Back to Einstein: how to include trapping processes in fluvial diffusion models?

James K. Pierce [1] **and Marwan A. Hassan**[1]

[1]Department of Geography
University of British Columbia

Key Points:

- We generalize the bedload diffusion model of Hans Albert Einstein to include the duration of sediment motion and the possibility of trapping.
- This derives three stages of bedload diffusion and unambiguously attributes physical mechanisms to each stage.

Corresponding author: James Kevin Pierce, `kpierce@alumni.ubc.ca`

Abstract

One approach to predict geomorphic change in rivers is to upscale from the motions of individual sediment grains. A difficulty with this approach is the wide range of individual motion characteristics that imply sediment diffusion, or the spreading out of grains through time. A more acute difficulty is that these motion characteristics apparently vary across temporal and spatial scales, implying multiple stages of sediment diffusion. We relate this multi-stage diffusion to within-channel trapping processes that impede the motions of individual grains. Grains can become stranded on high bars during floods, or they can become buried within the sedimentary bed, but these processes take a relatively long time to occur. Drawing on ideas from condensed matter physics and cellular biology, we describe the finite-velocity motions of individual grains within channels as random walkers subject to trapping. This derives three stages of bed-load diffusion and clarifies the underlying mechanism, providing a tool to link between scales.

1 Introduction

Models to predict the movement patterns of individual bedload sediment grains through rivers have been studied since at least H. A. Einstein (1937). Einstein originally used a random walk concept to describe these patterns statistically, and he applied his model to experimental data from a series of flume experiments. Individual movements are ultimately responsible for changes in the spatial organization of river channels (Hassan & Bradley, 2017), so predictive models of these patterns could benefit a vast set of environmental considerations, ranging from aquatic habitat restoration (Hauer et al., 2016) to the management of contaminated streams (Macklin et al., 2006) and artificial reservoirs (Schleiss, Franca, Juez, & De Cesare, 2016). Unfortunately, Einstein's approach does not provide a definitive solution to this problem, although its essential concepts lie at the base of the majority of contemporary approaches. The essential issue is that Einstein's original measurements on bedload transport characteristics are now known to be relevant only on a certain observation timescale (Nikora, Heald, Goring, & McEwan, 2001). Since H. A. Einstein (1937), details of bedload transport have come to light which are not accessible by his measurement techniques. There is a need for new models to account for these new details and predict the movements of individual grains.

When bedload grains transport downstream, each grain is imparted a unique sequence of forces by turbulent and steady components of the flow () and collisions with other moving and stationary grains (Gordon, Carmichael, & Isackson, 1972). As a result, grains spread apart from one another as they transport downstream: a phenomenon called diffusion. Diffusion is characterized by the variance of particle position $\sigma_x^2(t)$ (Furbish, Fathel, Schmeeckle, Jerolmack, & Schumer, 2017). An extreme diversity of transport phenomena show diffusion: it is not limited to the movement of grains within river channels, and it has been deeply studied in geology (Berkowitz, Cortis, Dentz, & Scher, 2006), chemistry (Shugard & Reiss, 1976), biology (Sokolov, 2012), hydrology (Yang & Wang, 2019), and many other contexts. The classic examples are pollen grains diffusing in response to molecular collisions while suspended in water, as first described mathematically by Hans Albert Einstein's father (A. Einstein, 1905) and sand within a pipe flow (Taylor, 1920). In these cases, individual particles (pollen, sand, gravel) spread apart at a rate proportional to the time: $\sigma_x^2 \propto t$. Such diffusion is said to be normal. This name connotes the idea that this diffusion is somehow particularly characteristic of natural phenomena.

In contrast to this nomenclature, researchers of transport phenomena across the physical sciences have realized that normal diffusion is far from ubiquitous (Shlesinger, Zaslavsky, & Klafter, 1993). When diffusion is not normal, it is called anomalous, a

term defined by a variance scaling $\sigma_x^2 \propto t^\gamma$, where $\gamma \neq 1$. Some examples of anomalous diffusion include transport of nutrients through lipid bilayers (e.g. Jeon, Monne, Javanainen, & Metzler, 2012; Molina-Garcia et al., 2018), contaminants through groundwater aquifers (e.g. Aarão Reis & Di Caprio, 2014; Yang & Wang, 2019), charge carriers through solids (e.g. Scher & Lax, 1973), and, indeed, sediment grains through rivers (Bradley, 2017; Hassan & Bradley, 2017; Martin, Jerolmack, & Schumer, 2012; Phillips, Martin, & Jerolmack, 2013). In this work, we revisit the approach of Einstein to bear on the problem of multiple-stage diffusion. We draw upon methods developed in biology and condensed matter physics to form a description of anomalous and multi-stage diffusion which is directly linked to the original formulation of Einstein. There is a need to review Hassan and Bradley (2017); Martin et al. (2012); Phillips et al. (2013); Wu et al. (2019); Zhang, Meerschaert, and Packman (2012) and write something meaningful involving them, then rewrite this introduction because it isn't good. Anyway, here's the model.

2 Bedload diffusion with trapping

To include motion, rest, and burial into a model of bedload diffusion, we leverage the mathematical formalism of multi-state random walks. The multi-state random walk was first considered by Weiss (1976) and is reviewed in Weiss (1994). Our development closely follows these works and involves two essential stages. First, we calculate the joint probabilities that a tracer's residence or sojourn within each state ends at time t having position x. Second, we calculate the joint probabilities that a tracer in each state has position x at time t regardless of whether or not a sojourn has just completed. These two sets of objects completely define the stochastic dynamics of tracers.

Tracers at rest may be trapped by burial. We consider burial to be permanent (e.g. Wu et al., 2019), and we characterize it probabilistically. Assuming that resting grains may be buried with constant probability per unit time κ, the probability that a resting grain is not trapped after a time t is given by a survival probability $\Phi(t) = e^{-\kappa t}$. Likewise, the probability that it is trapped after resting for a time t is the complement $1 - \Phi(t)$. Associating indices $i = 0, 1, 2$ with the trapping, resting, and moving states, we introduce $\omega_{1T}(x,t)$, $\omega_{1F}(x,t)$, and $\omega_2(x,t)$ as the joint probabilities to find a grain having just completed a sojourn at x, t. The subscript $1T$ denotes the completion of a rest sojourn due to trapping, while $1F$ denotes the completion of a rest sojourn due to movement. Similarly, the subscript 2 denotes the completion of a movement sojourn due to the onset of resting. We consider that only resting grains can be buried. Using an argument similar to Weiss (1994), we can write integral equations to link the ω's.

$$\omega_{1T}(x,t) = \theta_1\big[1 - \Phi(t)\big]g_1(x,t) + \int_0^x dx' \int_0^t dt'\omega_2(x',t')\big[1 - \Phi(t)\big]g_1(x - x', t - t') \quad (1)$$

$$\omega_{1F}(x,t) = \theta_1\Phi(t)g_1(x,t) + \int_0^x dx' \int_0^t dt'\omega_2(x',t')\Phi(t)g_1(x - x', t - t') \quad (2)$$

$$\omega_2(x,t) = \theta_2 g_2(x,t) + \int_0^x dx' \int_0^t dt'\omega_{1F}(x',t')g_2(x - x', t - t') \quad (3)$$

$$(4)$$

The factors $g_i(\xi, \tau)$ are called propagators in the state i. They encode the transfer of probability across a distance ξ and a time τ while the walker resides in the ith state. Specifying the propagators and solving these equations for the ω completes the first stage of the derivation.

The propagators are defined by the behavior of particles in each state. We consider that sojourns through motion and rest states last for exponentially distributed times. Since resting particles do not move, while moving particles translate with a
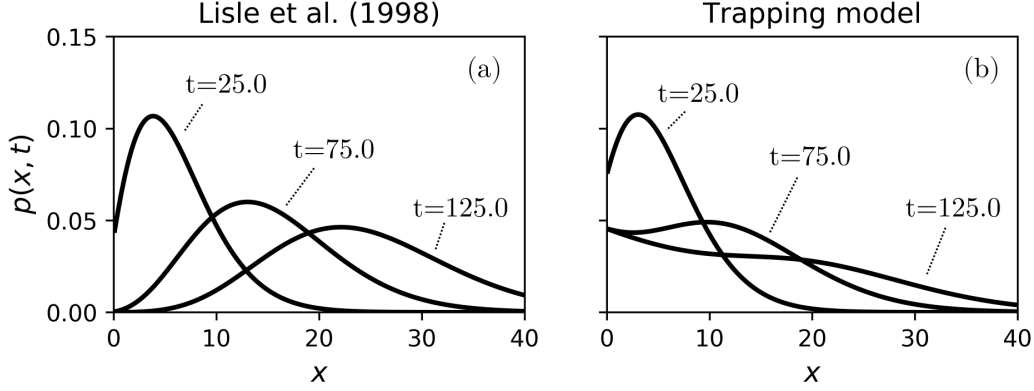
Figure 1. Joint distributions of a tracer being found at x t are impeded by the trapping process. Panel (a) shows the Lisle et al. (1998) model, and panel (b) shows the the distribution 12 which results from a trapping process occurring at rate κ. Cross-comparison of both panels shows that trapping redistributes probability to smaller values of x, and this redistribution becomes more important for larger t.

constant velocity v, we reason the propagators are

$$g_1(x,t) = \delta(x)k_1 e^{-k_1 t} \tag{5}$$

$$g_2(x,t) = \delta(x - vt)k_2 e^{-k_2 t}. \tag{6}$$

In this notation, $1/k_1$ is the mean resting time, while $1/k_2$ is the mean movement time.

The second stage of our development involves the joint probabilities of being in state regardless of whether a sojourn has just completed. These are denoted by p_0 (trapped), p_1 (rest), and p_2 (motion) and involve the ωs for their definition.

$$p_0(x,t) = \int_0^t dt' \omega_{1T}(x, t - t') \tag{7}$$

$$p_1(x,t) = \theta_1 G_1(x,t) + \int_0^x dx' \int_0^t dt' \omega_2(x',t') G_1(x - x', t - t') \tag{8}$$

$$p_2(x,t) = \theta_2 G_2(x,t) + \int_0^x dx' \int_0^t dt' \omega_{1F}(x',t') G_2(x - x', t - t'), \tag{9}$$

, while the total probability is $p(x,t) = p_0(x,t) + p_1(x,t) + p_2(x,t)$. When the propagators are chosen as Lisle et al. (1998), this has solution

$$p(x,t) = e^{-(\kappa + k_1)(t - x/v) - k_2 x/v} \theta(t - x/v) \tag{10}$$

$$\times \left[\frac{1}{v}\delta(t - x/v) + \frac{1}{v}\sqrt{\frac{k_1 k_2 x}{v(t - x/v)}} \mathcal{I}_1\left(2\sqrt{\frac{k_1 k_2 x}{v}\left(t - \frac{x}{v}\right)} \right) + \frac{k_2}{v}\mathcal{I}_0\left(2\sqrt{\frac{k_1 k_2 x}{v}\left(t - \frac{x}{v}\right)} \right) \right] \tag{11}$$

$$+ \kappa \frac{k_2}{v(\kappa + k_1)} \theta(t - x/v) e^{-k_2 x/v} \mathcal{P}_1\left(\frac{k_1 k_2 x}{v(\kappa + k_1)}, [\kappa + k_1][t - x/v] \right) \tag{12}$$

as shown in appendix A. The I_ν are modified Bessel functions of the first kind and $\mathcal{P}_1$ is a Marcum-Q function (Marcum, 1960; Temme, 1996). $\theta(x)$ is the Heaviside step function and we use the convention $\theta(x = 0) = 1$. This distribution is depicted in figure 1. The key tool in the analysis in the appendices is the double-Laplace transform of

functions which depend on space and time:

$$\hat{f}(\eta, s) = \int_0^\infty dx \int_0^\infty ds e^{-st-\eta x} f(x,t).$$

(13)

This unravels the convolution structure in the equations and provides an alternative approach to calculate the moments of x which avoids integrating (12). This approach is based on taking derivatives of $\hat{p}(\eta, s)$ as described in appendix B. The first two moments are

$$\langle x(t) \rangle = A_1 e^{(b-a)t} + B_1 e^{-(a+b)t} + C_1$$

(14)

$$\langle x^2(t) \rangle = A_2(t) e^{(b-a)t} + B_2(t) e^{-(a+b)t} + C_2,$$

(15)

where A_i and B_i are polynomials tabulated in (1). The operator $\langle \circ \rangle$ denotes the ensemble average (e.g. Kittel, 1958). In terms of these moments, the variance $\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2$ is

$$\sigma_x^2(t) = A(t) e^{(b-a)t} + B(t) e^{-(a+b)t} + C(t)$$

(16)

We have made no approximation to derive this expression, so it represents the phenomena of a Lisle et al. (1998)-type model generalized to include sediment trapping processes.

Table 1. Polynomials and transcendental functions used in the expressions of the mean (14), second moment (15) and variance (16) of bedload tracers.

$$A_1 = \frac{v}{2b}\left[1 + \frac{\kappa+k_1}{b-a}\right]$$
$$B_1 = -\frac{v}{2b}\left[1 - \frac{\kappa+k_1}{a+b}\right]$$
$$C_1 = -\frac{v}{2b}\left[\frac{\kappa+k1}{b-a} + \frac{\kappa+k1}{a+b}\right]$$
$$A_2(t) = \frac{v^2}{2b^3}\left[b + (b-a)[bt-1] + 2(\kappa+k_1)[bt-1] + \frac{(\kappa+k_1)^2}{(a-b)^2}[-abt+a+b(bt-2)]\right]$$
$$B_2(t) = \frac{v^2}{2b^3}\left[b - (a+b)[bt+1] + 2(\kappa+k_1)[bt+1] - \frac{(\kappa+k_1)^2}{(a+b)^2}[bt(a+b)+a+2b]\right]$$
$$C_2(t) = \frac{v^2}{2b^3}(\kappa+k_1)^2\left[\frac{a+2b}{(a+b)^2} + \frac{-a+2b}{(a-b)^2}\right]$$
$$A(t) = A_2(t) - 2A_1 C_1 + A_1^2 \exp[(b-a)t]$$
$$B(t) = B_2(t) - 2B_1 C_1 + B_1^2 \exp[-(a+b)t]$$
$$C(t) = C_2 - C_1^2 + 2A_1 B_1 \exp[-2at]$$

3 Discussion: new findings and foundational links

A plot of $\sigma_x^2(t)$ is shown in figure 2.

4 Conclusion

A Calculation of the distribution function with trapping

Taking double transforms gives

$$\tilde{\omega}_{1T}(\eta, s) = \theta_1 \tilde{g}_1(\eta, s) + \tilde{\omega}_2(\eta, s)\tilde{g}_1(\eta, s) - \tilde{\omega}_{1F}(\eta, s)$$

(A.1)

$$\tilde{\omega}_{1F}(\eta, s) = \theta_1 \tilde{g}_1(\eta, s+\kappa) + \tilde{\omega}_2(\eta, s)\tilde{g}_1(\eta, s+\kappa)$$

(A.2)

$$\tilde{\omega}_2(\eta, s) = \theta_2 \tilde{g}_2(\eta, s) + \tilde{\omega}_{1F}(\eta, s)\tilde{g}_2(\eta, s)$$
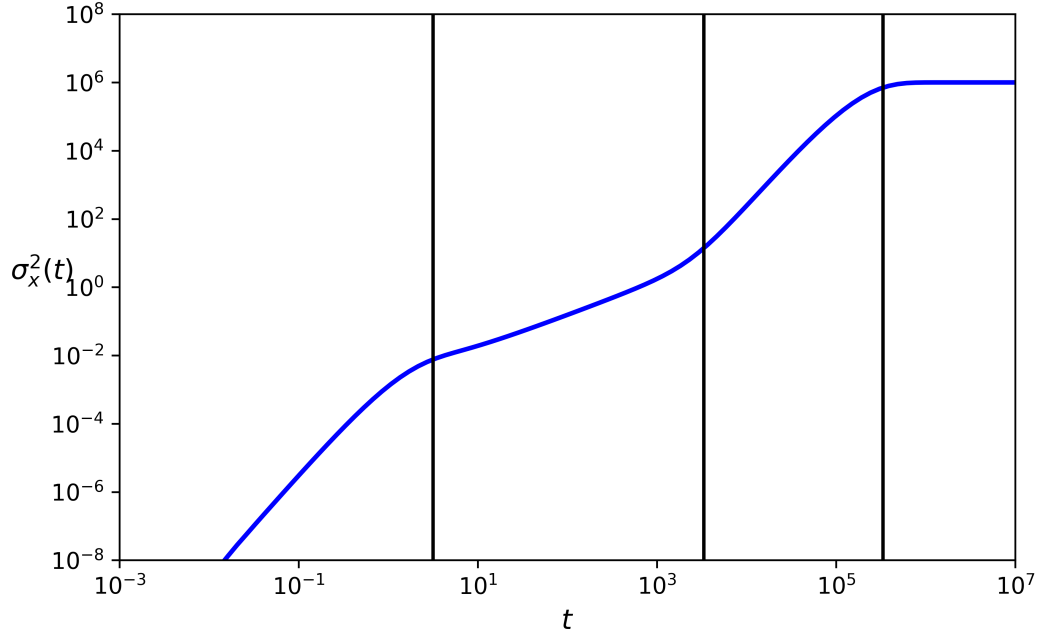
(A.3)

Figure 2. The variance of tracer position exhibits four distinct scaling regions as time increases: local, intermediate, global, and finally a non-diffusive range characterized by the eventual trapping of all grains. Local and global ranges show super-diffusion $\sigma_x^2 \propto t^3$ while the intermediate range shows normal diffusion. These conclusions hold only for $\kappa \ll k_1 \ll k_2$, meaning motion intervals are generally much shorter than resting intervals, which are in turn are generally much smaller than the time required for resting particles to become buried.

This system solves for

$$\tilde{\omega}_{1T}(\eta, s) = \frac{\theta_1 + \theta_2 \tilde{g}_2(\eta, s)}{1 - \tilde{g}_1(\eta, s + \kappa)\tilde{g}_2(\eta, s)} \{\tilde{g}_1(\eta, s) - \tilde{g}_1(\eta, s + \kappa)\} \tag{A.4}$$

$$\tilde{\omega}_{1F}(\eta, s) = \frac{\theta_1 + \theta_2 \tilde{g}_2(\eta, s)}{1 - \tilde{g}_1(\eta, s + \kappa)\tilde{g}_2(\eta, s)} \tilde{g}_1(\eta, s + \kappa) \tag{A.5}$$

$$\tilde{\omega}_2(\eta, s) = \frac{\theta_2 + \theta_1 \tilde{g}_1(\eta, s + \kappa)}{1 - \tilde{g}_1(\eta, s + \kappa)\tilde{g}_2(\eta, s)} \tilde{g}_2(\eta, s) \tag{A.6}$$

$$\tag{A.7}$$

The probabilities of being in state 0 (trapped), 1 (rest), and 2 (motion) are

$$p_0(x, t) = \int_0^t dt' \omega_{1T}(x, t - t') \tag{A.8}$$

$$p_1(x, t) = \theta_1 G_1(x, t) + \int_0^x dx' \int_0^t dt' \omega_2(x', t') G_1(x - x', t - t') \tag{A.9}$$

$$p_2(x, t) = \theta_2 G_2(x, t) + \int_0^x dx' \int_0^t dt' \omega_{1F}(x', t') G_2(x - x', t - t'), \tag{A.10}$$

Double transforming gives

$$\tilde{p}_0(\eta, s) = \frac{1}{s} \tilde{\omega}_{1T}(\eta, s) \tag{A.11}$$

$$\tilde{p}_1(\eta, s) = \theta_1 \tilde{G}_1(\eta, s) + \tilde{\omega}_2(\eta, s)\tilde{G}_1(\eta, s) \tag{A.12}$$

$$\tilde{p}_2(\eta, s) = \theta_2 \tilde{G}_2(\eta, s) + \tilde{\omega}_{1F}(\eta, s)\tilde{G}_2(\eta, s) \tag{A.13}$$

$$\tag{A.14}$$

The total probability is $p(x, t) = p_0(x, t) + p_1(x, t) + p_2(x, t)$ or

$$\tilde{p}(\eta, s) = \frac{1}{s} \frac{\theta_1 + \theta_2 \tilde{g}_2(\eta, s)}{1 - \tilde{g}_1(\eta, s + \kappa)\tilde{g}_2(\eta, s)} \{\tilde{g}_1(\eta, s) - \tilde{g}_1(\eta, s + \kappa)\}$$
$$+ \frac{\theta_1 [\tilde{G}_1(\eta, s) + \tilde{g}_1(\eta, s + \kappa)\tilde{G}_2(\eta, s)] + \theta_2 [\tilde{G}_2(\eta, s) + \tilde{g}_2(\eta, s)\tilde{G}_1(\eta, s)]}{1 - \tilde{g}_1(\eta, s + \kappa)\tilde{g}_2(\eta, s)} \tag{A.15}$$

Using the identities $\tilde{g}_i(0, s) = \tilde{\psi}_i(s)$ and $\tilde{G}_i(0, s) = (1 - \tilde{\psi}_i(s))/s$, it follows that the joint distribution is normalized in space: $\tilde{p}(0, s) = \mathcal{L}\{\int_0^\infty dx p(x, t); s\} = 1/s$.

After a lot of work which is in your notebook, this becomes

$$p(x, t) = e^{-(\kappa + k_1)(t - x/v) - k_2 x/v} \left[\frac{1}{v} \mathcal{L}_{s \to t - x/v}^{-1} \left\{ \exp\left[\frac{k_1 k_2}{vs} x \right] \right\} \right. \tag{A.16}$$

$$+ \frac{k_2}{v} \mathcal{L}_{s \to t - x/v}^{-1} \left\{ \frac{1}{s} \exp\left[\frac{k_1 k_2}{vs} x \right] \right\} \tag{A.17}$$

$$\left. + \frac{\kappa k_2}{v} \mathcal{L}_{s \to t - x/v}^{-1} \left\{ \frac{1}{(s - \kappa - k_1)s} \exp\left[\frac{k_1 k_2}{vs} x \right] \right\} \right] \tag{A.18}$$

$$p(x, t) = e^{-(\kappa + k_1)(t - x/v) - k_2 x/v} \left[\frac{1}{v} \delta(t - x/v) + \frac{1}{v} \sqrt{\frac{k_1 k_2 x}{v(t - x/v)}} \theta(t - x/v)\mathcal{I}_1\left(2\sqrt{\frac{k_1 k_2 x}{v}\left(t - \frac{x}{v} \right)} \right) \right. \tag{A.19}$$

$$+ \frac{k_2}{v} \theta(t - x/v)\mathcal{I}_0\left(2\sqrt{\frac{k_1 k_2 x}{v}\left(t - \frac{x}{v} \right)} \right) \tag{A.20}$$

$$\left. + \kappa \frac{k_2}{v(\kappa + k_1)} e^{(\kappa + k_1)(t - x/v)} \theta(t - x/v)\mathcal{F}(x, t) \right] \tag{A.21}$$

where the function $\mathcal{F}$ is

$$\mathcal{F}(x,t) = \sum_{j=0}^{\infty} \frac{\left[\frac{k_1 k_2 x}{v(\kappa+k_1)}\right]^j}{j!j!} \gamma\big(j+1, [\kappa+k_1][t-x/v]\big), \tag{A.22}$$

where the $\mathcal{I}_\nu(z)$ are modified Bessel functions of the first kind, and $\gamma(\alpha, z)$ is the lower incomplete gamma function. This function $\mathcal{F}(x,t)$ is the Marcum Q-function (Temme, 1996), given by

$$\mathcal{Q}_1(x,y) = 1 - e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} \frac{\gamma(1+n, y)}{\Gamma(n+1)} \tag{A.23}$$

This function was originally introduced in relation to radar detection problems (e.g. Marcum, 1960). It has a representation as an infinite superposition of modified Bessel functions:

$$\mathcal{Q}_1(x,y) = 1 - \int_0^y dz\, e^{-z-x} \mathcal{I}_0(2\sqrt{xz}) \tag{A.24}$$

So we are not far from where we started with Lisle et al. (1998): Modified Bessel functions are the norm in this type of 1D diffusion problem (e.g. Lisle et al., 1998). Hence in summary

$$\mathcal{F}(x,t) = 1 - \mathcal{Q}_1\left(\frac{k_1 k_2 x}{v(\kappa+k_1)}, [\kappa+k_1][t-x/v]\right) \tag{A.25}$$

B Calculation of the moments

Often, Tauberian-type theorems are used to study the long or short time asymptotic scaling of first or second moments of random walks having any generality. This is because the mathematics get difficult, and Tauberian theorems provide a very powerful tool which leverages the stability of the Laplace transform. Unfortunately, this approach is insufficient for our purposes. We are unaware of an intermediate-regime Tauberian-type theorem, so we must pursue full solutions of the moments in order to discriminate the diffusive ranges and the full diffusion behavior.

$$\partial_\eta \tilde{p}(\eta, s) = -v \frac{1}{s} \frac{[(s+\kappa+k')s + \kappa k_2][s+\kappa+k_1]}{[\eta v(s+\kappa+k_1) + (s+\kappa+k')s + \kappa k_2]^2} \tag{B.1}$$

$$\partial_\eta^2 \tilde{p}(\eta, s) = 2v^2 \frac{1}{s} \frac{[(s+\kappa+k')s + \kappa k_2][s+\kappa+k_1]^2}{[\eta v(s+\kappa+k_1) + (s+\kappa+k')s + \kappa k_2]^3} \tag{B.2}$$

$$\frac{\langle \tilde{x} \rangle}{v} = \frac{1}{s} \frac{s+\kappa+k_1}{(s+\kappa+k')s + \kappa k_2} \tag{B.3}$$

$$\frac{\langle \tilde{x}^2 \rangle}{2v^2} = \frac{1}{s} \frac{[s+\kappa+k_1]^2}{[(s+\kappa+k')s + \kappa k_2]^2} \tag{B.4}$$

A similar approach provides

$$\frac{\langle x^2 \rangle}{2v^2} = \left(\frac{d}{dt} \circ + \circ \Big|_{t=0} + 2(\kappa+k_1) \circ + (\kappa+k_1)^2 \int_0^t dt \circ \right) \mathcal{L}^{-1}\left\{\frac{1}{[(s+a)^2 - b^2]^2}; t\right\} \tag{B.5}$$

$$= \left(\frac{d}{dt} \circ + \circ \Big|_{t=0} + 2(\kappa+k_1) \circ + (\kappa+k_1)^2 \int_0^t dt \circ \right) \mathcal{L}^{-1}\left\{\frac{1}{[s^2 - b^2]^2}; t\right\} e^{-at} \tag{B.6}$$

$$= \left(\frac{d}{dt} \circ + \circ \Big|_{t=0} + 2(\kappa+k_1) \circ + (\kappa+k_1)^2 \int_0^t dt \circ \right) e^{-at} \frac{1}{2b^3}\left[bt \cosh(bt) - \sinh(bt)\right] \tag{B.7}$$

using Prudnikov 2.1.5.6. This becomes

$$\frac{\langle x^2 \rangle}{2v^2} = \frac{t}{b} \sinh(bt) + \frac{(\kappa + k_1)}{b^3} \big[bt \cosh(bt) - \sinh(bt) \big] \tag{B.8}$$

$$+ e^{-at} \frac{b(b^2(at-2)) - a^3 t)\cosh(bt) + (a^3 - a^2 b^2 t - 3ab^2 + b^4 t)\sinh(bt)}{(a-b)^2(a+b)^2} \tag{B.9}$$

$$\frac{\langle x \rangle}{v} = \mathcal{L}^{-1} \Big\{ \frac{1}{s} \frac{s + \kappa + k_1}{(s + \kappa + k')s + \kappa k_2}; t \Big\} \tag{B.10}$$

$$= \mathcal{L}^{-1} \Big\{ \frac{1}{\big[s + \frac{\kappa + k'}{2} \big]^2 + \kappa k_2 - \big[\frac{\kappa + k'}{2} \big]^2}; t \Big\} + \int_0^t du \mathcal{L}^{-1} \Big\{ \frac{\kappa + k_1}{\big[s + \frac{\kappa + k'}{2} \big]^2 + \kappa k_2 - \big[\frac{\kappa + k'}{2} \big]^2}; u \Big\} \tag{B.11}$$

$$= e^{-(\kappa + k')t/2} \mathcal{L}^{-1} \Big\{ \frac{1}{s^2 - b^2}; t \Big\} + (\kappa + k_1) \int_0^t du e^{-(\kappa + k')u/2} \mathcal{L}^{-1} \Big\{ \frac{1}{s^2 - b^2}; u \Big\} \tag{B.12}$$

here $b^2 = -\kappa k_2 + \big[\frac{\kappa + k'}{2} \big]^2$ Then with Prudnikov 2.1.5.4:

$$\frac{\langle x \rangle}{v} = \frac{1}{b} e^{-(\kappa + k')t/2} \sinh(bt) + \frac{(\kappa + k_1)}{b} \int_0^t du e^{-(\kappa + k')u/2} \sinh(bu) \tag{B.13}$$

$$= \frac{1}{b} e^{-at} \sinh(bt) + \frac{\kappa + k_1}{2b} \Big[\frac{1}{b-a} \big(e^{(b-a)t} - 1 \big) + \frac{1}{a+b} \big(e^{-(a+b)t} - 1 \big) \Big] \tag{B.14}$$

where $a = (\kappa + k')/2$ and $b = \sqrt{a^2 - \kappa k_2}$. We are interested in the domain that $a \geq \sqrt{\kappa k_2}$ so that $b \geq a$. The limit of $\kappa \to 0$ provides

$$\frac{k'^2}{v} \langle x \rangle = k_2 (1 - e^{-k't}) + k_1 k' t \tag{B.15}$$

which aligns perfectly with earlier results.

C Links to H. A. Einstein (1937) and Lisle et al. (1998)

References

Aarão Reis, F. D., & Di Caprio, D. (2014). Crossover from anomalous to normal diffusion in porous media. *Physical Review E - Statistical, Nonlinear, and Soft Matter Physics*, *89*(6), 1–10. doi: 10.1103/PhysRevE.89.062126

Berkowitz, B., Cortis, A., Dentz, M., & Scher, H. (2006). Modeling Non-fickian transport in geological formations as a continuous time random walk. *Reviews of Geophysics*, *44*(2), 1–49. doi: 10.1029/2005RG000178

Bradley, N. D. (2017). Direct Observation of Heavy-Tailed Storage Times of Bed Load Tracer Particles Causing Anomalous Superdiffusion. *Geophysical Research Letters*, *44*(24), 12,227–12,235. doi: 10.1002/2017GL075045

Einstein, A. (1905). Brownian Movement. *Biotropica*, *1*. doi: 10.5167/uzh-53657

Einstein, H. A. (1937). *Bed-load transport as a probability problem* (Unpublished doctoral dissertation). ETH Zurich.

Furbish, D. J., Fathel, S. L., Schmeeckle, M. W., Jerolmack, D. J., & Schumer, R. (2017). The elements and richness of particle diffusion during sediment

transport at small timescales. *Earth Surface Processes and Landforms*, *42*(1), 214–237. doi: 10.1002/esp.4084

Gordon, R., Carmichael, J. B., & Isackson, F. J. (1972). Saltation of Plastic Balls in a 'One-Dimensional' Flume. *Water Resources Research*, *8*(2), 444–458. doi: 10 .1029/WR008i002p00444

Hassan, M. A., & Bradley, D. N. (2017). Geomorphic controls on tracer particle dispersion in gravel bed rivers. In *Gravel-bed rivers: Processes and disasters* (pp. 159–184). New York, NY: John Wiley & Sons Ltd. doi: 10.16719/j.cnki.1671 -6981.2015.03.007

Hauer, F. R., Locke, H., Dreitz, V. J., Hebblewhite, M., Lowe, W. H., Muhlfeld, C. C., . . . Rood, S. B. (2016). Gravel-bed river floodplains are the ecological nexus of glaciated mountain landscapes. *Science Advances*, *2*(6), 1–14. doi: 10.1126/sciadv.1600026

Jeon, J. H., Monne, H. M. S., Javanainen, M., & Metzler, R. (2012). Anomalous diffusion of phospholipids and cholesterols in a lipid bilayer and its origins. *Physical Review Letters*, *109*(18), 1–5. doi: 10.1103/PhysRevLett.109.188103

Kittel, C. (1958). *Elementary Statistical Physics.* R.E. Krieger Pub. Co.

Lisle, I. G., Rose, C. W., Hogarth, W. L., Hairsine, P. B., Sander, G. C., & Parlange, J.-Y. (1998). Stochastic sediment transport in soil erosion. *Journal of Hydrology*, *204*, 217–230.

Macklin, M. G., Brewer, P. A., Hudson-Edwards, K. A., Bird, G., Coulthard, T. J., Dennis, I. A., . . . Turner, J. N. (2006). A geomorphological approach to the management of rivers contaminated by metal mining. *Geomorphology*, *79*(3-4), 423–447. doi: 10.1016/j.geomorph.2006.06.024

Marcum, J. (1960). A statistical theory of target detection by pulse radar. *IRE Trans. Inform. Theory*, *6*, 59–268.

Martin, R. L., Jerolmack, D. J., & Schumer, R. (2012). The physical basis for anomalous diffusion in bed load transport. *Journal of Geophysical Research: Earth Surface*, *117*(1), 1–18. doi: 10.1029/2011JF002075

Molina-Garcia, D., Sandev, T., Safdari, H., Pagnini, G., Chechkin, A., & Metzler, R. (2018). Crossover from anomalous to normal diffusion: Truncated power-law noise correlations and applications to dynamics in lipid bilayers. *New Journal of Physics*, *20*(10). doi: 10.1088/1367-2630/aae4b2

Nikora, V., Heald, J., Goring, D., & McEwan, I. (2001). Diffusion of saltating particles in unidirectional water flow over a rough granular bed. *Journal of Physics A: Mathematical and General*, *34*(50). doi: 10.1088/0305-4470/34/50/103

Phillips, C. B., Martin, R. L., & Jerolmack, D. J. (2013). Impulse framework for unsteady flows reveals superdiffusive bed load transport. *Geophysical Research Letters*, *40*(7), 1328–1333. doi: 10.1002/grl.50323

Scher, H., & Lax, M. (1973). Stochastic transport in a disordered solid. I. Theory. *Physical Review B*, *7*(10), 4491–4502. doi: 10.1103/PhysRevB.7.4491

Schleiss, A., Franca, M., Juez, C., & De Cesare, G. (2016). Reservoir sedimentation. *Journal of Hydraulic Research*, *54*(6). doi: 10.1080/00221686.2016.1225320

Shlesinger, M. F., Zaslavsky, G. M., & Klafter, J. (1993). Strange kinetics. *Nature*, *363*(6424), 31–37. doi: 10.1038/363031a0

Shugard, W., & Reiss, H. (1976). Transient nucleation in H20-H2S04 mixtures: A stochastic approach. *Journal of Chemical Physics*, *65*, 2827– 2840.

Sokolov, I. M. (2012). Models of anomalous diffusion in crowded environments. *Soft Matter*, *8*(35), 9043–9052. doi: 10.1039/c2sm25701g

Taylor, G. (1920). DIFFUSION BY CONTINUOUS MOVEMENTS. , *6*.

Temme, N. M. (1996). *Special functions: an introduction to the classical functions of mathematical physics.* John Wiley & Sons Ltd.

Weiss, G. H. (1976). The two-state random walk. *Journal of Statistical Physics*, *15*(2), 157–165. doi: 10.1007/BF01012035

Weiss, G. H. (1994). *Aspects and applications of the random walk.* Amsterdam:

North Holland.

Wu, Z., Foufoula-Georgiou, E., Parker, G., Singh, A., Fu, X., & Wang, G. (2019). Analytical Solution for Anomalous Diffusion of Bedload Tracers Gradually Undergoing Burial. *Journal of Geophysical Research: Earth Surface*, *124*(1), 21–37. doi: 10.1029/2018JF004654

Yang, X. R., & Wang, Y. (2019). Ubiquity of anomalous transport in porous media: Numerical evidence, continuous time random walk modelling, and hydrodynamic interpretation. *Scientific Reports*, *9*(1), 1–11. Retrieved from `http://dx.doi.org/10.1038/s41598-019-39363-3` doi: 10.1038/s41598-019-39363-3

Zhang, Y., Meerschaert, M. M., & Packman, A. I. (2012). Linking fluvial bed sediment transport across scales. *Geophysical Research Letters*, *39*(20), 1–6. doi: 10.1029/2012GL053476