The Lisle model

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The Lisle distribution in Laplace space is

$$\tilde{p}(\eta, s) = \frac{1}{s} \frac{k' + s + \theta_1 \eta v}{\eta v(s + k_1) + s(s + k')} \tag{1}$$

It inverts to

$$p(x,t) = \theta_1 \delta(x) e^{-k_1 t} + \frac{1}{v} e^{-\tau-\xi} \Big\{ \theta_1 \Big[k_1 \mathcal{I}_0\Big(2\sqrt{\xi\tau}\Big) + k_2 \sqrt{\frac{\tau}{\xi}} \mathcal{I}_1\Big(2\sqrt{\xi\tau}\Big) \Big] \\ + \theta_2 \Big[k_1 \delta(\tau) + k_2 \mathcal{I}_0\Big(2\sqrt{\xi\tau}\Big) + k_1 \sqrt{\frac{\xi}{\tau}} \mathcal{I}_1\Big(2\sqrt{\xi\tau}\Big) \Big] \Big\}. \tag{2}$$

Taking derivatives of it implies Laplace moments

$$\frac{\langle \tilde{x} \rangle}{v} = \frac{k_1 + \theta_2 s}{s^2(s + k')} \tag{3}$$

$$\frac{\langle \tilde{x}^2 \rangle}{2v^2} = \frac{(s + k_1)(\theta_2 s + k_1)}{s^3(s + k')^2}. \tag{4}$$

Inverting the first equation provides the mean

$$\frac{k'^2 \langle x \rangle}{v} = k_1 k' t + (\theta_2 k_2 - \theta_1 k_1)(1 - e^{-k't}). \tag{5}$$

Inverting the second provides the second moment

$$\frac{k'^4}{2v^2} \langle x^2 \rangle = \theta_2 k'^2 (1 - (1 + k't) e^{-k't}) \\ + k_1 k'(1 + \theta_2)((1 + e^{-k't}) k't - 2(1 - e^{-k't})) \\ + \frac{k_1^2}{k'^4} (3(1 - e^{-k't}) - k't(2 + e^{-k't}) + \frac{(k't)^2}{2}). \tag{6}$$

This rearranges to

$$\frac{k'^4}{2v^2} \langle x^2 \rangle = \theta_2 \Big[k_1 k'(k't - 1 + e^{-k't}) + k_2^2 (1 - (1 + k't) e^{-k't}) \Big] \\ + k_1 k'((1 + e^{-k't}) k't - 2(1 - e^{-k't})) \\ + \frac{k_1^2}{k'^4} (3(1 - e^{-k't}) - k't(2 + e^{-k't}) + \frac{(k't)^2}{2}). \tag{7}$$

This doesn't really simplify, nor does the variance. It only gets worse. Anyway, you can use this to check your results for the more general model.