Marcum Q-Function: Laplace transforms for 2SRW paper

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August 7, 2019

For computing the distribution function for the two-state random walk with trapping from the rest state, a number of Laplace transforms arise which are difficult applications of special functions. These are:

$$\mathcal{T}_1(t;a,c) = \mathcal{L}^{-1}\Big\{\frac{1}{(s-a)}\exp(c/s);t\Big\} \tag{1}$$

$$\mathcal{T}_2(t;a,c) = \mathcal{L}^{-1}\Big\{\frac{1}{(s-a)s}\exp(c/s);t\Big\} \tag{2}$$

$$\mathcal{T}_3(t;a,b,c) = \mathcal{L}^{-1}\Big\{\frac{1}{(s-a)(s-b)s}\exp(c/s);t\Big\} \tag{3}$$

$$\mathcal{T}_4(t;a,b,c) = \mathcal{L}^{-1}\Big\{\frac{1}{(s-a)(s-b)s^2}\exp(c/s);t\Big\} \tag{4}$$

$$\tag{5}$$

There isn't much info on these. However there are a few key starting points:

$$\mathcal{K}_1(t;c) = \mathcal{L}^{-1}\Big\{\exp(c/s);t\Big\} = \delta(t) + \sqrt{\frac{c}{t}}\mathcal{I}_1\big(2\sqrt{ct}\big) \tag{6}$$

$$\mathcal{K}_2(t;c) = \mathcal{L}^{-1}\Big\{\frac{1}{s}\exp(c/s);t\Big\} = \mathcal{I}_0\big(2\sqrt{ct}\big). \tag{7}$$

One definition of the modified Bessel function is

$$\mathcal{I}_\nu(x) = \sum_{k=0}^{\infty}\frac{(x/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)}. \tag{8}$$

This satisfies the property $\mathcal{I}_\nu(z) = \mathcal{I}_{-\nu}(z)$. More generally, it obeys the recursion relation

$$\mathcal{I}_{\nu-1}(z) - \mathcal{I}_{\nu+1}(z) = \frac{2\nu}{z}\mathcal{I}_\nu(z), \tag{9}$$

and has derivatives

$$\mathcal{I}'_\nu(z) = \mathcal{I}_{\nu-1}(z) - \frac{\nu}{z}\mathcal{I}_\nu(z) = \mathcal{I}_{\nu+1}(z) + \frac{\nu}{z}\mathcal{I}_\nu(z). \tag{10}$$

The two Laplace transforms above can be verified from this definition and linked using the property (from *Prudnikov et al.* [1992]):

$$\mathcal{L}^{-1}\Big\{\frac{F(s)}{s-a}\Big\} = e^{at}\int_0^t du e^{-au}f(u). \tag{11}$$

A link to the (complementary) Marcum Q-function also appears. This is defined by

$$\mathcal{P}_\mu(x,y) = \int_0^y dz\Big(\frac{z}{x}\Big)^{\frac{1}{2}(\mu-1)}e^{-z-x}\mathcal{I}_{\mu-1}\big(2\sqrt{xz}\big), \tag{12}$$

and the central reference for this function is *Temme* [1996]. The complementary function obeys the recursion

$$\mathcal{P}_{\mu+1}(x,y) = \mathcal{P}_\mu(x,y) - \Big(\frac{y}{x}\Big)^{\mu/2}e^{-x}\mathcal{I}_\mu\big(2\sqrt{xz}\big) \tag{13}$$

1 The transform $\mathcal{T}_1$:

Using (11) and (6) obtains

$$e^{-at}\mathcal{T}_1(t;a,c) - 1 = \int_0^t dte^{-au}\sqrt{\frac{c}{u}}\mathcal{I}_1\big(2\sqrt{cu}\big) \tag{14}$$

$$\tag{15}$$

The derivative recursion formula (10) provides $\partial_u \mathcal{I}_0(2\sqrt{cu}) = \sqrt{(c/u)}\mathcal{I}_1(2\sqrt{cu})$. Using this relation and integrating by parts leads to

$$e^{-at}\mathcal{T}_1(t;a,c) - 1 = e^{-at}\mathcal{I}_0(2\sqrt{ct}) - 1 + a\int_0^t due^{-au}\mathcal{I}_0(2\sqrt{cu}), \tag{16}$$

noting the value $\mathcal{I}_0(0) = 1$. Rearranging gives

$$\mathcal{T}_1(t;a,c) = \mathcal{I}_0(2\sqrt{ct}) + ae^{at}\int_0^t due^{-au}\mathcal{I}_0(2\sqrt{cu}). \tag{17}$$

Setting $z = au$ and multiplying and dividing by $e^{c/a}$ provides

$$\mathcal{T}_1(t;a,c) = \mathcal{I}_0(2\sqrt{ct}) + e^{at+c/a}\int_0^{at} dze^{-z-c/a}\mathcal{I}_0(2\sqrt{(c/a)z}), \tag{18}$$

which is identified as

$$\mathcal{T}_1(t;a,c) = \mathcal{L}^{-1}\Big\{\frac{1}{(s-a)}\exp(c/s);t\Big\} = \mathcal{I}_0(2\sqrt{ct}) + e^{at+c/a}\mathcal{P}_1(c/a,at). \tag{19}$$

This becomes $\mathcal{K}_2(t;c)$ in the limit $a \to 0$ as expected.

2 The transform $\mathcal{T}_2$:

Using (11) and (7) provides

$$\mathcal{T}_2(t;a,c) = e^{at}\int_0^t due^{-au}\mathcal{I}_0(2\sqrt{cu}), \tag{20}$$

which rearranges to

$$\mathcal{T}_2(t;a,c) = \mathcal{L}^{-1}\Big\{\frac{1}{(s-a)s}\exp(c/s);t\Big\} = \frac{1}{a}e^{at+c/a}\mathcal{P}_1(c/a,at). \tag{21}$$

3 The transform $\mathcal{T}_3$:

Leveraging the partial fractions expansion

$$\frac{1}{(s-a)(s-b)} = \frac{1}{b-a}\Big[\frac{-1}{s-a} + \frac{1}{s-b}\Big] \tag{22}$$

with (7) provides

$$\mathcal{T}_3(t;a,b,c) = \frac{1}{b-a}\Big[-\mathcal{T}_2(t;a;c) + \mathcal{T}_2(t;b,c)\Big] \tag{23}$$

or

$$\mathcal{T}_3(t;a,b,c) = \mathcal{L}^{-1}\Big\{\frac{1}{(s-a)(s-b)s}\exp(c/s);t\Big\} = \frac{1}{b-a}\Big[-\frac{1}{a}e^{at+c/a}\mathcal{P}_1(c/a,at) \ + \frac{1}{b}e^{bt+c/b}\mathcal{P}_1(c/b,bt)\Big]. \tag{24}$$

4 The transform $\mathcal{T}_4$:

Using the partial fractions expansion

$$\frac{1}{(s-a)(s-b)s^2} = \frac{a+b}{a^2b^2s} + \frac{1}{a^2(a-b)(s-a)} + \frac{1}{b^2(b-a)(s-b)} + \frac{1}{abs^2} \tag{25}$$

gives

$$\mathcal{T}_4(t;a,b,c) = \frac{a+b}{a^2b^2}\mathcal{K}_2(t;c) + \frac{1}{a^2(a-b)}\mathcal{T}_1(t;a,c) + \frac{1}{b^2(b-a)}\mathcal{T}_1(t;b,c) + \frac{1}{ab}\mathcal{K}_3(t;c), \tag{26}$$

where

$$\mathcal{K}_3(t;c) = \int_0^t du \mathcal{I}_0(2\sqrt{cu}) = \frac{1}{c}\sum_{k=0}^{\infty}\frac{(ct)^{k+1}}{k!(k+1)!} = \sqrt{\frac{t}{c}}\mathcal{I}_1(2\sqrt{ct}). \tag{27}$$

That should do it ...

Probably it remains to check these by taking direct transforms.

References

Prudnikov, A., Y. A. Brychkov, and O. Marichev, *Integrals and Series: Volume 5: Inverse Laplace Transforms*, Gordon and Breach Science Publishers, 1992.

Temme, N. M., *Special functions: an introduction to the classical functions of mathematical physics*, John Wiley & Sons Ltd., 1996.