Bedload diffusion theory

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I consider a three-state random walk to describe the diffusion of sediment tracers undergoing burial within a river channel. The first state is motion, the second is resting on the bed, and the third is burial.

Let $\omega_2(x,t)$ be the probability that an (unburied) tracer just transitioned into motion having position x at time t, and let $\omega_1(x,t)$ be the probability that an unburied tracer just transitioned to rest having position x at time t. Suppose unburied tracers become buried tracers with constant probability κ per time. Then the probability a tracer does not become buried after resting for duration t is

$$\Phi(t) = e^{-\kappa t}. \tag{1}$$

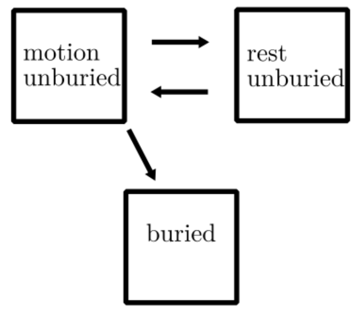

Figure 1: Schematic depiction of the three state process

1 Probabilities that a transition just occurred

Let the initial probabilities for a tracer being at rest or in motion be θ_1 and θ_2; let $\omega_1(x,t)$ be the joint distribution of finding the particle trapped at x,t just at the completion of a rest sojourn; let $\omega_1(x,t)$ be the joint distribution of finding the particle free at x,t just at the completion of a rest sojourn; and let $\omega_2(x,t)$ be the joint distribution to find the tracer at x,t just at the completion of a motion sojourn. Using arguments analogous to the two-state random walk [e.g. *Weiss*, 1976, 1994] and to reaction-diffusion problems [e.g. *Schmidt et al.*, 2007], we have

$$\omega_1(x,t) = \theta_1 g_1(x,t)\Phi(t) + \int_0^t dt' \int_0^x dx' \omega_2(x',t')\Phi(t-t')g_1(x-x',t-t'), \tag{2}$$

$$\sigma_1(x,t) = \theta_1 g_1(x,t)\big[1-\Phi(t)\big] + \int_0^t dt' \int_0^x dx' \omega_2(x',t')\big[1-\Phi(t-t')\big]g_1(x-x',t-t'), \tag{3}$$

$$\omega_2(x,t) = \theta_2 g_2(x,t) + \int_0^t dt' \int_0^x dx' \omega_1(x',t') g_2(x-x',t-t') \tag{4}$$

The double laplace transforms are

$$\hat{\omega}_1(\eta,s) = \theta_1 \hat{g}_1(\eta,s) \tag{5}$$

$$\hat{\sigma}_1(\eta,s) = \theta_1\big[\hat{g}_1(\eta,s) - \hat{g}_1(\eta,s+\kappa)\big] - \tag{6}$$

$$\hat{\omega}_2(\eta,s) = \theta_2 \hat{g}_2(\eta,s) \tag{7}$$

$$\tag{8}$$

and this now purely algebraic system has solutions

$$\hat{\omega}_0 = \frac{\theta_1 \hat{g}_1(\eta,s) + \theta_2 \hat{g}_1(\eta,s)\hat{g}_2(\eta,s)}{1-\hat{g}_1(\eta,s)\hat{g}_2(\eta,s+\kappa)}\big[\hat{g}_0(\eta,s) - \hat{g}_0(\eta,s+\kappa)\big] \tag{9}$$

$$\hat{\omega}_1 = \frac{\theta_1 \hat{g}_1(\eta,s) + \theta_2 \hat{g}_1(\eta,s)\hat{g}_2(\eta,s)}{1-\hat{g}_1(\eta,s)\hat{g}_2(\eta,s+\kappa)} \tag{10}$$

$$\hat{\omega}_2 = \frac{\theta_2 \hat{g}_2(\eta,s) + \theta_1 \hat{g}_1(\eta,s)\hat{g}_2(\eta,s+\kappa)}{1-\hat{g}_1(\eta,s)\hat{g}_2(\eta,s+\kappa)} \tag{11}$$

2 Probabilities away from transition points

Define

$$G_i(x,t) = f_j(x,t)\Psi_i(t) \tag{12}$$

where $\Psi_i(t) = \int_t^\infty \psi_i(t)dt$. This is the probability density that a sojourn in state i lasts longer than time t and the walker undergoes a displacement x during this time. Denote by $p_i(x,t)$ the probability that a tracer is found in state i having position x and t. These are related to the $\omega_i(x,t)$ by

$$p_0(x,t) = \int_0^x dx' \int_0^t dt' \omega_1(x',t')\big[1-\Phi(t-t')\big]G_0(x-x',t-t') \tag{13}$$

$$p_1(x,t) = \theta_1 G_1(x,t) + \int_0^x dx' \int_0^t dt' \omega_2(x',t') G_1(x-x',t-t') \tag{14}$$

$$p_2(x,t) = \theta_2 G_2(x,t) + \int_0^x dx' \int_0^t dt' \omega_1(x',t')\Phi(t-t') G_2(x-x',t-t') \tag{15}$$

$$\tag{16}$$

The total probability to be at x,t is

$$p(x,t) = p_0(x,t) + p_1(x,t) + p_2(x,t). \tag{17}$$

Taking double transforms provides

$$\hat{p}_0(\eta,s) = \hat{\omega}_1(\eta,s)\big[\hat{G}_0(\eta,s) - \hat{G}_0(\eta,s+\kappa)\big] \tag{18}$$

$$\hat{p}_1(\eta,s) = \theta_1 \hat{G}_1(\eta,s) + \hat{\omega}_2(\eta,s)\hat{G}_1(\eta,s) \tag{19}$$

$$\hat{p}_2(\eta,s) = \theta_2 \hat{G}_2(\eta,s) + \hat{\omega}_1(\eta,s)\hat{G}_2(\eta,s+\kappa) \tag{20}$$

$$\tag{21}$$

The total probability is

$$p(x,t) = \tag{22}$$

3 Propagators

In all cases, $g_0(x,t) = \delta(x)\theta(t)$, reflecting the fact that buried tracers do not move and so are equally probable to be found at any future time exclusively at the location x. The other propagators g_1 and g_2 have been previously considered by *Einstein* [1937] and *Lisle et al.* [1998]. Einstein's propagators are $g_1(x,t) = \delta(x)k_1 e^{-k_1 t}$ and $g_2(x,t) = k_2 e^{-k_2 x}\delta(t)$ reflecting instantaneous steps and exponentially distributed steps and rests of characteristic scales k_1 and k_2. Lisle's propagators are $g_1(x,t) = \delta(x)k_1 e^{-k_1 t}$ and $g_2(x,t) = \delta(x-vt)k_2 e^{-k_2 t}$ reflecting motions of characteristic timescale k_1 having velocity v and rests having characteristic timescale k_2.

3.1 Einstein propagators

Einstein's assumptions in the double transformed representation are

$$\hat{g}_0(\eta, s) = \frac{1}{s} \tag{23}$$
$$\hat{g}_1(\eta, s) = \frac{k_1}{k_1 + s} \tag{24}$$
$$\hat{g}_2(\eta, s) = \frac{k_2}{k_2 + \eta} \tag{25}$$
$$\tag{26}$$

and because of the exponential sojourn times,

$$\hat{G}_0(\eta, s) = \tag{27}$$
$$\hat{G}_1(\eta, s) = \frac{1}{k_1 + s} \tag{28}$$
$$\hat{G}_2(\eta, s) = \frac{1}{k_2 + \eta} \tag{29}$$
$$\tag{30}$$

3.2 Lisle propagators

Lisle's are similarly

$$\hat{g}_0(\eta, s) = \frac{1}{s} \tag{31}$$
$$\hat{g}_1(\eta, s) = \frac{k_1}{k_1 + s} \tag{32}$$
$$\hat{g}_2(\eta, s) = \frac{k_2}{k_2 + \eta v + s} \tag{33}$$

and

$$\hat{G}_0(\eta, s) = \tag{34}$$
$$\hat{G}_1(\eta, s) = \frac{1}{k_1 + s} \tag{35}$$
$$\hat{G}_2(\eta, s) = \frac{1}{k_2 + \eta v + s} \tag{36}$$
$$\tag{37}$$

References

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