Back to Einstein: how to include burial in fluvial sediment diffusion models?

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Key Points:

- We develop a random-walk model of objects in intermittent transport through an environment with traps
- Its solution provides three ranges of diffusion, two of which are anomalous
- We apply the model to sediment transport in rivers to clarify the scale-dependence of bedload diffusion

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Abstract

In gravel-bed rivers, sediment grains transport through a sequence of motions and rests. When grains rest on the surface, they can be buried by material deposited from upstream. Buried grains are not exposed to the fluid flow, so these grains are immobile for relatively long periods, only becoming mobile once they're uncovered. This crucially affects the transport characteristics of sediment. Despite the importance of sediment burial to diffusion, very few models have accounted for it. In this letter, we develop a random-walk model to incorporate sediment burial. The model predicts three ranges of sediment diffusion with distinct characteristics in each range. We express the crossover times between these ranges in terms of measurable transport parameters, and explain each range in terms of underlying physical processes. This provides new geophysical understanding of the scale-dependent phenomenon of fluvial sediment transport.

1 Introduction

Anomalous diffusion has been the topic of intense research lately, since it appears in contexts as diverse and important as the transport of cholesterols through lipid bilayers (Jeon, Monne, Javanainen, & Metzler, 2012; Molina-Garcia et al., 2018), contaminants through soils (Berkowitz, Cortis, Dentz, & Scher, 2006; X. R. Yang & Wang, 2019), and pollinator insects through ecosystems (Reynolds & Rhodes, 2009; Vallaeys, Tyson, Lane, Deleersnijder, & Hanert, 2017). Emerging research in fluvial geomorphology has come to anomalous diffusion as well, since coarse sediment grains transporting through river channels apparently display it (Bradley, 2017; Martin, Jerolmack, & Schumer, 2012). H. A. Einstein (1937) developed the first model of fluvial bedload diffusion, which is the spreading apart of grains as they transport downstream. Diffusion is induced by differences in the transport characteristics of one grain and the next, and it is usually quantified by the time-dependence of the positional variance $\sigma_x^2(t)$ of a population. When $\sigma_x^2 \propto t$, the diffusion is said to be normal, since this is found in the classic diffusion problems (e.g. A. Einstein, 1905; Taylor, 1920). However, many transport phenomena show $\sigma_x^2 \propto t^\gamma$, with $\gamma \neq 1$. This diffusion is said to be anomalous (Sokolov, 2012). If $\gamma > 1$, it is said to be superdiffusive; while if $\gamma < 1$, it is said to be subdiffusive (Metzler & Klafter, 2000).

Researchers after Einstein have come to recognize that coarse sediment moving through river channels can show either anomalous or normal diffusion depending on the timescale of observation (Nikora, Habersack, Huber, & McEwan, 2002). This is a significant issue since it implies diffusion models should be scale dependent, and it renders experimental data contingent on their observation timescales. Nikora et al. (2002); Nikora, Heald, Goring, and McEwan (2001) first identified three ranges of bedload diffusion from simulations and experimental data, and they termed these local, intermediate, and global ranges in order of increasing scale. σ_x^2 scales with a different power of time in each range, and this scaling can be anomalous or normal. Their findings are supported by a wide set of contemporary data and numerical simulations that show up to three ranges of bedload diffusion (Bialik, Nikora, & Rowiński, 2012; Bradley, 2017; Fan, Singh, Guala, Foufoula-Georgiou, & Wu, 2016; Martin et al., 2012; Wu et al., 2019; Zhang, Meerschaert, & Packman, 2012). While earlier works have developed models of bedload diffusion, there are contradictions in the literature about the expected diffusion characteristics (super/normal/subdiffusion) of each range, and no model has been developed, to our knowledge, that derives all three diffusion ranges from process-based concepts of fluvial sediment transport.

In this letter, we present a model of bedload diffusion which describes local, intermediate, and global ranges by including the duration of motion and the possibility of trapping due to burial into the classic bedload diffusion model of H. A. Einstein (1937). Einstein's model has been highly influential in river geophysics and has fostered an entire paradigm of research that leverages and generalizes his stochastic methods (e.g. Gordon, Carmichael, & Isackson, 1972; Hubbell & Sayre, 1964; Nakagawa & Tsujimoto, 1976; C. T. Yang & Sayre, 1971; Yano, 1969). Its essential content is that individual grains move in instantaneous steps interrupted by durations of rest which lie on statistical distributions (Hassan, Church, & Schick, 1991). It predicts a single range of normal diffusion (H. A. Einstein, 1937; Nakagawa & Tsujimoto, 1976). Essentially, the Einstein model is a special case of the continuous time random walk (CTRW) (Montroll & Weiss, 1965), originally developed to describe anomalous diffusion of charge carriers in solids. To include the duration of motion and the burial process we leverage the multi-state generalization of the CTRW developed by Weiss (1976, 1994). Our development allows us to derive three ranges of diffusion and the scaling behavior of σ_x^2 in each range. The model is analytically solvable, and we determine the cross-over times between diffusion ranges. Several progressive works have discriminated multiple ranges of bedload diffusion due to the duration of motion (Lisle et al., 1998) and sediment burial (Wu et al., 2019), and multiple ranges of diffusion have been shown in models of other transport phenomena (e.g. Aarão Reis & Di Caprio, 2014; Balakrishnan & Chaturvedi, 1988; Bena, 2006; Flekkøy, 2017). However, we believe our proposed model is the first analytically solvable model of three diffusion ranges. We present the model in section 2 and solve it in section 3. We use the solution in section 4 to directly attribute physical processes to each diffusion range, to discern the crossover times between ranges in terms of empirically measurable transport parameters, and to derive earlier works as limiting cases (e.g. H. A. Einstein, 1937; Lisle et al., 1998; Wu et al., 2019).

2 Bedload diffusion with burial as a multi-state random walk

We use a three-state random walk where the states are motion, rest, and burial. We label these as $i = 2$ (motion), $i = 1$ (rest), and $i = 0$ (burial). Our development of the governing equations for the three-state walk closely mimics Weiss (1994), and our approach to incorporating the trapping process closely mirrors Schmidt, Sagués, and Sokolov (2007). In our model, residence times in motion and rest states are random variables characterized by exponential distributions and motions are characterized by a constant velocity v. We consider burial to be a permanent condition which has some probability to occur when resting grains are covered by transported sediment. The probability of burial per unit time (burial rate) is considered constant, so the probability of trapping increases with the time grains spend resting.

A central concept in our derivation is the idea of a sojourn in the state i (Weiss, 1994). When a grain enters a state i at some time t_0 and position x_0, then leaves a state at some other time t_1 and position x_1, we say that the grain has completed a sojourn in the state i. The joint probability density for a complete sojourn through the state i of time $t = t_1 - t_0$ and displacement $x = x_1 - x_0$ is denoted $g_i(x, t)$. Similarly, we can consider incomplete sojourns. If a grain begins a sojourn in the state i at (t_0, x_0) and the sojourn is still on-going, the joint probability density to find the grain at (x_1, t_1) is $G_i(x, t)$. The g_i and G_i can be further decomposed when time and space components of the motion are independent (Weiss, 1994). We refer to g_i and G_i as the complete and incomplete propagators, since they move probability through space-time and are associated respectively with complete and incomplete sojourns.

Our target is the probability distribution $p(x, t)$ to find a grain at x, t if we know it started at $(x, t) = (0, 0)$, that is, with the initial distribution $p(x, 0) = \delta(x)$. We denote the initial probabilities to be at rest or in motion as θ_1 and θ_2. Normalization requires $\theta_1 + \theta_2 = 1$. Our derivation has two main steps. First, we introduce and solve for a set of joint probabilities associated with transitions of a grain from one state to another. Second, we use these quantities to solve for the probabilities that a grain is in state i having position x at time t. Afterward, we sum the these distributions over all states i to derive the joint probability that a grain is in any state at (x, t).

Now we begin the first stage of the derivation. Grains at rest may be trapped by burial. We consider burial to be permanent (e.g. Wu et al., 2019), and we assume resting grains may be buried with constant probability per unit time κ. Equivalently, we could say the mean time required for a resting grain to become buried is $1/\kappa$. From this assumption, the probability that a grain is not trapped after a time t at rest is given by a survival function $\Phi_F(t) = e^{-\kappa t}$ (F is for "free"). Likewise, the probability that it is trapped after resting for a time t is the complement $\Phi_T(t) = 1 - \Phi_F(t)$ (T is for "trapped"). We introduce $\omega_{1T}(x, t)$, $\omega_{1F}(x, t)$, and $\omega_2(x, t)$ as the joint probabilities to find a grain at (x, t) having just completed a sojourn. The subscript $1T$ denotes the completion of a rest sojourn due to trapping by burial, while $1F$ denotes the completion of a rest sojourn due to motion. Similarly, the subscript 2 denotes the completion of a motion sojourn due to resting. Using an argument similar to Weiss (1994), we write integral equations to link the ω's:

$$\omega_{1T}(x,t) = \theta_1 \Phi_T(t) g_1(x,t) + \int_0^x dx' \int_0^t dt' \omega_2(x',t') \Phi_T(t-t') g_1(x-x', t-t'), \tag{1}$$

$$\omega_{1F}(x,t) = \theta_1 \Phi_F(t) g_1(x,t) + \int_0^x dx' \int_0^t dt' \omega_2(x',t') \Phi_F(t-t') g_1(x-x', t-t'), \tag{2}$$

$$\omega_2(x,t) = \theta_2 g_2(x,t) + \int_0^x dx' \int_0^t dt' \omega_{1F}(x',t') g_2(x-x', t-t'). \tag{3}$$

The first equation can be understood as follows: $\omega_{1T}(x, t)$ describes the probability that a sojourn in the state 1 ends due to trapping at (x, t). This quantity has two contributions. The first contribution represents the possibility that the grain started at $(x, t) = (0, 0)$ in the $i = 1$ state (with probability θ_1), propagated a distance x and a time t in the $i = 1$ state (with probability density $g_1(x, t)$), was trapped (with probability $\Phi_T(t)$), and is now at (x, t). The second contribution describes the possibility that the grain was in a motion sojourn which ended at x', t' when it came to rest. From here, it propagated from x', t' to x, t at rest (with probability density $g_1(x - x', t - t')$) and was trapped during this sojourn (with probability $\Phi_T(t - t')$). The other equations can be reasoned similarly; these arguments are analogous to Weiss (1994). Once the propagators are specified, we can solve (1-3) for the ω's. This completes the first stage of the derivation.

The second stage of our derivation involves the joint probabilities of being in state regardless of whether a sojourn has just completed. These are denoted by $p_0(x, t)$ (trapped), $p_1(x, t)$ (rest), and $p_2(x, t)$ (motion), and they involve the ω's for their definition:

$$p_0(x,t) = \int_0^t dt' \omega_{1T}(x, t-t'), \tag{4}$$

$$p_1(x,t) = \theta_1 G_1(x,t) + \int_0^x dx' \int_0^t dt' \omega_2(x',t') G_1(x-x', t-t'), \tag{5}$$

$$p_2(x,t) = \theta_2 G_2(x,t) + \int_0^x dx' \int_0^t dt' \omega_{1F}(x',t') G_2(x-x', t-t'). \tag{6}$$

Equation (4) says that grains buried at any (x, t) arrived there due to trapping at x at any time leading up to t. The reasoning in (5-6) is the same as for (1-3), except we use the propagators for incomplete sojourns. These equations can be solved once the propagators are specified and the ω's are known from (1-3). Finally, we form the total probability density for a grain to be found at (x, t) in any state. This is simply

$$p(x,t) = p_0(x,t) + p_1(x,t) + p_2(x,t). \tag{7}$$

This joint density is completely determined once (1-6) are solved.

3 Specification of propagators and solution of model

We consider sojourns in the rest state to occur for an exponentially distributed time interval given by the distribution $\psi_1(t) = k_1 e^{-k_1 t}$. The probability that a sojourn in this state lasts for at least a time t is then $\Psi_1(t) = \int_t^\infty \psi_1(t)dt = e^{-k_1 t}$. $1/k_1$ is the mean duration of a single rest. Since grains do not move in the rest sojourn, the probability density that a grain is displaced by a distance x in the rest sojourn in $\delta(x)$. Hence the complete propagator for rest sojourns is $g_1(x,t) = \delta(x)\psi_1(t)$, or

$$g_1(x,t) = \delta(x) k_1 e^{-k_1 t}. \tag{8}$$

Likewise, the incomplete propagator for a rest sojourn is $G_1(x,t) = \delta(x)\Psi_1(t) = \delta(x)e^{-k_1 t} = g_1(x,t)/k_1$. The motion propagators are reasoned similarly. We consider motions to occur with a constant velocity v and to have an exponentially distributed duration given by $\psi_2(t) = k_2 e^{-k_2 t}$. $1/k_2$ is the mean duration of a single motion. Since the velocity of motions is deterministic, the probability to find a grain at position x in a motion sojourn is $\delta(x - vt)$, and the complete propagator for motion sojourns is

$$g_2(x,t) = \delta(x - vt) k_2 e^{-k_2 t}, \tag{9}$$

while the incomplete propagator is $G_2(x,t) = g_2(x,t)/k_2$ as before.

Having defined the propagators, we set out to solve (1-6) and understand bedload diffusion with a finite motion interval and when subject to trapping by burial. The convolution structure of equations (1-6) presents a formidable problem. Luckily, we have the device of Laplace transforms. These project integro-differential equations into an alternate space in which convolutions are unraveled (e.g. Arfken, 1985). The double Laplace transform of a joint probability distribution $p(x,t)$ is defined by

$$\tilde{p}(\eta, s) = \int_0^\infty dx e^{-\eta x} \int_0^\infty dt e^{-st} p(x,t). \tag{10}$$

The Laplace-transformed moments of x are linked to derivatives of the double-transformed distribution (10) (cf. Berezhkovskii & Weiss, 2002). From equation (10) it's clear that

$$\langle \tilde{x}(s)^k \rangle = (-)^k \partial_\eta^k \tilde{p}(\eta, s)\Big|_{\eta=0}. \tag{11}$$

The operator $\langle \circ \rangle$ denotes the ensemble average (e.g. Kittel, 1958). This means we can compute the variance of position as $\sigma_x^2(t) = \langle x^2 \rangle - \langle x \rangle^2 = \mathcal{L}^{-1}\{\langle \tilde{x}^2 \rangle; t\} - \mathcal{L}^{-1}\{\langle \tilde{x} \rangle; t\}^2$, where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform (e.g. Arfken, 1985). This is a powerful tool, since we can use it to derive the positional variance without integrating the distribution in equation (7).

Using the propagators (8-9) and this transform calculus, the joint distribution $p(x,t)$ is derived in appendix A, while the moments $\langle x \rangle$ and $\langle x^2 \rangle$ and ultimately the variance of position $\sigma_x^2(t)$ are derived in appendix B. With the shorthand notations $\xi = k_2 x/v$, $\tau = k_1(t - x/v)$, and $\Omega = (\kappa + k_1)/k_1$ (cf. Lisle et al., 1998), the joint distribution to find a grain at position x at time t is

$$\begin{aligned}
p(x,t) = {} & \theta_1 \mathcal{H}(\xi)\mathcal{H}(\tau)\Big[1 - \frac{k_1}{\kappa + k_1}\Big(1 - e^{-(\kappa+k_1)t}\Big)\Big]\delta(x) \\
& + \frac{1}{v} e^{-\Omega\tau - \xi} \mathcal{H}(\xi)\mathcal{H}(\tau)\Big(\theta_1\Big[k_1 \mathcal{I}_0\big(2\sqrt{\xi\tau}\big) + k_2\sqrt{\frac{\tau}{\xi}}\mathcal{I}_1\big(2\sqrt{\xi\tau}\big)\Big] \\
& + \theta_2\Big[k_1\delta(\tau) + k_2 \mathcal{I}_0\big(2\sqrt{\xi\tau}\big) + k_1\sqrt{\frac{\xi}{\tau}}\mathcal{I}_1\big(2\sqrt{\xi\tau}\big)\Big]\Big) \\
& + \frac{1}{v}\frac{\kappa k_2}{\kappa + k_1} e^{-\kappa\xi/(\kappa+k_1)} \mathcal{H}(\xi)\mathcal{H}(\tau)\Big((\theta_1/\Omega)\mathcal{P}_2(\xi/\Omega, \Omega\tau) + \theta_2 \mathcal{P}_1(\xi/\Omega, \Omega\tau)\Big).
\end{aligned} \tag{12}$$

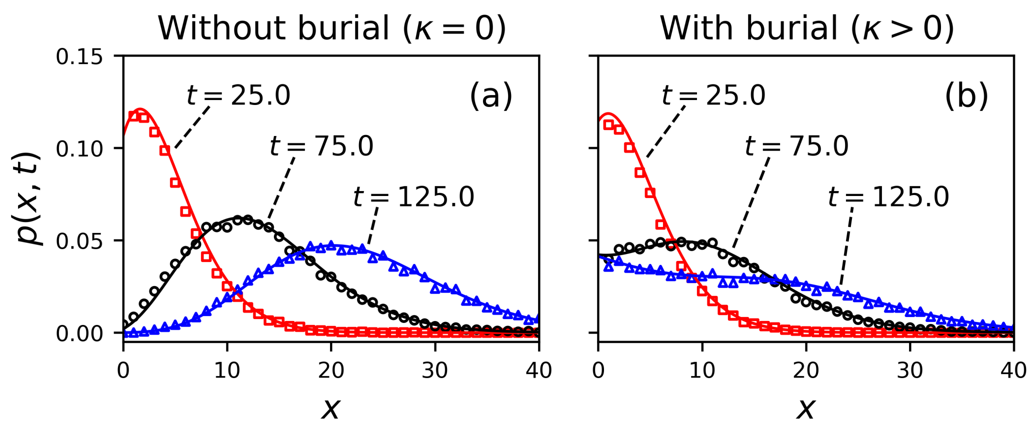

Figure 1. Joint distributions for a grain to be at position x at time t are displayed for the choice $k_1 = 0.1$, $k_2 = 1.0$, $v = 2.0$. Grains are considered initially at rest ($\theta_1 = 1$, $\theta_2 = 0$). The solid lines are the analytical distribution in equation (12), while the points are simulation results to show mathematical correctness. Colors pertain to different times. Units are unspecified, since our aim is to demonstrate the general characteristics of $p(x,t)$. Panel (a) shows the case $\kappa = 0$ – the absence of burial. In this case, the joint distribution tends toward Gaussian at large times (e.g. H. A. Einstein, 1937; Lisle et al., 1998). Panel (b) shows the case when grains have rate $\kappa = 0.01$ to become buried while resting. Because of burial, the joint distribution tends toward a more uniform distribution than Gaussian. This shows a redistribution of probability to smaller values of x due to the burial process (cf. Wu et al., 2019). The redistribution is encoded mathematically by the Marcum Q-function terms in equation (12). A similar tendency is seen in field studies of tracer dispersion in gravel bed rivers (e.g. Hassan & Church, 1994).

$\mathcal{H}$ is the Heaviside step function and we use the convention $\mathcal{H}(0) = 1$. The $\mathcal{I}_\nu$ are modified Bessel functions of the first kind, and the $\mathcal{P}_\mu$ are generalized Marcum Q-functions defined by $\mathcal{P}_\mu(x,y) = \int_0^y e^{-z-x}(z/x)^{(\mu-1)/2}\mathcal{I}_{\mu-1}(2\sqrt{xz})dz$ (Temme, 1996). Modified Bessel functions are common in one-dimensional diffusion problems (e.g. Daly & Porporato, 2010; H. A. Einstein, 1937; Giddings & Eyring, 1955).

The Marcum Q-functions are convolutions between modified Bessel functions and decaying exponentials. They were originally devised in relation to radar detection problems (Marcum, 1960). Conceptually, the Q-functions emerge in our context from the sediment burial process. According to our assumptions, resting grains can become buried in some interval of time with an exponential probability. Meanwhile, the probability that grains are resting follows a modified Bessel distribution (e.g. H. A. Einstein, 1937; Lisle et al., 1998). As a result, the probability that sediment is resting and becomes buried involves the convolution structure of the Marcum Q-functions. Consistent with this interpretation, the terms involving these convolutions vanish when the burial rate is taken to zero ($\kappa \to 0$). The distribution (12) is depicted in figure 1 alongside simulations generated by a direct method. This method is based on evaluating the cumulative transition probabilities between states on a small timestep (cf. Barik, Ghosh, & Ray, 2006). A link to the simulation code, which includes descriptive comments, is available in the acknowledgments.

The first two moments and the positional variance are derived in appendix B. The moments are

$$\langle x(t)\rangle = A_1 e^{(b-a)t} + B_1 e^{-(a+b)t} + C_1, \tag{13}$$
$$\langle x^2(t)\rangle = A_2(t) e^{(b-a)t} + B_2(t) e^{-(a+b)t} + C_2. \tag{14}$$

In these equations, $a = (\kappa + k_1 + k_2)/2$ and $b = \sqrt{a^2 - \kappa k_2}$ are effective rates having dimensions of inverse time. The A_i and B_i are polynomials available in table 1. The variance is

$$\sigma_x^2(t) = A(t) e^{(b-a)t} + B(t) e^{-(a+b)t} + C(t). \tag{15}$$

A, B, and C are transcendental functions available in table 1. This equation represents the scale-dependence of bedload diffusion for sediment gradually undergoing burial.

Table 1. Polynomials and transcendental functions used in the expressions of the mean (13), second moment (14) and variance (15) of bedload tracers.

$A_1 = \frac{v}{2b}\left[\theta_2 + \frac{k_1 + \theta_2\kappa}{b-a}\right]$
$B_1 = -\frac{v}{2b}\left[\theta_2 - \frac{k_1 + \theta_2\kappa}{a+b}\right]$
$C_1 = -\frac{v}{2b}\left[\frac{k_1 + \theta_2\kappa}{b-a} + \frac{k_1 + \theta_2\kappa}{a+b}\right]$
$A_2(t) = \frac{v^2}{2b^3}\Big[(bt-1)[k_1 + \theta_2(2\kappa + k_1 + b - a)] + \theta_2 b + \frac{(\kappa + k_1)(\theta_2\kappa + k_1)}{(b-a)^2}[(bt-1)(b-a) - b]\Big]$
$B_2(t) = \frac{v^2}{2b^3}\Big[(bt+1)[k_1 + \theta_2(2\kappa + k_1 - a - b)] + \theta_2 b - \frac{(\kappa + k_1)(\theta_2\kappa + k_1)}{(a+b)^2}[(bt+1)(a+b) + b]\Big]$
$C_2 = \frac{v^2}{2b^3}(\kappa + k_1)(\theta_2\kappa + k_1)\left[\frac{2b-a}{(b-a)^2} + \frac{a+2b}{(a+b)^2}\right]$
$A(t) = A_2(t) - 2A_1C_1 - A_1^2 \exp[(b-a)t]$
$B(t) = B_2(t) - 2B_1C_1 - B_1^2 \exp[-(a+b)t]$
$C(t) = C_2 - C_1^2 - 2A_1B_1 \exp[-2at]$

The positional variance is plotted in figure 2 for conditions $\theta_1 = 1$ and $k_2 \gg k_1 \gg \kappa$. These conditions mean that all grains are initially at rest (cf. Wu et al., 2019), motion intervals are typically much shorter than rests (cf. H. A. Einstein, 1937), and that sediment burial requires a much longer time than typical rests. We concentrate on these conditions in the discussion which follows as they are most relevant to bedload diffusion in gravel-bed rivers. Under typical conditions, sediment transport is mostly rarefied and intermittent (Frey, 2014) and the process of grains becoming buried while they rest on the surface is relatively slow (e.g. Hassan & Church, 1994). Figure 2 demonstrates that under these conditions the variance (15) shows three ranges of bedload diffusion with approximate power law scaling ($\sigma_x^2 \propto t^\gamma$) and a fourth range with no diffusion ($\sigma_x^2 = \text{const}$) that results from the burial of all sediment grains. We identify the first three ranges as similar to the local, intermediate, and global ranges proposed by Nikora et al. (2002, 2001). We suggest to call the fourth range the geomorphic range, since any further transport in this range would require scour of the bed (i.e., a geomorphic change) to expose buried grains to the flow (cf. Martin, Purohit,

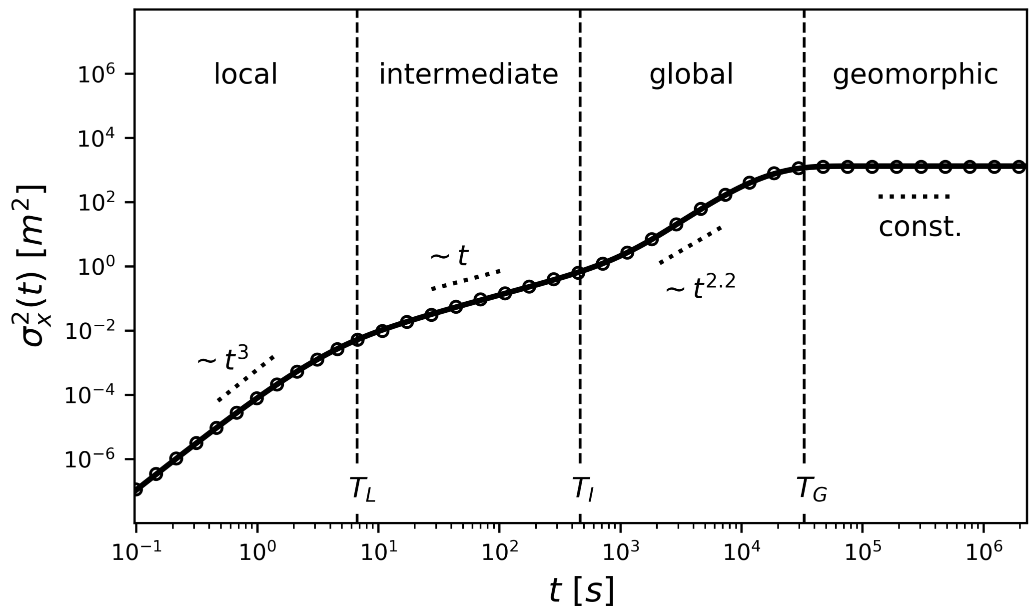

Figure 2. The variance equation (15) is plotted for the parameters $1/k_2 = 1.5$s, $1/k_1 = 30.0$s, and $v = 0.1$m/s. These values are comparable to laboratory flume experiments transporting small ($\sim$ 5mm) gravels (cf. Lajeunesse et al., 2010; Martin et al., 2012). The timescale of burial is set to $1/\kappa = 7200.0$s (two hours), and the initial condition is rest ($\theta_1 = 1$). The solid line is equation (15) while the points are directly simulated. When $k_2 \gg k_1 \gg \kappa$, as as is the case in this plot, there are four distinct scaling ranges of σ_x^2: local, intermediate, global, and geomorphic. Within each range, a slope key is added to demonstrate the scaling $\sigma_x^2 \propto t^\gamma$. There are three crossovers between these ranges, denoted on the figure by vertical lines. Crossover times T_L, T_I, and T_G are indicated at the bottom of the plot. They are given in equations (16-18) in terms of model parameters.

& Jerolmack, 2014; Nakagawa & Tsujimoto, 1980; Voepel, Schumer, & Hassan, 2013). Using this terminology, we summarize that when $k_2 \gg k_1 \gg \kappa$ and $\theta_1 = 1$, the variance in equation (15) expresses four scaling ranges of bedload diffusion: local, intermediate, global, and geomorphic.

4 Discussion of scale-dependent bedload diffusion

The model we have presented involves four parameters. These are the characteristic velocity v of moving sediment and three key timescales: $1/k_2$, $1/k_1$, and $1/\kappa$. These timescales represent the mean duration of motion, the mean duration of rest, and the mean duration of rest before burial occurs. As shown in figure 2, the diffusion in the first three ranges is approximately characterized by scaling laws $\sigma_x^2 \propto t^\gamma$. The exponents γ in each range result from competition between different terms in equation (15). Between these ranges, there are three crossover regions where the scaling is not purely a power law. Because these crossover regions are relatively narrow, we can approximately characterize them with crossover times T_L, T_I, and T_G. Then the diffusion ranges can be divided into $0 < t < T_L$ (local), $T_L < t < T_I$ (intermediate), $T_I < t < T_G$ (global), and $T_G < t$ (geomorphic). These crossover times are depicted as vertical lines in figure 2. To understand the scale dependence of bedload diffusion expressed by equation 15, we need to determine the exponents γ of each range and express the crossover times T_L, T_I, and T_G in terms of the model parameters v, k_1, k_2, and κ.

The diffusion exponents γ in the local, intermediate, and global ranges can be determined from two limiting cases of equation (15): (1) $t \ll 1/\kappa$, and (2) $1/k_2 \to 0$ while $vk_2 = \text{const}$. Limit (1) corresponds to times so short that a negligible amount of sediment burial has occurred, while limit (2) corresponds to times so long that individual motions appear as instantaneous steps (having mean step length $l = vk_2$) (cf. H. A. Einstein, 1937). We evaluate these limits in appendix C and conclude the scaling exponent in the local range is $2 \leq \gamma \leq 3$ depending on the initial conditions θ_i; $\gamma = 1$ in the intermediate range; and $2 \leq \gamma \leq 3$ within the global range, depending on the parameters k_2, k_1, and κ. To summarize, when the timescale parameters satisfy $k_2 \gg k_1 \gg \kappa$, the model predicts local range superdiffusion, intermediate range normal diffusion, and global range superdiffusion.

The crossover times between local, intermediate, global, and geomorphic ranges depicted in figure 2 are defined by

$$T_L = \sqrt{\frac{1}{k_1}\frac{1}{k_2}}, \tag{16}$$

$$T_I = \sqrt{\frac{1}{k_1}\frac{1}{\kappa}}, \tag{17}$$

$$T_G = \sqrt{\frac{1}{\kappa}\frac{k_1+k_2}{\kappa k_1}}. \tag{18}$$

Each of these is a geometric mixture of two fundamental timescales of the model. The crossover T_L between local and intermediate ranges is a mixture of the mean times of a single rest and a single motion, while the crossover T_I between intermediate and global ranges is a mixture of the mean time of a single rest and the mean time required for a resting grain to become buried. The crossover T_G between global and geomorphic ranges is more difficult to understand. We came to equation (18) by reasoning that the first grains to trap will be those which have almost always been at rest since $t = 0$. These will trap at $t \sim 1/\kappa$, which is the first timescale in equation (18). The last grains to trap will be those which are likely to be in motion at relatively large times, since this means they have not been in the resting state in which they are susceptible to burial. By analogy to the two-state mobile-immobile model of Ancey, Böhm, Jodeau, and Frey (2006), we reason that if grains have been in transport a relatively long time and has not been buried, the fraction of time they spend in motion will be $(1/k_2)/(1/k_1 + 1/k_2) = k_1/(k_1 + k_2)$. For these grains, the trapping rate is effectively reduced to $\kappa k_1/(k_1 + k_2)$, which is the second timescale in equation (18). We have visually tested the crossover time expressions (16-18) for many parameter choices. As long as they satisfy $k_2 \gg k_1 \gg \kappa$, these expressions provide an adequate characterization of the crossover locations between diffusion regimes. Equations (16) and (17) are especially trustworthy. However, by contriving large values of k_2, it is possible to drive T_G to ever larger values, in effect making the crossover region between global and geomorphic ranges arbitrarily wide. Ultimately, we suggest these expressions be used with care and in consideration of the idealization being made in representing a crossover region of finite width by a single characteristic time. We suggest the width and characteristics of the crossover regions between the power-law ranges of bedload transport as an open research problem in need of clarification.

We have extended earlier bedload diffusion models to describe four ranges of diffusion instead of one or two. Earlier models can be accessed through the two simplified limits used earlier to extract the scaling exponents γ. As discussed in appendix C, limit (1) leads directly to the model developed by Lisle et al. (1998) to describe soil transport within a sheet flow, while limit (2) implies identical diffusion characteristics as the active layer formulation of bedload transport gradually undergoing burial developed by Wu et al. (2019). Each of these subsequently limits to the original descriptions of H. A. Einstein (1937) and his followers (e.g. Hubbell & Sayre, 1964; Nakagawa & Tsujimoto, 1976; Shen, 1980; Yano, 1969). Lisle et al. (1998) generalized the Einstein model, which is an infinite velocity diffusion model involving instantaneous steps, to include a finite velocity and duration of motion. Although Lisle et al. (1998) only mention this anecdotally, their generalization leads to two stages of diffusion, in constrast to the single range of H. A. Einstein (1937) and followers (e.g. Nakagawa & Tsujimoto, 1976; Yano, 1969). One is local range superdiffusion and the other is an intermediate range normal diffusion. This is shown in appendix C. Wu et al. (2019) developed an active layer formulation of fluvial bedload transport where sediment grains can be transferred from the active layer (surface) to a substrate layer (burial) with a constant probability per unit time. They simplified the problem by considering permanent burial (as we have) and motion by instantaneous steps, deriving two stages of diffusion that we identify as an intermediate range normal diffusion and a global range superdiffusion. Our model extends these two works and links them into the formalism of multi-state CTRWs (e.g. Weiss, 1994) that was implicitly applied by H. A. Einstein (1937). Further developments may benefit from staying close to these unifying ideas from the physics literature.

We identify two key implications of the model we've presented in this letter. First, this work is the first to confine the range over which normal diffusion of bedload is expected. We can infer from the classic flume tracer studies that normal diffusion is expected of bedload when the timescale of observation is neither sufficiently small to resolve individual motions nor sufficiently large to resolve the trapping of tracers by sediment burial (H. A. Einstein, 1937, 1942; Nakagawa & Tsujimoto, 1976; Yano, 1969; Yano, Tsuchiya, & Michiue, 1969). We have quantified this observation. When the timescale under consideration satisfies $T_L < t < T_I$, with T_L and T_I given by equations (16) and (17), we expect normal bedload diffusion $\sigma_x^2 = D_d t$. This heuristic may be useful to discern when and if simple stochastic models of bedload transport may be relevant to environmental science considerations such as contaminant transport (e.g Macklin et al., 2006; Malmon, Reneau, Dunne, Katzman, & Drakos, 2005) or aquatic habitat restoration (e.g. Gaeuman, Stewart, Schmandt, & Pryor, 2017). Second, the model links bedload transport understanding across scales. In practice, we might measure sediment diffusion within a channel on one timescale with intent to apply our knowledge at longer or shorter timescales. Often, the measurement timescale is limited by practical reasons. For example, we could determine k_1 and k_2 by measuring the virtual velocity and diffusivity of sediment tracers in the intermediate range (e.g. Hassan & Bradley, 2017). With an estimate of the trapping rate κ, equation 15 can predict diffusion characteristics on the global range $T_I < t < T_G$, which is more difficult to study experimentally.

Finally, we highlight some limitations of our model and suggest some ideas for future research. Perhaps the most obvious limitation is that we have considered burial to be a permanent condition, which led to no transport or diffusion in the geomorphic range of timescales. In actuality, sediment burial is a temporary process relating to the scour and fill of the sedimentary bed (e.g. Hassan & Church, 1994; Martin et al., 2014; Voepel et al., 2013; Wong, Parker, DeVries, Brown, & Burges, 2007), and there is a need to better quantify the timescales of this process. In particular, there have been no measurements so far of the sediment burial rate κ (or χ in the notation of Wu et al. (2019)). In principle, this could be extracted from flume experiments by timing the duration that tracers can rest on the surface before they become buried. Furthermore, existing theoretical studies have attributed sediment resting periods to any resting sediment, whether buried or not (e.g. Martin et al., 2014; Voepel et al., 2013). We argue this is a process-independent concept of sediment resting. This concept implies heavy-tailed sediment resting time distributions (e.g. Hassan, Voepel, Schumer, Parker, & Fraccarollo, 2013; Martin et al., 2012; Olinde & Johnson, 2015). However, one can envision multiple trapping processes in coexistence within channels, such as sediment trapping by burial (e.g. Ferguson & Hoey, 2002; Haschenburger, 2013; Hassan & Church, 1994) and by stranding on top of bars (e.g. Bradley, 2017; Ferguson & Hoey, 2002) and floodplains (e.g. Malmon, Dunne, & Reneau, 2003). We suggest it may be useful to treat sediment resting intervals of sediment exposed on the surface differently than resting intervals of trapped sediment. This opens up the possibility to include multiple trapping processes which may act simultaneously and on different timescales into fluvial sediment diffusion models. A second limitation we have already alluded to: this limitation is shared by all stochastic bedload transport models we are aware of. This is that the role of channel morphology and its evolution on statistical characteristics of sediment transport are experimentally well-known and appreciable (e.g. Hassan & Bradley, 2017; Kasprak, Wheaton, Ashmore, Hensleigh, & Peirce, 2014; Pyrce & Ashmore, 2005), but this appears to us to have been essentially neglected in models. There is a need to better understand the statistical characteristics of bedload transport in a context of channel morphology. Presumably, this imparts spatial and temporal variations into these statistics, or it possibly renders them ill-defined, which would be an unfortunate state of affairs. This interrelationship between morphology and transport deserves ever more attention. The connections are apparently more intimate than some classical researchers realized, if the recent observations of stick-slip motions of bars are any indication (Dhont & Ancey, 2018). Finally, with regard to these limitations we highlight that the random walk formalism from the physics literature we have used in this letter (Weiss, 1994) has been carefully extended to conditions where mobility characteristics vary across space and time. This is called a random walk in a random environment and is a popular concept in biological transport (e.g. Codling, Plank, & Benhamou, 2008). Perhaps there are lessons in this literature for river geophysics. Can the morphologically active river channel be treated as a random environment while bedload grains move in random walks? More openness to interdisciplinary research might provide new traction on the problem of appropriately describing bedload transport from local to geomorphic scales.

5 Conclusion

We have developed a random walk model between alternating mobile and immobile states with a possibility of trapping from the immobile state, and we have used it to describe the diffusion of bedload sediment transporting through a river channel as it gradually becomes buried. This ultimately extends the original diffusion model of H. A. Einstein (1937). To our knowledge, this model is the first analytical description of bedload diffusion across local, intermediate, global, and geomorphic timescales. Pushing the ideas of Nikora et al. (2002) somewhat further, we have suggested a geomorphic range of diffusion to describe the range at timescales larger than the global range that is not properly characterized by existing models of bedload diffusion that do not account for morphodynamic processes. Our model implies a possibility to use information from experiments conducted on one timescale to understand sediment diffusion phenomena and related problems such as channel morphology (e.g. Church, 2006; Hassan & Bradley, 2017) and environmental issues (e.g. Gaeuman et al., 2017; Macklin et al., 2006) on other timescales. The next step from this work is to incorporate the bed scour process which re-exposes buried sediment into a bedload diffusion model. This will overcome the approximation of permanent burial. The multi-state random walk formalism we used here should readily accommodate this extension (e.g. Weiss, 1994). This extension would ultimately improve on our efforts and those of Wu et al. (2019) to better describe bedload diffusion across the global and geomorphic ranges. while furthering the research trend of incorporating more physically-based ideas into the research paradigm of Einstein.

A Calculation of the distribution function

Double transforming (1-3) using the definition (10) gives

$$\tilde{\omega}_{1T}(\eta,s) = \theta_1\tilde{g}_1(\eta,s) + \tilde{\omega}_2(\eta,s)\tilde{g}_1(\eta,s) - \tilde{\omega}_{1F}(\eta,s), \tag{A.1}$$
$$\tilde{\omega}_{1F}(\eta,s) = \theta_1\tilde{g}_1(\eta,s+\kappa) + \tilde{\omega}_2(\eta,s)\tilde{g}_1(\eta,s+\kappa), \tag{A.2}$$
$$\tilde{\omega}_2(\eta,s) \;\; = \theta_2\tilde{g}_2(\eta,s) + \tilde{\omega}_{1F}(\eta,s)\tilde{g}_2(\eta,s). \tag{A.3}$$

This purely algebraic system solves for

$$\tilde{\omega}_{1T}(\eta,s) = \frac{\theta_1 + \theta_2\tilde{g}_2(\eta,s)}{1-\tilde{g}_1(\eta,s+\kappa)\tilde{g}_2(\eta,s)}\big\{\tilde{g}_1(\eta,s) - \tilde{g}_1(\eta,s+\kappa)\big\}, \tag{A.4}$$
$$\tilde{\omega}_{1F}(\eta,s) = \frac{\theta_1 + \theta_2\tilde{g}_2(\eta,s)}{1-\tilde{g}_1(\eta,s+\kappa)\tilde{g}_2(\eta,s)}\tilde{g}_1(\eta,s+\kappa), \tag{A.5}$$
$$\tilde{\omega}_2(\eta,s) \;\; = \frac{\theta_2 + \theta_1\tilde{g}_1(\eta,s+\kappa)}{1-\tilde{g}_1(\eta,s+\kappa)\tilde{g}_2(\eta,s)}\tilde{g}_2(\eta,s). \tag{A.6}$$

Double transforming (4-6) gives

$$\tilde{p}_0(\eta,s) = \frac{1}{s}\tilde{\omega}_{1T}(\eta,s), \tag{A.7}$$
$$\tilde{p}_1(\eta,s) = \theta_1\tilde{G}_1(\eta,s) + \tilde{\omega}_2(\eta,s)\tilde{G}_1(\eta,s), \tag{A.8}$$
$$\tilde{p}_2(\eta,s) = \theta_2\tilde{G}_2(\eta,s) + \tilde{\omega}_{1F}(\eta,s)\tilde{G}_2(\eta,s). \tag{A.9}$$

The total probability is $p(x,t) = p_0(x,t)+p_1(x,t)+p_2(x,t)$. Using equations (A.4-A.9) this becomes, in the double Laplace representation,

$$\begin{aligned}\tilde{p}(\eta,s) = &\frac{1}{s}\frac{\theta_1 + \theta_2\tilde{g}_2(\eta,s)}{1-\tilde{g}_1(\eta,s+\kappa)\tilde{g}_2(\eta,s)}\big\{\tilde{g}_1(\eta,s) - \tilde{g}_1(\eta,s+\kappa)\big\}\\ &+ \frac{\theta_1\big[\tilde{G}_1(\eta,s) + \tilde{g}_1(\eta,s+\kappa)\tilde{G}_2(\eta,s)\big] + \theta_2\big[\tilde{G}_2(\eta,s) + \tilde{g}_2(\eta,s)\tilde{G}_1(\eta,s)\big]}{1-\tilde{g}_1(\eta,s+\kappa)\tilde{g}_2(\eta,s)}.\end{aligned} \tag{A.10}$$

Plugging the propagators outlined in equations (8-9) into equation (A.10) gives

$$\tilde{p}(\eta,s) = \frac{1}{s}\frac{(s+\kappa+k')s + \theta_1(s+\kappa)\eta v + \kappa k_2}{(s+\kappa+k_1)\eta v + (s+\kappa+k')s + \kappa k_2}. \tag{A.11}$$

In this equation, $k' = k_1 + k_2$, and we have used the normalization requirement of the initial probabilities $\theta_1 + \theta_2 = 1$. The double inverse transform of this equation provides the distribution $p(x,t)$. It is convenient to invert the transform over η first. Using the results 15.103 (transform of exponential), 15.123 (transform of derivative), and 15.141 (transform of Dirac delta function) from Arfken (1985) provides

$$\begin{aligned}\tilde{p}(x,s) = \theta_1\frac{s+\kappa}{s(s+\kappa+k_1)}\delta(x) + \frac{1}{v}\Big(&\frac{(s+\kappa+k')s + \kappa k_2}{s(s+\kappa+k_1)}\\ &- \frac{\theta_1(s+\kappa)[s(s+\kappa+k_1)+\kappa k_2]}{s(s+\kappa+k_1)^2}\Big)\exp\Big[-\frac{(s+\kappa+k')s+\kappa k_2}{s+\kappa+k_1}\frac{x}{v}\Big].\end{aligned} \tag{A.12}$$

Taking the remaining inverse transform over s, applying results 15.152 (substitution), 15.164 (translation), and 15.175 (transform of te^{kt}) from Arfken (1985), and defining the shorthand notations $\tau = k_1(t - x/v)$, $\xi = k_2 x/v$, and $\Omega = (\kappa + k_1)/k_1$, gives the simpler form

$$\begin{aligned}p(x,t) = &\theta_1\Big[1 - \frac{k_1}{\kappa+k_1}\big(1-e^{-(\kappa+k_1)t}\big)\Big]\delta(x) + \frac{1}{v}\exp[\Omega\tau - \xi]\\ &\times\mathcal{L}^{-1}\Big\{\Big(\theta_2 + \frac{\theta_1 k_1 + \theta_2 k_2}{s} + \frac{\theta_1 k_1 k_2}{s^2} + \frac{\theta_2\kappa k_2}{s(s-\kappa-k_1)} + \frac{\theta_1\kappa k_1 k_2}{s^2(s-\kappa-k_1)}\Big)\\ &\qquad\times\exp\big[\frac{k_1\xi}{s}\big];\tau/k_1\Big\}.\end{aligned} \tag{A.13}$$

Using entries 2.2.2.1, 2.2.2.8, and 1.1.1.13 from Prudnikov, Brychkov, and Marichev (1992) in conjunction with the definition of the Marcum Q-function $\mathcal{P}_\mu(x,t)$ (e.g. Temme, 1996), and inserting the Heaviside functions to account for the fact that grains can neither travel backwards nor at speeds exceeding v, we finally arrive at equation 12 for the joint distribution $p(x,t)$.

B Calculation of the moments

We employ equation 11 to compute the first two moments of position x and ultimately its variance. The first two derivatives of the double Laplace transformed distribution (A.11) are

$$\partial_\eta \tilde{p}(\eta,s) = -v\frac{1}{s}\frac{[(s+\kappa+k')s+\kappa k_2][\theta_2(s+\kappa)+k_1]}{[\eta v(s+\kappa+k_1)+(s+\kappa+k')s+\kappa k_2]^2}, \tag{B.1}$$

$$\partial^2_\eta \tilde{p}(\eta,s) = 2v^2\frac{1}{s}\frac{(s+\kappa+k_1)[(s+\kappa+k')s+\kappa k_2][\theta_2(s+\kappa)+k_1]}{[\eta v(s+\kappa+k_1)+(s+\kappa+k')s+\kappa k_2]^3}. \tag{B.2}$$

Evaluating these at $\eta = 0$ and applying equation (11) provides the Laplace transformed moments

$$\frac{\langle \tilde{x}(s)\rangle}{v} = \frac{1}{s}\frac{\theta_2(s+\kappa)+k_1}{(s+\kappa+k')s+\kappa k_2} = \frac{1}{s}\frac{\theta_2(s+\kappa)+k_1}{(s+a+b)(s+a-b)}, \tag{B.3}$$

$$\frac{\langle \tilde{x}^2(s)\rangle}{2v^2} = \frac{1}{s}\frac{(s+\kappa+k_1)(\theta_2(s+\kappa)+k_1)}{[(s+\kappa+k')s+\kappa k_2]^2} = \frac{1}{s}\frac{(s+\kappa+k_1)(\theta_2(s+\kappa)+k_1)}{(s+a+b)^2(s+a-b)^2}. \tag{B.4}$$

The parameters $a = (\kappa+k')/2$ and $b^2 = a^2 - \kappa k_2$ were introduced to factorize the denominators. These equations can be inverted using the properties 15.164 (translation), 15.11.1 (integration), and 15.123 (differentiation) from Arfken (1985) after expansion in partial fractions. For the mean, the calculation is

$$\frac{2b}{v}\langle x\rangle = \Big[\theta_2 + (k_1+\theta_2\kappa)\int_0^t dt\Big]\mathcal{L}^{-1}\Big\{\frac{1}{s+a-b} - \frac{1}{s+a+b};t\Big\} \tag{B.5}$$

$$= \Big[\theta_2 + \frac{k_1+\theta_2\kappa}{b-a}\Big]e^{(b-a)t} - \Big[\theta_2 - \frac{k_1+\theta_2\kappa}{a+b}\Big]e^{-(a+b)t} - \Big[\frac{k_1+\theta_2\kappa}{b-a} + \frac{k_1+\theta_2\kappa}{a+b}\Big]. \tag{B.6}$$

This equation rearranges to (13). The second moment (B.4) is

$$\frac{2b^2}{v^2}\langle x^2\rangle = \Big[\theta_2(\delta(t)+\partial_t) + (\theta_2(2\kappa+k_1)+k_1) + (\kappa+k_1)(\theta_2\kappa+k_1)\int_0^t dt\Big] \times \mathcal{L}^{-1}\Big\{\frac{1}{(s+a-b)^2} + \frac{1}{(s+a+b)^2} - \frac{1}{b(s+a-b)} + \frac{1}{b(s+a+b)};t\Big\}. \tag{B.7}$$

This becomes

$$\frac{2b^3}{v^2}\langle x^2\rangle = \Big[\theta_2\partial_t + [\theta_2(2\kappa+k_1)+k_1] + (\kappa+k_1)(\theta_2\kappa+k_1)\int_0^t dt\Big] \times\Big((bt-1)e^{(b-a)t} + (bt+1)e^{-(a+b)t}\Big), \tag{B.8}$$

which evaluates to equation (14). The variance in equation (15) follows from $\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2$.

C Limiting behavior of the moments

The easiest approach to extract the earlier results of Wu et al. (2019), Lisle et al. (1998), and H. A. Einstein (1937) for the moments and positional variance as limiting cases is to use the Laplace transformed moments (B.3) and (B.4) as a starting point. An alternate approach is to take limits in the moments (13) and (14) directly, but this is somewhat challenging. First we obtain the Wu et al. (2019) model as a limit case. These authors accounted for sediment burial considering motions to be instantaneous. They characterized sediment movement by a thin-tailed step length distribution. This implicitly involves an infinite motion velocity. To obtain their conclusions on bedload diffusion, we send the mean duration of motion to zero, the velocity to infinity ($1/k_2 \to 0$ and $v \to \infty$), and we hold the mean step distance $l = v/k_2$ constant. We use the initial condition that grains start at rest ($\theta_1 = 1$). Enacting these limits in equations (B.3) and (B.4) provides

$$\langle \tilde{x} \rangle = k_1 l \frac{1}{s(s+\kappa)}, \tag{C.1}$$

$$\langle \tilde{x}^2 \rangle = 2l^2 k_1 \frac{s+\kappa+k_1}{s(s+\kappa)^2}. \tag{C.2}$$

Inverting these equations and introducing the variables $c = lk_1$ (an effective velocity) and $D_d = l^2 k_1$ (a diffusivity) provides positional variance

$$\sigma_x^2(t) = \frac{2D_d(1-e^{-\kappa t})}{\kappa} + \frac{(1-e^{-2\kappa t} - 2e^{-\kappa t}\kappa t)c^2}{\kappa^2}. \tag{C.3}$$

This is mathematically identical to the key result of Wu et al. (2019), providing two ranges of diffusion when $\kappa \ll k_1$. One is normal and the other, induced by sediment burial, is superdiffusive. From here, the H. A. Einstein (1937) result can be obtained by turning off the burial process ($\kappa \to 0$):

$$\sigma_x^2(t) = 2D_d t. \tag{C.4}$$

This represents a single range of normal diffusion (cf. Hubbell & Sayre, 1964; Nakagawa & Tsujimoto, 1976).

The Lisle et al. (1998) result can be obtained similarly. For this case, we neglect the burial process ($\kappa \to 0$) but retain the finite velocity and motion duration in equations (B.3) and (B.4). For the initial condition $\theta_1 = 1$, this provides

$$\langle \tilde{x} \rangle = vk_1 \frac{1}{s^2(s+k')}, \tag{C.5}$$

$$\langle \tilde{x}^2 \rangle = 2v^2 k_1 \frac{s+k_1}{s^3(s+k')^2}. \tag{C.6}$$

Inverting these equations provides the variance when $t \ll 1/\kappa$:

$$\sigma_x^2 = 2v^2 \frac{k_1}{k'^4}\Big(k_1\big[\frac{1}{2} - k'te^{-k't} - \frac{1}{2}e^{-2k't}\big] + k_2\big[-2 + k't + (2+k't)e^{-k't}\big]\Big). \tag{C.7}$$

This result encodes two stages of diffusion: at small times there is superdiffusion $\sigma_x^2 \propto t^\gamma$ with $\gamma = 2-3$ depending on the initial condition, and at longer times there is normal diffusion. This links back to the Einstein model in the limit of instantaneous steps: $1/k_2 \to 0$ and $v \to \infty$ while $v/k_2 = l$.

Finally, we examine the limit $t \to 0$ while leaving $\kappa \neq 0$. This will highlight the effect of initial conditions on the local range superdiffusion. By applying Tauberian theorems, we can argue the $t \to 0$ variance is determined by the $s \to \infty$ limits of (B.3) and (B.4) (e.g. Weeks & Swinney, 1998; Weiss, 1994). Expanding these equations in powers of $1/s \ll 1$ and inverting the resulting transforms gives

$$\langle x \rangle = v\theta_2 t + \frac{1}{2}v(\theta_1 k_1 - \theta_2 k_2)t^2 + O(t^3), \tag{C.8}$$

$$\langle x^2 \rangle = v^2\theta_2 t^2 + \frac{1}{3}v^2(\theta_1 k_1 - 2\theta_2 k_2)t^3 + O(t^4). \tag{C.9}$$

Taking only leading order terms for each option of θ_1 and θ_2, these equations provide the asymptotic ($t \to 0$) variance

$$\sigma_x^2(t) \sim v^2\theta_1\theta_2 t^2 + \frac{1}{3}v^2(\theta_1 k_1 + \theta_2 k_2)t^3. \tag{C.10}$$

This equation shows the effect of initial conditions on the diffusion characteristics of the local range.

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