Non-power-law-scale properties of rainfall in space and time

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Received 18 November 2004; revised 11 May 2005; accepted 18 May 2005; published 17 August 2005.

[1] The identification of general relationships linking statistical properties of rainfall aggregated at different temporal and spatial scales possesses clear theoretical and practical relevance. Among other properties it is important to characterize the scale dependence of rainfall variability, which may, for many purposes, be summarized by its variance. This paper presents theoretical results and observational analyses connecting the variance of temporal and spatial rainfall to the scale of observation under an assumption of second-order stationarity. Previous results regarding the theoretical form of the temporal variance as a function of aggregation are refined and extended to the spatial case. It is shown that the variance of aggregated rainfall exhibits an overall nonscaling form within the range of scales of usual hydrological interest. It is further shown that the theoretically derived form of the variance as a function of scale is incompatible with power law relations usually assumed for the second-order statistical moment within multiscaling approaches to rainfall modeling. Predictions of the theoretical derivations are validated by use of observations representing a wide variety of resolutions (from 2 s to 1 hour in time and 4 km in space), climates, and measurement instruments. The validation shows a good agreement between observations and the theoretically predicted forms of the variance as a function of aggregation. The proposed theoretical framework thus provides a useful context for rainfall analysis, modeling, and downscaling in space and time and suggests careful reexaminations of usual multiscaling assumptions.

Citation: Marani, M. (2005), Non-power-law-scale properties of rainfall in space and time, *Water Resour. Res.*, *41*, W08413, doi:10.1029/2004WR003822.

1. Introduction

[2] The identification of a general space-time statistical structure of rainfall is an issue of central importance in the hydrological and atmospheric sciences and is the subject of a wide literature [e.g., *Austin and Houze*, 1972; *Zawadzki*, 1973; *Rodriguez-Iturbe et al.*, 1984, 1998; *Lovejoy and Mandelbrot*, 1985; *Schertzer and Lovejoy*, 1987; *Crane*, 1990; *Gupta and Waymire*, 1993; *Over and Gupta*, 1996]. The aspiration to a unified theoretical description of rainfall properties observed at different spatial/temporal scales is in particular driven by the wide range of scales of hydrological/climatological interest. The need of linking rainfall statistical properties at different scales of aggregation has, in the last twenty years, found answers in the theory of multifractal processes through which much of the recent observational work is interpreted [e.g., *Lovejoy and Schertzer*, 1985; *Gupta and Waymire*, 1990; *Fraedrich and Larnder*, 1993; *Tessier et al.*, 1993; *Davis et al.*, 1994; *Georgakakos et al.*, 1994; *Olsson*, 1995; *Fabry*, 1996; *Onof et al.*, 1996; *Menabde et al.*, 1997; *de Lima and Grasman*, 1999; *Deidda*, 2000; *Veneziano and Furcolo*, 2002; *Deidda et al.*, 2004]. Correspondingly, an entire family of rainfall models exhibiting fractal and multifractal properties has been developed [e.g., *Lovejoy and Mandelbrot*, 1985; *Gupta and Waymire*, 1993; *Over and Gupta*, 1996; *Schertzer and Lovejoy*, 1987; *Schmitt et al.*, 1998; *Veneziano and Iacobellis*, 2002], complementing the more traditional family of stochastic models based on occurrence processes mimicking physical properties of storms and rainfall cells [e.g., *Katz*, 1977; *Waymire and Gupta*, 1981a, 1981b, 1981c; *Rodriguez-Iturbe et al.*, 1987, 1988; *Cox and Isham*, 1988; *Cowpertwait*, 1994; *Katz and Parlange*, 1995]. One of the most appealing characteristics of the multifractal framework is probably the simplicity of the relationship postulated to exist between statistical moments computed at different scales of aggregation. A power law dependence of all statistical moments on the scale of aggregation is assumed to hold, which allows the straightforward connection of rainfall statistics between any aggregation scales, e.g., for downscaling purposes. *Marani* [2003] proposes an alternative approach, which, rather than postulating a power law form for all statistical moments of aggregated rainfall, assumes second-order stationarity of the continuous process, which, aggregated by observation procedures, generates measured rainfall in space and/or time. This approach allows the derivation of simple, yet non-power law, scale relations between the variance of temporal rainfall at different observation scales, showing that, under the above assumptions, scaling regimes (i.e., power law) are unlikely to be of interest for their asymptotic nature and for being confined to very large scales of aggregation. *Marani* [2003] further suggested

0043-1397/05/2004WR003822$09.00

that the interval of observational scales of hydrological interest is characterized by a transition regime in which the non-power law dependence of variance on scale can be explicitly derived when a form autocorrelation function of the continuous rainfall process is specified. In support and extension of the results of *Marani* [2003] this paper presents (1) an extensive validation of the theory based on observations from a wide variety of locations, climates, and observation instruments, (2) a more robust identification of the limits of the regimes characterized by different forms of scale relations, (3) a derivation and discussion of the links between the proposed theory and the formalism of the fractal and multifractal framework, illustrating possible incompatibilities, and (4) an extension of the temporal theory to a spatial context with a validation based on radar rainfall estimates.

[3] The statistical properties of rainfall aggregated in space and time are analyzed in the following. Let $V(\mathbf{x}_j, t_k, L_1, L_2, T)$ be the rainfall volume falling over a domain $S_j = \{(x^{(1)}, x^{(2)}) : x_j^{(1)} < x^{(1)} < x_j^{(1)} + L_1; x_j^{(2)} < x^{(2)} < x_j^{(2)} + L_2\}$ during the time interval $[t_k, t_k + T]$:

$$V\left(\mathbf{x}_j, t_k, L_1, L_2, T\right) = \int_{S_j} d\mathbf{x} \int_{t_k}^{t_k+T} i(\mathbf{x}, t) dt \tag{1}$$

where $i(\mathbf{x}, t)$ is the local and instantaneous rainfall intensity (dimensions: *length*/*time*). Equation (1) defines quite generally any rainfall observation, which is necessarily aggregated both in space and time.

[4] To address the properties of "point" rainfall aggregated in time one may consider a fixed area in space, S_j (e.g., rain gauge area or the projection of the observation volume in the case of a radar instrument), and, by dividing as customary equation (1) by S_j, introduce the spatial average rainfall depth:

$$H(t_k, T) = \int_{t_k}^{t_k+T} i(t) dt \tag{2}$$

where

$$i(t) = \frac{1}{S_j} \int_{S_j} i(\mathbf{x}, t) d\mathbf{x} \tag{3}$$

is the local average rainfall intensity.

[5] Some observations are suggested to indicate that $H(t_k, T)$ is multiscaling [e.g., *Falconer*, 1990], i.e., to exhibit, over some range of aggregation scales, the property [e.g., *Lovejoy and Schertzer*, 1985; *Gupta and Waymire*, 1990; *Fraedrich and Larnder*, 1993; *Davis et al.*, 1994; *Fabry*, 1996]

$$E[H(t_k, T)^q] \propto T^{\nu(q)} \tag{4}$$

where the constant of proportionality is independent of t_k, $E[\ ]$ is the expectation operator and $\nu(q)$ a nonlinear function of q (when $\nu(q)$ is linear $H(t_k, T)$ is said to be *simple scaling*). Notice that equation (4) implies a form of stationarity (in the sense of the statistical moments [e.g., *Gupta and Waymire*, 1990]) of the stochastic process generating $H(t_k, T)$ as the scaling of qth moments does not depend on t_k.

[6] The discussion of rainfall spatial scaling properties may be based on the space-dependent rainfall depth defined by fixing the time interval $[t_k, t_k + T]$:

$$h(\mathbf{x}) = \int_{t_k}^{t_k+T} i(\mathbf{x}, t) dt \tag{5}$$

[7] The rainfall volume given by equation (1) now only depends on the spatial parameters. If, for simplicity, one takes $L_1 = L_2 = L$, equation (1) becomes

$$V\left(\mathbf{x}_j, L\right) = \int_{S_j} h(\mathbf{x}) d\mathbf{x} \tag{6}$$

[8] Some observations have been suggested to indicate that also $V(\mathbf{x}_j, L)$ is multiscaling, i.e., that

$$E\left[V\left(\mathbf{x}_j, L\right)^q\right] \propto L^{\chi(q)} \tag{7}$$

holds within a range of observational spatial scales [e.g., *Gupta and Waymire*, 1990; *Tessier et al.*, 1993; *Onof et al.*, 1996; *Deidda et al.*, 2004]. Notice that spatial homogeneity of $V(\mathbf{x}_j, L)$ is assumed in (7), because its scaling characters are not allowed to depend on $\mathbf{x}_j$. Isotropy is also implied, as there is no distinction between the two coordinate directions, though this assumption may be relaxed if necessary [e.g., *Schertzer and Lovejoy*, 1987].

[9] The scaling characters of point or spatial rainfall is sometimes studied through and alternative framework based on the definition of generalized structure functions [*Vainshtein et al.*, 1994], e.g., for purely temporal rainfall:

$$E[|i(t+\tau) - i(t)|^q] \propto \tau^{\zeta(q)} \tag{8}$$

Observed rainfall has been suggested to satisfy equation (8) as well as its spatial counterpart [e.g., *Ferraris et al.*, 2003]. The use of generalized structure functions is equivalent to the use of statistical moments, and exponents $\nu(q)$ and $\zeta(q)$ have been shown to be strictly related [*Menabde et al.*, 1997].

[10] The second-order moment of rainfall is easily related to the variance as a function of temporal or spatial aggregation, $\sigma_H^2(T) = E\{[H(t_k, T) - E[H(t_k, T)]]^2\} = E[H(t_k, T)^2] - E[H(t_k, T)]^2$, with an analogous relationship holding for the spatial case.

[11] Noting that, in the assumed stationarity hypothesis,

$$E[H(t_k, T)] = \int_{t_k}^{t_k+T} E[i(t)] dt = \mu_i \cdot T \tag{9}$$

where μ_i is the expected value of $i(t)$, one obtains

$$E\left[H(t_k, T)^2\right] = \sigma_H^2(T) + \mu_i^2 \cdot T^2 \tag{10}$$

linking the second moment and the variance of the process aggregated at different scales. Quite straightforwardly for the spatial case and under the isotropic hypothesis one finds

$$E\left[V\left(\mathbf{x}_j, L\right)^2\right] = \sigma_V^2(L) + \mu_h^2 \cdot L^4 \tag{11}$$

where $\sigma_V^2(L)$ is the variance of $V(\mathbf{x}_j, L)$ and μ_h is the expected value of $h(\mathbf{x})$. Relationships (10) and (11) will be useful later to determine the scaling (or nonscaling) properties of $H(t_k, T)$ and $V(\mathbf{x}_j, L)$.

2. Point Rainfall Aggregated in Time

2.1. Theoretical Analysis

[12] Under the hypothesis of second-order stationarity of $i(t)$ the temporal variance $\sigma_H^2(T)$ of $H(t_k, T)$ is a function of T only and may be expressed, by use of (1) and after some manipulations, as [*Vanmarke*, 1983; *Marani*, 2003]

$$\sigma_H^2(T) = 2\,\sigma_i^2 \int_0^T (T-\tau)\rho_i(\tau)d\tau \tag{12}$$

where σ_i and $\rho_i(\tau)$ are the standard deviation and autocorrelation function of $i(t)$ respectively. Notice that this second-order stationarity hypothesis is different from the stationarity hypothesis postulated to define scaling processes. Here the stationarity hypothesis only refers to the first and second-order moments of the continuous process $i(t)$, whereas equation (4) (or its structure function counterpart) requires the stationarity of all qth moments of the process aggregated at resolution T. If the integral scale [*Taylor*, 1921]

$$I = \int_0^\infty \rho_i(\tau)d\tau \tag{13}$$

exists (and thus $\rho_i(\tau)\cdot\tau \to 0$ when $\tau \to \infty$) one may see [e.g., *Marani*, 2003] that asymptotically $\sigma_H^2(T) \propto T$ for $T \to \infty$ and the process may be termed as having a *finite* memory. In this case, due to (10), it is $E[H(t_k, T)^2] \propto T^2$ for $T \to \infty$.

[13] When, on the contrary, I does not exist because $\rho_i(\tau) \xrightarrow{\tau\to\infty} \tau^{-\alpha}$ with $\alpha < 1$, one finds that, asymptotically, $\sigma_H^2(T) \to T^{2-\alpha}$ for $T \to \infty$ and the process may be said to have an infinite memory. The asymptotic behavior of $\sigma_H^2(T)$ is thus representative of the correlation properties of the stochastic process generating the observed rainfall, and in particular of its memory. This is a rather important property since a power law decay of the autocorrelation function implies long-term correlation and persistence (e.g., the Hurst phenomenon [*Hurst*, 1951; *Mandelbrot and Wallis*, 1968]). Interestingly, equation (10) indicates that, asymptotically, the second moment is again proportional to T^2 as in the finite memory case.

[14] Because the theoretical power law forms described above are only valid in the limit for large T, it is important to the determine the behavior of $\sigma_H^2(T)$ for "small" values of T, outside this asymptotic scaling regime. *Marani* [2003] derives the properties of $\sigma_H^2(T)$ near the origin by determining the Taylor expansion of (12) at $T = 0$. By differentiation,

$$\begin{aligned}\frac{d\sigma_H^2(T)}{dT} &= 2\sigma_i^2\int_0^T \rho(\tau)d\tau\\ \frac{d^2\sigma_H^2(T)}{dT^2} &= 2\sigma_i^2\rho(T)\\ \frac{d^3\sigma_H^2(T)}{dT^3} &= 2\sigma_i^2\frac{d\rho(T)}{dT}\end{aligned} \tag{14}$$

Given that (12) and the first expression in (14) are zero for $T = 0$, the following approximation is valid near the origin:

$$\sigma_H^2(T) = \sigma_i^2\,T^2 + \frac{1}{3}\,\sigma_i^2\,\rho'(0)\,T^3 + \ldots \tag{15}$$

(in fact $\rho(0) = 1$). Therefore, at the lowest order and independently of the correlation properties of $i(t)$ and of the existence of its integral scale, the variance of the aggregated process is a parabolic function of the length of the aggregation interval near the origin. It should also be noticed that $\rho(\tau)$ must be a nonincreasing function in $\tau = 0$ because the autocorrelation has a maximum there. Therefore it must be $\rho'(0) \le 0$ and (15) indicates that $\sigma_H^2(T)$ tends to a parabolic form from below when $T \to 0$. The region in which the parabolic approximation to $\sigma_H^2(T)$ holds will be termed "inner regime."

[15] It should be noticed that, due to (10) and (15), the second moment must be $E[H(t_k, T)^2] \propto (\sigma_i^2 + \mu_i^2)T^2$ for small values of T, irrespective of the correlation characteristics of the underlying continuous process.

[16] The analysis of the variance of the aggregated process thus shows that, because $\sigma_H^2(T)$ behaves as T^β, with $1 \le \beta < 2$, for large T and as T^2 for $T \to 0$, a transition regime must exist, linking the inner and the scaling regimes, in which $\sigma_H^2(T)$ is not a power law [*Marani*, 2003].

[17] This result has implications with reference to the possible power law form of the second statistical moment often assumed in analyses and modeling of rainfall. In fact, in view of the above discussion, $E[H(t_k, T)^2]$ must be proportional to T^2 both in the inner and in the asymptotic scaling regime. Nevertheless, this does not imply that $E[H(t_k, T)^2]$ must everywhere be quadratic in T because, in the transition regime, $\sigma_H^2(T)$ is no longer proportional to T^2, and its slope in a log-log plane, though not a constant, must be smaller than two. Therefore, from equation (10), $E[H(t_k, T)^2]$ is not a parabolic function in the transition regime and may here appear to exhibit a log-log slope between 1 and 2, before asymptotically returning to a quadratic function of T in the scaling regime. This circumstance may provide an explanation for the examples of nontrivial power law scaling of the second moment reported in the literature.

[18] This theoretical analysis indicates that downscaling procedures of rainfall time series, often using power law assumptions for statistical moments, may be inappropriate. If the transition regime is confined to very small aggregation scales (e.g., smaller than a few minutes), then the scaling regime will hold for the time scales of common hydrologic interest and power law assumptions for the variance may be valid. On the contrary, if the transition regime is located, say, at the hourly to daily time scales, one must conclude that power law assumptions for the variance are unsuitable to describe real precipitation processes. In either case, a power law may, at best, be an approximation for the second-order moment and the second-order structure function in their transition regime, as they must be quadratic in the inner and scaling regimes.

[19] It is interesting to note that also the $\sigma_H^2(T)$ functions obtained from stochastic models of rainfall necessarily exhibit distinct regimes as a function of the aggregation, with relevant consequences for model identification, cali-

Table 1. Characteristics of the Stations and of the Observation Periods Analyzed

Station	Source	Instrument	Δt, s	Duration, days	Period
Iowa City (Iowa, USA)	Iowa University	optical rain gauge	2	4	May 1996
Middle Wallop (UK)	NERC	hydragauge	15	23	May 1995
Lebanon (Indiana, USA)	NOAA	tipping bucket	900	31	1971–1992
Ashover (UK)	NERC	tipping bucket	3600	31	1983–1988
Kwajalein (Marshall Isl.)	TRMM	disdrometer	60	30	Sept 1999
Holopaw (Florida, USA)	TRMM	disdrometer	60	22	Aug 1998
Rondonia (Brazil)	TRMM	disdrometer	60	14	Feb 1999
Red River (Arkansas, USA)	NOAA	radar	3600	31	Dec 2003

bration and validation. For example, a model which produces multiscaling rainfall sequences also will exhibit inner, transition, and scaling regimes, with different characteristic exponents. On the contrary, stochastic models based on clustered Poissonian processes, which, by definition, are characterized by an exponential autocorrelation [e.g., *Rodriguez-Iturbe et al.*, 1987] and thus by $\sigma_H^2(T)$ asymptotically linear, may seem to exhibit a scaling exponent for $\sigma_H^2(T)$ between 1 and 2 due to the presence of the transition and inner regimes.

[20] Because of the contrasting properties of the second-order moments (or variance or structure functions) derived under the hypotheses of *Marani* [2003] and within the multifractal framework, it is worth to recall the differences in the assumptions on which the two approaches are based. Hence, while multifractality does not require the variance σ_i^2 to exist, it does require equation (4) to hold for all values of q. The assumptions underlying the two frameworks are thus different (and one can hardly be considered to be more general than the other) and their merits should be evaluated on the basis of their ability of interpreting the observations.

2.2. Observations

[21] *Marani* [2003] used rainfall observations with resolutions of 15 min and 1 hour at four different sites and found the boundary between transition and scaling regimes to be located, depending on site and season, between T = 20 hours and T = 80 hours. In those cases the transition regime was thus found to cover a range of aggregation scales of common hydrologic interest, but the position of the inner regime remained undetermined, due to the relatively coarse resolution of the time series used. In order to obtain a detailed validation of the theoretical results described in the previous sections, they are here tested using rainfall observations representing a wider variety of time resolutions, climatic properties and measurement techniques. Sampling different climatic regimes allows to address the possible impact of different rainfall correlation properties on the results. High time resolutions allow the exploration of rainfall properties for small aggregation intervals, possibly within the inner regime. The use of data acquired through different techniques is important to dissipate possible doubts regarding spurious effects induced by the measuring instrument [e.g., *Fraedrich and Larnder*, 1993; *de Lima and Grasman*, 1999].

[22] The main characteristics of the time series analyzed are described in Table 1, while the geographic locations of the stations and the relative mean monthly rainfall values are summarized in Figure 1. The set of instruments includes traditional devices, such as tipping bucket gauges and a C band meteorological radar, and less standard ones, such as optical rain gauges, the hydragauge, and disdrometers.

[23] The tipping bucket time series considered are extracted from observations at Ashover (UK, from NERC-BADC database, 1 hour resolution) and Lebanon (Indiana, USA, from the NCDC database, 15-min resolution).

[24] The optical rain gauge measures precipitation on the basis of the scintillation induced by rain drops falling through an infrared beam. Rainfall intensity is estimated from scintillation intensity with an accuracy of ±5%, a maximum discretization of 0.001 mm and maximum time resolution of 2 s (http://www.isro.org/nmrf/org.htm). The observations analyzed here were performed at Iowa City (Iowa, USA), with a 2-s resolution.

[25] The Hydragauge is a device which converts the collected rainfall into a series of constant dimension water drops, which cause the closure of an electric circuit allowing counting of the drops and thus the computation of rainfall intensity. The maximum time resolution is 15 s and the maximum measurable intensity is 200 mm/h (http://badc.nerc.ac.uk/data/hyrex/refs.html). The time series considered was observed at Middle Wallop (UK, NERC-BADC database, 15-s resolution).

[26] The disdrometer, or Joss-Waldvogel disdrometer, transforms the vertical momentum of drops impacting the surface of the device into an electric pulse whose amplitude is proportional to the drop diameter. The instrument then infers the frequency distribution of drops hitting it during a fixed time interval and, by estimating their fall velocity, derives rainfall intensity. The accuracy is ±15% and the maximum temporal resolution is 10 s (www.distromet.com). The observations used here were performed at Kwajalein (Marshall Islands), Holopaw (Florida, USA) and Rondonia (Brazil, all data were produced under the TRMM NASA/JAXA Mission, 1-min resolution).

[27] Analyses of temporal rainfall were also performed on meteorological radar observations. Temporal statistics are in the following computed on hourly rainfall estimates in the period 1997–2003 for a randomly selected pixel (of size 4 km by 4 km) within a radar field of size 1340 km by 636 km covering the Arkansas Red River basin (USA) Arkansas (USA, http://www.srh.noaa.gov/abrfc/).

2.3. Data Analysis

[28] The theoretical framework described above is based on a temporal stationarity assumption. Correspondingly, and similar to the approach of *Marani* [2003], time series with maximum length of 31 days will be considered in the analyses to assure that seasonal nonstationarities are excluded. Possible effects of daily cycles will be evalu-

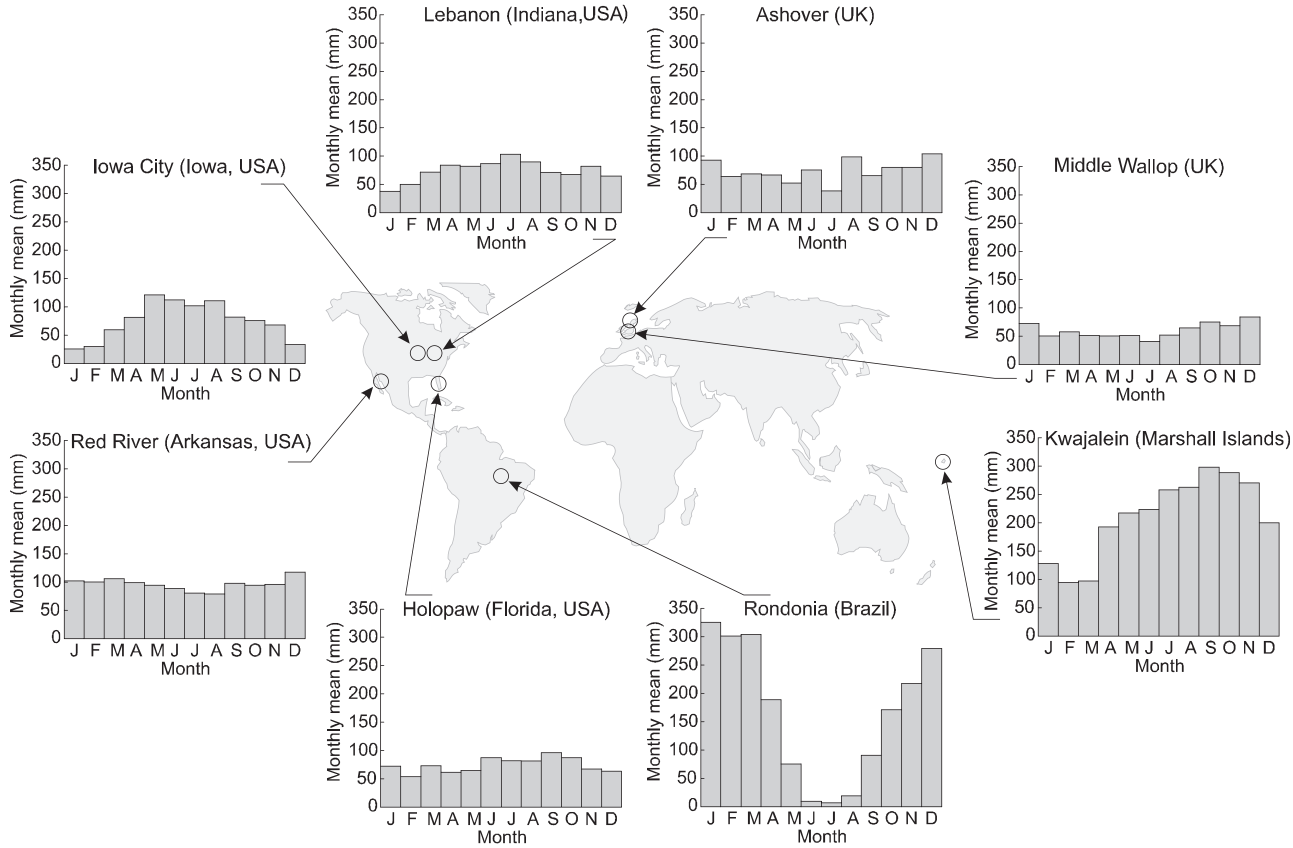

Figure 1. Location of the observation sites considered. The different climatic characteristics of the sites are summarized by the observed mean monthly rainfall values.

ated on the basis of the correspondence between theory and observations. The experimental form of $\sigma_H^2(T)$ is obtained by aggregating rainfall time series over intervals of increasing length T and by computing sample variance for each aggregation level. The results of this procedure applied to the data described in Figure 1 and Table 1 are shown in the doubly logarithmic plot of Figure 2, which clearly reveals the presence of different regimes. The tendency of all the experimental $\sigma_H^2(T)$ functions toward a quadratic law for small values of T, as predicted by equation (15), is evidenced by the reference lines with slope equal to 2 (which represents, in a log-log plot, a power law with exponent equal to 2). Such tendency is more clearly evident for data series having higher time resolutions, and indicates that the boundary between the inner and the transition regime may be located between about $T = 10 \div 15$ min in all cases. A scaling regime may also be identified at the largest aggregation scales whose lower boundary location is consistent with previous estimates of $T = 20 \div 80$ hours [*Marani*, 2003]. Therefore, in the wide set of different locations, climates and observation devices considered, the transition regime is located at the temporal scales of usual hydrologic interest. Notice that the shape of the variance curve in the transition regime is not always ‘‘smooth’’ as it depends on the shape of the autocorrelation function characterizing rainfall at each specific site. Interestingly, possible nonstationarities, e.g., due to daily cycles, in all cases do not seem to affect the shape of the variance as a function of aggregation in the inner regime nor the existence of the scaling regime. These evidences corroborate the theory and seem to indicate a more general validity of its predictions with respect to the stationarity hypothesis assumed here to derive it.

[29] The theoretical predictions for the second-order moment as a function of scale also appear to be confirmed by Figure 3. The second-order moments of observed time series indeed show a quadratic inner regime, a non-power law transition regime and, in most cases, a tendency toward an again quadratic ‘‘scaling’’ regime, as predicted by the discussion in section 2 and by equation (10). Interestingly, these observational results are consistent with previously unnoticed features of experimental second-order moments from existing literature [*Onof et al.*, 1996], which will be addressed in section 4.

[30] It must therefore be concluded that the above theoretical and observational analysis does not support the notion of a power law second-order moment (as expressed by (4) with $q = 2$) nor, by virtue of the equivalence of the two approaches [*Menabde et al.*, 1997], of the second-order generalized structure function (equation (8) with $q = 2$). This finding suggests a careful reconsideration of the general validity of multiscaling assumptions in temporal rainfall.

3. Space-Aggregated Rainfall

3.1. Theoretical Analysis

[31] In the hypothesis of spatial homogeneity of the stochastic field $h(\mathbf{x})$ [e.g., *Zawadzki*, 1973; *Crane*, 1990; *Barancourt et al.*, 1992] the spatial variance, say $\sigma_V^2(L_1, L_2)$,

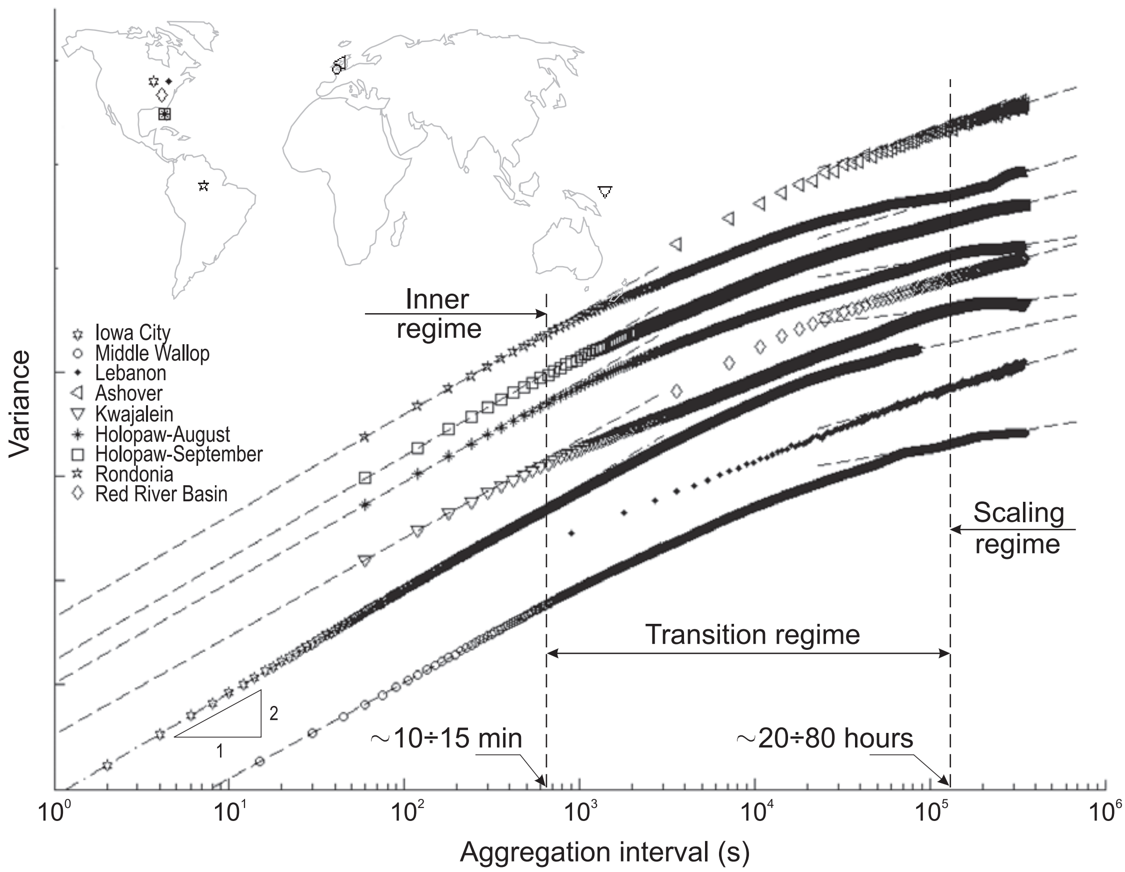

Figure 2. Experimental variance as a function of aggregation for the set of time series analyzed. Straight lines representing different regimes in these log-log plots were obtained by fitting power laws on subsets of the observations. The data points in the subsets were selected such that the resulting R^2 was greater than 0.9. Note that the values of observational variances were arbitrarily shifted vertically in order to avoid overlapping of the data series.

of $V(\mathbf{x}_j, L_1, L_2)$ (see equation (1)) may be expressed, similarly to (12), as

$$\sigma_V^2(L_1, L_2) = 4\,\sigma_h^2 \int_0^{L_1} ds_1 \int_0^{L_2} ds_2 (L_1 - s_1)(L_2 - s_2)\rho_h(s_1, s_2) \tag{16}$$

where σ_h^2 and $\rho_h(s_1, s_2)$ are variance and autocorrelation function of $h(\mathbf{x})$.

[32] In order to determine the behavior of $\sigma_V^2(L_1, L_2)$ near the origin one may study the Taylor expansion of (16) around ($L_1 = 0$, $L_2 = 0$). The calculation of the first terms of the expansion shows that all the partial derivatives of $\sigma_V^2(L_1, L_2)$ with respect to L_1 and L_2 up to the third order (included) are equal to zero in ($L_1 = 0$, $L_2 = 0$). The leading term in the expansion is therefore

$$\left.\frac{\partial^4 \sigma^2}{\partial L_1^2 \partial L_2^2}\right|_{(0,0)} = 4\,\sigma_h^2 \rho_h(0, 0) \tag{17}$$

Near the origin the following approximation is thus valid:

$$\sigma_V^2(L_1, L_2) \cong \sigma_h^2 L_1^2 L_2^2 \tag{18}$$

as $\rho_h(0, 0) = 1$.

[33] To determine an approximation to $\sigma_V^2(L_1, L_2)$ when $L_1, L_2 \to \infty$, one may notice that, in this limit,

$$\begin{aligned} \int_0^{L_1} ds_1 \int_0^{L_2} ds_2 (L_1 - s_1)(L_2 - s_2)\rho_h(s_1, s_2) &= \int_0^{L_{01}} ds_1 \int_0^{L_{02}} ds_2 \\ \cdot (L_1 - s_1)(L_2 - s_2)\rho_h(s_1, s_2) &+ \int_{L_{01}}^{L_1} ds_1 \int_{L_{02}}^{L_2} ds_2 (L_1 - s_1) \\ \cdot (L_2 - s_2)\rho_h(s_1, s_2) \cong const. &+ \int_{L_{01}}^{L_1} ds_1 \int_{L_{02}}^{L_2} ds_2 L_1 L_2 \rho_h(s_1, s_2) \end{aligned} \tag{19}$$

where L_{01} and L_{02} are two large, but finite, values of L_1, L_2. The approximate equality in (19) is justified by observing that $(L_1 - s_1)(L_2 - s_2) \cong L_1 L_2$ when $L_1, L_2 \gg s_1, s_2$ and that $\rho_h(s_1, s_2) \to 0$ when $s_1, s_2 \to L_1, L_2 \to \infty$.

[34] If the spatial integral scale, I_S, defined as

$$I_S = \int_0^{\infty} ds_1 \int_0^{\infty} ds_2 \rho_h(s_1, s_2) \tag{20}$$

exists (finite spatial memory), then the asymptotic form of $\sigma_V^2(L_1, L_2)$ for $L_1, L_2 \to \infty$ is

$$\sigma_V^2(L_1, L_2) \propto L_1^2 L_2^2 \int_{L_{01}}^{\infty} ds_1 \int_{L_{02}}^{\infty} ds_2 \rho_h(s_1, s_2) \propto L_1^2 L_2^2 \tag{21}$$

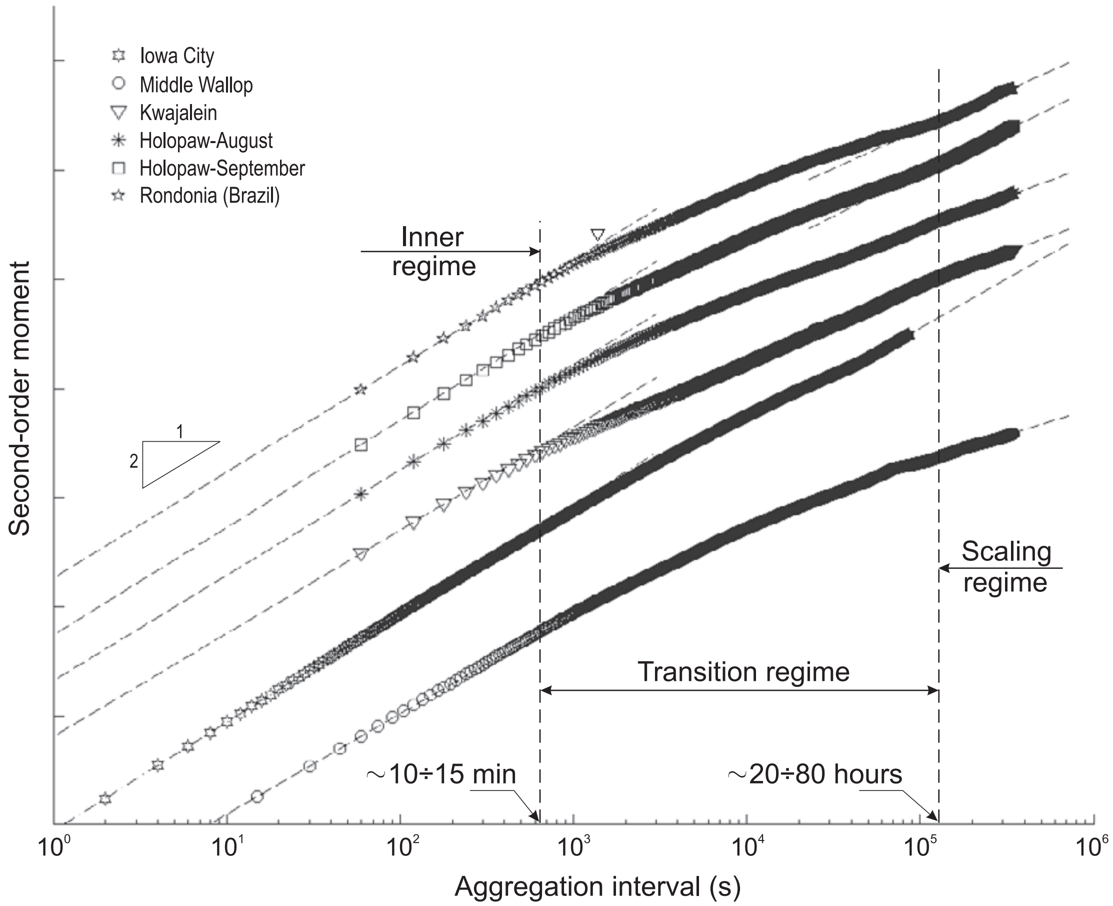

Figure 3. Experimental second-order moments as a function of aggregation. Only the highest-resolution data sets have here been considered because they allow the analysis of all three predicted regimes. Straight lines representing different regimes in these log-log plots were obtained by fitting power laws on subsets of the observations. The data points in the subsets were selected such that the resulting R^2 was greater than 0.9. Note that the values of observational variances were arbitrarily shifted vertically in order to avoid overlapping of the data series.

If, on the contrary, the integral scale of the process does not exist, because asymptotically $\rho_h(s_1, s_2) \propto s_1^{-\beta} s_2^{-\gamma}$ with $\beta, \gamma < 1$ (infinite spatial memory), equation (16) yields, for $L_1, L_2 \to \infty$, $\sigma_V^2(L_1, L_2) \propto L_1^{2-\beta} L_2^{2-\gamma}$.

[35] If the process generating $h(\mathbf{x})$ is assumed to be isotropic one may choose, for simplicity sake, $L_1 = L_2 = L$ and note that, because of isotropy, $\beta = \gamma$. Hence, in the case of an isotropic process and in the scaling regime $\sigma_V^2(L) \propto L^4$ for a finite memory process, while $\sigma_V^2(L) \propto L^{4-2\beta}$ for an infinite memory one. Therefore, also in the case of the spatial properties of precipitation, similarly to the case of its temporal properties, the variance of the process aggregated at different scales shows the presence of three distinct regimes: inner, transition, and scaling regime. The properties of the last two regimes are dependent on the specific correlation characters of the rainfall process, while those of the inner regime are of general validity under the assumptions made.

[36] The properties determined for $\sigma_V^2(L)$ have implications for the second-order moment in view of (11). In particular, and similarly to the temporal case, $E[V(\mathbf{x}_j, L)^2]$ must be a power law $\propto L^4$ in the inner regime, must exhibit a non-power law shape in the transition regime, and should again tend to a $\propto L^4$ form in the scaling regime.

[37] As noted above, the analyses described also have implications for scaling characterizations based on generalized structure functions because these are strictly equivalent to moment characterizations [*Vainshtein et al.*, 1994].

3.2. Data Analysis

[38] The spatial variance (16) was computed at different resolutions for the Arkansas Red River basin radar rainfall estimates (which are produced by composing observations from several radars in the area). The values of the spatial variance were computed for each hourly spatial distribution and for different aggregation scales between $L = 4$ km and $L = 80$ km. The results obtained for each aggregation scale were then ensemble-averaged over all available hourly intervals for each month of the year. The analyses were conducted separately for each month in order not to violate the assumption of stationarity and the ensemble estimates of the variance as a function of aggregation are represented in Figure 4. The experimental variance as a function of the aggregation interval, coherently with theory, shows a clear

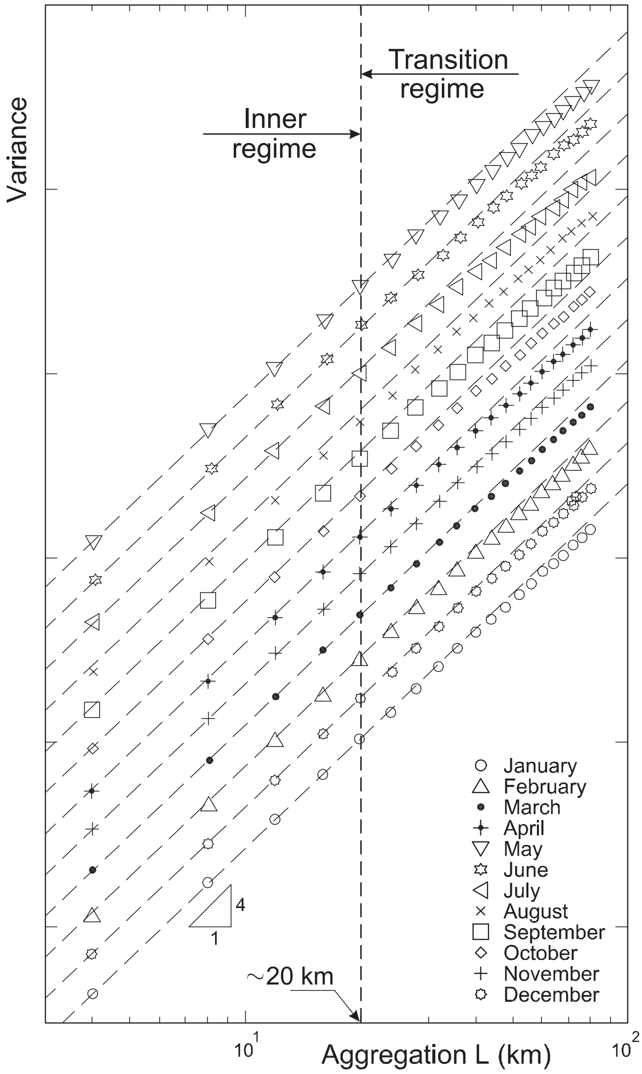

Figure 4. Spatial variance of Arkansas Red River Basin radar rainfall estimates as a function of aggregation scale. Variance values for different months are shifted vertically to improve readability. The straight lines representing the inner regime in these log-log plots were obtained by fitting power laws on subsets of the observations. The data points in the subsets were selected such that the resulting R^2 was greater than 0.9. Note that the values of observational variances were arbitrarily shifted vertically in order to avoid overlapping of the data series.

tendency to scale as L^4 for resolutions smaller than $L = 20 \div 30$ km, marking the boundary between the inner and the transition regimes. The largest scales of aggregation which could be explored with the available data did not allow the detection of a scaling regime, which would probably require larger domains of observation (provided that the spatial homogeneity assumption is respected). Indeed the analyses performed suggest the existence of an inner and a transition regime also in the spatial case.

4. Discussion and Conclusions

[39] The theoretical framework proposed to describe the variance of rainfall aggregated in time or space indicates the existence of a range of scales, the transition regime, in which the variance is not a power law function of the aggregation, and of two regimes in which the variance is indeed a power law function. The analysis of a wide set of rainfall observations representative of different climates and measurement techniques supports these theoretical predictions. In particular, observational analyses show that the variance exhibits non-power law forms for aggregation time scales of usual hydrological interest (15 min $\div$ 80 hours) and that space rainfall exhibits, in the case considered, a power law (inner) regime for spatial scales up to about 20 km.

[40] The theoretical analysis proposed also indicates that the form of the second-order moment is linked to the form of the variance as a function of aggregation both in the temporal and in the spatial case. This implies that the results presented, supporting the existence of inner, transition and scaling regimes, are not consistent with the assumption of a simple power law form of the second-order moment and thus do not support the notion of multiscaling in rainfall, be it qualified through the scaling of moments or of generalized structure functions.

[41] It is important to note that the quadratic form of the second moment as a function of aggregation in the scaling regime, indicated by theory and observations, is also consistent with previous observations. For example, *Onof et al.* [1996] study the moments of rainfall intensity $R(t_k, T) = H(t_k, T)/T$, whose second-order moment is $E[R(t_k, T)^2] = E[H(t_k, T)^2] \cdot T^{-2}$. The asymptotic form $E[H(t_k, T)^2] \propto T^2$ predicted in the scaling regime by the theory introduced here corresponds to $E[R(t_k, T)^2] \rightarrow const.$, consistent with the clearly decreasing slope of the second-order moment in most of *Onof et al.*'s Figures 6a–6f for large values of aggregations.

[42] Furthermore, the position of the transition regime identified here may provide an explanation for breaks in the scaling of the power spectrum of observed rainfall between 15 min and 3 days suggested by other authors [e.g., *Fraedrich and Larnder*, 1993; *Olsson*, 1995] and previously attributed to instrument limitations.

[43] In principle multiscaling does not require the existence of the variance of the underlying continuous process, as do equations (12) and (16). Nevertheless, it should be noted that multiscaling theory a priori assumes power law forms of the statistical moments, whereas the theory proposed here has the advantage of deducing, under a fairly general and reasonable stationarity hypothesis, a theoretical form of the variance as a function of aggregation. Well-defined properties derived from such theoretical expressions have indeed been validated by observations. Even though the analysis was limited to second-order moments, the fact that scaling approaches are found to be unsuitable to describe rainfall second-order statistics over wide aggregation scale ranges of usual interest suggests doubts regarding the general validity of simple scaling or multiscaling approaches in rainfall studies.

[44] In further support to the proposed theoretical framework, the existence of scaling and nonscaling regimes may provide interpretations for limits of power law scaling intervals previously reported and of uncertain explanation [e.g., *Olsson*, 1995; *de Lima and Grasman*, 1999; *Onof et al.*, 1996] and an explanation of why clustered poisson models have been found to reproduce persistence [e.g.,

Onof et al., 1996; *Olsson and Burlando*, 2002] even though they are characterized by a finite memory (and thus by an asymptotically linear variance). It must in fact be considered that the model $\sigma_H^2(T)$ in the transition regime must have a form which is intermediate between T^2 and T, and thus may be approximated, within a limited range of scales, by a power law form T^α with $\alpha > 1$. This is quite instructive and has consequences for the calibration and use of point or space-time rainfall models, be them based on multiplicative schemes (e.g., cascade models) or on marked poisson processes. In fact, model calibration and use involve a whole range of scales (typically between, say, $T = 1$ hour and $T = 1$ month) and thus a match between the $\sigma_H^2(T)$ functions of the observations and of the model is required within this entire scale interval and not just at a few calibration scales. If this is not carefully ensured, inadequate reproduction of rainfall variability at some scales will occur, as has indeed been observed [*Foufoula-Georgiou and Guttorp*, 1986].

[45] In contrast with the complex dependence of variance properties on aggregation scale observed for rainfall in time, the analyses of spatial rainfall presented indicate that the theoretically derived L^4 dependence of the variance satisfactorily holds over the spatial scales of usual hydrologic interest for the case considered, possibly suggesting considerable simplifications for interpretations and modeling.

[46] Relevant consequences follow for downscaling procedures as well. The theoretical and observational discussion presented indicates that temporal downscaling relying on power law scaling assumptions (based on moments, centered moments or structure functions) may result into gross extrapolation errors due to the presence of the transition regime. For example, it may be expected that, if a stochastic model is calibrated at the daily and higher time scales for donwscaling purposes, it will not reproduce hourly rainfall variability due to inevitable differences between model and actual $\sigma_H^2(T)$ functions. To prevent this, downscaling procedures should instead be based on theoretical expressions of $\sigma_H^2(T)$ obtained from (12) and (16) by assuming suitable autocorrelation forms, as suggested by *Marani* [2003].

[47] On the contrary, the proposed theoretical framework supports the suitability of a power law assumption for the spatial downscaling of the second moment of a rainfall field. The extension of the inner regime to scales of typical practical interest, e.g., when downscaling results from numerical weather forecasts (usually computed at scales of about L = 10 km) to scales of hydrological interest (usually down to about L = 100 m) in fact ensures that the variance be proportional to L^4, much simplifying downscaling relationships.

[48] In closing, it is interesting to note that the theoretical framework presented may have some wider relevance, as it applies to any aggregated stochastic field with finite mean and variance, and thus potentially to a large set of observed physical variables.

[49] **Acknowledgments.** This research was funded by the Aquaterra EU integrated project (contract 505428-GOCE). The author thanks Roberto Magarotto for his help with numerical processing of the data and the Hydrometeorology Laboratory (HML) at the Iowa Institute of Hydraulic Research (IIHR), Iowa University, NERC (UK), and NOAA-NCDC (USA) for making the data available. Furthermore, the author wishes to thank Sandra Yuter, Department of Atmospheric Science, University of Washington, for the production of the data from the KWAJEX experiment and the Distributed Active Archive Center (Code 902) at the Goddard Space Flight Center, Greenbelt, Maryland, which archives and distributes them under sponsorship of NASA's Earth Science Enterprise.

References

Austin, P. M., and R. A. Houze Jr. (1972), Analysis of the structure of precipitation patterns in New England, *J. Appl. Meteorol.*, *11*, 926–934.
Barancourt, C., J. D. Creutin, and J. Rivoirard (1992), A method for delineating and estimating rainfall fields, *Water Resour. Res.*, *28*, 1133–1144.
Cowpertwait, P. (1994), A generalized point process model for rainfall, *Proc. R. Soc. London, Ser. A*, *447*, 23–37.
Cox, D. R., and V. Isham (1988), A simple spatial-temporal model of rainfall, *Proc. R. Soc. London, Ser. A*, *415*, 317–328.
Crane, R. K. (1990), Space-time structure of rain rate fields, *J. Geophys. Res.*, *95*, 2001–2020.
Davis, A., A. Marshak, W. Wiscombe, and R. Cahalan (1994), Multifractal characterizations of nonstationarity and intermittency in geophysical fields: Observed, retrieved, or simulated, *J. Geophys. Res.*, *99*, 8055–8072.
Deidda, R. (2000), Rainfall downscaling in a space-time multifractal framework, *Water Resour. Res.*, *36*, 1779–1794.
Deidda, R., M. G. Badas, and E. Piga (2004), Space-time scaling in high-intensity Tropical Ocean Global Atmosphere Coupled Ocean-Atmosphere Response Experiment (TOGA-COARE) storms, *Water Resour. Res.*, *40*, W02506, doi:10.1029/2003WR002574.
de Lima, M. I. P., and J. Grasman (1999), Multifractal analysis of 15-min and daily rainfall from a semi-arid region in Portugal, *J. Hydrol.*, *220*, 1–11.
Fabry, F. (1996), On the determination of scale ranges for precipitation fields, *J. Geophys. Res.*, *101*, 12,819–12,826.
Falconer, K. (1990), *Fractal Geometry: Mathematical Foundations and Applications*, John Wiley, Hoboken, N. J.
Ferraris, L., S. Gabellani, N. Rebora, and A. Provenzale (2003), A comparison of stochastic models for spatial rainfall downscaling, *Water Resour. Res.*, *39*(12), 1368, doi:10.1029/2003WR002504.
Foufoula-Georgiou, E., and P. Guttorp (1986), Compatibility of continuous rainfall occurrence models with discrite rainfall observations, *Water Resour. Res.*, *22*, 1316–1322.
Fraedrich, K., and C. Larnder (1993), Scaling regimes of composite rainfall time series, *Tellus, Ser. A*, *45*, 289–298.
Georgakakos, K. P., A. A. Carsteanu, P. L. Sturdevant, and J. A. Cramer (1994), Observation and analysis of Midwestern rain rates, *J. Appl. Meteorol.*, *33*(1), 1433–1444.
Gupta, V. K., and E. C. Waymire (1990), Multiscaling properties of spatial rainfall and river flow distributions, *J. Geophys. Res.*, *95*, 1999–2009.
Gupta, V. K., and E. C. Waymire (1993), A statistical analysis of mesoscale rainfall as a random cascade, *J. Appl. Meteorol.*, *32*, 251–267.
Hurst, H. E. (1951), Long-term storage capacity of reservoirs, *Trans. Am. Soc. Civ. Eng.*, *116*, 770–808.
Katz, R. W. (1977), Precipitation as a chain-dependent process, *J. Appl. Meteorol.*, *16*, 671–676.
Katz, R. W., and M. B. Parlange (1995), Generalizations of chain-dependent processes: Application to hourly precipitation, *Water Resour. Res.*, *31*, 1331–1341.
Lovejoy, S., and B. Mandelbrot (1985), Fractal properties of rain, and a fractal model, *Tellus, Ser. A*, *37*, 209–232.
Lovejoy, S., and D. Schertzer (1985), Generalized scale invariance in the atmosphere and fractal models of rain, *Water Resour. Res.*, *21*, 1233–1250.
Mandelbrot, B. B., and J. R. Wallis (1968), Noah, Joseph and operational hydrology, *Water Resour. Res.*, *4*, 909–918.
Marani, M. (2003), On the correlation structure of point rainfall, *Water Resour. Res.*, *39*(5), 1128, doi:10.1029/2002WR001456.
Menabde, M., D. Harris, A. Seed, G. Austin, and D. Stow (1997), Multiscaling properties of rainfall and bounded random cascades, *Water Resour. Res.*, *33*, 2823–2830.
Olsson, J. (1995), Limits and characteristics of the multifractal behaviour of a high resolution rainfall time-series, *Nonlinear Processes Geophys.*, *2*, 23–29.
Olsson, J., and P. Burlando (2002), Reproduction of temporal scaling by a rectangular pulses rainfall model, *Hydrol. Processes*, *16*, 611–630.
Onof, C., P. Northrop, H. S. Wheater, and V. Isham (1996), Spatiotemporal storm structure and scaling property analysis for modeling, *J. Geophys. Res.*, *101*, 26,415–26,425.

Over, T., and V. Gupta (1996), A space-time theory of mesoscale rainfall using random cascades, *J. Geophys. Res.*, *101*, 26,319–26,331.

Rodriguez-Iturbe, I., V. K. Gupta, and E. Waymire (1984), Scale considerations in the modeling of temporal rainfall, *Water Resour. Res.*, *20*, 1611–1619.

Rodriguez-Iturbe, I., D. R. Cox, and V. Isham (1987), Some models for rainfall based on stochastic point processes, *Proc. R. Soc. London, Ser. A*, *410*, 269–288.

Rodriguez-Iturbe, I., D. R. Cox, and V. Isham (1988), A point process model for rainfall: Further developments, *Proc. R. Soc. London, Ser. A*, *417*, 283–298.

Rodriguez-Iturbe, I., M. Marani, P. D'Odorico, and A. Rinaldo (1998), On space-time scaling of cumulated rainfall fields, *Water Resour. Res.*, *34*, 3461–3469.

Schertzer, D., and S. Lovejoy (1987), Physical modelling and analysis of rain and clouds by anisotropic scaling multiplicative processes, *J. Geophys. Res.*, *92*, 9693–9714.

Schmitt, F., S. Vannitsem, and A. Barbosa (1998), Modelling of rainfall time series using two-state renewal processes and multifractal, *J. Geophys. Res.*, *103*, 23,181–23,194.

Taylor, G. I. (1921), Diffusion by continuous movements, *Proc. London Math. Soc.*, *20*, 196–211.

Tessier, Y., S. Lovejoy, and D. Schertzer (1993), Universal multifractals: Theory and observations for rain and clouds, *J. Appl. Meteorol.*, *32*, 223–250.

Vainshtein, S. I., K. R. Sreenivasan, R. T. Pierrehumbert, V. Kashyap, and A. Juneja (1994), Scaling exponents for turbulence and other random processes and their relationships with multifractal structure, *Phys. Rev. E*, *50*, 823–835.

Vanmarke, E. (1983), *Random Fields: Analysis and Synthesis*, MIT Press, Cambridge, Mass.

Veneziano, D., and P. Furcolo (2002), Multifractality of rainfall and scaling of intensity-duration-frequency curves, *Water Resour. Res.*, *38*(12), 1306, doi:10.1029/2001WR000372.

Veneziano, D., and V. Iacobellis (2002), Multiscaling pulse representation of temporal rainfall, *Water Resour. Res.*, *38*(8), 1138, doi:10.1029/2001WR000522.

Waymire, E., and V. K. Gupta (1981a), The mathematical structure of rainfall representations: 1. A review of the stochastic rainfall models, *Water Resour. Res.*, *17*, 1261–1272.

Waymire, E., and V. K. Gupta (1981b), The mathematical structure of rainfall representations: 2. A review of the theory of point processes, *Water Resour. Res.*, *17*, 1273–1285.

Waymire, E., and V. K. Gupta (1981c), The mathematical structure of rainfall representations: 3. Some applications of the point process theory to rainfall processes, *Water Resour. Res.*, *17*, 1287–1294.

Zawadzki, I. I. (1973), Statistical properties of precipitation patterns, *J. Appl. Meteorol.*, *12*, 459–472.

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