

The Lisle model

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The *Lisle et al.* [1998] model in double Laplace space is

$$\tilde{p}(\eta, s) = \frac{1}{s} \frac{k' + s + \theta_1 \eta v}{\eta v(s + k_1) + s(s + k')}. \quad (1)$$

This follows from the $\kappa \rightarrow 0$ limit of equation A.11 in the main text. This inverts to

$$p(x, t) = \theta_1 \delta(x) e^{-k_1 t} + \frac{1}{v} e^{-\tau - \xi} \left\{ \theta_1 \left[k_1 \mathcal{I}_0(2\sqrt{\xi\tau}) + k_2 \sqrt{\frac{\tau}{\xi}} \mathcal{I}_1(2\sqrt{\xi\tau}) \right] \right. \\ \left. + \theta_2 \left[k_1 \delta(\tau) + k_2 \mathcal{I}_0(2\sqrt{\xi\tau}) + k_1 \sqrt{\frac{\xi}{\tau}} \mathcal{I}_1(2\sqrt{\xi\tau}) \right] \right\}. \quad (2)$$

This is the $\kappa \rightarrow 0$ limit of equation 12 in the main text. Taking derivatives of (1) implies Laplace moments

$$\frac{\langle \tilde{x} \rangle}{v} = \frac{k_1 + \theta_2 s}{s^2(s + k')} \quad (3)$$

$$\frac{\langle \tilde{x}^2 \rangle}{2v^2} = \frac{(s + k_1)(\theta_2 s + k_1)}{s^3(s + k')^2}. \quad (4)$$

Inverting the first moment provides the mean

$$\frac{k'^2 \langle x \rangle}{v} = k_1 k' t + (\theta_2 k_2 - \theta_1 k_1)(1 - e^{-k' t}). \quad (5)$$

Inverting the second moment provides

$$\frac{k'^4}{2v^2} \langle x^2 \rangle = \theta_2 k'^2 (1 - (1 + k' t) e^{-k' t}) \\ + k_1 k' (1 + \theta_2) ((1 + e^{-k' t}) k' t - 2(1 - e^{-k' t})) \\ + \frac{k_1^2}{k'^4} (3(1 - e^{-k' t}) - k' t (2 + e^{-k' t}) + \frac{(k' t)^2}{2}). \quad (6)$$

This rearranges to

$$\frac{k'^4}{2v^2} \langle x^2 \rangle = \theta_2 \left[k_1 k' (k' t - 1 + e^{-k' t}) + k_2^2 (1 - (1 + k' t) e^{-k' t}) \right] \\ + k_1 k' ((1 + e^{-k' t}) k' t - 2(1 - e^{-k' t})) \\ + \frac{k_1^2}{k'^4} (3(1 - e^{-k' t}) - k' t (2 + e^{-k' t}) + \frac{(k' t)^2}{2}). \quad (7)$$

The difference $\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2$ provides the variance

$$\sigma_x^2(x) = \quad (8)$$

References

Lisle, I. G., C. W. Rose, W. L. Hogarth, P. B. Hairsine, G. C. Sander, and J.-Y. Parlange, Stochastic sediment transport in soil erosion, *Journal of Hydrology*, 204, 217–230, 1998.