

1      **Joint stochastic theory of bedload transport and bed**  
2      **elevation: derivation of heavy-tailed fluvial sediment**  
3      **resting times**

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6      **Key Points:**

- 7      • We model fluvial bedload activity and local bed elevation as a two-species stochastic  
8      birth-death process.
- 9      • Computations show heavy-tailed power-law distributions of resting times for sed-  
10     iment undergoing burial.
- 11     • Scaling provides a universal tail parameter  $\alpha \approx 1.1$ , with implications for tracer  
12     statistics.

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13 **Abstract**

14 A consensus has formed that fluvial bedload resting times lie on heavy-tailed statisti-  
 15 cal distributions, and these may result from sediment burial. Owing to observational dif-  
 16 ficulties, only a handful of experiments have resolved these distributions, and there have  
 17 been few theoretical attempts to build understanding, leaving their generating mecha-  
 18 nism and specific characteristics uncertain. To address these topics, we present a new  
 19 theory describing bedload transport and bed elevation changes as a joint stochastic pro-  
 20 cess, and we derive resting time distributions for sediment undergoing burial from these  
 21 joint dynamics. Our theory predicts heavy-tailed power-law distributions of resting times  
 22 with universal tail behavior completely characterized by the mean erosion rate and its  
 23 scaling with bed elevation changes. This result might be used to determine if burial is  
 24 a dominant generating mechanism of anomalous diffusion in gravel-bed rivers.

25 **1 Introduction**

26 The majority of classic studies into fluvial sediment transport have attempted to  
 27 relate the bulk downstream flux of bedload to characteristics of the hydraulic forcing (e.g.  
 28 Yalin, 1972), yet the relevance of this approach to environmental problems is limited,  
 29 as many contemporary issues require knowledge of differences between motions of indi-  
 30 vidual grains, and not just average characteristics. For example, the export of contam-  
 31 inants from channels (e.g. Malmon, Reneau, Dunne, Katzman, & Drakos, 2005) and the  
 32 morphological response of channels to ecological restoration efforts (e.g. Gaeuman, Stew-  
 33 art, Schmandt, & Pryor, 2017) or changes in hydrology or sediment supply (e.g. Hassan  
 34 & Bradley, 2017) are not described by bulk fluxes, highlighting understanding of indi-  
 35 vidual sediment motions as an important topic for geophysics research.

36 A significant complication is that individual grains transport within a noisy envi-  
 37 ronment, and the noise sources range across spatial and temporal scales from fluid tur-  
 38 bulence (Celik, Diplas, & Dancey, 2014) and the variable arrangement of bed surface grains  
 39 (Gordon, Carmichael, & Isackson, 1972), to channel morphology changes (Hassan & Bradley,  
 40 2017) and unsteady flows (Phillips, Martin, & Jerolmack, 2013). As a result, the trans-  
 41 port characteristics of individual grains within a channel are not deterministic (e.g. Ein-  
 42 stein, 1937), and this holds even in the most controlled laboratory experiments (e.g. Böhm,  
 43 Ancey, Frey, Reboud, & Ducottet, 2004; Charru, Mouilleron, & Eiff, 2004; Fathel, Fur-  
 44 bish, & Schmeeckle, 2015; Heyman, Bohorquez, & Ancey, 2016).

45 This realization inspired researchers to create probabilistic theories of individual  
 46 motions based on random walk concepts, wherein bedload motions are approximated as  
 47 alternating sequences of steps and rests, with step lengths and resting times treated as  
 48 random variables drawn from statistical distributions (N. D. Bradley & Tucker, 2012;  
 49 Einstein, 1937; Hassan, Church, & Schick, 1991; Nakagawa & Tsujimoto, 1976; Yano,  
 50 1969). In these theories, differences between the random motions of one grain and the  
 51 next imply bedload diffusion, or a spreading out of grains through time. The diffusion  
 52 characteristics predicted by these models critically differ depending on whether the step  
 53 length and resting time distributions have light or heavy tails (e.g. D. N. Bradley, 2017).

54 Heavy-tailed distributions have exceedance functions  $P(X > x) \sim x^{-\alpha}$  with tail  
 55 parameters  $\alpha < 2$ , meaning large values of  $x$  are relatively common, while light-tailed  
 56 distributions have  $\alpha \geq 2$ , meaning large values of  $x$  are relatively rare. If both resting  
 57 time and step distance distributions have light tails, the diffusion is said to be normal  
 58 or Fickian, with a variance of particle positions  $\sigma_x^2$  scaling with time  $t$  as  $\sigma_x^2 \propto t$ . How-  
 59 ever, if either distribution has a heavy-tail, the diffusion is called anomalous, with a vari-  
 60 ance of particle position scaling as  $\sigma_x^2 \propto t^\gamma$ , where  $\gamma \neq 1$ . In this expression,  $\gamma < 1$  is  
 61 called sub-diffusion and  $\gamma > 1$  is super-diffusion. In strongly asymmetric random walks  
 such as bedload transport, heavy-tailed step lengths imply super-diffusion, while heavy-

tailed resting times imply either super or sub-diffusion, depending on  $\alpha$  (Weeks & Swinney, 1998; Weeks, Urbach, & Swinney, 1996).

Tracer experiments in gravel bed rivers show anomalous bedload diffusion (D. N. Bradley, 2017; Phillips et al., 2013), light-tailed step lengths (N. D. Bradley & Tucker, 2012; Hassan, Voepel, Schumer, Parker, & Fraccarollo, 2013), and heavy-tailed resting times (D. N. Bradley, 2017; Olinde & Johnson, 2015; Pretzlav, 2016; Voepel, Schumer, & Hassan, 2013), forming a coherent experimental picture of super-diffusive bedload transport at long observation timescales (e.g. Martin, Jerolmack, & Schumer, 2012; Nikora, 2002). However, field studies do not resolve the mechanism generating heavy-tailed resting times (e.g. D. N. Bradley, 2017), and the experimental distributions display clear differences in form and characteristics, including different tail parameters (e.g. Olinde & Johnson, 2015) and sometimes truncation (e.g. D. N. Bradley, 2017) or tempering to light tails at large resting times (e.g. Voepel et al., 2013). The mechanism generating heavy tails and the differences between experimental data deserve attention.

A predominant hypothesis is that heavy-tailed resting times and anomalous diffusion originate from sediment burial (Martin, Purohit, & Jerolmack, 2014; Voepel et al., 2013; Wu et al., 2019). Conceptually, when grains rest on the bed surface, material transported from upstream can deposit on top of them, preventing entrainment until it's removed, driving up resting times and imparting a heavy tail to the distribution. Martin et al. (2014) have provided the only direct support for this hypothesis. They traced grains in a narrow flume with clear sidewalls, directly resolving burial as the generator of heavy-tailed resting times, and they described their results with a theoretical model, formally similar to an earlier effort by Voepel et al. (2013).

The models of Voepel et al. (2013) and Martin et al. (2014) treat bed elevations as a random walk and interpret resting times as return periods from above in the bed elevation time-series (e.g. Redner, 2007). Both models are successful in describing different experimental resting time distributions. However, the assumptions and results of these models are inconsistent with one another, and their treatment of bed elevations as a process independent of sediment transport is questionable, since erosion and deposition are the source of bed elevation changes (e.g. Wong, Parker, DeVries, Brown, & Burges, 2007), suggesting further study is necessary.

In this work, we approach the problem from a different angle, making an extension of the stochastic bedload transport theory of Ancey, Davison, Böhm, Jodeau, and Frey (2008) to link bed elevation changes to the erosion and deposition events of individual grains, and we derive resting times as a consequence of this theory. The key assumptions of our model are: (1) bedload erosion and deposition can be characterized by probabilities per unit time, or rates (e.g. Ancey et al., 2008; Einstein, 1950); and (2) these rates are contingent on the local bed elevation, encoding the property that the erosion of sediment is emphasized from regions of exposure, while deposition is emphasized in regions of shelter (e.g. Sawai, 1987; Wong et al., 2007). Our theory generates heavy-tailed distributions with no tempering and a universal tail parameter  $\alpha \approx 1.1$  for a particular non-dimensionalization of the resting time, showing close correspondence to the results of Martin et al. (2014) and suggesting a correction to some imperfections in their results. We conclude by discussing how our results could be used to determine if burial drives anomalous diffusion in gravel-bed rivers.

## 2 Stochastic theory

We define a volume of downstream length  $L$  which contains some number  $n$  of moving particles in the water flow and some number  $m$  of stationary particles composing the bed at some time  $t$ . For simplicity, we consider all particles as approximately spherical with the same diameter  $2a$ , so their mobility and packing characteristics are similar. Fol-



**Figure 1.** The conceptual picture of a control volume containing  $n$  moving grains and  $m$  resting grains. Migration in, entrainment, deposition, and migration out are represented by arrows, and the probability distribution of bed elevations is illustrated.

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owing Ancey et al. (2008), we prescribe four events which can occur at any instant to  
modify the populations  $n$  and  $m$ , and we characterize these events using probabilities  
per unit time, or rates. These are: (1) migration of a moving particle into the volume  
from upstream ( $n \rightarrow n + 1$ ); (2) the entrainment of a stationary particle into motion  
within the volume ( $m \rightarrow m - 1$  and  $n \rightarrow n + 1$ ); (3) the deposition of a moving par-  
ticle to rest within the volume ( $m \rightarrow m + 1$  and  $n \rightarrow n - 1$ ); and (4) the migration of  
a moving particle out of the volume to downstream ( $n \rightarrow n - 1$ ). As the events occur  
at random intervals, they set up a joint stochastic evolution of the populations  $n$  and  
 $m$  characterized by a joint probability mass function (pmf)  $P(n, m, t)$  having marginal  
pmfs  $P(n, t) = \sum_m P(n, m, t)$  and  $P(m, t) = \sum_n P(n, m, t)$  for the number of parti-  
cles in motion and rest in the volume at  $t$ . These concepts are depicted in figure 1.

The populations  $n$  and  $m$  yield the bedload transport rate  $q_s$  and the local bed el-  
evation  $z$ . The mean bedload transport rate is given by  $q_s \propto u_s \langle n \rangle$ , where  $u_s$  is the char-  
acteristic velocity of moving bedload and  $\langle n \rangle$  is the mean number of grains in motion  
(e.g. Ancey et al., 2008; Charru et al., 2004; Furbish, Haff, Roseberry, & Schmeeckle, 2012).  
The bed elevation is related to  $m$  though the packing geometry of the bed. To derive this,  
we prescribe a mean number of grains at rest  $m_0$  and introduce a packing fraction  $\phi$  of  
grains in the bed. Then considering a two-dimensional bed (e.g. Einstein, 1950; Paintal,  
1971), the deviation from the mean bed elevation is

$$z(m) = \frac{\pi a^2}{\phi L} (m - m_0) = z_1 (m - m_0). \quad (1)$$

The constant  $z_1 = \pi a^2 / (\phi L)$  is an important scale of the problem.  $z_1$  is the magnitude  
of bed elevation change (in an average sense across the control volume) associated with  
the addition or removal of a single grain. We write the rates for the four possible tran-  
sitions as (e.g. Ancey et al., 2008):

$$\nu \quad \text{migration in,} \quad (2)$$

$$\lambda(m) + \mu(m)n \quad \text{entrainment,} \quad (3)$$

$$\sigma(m)n \quad \text{deposition,} \quad (4)$$

$$\gamma n \quad \text{migration out.} \quad (5)$$

124 These rates are always independent of the past history of the process, only depending  
 125 on its current state  $(n, m)$ . This is the Markov hypothesis (e.g. Cox & Miller, 1965), which  
 126 implies the time intervals between subsequent transitions are exponentially distributed  
 127 (e.g. Gillespie, 2007).

128 In (2-5),  $\nu$  and  $\gamma$  are constants characterizing migration rates per grain into and  
 129 out of the volume. They lack any dependence on the populations  $n$  and  $m$ . In contrast,  
 130  $\lambda(m)$ ,  $\mu(m)$ , and  $\sigma(m)$ , characterizing entrainment, collective entrainment, and depo-  
 131 sition rates per grain depend on  $m$  through the bed elevation (1). Collective entrainment  
 132 has been discussed carefully in many studies (e.g. Ancey et al., 2008; Ancey & Heyman,  
 133 2014; Heyman, Ma, Mettra, & Ancey, 2014; Heyman, Mettra, Ma, & Ancey, 2013; Ma  
 134 et al., 2014), and we won't dwell on this topic. For our purposes, it's enough to state it  
 135 was introduced in Ancey et al. (2008) as a means to obtain bedload fluctuations of re-  
 136 alistic magnitude in birth-death theories of bedload transport, which can be viewed as  
 137 contemporary offshoots of the classic work of Einstein (1950).

138 Now we define the dependence of entrainment and deposition rates on  $z(m)$ . As  
 139 is well-known, bed elevation changes modify the likelihood of entrainment and deposi-  
 140 tion in a negative feedback (Sawai, 1987; Wong et al., 2007); that is, aggradation increases  
 141 the likelihood of entrainment, while degradation increases the likelihood of deposition.  
 142 Wong et al. (2007) concluded that bed elevation changes induce an exponential varia-  
 143 tion in entrainment and deposition probabilities, while Sawai (1987) concluded that the  
 144 variation is linear. For simplicity, we incorporate the scaling of Sawai (1987) and note  
 145 its equivalence to the Wong et al. (2007) scaling when bed elevation changes are small.  
 146 Because experimental distributions of bed elevations are usually not skewed (Martin et  
 147 al., 2014; Singh, Fienberg, Jerolmack, Marr, & Foufoula-Georgiou, 2009; Wong et al., 2007),  
 148 we expect the erosion and deposition feedbacks to be symmetrical.

Using these ideas, the entrainment and deposition rates can be written  $\chi(m) = \chi_0(1 \pm z_1 z(m)/(2l)^2)$ , where  $\chi = \lambda, \mu, \sigma$ , and entrainment parameters take the plus sign  
 while the deposition parameter takes the minus, and we have introduced a factor  $l$  as  
 the length scale of bed elevation change at which the entrainment and deposition rates  
 are significantly affected. With these substitutions, the entrainment and deposition rates  
 (3-4) become:

$$(\lambda_0 + \mu_0 n)(1 + z_1 z(m)/(2l)^2) \quad \text{entrainment,} \quad (6)$$

$$\sigma_0(1 - z_1 z(m)/(2l)^2)n \quad \text{deposition.} \quad (7)$$

In these expressions, the ratio  $z_1/l$  controls the sensitivity of the rates to the addition  
 or removal of a single grain. At  $z(m) = 0$ , these rates reduce to those of the Ancey et  
 al. (2008) theory. Away from this elevation, entrainment and deposition are alternatively  
 suppressed and accentuated, depending on the sign of  $z(m)$ , introducing a mean-reverting  
 character to bed elevation changes.

In terms of the transition rates (2), (5), (6), and (7), we set up the Master equa-  
 tion for the probability flow between discrete states  $(n, m)$  in the standard way (e.g. An-  
 cey et al., 2008; Cox & Miller, 1965; Gillespie, 1992):

$$\begin{aligned} \frac{\partial P}{\partial t}(n, m; t) = & \nu P(n-1, m; t) + \{\lambda(m+1) + [n-1]\mu(m+1)\}P(n-1, m+1; t) \\ & + [n+1]\sigma(m-1)P(n+1, m-1; t) + [n+1]\gamma P(n+1, m; t) \\ & - \{\nu + \lambda(m) + n\mu(m) + n\sigma(m) + n\gamma\}P(n, m; t). \end{aligned} \quad (8)$$

The solution  $P(n, m; t)$  of this equation provides the statistics of  $n$  and  $m$ , meaning it  
 provides the statistics of bedload flux and bed elevation. This process is a two-species  
 stochastic birth-death model (e.g. Cox & Miller, 1965) of a type common in ecological  
 population modeling (e.g. Pielou, 1977; Swift, 2002). In our case, the two species are the  
 moving and stationary grains in the control volume.

159 **3 Simulations**

160 Unfortunately, (8) does not admit an analytical solution, at least by methods ap-  
 161 plied to similar systems in the population ecology literature (e.g Swift, 2002) or to the  
 162 case where changes in bed elevation are ignored (Ancey et al., 2008). This difficulty re-  
 163 sults from the product terms between  $n$  and  $m$  in equation 8. To proceed and gain un-  
 164 derstanding of the stochastic evolution of the populations  $n$  and  $m$ , we turn to numer-  
 165 ical methods.

166 The simulation of birth-death type master equations has been extensively studied  
 167 for its relevance to chemical physics and population ecology. Balancing conceptual sim-  
 168 plicity against computational efficiency, we choose the classic Gillespie stochastic sim-  
 169 ulation algorithm (SSA) to simulate realizations of the random dynamics (Gillespie, 1977,  
 170 1992, 2007).

171 The SSA leverages the defining property of a Markov process: due to memoryless-  
 172 ness in the transition rates, the time interval between subsequent transitions is exponen-  
 173 tially distributed (e.g. Cox & Miller, 1965). Once a transition occurs, its type can be ran-  
 174 domly selected using the relative rates of all possible transitions. Accordingly, to step  
 175 our birth-death process through a single transition, we can obtain when it occurs by draw-  
 176 ing a random value from an exponential distribution, then we select its type by draw-  
 177 ing a random value between 0 and 1 from a uniform distribution. After this transition  
 178 is enacted, i.e., by stepping  $n$  and  $m$  by the shifts associated with the type of transition  
 179 which occurred, this two-stage selection is repeated as desired to form an exact realiza-  
 180 tion of the stochastic process of arbitrary length (Gillespie, 1977, 1992, 2007). One can  
 181 then compute any statistics of interest from these realizations, providing numerical ap-  
 182 proximation to (8).

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 185 **Table 1.** Parameters from Ancey et al. (2008)  
 186 experiments describing the rates of migration in,  
 187 entrainment, deposition, and migration out of  
 188 the control volume when  $z(m) = 0$ . All units are  
 189  $s^{-1}$  (probability/time).

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Flow	$\nu$	$\lambda_0$	$\mu_0$	$\sigma_0$	$\gamma$
(a)	5.45	6.59	3.74	4.67	0.77
(g)	7.74	8.42	4.34	4.95	0.56
(i)	15.56	22.07	3.56	4.52	0.68
(l)	15.52	14.64	4.32	4.77	0.48
(n)	15.45	24.49	3.64	4.21	0.36

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Using the SSA, we simulated 5 flow conditions from the Ancey et al. (2008) experiments, prescribing 8 different values of  $l$ , ranging from half of the granular radius ( $l = a/2$ ) to 4 granular diameters ( $l = 8a$ ) for a total of 40 simulations. Each simulation was run for 1500hrs of virtual time. The parameters used for our numerical experiments are taken from the flume trials in Ancey et al. (2008) and are summarized in table 1. They are labeled (a), (g), and so on, in order of increasing bedload transport rate. In all simulations, we take the packing fraction  $\phi = 0.6$ , a typical value for a pile of spheres (Bennett, 1972), and set  $\Delta x = 22.5\text{cm}$  and  $a = 0.3\text{cm}$  in accord with the Ancey et al. (2008) experiments.

202 One simulated realization of our  
 203 joint stochastic process for bed elevations and bedload transport is depicted in figure 2  
 204 (a). These realizations determine the joint statistics of  $n$  and  $m$ . The bedload transport  
 205 statistics are depicted for a subset of all simulation conditions in figure 2 (b). The An-  
 206 cey et al. (2008) theory predicts negative binomial distributions for the number of mov-  
 207 ing grains within the control volume, and the mathematical form of these distributions  
 208 is apparently not changed by our extension to account for feedbacks between bed ele-  
 209 vation changes and entrainment and deposition probabilities. We obtain an excellent neg-  
 210 ative binomial fit to the marginal probability distribution of  $n$ ,  $P(n) = \sum_m P(n, m, t)$ ,



**Figure 2.** Figure (a) depicts timeseries of particle activity and bed elevation over a 20 minute interval. Figures (b) and (c) display probability distribution functions of bed elevation (equation 1) and particle activity for a subset of the simulations. Colors represent flow conditions, while differing line styles represent different values of the differential mobility parameter  $l$ .

for all 40 of our simulation results. However, activity statistics, including the mean activity and its variance, are definitely shifted by the inclusion of differential mobility with bed elevation changes. Bed elevation changes appear to buffer the magnitude of bedload fluctuations by up to 30 percent, which is an expected effect of the model, since a rapid increase in the bedload rate induced by a series of many entrainments will lower the bed elevation and increase the probability of deposition, buffering the magnitude of the bedload rate increase.

Our bed elevation timeseries exhibit longer temporal correlations than related bedload activity series, evident in figure 2 (a). All 40 of our simulations develop clean unimodal distributions of bed elevations which are fit by Gaussian distributions with excellent correlation, and a subset of these marginal bed elevation pdfs with their Gaussian fits are displayed in figure 2 (c). The mean number of particles resting on the bed is  $m_0$ , corresponding to a relative elevation  $z(m_0) = 0$ . The variance of bed elevations is controlled by the differential mobility parameter  $l$ . Apparently, the simulations support the conclusion that  $\text{var}(m) = (l/z_1)^2$ . This conclusion is evident in figure 3, with generally excellent correspondence between this relationship and the simulation points, with some scatter we attribute to the finite duration of our simulations.

To compute the resting time distribution; at each elevation  $m$ , we extract the set of all departure times from this elevation, or times at which the bed was at this elevation and a deposition occurred; then we extract the set of all return times to this elevation, or times at which the bed was one increment above this elevation ( $m+1$ ) and an entrainment occurred. Taking differences between these two time-series returns the set of all return times from above marginal to the elevation  $m$ , which we binned across a 0.5s interval to compute the cumulative probabilities of return times at each elevation  $m$ ,  $P(T_r > t|m)$ . Following earlier investigators, we computed the unconditional or overall rest time distribution as the convolution of these conditional distributions over all bed elevations (e.g. Voepel et al., 2013; Yang & Sayre, 1971):

$$P(T_r > t) = \sum_m P(m)P(T_r > t|m), \quad (9)$$

228 where  $P(m)$  is the pdf of bed elevation like those depicted in figure 2 (b) and the sum  
229 is over all bed elevations attained during the simulation.

230 This analysis derives unconditional  
231 exceedance probabilities of resting times  
232 with heavy power-law tails. A subset of  
233 all of our simulation results are depicted  
234 in figure 4 (a)-(d). Apparently, for suit-  
235 ably long times, the tail parameter  $\alpha$  of  
236 these resting time distributions is inde-  
237 pendent of flow conditions or the differ-  
238 ential mobility parameter  $l$ . However,  
239 the timescale at which particle resting  
240 transitions from exponential to power-  
241 law scaling shifts with flow conditions  
242 and  $l$ . Martin et al. (2014) obtained an  
243 approximate collapse at the tails of their  
244 experimental resting time distributions  
245 using the reciprocal of the rate of entrain-  
246 ment or deposition events occurring. They  
247 denoted this rate by  $a$ , so that their timescale  
248 was  $1/a$ . Scaling the resting times by  $1/a$   
249 provides an incomplete collapse of the  
250 tails of our simulated resting time dis-  
251 tributions, which may describe the in-  
252 complete collapse of the experimental data  
253 of Martin et al. (2014). It collapses the  
254 tails across flow conditions when  $l$  (the  
255 standard deviation of bed elevation) is  
256 fixed, i.e., it leads to the collapse seen  
257 between figures 4 (a) and 4 (b), but if  
258  $l$  (which is the standard deviation of bed elevation) varies with flow, then the power-law  
259 scaling of resting times is no longer controlled by  $1/a$  alone. Instead, we must include  
260 a factor representing the differential entrainment and deposition characteristics of grains  
261 as the bed elevation changes. The timescale which provides universal collapse of the power-  
262 law tails of all our simulations is  $T_0 = l/(z_1 E)$ , where  $E$  is the entrainment rate.

263 We can understand  $T_0$  with a physical argument. According to figure 3, the typ-  
264 ical length scale of bed elevation fluctuations is  $l$ , and as mentioned the length scale  $z_1$   
265 is the magnitude of bed elevation change enacted by the entrainment or deposition of  
266 a single particle. In equilibrium bedload transport, Einstein (1950) tells us the condi-  
267 tion  $E = D$  holds: this is a statement of mass conservation. Since  $E$  represents the mean  
268 number of particles removed from the bed in a unit of time, the product  $Ez_1$  can be in-  
269 terpreted as a representative velocity scale of bed return. It is the distance the bed low-  
270 ers with the removal of a single particle divided by the mean time required to remove  
271 it. Hence we extract our timescale as a key distance scale over a key velocity scale:  $T_0 =$   
272  $l/(z_1 E)$ . Scaling the resting time as  $T_r/T_0$  exhibits a consistent power-law tail across all  
273 of our simulation results.

## 274 4 Discussion

275 Our theory of bed elevations derives results similar to Martin et al. (2014) using  
276 bedload transport as a starting point, and it also provides a statistical characterization  
277 of bedload transport. Our assumptions derive a heavy-tailed power-law distribution of  
278 resting times with a tail parameter  $\alpha \approx 1$  which displays differences across flow con-  
279 ditions which partially collapse upon scaling by an activity timescale.



Figure 3. The standard deviation of bed elevation scales one-to-one with the differential mobility parameter ( $l/z_1$ ), indicating the ratio  $l/z_1$  controls the magnitude of bed elevation fluctuations.



**Figure 4.** This figure summarizes the resting time exceedance distributions for a subset of all simulations. Part (a) displays resting time distributions for a range of flow conditions at fixed  $l$ , while part (b) displays the collapse obtained by scaling  $T_r$  by  $T_0$ . The collapse between (a) and (b) is analogous to Martin et al. (2014), and it is induced by the factor of  $1/E$  within  $T_0$ . Parts (c) and (d) display a similar collapse for a fixed flow condition at variable  $l$ . In this case, collapse is driven by the factor of  $z_1/l$  within  $T_0$ , and this influence of differential mobility in resting statistics has not to our knowledge been noticed up to now.

However, in extension of the Martin et al. (2014) theory, our model reveals another timescale which fully collapses the power-law tails of the resting time distributions, suggesting a universal power-law should characterize the asymptotic resting times of sediment undergoing burial if the assumptions of our model are correct. This timescale includes an additional factor characterizing the dependence of entrainment and deposition probabilities on changes in local bed elevations. We hypothesize this new factor may explain some of the differences between field (e.g. Olinde & Johnson, 2015) and laboratory (e.g. Martin et al., 2014) observations of resting time distributions, providing additional information to determine whether burial is the dominant mechanism of the heavy-tailed sediment resting times observed in natural channels.

Pierce The model describes why Martin et al (2014) obtained incomplete collapse between their resting time distributions. They didn't include the  $(z_1/l)^2$  type factor in their scaling time-scale It is a first joint stochastic description of bed elevations and bed-load transport, mixing martin 2014 and ancey 2008. It provides a mechanism for heavy-tailed resting times and implies super-diffusion of bedload and a virtual velocity of sediment which decreases toward zero with time. what else? help

Hassan 1. Discuss what is special in the model and how it works. (most of point number 2)

- 298      2. Your point number 1
- 299      3. The last paragraph from the introduction
- 300      4. Your point number 3 with implications (What this means for the study of bed-load (and channel morphology in streams). The implications should be short.

302      **5 Conclusion**

303      **Acknowledgments**

304      The computer code used to simulate the presented model is available upon request from  
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307      **References**

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