distributions (figure 3b), consistent with our assumptions about the symmetric scaling of erosion and deposition rates with bed elevation changes (e.g. Wong et al., 2007).

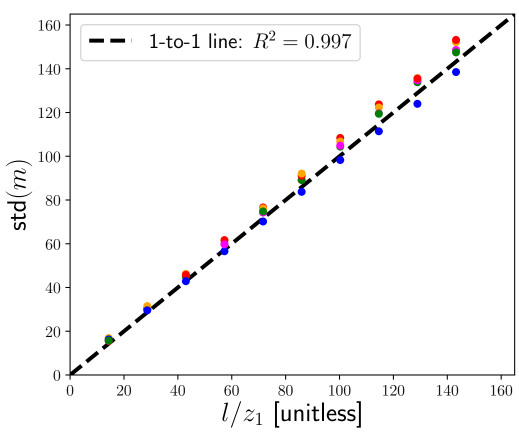

Figure 2. Data from all simulations is plotted to show that l controls deviations of bed elevations: $\mathrm{var}(m) = (l/z_1)^2$.

From the marginal distributions, we calculate means and variances of bedload activity (n) and elevation (m). The mean bed elevation is m_0, the parameter in (1). m fluctuates around this value because it sets the equilibrium position of the elevation-related feedbacks within (1). The variance of m seems given by $z_1^2 \mathrm{var}(m) = l^2$, as indicated in figure 2, consistent with our interpretation of l as a measure of bed elevation fluctuations. The moments of n shift with the ratio z_1/l as a result of the feedbacks between bed elevation changes and bedload transport. As $z_1/l \to 0$ the feedbacks are turned off so the Ancey et al. (2008) moments are reproduced, but for non-zero z_1/l, more complicated behavior emerges. The moments can be either suppressed or accentuated depending on the

Now we describe the analysis of bedload resting times from time-series of m. Following Voepel et al. (2013) and Martin et al. (2014), we concentrate on a particular bed elevation

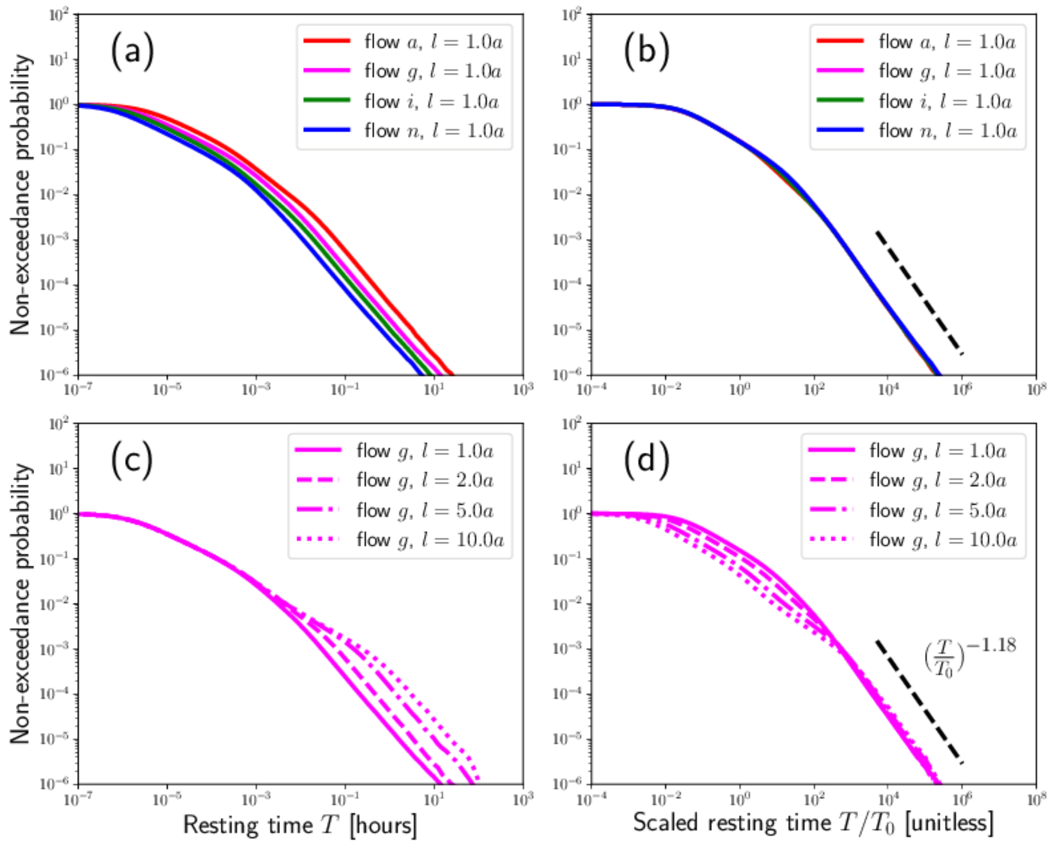

Figure 4. Resting time statistics scale differently with transport conditions and the bed elevation variance. Panel (a) shows differing flow conditions at a fixed l value, while panel (c) shows fixed flow conditions at differing l. When scaled by T_0 (10), both types of difference collapse in the tails of the distributions, as shown in panels (b) and (d). In panels (b) and (d), the black dotted lines indicate a power law decay of the collapsed tails having parameter $\alpha \approx 1.18$.

m', and find all time intervals separating deposition events at $m = m'$ from erosion events at $m = m' + 1$. These are the return times from above of the bed conditional to the elevation m'. Binning these conditional return times (with logarithmic bins) and counting the occurrences in each bin, we obtain a non-exceedance distribution of return times t_r held conditional to the elevation m': $P(T > t_r|m')$. Using the marginal probability distribution of bed elevations, we derive the unconditional non-exceedance distribution of resting times as a sum over all elevations (Martin et al., 2014; Nakagawa & Tsujimoto, 1980; Voepel et al., 2013; Yang & Sayre, 1971):

$$P(T > t_r) = \sum_{m'} P(m')P(T > t_r|m'). \tag{9}$$

Some of these results are displayed in figure 4. In contrast to earlier works our analysis does not require binning over the elevation, since our elevation series is discrete (multiples of z_1). Comparing panels (a) and (c) shows the resting time distributions scale with the intensity of bedload transport and the standard deviation of bed elevations (l) differently. However, as shown in panels (b) and (d), a characteristic timescale T_0 can be found to collapse away both of these differences.

We can obtain T_0 heuristically by finding a characteristic speed of bed elevation change. Formally, the mean erosion rate is $E = \sum_{n,m} R_E(n, m)P(n, m)$. This is the number of grains leaving the bed per unit time. Since the removal of a single grain changes the bed elevation by z_1, bed elevations change with a characteristic speed $z_1 E$. Since the characteristic deviation of elevation is l, the time required for the bed to shift through a characteristic deviation is

$$T_0 = \frac{l}{z_1 E}. \tag{10}$$

When scaling the resting time by this T_0, we obtain the collapse shown in figure 4. Using the log-likelihood estimation described by Newman (2005), we estimate for all return times satisfying $T/T_0 > 10^3$, the resting time non-exceedance distributions decay as a heavy-tailed power law with parameter $\alpha = 1.18 \pm 0.32$.

5 Discussion

Einstein (1937) created the first theory of individual bedload motions and bedload diffusion, and his ideas can be viewed as the historical nexus of an entire paradigm of research which extends to the present day (e.g. Ancey et al., 2008; Hassan et al., 1991; Hubbell & Sayre, 1964; Nakagawa & Tsujimoto, 1976; Wu et al., 2019). Works in this paradigm attempt to understand properties of bedload transport by applying stochastic concepts to individual sediment motions. With a few exceptions (e.g. Nakagawa & Tsujimoto, 1980; Wu et al., 2019; Yang & Sayre, 1971), existing theories are spatially one-dimensional, concentrating on the motion of grains in the downstream direction but neglecting the vertical dimension wherein local bed elevation changes imply sediment burial (e.g. Martin et al., 2014; Voepel et al., 2013) and modify the mobility of surface grains (e.g. Nakagawa & Tsujimoto, 1980; Yang & Sayre, 1971).

In this work, we've included this vertical dimension into the earlier bedload transport theory of Ancey et al. (2008) to create a new joint description of bedload transport and bed elevation changes. We find negative binomial distributions of bedload activity and normal distributions of bed elevations, reproducing a wide set of experimental findings (Ancey et al., 2008; Heyman et al., 2016; Martin et al., 2014; Singh et al., 2009; Wong et al., 2007). More importantly, interpreting resting times of sediment undergoing burial as return times from above in the bed elevation time series (e.g. Martin et al., 2014; Voepel et al., 2013), we've predicted the form and characteristics of this distribution, which is otherwise poorly understood, difficult to measure, and important for bedload diffusion (e.g. Bradley, 2017; Martin et al., 2014; Voepel et al., 2013). We now discuss the relationship of our work to earlier theory, highlight the key implications of our findings, and suggest shortcomings and possible extensions of our approach.

Our joint theory reproduces the descriptions of bedload activities by Ancey et al. (2008) and bed elevations by Martin et al. (2014) in particular limits of (8). The Ancey et al. (2008) bedload theory is obtained when bed elevation fluctuations δm are small: $m \approx m_0$. Taking account of this change in (8) derives the master equation of Ancey et al. (2008) for the bedload activity distribution $P(n,t)$. Similarly, the Martin et al. (2014) bed elevation theory is obtained when bedload activity fluctuations δn are small: $n \approx \langle n \rangle$. In this case, identifying the mean entrainment and deposition rates as $E = \lambda_0 + \mu_0 \langle n \rangle$ and $D = \sigma_0 \langle n \rangle$, and using the steady-state transport condition $E = D$ (e.g. Einstein, 1950) provides

$$\frac{\partial}{\partial t} P(m,t) = E \Big\{ \Big[1 + \Big(\frac{z_1}{2l} \Big)^2 m \Big] P(m+1,t) + \Big[1 - \Big(\frac{z_1}{2l} \Big)^2 m \Big] P(m-1,t) - 2P(m,t) \Big\}, \tag{11}$$

This is a discrete state analogue of the mean-reverting random walk used by Martin et al. (2014) to model bed elevation changes. In general, the behavior of our joint theory is richer than either of the Ancey et al. (2008) and Martin et al. (2014) theories because of the feedbacks we've incorporated between local bed elevations and transport. However,

Following Voepel et al. (2013) and Martin et al. (2014) to interpret resting times of sediment undergoing burial as return times from above in the bed elevation time series, Incorporating feedbacks between local bed elevations and the instantaneous erosion and deposition rates (e.g. Wong et al., 2007) into the stochastic theory of bedload transport by Ancey et al. (2008), we've obtained We've also provided a new theoretical framework to study the resting time distributions for sediment undergoing burial, which are poorly understood, difficult to measure, and important for bedload diffusion

This theory provides a new theoretical prediction of resting time distributions for sediment undergoing burial and provides a base for further developments in the Einstein paradigm which include a vertical dimension of bed elevation dynamics.

As we'll discuss, the key implication of this work is that sediment burial is capable of driving long-term super-diffusion in bedload transport.

Differences between our joint theory and the independent treatments of

The computed resting time distributions (figure 4) show asymptotic power law tails with parameter $\alpha = 1.18 \pm 0.32$ after scaling by T_0. For a general power law, if $\alpha > 1$, neither the mean or variance of the resting time will converge, while if $1 < \alpha < 2$, the mean will converge while the variance will not (e.g. Bradley, 2017). Within our computational uncertainty, which stems from the finite duration of our simulations and the log-likelihood estimation of α (e.g. Newman, 2005), we are unable to conclude whether the mean resting time will diverge, but we can conclude the variance will diverge. According to Weeks and Swinney (1998), if the step length distribution has a light tail (e.g. Hassan et al., 2013), our computed power-law resting times are sufficiently heavy-tailed to imply diffusion scaling as either $\sigma_x^2 \propto t^{3-\alpha} \approx t^{\{1.82\pm0.32\}}$ or $\sigma_x^2 \propto t^{2\alpha} \approx t^{\{3.64\pm0.45\}}$ at asymptotically large times. In either case, the process of sediment burial induces an extreme super-diffusion of bedload: at long timescales, some grains will continue to transport downstream in alternate motion-rest sequences while others will become trapped under the bed for relatively long periods of time, driving a rapid spreading of the population.

Finally, we propose a possible extension of the joint theory (8) by following its connection to Ancey et al. (2008) and follow-ups (e.g. Ancey & Heyman, 2014; Heyman, Bohórquez, & Ancey, 2015; Heyman et al., 2014). These works are based on chaining many Ancey et al. (2008) single-cell models together along a line, with migration out of one cell being migration into another. In this way, they provide a framework to study spatial correlations in bedload transport (e.g. Heyman et al., 2015, 2014). Similar approaches have been used to formulate reaction-diffusion and flow problems in stochastic physics (e.g. Gardiner, 1983). One can imagine using the model (8) in the same way, chaining an array of volumes (as in figure 1) together along a line to generate a fluvial morphodynamics theory which is ultimately rooted in a stochastic concept of individual motions. Such a theory could provide spatial correlations in bed elevation changes and bedload transport while taking account of their inherent granularity. Given the increasing realization that granular physics phenomena initiated by individual grains, such as avalanches and jamming, play a non-negligible role in fluvial processes (e.g. Dhont & Ancey, 2018; Saletti, 2016), we speculate such a theory, if suitably extended, might provide unique traction on future research problems centered around processes initiated by individual grains.

6 Conclusion

We predict asymptotic heavy-tailed power law resting times with parameter $\alpha = 1.18 \pm 0.32$ as a result of sediment burial. The tails of these distributions show near-perfect collapse upon scaling resting times by the factor $T_0 = l/(z_1 E)$, a characteristic timescale of bed elevation change, and tails occur for $T/T_0 > 10^3$.

Many open questions remain around the problem of anomalous bedload diffusion. In this paper, we have made incremental steps forward, by framing sediment burial directly in terms of the sediment transport process. When burial is a dominant process affecting the motions of individual grains, our results should serve as a prototype of this process, as they're derived under the assumptions of uniform grain size and equilibrium sediment transport in the spirit of Einstein (1950). We predict anomalous bedload diffusion $\sigma_x^2 \propto t^{1.85}$ and a convergent mean, corroborating a subset of earlier studies. Most directly, we have derived the Martin et al. (2014) model as an approximate limit of our more general theory, while we found no connection to the Voepel et al. (2013) model. In closing, we highlight that other return processes in nature could impart heavy tails to fluvial bed material resting times (e.g. Bradley, 2017). To truly understand anomalous diffusion, we need to pin down these processes, quantify their relative importance, and learn to mix their effects.

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