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Key Points:
- An Exner equation yields a Master Equation for tracer pebble dispersion
- This Master Equation captures the effect of mixing in the vertical
- Vertical dispersion is nonlocal but cannot be expressed with fractional derivatives

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Exner-Based Master Equation for transport and dispersion of river pebble tracers: Derivation, asymptotic forms, and quantification of nonlocal vertical dispersion

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Abstract Ideas deriving from the standard formulation for continuous time random walk (CTRW) based on the Montroll-Weiss Master Equation (ME) have been recently applied to transport and diffusion of river tracer pebbles. CTRW, accompanied by appropriate probability density functions (PDFs) for walker step length and waiting time, yields asymptotically the standard advection-diffusion equation (ADE) for thin-tailed PDFs and the fractional advection-diffusion equation (fADE) for heavy-tailed PDFs, the latter allowing the possibilities of subdiffusion or superdiffusion. Here we show that the CTRW ME is inappropriate for river pebbles moving as bed load: a deposited particle raises local bed elevation, and an entrained particle lowers it so that particles interact with the "lattice" of the sediment-water interface. We use the Parker-Paola-Leclair framework, which is a probabilistic formulation of the Exner equation of sediment conservation, to develop a new ME for tracer transport and dispersion for alluvial morphodynamics. The formulation is based on the existence of a mean bed elevation averaged over fluctuations. The new ME yields asymptotic forms for ADE and fADE that differ significantly from CTRW. It allows vertical as well as streamwise advection-diffusion. Vertical dispersion is nonlocal but cannot be expressed with fractional derivatives. In order to illustrate the new model, we apply it to the restricted case of vertical dispersion only, with both thin and heavy tails for relevant PDFs. Vertical dispersion shows a subdiffusive behavior.

1. Introduction

The erosion, transport, and deposition of pebbles in rivers have often been studied by considering the motion of tracer particles. Such studies have been staple components of field research [e.g., *Ferguson and Hoey*, 2002; *Ferguson et al.*, 2002; *Hassan et al.*, 2013] as well as experimental investigations [e.g., *Wong et al.*, 2007; *Martin et al.*, 2012]. The theoretical basis for the study of the dispersal of sediment tracer particles was delineated by *Einstein* [1950], who formulated the problem in terms of a standard 1-D random walk in which each particle moves in a series of steps punctuated by waiting times [see also *Nakagawa and Tsujimoto*, 1976; *Tsujimoto*, 1978]. More specifically, each particle moves a step of length r [L] after waiting time τ [T], the statistics of which govern tracer particle dispersal.

Let $p_s(r)$ [1/L] and $p_w(\tau)$ [1/T] denote the probability density functions (PDFs) of step length and waiting time, respectively. When both these PDFs have thin tails, such that $p_s(r)$ decays exponentially as $r \to \infty$ and $p_w(\tau)$ decays exponentially as $\tau \to \infty$, the formulation can be reduced asymptotically to a standard advection-diffusion equation (ADE), according to which the streamwise spatial standard deviation σ [L] of a patch of tracer particles increases with the square root of time t [T], i.e., as $t^{1/2}$.

Subsequent to Einstein's original work on tracers, the study of random walks has been extended to the case of continuous time random walks (CTRW) [*Montroll and Weiss*, 1965]. This more general formulation, which derives from a Master Equation (ME) governing the statistics of a walker, leads to a much richer range of behaviors. More specifically, the CTRW formulation allows exploration of the consequences of heavy-tailed PDFs for $p_s(r)$ or $p_w(\tau)$, i.e., PDFs that decay in r or τ according to a power law rather than exponentially. In such cases, moments above some value fail to exist. The asymptotic consequence of such formulation is a fractional advection-diffusion equation (fADE) allowing for the possibility of anomalous diffusion, such that $\sigma \sim t^{\chi/2}$, where χ can deviate from unity. The case encompassing anomalously long step length r corresponding to heavy-tailed $p_s(r)$ gives rise to superdiffusion, for which $\chi > 1$, and the case encompassing anomalously long waiting time τ corresponding to heavy-tailed $p_w(\tau)$ gives rise to subdiffusion, for which $\chi < 1$ [e.g., *Schumer et al*., 2009]. Various combinations of heavy-tailed jump length and waiting time distributions can lead to nonintuitive ballistic, superdiffusive, or subdiffusive behavior, particularly in the case of asymmetric random walks [*Weeks et al*., 1996].

In recent years, the concepts of CTRW and fADE have filtered into the study of tracer sediment transport in rivers, as well as the study of particle tracer transport in the more general context of Earth surface processes [e.g., *Nikora et al*., 2002; *Schumer et al*., 2009; *Furbish et al*., 2009; *Bradley et al*., 2010; *Ganti et al*., 2010; *Furbish et al*., 2012; *Martin et al*., 2012; *Zhang et al*., 2012]. To provide context for these applications, we summarize here some results of *Schumer et al*. [2009] pertaining to Montroll-Weiss CTRW. The standard random walk model with thin-tailed functions for $p_s(r)$ and $p_w(\tau)$ applied in the context of CTRW gives rise to ADE, i.e.,

$$\frac{\partial f_a}{\partial t} + c\frac{\partial f_a}{\partial x} = D_d\frac{\partial^2 f_a}{\partial x^2} \tag{1}$$

In the above equation, x [L] denotes the streamwise coordinate, t [T] denotes time, $f_a(x,t)$ denotes the fraction of particles within some reservoir layer near the bed surface (active layer [see *Ganti et al*., 2010]) that are tracers at (x, t), c [L/T] denotes a particle advection velocity, and D_d [L^2/T] denotes a particle diffusivity (or more properly, dispersivity).

When $p_s(r)$ is heavy tailed such that it has a mean but no standard deviation, i.e.,

$$p_s(r) \sim r^{-\alpha} \tag{2a}$$

$$1 < \alpha < 2 \tag{2b}$$

or $p_w(\tau)$ is heavy tailed such that it has no mean, i.e.,

$$p_w(\tau) \sim \tau^{-\gamma} \tag{3a}$$

$$0 < \gamma < 1 \tag{3b}$$

the relation governing tracer particle dispersal obtained from the ME of CTRW is no longer equation (1) but rather the more general fADE formulation:

$$\frac{\partial^\gamma f_a}{\partial t^\gamma} + c\frac{\partial f_a}{\partial x} = D_d\frac{\partial^\alpha f_a}{\partial x^\alpha} \tag{4}$$

Strictly speaking, in the above equations c is no longer an advection velocity and D_d is no longer a diffusivity, because the respective dimensions are L/T$^\gamma$ and L$^\alpha$/T$^\gamma$, but they can be treated as such in a general sense. It can be found from equation (4) that the growth rate of the streamwise standard deviation σ of a patch of tracers now obeys the relation:

$$\sigma \sim t^{\frac{\gamma}{\alpha}} \tag{5}$$

The case of standard ADE is captured by the choices $\gamma = 1$ and $\alpha = 2$. In the anomalous formulation, the derivatives are fractional; the choices $\gamma = 1$ and $\alpha < 2$ lead to superdiffusion, and the choices $\gamma < 1$ and $\alpha = 2$ lead to subdiffusion. More generally, superdiffusion prevails when $\gamma/\alpha > 0.5$ and subdiffusion prevails when $\gamma/\alpha < 0.5$.

Superdiffusive behavior of particle tracer dispersion might be generated by mechanisms which allow for some particles to travel very long distances in a single step. One example of such a mechanism is that of preferential-connected lanes of transport [*Parker*, 2008; *Ganti et al*., 2010] presented another example associated with step length variation in grain size mixtures. Subdiffusion might be generated by burial of particles in zones where reexhumation is unlikely [e.g., *Voepel et al*., 2013; *Stark et al*., 2009] have considered related problem in which long residence time of alluvium inhibits bedrock incision. Both these behaviors can be studied directly by analyzing data for dispersal patterns, without invoking either the framework of CTRW or a governing Master Equation [e.g., *Nikora et al*., 2002; *Bradley et al*., 2010; *Martin et al*., 2012].

The above notwithstanding, a deeper understanding of tracer particle dispersion in the context of CTRW requires the delineation of a ME suitable to the problem. To date, there have been two notable attempts to do so for the case of sediment transport in rivers, i.e., those of *Ganti et al*. [2010] and *Furbish et al*. [2012]. Both of these expositions helped motivate the research presented here. This, notwithstanding, neither include the degree of freedom associated with particle deposition and entrainment from an arbitrary bed elevation. Here we tackle the problem of delineating a generalized ME for the case of bed load transport in rivers. Our model, the Exner-based Master Equation (EBME) encompasses both thin- and heavy-tailed step length behavior. More importantly, it considers the entrainment and deposition of particles on an elevation-specific basis, a key feature needed to describe the exchange of particles in the vertical direction as they disperse downstream. This feature is needed to describe the advective slowdown of tracer particles described by *Ferguson and Hoey* [2002] as particles are buried ever more deeply.

As in the case of the model of *Ganti et al.* [2010], the model presented here considers only the statistics of step length. The statistics of waiting time are replaced by a specified mean frequency of entrainment of bed particles into bed load transport. The model, however, can be extended to include the statistics of waiting time.

2. Master Equation for the Standard CTRW Model

As noted above, fADE was originally derived in terms of a specific Master Equation governing CTRW. This Master Equation, while of historical value in the development of CTRW, is inappropriate for the description of bed load tracer particle dispersion in rivers. In order to illustrate this, it is useful to briefly review the formulation. In the process of doing so, we introduce the tools necessary to develop our Exner-based Master Equation (EBME). The analysis presented here mostly follows that of *Schumer et al.* [2009]; standard results from fractional calculus are used without specific citation.

In the standard 1-D CTRW formulation [*Montroll and Weiss*, 1965; *Klafter and Silbey*, 1980; *Klafter et al.*, 1987] the ME takes the following form:

$$\rho(x,t) = \int_0^t \int_0^\infty \rho(r,\tau) p_s(x-r) p_w(t-\tau) \mathrm{d}r \mathrm{d}\tau + \left[1 - \int_0^t p_w(\tau)\mathrm{d}t\right]\delta(x) \tag{6}$$

where $\rho(x,t)$ [1/L] denotes the probability density that a particle is at x at time t.

In fact, equation (6) involves two simplifications of *Klafter et al.* [1987]: (a) step length and waiting time are taken to be uncoupled processes and (b) particles are assumed to move only downstream so that $p_s(r)$ vanishes for $r < 0$. The above equation is nonlocal in so far as the kernel in the convolution integral is not concentrated at a single point [*Du et al.*, 2012]. Here we distinguish between two kinds of nonlocality: simple nonlocality associated with a thin-tailed form for p_s or p_w and asymptotic nonlocality associated with the corresponding heavy-tailed form. At $t = 0$, equation (6) reduces to

$$\rho(x,0) = \delta(x) \tag{7}$$

so that a particle originates from the origin.

Laplace transforms in time and Fourier transforms in space are used to reduce equations (6) and (7). Where $A(t)$ is any function of time and $B(x)$ is any function of space, their Laplace and Fourier transforms are given respectively as

$$\widetilde{A}(s) = \int_0^\infty A(t) e^{-st} \mathrm{d}t \tag{8a}$$

$$\hat{B}(k) = \int_{-\infty}^\infty B(x) e^{-ikx} \mathrm{d}x \tag{8b}$$

Applying equation (8b) to equation (7) yields

$$\hat{\rho}(k,0) = 1 \tag{9}$$

Applying equations (8a) and (8b) to equation (6) yields

$$\widetilde{\hat{\rho}}(k,s) = \frac{1-\widetilde{p}_w}{s} \frac{1}{1-\hat{p}_s \widetilde{p}_w} \tag{10}$$

corresponding to equation (21) of *Klafter et al.* [1987] and equation (30) of *Schumer et al.* [2009].

The case of thin tails is considered first. Taylor expansions of $\hat{p}_s(k)$ to second order and $\widetilde{p}_w(s)$ to first order give

$$\hat{p}_s(k) \cong 1 - ik\bar{r} + \frac{1}{2}(ik)^2 \mu_2 + \cdots \tag{11a}$$

$$\widetilde{p}_w(s) = 1 - \bar{\tau} s + \cdots \tag{11b}$$

where $\bar{r}$ [L] and $\bar{\tau}$ [T] denote mean step length and waiting time and μ_2 [L^2] is the second moment of p_s becomes:

$$\bar{r} = \int_0^\infty r p_s(r) \mathrm{d}r \tag{12a}$$

$$\mu_2 = \int_0^\infty r^2 p_s(r) \mathrm{d}r \tag{12b}$$

$$\bar{\tau} = \int_0^\infty \tau p_w(\tau)\mathrm{d}\tau \tag{12c}$$

Substituting equations (11a) and (11b) into equation (10), using the following properties of Fourier and Laplace transforms,

$$\widetilde{\frac{\mathrm{d}A}{\mathrm{d}t}} = s\widetilde{A} - A(0) \tag{13a}$$

$$\widehat{\frac{\mathrm{d}^n B}{\mathrm{d}x^n}} = (ik)^n \hat{B} \tag{13b}$$

reducing with equation (9) and truncating the expansions yields the formulation:

$$\frac{\partial \rho}{\partial t} + c\frac{\partial \rho}{\partial x} = D_d \frac{\partial^2 \rho}{\partial x^2} + \bar{r}\frac{\partial^2 \rho}{\partial x \partial t} \tag{14a}$$

$$c = \frac{\bar{r}}{\bar{\tau}} \tag{14b}$$

$$D_d = \frac{\mu_2}{2\bar{\tau}} \tag{14c}$$

Writing $\bar{r} = c\,\bar{\tau}$ and assuming that c and D_d remain finite in the limit as $\bar{\tau} \rightarrow 0$ allows the cross derivative in x and t to be dropped, so resulting in the ADE formulation. In the analysis below, the cross derivative is retained for generality.

For the case of heavy tails, we now replace equations (11a) and (11b) with the forms (resulting from fractional Taylor expansion)

$$\hat{p}_s(k) \cong 1 - ik\bar{r} + c_\alpha (ik)^\alpha \tag{15a}$$

$$\widetilde{p}_w(s) = 1 - c_\gamma s^\gamma + \cdots \tag{15b}$$

where $0 < \gamma < 1$ (subdiffusive waiting time) and $1 < \alpha < 2$ (superdiffusive step length). Substituting equations (15a) and (15b) into equation (10), using the following properties of fractional Fourier and Laplace transforms,

$$\widetilde{\frac{\mathrm{d}A^\gamma}{\mathrm{d}t}} = s^\gamma \widetilde{A} - A(0) \tag{16a}$$

$$\widehat{\frac{\mathrm{d}^\alpha B}{\mathrm{d}x^\alpha}} = (ik)^\alpha \hat{B} \tag{16b}$$

reducing with equation (9) and truncating the expansions yields the formulation

$$\frac{\partial^\gamma \rho}{\partial t^\gamma} + c\frac{\partial \rho}{\partial x} = \bar{r}\frac{\partial}{\partial x}\frac{\partial^\gamma \rho}{\partial t^\gamma} + D_d \frac{\partial^\alpha \rho}{\partial x^\alpha} \tag{17a}$$

$$c = \frac{\bar{r}}{c_\gamma} \tag{17b}$$

$$D_d = \frac{c_\alpha}{c_\gamma} \tag{17c}$$

Dropping the cross derivative in x and t yields the fADE formulation.

The context in which equations (14a), (14b), (14c) and (17a), (17b), (17c) might be applicable to the case of bed load tracers is limited by the fact that they do not encompass vertical advection and diffusion of such particles. An appropriate candidate for a framework that does not include vertical processes is a restricted version of the active layer formulation, according to which (a) bed load particles exchange only with an "active layer" of bed material at the surface of thickness L_a [L] [*Parker*, 2008; *Ganti et al.*, 2010] and (b) the bed undergoes no aggradation or degradation, i.e., change in mean bed elevation η (averaged over an appropriate window). Let N_{tr} denote the total number of tracer particles released, λ_p denote the porosity of the bed, V_p [L^3] denote particle volume, and B [L] denote the width of the channel. The fraction of grains $f_a(x,t)$ that are tracers at (x,t) is then given as

$$f_a = \frac{N_{tr} V_p}{L_a B(1-\lambda_p)}\rho(x,t) \tag{18}$$

It should be noted that in the analysis presented here, (a) the effect of varying particle shape has been neglected and (b) only uniform sediment is considered.

3. Problems Associated With Application of the CRTW ME to Bed Load Tracers

CRTW was originally derived as the continuous limit for a random walker on a lattice. The lattice simply defines streamwise locations where the particle might come to rest or pass through. The particle does not interact with the lattice.

Bed load transport in a river functions differently. Sediment transport can be divided into two components. Bed material load interacts with the bed by changing its elevation as each grain deposits or erodes. Wash load or throughput load either (a) passes through the reach of interest without changing bed elevation or (b) exchanges between the water column and the bed only via the pores of the bed material, again without changing bed elevation. Here we consider the case of bed load moving as bed material load.

In the case of bed load, the lattice has vertical as well as streamwise locations, and its structure interacts strongly with the particles. A previously moving particle that comes to rest (deposits) raises the bed, and a previously resting particle that moves (is entrained) lowers the bed. Since bed load transport itself is a quasi-random process, the lattice structure through which particles move, and in particular bed elevation at a lattice point, also becomes a quasi-random variable. The ME of CTRW is incapable of handling this interaction.

The starting point for the Exner-based Master Equation (EBME) is the analysis of *Parker et al*. [2000], here referred to as the Parker-Paola-Leclair (PPL) framework. This framework provides a probabilistic framework for sediment conservation that captures the vertical structure of bed elevation variation as particles erode and deposit. Integration of this equation in the vertical yields the standard Exner equation of sediment mass conservation. As opposed to the formulation in *Ganti et al*. [2010], the formulation of *Parker et al*. [2000] does not invoke the simplification of an active layer.

4. PPL Framework for Exner Equation of Sediment Continuity

The classical entrainment form of Exner equation of sediment conservation can be written as [e.g., *Tsujimoto*, 1978; *Parker et al*., 2000; *Ganti et al*., 2010; *Pelosi and Parker*, 2014]

$$\left(1-\lambda_p\right)\frac{\partial \eta(x,t)}{\partial t} = -E(x,t) + D(x,t) \tag{19}$$

where λ_p denotes the bed porosity, η [L] is the mean bed elevation (as defined in terms of spatial or temporal averaging over fluctuations associated with the entrainment or deposition of individual particles), D [L/T] is the mean volume rate of deposition of bed load per unit area per unit time, and E [L/T] is the mean volume rate of entrainment of bed sediment into bed load transport per unit area per unit time. Here we apply equation (19) in the context of bed load transport of particles with uniform size D_p [L] and material density.

The formulation of equation (19) is predicated on the assumption that bed elevation, while fluctuating locally as particles are entrained or deposited, does indeed have a mean value η (averaged over an appropriate window). In the analysis presented here, the details of the statistics of waiting time (as outlined for the case of bed load transport by *Voepel et al*. [2013]) are replaced with a single parameter J [1/T] corresponding to the frequency at which a bed particle is entrained.

It is assumed that (a) the removal from the bed of a cube with volume ${D_p}^3$, where D_p is particle size, corresponds to the removal of a volume of sediment equal to $(1-\lambda_p)\,{D_p}^3$ and (b) the removal or deposition of one particle of diameter D_p results in a change in vertical elevation of D_p. The frequency J of particle entrainment can be used to define a characteristic waiting time τ_{char} [T] as follows:

$$\tau_{\mathrm{char}} = \frac{1}{J} \tag{20}$$

The frequency of entrainment may slowly vary in time, in so far as it correlates with mean flow parameters. The entrainment rate of sediment volume is then given as follows:

$$E(x,t) = \left(1-\lambda_p\right)D_p J(x,t) \tag{21}$$

The deposition rate D can in turn be related to the entrainment rate, E, by means of the probability density of the particle step length, $p_s(r)$ [*Tsujimoto*, 1978; *Parker et al*., 2000; *Ganti et al*., 2010]:

$$D(x,t) = \int_0^\infty E(x-r,t)\,p_s(r)\,\mathrm{d}r \tag{22}$$

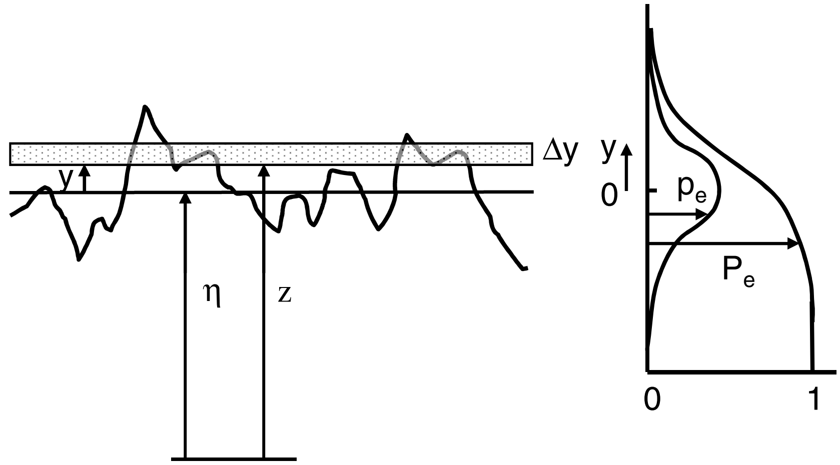

Figure 1. Definition diagram for the Parker-Paola-Leclair framework for sediment and sediment tracer conservation. Here η is the local mean bed elevation, $P_e(x, z, t)$ is the probability that a point at elevation z (or y) is in the sediment bed, and $p_e\ (x,y,t)$ the probability density that instantaneous bed elevation is at level z (or y).

From equations (21) and (22), the deposition rate takes the form

$$D(x,t) = (1-\lambda_p)D_p\int_0^\infty J(x-r,t)p_s(r)\mathrm{d}r \tag{23}$$

Combining equations (19), (21), and (23) yields the integral form of the Exner equation

$$\frac{\partial\eta(x,t)}{\partial t} = -D_pJ(x,t) + D_p\int_0^\infty Jp_s(r)\mathrm{d}r \tag{24}$$

Pelosi and Parker [2014] have studied the behavior of equation (24) in the context of the ratio of mean step length $\bar{r}$ to the length of the reach under consideration. They considered both thin-tailed and heavy-tailed PDFs p_s for step length.

When extended to the case of tracer particles in uniform sediment or to mixtures of grain sizes, equation (24) is usually implemented in the context of an active layer of thickness L_a, as described above [*Parker*, 2008; *Ganti et al.*, 2010]. Thus, the erodible bed is ideally divided into layers: (i) an upper layer (active layer), which has no vertical structure and actively exchanges with the bed load, and (ii) a deeper layer (substrate) that exchanges with the bed load only when aggradation or degradation occurs [e.g., *Viparelli et al.*, 2011].

Parker et al. [2000], however, specified a general probabilistic formulation of the Exner equation of sediment continuity with no discrete layers. It is able to capture vertical exchange of sediment particles, and specifically tracer particles, with no need of the relatively heavy-handed assumption of an active layer.

In the following, we refer to local mean bed elevation as the bed elevation averaged over bed fluctuations.

Let z [L] denote a coordinate oriented upward normal to the local mean bed and $P_e(x,z,t)$ denote the probability that a point at elevation z is in the sediment bed (rather than the water above it; Figure 1). Thus, $P_e(x,z,t)$ approaches unity when $z \to -\infty$ (deep in the deposit) and zero when $z \to \infty$ (in the water column). Because of its definition, P_e also corresponds to the probability that the bed surface elevation is higher than z; hence, the probability density p_e [1/L] that instantaneous bed elevation is at level z is

$$p_e(z) = -\frac{\partial P_e}{\partial z} \tag{25a}$$

$$\int_{-\infty}^{\infty} p_e(z)\mathrm{d}z = 1 \tag{25b}$$

A new vertical coordinate system can now be introduced in terms of the variable y [L], representing the deviation from the mean bed elevation η

$$y = z - \eta(x,t) \tag{26}$$

Consequently, P_e becomes a function of y and equation (25a) takes the form:

$$p_e(y) = -\frac{\partial P_e}{\partial y} \tag{27}$$

Now let p_{JO} (y) [1/L] be the probability density that a particle that is entrained into bed load comes from level y and p_{JI} (y) [1/L] be the probability density that a particle that is deposited is emplaced in the bed at level y. As shown in Figure 1, the volume of sediment per unit length and width contained in a strip with height dy is given as $(1-\lambda_p)P_e\text{d}y$, and the entrainment and deposition rates within this strip are given, respectively, as $(1-\lambda_p)\, p_{JO}\, J\, D_p\text{d}y$ and $(1-\lambda_p)\, p_{JI}\, J\, D_p\text{d}y$. A formulation of mass balance in correspondence with (19) then yields the PPL elevation-specific form of the Exner equation of mass balance

$$\frac{\partial P_e}{\partial t} = -D_pJ(x,t)p_{JO}(y) + p_{JI}(y)D_p\int_0^\infty J(x-r,t)p_s(r)\text{d}r \tag{28}$$

Thus, particles can entrain from any elevation y with probability p_{JO} and deposit into the bed at any elevation y with probability p_{JI}, after having been entrained at any distance r upstream from any level y. The above equation involves a simplification, in that it assumes that the elevation of deposition is uncorrelated with step length.

In general, P_e is a function of x, y, and t, where y is according to equation (26), the elevation relative to the mean bed. Thus, $P_e = P_e(x, z-\eta(t), t)$. Note that P_e can vary in time in two ways; the structure of P_e itself can vary in time, and the value of P_e can change at a given elevation due to bed aggradation. The chain rule applied to equations (26) and (27) yields

$$\frac{\partial P_e[x,y(t),t]}{\partial t} = \frac{\partial P_e(x,y,t)}{\partial t} - \frac{\partial P_e(x,y,t)}{\partial y}\frac{\partial \eta}{\partial t} = \frac{\partial P_e}{\partial t} + p_e(y)\frac{\partial \eta}{\partial t} \tag{29}$$

which substituted in equation (28) gives

$$\frac{\partial P_e}{\partial t} + p_e\frac{\partial \eta}{\partial t} = -D_pJ(x,t)p_{JO}(y) + p_{JI}(y)D_p\int_0^\infty J(x-r,t)p_s(r)\text{d}r \tag{30}$$

Integrating equation (30) in y recovers the Exner equation (24) [*Blom and Parker*, 2004].

Let η_{inst} denotes instantaneous bed elevation. In order for a particle to be eroded from level y, η_{inst} must cross this level downward, and in order for a particle to be deposited at level y, η_{inst} must cross this level upward. Assuming that the frequency of upward crossing is equal to that of downward crossing, it follows that

$$p_{JI}(y) = p_{JO}(y) = p_J(y) \tag{31}$$

so that equation (30) becomes

$$\frac{\partial P_e}{\partial t} + p_e\frac{\partial \eta}{\partial t} = -D_pJ(x,t)p_J(y) + p_J(y)D_p\int_0^\infty J(x-r,t)p_s(r)\text{d}r \tag{32}$$

The assumptions leading to (32) are based on a bed that is in macroscopic equilibrium (constant η) and represent first-order approximations for a bed that is only slowly aggrading or degrading (in which case bed elevation variation is driven by slow spatiotemporal variation in J).

5. Exner-Based Master Equations for Rivers Carrying Bed Load

Equation (32) is, in and of itself, a fairly trivial extension of the Exner formulation. Its true value becomes apparent when applied to bed load tracers. The PPL framework is now applied to this case.

Let $f(x,y,t)$ [1/L] denote the fraction density of tracers at elevation y, such that $f\text{d}y$ defines the fraction of bed particles that are tracers between elevations y and $y+\text{d}y$. The same formulation that yields equation (32) gives the following result for tracer conservation:

$$\begin{aligned} P_e\left(\frac{\partial f}{\partial t} - \frac{\partial f}{\partial y}\frac{\partial \eta}{\partial t}\right) + f\left(\frac{\partial P_e}{\partial t} + p_e\frac{\partial \eta}{\partial t}\right) &= -\,p_J(y)D_pJ(x,t)f(x,y,t) \\ &\quad + p_J(y)D_p\int_{-\infty}^{\infty}\int_0^\infty J(x-r,t)f(x-r,y',t)p_J(y')\,p_s(r)\text{d}r\text{d}y' \end{aligned} \tag{33}$$

Equation (33) defines the Exner-based Master Equation (EBME) for the tracer problem obtained from the PPL framework. Note that this formulation can be nonlocal in x and y [*Du et al.*, 2012]. According to the above relation, a tracer particle may be entrained from any level y' and deposited at a different level y. The model thus incorporates vertical exchange of tracers, a feature that is captured in neither the ME of CTRW nor the active layer formulation of the Exner equation. In addition, tracers are conserved as the bed aggrades and degrades, allowing burial and exhumation to be driven not only by random processes inherent in the density $p_J(y)$ but also through mean bed elevation variation. The formulation does not, however, capture the statistics of waiting time.

For the sake of comparison, it is useful to delineate the Master Equation associated with the active layer formulation used in *Ganti et al*. [2010]. The general form is given in *Parker et al*. [2000] but can also be obtained directly from (33) under the following assumptions: (a) only particles in the active layer exchange with the bed load, (b) the active layer has specified thickness L_a and extends from $y = -L_a$ to 0, (c) $P_e = 1$ within the active layer and 0 outside of it, and (d) $p_e = p_J = 1/L_a$ within the active layer and 0 outside of it. Equation (33) reduces to

$$f_I \frac{\partial \eta(x,t)}{\partial t} + L_a \frac{\partial f_a(x,t)}{\partial t} = -D_p J(x,t) f_a(x,t) + D_p \int_0^\infty J(x-r,t) f_a(x,t) p_s(r) \mathrm{d}r \tag{34}$$

where f_I denotes the fraction of tracers in the bed material that is exchanged between the active layer and substrate as the bed aggrades or degrades.

We refer to the above relation as EBME-A (Exner-based Master Equation, Active layer formulation) below.

6. Vertical and Streamwise Dispersal of Tracers Within an Equilibrium Bed

The physical contents of the above two formulations are best grasped in the context of macroscopic equilibrium conditions, for which the bed neither aggrades nor degrades. For this case, $\partial P_e/\partial t = \partial \eta/\partial t = 0$ and the parameters J, $p_J(y)$ and $p_s(r)$ are assumed to change neither in x nor in t. The respective forms for EBME and EBME-A are as follows. From (34), EBME-A becomes

$$L_a \frac{\partial f_a(x,t)}{\partial t} = -D_p J f_a(x,t) + D_p J \int_0^\infty f_a(x-r,t) p_s(r) \mathrm{d}r \tag{35}$$

From equation (33), EBME becomes

$$P_e(y) \frac{\partial f(x,y,t)}{\partial t} = -D_p J f(x,y,t) p_J(y) + D_p J p_J(y) \int_{-\infty}^\infty \int_0^\infty f(x-r,y',t) p_J(y') p_S(r) \mathrm{d}y' \mathrm{d}r \tag{36}$$

The difference between the EBME and EBME-A formulations is best illustrated by writing equation (36) in terms of a form that is as close as possible to equation (35) and then balancing it with residual terms. In this way, equation (36) can be expressed as

$$\begin{aligned} P_e(y) \frac{\partial f(x,y,t)}{\partial t} = &- D_p J f(x,y,t) p_J(y) + D_p J p_J(y) \int_0^\infty f(x-r,y,t) p_S(r) \mathrm{d}r \\ &- D_p J p_J(y) \int_0^\infty \varphi(x-r,y',t) p_S(r) \mathrm{d}r \end{aligned} \tag{37a}$$

where

$$\varphi(x,y,t) = f(x,y,t) - \langle f \rangle \tag{37b}$$

$$\langle f \rangle = \int_{-\infty}^\infty f(x,y,t) p_J(y) \mathrm{d}y \tag{37c}$$

Here $\langle f \rangle$ denotes the jump-averaged value of f, i.e., the value of f averaged over the probability density function of elevation from which particles jump (are entrained) p_J, and φ is a deviatoric tracer fraction density. A comparison of equations (35) and (37a) reveals that equation (37a) captures a nonlocal feature that equation (35) cannot, i.e., the vertical dispersal of tracers. That is, the term $\partial f/\partial t$ is driven by (among other things) the difference $\varphi = f - \langle f \rangle$ between local tracer fraction and jump-averaged fraction. The minus sign in front of the terms containing the deviatoric term φ in equation (37a) ensures that particles disperse in the vertical from zones that are higher than the jump-averaged mean to those that are lower than it. The evolving distribution of the fraction of tracers at any given level is thus a function of the fraction of tracers at all levels.

7. Asymptotic Fractional Formulations

The convolution forms of EBME-A and EBME are easy to solve numerically in a straightforward way. Their corresponding streamwise fractional forms associated with asymptotic behavior, however, are not easily solved numerically. This notwithstanding, the relevant forms are presented here for the purpose of illustration and comparison with the CTRW ME formulation. The results given below follow from the analysis given in section 2.

EBME-A yields the following asymptotic result for the thin- and heavy-tailed cases, respectively, based on the spatial Fourier transform (8b) and the expansions (11a) or (15a) for p_s. The thin-tailed or ADE form is

$$\frac{\partial f}{\partial t}+c\frac{\partial f}{\partial x}=D_d\frac{\partial^2 f}{\partial x^2} \tag{38a}$$

$$c=\frac{D_p J\bar{r}}{L_a} \tag{38b}$$

$$D_d=\frac{1}{2}\frac{D_c J\mu_2}{L_a} \tag{38c}$$

and the heavy tailed or fADE form is

$$\frac{\partial f}{\partial t}+c\frac{\partial f}{\partial x}=D_d\frac{\partial^\alpha f}{\partial x^\alpha} \tag{39a}$$

$$c=\frac{D_p J\bar{r}}{L_a} \tag{39b}$$

$$D_d=\frac{1}{2}\frac{D_c J c_\alpha}{L_a} \tag{39c}$$

The above relations correspond to the results of *Ganti et al.* [2010].

Similarly, reducing EBME yields the thin-tailed form

$$\begin{aligned}\frac{\partial f}{\partial t}&=-c(y)\frac{\partial f}{\partial x}+D_d(y)\frac{\partial^2 f}{\partial x^2}\\&-K(f-\langle f\rangle)+c(y)\frac{\partial (f-\langle f\rangle)}{\partial x}-D_d(y)\frac{\partial^2 (f-\langle f\rangle)}{\partial x^2}\end{aligned} \tag{40a}$$

$$c(y)=K(y)\bar{r} \tag{40b}$$

$$D_d(y)=\frac{1}{2}K(y)\mu_2 \tag{40c}$$

$$K(y)=\frac{D_p J p_J(y)}{P_e(y)} \tag{40d}$$

$$\langle f\rangle=\int_{-\infty}^{\infty} f p_J \mathrm{d}y \tag{40e}$$

and the heavy-tailed form

$$\begin{aligned}\frac{\partial f}{\partial t}&=-c(y)\frac{\partial f}{\partial x}+D_d(y)\frac{\partial^\alpha f}{\partial x^\alpha}\\&-K(f-\langle f\rangle)+c(y)\frac{\partial (f-\langle f\rangle)}{\partial x}-D_d(y)\frac{\partial^\alpha (f-\langle f\rangle)}{\partial x^\alpha}\end{aligned} \tag{41a}$$

where c and K are given in equations (40b) and (40d) and

$$D_d(y)=K(y)c_\alpha \tag{41b}$$

Now equation (40a) does not define a standard ADE, and equation (41a) does not define a standard fADE. More specifically, the advection speed and diffusivity are functions of the vertical coordinate y in the case of EBME. In addition, vertical mixing is driven by the deviatoric term $\varphi=f-\langle f\rangle$. This deviatoric term, i.e., the third term on the right-hand side (RHS) of equation (41a), captures mixing in the vertical, driven by the difference between the local value f and its jump-averaged value $\langle f\rangle=\int_{-\infty}^{\infty} f(x,y,t)p_J(y)\mathrm{d}y$. In addition, φ can be both advected downstream, in accordance with the fourth term on the RHS of equation (41a), and diffused, in accordance with the fifth term on the RHS of the same equation. This diffusion can be either normal or anomalous, as clearly expressed by the fifth term on the RHS of equation (41a).

8. Simplified Model for Vertical Dispersion: Thin Versus Heavy-Tailed Probability Densities for Elevation From Which a Particle Jumps

The EBME formulation based on Parker-Paola-Leclair results in asymptotic fractional ADEs that show substantial differences from the CTRW ME formulation due to the vertical advection and dispersion terms. We

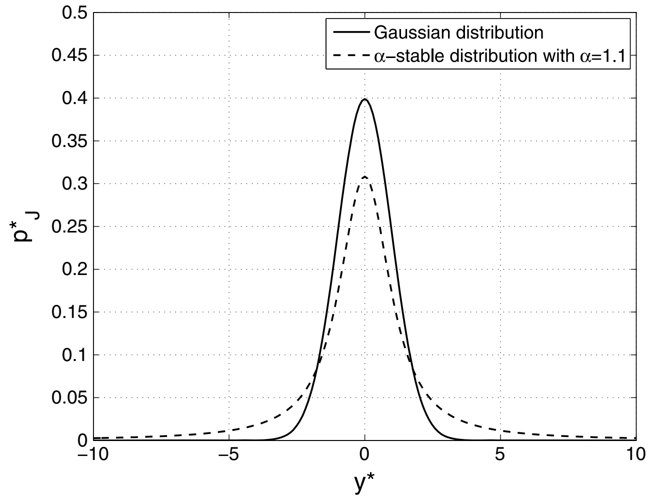

Figure 2. Comparison of thin-tailed (Gaussian) and heavy-tailed PDFs for $p*_J$. The heavy-tailed distribution corresponds to a Lévy α-stable distribution with $\alpha = 1.1$.

do not implement the complete formulation herein. Instead, we show several simplified examples that illustrate the effect of nonlocal vertical dispersion in EBME. We do this by neglecting all the terms on the right-hand side of equation (40a) except the term $K(f - \langle f \rangle)$:

$$P_e \frac{\partial f}{\partial t} = -D_c J p_J(y)[f(y,t) - \langle f \rangle] \tag{42a}$$

$$\langle f \rangle = \int_{-\infty}^{\infty} f(y',t) p_J(y') \mathrm{d}y \tag{42b}$$

Note that the form of the equation is nonlocal, in that the values of f at all elevations y' contribute to the time rate of change of f at any given elevation y. In addition, $p_J(y)$ may be thin tailed or heavy tailed, but there is no obvious way to convert the governing equation into an asymptotic form involving fractional derivatives.

It is important to keep in mind that the conserved quantity in equation (42a) is not the density of fraction of tracers in the sediment $f(y,t)$, but rather the density of fraction of tracers

$$f_{\mathrm{sw}} = P_e f \tag{43}$$

averaged along a line of constant y that includes portions that are instantaneously in sediment and other portions that are instantaneous in the water column (Figure 1).

We first cast the problem in dimensionless form. Let γ^* denote an appropriate length scale (as specified below). Defining

$$f^* = \gamma^* f \tag{44a}$$

$$y^* = \frac{y}{\gamma^*} \tag{44b}$$

$$t^* = \frac{D_p}{\gamma^*} J t \tag{44c}$$

$$p_J = \frac{1}{\gamma^*} p_J^* \tag{44d}$$

it is found that equations (42a) and (42b) reduce to

$$\frac{\partial f}{\partial t^*} = \frac{p_J^*}{P_e}(f - \langle f \rangle) \tag{45a}$$

$$\langle f \rangle = \int_{-\infty}^{\infty} f p_J^* \mathrm{d}y^* \tag{45b}$$

We specify the length scale in terms of two alternatives. To study thin-tailed behavior, we consider a Gaussian distribution for p_J, in which case γ^* is the standard deviation σ_J of p_J and

$$p_J^*(y^*) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{(y^*)^2}{2}\right) \tag{46}$$

To study heavy-tailed behavior, we let γ^* correspond to the scale parameter γ' of a Lévy α-stable distribution, which is defined by its Fourier transform as [*Nolan*, 1997]

$$\widehat{p_J^*}(k) = \left\{-i\delta' k - |\gamma' k|^{\alpha}\left[1 + i\beta\,\mathrm{sgn}(k)\tan\left(\frac{\pi\alpha}{2}\right)\right]\right\} \tag{47}$$

where k is an integer value which defines the type of parameterization (here equal to 1), α denotes the stability parameter (here chosen equal to 1.1), and β denotes the skewness parameter (here set equal to zero).

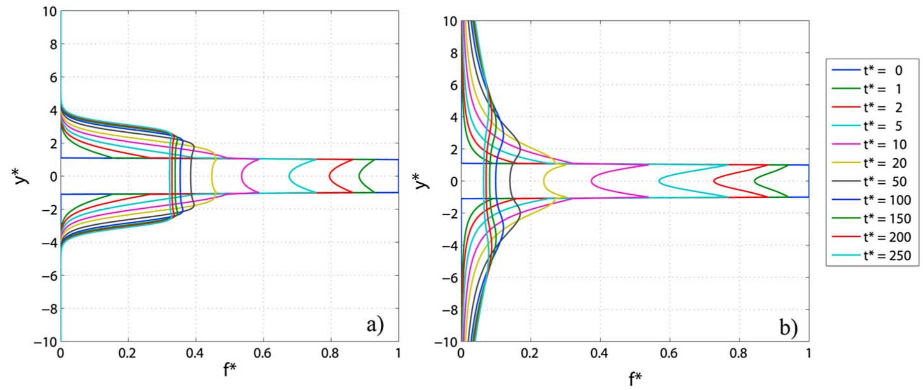

Figure 3. Evolution in time of distribution of tracer fraction f^* for the case of (a) a thin-tailed (i.e., Gaussian) form for the PDF p_J^* describing the probability that an entrained particle jumps from elevation y^* and (b) a heavy-tailed (i.e., α-stable, with $\alpha = 1.1$) form for the PDF p_J^*. Here $P_e = 1$, resulting in a symmetric pattern.

In addition, the location and scale parameters δ' and γ' are, respectively, set equal to 0 and 1, so as to obtain correspondence with equation (46). Note that the Gaussian distribution is a subset of α-stable distributions obtained when $\alpha = 2$. Both these distributions are shown in Figure 2.

The first case we consider is one for which $P_e = 1$. This case does not correspond to a defined bed (interface between water and sediment). Instead, it is predicated on a vertical lattice that is filled with particles. Particle pairs at elevations y and y' exchange according to the PDF $p_J(y)$. In addition, $f_{sw} = f$ according to equation (43). The initial condition used in the calculations was arbitrarily set to the following top-hat distribution:

$$f^*(y^*, 0) = \begin{cases} 1 & , \quad -1 \leq y^* \leq 1 \\ 0 & , \quad |y^*| > 1 \end{cases} \tag{48}$$

Results of the calculations are shown in Figure 3a for the Gaussian distribution and Figure 3b for the α-stable distribution.

Figures 3a and 3b both illustrate the tendency for tracer particles to be dispersed from high to low concentration in the vertical. The pattern of dispersion is non-Fickian, as evidenced by the tendency for the formation of a depression in tracer fraction at $y^* = 0$. The effect of heavy- versus thin-tailed PDFs for p_J is readily apparent; the heavy-tailed case of Figure 3b shows much more rapid and much more far-reaching dispersal than the thin-tailed case of Figure 3a.

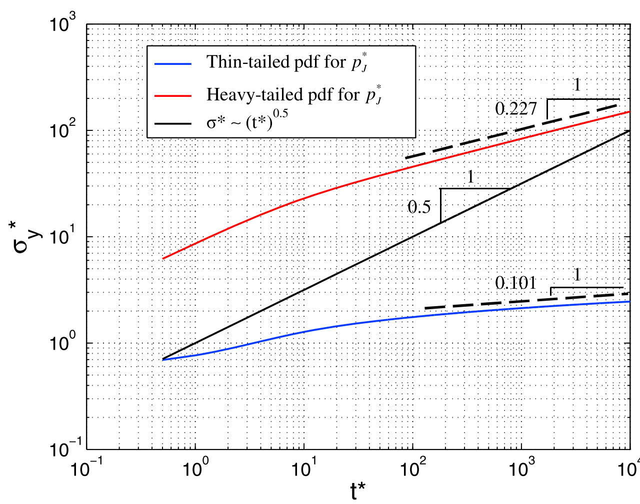

Figure 4. Evolution in time of the vertical standard deviation σ_y^* for the thin- and heavy- tailed case for p_J, with $P_e = 1$. Also plotted is the line $\sigma_y^* \sim (t^*)^{1/2}$ corresponding to normal diffusion.

The mean elevation $\overline{y^*}$ and vertical standard deviation σ_y^* of the tracer particle distribution are given by the respective relations

$$\overline{y^*} = \frac{\int_{-\infty}^{\infty} y^* f_{sw}(y^*) dy^*}{\int_{-\infty}^{\infty} f_{sw}(y^*) dy^*} \tag{49a}$$

$$\left(\sigma_y^*\right)^2 = \frac{\int_{-\infty}^{\infty} (y^* - \overline{y}^*)^2 f_{sw}(y^*) dy^*}{\int_{-\infty}^{\infty} f_{sw}(y^*) dy^*} \tag{49b}$$

In the case $P_e = 1$ with the initial condition (48), the problem is symmetric so that $\overline{y^*} = 0$. Figure 4 shows plots of σ_y^* versus t^* for the thin- and heavy-tailed case. Also plotted is the scale relation $\sigma_y^* \sim (t^*)^{1/2}$ corresponding to normal diffusion. In the thin-tailed case, the vertical dispersion is subdiffusive

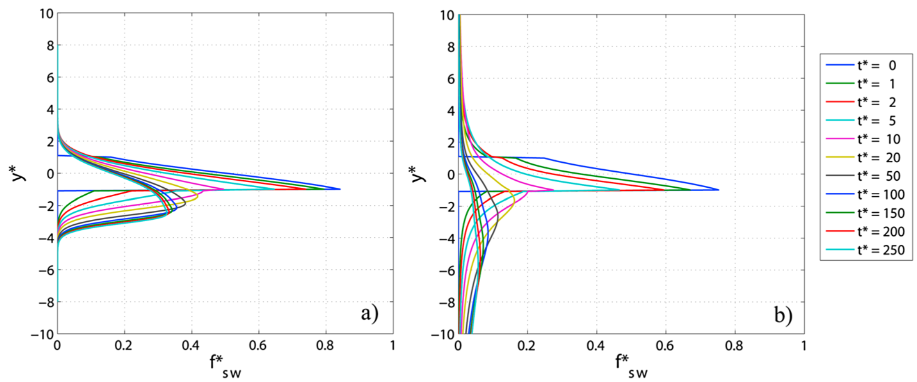

Figure 5. Evolution in time of the distribution of tracer fraction f_{sw}*, for the case of vertically varying P_e corresponding to a fluctuating water-sediment interface (riverbed). (a) Thin-tailed (i.e., Gaussian) form for the PDF p_J* describing the probability that an entrained particle jumps from elevation y* and (b) heavy-tailed (i.e., α-stable, with $\alpha = 1.1$) form for the PDF p_J*.

over the entire range of modeled times, while for the heavy-tailed case dispersion is first nearly normal and then asymptotically becomes subdiffusive.

We now consider solutions to equation (45a) for vertically varying $P_e(y)$ corresponding to the statistics of a fluctuating sediment-water interface (Figure 1). Here we approximate p_e with p_J, as specified in equations (46) and (47), because in order for instantaneous bed elevation η_{inst} to cross elevation y (either upward or downward), it must attain the same value. We then evaluate P_e from equation (27). Otherwise, we solve the problem in the same way as the case $P_e = 1$.

In so far as the conserved quantity in equation (42a) is f_{sw}, rather than f, plots of

$$f^*_{sw} = P_e f^* \tag{50}$$

versus y^* at various times are shown in Figure 5a for the thin-tailed case and Figure 5b for the heavy-tailed case. The tendency for tracer particles to be dispersed from high to low concentration in the vertical is similar to the case $P_e = 1$ shown above but is now modulated by the form of P_e which takes an ever smaller value for large positive values of y^* and approaches unity for large negative value of y^*. This pattern results in preferentially downward migration of particles in the deposit. The effect of heavy- versus thin-tailed PDFs for p_J is still very evident: the heavy-tailed case of Figure 5b again shows much more rapid and much more far-reaching dispersion than the thin-tailed case of Figure 5a.

Figure 6 shows the variation in time of the mean elevation $\overline{y^*}$ of tracer pebble distribution. Particles are transported downward at a much more rapid rate in the heavy-tailed case. The rate of downstream migration of $\overline{y^*}$ slows in time as pebbles are emplaced deep in the bed, where the probability of entrainment and deposition asymptotically approaches 0. This downward vertical transport is likely responsible for the long-term slowdown of the streamwise advection speed of river tracer pebbles observed by *Ferguson and Hoey* [2002]. Verification of this requires a full implementation of equation (36).

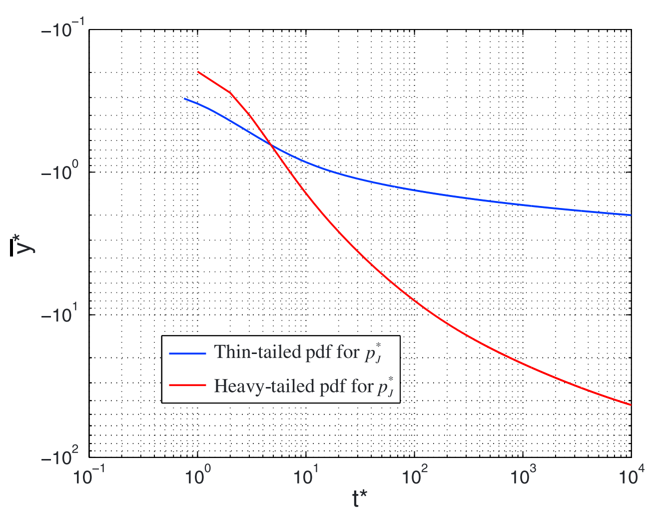

Figure 6. Evolution in time of the mean elevation $\bar{y}^*$ of tracer particle distribution for the thin- and heavy- tailed cases. Here a vertically varying form P_e corresponding to a fluctuating water-sediment interface (riverbed) is used.

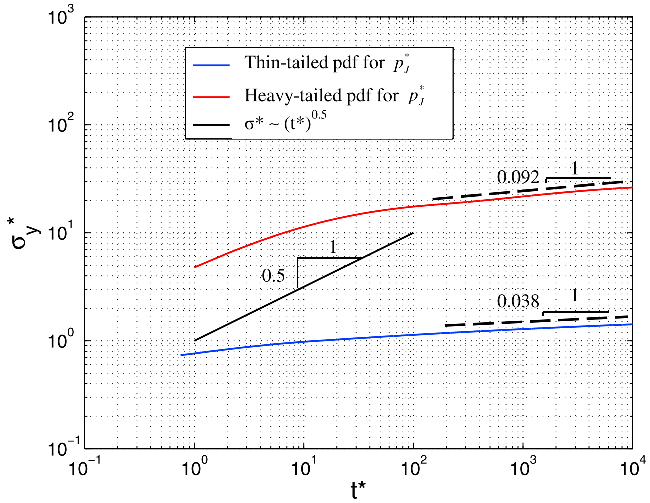

Figure 7. Evolution in time of the vertical standard deviation σ_y* of the tracer distribution for the thin- and heavy- tailed cases. Also plotted is the line $\sigma_y* \sim (t*)^{1/2}$ corresponding to normal diffusion. Here a vertically varying form P_e corresponding to a fluctuating water-sediment interface (riverbed) is used.

Finally, Figure 7 shows plots of σ_y^* versus t* for the thin- and heavy-tailed case with varying P_e. Strongly subdiffusive behavior is seen in both cases at large t*. This behavior is stronger than the case of constant P_e. The heavy-tailed case is less subdiffusive than the thin-tailed case. This reflects the higher probability of a particle migrating to a deep elevation in the heavy-tailed case.

The asymptotic behavior of equation (45a) is readily apparent from the equation itself: f, and thus $< f >$, converges to 0 as $t \to \infty$, as the particles progressively disperse in the vertical. The approach to 0 becomes ever slower with time, however, because p_J declines either exponentially (thin tail) or as a power law (heavy tail) with increasing $|y|$.

9. Conclusions

The principal objective of the paper is to define a generalized Master Equation for the case of bed load transport (moving as bed material load) in rivers, so as to include PDFs of particle step length and particle waiting time, as well as vertical exchange of particles. It is shown here that the *Montroll and Weiss*, [1965] Master Equation for the continuous time random walk (CTRW) model does not apply to the case where the walker (sediment particle) interacts with the lattice by causing the sediment-water interface to change in elevation. The Master Equation forms proposed here are derived from the probabilistic Exner equation of sediment mass conservation of *Parker et al*. [2000]; they are substantially different from the Master Equation of CTRW. Our formulation characterizes vertical dispersion, as well as streamwise advection-diffusion of tracer particles.

Two forms for the Exner-based Master Equation are presented. The form EBME (Exner-based Master Equation) captures vertical as well as streamwise advection and diffusion of tracer pebbles. Under constraints which preclude vertical advection and diffusion, the form EBME reduces to the form EBMA-A associated with the active layer assumption of *Parker et al*. [2000] and *Ganti et al*. [2010]. Forms of these two Master Equations that are asymptotic in streamwise coordinate x are derived so as to allow comparison with the standard ADE (normal advection-diffusion equation) and fADE (fractional advection-diffusion equation). Although they include streamwise advection and diffusion, the Exner-based Master Equations take forms that differ substantially from standard ADE and fADE. The key differences are as follows: (a) advection and diffusion coefficients vary in the vertical and (b) a nonlocal, asymptotically nonfractional dispersion term mixes tracers in the vertical.

In order to illustrate the key aspect of vertical dispersion, a simplified version of EBME, in which streamwise variation in neglected, is solved numerically. The key statistical parameter in this model is $p_J(y)$ corresponding to the probability that when a particle jumps, it jumps from elevation y relative to the mean bed. Both thin-tailed and heavy-tailed PDFs for p_J are considered. We consider two cases, one modeling a vertical lattice that is invariant in the vertical direction and one modeling a lattice characterizing a sediment bed, such that the probability of the bed being at elevation y decreases upward and increases downward. Asymptotically, both the thin- and the heavy-tailed cases show subdiffusive vertical dispersion of pebbles. This subdiffusive behavior is enhanced for the thin-tailed case: when p_J is heavy tailed, particles can be dispersed more rapidly in the vertical direction.

For the case of a sediment bed, i.e., when the probability of the bed being at a given elevation increases downward, tracer particles migrate downward as they disperse. The rate of downward migration decreases in time, as particles reach locations so deep that their probability of entrainment is asymptotically low. This downward migration is the likely reason for the slowdown of streamwise advection of tracer pebbles observed in the field by *Ferguson and Hoey* [2002].

The Master Equations given here provide a basis for future calculations in which (a) streamwise advection and diffusion are included and (b) specific PDFs for bed elevation $p_e(y)$, jump elevation $p_J(y)$, and step length $p_s(r)$ based on laboratory and field data are used. The work of *Wong et al.* [2007], *Hassan et al.* [2013], and *Voepel et al.* [2013] provide information concerning these PDFs. One exciting goal for the future is a further clarification of the phenomenon of streamwise advective slowdown of tracer particles. Another similarly exciting goal is the extension of the Exner-based Master Equation formulation to encompass the statistics of waiting time.

Notation

$\mathcal{C}$	particle advection velocity, [L/T].
C	tracer concentration.
D	volume rate of deposition of bed load per unit area per unit time, [L/T].
D_d	particle dispersivity, [L^2/T].
D_p	particle diameter, [L].
E	volume rate of entrainment of bed sediment into bed load transport per unit area per unit time, [L/T].
$f(x,y,t)$	fraction density of tracers in sediment at elevation y, [1/L].
f^*	dimensionless fraction density of tracers in sediment at elevation y.
$\langle f \rangle$	jump-averaged value of $f\,(x,y,t)$, [1/L].
$f_a\,(x,t)$	fraction of particles within the active layer.
f_I	fraction of tracers in the bed material that is exchanged as the bed aggrades or degrades.
$f_{sw} = P_e f$	fraction density of tracers along a line at elevation y, averaged to include water and sediment regions.
$f_{sw}{}^*$	dimensionless fraction density of tracers along a line at elevation y, averaged to include water and sediment regions.
$J\,(x, t)$	frequency of entrainment, [1/T].
L_a	thickness of the active layer, [L].
$P_e(x, z, t)$	probability that a point at elevation z (or y) is in the sediment bed.
$p_e\,(x,y,t)$	probability density that instantaneous bed elevation is at level z (or y), [1/L].
p_J	probability density function of elevation from which particles jump (are entrained), $p_J = p_{JI} = p_{JO}$, [1/L].
p_J^*	dimensionless p_J.
$\hat{p}_J^*(k)$	Fourier transform of p_J.
$p_{JI}\,(y)$	probability density that a particle that is deposited is emplaced at level y, [1/L].
$p_{JO}\,(y)$	probability density that a particle that is entrained into bed load comes from level y, [1/L].
$p_s\,(r)$	probability density function (density) of step length, [1/L].
$\hat{p}_s(k)$	Fourier transform of p_s.
$p_w\,(\tau)$	probability density function of waiting time (marginal distribution of waiting time), [1/T].
$p_w\,(\tau\vert\, y)$	elevation-specific probability density of waiting time (conditional probability distribution of waiting time given the elevation y), [1/T].
$\widetilde{p}_w(s)$	Laplace transform of p_w.
r	particle step length, [L].
$\bar{r}$	mean step length, [L].
t	time, [T].
t^*	dimensionless time.
x	streamwise coordinate, [L].
y	deviation from the mean bed elevation η,[L].
y^*	dimensionless vertical coordinate.
$\bar{y}^*$	dimensionless mean elevation.
z	coordinate oriented upward normal to the local mean bed, [L].
α	stability parameter of α-stable distribution.
β	skewness parameter of the α-stable distribution p_J.
δ'	location parameter of the α-stable distribution p_J, [L].

δ (x) probability density that a particle is at x at time $t = 0$, [1/L].
γ' scale parameter of the α-stable distribution p_J, [L].
γ^* length scale, equal to γ'.
η local mean bed elevation, [L].
λ_p porosity of the bed.
μ_2 second moment of p_s, [L^2].
ρ (x,t) probability density that a particle is at x at time t, [1/L].
σ streamwise spatial standard deviation of a patch of tracer particles, [L].
σ_J standard deviation of the Gaussian distribution p_J, [L].
σ_y^* dimensionless vertical standard deviation.
τ_{char} characteristic waiting time = $1/J$, [T].

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