

# Back to Einstein: how to include trapping processes in fluvial diffusion models?

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## Key Points:

- We generalize the bedload diffusion model of Hans Albert Einstein to include periods of motion and sediment trapping processes.
- This derives three stages of bedload diffusion and unambiguously attributes physical mechanisms to each stage.

## Abstract

One approach to predict geomorphic change in rivers is to upscale from the motions of individual sediment grains. A difficulty with this approach is the wide range of individual motion characteristics that imply sediment diffusion, or the spreading out of grains through time. A more acute difficulty is that these motion characteristics apparently vary across temporal and spatial scales, implying multiple stages of sediment diffusion. We relate this multi-stage diffusion to within-channel trapping processes that impede the motions of individual grains. Grains can become stranded on high bars during floods, or they can become buried within the sedimentary bed, but these processes take a relatively long time to occur. Drawing on ideas from condensed matter physics and cellular biology, we describe the motions of individual grains within channels as random walkers subject to trapping. This derives multi-stage diffusion and clarifies its mechanism, providing a tool to link between scales.

## 1 Introduction

Models to predict the movement patterns of individual bedload sediment grains through rivers have been studied since at least H. A. Einstein (1937). Einstein originally used a random walk concept to describe these patterns statistically, and he applied his model to experimental data from a series of flume experiments. Individual movements are ultimately responsible for changes in the spatial organization of river channels (Hassan & Bradley, 2017), so predictive models of these patterns could benefit a vast set of environmental considerations, ranging from aquatic habitat restoration (Hauer et al., 2016) to the management of contaminated streams (Macklin et al., 2006) and artificial reservoirs (Schleiss, Franca, Juez, & De Cesare, 2016). Unfortunately, Einstein’s approach does not provide a definitive solution to this problem, although its essential concepts lie at the base of the majority of contemporary approaches. The essential issue is that Einstein’s original measurements on bedload transport characteristics are now known to be relevant only on a certain observation timescale (Nikora, Heald, Goring, & McEwan, 2001). Since H. A. Einstein (1937), details of bedload transport have come to light which are not accessible by his measurement techniques. There is a need for new models to account for these new details and predict the movements of individual grains.

When bedload grains transport downstream, each grain is imparted a unique sequence of forces by turbulent and steady components of the flow () and collisions with other moving and stationary grains (Gordon, Carmichael, & Isackson, 1972). As a result, grains spread apart from one another as they transport downstream: a phenomenon called diffusion. Diffusion is characterized by the variance of particle position  $\sigma_x^2(t)$  (Furbish, Fathel, Schmeckle, Jerolmack, & Schumer, 2017). An extreme diversity of transport phenomena show diffusion: it is not limited to the movement of grains within river channels, and it has been deeply studied in geology (Berkowitz, Cortis, Dentz, & Scher, 2006), chemistry Shugard1976, biology (Sokolov, 2012), hydrology (Yang & Wang, 2019), and many other contexts. The classic examples are pollen grains diffusing in response to molecular collisions while suspended in water, as first described mathematically by Hans Albert Einstein’s father (A. Einstein, 1905) and sand within a pipe flow (Taylor, 1920). In these cases, individual particles (pollen, sand, gravel) spread apart at a rate proportional to the time:  $\sigma_x^2 \propto t$ . Such diffusion is said to be normal. This name connotes the idea that this diffusion is somehow particularly characteristic of natural phenomena.

In contrast to this nomenclature, researchers of transport phenomena across the physical sciences have realized that normal diffusion is far from ubiquitous (Shlesinger, Zaslavsky, & Klafter, 1993). When diffusion is not normal, it is called anomalous, a term defined by a variance scaling  $\sigma_x^2 \propto t^\gamma$ , where  $\gamma \neq 1$ . Some examples of anomalous

diffusion include transport of nutrients through lipid bilayers (e.g. Jeon, Monne, Javanainen, & Metzler, 2012; Molina-Garcia et al., 2018), contaminants through ground-water aquifers (e.g. Aarão Reis & Di Caprio, 2014; Yang & Wang, 2019), charge carriers through solids (e.g. Scher & Lax, 1973), and, indeed, sediment grains through rivers (Bradley, 2017; Hassan & Bradley, 2017; Martin, Jerolmack, & Schumer, 2012; Phillips, Martin, & Jerolmack, 2013). In this work, we revisit the approach of Einstein to bear on the problem of multiple-stage diffusion. We draw upon methods developed in biology and condensed matter physics to form a description of anomalous and multi-stage diffusion which is directly linked to the original formulation of Einstein. There is a need to review Hassan and Bradley (2017); Martin et al. (2012); Phillips et al. (2013); Wu et al. (2019); Zhang, Meerschaert, and Packman (2012) and write something meaningful involving them, then rewrite this introduction because it isn't good. Anyway, here's the model.

## 2 Bedload diffusion with trapping

To include motion, rest, and trapping, and to keep track of grains which fall into the trapped state, we need to introduce a three-state random walk. The multistate random walk was first considered by Weiss (1976) and it is reviewed in Weiss (1994).

$$\omega_{1T}(x, t) = \theta_1 [1 - \Phi(t)] g_1(x, t) + \int_0^x dx' \int_0^t dt' \omega_2(x', t') [1 - \Phi(t)] g_1(x - x', t - t') \quad (1)$$

$$\omega_{1F}(x, t) = \theta_1 \Phi(t) g_1(x, t) + \int_0^x dx' \int_0^t dt' \omega_2(x', t') \Phi(t) g_1(x - x', t - t') \quad (2)$$

$$\omega_2(x, t) = \theta_2 g_2(x, t) + \int_0^x dx' \int_0^t dt' \omega_{1F}(x', t') g_2(x - x', t - t') \quad (3)$$

$$(4)$$

The probabilities are The probabilities of being in state 0 (trapped), 1 (rest), and 2 (motion) are

$$p_0(x, t) = \int_0^t dt' \omega_{1T}(x, t - t') \quad (5)$$

$$p_1(x, t) = \theta_1 G_1(x, t) + \int_0^x dx' \int_0^t dt' \omega_2(x', t') G_1(x - x', t - t') \quad (6)$$

$$p_2(x, t) = \theta_2 G_2(x, t) + \int_0^x dx' \int_0^t dt' \omega_{1F}(x', t') G_2(x - x', t - t'), \quad (7)$$

, while the total probability is  $p(x, t) = p_0(x, t) + p_1(x, t) + p_2(x, t)$ . When the propagators are chosen as Lisle et al. (1998), this has solution

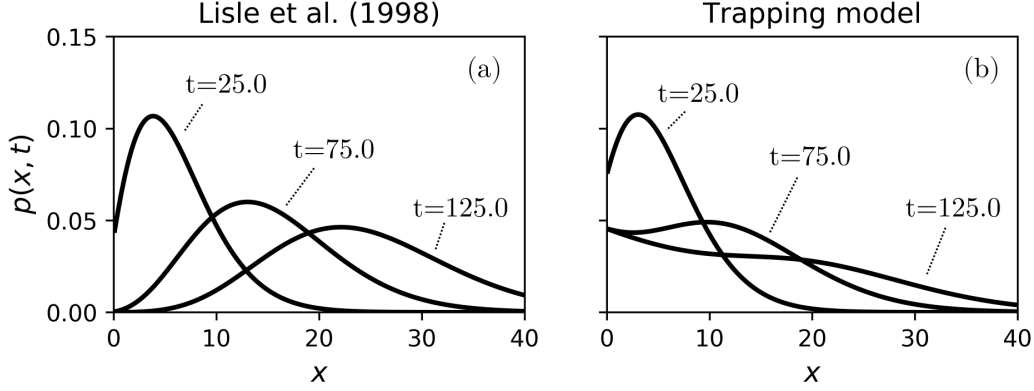
$$p(x, t) = e^{-(\kappa + k_1)(t - x/v) - k_2 x/v} \theta(t - x/v) \quad (8)$$

$$\times \left[ \frac{1}{v} \delta(t - x/v) + \frac{1}{v} \sqrt{\frac{k_1 k_2 x}{v(t - x/v)}} \mathcal{I}_1 \left( 2 \sqrt{\frac{k_1 k_2 x}{v}} \left( t - \frac{x}{v} \right) \right) + \frac{k_2}{v} \mathcal{I}_0 \left( 2 \sqrt{\frac{k_1 k_2 x}{v}} \left( t - \frac{x}{v} \right) \right) \right] \quad (9)$$

$$+ \kappa \frac{k_2}{v(\kappa + k_1)} \theta(t - x/v) e^{-k_2 x/v} \mathcal{P}_1 \left( \frac{k_1 k_2 x}{v(\kappa + k_1)}, [\kappa + k_1][t - x/v] \right) \quad (10)$$

as shown in appendix A. The  $I_\nu$  are modified Bessel functions of the first kind and  $\mathcal{P}_1$  is a Marcum-Q function (Marcum, 1960; Temme, 1996).  $\theta(x)$  is the Heaviside step function and we use the convention  $\theta(x = 0) = 1$ . This distribution is depicted in figure 1. The key tool in the analysis in the appendices is the double-Laplace transform of functions which depend on space and time:

$$\hat{f}(\eta, s) = \int_0^\infty dx \int_0^\infty dt e^{-st - \eta x} f(x, t). \quad (11)$$



**Figure 1.** Joint distributions of a tracer being found at  $x$   $t$  are impeded by the trapping process. Panel (a) shows the Lisle et al. (1998) model, and panel (b) shows the the distribution 10 which results from a trapping process occurring at rate  $\kappa$ . Cross-comparison of both panels shows that trapping redistributes probability to smaller values of  $x$ , and this redistribution becomes more important for larger  $t$ .

This unravels the convolution structure in the equations and provides an alternative approach to calculate the moments of  $x$  which avoids integrating (10). This approach is based on taking derivatives of  $\hat{p}(\eta, s)$  as described in appendix B. The first two moments are

$$\langle x(t) \rangle = A_1 e^{(b-a)t} + B_1 e^{-(a+b)t} + C_1 \quad (12)$$

$$\langle x^2(t) \rangle = A_2(t) e^{(b-a)t} + B_2(t) e^{-(a+b)t} + C_2, \quad (13)$$

where  $A_i$  and  $B_i$  are polynomials tabulated in (1). The operator  $\langle \circ \rangle$  denotes the ensemble average (e.g. Kittel, 1958). In terms of these moments, the variance  $\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2$  is

$$\sigma_x^2(t) = A(t) e^{(b-a)t} + B(t) e^{-(a+b)t} + C(t) \quad (14)$$

We have made no approximation to derive this expression, so it represents the phenomena of a Lisle et al. (1998)-type model generalized to include sediment trapping processes.

**Table 1.** Polynomials and transcendental functions used in the expressions of the mean (12), second moment (13) and variance (14) of bedload tracers.

$A_1 = \frac{v}{2b} \left[ 1 + \frac{\kappa + k_1}{b-a} \right]$ $B_1 = -\frac{v}{2b} \left[ 1 - \frac{\kappa + k_1}{a+b} \right]$ $C_1 = -\frac{v}{2b} \left[ \frac{\kappa + k_1}{b-a} + \frac{\kappa + k_1}{a+b} \right]$	
$A_2(t) = \frac{v^2}{2b^3} \left[ b + (b-a)[bt-1] + 2(\kappa + k_1)[bt-1] + \frac{(\kappa + k_1)^2}{(a-b)^2} [-abt + a + b(bt-2)] \right]$ $B_2(t) = \frac{v^2}{2b^3} \left[ b - (a+b)[bt-1] + 2(\kappa + k_1)[bt+1] - \frac{(\kappa + k_1)^2}{(a+b)^2} [bt(a+b) + a + 2b] \right]$	
$C_2(t) = \frac{v^2}{2b^3} (\kappa + k_1)^2 \left[ \frac{a+2b}{(a+b)^2} + \frac{-a+2b}{(a-b)^2} \right]$ $A(t) = A_2(t) - 2A_1C_1 + A_1^2 \exp[(b-a)t]$ $B(t) = B_2(t) - 2B_1C_1 + B_1^2 \exp[-(a+b)t]$ $C(t) = C_2 - C_1^2 + 2A_1B_1 \exp[-2at]$	

### 3 Discussion: new findings and foundational links

### 4 Conclusion

### A Calculation of the distribution function with trapping

Taking double transforms gives

$$\tilde{\omega}_{1T}(\eta, s) = \theta_1 \tilde{g}_1(\eta, s) + \tilde{\omega}_2(\eta, s) \tilde{g}_1(\eta, s) - \tilde{\omega}_{1F}(\eta, s) \quad (\text{A.1})$$

$$\tilde{\omega}_{1F}(\eta, s) = \theta_1 \tilde{g}_1(\eta, s + \kappa) + \tilde{\omega}_2(\eta, s) \tilde{g}_1(\eta, s + \kappa) \quad (\text{A.2})$$

$$\tilde{\omega}_2(\eta, s) = \theta_2 \tilde{g}_2(\eta, s) + \tilde{\omega}_{1F}(\eta, s) \tilde{g}_2(\eta, s) \quad (\text{A.3})$$

This system solves for

$$\tilde{\omega}_{1T}(\eta, s) = \frac{\theta_1 + \theta_2 \tilde{g}_2(\eta, s)}{1 - \tilde{g}_1(\eta, s + \kappa) \tilde{g}_2(\eta, s)} \{ \tilde{g}_1(\eta, s) - \tilde{g}_1(\eta, s + \kappa) \} \quad (\text{A.4})$$

$$\tilde{\omega}_{1F}(\eta, s) = \frac{\theta_1 + \theta_2 \tilde{g}_2(\eta, s)}{1 - \tilde{g}_1(\eta, s + \kappa) \tilde{g}_2(\eta, s)} \tilde{g}_1(\eta, s + \kappa) \quad (\text{A.5})$$

$$\tilde{\omega}_2(\eta, s) = \frac{\theta_2 + \theta_1 \tilde{g}_1(\eta, s + \kappa)}{1 - \tilde{g}_1(\eta, s + \kappa) \tilde{g}_2(\eta, s)} \tilde{g}_2(\eta, s) \quad (\text{A.6})$$

$$(\text{A.7})$$

The probabilities of being in state 0 (trapped), 1 (rest), and 2 (motion) are

$$p_0(x, t) = \int_0^t dt' \omega_{1T}(x, t - t') \quad (\text{A.8})$$

$$p_1(x, t) = \theta_1 G_1(x, t) + \int_0^x dx' \int_0^t dt' \omega_2(x', t') G_1(x - x', t - t') \quad (\text{A.9})$$

$$p_2(x, t) = \theta_2 G_2(x, t) + \int_0^x dx' \int_0^t dt' \omega_{1F}(x', t') G_2(x - x', t - t'), \quad (\text{A.10})$$

Double transforming gives

$$\tilde{p}_0(\eta, s) = \frac{1}{s} \tilde{\omega}_{1T}(\eta, s) \quad (\text{A.11})$$

$$\tilde{p}_1(\eta, s) = \theta_1 \tilde{G}_1(\eta, s) + \tilde{\omega}_2(\eta, s) \tilde{G}_1(\eta, s) \quad (\text{A.12})$$

$$\tilde{p}_2(\eta, s) = \theta_2 \tilde{G}_2(\eta, s) + \tilde{\omega}_{1F}(\eta, s) \tilde{G}_2(\eta, s) \quad (\text{A.13})$$

$$(\text{A.14})$$

The total probability is  $p(x, t) = p_0(x, t) + p_1(x, t) + p_2(x, t)$  or

$$\begin{aligned} \tilde{p}(\eta, s) = & \frac{1}{s} \frac{\theta_1 + \theta_2 \tilde{g}_2(\eta, s)}{1 - \tilde{g}_1(\eta, s + \kappa) \tilde{g}_2(\eta, s)} \{ \tilde{g}_1(\eta, s) - \tilde{g}_1(\eta, s + \kappa) \} \\ & + \frac{\theta_1 [\tilde{G}_1(\eta, s) + \tilde{g}_1(\eta, s + \kappa) \tilde{G}_2(\eta, s)] + \theta_2 [\tilde{G}_2(\eta, s) + \tilde{g}_2(\eta, s) \tilde{G}_1(\eta, s)]}{1 - \tilde{g}_1(\eta, s + \kappa) \tilde{g}_2(\eta, s)} \end{aligned} \quad (\text{A.15})$$

Using the identities  $\tilde{g}_i(0, s) = \tilde{\psi}_i(s)$  and  $\tilde{G}_i(0, s) = (1 - \tilde{\psi}_i(s))/s$ , it follows that the joint distribution is normalized in space:  $\tilde{p}(0, s) = \mathcal{L}\{\int_0^\infty dx p(x, t); s\} = 1/s$ .

After a lot of work which is in your notebook, this becomes

$$p(x, t) = e^{-(\kappa + k_1)(t - x/v) - k_2 x/v} \left[ \frac{1}{v} \mathcal{L}_{s \rightarrow t - x/v}^{-1} \left\{ \exp \left[ \frac{k_1 k_2}{vs} x \right] \right\} \right. \quad (\text{A.16})$$

$$\left. + \frac{k_2}{v} \mathcal{L}_{s \rightarrow t - x/v}^{-1} \left\{ \frac{1}{s} \exp \left[ \frac{k_1 k_2}{vs} x \right] \right\} \right. \quad (\text{A.17})$$

$$\left. + \frac{\kappa k_2}{v} \mathcal{L}_{s \rightarrow t - x/v}^{-1} \left\{ \frac{1}{(s - \kappa - k_1)s} \exp \left[ \frac{k_1 k_2}{vs} x \right] \right\} \right] \quad (\text{A.18})$$

$$p(x, t) = e^{-(\kappa+k_1)(t-x/v)-k_2x/v} \left[ \frac{1}{v} \delta(t-x/v) + \frac{1}{v} \sqrt{\frac{k_1 k_2 x}{v(t-x/v)}} \theta(t-x/v) \mathcal{I}_1 \left( 2\sqrt{\frac{k_1 k_2 x}{v} \left( t - \frac{x}{v} \right)} \right) \right. \quad (\text{A.19})$$

$$\left. + \frac{k_2}{v} \theta(t-x/v) \mathcal{I}_0 \left( 2\sqrt{\frac{k_1 k_2 x}{v} \left( t - \frac{x}{v} \right)} \right) \right. \quad (\text{A.20})$$

$$\left. + \kappa \frac{k_2}{v(\kappa+k_1)} e^{(\kappa+k_1)(t-x/v)} \theta(t-x/v) \mathcal{F}(x, t) \right] \quad (\text{A.21})$$

where the function  $\mathcal{F}$  is

$$\mathcal{F}(x, t) = \sum_{j=0}^{\infty} \frac{\left[ \frac{k_1 k_2 x}{v(\kappa+k_1)} \right]^j}{j! j!} \gamma(j+1, [\kappa+k_1][t-x/v]), \quad (\text{A.22})$$

where the  $\mathcal{I}_\nu(z)$  are modified Bessel functions of the first kind, and  $\gamma(\alpha, z)$  is the lower incomplete gamma function. This function  $\mathcal{F}(x, t)$  is the Marcum Q-function (Temme, 1996), given by

$$\mathcal{Q}_1(x, y) = 1 - e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} \frac{\gamma(1+n, y)}{\Gamma(n+1)} \quad (\text{A.23})$$

This function was originally introduced in relation to radar detection problems (e.g. Marcum, 1960). It has a representation as an infinite superposition of modified Bessel functions:

$$\mathcal{Q}_1(x, y) = 1 - \int_0^y dz e^{-z-x} \mathcal{I}_0(2\sqrt{xz}) \quad (\text{A.24})$$

So we are not far from where we started with Lisle et al. (1998): Modified Bessel functions are the norm in this type of 1D diffusion problem (e.g. Lisle et al., 1998). Hence in summary

$$\mathcal{F}(x, t) = 1 - \mathcal{Q}_1 \left( \frac{k_1 k_2 x}{v(\kappa+k_1)}, [\kappa+k_1][t-x/v] \right) \quad (\text{A.25})$$

## B Calculation of the moments

Often, Tauberian-type theorems are used to study the long or short time asymptotic scaling of first or second moments of random walks having any generality. This is because the mathematics get difficult, and Tauberian theorems provide a very powerful tool which leverages the stability of the Laplace transform. Unfortunately, this approach is insufficient for our purposes. We are unaware of an intermediate-regime Tauberian-type theorem, so we must pursue full solutions of the moments in order to discriminate the diffusive ranges and the full diffusion behavior.

$$\partial_\eta \tilde{p}(\eta, s) = -v \frac{1}{s} \frac{[(s+\kappa+k')s + \kappa k_2][s + \kappa + k_1]}{[\eta v(s+\kappa+k_1) + (s+\kappa+k')s + \kappa k_2]^2} \quad (\text{B.1})$$

$$\partial_\eta^2 \tilde{p}(\eta, s) = 2v^2 \frac{1}{s} \frac{[(s+\kappa+k')s + \kappa k_2][s + \kappa + k_1]^2}{[\eta v(s+\kappa+k_1) + (s+\kappa+k')s + \kappa k_2]^3} \quad (\text{B.2})$$

$$\frac{\langle \tilde{x} \rangle}{v} = \frac{1}{s} \frac{s + \kappa + k_1}{(s + \kappa + k')s + \kappa k_2} \quad (\text{B.3})$$

$$\frac{\langle \tilde{x}^2 \rangle}{2v^2} = \frac{1}{s} \frac{[s + \kappa + k_1]^2}{[(s + \kappa + k')s + \kappa k_2]^2} \quad (\text{B.4})$$

A similar approach provides

$$\frac{\langle x^2 \rangle}{2v^2} = \left( \frac{d}{dt} \circ + \circ \Big|_{t=0} + 2(\kappa + k_1) \circ + (\kappa + k_1)^2 \int_0^t dt \circ \right) \mathcal{L}^{-1} \left\{ \frac{1}{[(s+a)^2 - b^2]^2}; t \right\} \quad (\text{B.5})$$

$$= \left( \frac{d}{dt} \circ + \circ \Big|_{t=0} + 2(\kappa + k_1) \circ + (\kappa + k_1)^2 \int_0^t dt \circ \right) \mathcal{L}^{-1} \left\{ \frac{1}{[s^2 - b^2]^2}; t \right\} e^{-at} \quad (\text{B.6})$$

$$= \left( \frac{d}{dt} \circ + \circ \Big|_{t=0} + 2(\kappa + k_1) \circ + (\kappa + k_1)^2 \int_0^t dt \circ \right) e^{-at} \frac{1}{2b^3} [bt \cosh(bt) - \sinh(bt)] \quad (\text{B.7})$$

using Prudnikov 2.1.5.6. This becomes

$$\frac{\langle x^2 \rangle}{2v^2} = \frac{t}{b} \sinh(bt) + \frac{(\kappa + k_1)}{b^3} [bt \cosh(bt) - \sinh(bt)] \quad (\text{B.8})$$

$$+ e^{-at} \frac{b(b^2(at - 2)) - a^3t \cosh(bt) + (a^3 - a^2b^2t - 3ab^2 + b^4t) \sinh(bt)}{(a - b)^2(a + b)^2} \quad (\text{B.9})$$

$$\frac{\langle x \rangle}{v} = \mathcal{L}^{-1} \left\{ \frac{1}{s} \frac{s + \kappa + k_1}{(s + \kappa + k')s + \kappa k_2}; t \right\} \quad (\text{B.10})$$

$$= \mathcal{L}^{-1} \left\{ \frac{1}{[s + \frac{\kappa + k'}{2}]^2 + \kappa k_2 - [\frac{\kappa + k'}{2}]^2}; t \right\} + \int_0^t du \mathcal{L}^{-1} \left\{ \frac{\kappa + k_1}{[s + \frac{\kappa + k'}{2}]^2 + \kappa k_2 - [\frac{\kappa + k'}{2}]^2}; u \right\} \quad (\text{B.11})$$

$$= e^{-(\kappa + k')t/2} \mathcal{L}^{-1} \left\{ \frac{1}{s^2 - b^2}; t \right\} + (\kappa + k_1) \int_0^t du e^{-(\kappa + k')u/2} \mathcal{L}^{-1} \left\{ \frac{1}{s^2 - b^2}; u \right\} \quad (\text{B.12})$$

here  $b^2 = -\kappa k_2 + [\frac{\kappa + k'}{2}]^2$ . Then with Prudnikov 2.1.5.4:

$$\frac{\langle x \rangle}{v} = \frac{1}{b} e^{-(\kappa + k')t/2} \sinh(bt) + \frac{(\kappa + k_1)}{b} \int_0^t du e^{-(\kappa + k')u/2} \sinh(bu) \quad (\text{B.13})$$

$$= \frac{1}{b} e^{-at} \sinh(bt) + \frac{\kappa + k_1}{2b} \left[ \frac{1}{b - a} (e^{(b-a)t} - 1) + \frac{1}{a + b} (e^{-(a+b)t} - 1) \right] \quad (\text{B.14})$$

where  $a = (\kappa + k')/2$  and  $b = \sqrt{a^2 - \kappa k_2}$ . We are interested in the domain that  $a \geq \sqrt{\kappa k_2}$  so that  $b \geq a$ . The limit of  $\kappa \rightarrow 0$  provides

$$\frac{k'^2}{v} \langle x \rangle = k_2(1 - e^{-k't}) + k_1 k' t \quad (\text{B.15})$$

which aligns perfectly with earlier results.

## C Links to H. A. Einstein (1937) and Lisle et al. (1998)

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