

Stochastic birth-death model of heavy-tailed fluvial sediment resting times

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Key Points:

- We model fluvial bedload activity and local bed elevation as a two-species stochastic birth-death process.
- The resulting timescales of sediment storage by burial lie on heavy-tailed power-law distributions.
- Our model describes a virtual velocity of tracer particles may decline indefinitely toward zero as their lifetime in the channel increases.

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Abstract

A consensus has formed that fluvial sediment expresses heavy-tailed statistical distributions of resting times due to the effect of sediment burial on its mobility, and this has key implications for the diffusion of sediment and evacuation of contaminants from river channels. Owing to observational difficulties, only a handful of experiments have resolved these heavy-tailed resting time distributions, and there have been few theoretical attempts to build understanding. These studies do not agree on the form or properties of resting time distributions. In this work, we develop a new stochastic theory which jointly describes bedload transport and bed elevation changes, and we derive resting time distributions for sediment undergoing burial in terms of the joint dynamics. Our theory corroborates a subset of existing experimental and theoretical work, expressing resting time distributions for sediment undergoing burial with heavy power-law tails, implying super-diffusion of bedload. Our key finding is that the largest bedload resting times are controlled not only by the erosion and deposition rates, as suggested by earlier studies, but also by the scaling of erosion and deposition rates with local changes in bed elevation.

1 Introduction

Classical theories of bedload transport in rivers relate the bulk downstream flux of sediment to characteristics of the hydraulic forcing (e.g. Yalin, 1972). Although bulk fluxes are often described using deterministic models (Wilcock & Crowe, 2003, e.g.), the individual motions constituting them are more like a random walk, being an alternating sequence of motions and rests which are best characterized statistically (e.g. H. A. Einstein, 1937). The statistical properties of individual motions generate differences in the virtual velocities of grains, and these differences imply diffusion, or a spreading out of grains as they transport downstream (H. A. Einstein, 1937; Nakagawa & Tsujimoto, 1976; Yano, 1969). Diffusion is important in applications where the downstream transport of only a subset of grains is of interest, as in contaminant evacuation (Malmon, Reneau, Dunne, Katzman, & Drakos, 2005), river restoration (Gaeuman, Stewart, Schmandt, & Pryor, 2017), and bed texture adjustment (Hassan & Bradley, 2017), highlighting bedload diffusion as an important topic for study.

In the classic diffusion problems, such as the Brownian motion of pollen grains in water (e.g. A. Einstein, 1905), the variance of particle positions σ_x^2 scales with time t as $\sigma_x^2 \propto t$, a rate of spreading called Fickian or normal diffusion. Bedload diffusion is more nuanced. The nature of bedload diffusion is controlled by the probability distributions of step length and resting time. If either of these distributions has a heavy-tail, the diffusion is called anomalous instead of normal. A heavy-tailed step length or resting time distribution has an exceedance distribution $P(X > x) \sim x^{-\alpha}$ with tail parameter $\alpha < 2$. Anomalous diffusion is characterized by a non-linear scaling of the variance of particle position with time, as in $\sigma_x^2 \propto t^\gamma$ with $\gamma \neq 1$. In this expression, $\gamma < 1$ is called sub-diffusion and $\gamma > 1$ is called super-diffusion. In a strongly asymmetric random walk as in bedload transport, heavy-tailed step length distributions imply super-diffusion, while heavy-tailed resting time distributions imply either super or sub-diffusion depending on the value of the tail parameter α (Weeks & Swinney, 1998).

A consensus has formed that step lengths in gravel-bed rivers are thin-tailed (Bradley, 2017; Hassan, Voepel, Schumer, Parker, & Fraccarollo, 2013) while bedload diffusion can be anomalous (Bradley, Tucker, & Benson, 2010; Fan, Xie, & Nie, 2017; Martin, Jerolmack, & Schumer, 2012; Nikora, 2002; Phillips, Martin, & Jerolmack, 2013; Zhang, Meer-schaert, & Packman, 2012), suggesting heavy-tailed resting times are the source of anomalous diffusion (Bradley, 2017; Martin et al., 2012). This perspective highlights the mechanism of heavy-tailed resting times as an important consideration. A dominant hypothesis is that heavy-tailed resting times stem from sediment burial (e.g. Bradley, 2017; Martin, Purohit, & Jerolmack, 2014; Voepel, Schumer, & Hassan, 2013). Conceptually, when

grains rest on the bed surface, material transported from upstream can deposit over top of them, burying them away from the flow and preventing their entrainment until the overlying material is removed, increasing sediment resting times and imparting a heavy tail to the distribution.

Although a handful of field studies have resolved heavy-tailed resting times (Bradley, 2017; Olinde & Johnson, 2015; Pretzlav, 2016; Voepel et al., 2013), these do not resolve sediment burial and exhumation during floods, so they cannot determine if burial is the mechanism generating heavy-tailed resting times. To our knowledge, only Voepel et al. (2013) and Martin et al. (2014) have provided direct support for the hypothesis that burial generates heavy-tailed resting times, although details of these works are inconsistent with one another. Martin et al. (2014) performed experiments in a narrow flume configured so grains formed a single layer in the cross-stream direction, providing direct resolution of sediment burial. Across a range of flow conditions, they obtained heavy-tailed power-law resting time distributions with a tail parameter $\alpha \approx 1$ which they directly attributed to sediment burial. Different flow conditions shifted the resting time above which the asymptotic power-law scaling applied, but did not obviously change the value of α . Using a bed activity timescale related to the rate of entrainment or deposition events, they managed to partially collapse their experimental distributions.

The theoretical component of the Martin et al. (2014) work is similar to that of Voepel et al. (2013). Both of these models treat local bed elevations as a bounded random walk, where the bed elevation steps randomly up or down with some probability, although they differ in that Voepel et al. (2013) introduces upper and lower bounds to the local bed elevation with hard reflecting barriers, while Martin et al. (2014) introduces them with a softer mean-reverting tendency. Both of these models are formally similar to classic stochastic theories of the resting times of sand grains on a duned bed (e.g. Nakagawa & Tsujimoto, 1980; Yang & Sayre, 1971). Within these models, the timescales of sediment burial are interpreted as the return time from above in the bed elevation time-series. From their assumptions, Voepel et al. (2013) conclude the resting time distribution should be a so-called tempered Pareto distribution, with an initial power-law decay which transitions to take on a thin-tailed exponential scaling, and they note agreement between their predictions and the field data of Habersack (2001). In contrast, Martin et al. (2014) predicted a heavy-tailed power-law distribution of resting times which does not temper to an exponential distribution, and they successfully calibrated this model to their extensive set of flume experiments.

We believe the differences between these models deserve further theoretical clarification, and we note that bedload transport is the process driving changes in bed elevation, but neither of these models directly incorporate it. In this paper, we leverage existing knowledge about bedload transport to study the resting times of coarse sediment under burial. We extend the stochastic bedload transport theory of Ancey, Davison, Böhm, Jodeau, and Frey (2008) to include bed elevation changes, and we use this model to compute the resting time distribution for sediment undergoing burial. Our theory of bed elevations derives results similar to Martin et al. (2014) using bedload transport as a starting point, and it also provides a statistical characterization of bedload transport. Our assumptions derive a heavy-tailed power-law distribution of resting times with a tail parameter $\alpha \approx 1$ which displays differences across flow conditions which partially collapse upon scaling by an activity timescale.

However, in extension of the Martin et al. (2014) theory, our model reveals another timescale which fully collapses the power-law tails of the resting time distributions, suggesting a universal power-law should characterize the asymptotic resting times of sediment undergoing burial if the assumptions of our model are correct. This timescale includes an additional factor characterizing the dependence of entrainment and deposition probabilities on changes in local bed elevations. We hypothesize this new factor may explain some of the differences between field (e.g. Olinde & Johnson, 2015) and laboratory

(e.g. Martin et al., 2014) observations of resting time distributions, providing additional information to determine whether burial is the dominant mechanism of the heavy-tailed sediment resting times observed in natural channels.

2 Theory

We define a control volume of downstream length Δx which contains some number n of moving particles within the water flow and some number m of stationary particles composing the bed at some instant. For simplicity we consider that all particles are approximately spherical with the same diameter $2a$, so that their mobility and packing characteristics are similar from one particle to the next. Within the control volume, we follow Ancey et al. (2008) and prescribe events which can occur from one instant to the next to modify the populations n and m , and we characterize these events using probabilities per unit time, or rates. These are: (1) migration of a moving particle into the control volume from upstream; (2) the entrainment of a stationary particle into motion; (3) the deposition of a moving particle into rest; and (4) the migration of a moving particle out of the control volume to downstream. We assume the time intervals between transitions of any kind are exponentially distributed in accord with the Markov hypothesis, equivalent to the assumption that the current transition rates are independent of their history (e.g. Cox & Miller, 1965). These concepts are depicted in figure 1.

If the mean number of particles at rest is m_0 and the particles have a packing fraction ϕ , the deviation from the mean bed elevation given a population m of particles at rest within the control volume is

$$z(m) = \frac{\pi a^2}{\phi \Delta x} (m - m_0) = z_1 (m - m_0). \quad (1)$$

The constant $z_1 = \pi a^2 / (\phi \Delta x)$ is an important scale of the problem. z_1 is the magnitude of bed elevation change (in an average sense across the control volume) associated with the addition or removal of a single particle. Following Ancey et al. (2008) almost verbatim we write the probabilities associated with each transition in a small time increment dt as

$$R_{MI}(n+1, m|n, m) = \nu + O(dt) \quad \text{migration in,} \quad (2)$$

$$R_E(n+1, m-1|n, m) = \lambda(m) + \mu(m)n + O(dt) \quad \text{entrainment,} \quad (3)$$

$$R_D(n-1, m+1|n, m) = \sigma(m)n + O(dt) \quad \text{deposition,} \quad (4)$$

$$R_{MO}(n-1, m+1|n, m) = \gamma n + O(dt) \quad \text{migration out,} \quad (5)$$

but we modify these equations by introducing an m dependence – through the local bed elevation $z(m)$ – into the entrainment and deposition rates. In writing these relations we have used Landau’s big O notation to indicate additional terms of order of dt or smaller.

When entrainment or deposition occurs, they change the local bed elevation which is known to modify subsequent probabilities of entrainment and deposition in negative feedback (Sawai, 1987; Wong, Parker, DeVries, Brown, & Burges, 2007). Wong et al. (2007) concluded from their experiments that bed elevation changes induce an exponential variation in entrainment and deposition probabilities, while Sawai (1987) concluded from his own experiments that the variation is linear – a trend also assumed by Martin et al. (2014). For simplicity, we incorporate the scaling of Sawai (1987) and Martin et al. (2014), and we note its equivalence to the Wong et al. (2007) scaling when bed elevation changes are small. These negative feedbacks can be incorporated into the entrainment and deposition rates as $\chi(m) = \chi_0(1 \pm z_1 z(m)/(2l)^2)$, where $\chi = \lambda, \mu, \sigma$, and entrainment parameters take the plus sign while the deposition parameter takes the minus, so that entrainment is accentuated when $z(m) > 0$ and deposition is accentuated

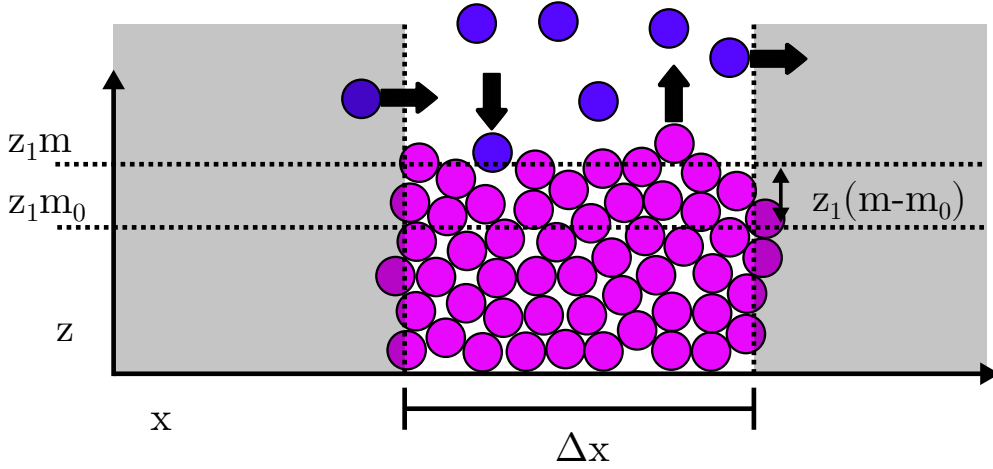


Figure 1. The conceptual picture of a control volume containing n moving particles and m resting particles. Migration in, entrainment, deposition, and migration out are represented by arrows, and the probability distribution of bed elevations is illustrated.

when $z(m) < 0$. With these substitutions, the entrainment and deposition rates become:

$$R_E(n+1, m-1|n, m) = (\lambda_0 + \mu_0 n)(1 + z_1 z(m)/(2l)^2) + O(dt), \quad (6)$$

$$R_D(n-1, m+1|n, m) = \sigma_0(1 - z_1 z(m)/(2l)^2)n + O(dt). \quad (7)$$

At $z(m) = 0$, these reduce to the transition rates of the Ancy et al. (2008) theory. Away from this elevation, entrainment and deposition are alternatively suppressed and accentuated, depending on the sign of $z(m)$, introducing a mean-reverting character to bed elevations similar to the Martin et al. (2014) theory. In these equations, l is a length scale of bed elevation change at which the entrainment and deposition rates are significantly affected, and the ratio z_1/l controls the sensitivity of these feedbacks to the addition or removal of a single particle. The feedbacks between bed elevation changes and entrainment and deposition are strong if z_1/l is large, and they're weak if z_1/l is small. These equations reflect the trend that entrainment is more likely from regions of exposure, while deposition is more likely in regions of shelter (e.g. Wong et al., 2007).

Within our model, a transition from a state n, m to a state n', m' occurs in a small increment dt of time with probability

$$\begin{aligned} R(n', m'|n, m) = & R_{MI}(n', m'|n, m)\delta_{n', n+1}\delta_{m', m} \\ & + R_E(n', m'|n, m)\delta_{n', n+1}\delta_{m', m-1} \\ & + R_D(n', m'|n, m)\delta_{n', n-1}\delta_{m', m+1} \\ & + R_E(n', m'|n, m)\delta_{n', n-1}\delta_{m', m+1} + O(dt), \end{aligned} \quad (8)$$

being zero for all transitions except the allowed four: (1) migration in, (2) entrainment, (3) deposition, and (4) migration out. The forward Kolmogorov equation $P(n, m; t + dt) = \sum_{n', m'} R(n', m'|n, m)P(n, m; t)$ describing the flow of probability at the state n, m between times t and $t + dt$ is

$$\begin{aligned} \frac{\partial P}{\partial t}(n, m; t) = & \nu P(n-1, m; t) + \{\lambda(m+1) + [n-1]\mu(m+1)\}P(n-1, m+1; t) \\ & + [n+1]\sigma(m-1)P(n+1, m-1; t) + [n+1]\gamma P(n+1, m; t) \\ & - \{\nu + \lambda(m) + n\mu(m) + n\sigma(m) + n\gamma\}P(n, m; t) \end{aligned} \quad (9)$$

in the $dt \rightarrow 0$ limit (Cox & Miller, 1965; Gillespie, 1992). This is the master equation for the probability distribution $P(n, m; t)$ that there are n moving particles and m stationary particles at time t within the control volume. Its solution provides probability distributions for the particle activity, a proxy for the bedload rate (e.g. Ancey et al., 2008; Furbish, Ball, & Schmeeckle, 2012), and the number of resting particles, linked through equation 1 to the local bed elevation. This model is a two-species stochastic birth-death theory (e.g. Pielou, 1977) for the joint dynamics of bedload transport and bed elevations.

3 Simulations

Equation 9 does not admit an analytical solution, at least not by the methods applied to similar systems in the population ecology literature (e.g. Swift, 2002). In response to this difficulty we pursue a numerical simulation. Simulation of these birth-death type master equations has been extensively studied for its relevance to chemical physics and population ecology. Balancing conceptual simplicity against computational efficiency, we choose the classic Gillespie algorithm (Gillespie, 1977, 1992, 2007). The Gillespie algorithm leverages the defining property of Markov processes that the time interval between any subsequent transitions is exponentially distributed. When a transition occurs, its type is randomly selected depending on the relative rates of all possibilities. Accordingly, to step our birth-death process through a single transition, we can select the time to the next transition by drawing a random value from an exponential distribution, then we select the type of transition which occurs by selecting a random value from a uniform distribution. After this transition is enacted, i.e., by stepping n and m by the shifts associated with the type of transition which occurred, this two-stage selection is repeated indefinitely to form an exact realization of the stochastic process (Gillespie, 1977, 1992, 2007). Computing the joint statistics of n and m from these simulations provides a numerical approximation of equation 9.

Table 1. Parameters from Ancey et al. (2008) experiments describing the rates of migration in, entrainment, deposition, and migration out of the control volume when $z(m) = 0$. All units are s^{-1} (probability/time).

Flow	ν	λ_0	μ_0	σ_0	γ
(a)	5.45	6.59	3.74	4.67	0.77
(g)	7.74	8.42	4.34	4.95	0.56
(i)	15.56	22.07	3.56	4.52	0.68
(l)	15.52	14.64	4.32	4.77	0.48
(n)	15.45	24.49	3.64	4.21	0.36

Using Gillespie’s stochastic simulation algorithm, we simulated 5 flow conditions from the Ancey et al. (2008) experiments, prescribing 8 different values of the parameter l , ranging from half of the particle radius ($l = a/2$) to 4 particle diameters ($l = 8a$) for a total of 40 simulations. The parameters used for our simulations are taken from the experiments of Ancey et al. (2008) and are summarized in table 1. They are labeled (a), (g), and so on, in order of increasing bedload transport rate. Depending on the flow condition, exact stochastic trajectories consistent with equation 9 were simulated for either a 500 hr or 1,000 hr duration. In all simulations, we take the packing fraction $\phi = 0.6$, a typical value for a pile of spheres (Bennett, 1972), and set $\Delta x = 22.5cm$ and $a =$

0.3cm in accord with the Ancey et al. (2008) experiments.

One simulated realization of our joint stochastic process for bed elevations and bedload transport is depicted in figure 2 (a). These realizations determine the joint statistics of n and m . The bedload transport statistics are depicted for a subset of all simulation conditions in figure 2 (b). The Ancey et al. (2008) theory predicts negative binomial distributions for the number of moving particles within the control volume, and the mathematical form of these distributions is apparently not changed by our extension to

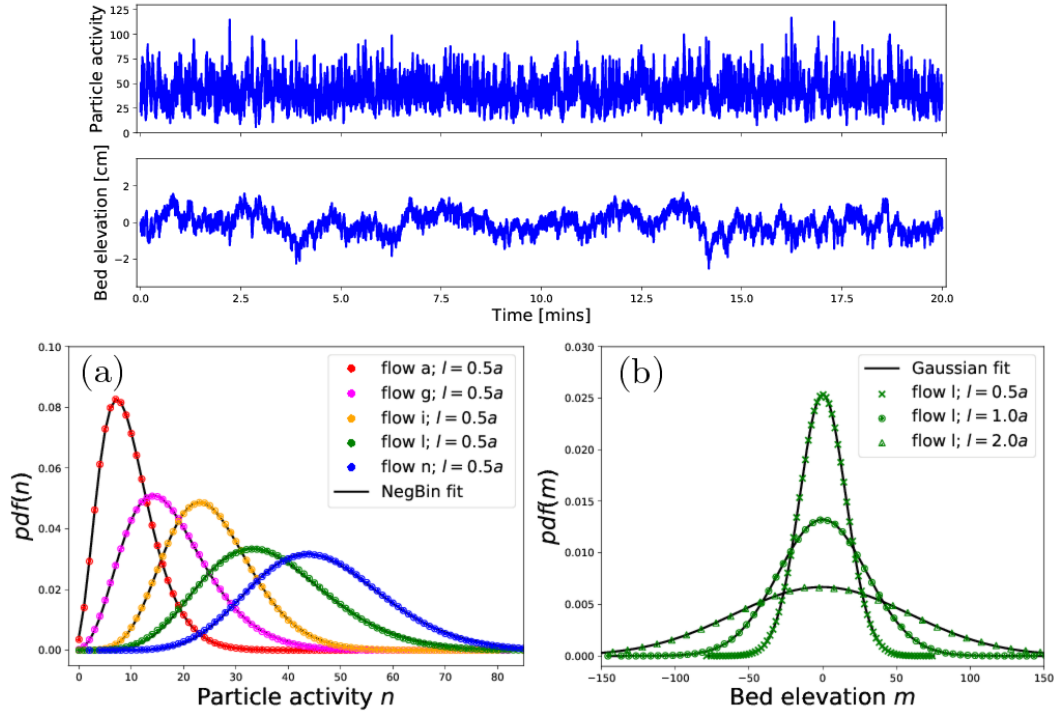


Figure 2. Figure (a) depicts timeseries of particle activity and bed elevation over a 20 minute interval. Figures (b) and (c) display probability distribution functions of bed elevation (equation 1) and particle activity for a subset of the simulations. Colors represent flow conditions, while differing line styles represent different values of the differential mobility parameter l .

account for feedbacks between bed elevation changes and entrainment and deposition probabilities. We obtain an excellent negative binomial fit to the marginal probability distribution of n , $P(n) = \sum_m P(n, m, t)$, for all 40 of our simulation results. However, particle activity statistics, including the mean particle activity and the variance of particle activity, are definitely shifted by the inclusion of differential mobility with bed elevation changes. Bed elevation changes appear to buffer the magnitude of bedload fluctuations by up to 30 percent, which is an expected effect of the model, since a rapid increase in the bedload rate induced by a series of many entrainments will lower the bed elevation and increase the probability of deposition, buffering the magnitude of the bedload rate increase.

Our bed elevation timeseries exhibit longer temporal correlations than related bedload activity series, evident in figure 2 (a). All 40 of our simulations develop clean unimodal distributions of bed elevations which are fit by Gaussian distributions with excellent correlation, and a subset of these marginal bed elevation pdfs with their Gaussian fits are displayed in figure 2 (c). The mean number of particles resting on the bed is m_0 , corresponding to a relative elevation $z(m_0) = 0$. The variance of bed elevations is controlled by the differential mobility parameter l . Apparently, the simulations support the conclusion that $var(m) = (l/z_1)^2$. This conclusion is evident in figure 3, with generally excellent correspondence between this relationship and the simulation points, with some scatter we attribute to the finite duration of our simulations.

To compute the resting time distribution; at each elevation m , we extract the set of all departure times from this elevation, or times at which the bed was at this elevation and a deposition occurred; then we extract the set of all return times to this elevation, or times at which the bed was one increment above this elevation ($m+1$) and

an entrainment occurred. Taking differences between these two time-series returns the set of all return times from above marginal to the elevation m , which we binned across a 0.5s interval to compute the cumulative probabilities of return times at each elevation m , $P(T_r > t|m)$. Following earlier investigators, we computed the unconditional or over-all rest time distribution as the convolution of these conditional distributions over all bed elevations (e.g. Voepel et al., 2013; ?):

$$P(T_r > t) = \sum_m P(m)P(T_r > t|m), \quad (10)$$

where $P(m)$ is the pdf of bed elevation like those depicted in figure 2 (b) and the sum is over all bed elevations attained during the simulation.

This analysis derives unconditional exceedance probabilities of resting times with heavy power-law tails. A subset of all of our simulation results are depicted in figure 4 (a)-(d). Apparently, for suitably long times, the tail parameter α of these resting time distributions is independent of flow conditions or the differential mobility parameter l . However, the timescale at which particle resting transitions from exponential to power-law scaling shifts with flow conditions and l . Martin et al. (2014) obtained an approximate collapse at the tails of their experimental resting time distributions using the reciprocal of the rate of entrainment or deposition events occurring. They denoted this rate by a , so that their timescale was $1/a$. Scaling the resting times by $1/a$ provides an incomplete collapse of the tails of our simulated resting time distributions, which may describe the incomplete collapse of the experimental data of Martin et al. (2014). It collapses the tails across flow conditions when l (the standard deviation of bed elevation) is fixed, i.e., it leads to the collapse seen between figures 4 (a) and 4 (b), but if l (which is the standard deviation of bed elevation) varies with flow, then the power-law scaling of resting times is no longer controlled by $1/a$ alone. Instead, we must include a factor representing the differential entrainment and deposition characteristics of grains as the bed elevation changes. The timescale which provides universal collapse of the power-law tails of all our simulations is $T_0 = l/(z_1 E)$, where E is the entrainment rate.

We can understand T_0 with a physical argument. According to figure 3, the typical length scale of bed elevation fluctuations is l , and as mentioned the length scale z_1 is the magnitude of bed elevation change enacted by the entrainment or deposition of a single particle. In equilibrium bedload transport, H. A. Einstein (1950) tells us the condition $E = D$ holds: this is a statement of mass conservation. Since E represents the mean number of particles removed from the bed in a unit of time, the product Ez_1 can be interpreted as a representative velocity scale of bed return. It is the distance the bed lowers with the removal of a single particle divided by the mean time required to remove it. Hence we extract our timescale as a key distance scale over a key velocity scale: $T_0 = l/(z_1 E)$. Scaling the resting time as T_r/T_0 exhibits a consistent power-law tail across all of our simulation results.

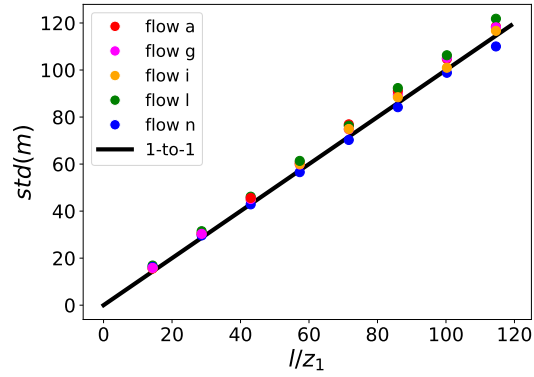


Figure 3. The standard deviation of bed elevation scales one-to-one with the differential mobility parameter (l/z_1), indicating the ratio l/z_1 controls the magnitude of bed elevation fluctuations.

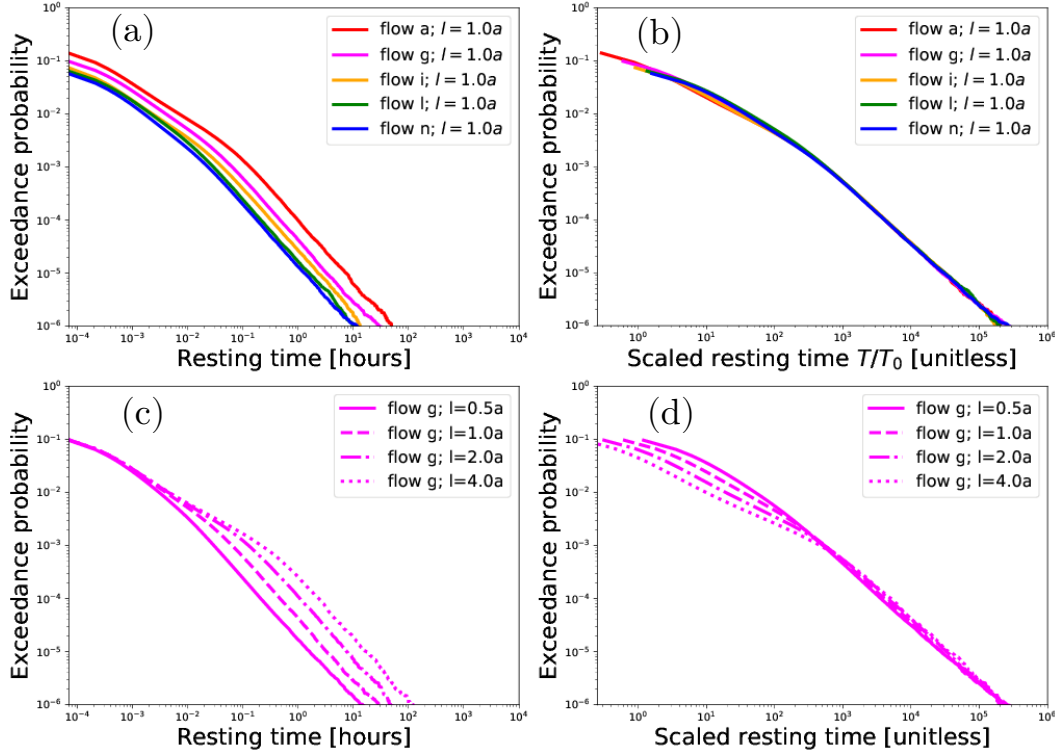


Figure 4. This figure summarizes the resting time exceedance distributions for a subset of all simulations. Part (a) displays resting time distributions for a range of flow conditions at fixed l , while part (b) displays the collapse obtained by scaling T_r by T_0 . The collapse between (a) and (b) is analogous to Martin et al. (2014), and it is induced by the factor of $1/E$ within T_0 . Parts (c) and (d) display a similar collapse for a fixed flow condition at variable l . In this case, collapse is driven by the factor of z_1/l within T_0 , and this influence of differential mobility in resting statistics has not to our knowledge been noticed up to now.

4 Discussion

I'm going to improve it. Any thoughts are appreciated!

5 Conclusion

Acknowledgments

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References

- Ancey, C., Davison, A. C., Böhm, T., Jodeau, M., & Frey, P. (2008). Entrainment and motion of coarse particles in a shallow water stream down a steep slope. *Journal of Fluid Mechanics*, 595(2008), 83–114. Retrieved from http://www.journals.cambridge.org/abstract{_}S0022112007008774 doi: 10.1017/S0022112007008774
- Bennett, C. H. (1972). Serially deposited amorphous aggregates of hard spheres. *Journal of Applied Physics*, 43(6), 2727–2734. doi: 10.1063/1.1661585
- Bradley, D. N. (2017). Direct Observation of Heavy-Tailed Storage Times of Bed

- Load Tracer Particles Causing Anomalous Superdiffusion. *Geophysical Research Letters*, 44(24), 12,212–227,235. doi: 10.1002/2017GL075045
- Bradley, D. N., Tucker, G. E., & Benson, D. A. (2010). Fractional dispersion in a sand bed river. *Journal of Geophysical Research*, 115, F00A09. Retrieved from <http://doi.wiley.com/10.1029/2009JF001268> doi: 10.1029/2009JF001268
- Cox, D. R., & Miller, H. (1965). *The Theory of Stochastic Processes*. London: Chapman and Hall.
- Einstein, A. (1905). Brownian Movement. *Biotropica*, 1. doi: 10.5167/uzh-53657
- Einstein, H. A. (1937). *Bed-load transport as a probability problem* (Unpublished doctoral dissertation). ETH Zurich.
- Einstein, H. A. (1950). *The Bed-Load Function for Sediment Transportation in Open Channel Flows* (Tech. Rep. No. 1026). Washington, DC: United States Department of Agriculture.
- Fan, N., Xie, Y., & Nie, R. (2017). Bed load transport for a mixture of particle sizes: Downstream sorting rather than anomalous diffusion. *Journal of Hydrology*, 553, 26–34. Retrieved from <http://dx.doi.org/10.1016/j.jhydrol.2017.07.012> doi: 10.1016/j.jhydrol.2017.07.012
- Furbish, D. J., Ball, A. E., & Schmeeckle, M. W. (2012). A probabilistic description of the bed load sediment flux: 4. Fickian diffusion at low transport rates. *Journal of Geophysical Research: Earth Surface*, 117(3), 1–13. doi: 10.1029/2012JF002356
- Gaeuman, D., Stewart, R., Schmandt, B., & Pryor, C. (2017). Geomorphic response to gravel augmentation and high-flow dam release in the Trinity River, California. *Earth Surface Processes and Landforms*, 42(15), 2523–2540. doi: 10.1002/esp.4191
- Gillespie, D. T. (1977). Exact stochastic simulation of coupled chemical reactions. *Journal of Physical Chemistry*, 81(25), 2340–2361. doi: 10.1021/j100540a008
- Gillespie, D. T. (1992). *Markov Processes: An Introduction For Physical Sciences*. Academic Press, Inc.
- Gillespie, D. T. (2007). Stochastic Simulation of Chemical Kinetics. *Annual Review of Physical Chemistry*, 58(1), 35–55. Retrieved from <http://www.annualreviews.org/doi/10.1146/annurev.physchem.58.032806.104637> doi: 10.1146/annurev.physchem.58.032806.104637
- Habersack, H. M. (2001). Radio-tracking gravel particles in a large braided river in New Zealand: A field test of the stochastic theory of bed load transport proposed by Einstein. *Hydrological Processes*, 15(3), 377–391. doi: 10.1002/hyp.147
- Hassan, M. A., & Bradley, D. N. (2017). Geomorphic controls on tracer particle dispersion in gravel bed rivers. In *Gravel-bed rivers: Processes and disasters* (pp. 159–184). New York, NY: John Wiley & Sons Ltd. doi: 10.16719/j.cnki.1671-6981.2015.03.007
- Hassan, M. A., Voepel, H., Schumer, R., Parker, G., & Fraccarollo, L. (2013). Displacement characteristics of coarse fluvial bed sediment. *Journal of Geophysical Research: Earth Surface*, 118(1), 155–165. doi: 10.1029/2012JF002374
- Malmon, D. V., Reneau, S. L., Dunne, T., Katzman, D., & Drakos, P. G. (2005). Influence of sediment storage on downstream delivery of contaminated sediment. *Water Resources Research*, 41(5), 1–17. doi: 10.1029/2004WR003288
- Martin, R. L., Jerolmack, D. J., & Schumer, R. (2012). The physical basis for anomalous diffusion in bed load transport. *Journal of Geophysical Research: Earth Surface*, 117(1), 1–18. doi: 10.1029/2011JF002075
- Martin, R. L., Purohit, P. K., & Jerolmack, D. J. (2014). Sedimentary bed evolution as a mean-reverting random walk: Implications for tracer statistics. *Geophysical Research Letters*, 41(17), 6152–6159. doi: 10.1002/2014GL060525
- Nakagawa, H., & Tsujimoto, T. (1976). On Probabilistic Characteristics of Motion

- of Individual Sediment Particles on Stream Beds. In *Hydraulic problems solved by stochastic methods: Second international iahr symposium on stochastic hydraulics*. Lund, Sweden.
- Nakagawa, H., & Tsujimoto, T. (1980). Sand bed instability due to bed load motion. *Journal of the Hydraulics Division-ASCE*.
- Nikora, V. (2002). On bed particle diffusion in gravel bed flows under weak bed load transport. *Water Resources Research*, 38(6), 1–9. Retrieved from <http://doi.wiley.com/10.1029/2001WR000513> doi: 10.1029/2001WR000513
- Olinde, L., & Johnson, J. P. L. (2015). Using RFID and accelerometer-embedded tracers to measure probabilities of bed load transport, step lengths, and rest times in a mountain stream. *Water Resources Research*, 51, 7572–7589. doi: 10.1002/2014WR016259
- Phillips, C. B., Martin, R. L., & Jerolmack, D. J. (2013). Impulse framework for unsteady flows reveals superdiffusive bed load transport. *Geophysical Research Letters*, 40(7), 1328–1333. doi: 10.1002/grl.50323
- Pielou, E. (1977). *Mathematical Ecology* (1st ed.). New York, NY: John Wiley & Sons Ltd.
- Pretzlav, K. L. (2016). *Armor development and bedload transport processes during snowmelt and flash floods using laboratory experiments, numerical modeling, and field-based motion-sensor tracers* (Unpublished doctoral dissertation). University of Texas.
- Sawai, K. (1987). Dispersion of bed load particles. *Bull. Disas. Prev. Res. Inst., Kyoto Univ.*, 37(Part 1, No. 323).
- Swift, R. J. (2002). A Stochastic Predator-Prey Model. *Bulletin of the Irish Mathematical Society*, 48, 57–63.
- Voepel, H., Schumer, R., & Hassan, M. A. (2013). Sediment residence time distributions: Theory and application from bed elevation measurements. *Journal of Geophysical Research: Earth Surface*, 118(4), 2557–2567. doi: 10.1002/jgrf.20151
- Weeks, E. R., & Swinney, H. L. (1998). Anomalous diffusion resulting from strongly asymmetric random walks. *Physical Review E - Statistical Physics, Plasmas, Fluids, and Related Interdisciplinary Topics*, 57(5), 4915–4920. doi: 10.1103/PhysRevE.57.4915
- Wilcock, P. R., & Crowe, J. C. (2003). Surface-based transport model for mixed-size sediment. *Journal of Hydraulic Engineering*, 129(2), 120–128. doi: 10.1061/(ASCE)0733-9429(2003)129:2(120)
- Wong, M., Parker, G., DeVries, P., Brown, T. M., & Burges, S. J. (2007). Experiments on dispersion of tracer stones under lower-regime plane-bed equilibrium bed load transport. *Water Resources Research*, 43(3), 1–23. doi: 10.1029/2006WR005172
- Yalin, M. S. (1972). *Mechanics of Sediment Transport*. Pergamon Press.
- Yang, C. T., & Sayre, W. W. (1971). Stochastic model for sand dispersion. *Journal of the Hydraulics Division, ASCE*, 97(HY2).
- Yano, K. (1969). Tracer Studies on the Movement of Sand and Gravel. In *Proceedings of the 12th congress iahr, vol 2*. (pp. 121–129). Kyoto, Japan.
- Zhang, Y., Meerschaert, M. M., & Packman, A. I. (2012). Linking fluvial bed sediment transport across scales. *Geophysical Research Letters*, 39(20), 1–6. doi: 10.1029/2012GL053476