Bedload diffusion theory

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1 The two-state random walk

The formalism of two-state random walks [e.g. *Weiss*, 1976, 1994; *Masoliver*, 2016; *Masoliver and Lindenberg*, 2017] ultimately composes all existing models of bedload diffusion [e.g. *Nikora et al.*, 2001; *Nikora*, 2002; *Zhang et al.*, 2012; *Fan et al.*, 2016] and provides a framework to build new models from. Assuming residence time distributions $\psi_i(t)$ in the states with associated survival probabilities $\Psi_i(t) = \int_t^\infty dt'\psi_i(t')$, probabilities of moving a distance x up to time t within a sojourn $f_i(x,t)$, denoting $g_i(x,t) = f_i(x,t)\psi_i(t)$ and $G_i(x,t) = f_i(x,t)\Psi_i(x,t)$, and letting $\omega_i(x,t)$ be the probability that a sojourn in state i ends at x,t provides

$$p_i(x,t) = \theta_i G_i(x,t) + \int_0^\infty dx' \int_0^\infty dt' \omega_{\bar{i}}(x',t') G_i(x-x',t-t') \tag{1}$$

and

$$\omega_i(x,t) = \theta_i g_i(x,t) + \int_0^\infty dx' \int_0^\infty dt' \omega_{\bar{i}}(x',t') g_i(x-x',t-t'). \tag{2}$$

Here θ_i are the initial probabilities of being in each state with $\theta_1 + \theta_2 = 1$. $\bar{i}$ is the opposite of i and $i = 1,2$. The probability of the two-state random walker being at position x at time t is

$$p(x,t) = p_1(x,t) + p_2(x,t). \tag{3}$$

Denoting the laplace transform with respect to the variable q as $\mathcal{L}_q$ and associating variables η and s with $\mathcal{L}_x$ and $\mathcal{L}_t$, taking $\mathcal{L}_x\mathcal{L}_t$ of (2) provides a much simpler algebraic problem for the probability p:

$$\tilde{\omega}_i = \frac{[\theta_i + \theta_{\bar{i}}\tilde{g}_{\bar{i}}]\tilde{g}_i}{1 - \tilde{g}_1\tilde{g}_2} \tag{4}$$

and (c.f. *Masoliver* [2016] eq. 20)

$$\tilde{p}_\pm = \Big(\theta_i + \tilde{\omega}_{\bar{i}}\Big)\tilde{G}_i = \frac{\theta_i + \theta_{\bar{i}}\tilde{g}_{\bar{i}}}{1 - \tilde{g}_1\tilde{g}_2}\tilde{G}_i. \tag{5}$$

Therefore the double transform of the joint PDF reads [c.f. *Weiss*, 1994, eq. 6.33 pg. 243]

$$\tilde{p}(\eta,s) = \frac{\theta_1[\tilde{G}_1 + \tilde{g}_1\tilde{G}_2] + \theta_2[\tilde{G}_2 + \tilde{g}_2\tilde{G}_1]}{1 - \tilde{g}_1\tilde{g}_2}. \tag{6}$$

This is a direct generalization of the famous Montroll-Weiss formula for a single state continuous-time random walk.

For the evaluation of this formula, a useful fact to take account of is

$$\mathcal{L}_t\{\Psi_\pm(t);s\} = \int_0^\infty dt e^{-st} \int_t^\infty dt' \psi_\pm(t') = \frac{1 - \tilde{\psi}(s)}{s}. \tag{7}$$

Finally, owing to the definition of the double Laplace transform of $p(x,t)$:

$$\tilde{p}(\eta,s) = \int_0^\infty dt e^{-st} \int_0^\infty dx e^{-\eta x} p(x,t) \tag{8}$$

we see the (double) inverse transform $p(x,t)$ is not necessary to study the moments $\langle x(t)^k \rangle = \int_0^\infty x^k p(x,t)$ of an ensemble of tracers since the moments follow from

$$\mathcal{L}_t\{\langle x(t)^k \rangle; s\} = (-)^k \partial_\eta^k \tilde{p}(\eta,s)\Big|_{\eta=0}. \tag{9}$$

1.1 Moments of a two-state random walk

Weeks and Swinney [1998] starts in motion, for future reference. In this section I will analyze the moments in generality.

2 The Einstein theory

Taking $g_1(x,t) = \delta(x)k_1 e^{-k_1 t}$ (rest) and $g_2(x,t) = k_2 e^{-k_2 x}\delta(t)$ (step) reproduces the *Einstein* [1937] diffusion theory. In this case the double transforms are:

$$\tilde{g}_1(\eta, s) = \frac{k_1}{k_1 + s} \tag{10}$$

$$\tilde{g}_2(\eta, s) = \frac{k_2}{k_2 + \eta}, \tag{11}$$

and the survival functions are

$$\Psi_1(t) = \int_t^\infty dt' k_1 e^{-k_1 t'} = e^{-k_1 t} \tag{12}$$

and

$$\Psi_2(t) = \int_t^\infty dt' \delta(t') = 0, \tag{13}$$

meaning $G_1(x,t) = \delta(x)e^{-k_1 t}$ and $G_2(x,t) = 0$, providing

$$\tilde{G}_1(\eta, s) = \frac{1}{k_1 + s}. \tag{14}$$

Taking $\theta_1 = 1$ and $\theta_2 = 0$, so the dynamics start at rest, the MW generalization (6) is

$$\tilde{p}(\eta, s) = \frac{\tilde{G}_1}{1 - \tilde{g}_1\tilde{g}_2} = \frac{1}{s + \frac{k_1\eta}{k_2+\eta}}. \tag{15}$$

The Laplace transform of the mean is

$$\langle\tilde{x}\rangle = \frac{k_1}{s^2 k_2} \tag{16}$$

so in real space it's $\langle x\rangle = k_- t/k_+$ as expected [e.g. *Einstein*, 1937; *Nakagawa and Tsujimoto*, 1976]. The Laplace transform of the second moment is

$$\langle\tilde{x}^2\rangle = 2\Big(\frac{k_1}{k_2}\Big)^2\Big[\frac{1}{k_1}\frac{1}{s^2} + \frac{1}{s^3}\Big], \tag{17}$$

implying a second moment $\langle x^2\rangle = (k_1/k_2)^2[2t/k_1 + t^2]$ and a variance exemplifying the normal diffusion of bedload:

$$\sigma_x^2 = \frac{2k_1}{k_2^2}t. \tag{18}$$

This is depicted in figure 1. Of course, for the Einstein theory a closed form solution of the pdf $p(x,t)$ is possible to obtain [e.g. *Einstein*, 1937; *Hubbell and Sayre*, 1964; *Daly and Porporato*, 2006; *Daly*, 2019]. The first transform in (15) inverts easily for

$$\tilde{p}(\eta, t) = \exp\Big\{-\frac{k_1\eta}{k_2+\eta}t\Big\}. \tag{19}$$

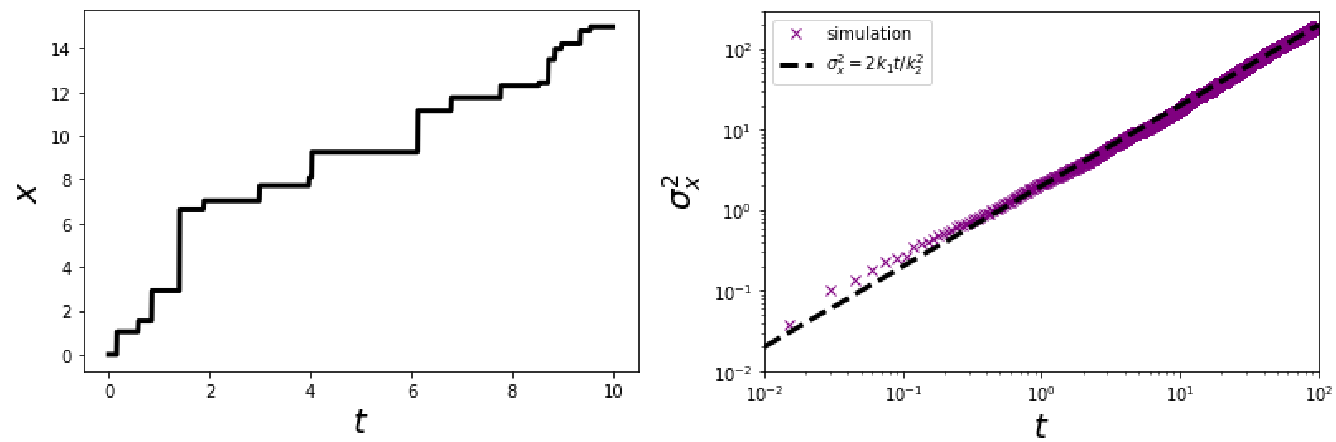

Figure 1: Left panel shows the Einstein walk in x-t space while the right panel shows the linear variance. There is a single range of normal diffusion when steps are instantaneous.

Incidentally this single Laplace transform of p provides the cumulant generating function $c(\eta, t) = \log \tilde{p}(-\eta, t)$ from which the variance follows from a more simple computation: $\sigma_x^2(t) = \partial_\eta^2 c(\eta, t)\big|_{\eta=0}$. The second inversion gives [e.g. *Daly*, 2019]

$$p(x,t) = e^{-k_1t-k_2x}\Big\{\sqrt{\frac{k_1k_2t}{x}}I_1\Big(2\sqrt{k_1k_2xt}\Big) + \delta(x)\Big\}. \tag{20}$$

This exact solution has been the benchmark theory of bedload diffusion for over 100 years. I have only derived it within a more general framework of multi-state random walks [e.g. *Weiss*, 1994].

A final note – taking the expression for $\tilde{p}(\eta, t)$ and inverting as in *Weiss* [1994] pg. 247 gives the equation

$$[k_1\partial_x + k_2\partial_t + \partial_x\partial_t]p = 0, \tag{21}$$

after some jangling. I'm surprised this is not the normal diffusion equation. I'm curious if it's correct and if the solution involving I_1 solves it. Decomposing $p = e^{-k_1t-k_2x}\pi$ gives

$$\partial_x\partial_t\pi = k_1k_2\pi. \tag{22}$$

Using the series representation

$$\pi(x,t) = k_1k_2\sum_{m=0}^{\infty}\frac{(k_1k_2)^m}{m!(m+1)!}t^{m+1}x^m \tag{23}$$

it's possible to see that the solution given earlier satisfies this differential equation. Presumably this would also follow from the Frobenius method.

3 The Lisle Theory

Apart from formulations of *Einstein* [1937] using different step length and resting time distributions than exponential [e.g. *Sayre and Hubbell*, 1965], the first significant advancement from *Einstein* [1937] was due to *Lisle et al.* [1998]. This type of random walk is depicted in figure 2.

I need to carefully investigate whether *Gordon et al.* [1972] did it too. They imparted a finite duration to bedload motions instead of considering them instantaneous like Einstein. In this way they derived two stages of bedload diffusion. This approach is closely related to the so-called persistent diffusion model [*Balakrishnan and Chaturvedi*, 1988; *Van Den Broeck*, 1990], the diffusion of a particle driven by dichotomous Markov noise [e.g. *Horsthemke and Lefever*, 1984; *Risken*, 1989; *Bena*, 2006]. The mathematics were essentially developed by Takacs (1957).

It is obtained by the choice $g_1(x,t) = \delta(x) k_1 e^{-k_1 t}$ (rest) and $g_2(x,t) = \delta(x - vt) k_2 e^{-k_2 t}$ (motion). Hence motions occur with velocity v for a duration characterized by an exponential distribution with mean $1/k_2$, while rests occur for a duration characterized by an exponential distribution with mean $1/k_1$. We consider each of the extreme initial conditions in turn.

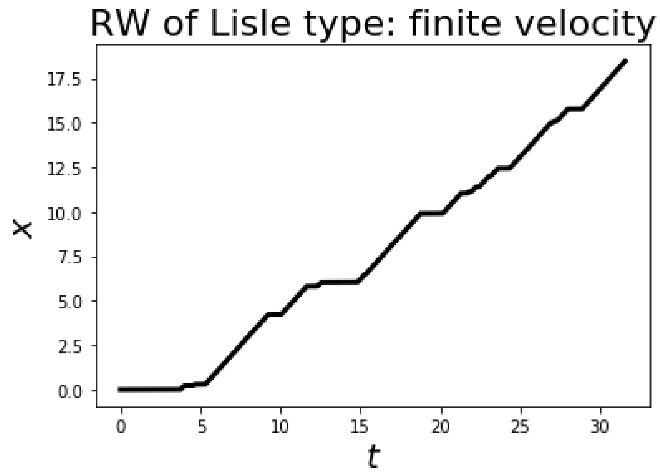

Figure 2: The *Lisle et al.* [1998] model with finite motion intervals. Motions appear as slanted lines, where the slope is the velocity v of motion. These types of walkers show super-diffusion at short timescales and normal diffusion at long timescales.

3.1 Rest initial state

If the process starts from rest, this means $\theta_1 = 1$ and $\theta_2 = 0$. In this case the Laplace transforms are

$$\tilde{g}_1(\eta, s) = \frac{k_1}{k_1 + s}, \tag{24}$$

$$\tilde{g}_2(\eta, s) = \frac{k_2}{k_2 + \eta v + s} \tag{25}$$

and

$$\tilde{G}_i(\eta, s) = \frac{1}{k_i} \tilde{g}_i(\eta, s). \tag{26}$$

Plugging these into (6) gives

$$\tilde{p}(\eta, s) = \frac{k + s + \eta v}{(k_1 + s)\eta v + (k + s)s}, \tag{27}$$

where $k = k_1 + k_2$. This inverts to

$$\tilde{p}(x, s) = \frac{k(k+s)}{v(k_1+s)^2} \exp\Big[-\frac{s(k+s)}{v(k_1+s)}x\Big] + \frac{1}{k_1+s}\delta(x). \tag{28}$$

Using the property $\mathcal{L}\{f(x+a); s\} = e^{as}\mathcal{L}\{f(x); s\}$ along with the transform of a modified Bessel function gives

$$p(x,t) = \delta(x)e^{-k_1 t} + \frac{k}{v}\exp\Big[-\frac{k_2 x}{v} - k_1\Big(t - \frac{x}{v}\Big)\Big]\Theta(t - x/v)$$
$$\times \Big\{ I_0\Big(2\sqrt{\frac{k_1 k_2 x}{v}}\big(t - \frac{x}{v}\big)\Big)$$
$$\sqrt{\frac{k_2 v(t - x/v)}{k_1 x}} I_1\Big(2\sqrt{\frac{k_1 k_2 x}{v}}\big(t - \frac{x}{v}\big)\Big)\Big\} \tag{29}$$

for the distribution of x at time t. The non-dimensionalization proposed by Lisle is $\xi = k_+ x/v$ and $\tau = k_-(t - x/v)$. In this notation the result appears as

$$p(\xi,\tau) = \frac{k_2}{v}\delta(\xi)e^{-\tau - k_1\xi/k_2} + \frac{k}{v}e^{-\xi-\tau}\Theta(\tau)\Theta(\xi)\Big\{I_0\big(2\sqrt{\xi\tau}\big) + \frac{k_2}{k_1}\sqrt{\frac{\tau}{\xi}}I_1\big(2\sqrt{\xi\tau}\big)\Big\} \tag{30}$$

3.1.1 Analytical solution of moments from rest

The first derivative gives

$$\langle\tilde{x}\rangle = vk_1\frac{1}{s^2(k+s)}, \tag{31}$$

while the second gives

$$\langle\tilde{x}^2\rangle = 2v^2k_1\frac{k_1+s}{s^3(k+s)^2}. \tag{32}$$

Therefore the mean follows from 83:

$$\frac{k^2}{vk_1}\langle x\rangle = e^{-kt} + kt - 1, \tag{33}$$

and the second moment follows from 84 and 85:

$$\frac{k^4}{2v^2k_1}\langle x^2\rangle = k_1\Big[\frac{(kt)^2}{2} - kt + 1 - e^{-kt}\Big] + k_2\Big[kt - 2 + (kt+2)e^{-kt}\Big] \tag{34}$$

Manipulating the mean provides

$$\frac{k^4}{2v^2k_1}\langle x\rangle^2 = k_1\Big(\frac{1}{2}e^{-2kt} + \frac{(kt)^2}{2} + \frac{1}{2} + (kt-1)e^{-kt} - kt\Big) \tag{35}$$

so the variance is

$$\frac{k^4}{2v^2k_1}\sigma_x^2 = k_1\Big[\frac{1}{2} - \frac{1}{2}e^{-2kt} - kte^{-kt}\Big] + k_2\Big[kt - 2 + (kt+2)e^{-kt}\Big]. \tag{36}$$

Taylor expanding shows asymptotic behavior

$$\sigma_x^2 \sim \begin{cases} \frac{1}{3}k_1v^2t^3, & t\to 0 \\ 2\frac{k_1k_2v^2}{k^3}t, & t\to\infty. \end{cases} \tag{37}$$

There is a cross-over from ballistic to normal diffusion.

3.1.2 Asymptotic solution of moments from rest

The $t\to\infty$ behavior comes from expanding 31 and 32 at $s\to 0$ and transforming. The expansions are

$$\langle\tilde{x}\rangle \sim vk_1\Big(\frac{1}{ks^2} - \frac{1}{k^2s}\Big) \tag{38}$$

$$\langle\tilde{x}^2\rangle \sim 2v^2k_1\Big(\frac{k_1}{k^2s^3} + \frac{k_2-k_1}{k^3s^2}\Big) \tag{39}$$

giving

$$\sigma_x^2 \sim \frac{2v^2k_1^2t^2}{2k^2} + \frac{2v^2k_1(k_2-k_1)t}{k^3} - \frac{v^2k_1^2t^2}{k^2} + \frac{2v^2k_1^2t}{k^3} \tag{40}$$

$$\sigma_x^2 \sim 2\frac{k_1k_2v^2}{k^3}t, \tag{41}$$

in agreement with 62. This was tricky to figure out. You have to keep the constant term in the mean. Similarly the $t \to 0$ behavior comes from expanding 31 and 32 at $1/s \to 0$ and transforming. The expansions are

$$\langle \tilde{x} \rangle \sim \frac{vk_1}{s^3}\Big(1 - \frac{k}{s}\Big) \tag{42}$$

$$\langle \tilde{x}^2 \rangle \sim \frac{2v^2k_1}{s^4}\Big(1 + [k_1 - 2k]\frac{1}{s}\Big) \tag{43}$$

giving asymptotic variance (at $t \to 0$)

$$\sigma_x^2 \sim \frac{1}{3}k_1 v^2 t^3 \tag{44}$$

in agreement with 62.

3.2 Motion initial state

Now I'll try using the opposite initial condition (the one chosen by *Lisle et al.* [1998]): $\theta_2 = 1$ and $\theta_1 = 0$. In this case

$$\tilde{p}(\eta, s) = \frac{k+s}{v(k_1+s)\eta + (k+s)s}, \tag{45}$$

and a first inverse transform gives

$$p(x, s) = \frac{k+s}{v(k_1+s)} \exp\Big[-\frac{(k+s)s}{v(k_1+s)}x\Big], \tag{46}$$

which is the same as the other case but without a delta function term at $x = 0$. This can be manipulated to

$$\begin{aligned}
p(x,t) &= \mathcal{L}^{-1}\Big\{\frac{k+s}{v(k_1+s)} \exp\Big[-\frac{(k+s)s}{v(k_1+s)}x\Big]; t\Big\} && (47)\\
&= e^{-k_1 t}\mathcal{L}^{-1}\Big\{\frac{k_2+s}{vs} \exp\Big[-\frac{(k_2+s)(s-k_1)}{vs}x\Big]; t\Big\} && (48)\\
&= e^{-k_1 t-(k_2-k_1)x/v}\mathcal{L}^{-1}\Big\{\frac{k_2+s}{vs} \exp\Big[\frac{k_1k_2}{vs}x - \frac{xs}{v}\Big]; t\Big\} && (49)\\
&= e^{-k_1 t-(k_2-k_1)x/v}\mathcal{L}^{-1}\Big\{\frac{k_2+s}{vs} \exp\Big[\frac{k_1k_2}{vs}x\Big]; t - x/v\Big\} && (50)\\
&= e^{-k_1 t-(k_2-k_1)x/v}\Big[\mathcal{L}^{-1}\Big\{\frac{k_2}{vs} \exp\Big[\frac{k_1k_2}{vs}x\Big]; t - x/v\Big\} + \mathcal{L}^{-1}\Big\{\frac{1}{v} \exp\Big[\frac{k_1k_2}{vs}x\Big]; t - x/v\Big\}\Big] && (51)\\
&= e^{-k_1 t-(k_2-k_1)x/v}\Big[\frac{k_2}{v}I_0\Big(2\sqrt{\frac{k_1k_2x}{v}\big(t-\frac{x}{v}\big)}\Big) + \frac{1}{v}\mathcal{L}^{-1}\Big\{\exp\Big[\frac{k_1k_2}{vs}x\Big] - 1; t - x/v\Big\} + \frac{1}{v}\delta(t - x/v)\Big] && (52)\\
&= e^{-k_1 t-(k_2-k_1)x/v}\Big[\frac{k_2}{v}I_0\Big(2\sqrt{\frac{k_1k_2x}{v}\big(t-\frac{x}{v}\big)}\Big) + \frac{1}{v}\sqrt{\frac{k_1k_2x}{v(t-x/v)}}I_1\Big(2\sqrt{\frac{k_1k_2x}{v}\big(t-\frac{x}{v}\big)}\Big) + \frac{1}{v}\delta(t - x/v)\Big]. && (53)
\end{aligned}$$

A key property here was $e^{-ax}f(x) = \mathcal{L}^{-1}\{f(s+a); x\}$, and those in the appendix. This type of math is not easy for me. In the non-dimensional variables this becomes

$$p(\xi, \tau) = e^{-\tau-\xi}\Big[\frac{k_2}{v}I_0(2\sqrt{\xi\tau}) + \frac{k_1}{v}\sqrt{\frac{\xi}{\tau}}I_1(2\sqrt{\xi\tau}) + \frac{k_1}{v}\delta(\tau)\Big]. \tag{54}$$

This appears totally aligned with *Lisle et al.* [1998].

3.2.1 moments having started in motion

Taking derivatives (it's easier this time) gives

$$\langle \tilde{x} \rangle = v \frac{k_1 + s}{s^2(s+k)} \tag{55}$$

$$\langle \tilde{x}^2 \rangle = 2v^2 \frac{(k_1+s)^2}{s^3(k+s)^2}. \tag{56}$$

Using 83 and 87 gives

$$\frac{k^2}{v} \langle x \rangle = k_1 k t + k_2 (1 - e^{-kt}). \tag{57}$$

Using 84, 85, and 86 gives

$$\frac{k^4}{2v^2} \langle x^2 \rangle = k_1^2 \frac{(kt)^2}{2} + k_1 k_2 \left[2kt - 2 + 2e^{-kt} \right] + k_2^2 \left[1 - (1+kt) e^{-kt} \right]. \tag{58}$$

Manipulating the mean to

$$\frac{k^4}{2v^2} \langle x \rangle^2 = k_1^2 \frac{(kt)^2}{2} + k_1 k_2 \left[\frac{1}{2} + \frac{1}{2} e^{-2kt} - e^{-kt} \right] + k_2^2 \left[kt - kte^{-kt} \right] \tag{59}$$

provides a variance

$$\frac{k^4}{2v^2 k_2} \sigma_x^2 = k_1 \left[kt + (2+kt) e^{-kt} - 2 \right] + k_2 \left[\frac{1}{2} - \frac{1}{2} e^{-2kt} - kte^{-kt} \right]. \tag{60}$$

Expanding this reveals asymptotic behavior

$$\sigma_x^2 \sim \begin{cases} \frac{1}{3} k_2 v^2 t^3, & t \to 0 \\ 2 \frac{k_1 k_2 v^2}{k^3} t, & t \to \infty. \end{cases} \tag{61}$$

The conclusion is initial condition does not affect the asymptotic scaling. This is still super-diffusion $\sigma_x^2 \propto t^3$ crossing to normal diffusion $\sigma_x^2 \propto t$. It only affects the coefficient of this scaling.

3.2.2 asymptotic approach to the moments starting in motion

Same story exactly. Not very interesting.

$$\sigma_x^2 \sim \begin{cases} \frac{1}{3} k_2 v^2 t^3, & t \to 0 \\ 2 \frac{k_1 k_2 v^2}{k^3} t, & t \to \infty. \end{cases} \tag{62}$$

with no hiccups.

3.3 Mixed initial state

With an arbitrary mixed state the double transformed density is

$$\tilde{p}(\eta, s) = \theta_1 \frac{k + s + \eta v}{(k_1+s)\eta v + (k+s)s} + \theta_2 \frac{k+s}{v(k_1+s)\eta + (k+s)s} \tag{63}$$

$$= \frac{k + s + \theta_1 \eta v}{(k_1+s)\eta v + (k+s)s} \tag{64}$$

with the identity $\theta_1 + \theta_2 = 1$.

3.3.1 analytical approach to the moments in an arbitrary mixed state

Taking one derivative gives

$$\partial_\eta \tilde{p}(\eta, s) = -v \frac{(\theta_2 s + k_1)(k+s)}{[(k_1+s)\eta v + (k+s)s]^2}, \tag{65}$$

and a second gives

$$\partial_\eta^2 \tilde{p}(\eta, s) = 2v^2 \frac{(\theta_2 s + k_1)(k+s)(k_1+s)}{[(k_1+s)\eta v + (k+s)s]^3}, \tag{66}$$

so the transformed first and second moments are

$$\langle \tilde{x} \rangle = v \frac{k_1 + \theta_2 s}{s^2(k+s)} \tag{67}$$

$$\langle \tilde{x}^2 \rangle = 2v^2 \frac{(k_1 + \theta_2 s)(k_1 + s)}{s^3(k+s)^2}. \tag{68}$$

These expressions seem correct since $\theta_2 = 0$ and $\theta_2 = 1$ cases provide the earlier expressions. Inverting the mean obtains

$$\frac{k^2}{v}\langle x \rangle = k_1\Big[\theta_1 e^{-kt} + kt - \theta_1\Big] + k_2\theta_2\Big[1 - e^{-kt}\Big] \tag{69}$$

$$= k_1\Big[(1-\theta_2)e^{-kt} + kt + (\theta_2 - 1)\Big] + k_2\theta_2\Big[1 - e^{-kt}\Big], \tag{70}$$

which still reduces to both earlier results. After more work the second moment becomes

$$\begin{aligned}\frac{k^4}{2v^2}\langle x^2 \rangle = k_1^2\Big[\frac{(kt)^2}{2} - kt + 1 - e^{-kt} + \theta_2\big\{kt - 1 + e^{-kt}\big\}\Big] \\ + k_1 k_2\Big[kt - 2 + (kt+2)e^{-kt} + \theta_2\big\{kt - kte^{-kt}\big\}\Big] \\ + k_2^2\theta_2\Big[1 - (1+kt)e^{-kt}\Big]\end{aligned} \tag{71}$$

which still reduces to earlier results. Proceeding from here gets very difficult and messy. It's possible.

3.3.2 asymptotic approach to the moments in an arbitrary mixed state

Expanding the earlier expressions for $s \to \infty$ gives

$$\langle \tilde{x} \rangle = v\Big[\frac{\theta_2}{s^2} + \frac{k_1 - \theta_2 k}{s^3} - \dots\Big] \tag{72}$$

$$\langle \tilde{x}^2 \rangle = 2v^2\Big[\frac{\theta_2}{s^3} + \frac{k_1\theta_2 + k_1 - 2k\theta_2}{s^4} + \dots\Big] \tag{73}$$

transforming to

$$\langle x \rangle \sim v\Big[\theta_2 t + (k_1 - \theta_2 k)\frac{t^2}{2}\Big] \tag{74}$$

$$\langle x^2 \rangle \sim 2v^2\Big[\frac{\theta_2 t^2}{2} + \frac{t^3}{3!}(k_1\theta_2 + k_1 - 2k\theta_2)\Big]. \tag{75}$$

The square of the mean is $\langle x \rangle^2 \sim v^2\theta_2^2 t^2 + v^2\theta_2(k_1 - \theta_2 k)t^3$, so the variance is (as $t \to 0$)

$$\sigma_x^2 \sim v^2\theta_1\theta_2 t^2 + \frac{1}{3}(\theta_1 k_1 + \theta_2 k_2)v^2 t^3. \tag{76}$$

This reproduces both earlier results and explains the link between *Lisle et al.* [1998] and my other investigations with the dichotomous Markov noise [e.g. *Horsthemke and Lefever*, 1984; *Bena*, 2006].

3.4 Summary of Lisle process

This process supports two stages of diffusion: short-time super-diffusion and long time normal-diffusion. The results are all correct and verified by simulations in figure 3.

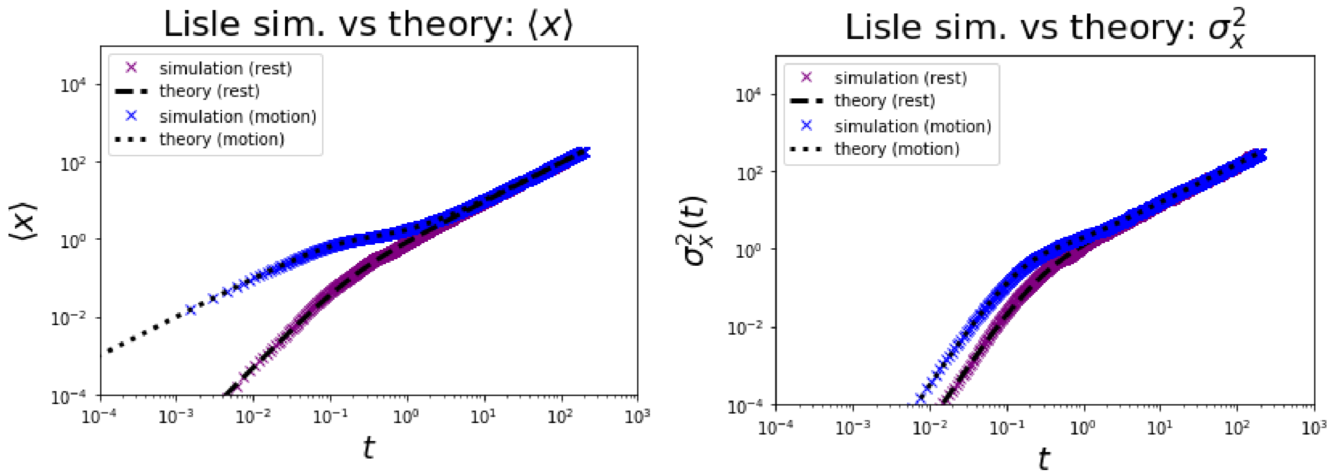

Figure 3: Variance and mean scaling compared to numerical simulations

The two stages of diffusion and their asymptotic behavior are indicated in figure 4. The initial ballistic diffusion is at least $\sigma_x^2 \propto t^2$ and at most $\sigma_x^2 \propto t^3$, and the crossover to normal diffusion occurs around $\max\{1/k_1, 1/k_2\}$.

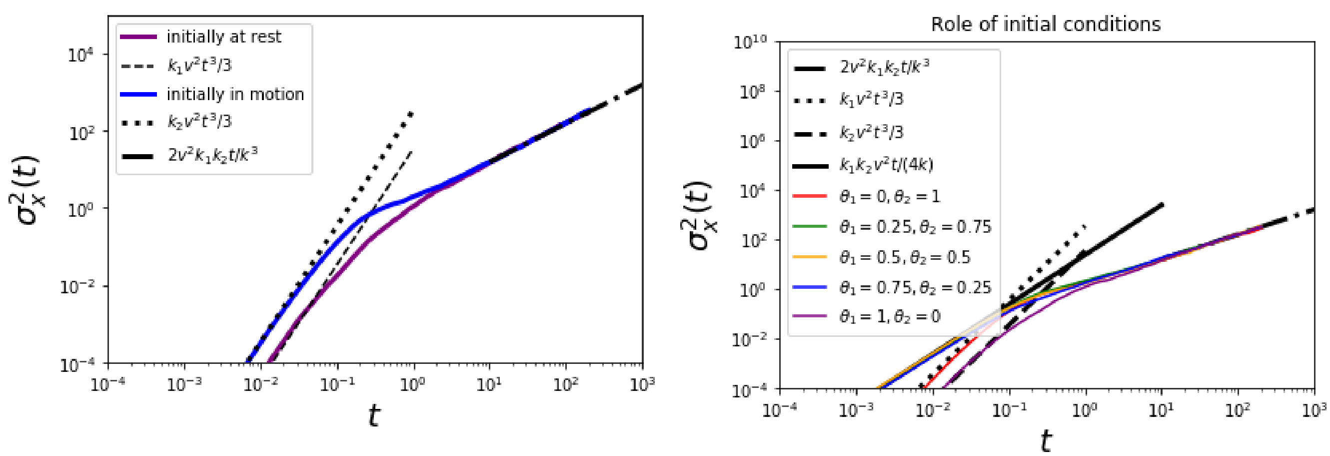

Figure 4: Scaling behavior of variance. Left panel shows pure initial conditions while the action-packed right panel shows mixed initial conditions. Mixed initial express a $\sigma_x^2 \propto t^2$ term.

4 Heavy-tailed resting time in a two-state walk

Set $\psi_1(t) = A_\alpha t^{-\alpha}$ for $t \geq t_\alpha$. $A_\alpha = (\alpha - 1)t_\alpha^{\alpha-1}$ [e.g. *Weeks et al.*, 1996, eq. 12]. Consider everything else the same as the *Lisle et al.* [1998] process. The laplace transform of $\psi_1(t)$ is

$$\tilde{\psi}(s) = A_\alpha s^{\alpha-1}\Gamma(1-\alpha, st_\alpha). \tag{77}$$

The incomplete gamma function is defined by

$$\Gamma(q,x) = \int_x^\infty t^{q-1} e^{-t} dt. \tag{78}$$

There's a need to expand this at small arguments. Crucially, *Weeks et al.* [1996] did not use the standard Pareto distribution notation.

4.1 $t \to \infty$ expected behavior

$\alpha < 1/2$ implies subdiffusion. $1/2 < \alpha < 2$ implies superdiffusion, $\alpha > 2$ implies normal diffusion.

5 A new generalization: Randomly stopped Lisle process

Consider the choice $\psi_1(t) = k_1 e^{-k_1 t}$, $f_1(x,t) = \delta(x)$, $\psi_2(t) = k_2 e^{-k_2 t}$, and

$$f_2(x,t|T) = \delta(x - vt)\Theta(T - t) + \delta(x)\Theta(t - T). \tag{79}$$

This describes a *randomly stopped* variant of the Lisle process. When $t > T$, the motion state becomes a second rest state: the motion is turned off. In this case, $g_1(\eta,s) = \frac{k_1}{k_1+s}$ and $G_1(\eta,s) = g_1(\eta,s)/k_1$ as before, while

$$g_2(\eta,s|T) = \frac{k_2}{k_2+s} e^{-(k_2+s)T} + \frac{k_2}{k_2+\eta v+s}\Big(1 - e^{-(k_2+\eta v+s)T}\Big) \tag{80}$$

and $G_2(\eta,s|T) = g_2(\eta,s|T)/k_2$. You can maybe solve this conditional to T to obtain $p(x,t|T)$. Then inputting a distribution for the trapping time T, the over-all distribution will be $p(x,t) = \int dT p(x,t|T) f(T)$.

Assuming the walker starts at rest, the double transformed probability is

$$\tilde{p}(\eta,s) = \tag{81}$$

Appendix: Laplace transforms

This is just a reference of useful Laplace transforms for these types of studies.

$$\mathcal{L}\Big\{\frac{1}{s^{k+1}}; s\Big\} = \frac{t^k}{k!}; \tag{82}$$

$$\mathcal{L}^{-1}\Big\{p^{-2}(p+a)^{-1}; x\Big\} = \frac{1}{a^2}(e^{-ax} + ax - 1), \tag{83}$$

[?, 2.1.2.33];

$$\mathcal{L}^{-1}\Big\{p^{-2}(p+a)^{-2}; x\Big\} = \frac{1}{a^3}[ax - 2 + (ax+2)e^{-ax}], \tag{84}$$

[?, 2.1.2.49];

$$\mathcal{L}^{-1}\Big\{p^{-3}(p+a)^{-2}; x\Big\} = \frac{1}{a^4}\Big[\frac{(ax)^2}{2} - 2ax - (ax+3)e^{-ax} + 3\Big], \tag{85}$$

derived from the previous result using the differentiation property;

$$\mathcal{L}^{-1}\Big\{p^{-1}(p+a)^{-2}; x\Big\} = \frac{1}{a^2}\Big(1 - (1+ax)e^{-ax}\Big), \tag{86}$$

[?, 2.1.2.47];

$$\mathcal{L}^{-1}\Big\{p^{-1}(p+a)^{-1}; x\Big\} = \frac{1}{a}(1-e^{-ax}), \tag{87}$$

[?, 2.1.2.31];

$$\mathcal{L}^{-1}\{e^{as}\tilde{f}(s); x\} = \mathcal{L}^{-1}\{\tilde{f}(s); x+a\}, \tag{88}$$

the shifting property;

$$\mathcal{L}^{-1}\{\frac{1}{s^{\nu}}\exp(a/s); t\} = \Big(\frac{t}{a}\Big)^{(\nu-1)/2} I_{\nu-1}\big(2\sqrt{at}\big), \tag{89}$$

valid for $\nu > 1$ from [?, 2.2.2.1];

$$\mathcal{L}^{-1}\{e^{a/p}-1; x\} = \sqrt{\frac{a}{x}} I_1(2\sqrt{ax}), \tag{90}$$

and [?, 2.2.2.8]. All of these Laplace transforms are verified from Wolfram Alpha so I expect no typos. Finally,

$$\mathcal{L}\Big\{\frac{a}{bs+c}; s\Big\} = \frac{a}{b}e^{-cx/b} \tag{91}$$

and

$$\mathcal{L}\Big\{\frac{as}{bs+c}; s\Big\} = \frac{a}{b}\Big[\delta(x) - \frac{c}{b}e^{-cx/b}\Big]. \tag{92}$$

The Laplace transform of a first derivative is

$$\mathcal{L}\{f'(t)\} = s\tilde{f}(s) - f(0), \tag{93}$$

while a second derivative is

$$\mathcal{L}\{f''(t)\} = s^2\tilde{f}(s) - sf(0) - f'(0). \tag{94}$$

Appendix: Asymptotics of stable laws

When a resting time distribution $\psi(t)$ has divergent moments, the behavior of its Laplace transform $\tilde{\psi}(s)$ for $s \to 0$ cannot be analyzed by simply expanding the exponential in its definition for small s as in

$$\tilde{\psi}(s) \sim \int_0^\infty \{1 - st + (st)^2/2 + \dots\}\psi(t), \tag{95}$$

since the moments of t involved in this expression are divergent. Instead, there is a different way (outlined around *Weiss* [1994] eq 2.95) leveraging the asymptotic form of $\psi(t)$ for $t \to \infty$:

$$\psi(t) \sim At^{-\alpha-1}. \tag{96}$$

Writing (using the normalization property of $\psi(t)$)

$$\tilde{\psi}(s) = 1 - (1-\tilde{\psi}(s)) = 1 - \int_0^\infty dt(1-e^{-st})\psi(t) \tag{97}$$

and setting $\tau = st$ gives

$$\tilde{\psi}(s) = 1 - \frac{1}{s}\int_0^\infty d\tau(1-e^{-\tau})\psi\big(\frac{\tau}{s}\big). \tag{98}$$

Clearly this integral is dominated by the asymptotic behavior of $\psi(t)$:

$$\tilde{\psi}(s) \sim 1 - Bs^{\alpha} \tag{99}$$

This is an important result for analyzing asymptotics of random walks involving power-law pausing time densities. The coefficient B is obtained (if necessary) by integrating the asymptotic power law using its coefficients and small-time cutoff. Such an argument is essential to the paper of *Weeks and Swinney* [1998]. They give B in terms of an incomplete Gamma function, which I could do if I wanted.

Appendix: Modified Bessel functions of the first kind

A cool property is

$$I_n(x) = T_n\Big(\frac{d}{dx}\Big) I_0(x) \tag{100}$$

where T_n is a Chebyshev polynomial of the first kind:

$$\begin{aligned} T_0(x) &= 1 &(101)\\ T_1(x) &= x &(102)\\ T_2(x) &= 2x^2 - 1 &(103)\\ &\vdots &(104) \end{aligned}$$

This is available at `http://mathworld.wolfram.com/ModifiedBesselFunctionoftheFirstKind.html`. In particular,

$$I_1(x) = I_0'(x). \tag{105}$$

The large x expansion ($x \gg |\nu^2 - 1/4|$) of the modified Bessel function of the first kind is

$$I_\nu(x) \approx \frac{e^x}{\sqrt{2\pi x}}\Big(1 + \frac{(1-2\nu)(1+2\nu)}{8x} + \dots\Big) \tag{106}$$

while the small x expansion ($0 \le x \le \sqrt{\nu+1}$) is

$$I_\nu(x) = \frac{1}{\Gamma(\nu+1)}\Big(\frac{x}{2}\Big)^\nu. \tag{107}$$

These are from `https://pdfs.semanticscholar.org/48dc/ca3cffb78de80ab37b84a992379c2f30bdda.pdf`. The recurrence relations are

$$I_{\nu-1}(x) - I_{\nu+1}(x) = \frac{2\nu}{x} I_\nu(x) \tag{108}$$

$$I_{\nu-1}(x) + I_{\nu+1}(x) = 2\frac{d}{dx} I_\nu(x). \tag{109}$$

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