Bedload diffusion theory

Kevin Pierce

July 19, 2019

I consider a two-state random walk with a reaction from one state. This is a model for alternate motion-rest switching of bedload tracers which can undergo burial when at rest. Consider un-buried tracers to be a population A, while buried tracers are a population B.

Let $\omega_2(x,t)$ be the probability that an (unburied) tracer just transitioned into motion having position x at time t, and let $\omega_1(x,t)$ be the probability that an unburied tracer just transitioned to rest having position x at time t. Suppose unburied tracers become buried tracers with constant probability κ per time. Then the probability that a resting tracer does not trap by time t is

$$\Phi(t) = 1 - e^{-\kappa t}. \tag{1}$$

Let the propagator of a particle through space and time be $g_1(x,t)$ in the rest state and $g_2(x,t)$ in the motion state. These propagators $g_i(x,t)$ characterize the probability that a particle will be found at position x at time t if it started its sojourn in the state i back at $x=0$ and $t=0$. A key point is that these propagators are asymmetric in space. Particles can only move in the direction of increasing x. Hence $g_i(x,t) = 0$ for $x < 0$. This reflects the asymmetry of river flow.

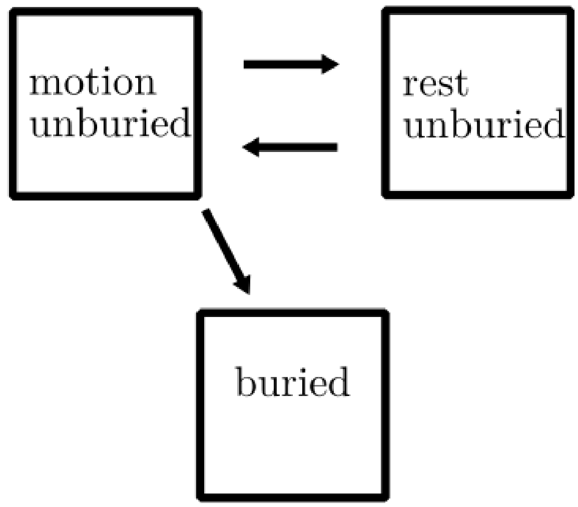

Figure 1: Schematic depiction of the three state process

1 Probabilities that a transition just occurred

Introducting the initial probabilities for a tracer being at rest or in motion as θ_1 and θ_2, and neglecting any possibility that tracers can start ($t=0$) buried, the governing equations can be developed by an argument analogous to that used to develop the multi-state continuous time random walk [e.g. *Weiss*, 1994]. The probabilities of being in an unburied state in motion or rest provided a transition just occurred are

$$\omega_1(x,t) = \theta_1 g_1(x,t)\Phi(t) + \int_0^t dt' \int_0^x dx' \omega_2(x',t') g_1(x-x',t-t')\Phi(t-t'), \tag{2}$$

$$\omega_2(x,t) = \theta_2 g_2(x,t) + \int_0^t dt' \int_0^x dx' \omega_1(x',t') g_2(x-x',t-t'). \tag{3}$$

In the limit of $\kappa \to 0$ so that no trapping ever occurs, these reduce to the theory of a two state random walk developed by *Weiss* [1976] and applied to soil transport by *Lisle et al.* [1998].

Taking the spatial Laplace transform of the more complicated expression gives

$$\hat{\omega}_1(\eta,t) = \theta_1 \hat{g}_1(\eta,t)e^{-\kappa t} + \int_0^t dt' \hat{\omega}_2(\eta,t')\hat{g}_1(\eta,t-t')e^{-\kappa(t-t')}. \tag{4}$$

Subsequently taking the temporal transform is more complex but luckily is not so bad because of the trapping-at-constant-rate assumption. Leveraging the Laplace transform shift property [e.g. *Arfken*, 1985]:

$$\hat{\omega}_1(\eta,s) = \theta_1 \hat{g}_1(\eta,s+\kappa) + \hat{\omega}_2(\eta,s)\hat{g}_1(\eta,s+\kappa) \tag{5}$$

$$\hat{\omega}_2(\eta,s) = \theta_2 \hat{g}_2(\eta,s) + \hat{\omega}_1(\eta,s)\hat{g}_2(\eta,s) \tag{6}$$

These solve for

$$\hat{\omega}_1(\eta,s) = \frac{\theta_1 + \theta_2\hat{g}_2(\eta,s)}{1-\hat{g}_1(\eta,s+\kappa)\hat{g}_2(\eta,s)}\hat{g}_1(\eta,s+\kappa) \tag{7}$$

$$\hat{\omega}_2(\eta,s) = \frac{\theta_2 + \theta_1\hat{g}_1(\eta,s+\kappa)}{1-\hat{g}_1(\eta,s+\kappa)\hat{g}_2(\eta,s)}\hat{g}_2(\eta,s) \tag{8}$$

1.1 Einstein propagators

Setting $g_1(x,t) = \delta(x)k_1e^{-k_1t}$ and $g_2(x,t) = k_2e^{-k_2x}\delta(t)$ gives

$$\hat{g}_1(\eta,s+\kappa) = \frac{k_1}{k_1+s+\kappa} \tag{9}$$

$$\hat{g}_2(\eta,s) = \frac{k_2}{k_2+\eta} \tag{10}$$

Starting tracers from rest gives

$$\hat{\omega}_1(\eta,s) = k_1\frac{k_2+\eta}{(k_1+\kappa+s)\eta + k_2(s+\kappa)} \tag{11}$$

$$\hat{\omega}_2(\eta,s) = \frac{k_1k_2}{(k_1+\kappa+s)\eta + k_2(s+\kappa)} \tag{12}$$

2 Probabilities away from transition points

Denoting the probability that a tracer is found unburied and at rest (i.e. in the 1 state) at x,t by $A_1(x,t)$, the probability that it is unburied and in motion (the 2 state) by $A_2(x,t)$, and the probability that it is found buried at x,t by $B(x,t)$, the next equations take the form (need to explain this way better)

$$A_1(x,t) = \theta_1 G_1(x,t)\Phi(t) + \int_0^t dt' \int_0^x dx'\omega_2(x',t')G_1(x-x',t-t')\Phi(t-t'). \tag{13}$$

$$A_2(x,t) = \theta_2 G_2(x,t) + \int_0^t dt' \int_0^x dx'\omega_1(x',t')G_2(x-x;,t-t'). \tag{14}$$

I still need to derive the equation for $B(x,t)$. The first two equations can be double transformed for

$$\omega_1 = \theta_1 g_1 \tag{15}$$

References

Arfken, G., *Mathematical Methods for Physicists*, Academic Press, Inc., doi:10.1063/1.3062258, 1985.

Lisle, I. G., C. W. Rose, W. L. Hogarth, P. B. Hairsine, G. C. Sander, and J.-Y. Parlange, Stochastic sediment transport in soil erosion, *Journal of Hydrology*, *204*, 217–230, 1998.

Weiss, G. H., The two-state random walk, *Journal of Statistical Physics*, *15*(2), 157–165, doi:10.1007/BF01012035, 1976.

Weiss, G. H., *Aspects and applications of the random walk*, North Holland, Amsterdam, 1994.