550.621 Final Examination, Spring 2013

Official Examination Policy

This take-home final examination for 550.621 Probability Theory II is due *no later than* **5:00 PM EDT, Thursday, May 16, 2013**. There are only three acceptable ways of turning in your exam: (1) Hand it to me in person, or (2) slide it under the door of my office, 306-F Whitehead Hall, or (3) submit it electronically to me at `jimfill@jhu.edu`. If you choose option (3), please *also* submit an electronic copy to Teaching Assistant Jason Matterer at `jmatter4@jhu.edu`.

You are not to discuss the exam with anyone except me before the due deadline. Conversation, even on the most casual level, about the exam will be considered a violation of the university's honor code.

Your work should be legibly written in complete English sentences (and paragraphs!). Every solution must be clearly explained. Be sure that your name is on every page of your solutions. Please write on only one side of each page.

You may consult your class notes and the course texts by Billingsley and Chung. You may also consult references on elementary probability and real analysis; these should not, however, make use of measure theory. *You may not use any other references of any kind.*

Point values for each question are shown in parentheses.

Your exam will be graded on a 300-point basis.

Examination Problems (300 points total)

1. **(100 total)** Let X_j, $j \geq 1$, be a sequence of independent random variables, and set $S_n := \sum_{j=1}^n X_j$. Suppose $\alpha \geq 1$ and

$$X_j = \begin{cases} \pm j^\alpha, & \text{with probability } \frac{1}{4j^{2(\alpha-1)}} \text{ each;} \\ 0, & \text{with probability } 1 - \frac{1}{2j^{2(\alpha-1)}}. \end{cases}$$

   (a) **(25)** For precisely which values of α does the sequence (S_n) converge almost surely to an almost surely finite limit?

   (b) **(25)** Calculate $s_n^2 := \sum_{j=1}^n \mathrm{Var}(X_j)$ for each n. Describe the associated normalized double array, and prove by considering variances that it is holospoudic for any value of α.

   (c) **(25)** For precisely which values of α is Lindeberg's condition satisfied?

   (d) **(25)** For precisely which values of α does S_n/s_n converge in distribution to standard normal? [Here's something to think about after the exam. Does S_n/s_n have *any* limiting distribution if not standard normal? If so, what?]

2. **(70)** Let $(\Omega, \mathcal{F}, P)$ be a probability space, let $\mathcal{F}_0$, $\mathcal{F}_1$, and $\mathcal{F}_2$ be three sub-σ-fields of $\mathcal{F}$, and suppose that $\mathcal{F}_0 \subset \mathcal{F}_1$. Show that

$$\forall A \in \mathcal{F}_1 \;\; \forall B \in \mathcal{F}_2\colon\;\; P(A \cap B|\mathcal{F}_0) = P(A|\mathcal{F}_0)\; P(B|\mathcal{F}_0) \;\text{ a.s.} \tag{1}$$

   if and only if

$$\forall B \in \mathcal{F}_2\colon\;\; P(B|\mathcal{F}_0) = P(B|\mathcal{F}_1) \;\text{ a.s.} \tag{2}$$

3. **(130 total)** Let $S_n = \sum_{j=1}^n X_j$, where the X_j's are independent random variables with a common distribution function F of the integer lattice type with span 1. Suppose that X_1

has mean 0 and variance $\sigma^2 \in (0, \infty)$. The main objective of this problem is to prove the *local central limit theorem* result that

$$\sigma\sqrt{n}\left[P\left\{\frac{S_n}{\sigma\sqrt{n}} = \frac{j}{\sigma\sqrt{n}}\right\} - \frac{1}{\sigma\sqrt{n}}\varphi\left(\frac{j}{\sigma\sqrt{n}}\right)\right] \longrightarrow 0 \text{ uniformly in } j \in \mathbf{Z} \tag{3}$$

as $n \to \infty$, where φ denotes the standard normal density function.

(a) **(25)** Grant (3) for the moment and use it to derive the *global central limit theorem* for S_n: for real numbers $-\infty < a < b < \infty$,

$$P\left\{a < \frac{S_n}{\sigma\sqrt{n}} \le b\right\} \longrightarrow P\{a < Z \le b\}$$

as $n \to \infty$, where Z has a standard normal distribution.

(b) **(40)** Let f denote the characteristic function of X_1. Begin the proof of (3) by showing that the expression there equals $1/(2\pi)$ times

$$\int_{|u|<\pi\sigma\sqrt{n}} \left[\left(f\left(\frac{u}{\sigma\sqrt{n}}\right)\right)^n - e^{-\frac{1}{2}u^2}\right] e^{-ji\frac{u}{\sigma\sqrt{n}}}\, du - \int_{|u|\ge\pi\sigma\sqrt{n}} e^{-\frac{1}{2}u^2} e^{-ji\frac{u}{\sigma\sqrt{n}}}\, du. \tag{4}$$

(c) **(10)** Show that the second integral in (4) tends to 0 uniformly in $j \in \mathbf{Z}$ as $n \to \infty$.

(d) **(55)** The first integral in (4) is dominated by the integral

$$\int_{|u|<\pi\sigma\sqrt{n}} \left|\left(f\left(\frac{u}{\sigma\sqrt{n}}\right)\right)^n - e^{-\frac{1}{2}u^2}\right| du.$$

Complete the proof of (3) by showing that this integral tends to 0 as $n \to \infty$. [HINT: Divide the range $|u| < \pi\sigma\sqrt{n}$ into the three regions $|u| \le M$, $M < |u| < \delta\sigma\sqrt{n}$, and $\delta\sigma\sqrt{n} \le |u| < \pi\sigma\sqrt{n}$ for suitably chosen constants M and δ. Show that each of the three corresponding integrals is small.]

I would make the following problem a 100-point exam problem except that its solution may be found both in Billingsley's book and in Chung's. I regret that I don't have a martingale problem to replace it on the exam!

4. (Convergence Theorem for Backward Martingales.) Let $(X_n)_{n\le 0}$ be a martingale with respect to $(\mathcal{F}_n)_{n\le 0}$. Then (with no conditions!) there exists an integrable random variable $X_{-\infty}$ such that X_n converges, both in L^1 and a.s., to $X_{-\infty}$ as $n \to -\infty$. Prove this.