

# Joint stochastic model of bedload transport and bed elevations: derivation of heavy-tailed resting times

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## Key Points:

- We model fluvial bedload activity and local bed elevation as a two-species stochastic birth-death process.
- Computations show heavy-tailed power-law distributions of resting times for sediment undergoing burial with tail parameter  $\alpha \approx 1.18$ .
- We discuss implications for bedload diffusion and propose a new theoretical framework for fluvial morphodynamics.

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## Abstract

We present a model of bedload transport and bed elevation changes as a joint stochastic process and use it to derive resting time distributions for sediment undergoing burial. Our model predicts heavy-tailed power-law distributions of resting times with tail behavior completely characterized by the mean erosion rate and its scaling with bed elevation changes. Resting time distributions of sediment undergoing burial are remarkably independent of bedload fluctuations. Drawing upon concepts from the stochastic physics literature, we hypothesize this results from the disparate characteristic timescales of bedload activity and bed elevation fluctuations.

## 1 Introduction

The classical approach to quantify sediment transport in rivers relates the bulk downstream flux of bedload to average characteristics of the sediment and flow (e.g. Yalin, 1972). This approach was originally developed for sand-bed channels, but its extension to gravel-bed rivers presents difficulties (e.g. Hassan, Voepel, Schumer, Parker, & Fraccarollo, 2013). Gravel-bed rivers can exhibit wide spatial and temporal variations in sediment and flow characteristics (e.g. Hassan et al., 2008), meaning local and average values may not correspond. In addition, these rivers are often in a state of active adjustment to episodic events and disturbances (e.g. Church & Ferguson, 2015), meaning average characteristics may be undefined. In this context, the bulk approach can perform poorly; studies showing errors of several orders of magnitude are well-known in the literature (e.g. Gomez & Church, 1989; Recking, Liébault, Peteuil, & Jolimet, 2012).

An alternative approach is to relate bedload transport to the displacement patterns of individual grains (e.g. Hassan et al., 2013). In principle, this approach circumvents issues with average characteristics of gravel-bed rivers and re-frames bedload transport in terms of deterministic (Newtonian) mechanics. In practice, measuring or predicting the forces acting on individual grains as they interact with the flow, bed, and other grains is not possible. This has motivated stochastic concepts of individual motions, whereby displacement patterns of grains are approximated as random walks (e.g. Weiss, 1994). In these models, displacement is the result of alternate step-rest sequences where step lengths and resting times are random variables drawn from statistical distributions (Bradley & Tucker, 2012; Einstein, 1937; Hassan, Church, & Schick, 1991; Nakagawa & Tsujimoto, 1976; Yano, 1969).

In the stochastic approach, differences between the random-walk motions of one grain and the next imply bedload diffusion, or a spreading apart of grains through time. Over long timescales, the diffusion characteristics predicted by these models critically differ depending on whether the step length and resting time distributions have light or heavy tails (e.g. Weeks & Swinney, 1998). Heavy-tailed distributions have exceedance functions  $P(X > x) \sim x^{-\alpha}$  with tail parameters  $\alpha < 2$ , meaning large values of  $x$  are relatively common, while light-tailed distributions have  $\alpha \geq 2$ , meaning large values of  $x$  are relatively rare. If both resting time and step distance distributions have light tails, the diffusion is said to be normal, with the variance of particle positions  $\sigma_x^2$  scaling with time  $t$  as  $\sigma_x^2 \propto t$ . However, if either distribution has a heavy-tail, the diffusion is called anomalous, with the variance of particle position scaling as  $\sigma_x^2 \propto t^\gamma$ , where  $\gamma \neq 1$ . In this expression,  $\gamma < 1$  is called sub-diffusion and  $\gamma > 1$  is super-diffusion. In strongly asymmetric random walks such as bedload transport, heavy-tailed step lengths imply super-diffusion, while heavy-tailed resting times imply either super or sub-diffusion, depending on  $\alpha$  (Bradley, 2017; Weeks & Swinney, 1998; Weeks, Urbach, & Swinney, 1996).

Tracer experiments in gravel bed rivers show anomalous bedload diffusion (Bradley, 2017; Phillips, Martin, & Jerolmack, 2013), light-tailed step lengths (Bradley & Tucker, 2012; Hassan et al., 2013), and heavy-tailed resting times (Bradley, 2017; Olinde & Johnson, 2015; Pretzlav, 2016; Voepel, Schumer, & Hassan, 2013), forming a coherent experimental picture

of super-diffusive bedload transport, at least at long observation timescales (e.g. Martin, Jerolmack, & Schumer, 2012; Nikora, 2002). However, field studies have not resolved the mechanism generating observed heavy-tailed resting times (e.g. Bradley, 2017), and empirical distributions display clear differences in their form and characteristics, with different tail parameters (e.g. Olinde & Johnson, 2015) and sometimes truncation (e.g. Bradley, 2017) or tempering to light tails at large resting times (e.g. Voepel et al., 2013). These differences and the mechanism generating heavy-tailed resting times deserve further research attention.

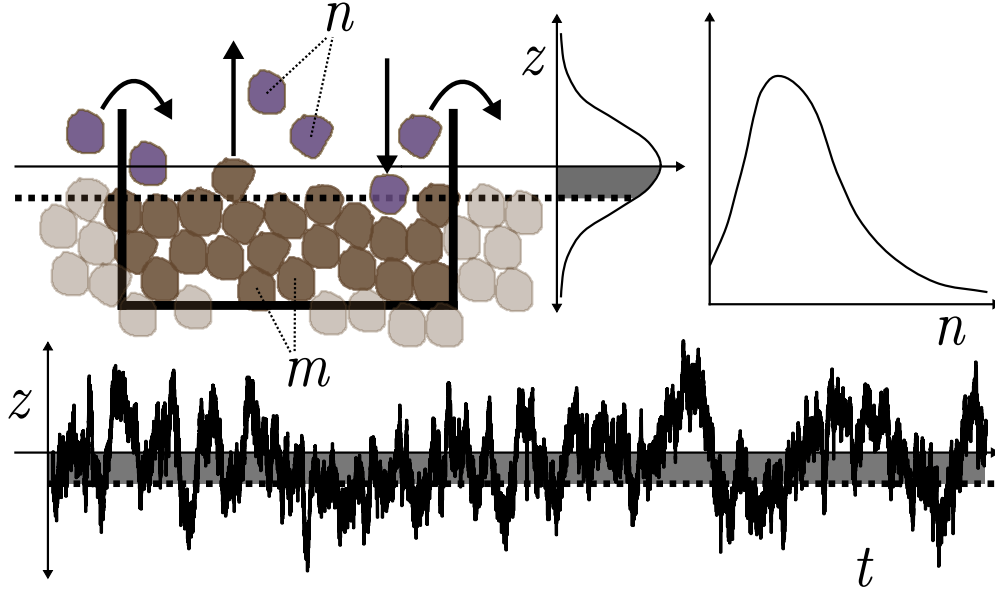
A predominant hypothesis is that heavy-tailed resting times and anomalous diffusion originate from sediment burial (Martin, Purohit, & Jerolmack, 2014; Voepel et al., 2013; Wu et al., 2019). Conceptually, when grains rest on the bed surface, material transported from upstream can deposit on top of them, preventing entrainment until it's removed, driving up resting times and imparting a heavy tail to the distribution. Martin et al. (2014) have provided the only direct support for this hypothesis. They traced grains in a narrow flume with clear sidewalls, directly resolving burial as the generator of heavy-tailed resting times, and they described their results with a theoretical model that is formally similar to an earlier effort by Voepel et al. (2013).

The models of Voepel et al. (2013) and Martin et al. (2014) consider bed elevations as a random walk and interpret resting times as return periods from above in the bed elevation time-series (e.g. Redner, 2007). Both models are successful in describing different experimental resting time distributions. However, the assumptions and results of these models are inconsistent with one another, and their treatment of bed elevations as a process independent of sediment transport is questionable at first glance, since the erosion and deposition of individual grains are the source of bed elevation changes (e.g. Wong, Parker, DeVries, Brown, & Burges, 2007).

In this work, we approach the problem from a different angle, making an extension of the stochastic bedload transport model of Ancey, Davison, Böhm, Jodeau, and Frey (2008) to link bed elevation changes to the erosion and deposition events of individual grains, and we derive resting times as a consequence of this model. The key assumptions of our model are: (1) bedload erosion and deposition can be characterized by probabilities per unit time, or rates (e.g. Ancey et al., 2008; Einstein, 1950); and (2) these rates are contingent on the local bed elevation, encoding the property that erosion of sediment is emphasized from regions of exposure, while deposition is emphasized in regions of shelter (e.g. Sawai, 1987; Wong et al., 2007). Our model generates heavy-tailed distributions with no tempering and a universal tail parameter  $\alpha \approx 1.18$  for a particular non-dimensionalization of the resting time, showing close correspondence to the findings of Martin et al. (2014) and suggesting a correction to some imperfections in their results. We conclude the paper by framing our work in relation to earlier ideas and discussing the implications of this work on open problems in individual bedload motions and anomalous diffusion.

## 2 Stochastic model

We define a volume of downstream length  $L$  that contains some number  $n$  of moving particles in the water flow and some number  $m$  of stationary particles composing the bed at time  $t$ . For simplicity, we consider all particles as approximately spherical with the same diameter  $2a$ , so their mobility and packing characteristics are similar. Following Ancey et al. (2008), we prescribe four events that can occur at any instant to modify the populations  $n$  and  $m$ , and we characterize these events using probabilities per unit time, or rates. These are: (1) migration of a moving particle into the volume from upstream ( $n \rightarrow n+1$ ); (2) the entrainment (i.e., erosion) of a stationary particle into motion within the volume ( $m \rightarrow m-1$  and  $n \rightarrow n+1$ ); (3) the deposition of a moving particle to rest within the volume ( $m \rightarrow m+1$  and  $n \rightarrow n-1$ ); and (4) the migration of a moving particle out of the volume to downstream ( $n \rightarrow n-1$ ). As the events occur at random intervals, they set up a joint stochastic evolution of the populations  $n$  and  $m$  characterized by a joint probability distribution  $P(n, m, t)$  having



**Figure 1.** Definition sketch of a control volume containing  $n$  moving grains and  $m$  resting grains. Migration, entrainment, and deposition processes are represented by curved arrows, and the bed elevation at some instant is depicted by dotted line. The bed is presented in a degraded state, where  $m < m_0$ . The distributions of  $n$  and  $m$  are indicated in the upper right panel, while the bottom panel is a time-series of bed elevations (1).

marginals  $P(n, t) = \sum_m P(n, m, t)$  and  $P(m, t) = \sum_n P(n, m, t)$  for the number of particles in motion and at rest in the volume at  $t$ . These concepts are depicted in figure 1.

The populations  $n$  and  $m$  provide the bulk bedload flux  $q_s$  and the local bed elevation  $z$ . The mean bedload transport rate is given by  $q_s \propto u_s \langle n \rangle$ , where  $u_s$  is the characteristic velocity of moving bedload and  $\langle n \rangle = \sum_{n,m} n P(n, m)$  is the mean number of grains in motion (e.g. Ancey et al., 2008; Charru, Mouilleron, & Eiff, 2004; Furbish, Haff, Roseberry, & Schmeeckle, 2012). The bed elevation is related to  $m$  through the packing geometry of the bed. To derive this, we prescribe a mean number of grains at rest  $m_0$  and introduce a packing fraction  $\phi$  of grains in the bed (Torquato, 2018). Considering a two-dimensional bed (e.g. Einstein, 1950; Paintal, 1971), the deviation from the mean bed elevation is

$$z(m) = \frac{\pi a^2}{\phi L} (m - m_0) = z_1 (m - m_0). \quad (1)$$

The constant  $z_1 = \pi a^2 / (\phi L)$  is an important scale of the problem.  $z_1$  is the magnitude of bed elevation change (in an average sense across the control volume) associated with the addition or removal of a single grain. We write the rates of the four possible transitions as (e.g. Ancey et al., 2008):

$$R_{MI}(n+1, m|n, m) = \nu \quad \text{migration in,} \quad (2)$$

$$R_E(n+1, m-1|n, m) = \lambda(m) + \mu(m)n \quad \text{entrainment,} \quad (3)$$

$$R_D(n-1, m+1|n, m) = \sigma(m)n \quad \text{deposition,} \quad (4)$$

$$R_{MO}(n-1, m|n, m) = \gamma n \quad \text{migration out.} \quad (5)$$

These rates are independent of the past history of the populations and depend only on the current populations  $(n, m)$ . As a result, the system is Markovian (e.g. Cox & Miller, 1965; van Kampen, 1992), meaning time intervals between subsequent transitions are exponentially distributed (e.g. Gillespie, 2007).

In (2-5),  $\nu$  and  $\gamma$  are constants characterizing migration rates of individual grains into and out of the volume. They lack any dependence on the populations  $n$  and  $m$ . In contrast,  $\lambda(m)$ ,  $\mu(m)$ , and  $\sigma(m)$ , characterizing the entrainment, collective entrainment (e.g. Ancey et al., 2008; Heyman, Ma, Mettra, & Ancey, 2014; Heyman, Mettra, Ma, & Ancey, 2013), and deposition rates of individual grains are considered to depend on  $m$ . As is well-known, bed elevation changes modify the likelihood of entrainment and deposition in a negative feedback (Sawai, 1987; Wong et al., 2007); that is, aggradation increases the likelihood of entrainment, while degradation increases the likelihood of deposition. Wong et al. (2007) concluded that bed elevation changes induce an exponential variation in entrainment and deposition probabilities, while Sawai (1987) concluded that the variation is linear. For simplicity, we incorporate the scaling of Sawai (1987) and note its equivalence to the Wong et al. (2007) scaling when bed elevation changes are small. Because experimental distributions of bed elevations are usually symmetrical, (Martin et al., 2014; Singh, Fienberg, Jerolmack, Marr, & Foufoula-Georgiou, 2009; Wong et al., 2007), we expect the erosion and deposition feedbacks to be anti-symmetrical. That is, as bed elevation changes drive up (down) erosion rates, so they drive down (up) deposition rates to the same degree.

Summarizing these ideas, the entrainment and deposition rates can be written  $\chi(m) = \chi_0(1 \pm z_1 z(m)/(2l)^2)$ , where  $\chi = \lambda, \mu, \sigma$ , and the entrainment parameters take the plus sign, while deposition takes the minus, and we have introduced a length scale  $l$ . As we'll see, the variance of bed elevation turns out to be given by  $\text{var}(z) = (l/z_1)^2$ . Accordingly,  $l$  characterizes the range of bed elevation variations, which could be interpreted as the active layer depth (e.g. Church, 2017). Another perspective is that  $l$  is the distance of bed elevation change at which the entrainment and deposition rates are significantly affected. With these substitutions, the local bed elevation-dependent entrainment and deposition rates (3-4) can be written:

$$R_E(n+1, m-1|n, m) = [\lambda_0 + \mu_0 n] \left[ 1 + \frac{z_1 z(m)}{(2l)^2} \right], \quad \text{entrainment,} \quad (6)$$

$$R_D(n-1, m+1|n, m) = \sigma_0 \left[ 1 - \frac{z_1 z(m)}{(2l)^2} \right] n, \quad \text{deposition.} \quad (7)$$

At  $z(m) = 0$ , the rates reduce to those of the Ancey et al. (2008) model. Away from this elevation, entrainment and deposition are alternatively suppressed and accentuated depending on the sign of  $z(m)$ .

In terms of the transition rates (2-7), we can obtain the Master equation for the probability flow using the forward Kolmogorov equation  $\partial P(n, m; t)/\partial t = \sum_{n', m'} R(n, m)P(n', m'; t)$  (e.g. Ancey et al., 2008; Cox & Miller, 1965; Gillespie, 1992) as

$$\begin{aligned} \frac{\partial P}{\partial t}(n, m; t) = & \nu P(n-1, m; t) + \{\lambda(m+1) + [n-1]\mu(m+1)\}P(n-1, m+1; t) \\ & + [n+1]\sigma(m-1)P(n+1, m-1; t) + [n+1]\gamma P(n+1, m; t) \\ & - \{\nu + \lambda(m) + n\mu(m) + n\sigma(m) + n\gamma\}P(n, m; t). \end{aligned} \quad (8)$$

The joint probability distribution  $P(n, m; t)$  solving this equation will fully characterize the statistics of  $n$  and  $m$ . We anticipate that solutions will adjust from the initial conditions to a steady-state distribution  $P_s(n, m)$ , independent of time, if the constant factors in the transition rates are representative of steady bedload transport conditions. This Master equation describes a two-species stochastic birth-death model (e.g. Cox & Miller, 1965) of a type well-known in the population ecology literature (e.g. Pielou, 1977; Swift, 2002). In our context, the two species are the moving and stationary grains in the volume.

### 3 Numerical simulations

Unfortunately, (8) does not appear to admit an analytical solution (but see Swift (2002) for a standard method that fails in this case). The difficulty stems from the product terms

between  $n$  and  $m$ . In response, we resort to numerical methods, simulating (8) with the Gillespie algorithm (Gillespie, 1977, 1992, 2007). The Gillespie algorithm leverages the defining property of a Markov process: when transition rates do not depend on the past, time intervals between transitions are exponentially distributed (e.g. Cox & Miller, 1965).

**Table 1.** Parameters from Ancey et al. (2008) experiments describing the rates of migration in, entrainment, deposition, and migration out when  $z(m) = 0$ . All units are  $s^{-1}$  (probability/time). In our model, bed elevation changes modulate these rates in accord with (2-7).

Flow	$\nu$	$\lambda_0$	$\mu_0$	$\sigma_0$	$\gamma$
(a)	5.45	6.59	3.74	4.67	0.77
(g)	7.74	8.42	4.34	4.95	0.56
(i)	15.56	22.07	3.56	4.52	0.68
(l)	15.52	14.64	4.32	4.77	0.48
(n)	15.45	24.49	3.64	4.21	0.36

parameters at each flow condition ( $\nu, \lambda_0, \mu_0, \sigma_0, \gamma$ ), we used values measured by Ancey et al. (2008) in a series of flume experiments. These are summarized in table 1. Flow conditions are labeled (a), (g), and so on, roughly in order of increasing bedload flux (see Ancey et al. (2008) for more details). However, our conclusions are not dependent on these parameters. In all simulations, we take the packing fraction  $\phi = 0.6$ , a typical value for a pile of spheres (e.g. Bennett, 1972), and set  $L = 22.5\text{cm}$  and  $a = 0.3\text{cm}$ , in accord with the Ancey et al. (2008) experiments. Each simulation was run for 1500hrs of virtual time, a period selected to ensure convergence of the resting time statistics.

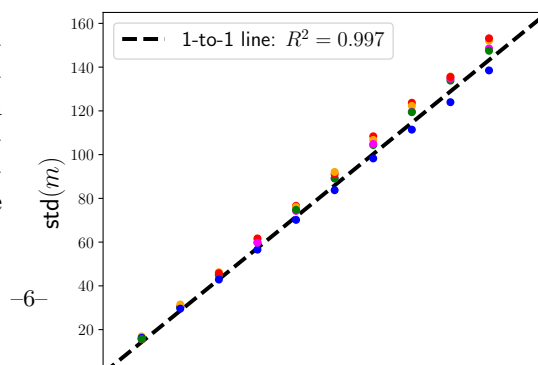
## 4 Results

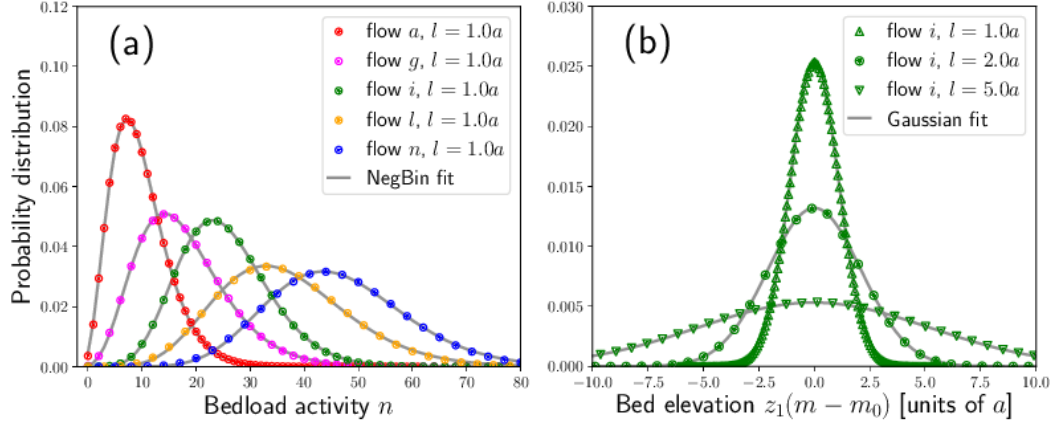
Our simulations show noisy time-series of bedload activities and bed elevations (as seen in the bottom panel of figure 1). From our chosen initial conditions, all simulations show a rapid attainment of steady state conditions followed by a steady-state stochastic dynamics of  $n$  and  $m$ , supporting a time-independent joint distribution  $P_s(n, m)$ . We compute this joint distribution by counting occurrences of the states  $(n, m)$  in the simulated time series. From this joint distribution we compute marginals  $P(n)$  and  $P(m)$  as explained in section 2.

Some of these marginal distributions are displayed in figure 3. Neglecting changes in bed elevation, Ancey et al. (2008) analytically derived negative binomial (NegBin) distributions for the bedload activity  $n$ , and this functional form seems preserved after including bed elevation changes, because all our computed distributions admit clean NegBin fits (figure

As a result, to step the Markov process through a single transition, it's enough to draw a random value from the exponential distribution of transition intervals to determine the time of the next transition. Then drawing another random value to choose the type of transition that occurs using the relative probabilities (2-7), the transition can be enacted by shifting  $t$ ,  $n$  and  $m$  by the appropriate values (i.e., entrainment is  $m \rightarrow m - 1$  and  $n \rightarrow n + 1$ , and so on). This procedure can be iterated to form an exact realization of the stochastic process (e.g. Gillespie, 2007).

In this way, we simulated 5 transport conditions with 10 different values of  $l$  taken across a range from  $l = a$  (a single radius) to  $l = 10a$  (10 radii). These values lie in the range exhibited by the majority of available experimental data (Martin et al., 2014; Singh et al., 2009; Wong et al., 2007). For the migration, entrainment, and deposition





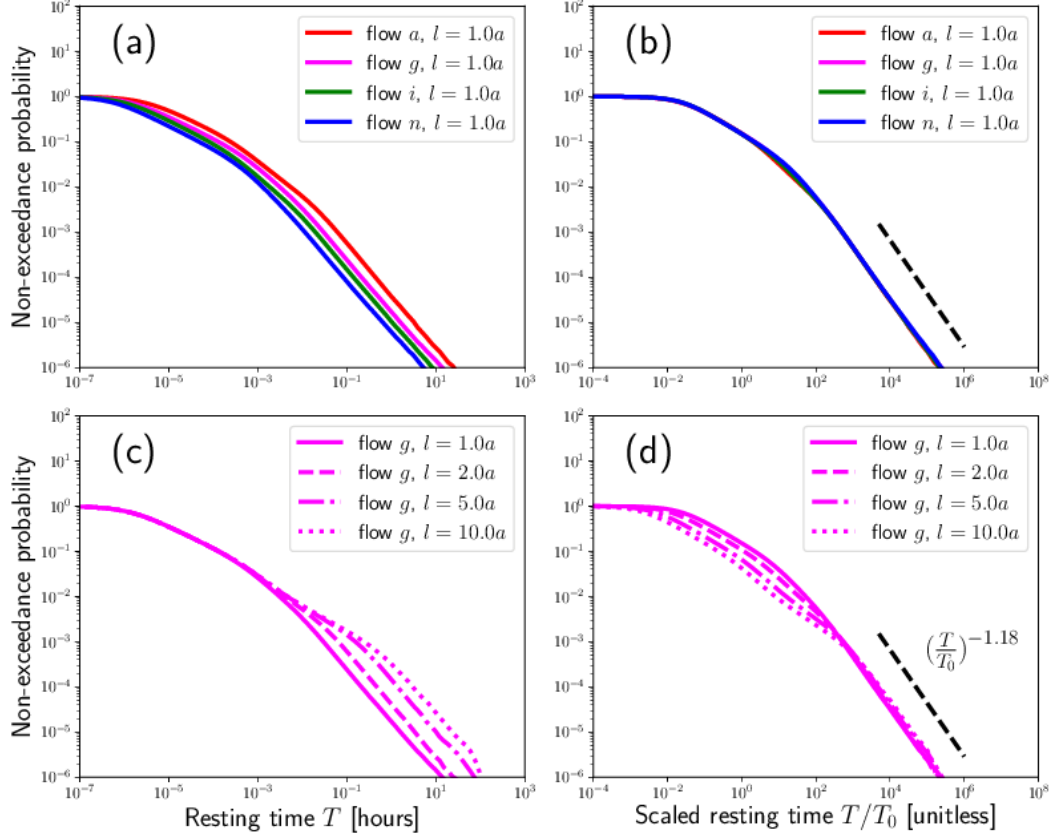
**Figure 3.** Marginal distributions of  $n$  and  $m$  for a subset of simulations. Some points have been omitted for clarity.

3a). For the bed elevation  $m$  (or  $z$ ), our computations show Gaussian distributions (figure 3b), consistent with our assumptions of a symmetric scaling of erosion and deposition rates with bed elevation changes (e.g. Wong et al., 2007).

From the marginal distributions, we calculate means and variances of bedload activity ( $n$ ) and bed elevation ( $m$ ). The mean bed elevation is  $m_0$ , the parameter in (1).  $m$  fluctuates around this value because it sets the equilibrium position of the elevation-related feedbacks within (1). The variance of  $m$  follows  $z_1^2 \text{var}(m) = l^2$ , as indicated in figure 2, consistent with our interpretation of  $l$  as a measure of bed elevation fluctuations. The variance of  $n$  ( $\text{var}(n)$ ), characterizing the magnitude of bedload activity fluctuations, has a more nuanced dependence on the coupling of transport to bed elevation changes. Generally speaking, our simulations show relatively small  $\text{var}(m)$  increases  $\text{var}(n)$  from its decoupled value (i.e., its  $l \rightarrow \infty$  value), while relatively large  $\text{var}(m)$  decreases  $\text{var}(n)$  from its decoupled value. Since  $\text{var}(m)$  reflects the coupling between entrainment/deposition rates and the local bed elevation, we see a relatively strong coupling (small  $l$ ) increases bedload fluctuations, while a relatively weak coupling (large  $l$ ) reduces them. In simpler terms, when bed elevation deviations are tightly linked to the mobility of surface grains, bedload fluctuations are enhanced.

Now we describe the analysis of resting times for sediment undergoing burial from the time-series of  $m$ . Following Voepel et al. (2013) and Martin et al. (2014), we concentrate on a particular bed elevation  $m'$ , and find all time intervals separating deposition events at  $m = m'$  from erosion events at  $m = m' + 1$ . These are the return times from above of the sedimentary bed conditional to the elevation  $m'$ . Binning these conditional return times (using logarithmic bins to reduce computational load) and counting the occurrences in each bin, we obtain a non-exceedance distribution of return times  $t_r$  held conditional to the elevation  $m'$ :  $P(T > t_r | m')$ . Using the marginal probability distribution of bed elevations  $P(m)$ , we derive the unconditional non-exceedance distribution of resting times as a sum over all elevations (Martin et al., 2014; Nakagawa & Tsujimoto, 1980; Voepel et al., 2013;





**Figure 4.** Resting time statistics scale differently with transport conditions and the bed elevation variance. Panel (a) shows differing flow conditions at a fixed  $l$  value, while panel (c) shows fixed flow conditions at differing  $l$ . When scaled by  $T_0$  (10), both types of difference collapse in the tails of the distributions, as shown in panels (b) and (d). In panels (b) and (d), the black dotted lines indicate a power law decay of the collapsed tails having parameter  $\alpha \approx 1.18$ .

Yang & Sayre, 1971):

$$P(T > t_r) = \sum_{m'} P(m') P(T > t_r | m'). \quad (9)$$

Some of these results are displayed in figure 4. In contrast to earlier works our analysis does not require an additional binning step over the elevation, since our elevation series is discrete (multiples of  $z_1$ ). This provides enhanced resolution of the resting time distributions. Comparing panels (a) and (c) shows the resting time distributions scale with the intensity of bedload transport and the standard deviation of bed elevations ( $l$ ) in different ways. However, as shown in panels (b) and (d), a characteristic timescale  $T_0$  is found to collapse away both of these types of differences.

We obtain  $T_0$  heuristically by finding a characteristic speed of bed elevation change. Formally, the mean erosion rate is  $E = \sum_{n,m} R_E(n+1, m-1|n, m) P_s(n, m)$ . This is the mean number of grains leaving the bed per unit time. Since the removal of a single grain changes the bed elevation by  $z_1$ , bed elevations change with a characteristic speed  $z_1 E$ . Since the range of elevation deviations is  $l$ , the time required for the bed to shift through this characteristic distance is

$$T_0 = \frac{l}{z_1 E}. \quad (10)$$



When scaling the resting time by this  $T_0$ , we obtain the collapse shown in figure 4. Using the log-likelihood estimation technique described by Newman (2005), we estimate the scaled resting time non-exceedance distributions decay as a heavy-tailed power law with parameter  $\alpha = 1.18 \pm 0.32$  for all return times satisfying  $T/T_0 > 10^3$ . As we'll discuss, these distributions are sufficiently heavy-tailed to violate the central limit theorem and drive anomalous super-diffusion of bedload, and this result provides support for an earlier study (e.g. Martin et al., 2014).

## 5 Discussion

Einstein (1937) developed the first model of individual bedload motions and bedload diffusion, and his ideas can be viewed as the historical nexus of an entire paradigm of research that extends into the present day (e.g. Ancey et al., 2008; Hassan et al., 1991; Hubbell & Sayre, 1964; Nakagawa & Tsujimoto, 1976; Wu et al., 2019). Works in this paradigm attempt to understand properties of bedload transport from applying a stochastic concept of individual sediment motions. With a few exceptions (e.g. Nakagawa & Tsujimoto, 1980; Wu et al., 2019; Yang & Sayre, 1971), existing descriptions are spatially one-dimensional, concentrating on the motion of grains in the downstream direction but neglecting the vertical dimension wherein local bed elevation changes imply sediment burial (e.g. Martin et al., 2014; Voepel et al., 2013) and modify the mobility of surface grains (e.g. Nakagawa & Tsujimoto, 1980; Yang & Sayre, 1971).

In this work, we've built upon earlier models (e.g. Ancey et al., 2008; Martin et al., 2014) to include this vertical dimension and provide a joint description of bedload transport and bed elevation changes. We find negative binomial distributions of bedload activity and normal distributions of bed elevations, reproducing a wide set of experimental findings (Ancey et al., 2008; Heyman, Bohorquez, & Ancey, 2016; Martin et al., 2014; Singh et al., 2009; Wong et al., 2007). More importantly, we've interpreted resting times of sediment undergoing burial as return times from above in the bed elevation time series (e.g. Martin et al., 2014; Voepel et al., 2013), and predicted the form and characteristics of this distribution, which are otherwise poorly understood, difficult to measure, and important for bedload diffusion (e.g. Bradley, 2017; Martin et al., 2014; Voepel et al., 2013). Of course, modeling complex geophysical phenomena (such as expressions of coupled fluid and granular phases) necessitates simplifying assumptions (e.g. Larsen et al., 2016), and our work is no exception. We believe the key limitations of our work are (1) our assumption that local (as opposed to non-local) deviations in bed elevation are the dominant control on the mobility of grains; and (2) those assumptions inherited from the underlying bedload transport model of Ancey et al. (2008), which essentially incorporates the earlier assumptions of Einstein (1950) into a stochastic framework. The first assumption can be somewhat justified under conditions in which the formation of organized bed structures is not favored (e.g. Hassan et al., 2008), while the second has been discussed in earlier works and appears justified in near-threshold transport conditions when the intermittent aspect of bedload transport is emphasized (e.g. Ancey et al., 2008) as long as organized bed structures are not present (e.g. Dhont & Ancey, 2018).

We now discuss the relationship of our work to the earlier theories it builds up (e.g. Ancey et al., 2008; Martin et al., 2014). Our joint model reproduces the descriptions of bedload activities by Ancey et al. (2008) and bed elevations by Martin et al. (2014) upon taking particular simplified limits of (8). The Ancey et al. (2008) bedload model is obtained when bed elevation fluctuations  $\delta m$  are considered small:  $m \approx m_0$ . Taking account of this change in (8) obtains the master equation of Ancey et al. (2008) for the bedload activity distribution  $P(n, t)$ . Hence the differences between our bedload activity statistics and those of the Ancey et al. (2008) model are induced by bed elevation fluctuations. When relatively large bed elevation fluctuations are possible (i.e.,  $l$  is large), bed elevation changes act to buffer bedload activity fluctuations. In contrast, when bed elevation changes are tightly linked to the mobility of moving and stationary grains (i.e.,  $l$  is small), bed elevation changes

enhance bedload fluctuations. We hypothesize this enhancement/suppression of bedload fluctuations is primarily due to the collective entrainment term in (7), since the Ancey et al. (2008) variance is sensitive to the collective entrainment process. However, more research will be required to clarify the linkage between bed elevation changes and bedload fluctuations. Given recent observations of sudden local elevation changes being induced by avalanches on the downstream face of bars (Dhont & Ancey, 2018), we identify the interplay between collective motions of bedload and bed elevation changes as an emerging research theme, and we suggest our joint description may hint toward a theoretical framework to address these issues.

The Martin et al. (2014) bed elevation model based upon the Ornstein-Uhlenbeck (OU) process is obtained in the converse limit when bedload activity fluctuations  $\delta n$  are small:  $n \approx \langle n \rangle$ . In this case, neglecting the migration terms and identifying the mean entrainment and deposition rates as  $E = \lambda_0 + \mu_0 \langle n \rangle$  and  $D = \sigma_0 \langle n \rangle$  before using the steady-state transport condition  $E = D$  (e.g. Einstein, 1950) gives

$$\frac{\partial}{\partial t} P(m, t) = E \left\{ \left[ 1 + \left( \frac{z_1}{2l} \right)^2 m \right] P(m+1, t) + \left[ 1 - \left( \frac{z_1}{2l} \right)^2 m \right] P(m-1, t) - 2P(m, t) \right\}, \quad (11)$$

This is a discrete state analogue of the OU process Martin et al. (2014) used to model bed elevation changes, and it provides excellent correspondence to the bed elevation statistics and resting time distributions computed from our joint model. Our resting time distributions of sediment undergoing burial essentially correspond with those of Martin et al. (2014) given our computational uncertainty. These authors proposed  $\alpha \approx 1$  from a continuum analogue of (11). Incidentally, they scale resting times by an "activity parameter"  $1/a$  which is equivalent to  $1/(2E)$  in our notation. Our work suggests the incomplete collapse displayed by Martin et al. (2014) may be rectified by including a bed elevation variance factor in their scaling as in (10), further justifying the correspondence of our model with theirs. In fact, we can see  $T_0$  is the correlation time of the limiting OU process (10) (e.g. Gardiner, 1983).

In light of the coupling between elevation fluctuations and the entrainment/deposition rates in (8), and the non-negligible elevation fluctuations our model produces (figure 3b), this correspondence to Martin et al. (2014) is initially surprising. However, we can understand it through the lens of "fast" and "slow" stochastic variables advanced by Haken (1983). Since appreciable bed elevation changes are the compound result of many bedload entrainment or deposition events ( $\sim l/z_1$  of them), bed elevation fluctuations persist for a typical timescale which is much longer than the timescale of bedload activity fluctuations, so  $n$  is a "fast" variable while  $m$  is a "slow" one. This statement could be formalized by computing the autocorrelation times of  $n$  and  $m$  (e.g. Gardiner, 1983). Accordingly, the value of  $m$  does not change appreciably during the period of time required for  $n$  to vary across a wide range, meaning the slow variable  $m$  is influenced by many  $n$  values during the course of its incremental evolution, justifying the mean-field limit (11) of Martin et al. (2014). This is a so-called slaving principle, whereby fast stochastic variables are coordinated to slow ones but not the converse. Hence (11) is justified from a more rigorous adiabatic approximation of (8) based on integrating out the fast variable  $n$  (e.g. Haken, 1983), and not only by an ad hoc mean-field limit. We believe such ideas may find widespread application to future river science considerations. Indeed, rivers encompass a wide set of temporally evolving attributes displaying apparent randomness and correlations on disparate scales (Chartrand, Jellinek, Hassan, & Ferrer-Boix, 2019), including hydraulic flow (e.g. Ferrer-Boix & Hassan, 2015), morphological structure (e.g. Dhont & Ancey, 2018), and sediment supply regime (e.g. Elgueta-Astaburuaga & Hassan, 2019), and investigators are only beginning to approach river science problems from a stochastic physics standpoint (e.g. Ancey et al., 2008; Furbish, Ball, & Schmeckle, 2012). In the present context, we have provided clean evidence that simpler one-dimensional models (e.g. Martin et al., 2014) can provide apt descriptions of more complex geophysical phenomena when a scale mismatch is present in their coupled components.

Now we discuss our computed resting time distributions (figure 4) and their implications for bedload diffusion. Our simulations show asymptotic power law tails with parameter  $\alpha = 1.18 \pm 0.32$  after scaling by  $T_0$  related to the characteristic speed of bed elevation changes. For a general power law, if  $\alpha > 1$ , neither the mean or variance of the resting time will converge, while if  $1 < \alpha < 2$ , the mean will converge while the variance will not (e.g. Bradley, 2017). Within our computational uncertainty, which stems from the finite duration of our simulations and the log-likelihood estimation of  $\alpha$  (e.g. Newman, 2005), we are unable to conclude whether the mean resting time will diverge, but we can conclude the variance will diverge. According to Weeks and Swinney (1998), if the step length distribution has a light tail (e.g. Hassan et al., 2013), our computed power-law resting times are sufficiently heavy-tailed to imply diffusion scaling as either  $\sigma_x^2 \propto t^{3-\alpha} \approx t^{\{1.82 \pm 0.32\}}$  or  $\sigma_x^2 \propto t^{2\alpha} \approx t^{\{3.64 \pm 0.45\}}$  at asymptotically large times. In either case, the process of sediment burial induces an extreme super-diffusion of bedload: at long timescales, some grains will continue to transport downstream in alternate motion-rest sequences while others will become trapped under the bed for relatively long periods of time, driving a rapid spreading of the population.

Finally, we propose a possible extension of the joint model (8) by following its connection to Ancey et al. (2008) and follow-ups (e.g. Ancey & Heyman, 2014; Heyman, Bohórquez, & Ancey, 2015; Heyman et al., 2014). These works are based on chaining many Ancey et al. (2008) single-cell models together along a line, with migration out of one cell being migration into another. In this way, they provide a framework to study spatial correlations in bedload transport (e.g. Heyman et al., 2015, 2014). Similar approaches have been used to formulate reaction-diffusion and flow problems in stochastic physics (e.g. Gardiner, 1983). One can imagine using the model (8) in the same way, chaining an array of volumes (as in figure 1) together along a line to generate a fluvial morphodynamics theory ultimately rooted in a stochastic concept of individual motions (e.g. Einstein, 1937). Such a theory could provide spatial correlations in bed elevation changes and bedload transport while taking account of their inherent granularity. Given the increasing realization that granular physics phenomena initiated by individual grains, such as avalanches and jamming, play a non-negligible role in fluvial processes (e.g. Dhont & Ancey, 2018; Saletti, 2016), we speculate such a model, if suitably extended, might provide unique traction on future research problems centered around processes initiated by individual grains.

## 6 Conclusion

We have created the first joint stochastic model of bedload transport and bed elevation changes based upon the dynamics of individual grains by fusing earlier works in the research paradigm of Einstein (1937). This is a two-population stochastic birth-death process of a type often encountered in mathematical ecology (e.g. Pielou, 1977) and chemical physics (e.g. Gardiner, 1983), and it reproduces empirical expectations for Gaussian bed elevation and Negative Binomial bedload activity distributions (e.g. Ancey et al., 2008; Wong et al., 2007). Interpreting resting times of sediment undergoing burial as return times from above in the bed elevation time series (e.g. Voepel et al., 2013), we predict asymptotic heavy-tailed power-law resting times with parameter  $\alpha = 1.18 \pm 0.32$ , corroborating the results of Martin et al. (2014). We conclude sediment burial is capable of inducing bedload super-diffusion at long timescales (e.g. ?). Finally, this work draws new concepts from the stochastic physics literature into river science for the first time and points out several difficult problems for future research. The limit of our description to the simpler description of Martin et al. (2014) provides a geophysical example of the slaving principle of Haken (1983) concerning the interaction of fast and slow stochastic variables, and its comparison to Ancey et al. (2008) suggests a nuanced linkage between collective transport and bed elevation changes which requires further study.

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