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Assignment 7
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Homework #7

(a) Prove that $\|X\|_p$ increases with $0 < p \leq \infty$.

Proof. Jensen's inequality states that if φ is a convex function and X is a random variable, we have

$$\varphi(\mathbb{E}[X]) \leq \mathbb{E}[\varphi(X)].$$

Let $0 < p < q < \infty$. Let $\varphi(x) = |x|^{q/p}$. It follows that φ is a convex function. Calculating, we see

$$\|X\|_p = (\mathbb{E}|X|^p)^{1/p} = (\varphi(\mathbb{E}|X|^p))^{1/q} \leq (\mathbb{E}\varphi(|X|^p))^{1/q} = (\mathbb{E}|X|^q)^{1/q} = \|X\|_q.$$

This shows that the L^p norm is increasing on $0 < p < \infty$. Now we consider $p = \infty$. Let $S = \{\omega : X(\omega) > \|X\|_\infty\}$. From the definition of essential supremum, it follows that $P(S) = 0$ or that S is a null set. For any p with $0 < p < \infty$, we have

$$\|X\|_p = (\int |X|^p)^{1/p} = (\int_{\Omega \setminus S} |X|^p)^{1/p} \leq (\int_{\Omega \setminus S} \|X\|_\infty^p)^{1/p} = (\|X\|_\infty^p P(\Omega \setminus S))^{1/p} = \|X\|_\infty.$$

Hence L^p norm is increasing on $0 < p \leq \infty$. □

(b) Under what conditions does it happen that $0 < r < s \leq \infty$ and $\|X\|_r = \|X\|_s < +\infty$?

(c) Prove that the spaces L^p decrease with $0 < p \leq \infty$. Under what conditions do L^r and L^s with $r < s$ contain exactly the same r.v.'s?

Proof. Let $0 < q < r \leq \infty$. Let $X \in L^r$. It follows that $\|X\|_r < \infty$. By the increasingness of the L^p norm, $\|X\|_q < \infty$. Hence $X \in L^q$. Therefore, $L^r \subset L^q$ and the L^p spaces are decreasing. □

(d) Let $S = \{p : 0 < p < \infty \text{ and } \|X\|_p < +\infty\}$. Show that S is of the form $S = (0, p_0)$ or $S = (0, p_0]$ for some $0 \leq p_0 \leq \infty$.

Proof. This follows from the increasingness of L^p. It cannot be any different. Examine S^c. By the increasingness of L^p, we have S^c must be of the form $(p_0, \infty]$ or $[p_0, \infty]$. Hence S must be of the form $(0, p_0)$ or $(0, p_0]$. □

(e) Prove that $\log(\|X\|_p^p)$ is convex in $p \in \text{interior}(S)$ and that $\|X\|_p$ is continuous in $p \in S$.

(f) Show that $\|X\|_\infty = \lim_{p\uparrow\infty} \uparrow \|X\|_p$.

Proof. We already know that the L^p norm increases. Let $\epsilon > 0$ and $S_\epsilon = \{\omega : \|X\|_\infty - \epsilon \le X(\omega) \le \|X\|_\infty\}$. By the normality and right-continuity of F_X, we have that $P(S_\epsilon) = \alpha > 0$. It follows that

$$\int |X|^p \ge \int_{S_\epsilon} |X|^p \ge \int_{S_\epsilon} (\|X\|_\infty - \epsilon)^p = \alpha(\|X\|_\infty - \epsilon)^p.$$

For sufficiently small ϵ and sufficiently large p, we have

$$\int |X|^p \ge (\|X\|_\infty - \epsilon)^p.$$

Taking the $1/p$-th power and the limit as p tends to infinity we have

$$\lim_{p\uparrow\infty} \uparrow \|X\|_p \ge \|X\|_\infty - \epsilon.$$

Since ϵ is arbitrary, we have shown $\|X\|_\infty = \lim_{p\uparrow\infty} \uparrow \|X\|_p$. $\square$

(g) Assume that $S \neq \emptyset$ and prove that

$$\lim_{p\downarrow 0} \downarrow \|X\|_p = \exp\{E \log |X|\},$$

with the understanding that $\exp\{-\infty\} := 0$.