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550.621 Probability
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Assignment 2
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Assignment 2

All the exercises on the transition probabilities handout

1. Let $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ be two measurable spaces and let $\pi_i : \Omega_1 \times \Omega_2 \to \Omega_i$ be the ith projection, $i = 1, 2$. Set

$$\mathcal{C} := \pi_1^{-1}(\mathcal{A}_1) \cup \pi_2^{-1}(\mathcal{A}_2), \text{ the class of } \textit{measurable cylinders,}$$
$$\mathcal{R} := \{A_1 \times A_2 : A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2\}, \text{ the class of } \textit{measurable rectangles,}$$
$$\mathcal{U} := \left\{ \sum_{j \in J} R_j : J \text{ finite}, R_j \in \mathcal{R} \text{ for each } j \right\}.$$

(a) Show $\mathcal{C}$ is closed under complementation.

Proof. Given an element C of $\mathcal{C}$, we have $C = A_1 \times \Omega_2$ or $C = \Omega_1 \times A_2$ for some $A_1 \in \mathcal{A}_1$ or $A_2 \in \mathcal{A}_2$. (As an aside, note since $\Omega_1 \in \mathcal{A}_1$ and $\Omega_2 \in \mathcal{A}_2$ by definition of σ-field, then $C \in \mathcal{R}$. Hence $\mathcal{C} \subset \mathcal{R}$). The complement of C is $C^c = A_1^c \times \Omega_2$ or $C^c = \Omega_1 \times A_2^c$. Since σ-fields are closed under complementation, $A_1^c \in \mathcal{A}_1$ and $A_2^c \in \mathcal{A}_2$. Therefore $C^c \in \mathcal{C}$. □

(b) Show $\mathcal{R}$ is a π-system.

Proof. Let $A_1 \times A_2 \in \mathcal{R}$ and $B_1 \times B_2 \in \mathcal{R}$ where A_i and B_i are elements of $\mathcal{A}_i$ for $i = 1, 2$. Taking the intersection of both sets, we have

$$(A_1 \times A_2) \cap (B_1 \times B_2) = (A_1 \cap B_1) \times (A_2 \cap B_2)$$

Since A_i and B_i are elements of $\mathcal{A}_i$ for $i = 1, 2$ it follows that $A_i \cap B_i \in \mathcal{A}_i$ for $i = 1, 2$ by closure under countable intersections of σ-fields. Thus $(A_1 \times A_2) \cap (B_1 \times B_2) = (A_1 \cap B_1) \times (A_2 \cap B_2) \in \mathcal{R}$. Hence $\mathcal{R}$ is a π-system. □

(c) Show $\mathcal{U}$ is the field generated by $\mathcal{C}$ (and by $\mathcal{R}$).

Proof. First, we show that $\mathcal{U}$ is a field.

Closure under binary intersection: Suppose $U_1 \in \mathcal{U}$ and $U_2 \in \mathcal{U}$. Then $U_1 = \sum_{j=1}^{J} R_j$ and let $U_2 = \sum_{k=1}^{K} S_k$ where $R_j \in \mathcal{R}$ and $S_k \in \mathcal{R}$ for each j and k and J and K are finite.

$$\begin{aligned} U_1 \cap U_2 &= \sum_{j=1}^{J} R_j \bigcap \sum_{k=1}^{K} S_k \\ &= \cup_{j=1}^{J} R_j \bigcap \cup_{k=1}^{K} S_k \\ &= \cup_{j=1}^{J} \left(R_j \bigcap \cup_{k=1}^{K} S_k \right) \\ &= \cup_{j=1}^{J} \left(\cup_{k=1}^{K} R_j \cap S_k \right) \end{aligned}$$

For each j and k, since $\mathcal{R}$ is a π-system, $R_j \cap S_k = T_{jk}$ is a rectangle. Furthermore, suppose $(i,j) \neq (i',j')$. Then

$$T_{jk} \cap T_{j'k'} = R_j \cap S_k \bigcap R_{j'} \cap S_{k'} = R_j \cap R_{j'} \bigcap S_k \cap S_{k'} = \emptyset$$

because $\{R_j\}$ are disjoint and $\{S_k\}$ are disjoint. Therefore,

$$U_1 \cap U_2 = \cup_{j=1}^{J} \left(\cup_{k=1}^{K} R_j \cap S_k \right) = \sum_{\substack{j \in \{1,\cdots,J\} \\ k \in \{1,\cdots,K\}}} T_{jk} \in \mathcal{U}$$

Thus $\mathcal{U}$ is closed under binary intersection.

Next we show closure under intersection of finite terms. The base case is already proven above ($K = 2$). Suppose for some integer $K > 2$

$$\cap_{k=1}^{K} U_k \in \mathcal{U}$$

where $U_k \in \mathcal{U}$ for each k. Then suppose $U_1, \cdots, U_{K+1}$ are members of $\mathcal{U}$. Calculating,

$$\cap_{k=1}^{K+1} U_k = \left(\cap_{k=1}^{K} U_k \right) \cap U_{K+1}$$

is the intersection of two elements of $\mathcal{U}$ by inductive hypothesis. Then by the base case their intersection is in $\mathcal{U}$. Hence

$$\cap_{k=1}^{K+1} U_k \in \mathcal{U}$$

By induction, the intersection of any k elements of $\mathcal{U}$ is a member of $\mathcal{U}$ for all positive integers k.

Closure under complementation: First a little lemma. Let $R = A_1 \times A_2$ be a rectangle in $\mathcal{R}$. Then

$$R^c = (A_1^c \times A_2) \cup (A_1 \times A_2^c) \cup (A_1^c \times A_2^c)$$

is a finite union of disjoint rectangles in $\mathcal{R}$. Hence $R^c \in \mathcal{U}$.

Suppose $U = \sum_{j=1}^{J} R_j \in \mathcal{U}$ where J is finite and $R_j \in \mathcal{R}$ for each j. Then

$$\begin{aligned} U^c &= \left(\sum_{j=1}^{J} R_j \right)^c \\ &= \left(\cup_{j=1}^{J} R_j \right)^c \\ &= \cap_{j=1}^{J} R_j^c \end{aligned}$$

Since $R_j^c \in \mathcal{U}$ by the little lemma and intersection of a finite number of members of $\mathcal{U}$ lies in $\mathcal{U}$ (as proved above with induction), it follows that

$$U^c = \cap_{j=1}^{J} R_j^c \in \mathcal{U}$$

Since closure of binary intersection is equivalent to closure of binary union under closure of complementation (Chung 17), it is demonstrated that $\mathcal{U}$ is a field.

$\mathcal{U}$ is the field generated by $\mathcal{R}$ and $\mathcal{C}$: It was shown in (a) that $\mathcal{C} \subset \mathcal{R}$. Since $\langle \mathcal{R} \rangle$ contains $\mathcal{R}$ (the angle brackets denote "field generated by"), it follows that

$$\mathcal{C} \subset \mathcal{R} \subset \langle \mathcal{R} \rangle \tag{1}$$

Consider $\langle \mathcal{C} \rangle$. Let $R \in \mathcal{R}$. Then $R = A_1 \times A_2$ for $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$. Then since $A_1 \times \Omega_2 \in \mathcal{C}$ and $\Omega_1 \times A_2 \in \mathcal{C}$ and since $\langle \mathcal{C} \rangle$ is closed under binary intersection,

$$R = A_1 \times A_2 = (A_1 \times \Omega_2) \cap (\Omega_1 \times A_2) \in \langle \mathcal{C} \rangle$$

Therefore $\mathcal{R} \subset \langle \mathcal{C} \rangle$. By definition of "field generated by," the field generated by a set is a subset of all other fields that contain that set. Thus

$$\mathcal{R} \subset \langle \mathcal{R} \rangle \subset \langle \mathcal{C} \rangle \qquad \text{and} \qquad \mathcal{C} \subset \langle \mathcal{C} \rangle \subset \langle \mathcal{R} \rangle \text{ by (1)}$$

Therefore $\langle \mathcal{C} \rangle = \langle \mathcal{R} \rangle$.

Let $R \in \mathcal{R}$, it is the case that R is the sum of one rectangle (itself). Thus $R \in \mathcal{U}$, and $\mathcal{R} \subset \mathcal{U}$. Since $\mathcal{U}$ is a field containing $\mathcal{R}$, by the same reasoning above,

$$\mathcal{R} \subset \langle \mathcal{R} \rangle \subset \mathcal{U} \tag{2}$$

Let $U = \sum_{j=1}^{J} R_j \in \mathcal{U}$ where J is finite and $R_j \in \mathcal{R}$ for each j. Then $U = \cup_{j=1}^{J} R_j$. Since closure of binary union implies closure of finite union (see Chung 17), then $\langle \mathcal{R} \rangle$ is closed under finite union. Hence

$$U = \sum_{j=1}^{J} R_j = \cup_{j=1}^{J} R_j \in \langle \mathcal{R} \rangle$$

Therefore $\mathcal{U} \subset \langle \mathcal{R} \rangle$. Combining that with (2), there is set equality, i.e.

$$\mathcal{U} = \langle \mathcal{R} \rangle$$

Thus

$$\mathcal{U} = \langle \mathcal{R} \rangle = \langle \mathcal{C} \rangle$$

since $\langle \mathcal{R} \rangle = \langle \mathcal{C} \rangle$ as shown above. □

2. Let Ω_1 and Ω_2 be two spaces; set $\Omega = \Omega_1 \times \Omega_2$. Let $X : \Omega \to \Psi$ (respectively, $A \subset \Omega$). The section of X (resp., of A) at $\omega_1 \in \Omega_1$ is defined to be the function $X_{\omega_1} : \Omega_2 \to \Psi$ (resp., the set $A_{\omega_1} \subset \Omega_2$) given by $X_{\omega_1}(\omega_2) = X(\omega_1, \omega_2)$ (resp., by $A_{\omega_1} = \{\omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in A\}$).

   Show that $(I_A)_{\omega_1} = I_{A_{\omega_1}}$ for $A \subset \Omega$ and $(X^{-1}(B))_{\omega_1} = X_{\omega_1}^{-1}(B)$ for $B \subset \Psi$.

   Proof. Let $A \subset \Omega$. Suppose $\omega_2 \in A_{\omega_1}$. Thus $(\omega_1, \omega_2) \in A$. Therefore, $I_{A_{\omega_1}}(\omega_2) = 1$ and $(I_A)_{\omega_1}(\omega_2) = I_A(\omega_1, \omega_2) = 1$. Both functions agree for $\omega_2 \in A_{\omega_1}$. Now suppose $\omega_2 \notin A_{\omega_1}$. Then $(\omega_1, \omega_2) \notin A$. Thus $I_{A_{\omega_1}}(\omega_2) = 0$ and $(I_A)_{\omega_1}(\omega_2) = I_A(\omega_1, \omega_2) = 0$. And we conclude $(I_A)_{\omega_1} = I_{A_{\omega_1}}$ for $A \subset \Omega$.

   For the second part, note for all $B \subset \Psi$

$$\begin{aligned}
(X^{-1}(B))_{\omega_1} &= \{\omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in X^{-1}(B)\} \\
&= \{\omega_2 \in \Omega_2 : X(\omega_1, \omega_2) \in B\} \\
&= \{\omega_2 \in \Omega_2 : X_{\omega_1}(\omega_2) \in B\} \\
&= (X_{\omega_1})^{-1}(B) \\
&= X_{\omega_1}^{-1}(B)
\end{aligned}$$

   as desired. □

3. Notations are the same as Problem 2. Let $i_{\omega_1} : \Omega_2 \to \Omega$ be the injection mapping defined by

$$i_{\omega_1}(\omega_2) = (\omega_1, \omega_2).$$

   Show that

$$A_{\omega_1} = i_{\omega_1}^{-1}(A), \qquad X_{\omega_1} = X \circ i_{\omega_1}.$$

   Proof. For the first part, notice

$$\begin{aligned}
A_{\omega_1} &= \{\omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in A\} \\
&= \{\omega_2 \in \Omega_2 : i_{\omega_1}(\omega_2) \in A\} \\
&= (i_{\omega_1})^{-1}(A) \\
&= i_{\omega_1}^{-1}(A)
\end{aligned}$$

   as desired. Also, for all $\omega_2 \in \Omega_2$ it is the case that

$$X_{\omega_1}(\omega_2) = X(\omega_1, \omega_2) = X(i_{\omega_1}(\omega_2)) = X \circ i_{\omega_1}(\omega_2).$$

   Hence $X_{\omega_1} = X \circ i_{\omega_1}$ as desired. □

4. Let Ψ be an uncountable set, and let $\mathcal{B}$ be the σ-field in Ψ generated by the singletons. ($\mathcal{B}$ consists of the countable and co-countable subsets of Ψ.) Take $(\Omega_1, \mathcal{A}_1) = (\Psi, \mathcal{B}) = (\Omega_2, \mathcal{A}_2)$. Consider the diagonal $\Delta := \{(\psi, \psi) : \psi \in \Psi\}$ of $\mathcal{A} = \mathcal{B} \otimes \mathcal{B}$. Show that every section of Δ is in $\mathcal{B}$, but $\Delta \notin \mathcal{A}$.

   Proof. For all $\psi_1 \in \Psi = \Omega_1$, the section

$$\Delta_{\psi_1} = \{\psi \in \Psi = \Omega_2 : (\psi_1, \psi) \in \Delta\} = \{\psi_1\}$$

   is an element of $\mathcal{B}$, since $\mathcal{B}$ contains the singletons. Similarly, for all $\psi_2 \in \Psi = \Omega_2$, the section

$$\Delta_{\psi_2} = \{\psi \in \Psi = \Omega_1 : (\psi, \psi_2) \in \Delta\} = \{\psi_2\}$$

   is an element of $\mathcal{B}$. Thus every section of Δ is in $\mathcal{B}$.

   Show $\Delta \notin \mathcal{A}$: Let

$$\begin{aligned} S_1 &= \{\{\psi\} \times \Psi : \psi \in \Psi\} \\ S_2 &= \{\Psi \times \{\psi\} : \psi \in \Psi\} \\ \mathcal{S} &= S_1 \cup S_2 \end{aligned}$$

   Let $\mathcal{C}$ be the class of cylinders of $\mathcal{A}$ as defined in problem 1. From the class notes, $\sigma(\mathcal{C}) = \mathcal{A}$. We show that $\sigma(\mathcal{C}) = \sigma(\mathcal{S})$.

   Let $S \in \mathcal{S}$. Then $S = \{\psi\} \times \Psi$ or $S = \Psi \times \{\psi\}$ for some $\psi \in \Psi$. Since $\mathcal{B}$ is generated by the singletons, $\{\psi\} \in \mathcal{B}$. Therefore both $\{\psi\} \times \Psi$ and $\Psi \times \{\psi\}$ are cylinders (members of $\mathcal{C}$). Furthermore, both $\{\psi\} \times \Psi$ and $\Psi \times \{\psi\}$ are members of $\sigma(\mathcal{C})$ since $\mathcal{C} \subset \sigma(\mathcal{C})$. Since S was arbitrary in $\mathcal{S}$,

$$\mathcal{S} \subset \mathcal{C} \subset \sigma(\mathcal{C})$$

   Because $\sigma(\mathcal{S})$ is a subset of any σ-field that contains $\mathcal{S}$, it follows that

$$\sigma(\mathcal{S}) \subset \sigma(\mathcal{C}).$$

   Define $\mathcal{G}_1 = \{B : B \times \Psi \in \sigma(\mathcal{S})\}$. We show $\mathcal{G}_1$ is a σ-field. Since $\sigma(\mathcal{S})$ is a σ-field, $\Psi \times \Psi \in \sigma(\mathcal{S})$. Therefore, $\Psi \in \mathcal{G}_1$. Let $B \in \mathcal{G}_1$. Then $B \times \Psi \in \sigma(\mathcal{S})$. Since $\sigma(\mathcal{S})$ is a σ-field,

$$\begin{aligned} (B \times \Psi)^c &= (B^c \times \Psi) \cup (B \times \Psi^c) \cup (B^c \times \Psi^c) \\ &= (B^c \times \Psi) \cup (B \times \emptyset) \cup (B^c \times \emptyset) \\ &= (B^c \times \Psi) \cup \emptyset \cup \emptyset \\ &= B^c \times \Psi \end{aligned}$$

is a member of $\sigma(\mathcal{S})$. Hence $B^c \in \mathcal{G}_1$. Let $B_1, B_2, \cdots$ be a sequence of sets in $\mathcal{G}_1$. Then $(B_1 \times \Psi), (B_2 \times \Psi), \cdots$ is a sequence of sets in $\sigma(\mathcal{S})$. Thus

$$\left(\bigcup_{i=1}^{\infty}(B_i \times \Psi)\right) = \left(\bigcup_{i=1}^{\infty} B_i\right) \times \Psi \in \sigma(\mathcal{S})$$

Hence $\cup_{i=1}^{\infty} B_i \in \mathcal{G}_1$. Therefore $\mathcal{G}_1$ is a σ-field. Note for all $\psi \in \Psi$ it is the case that $\{\psi\} \times \Psi \in \sigma(\mathcal{S})$ because $\{\psi\} \times \Psi \in S_1$ and $S_1 \subset \mathcal{S} \subset \sigma(\mathcal{S})$. It follows that $\mathcal{G}_1$ contains all the singletons. Since $\mathcal{B}$ is a subset of all σ-fields that contain the singletons, we have $\mathcal{B} \subset \mathcal{G}_1$.

Similarly, define $\mathcal{G}_2 = \{B : \Psi \times B \in \sigma(\mathcal{S})\}$. By similar reasoning to what comes above for $\mathcal{G}_1$ (substitute $B \times \Psi$ by $\Psi \times B$ and S_1 by S_2), it follows that $\mathcal{B} \subset \mathcal{G}_2$.

Let $C \in \mathcal{C}$. **Case 1:** For some $B \in \mathcal{B}$, we have $C = (B \times \Psi) \in \sigma(\mathcal{S})$ because $B \in \mathcal{B} \subset \mathcal{G}_1$. **Case 2:** For some $B \in \mathcal{B}$, we have $C = (\Psi \times B) \in \sigma(\mathcal{S})$ because $B \in \mathcal{B} \subset \mathcal{G}_2$. In both cases $C \in \sigma(\mathcal{S})$. Since C was arbitrary, $\mathcal{C} \subset \sigma(\mathcal{S})$. Note $\sigma(\mathcal{C})$ is a subset of all σ-fields that contain $\mathcal{C}$. Hence

$$\sigma(\mathcal{C}) \subset \sigma(\mathcal{S})$$

Since set inclusion is proved in both directions, it follows that $\sigma(\mathcal{C}) = \sigma(\mathcal{S})$.

Note $\mathcal{A} = \sigma(\mathcal{C}) = \sigma(\mathcal{S})$. Suppose, by way of contradiction, $\Delta \in \mathcal{A}$. Problem 2.9 in Billingsley states

Theorem. *If $B \in \sigma(\mathscr{A})$, then there exists a countable subclass $\mathscr{A}_B$ of $\mathscr{A}$ such that $B \in \sigma(\mathscr{A}_B)$.*

(A proof is given at the end of this assignment).

Since $\mathcal{A} = \sigma(\mathcal{S})$, we have $\Delta \in \sigma(\mathcal{S})$. By problem 2.9, there exists some countable sublass $\mathcal{S}_\Delta$ of $\mathcal{S}$ such that $\Delta \in \sigma(\mathcal{S}_\Delta)$. We conclude

$$\mathcal{S}_\Delta = \{\{\psi_i\} \times \Psi : i \in I_1\} \cup \{\Psi \times \{\psi_i\} : i \in I_2\},$$

Where I_1 and I_2 are at most countable indexing sets and could possibly be empty. Let $T = \{\psi_i : \psi_i \in \Psi \text{ and } i \in I_1 \cup I_2\}$ so that T is the set of all elements of Ψ that form a singleton cylinder in $\mathcal{S}_\Delta$.

Let $\mathcal{P}_1$ be the class containing T^c and the singletons $\{\psi_i\}$ where $i \in I_1 \cup I_2$. Hence $\mathcal{P}_1$ is a countable partition of Ψ. Let $\mathcal{P}_2 = [P_1 \times P_2 : P_1, P_2 \in \mathcal{P}_1]$. Hence $\mathcal{P}_2$ is a countable partition of $\Psi \times \Psi$. Let $\mathscr{U}$ (script "U") be the class of countable unions of sets (or blocks) in the partition $\mathcal{P}_2$, and let $\mathscr{U}$ include the empty set. An example, for demonstration, of an element in $\mathscr{U}$ is the set $(\{\psi_1\} \times \{\psi_1\}) \cup (\{\psi_2\} \times T^c)$. We claim (i) $\mathscr{U}$ is a σ-field and (ii) $\sigma(\mathcal{S}_\Delta) \subset \mathscr{U}$.

(i) By definition, $\mathscr{U}$ includes the empty set. Since $\mathcal{P}_2$ is countable and since the union of the blocks of $\mathcal{P}_2$ is $\Psi \times \Psi$, it follows that $\Psi \times \Psi$ is contained in $\mathscr{U}$. A set of $\mathscr{U}$ is the countable union of blocks of the partition, and that set's complement is the countable union of all the other blocks. Thus $\mathscr{U}$ is closed under complementation. Finally, $\mathscr{U}$ is closed under countable union by definition. Hence $\mathscr{U}$ has the three properties of a σ-field.

(ii) The elements of $\mathcal{S}_\Delta$ are contained in $\mathscr{U}$ since $\{\psi_i\} \in \mathcal{P}_1$ where $i \in I_1 \cup I_2$, and

$$\{\psi_i\} \times \Psi = \bigcup_{P \in \mathcal{P}_1} (\{\psi_i\} \times P) = \{\psi_i\} \times \Psi \in \mathscr{U} \quad \text{and similarly}$$
$$\Psi \times \{\psi_i\} = \bigcup_{P \in \mathcal{P}_1} (P \times \{\psi_i\}) = \Psi \times \{\psi_i\} \in \mathscr{U}.$$

By (i), $\mathscr{U}$ is a σ-field containing $\mathcal{S}_\Delta$. Thus $\sigma(\mathcal{S}_\Delta) \subset \mathscr{U}$ and $\Delta \in \mathscr{U}$.

Now we seek the contradiction. Set

$$D = \Delta \setminus \left(\bigcup_{\psi \in T} (\psi, \psi) \right) = \bigcup_{\psi \in T^c} (\psi, \psi) \subset T^c \times T^c.$$

Note $\cup_{\psi \in T}(\psi, \psi) = \cup_{\psi \in T}(\{\psi\} \times \{\psi\}) \in \mathscr{U}$ since it is a countable union of elements in $\mathcal{P}_2$. Since $\Delta \in \mathscr{U}$, we have $D \in \mathscr{U}$ by the closure properties of $\mathscr{U}$. Since T^c is uncountable, there exist ψ_1 and ψ_2 in T^c with $\psi_1 \neq \psi_2$. It follows that (ψ_1, ψ_2) is an element of $T^c \times T^c$ but it is not an element of D. Thus D is a strict subset of $T^c \times T^c$. Since $D \in \mathscr{U}$, we have D is equal to U, some countable union of sets in $\mathcal{P}_2$. One of the sets in the countable union U must be $T^c \times T^c$ (if not, then D is not a subset of U). In that case, however, D is a strict subset of $T^c \times T^c \subset U$. Thus $D \neq U$. This is our contradiction. Therefore $\Delta \notin \mathcal{A}$ as desired. □

5. A transition probability is defined as follows:

A mapping $T : \Omega_1 \times \mathcal{A}_2 \to [0, 1]$ is called a *transition probability* from $(\Omega_1, \mathcal{A}_1)$ to $(\Omega_2, \mathcal{A}_2)$ (briefly, from Ω_1 to Ω_2) if

(1) $T(\omega_1;\ \cdot\ )$ is a probability measure on $(\Omega_2, \mathcal{A}_2)$ for each $\omega_1 \in \Omega_1$, and

(2) $T(\ \cdot\ ; A_2)$ is $\mathcal{A}_1$-measurable for each $A_2 \in \mathcal{A}_2$.

Show that given (1), then (2) holds if $T(\ \cdot\ ; A_2)$ is $\mathcal{A}_1$-measurable for all A_2 in some π-system generating $\mathcal{A}_2$.

Proof. Let (1) be true. Let $T(\ \cdot\ ; A_2)$ be $\mathcal{A}_1$-measurable for all A_2 in some π-system $\mathscr{P}$ generating $\mathcal{A}_2$. Define

$$\mathscr{G} = \{A_2 \in \mathcal{A}_2 : T(\ \cdot\ ; A_2) \text{ is } \mathcal{A}_1\text{-measurable } \}$$

We now show that $\mathscr{G}$ is a λ-system.

Contains Ω_2: By (1), we have $T(\omega_1; \Omega_2) = 1$ for all $\omega_1 \in \Omega_1$. Thus $T(\,\cdot\,; \Omega_2)$ is a constant function. Hence it is $\mathcal{A}_1$-measurable, and $\Omega_2 \in \mathscr{G}$.

Closure under proper difference: Now suppose B_1 and B_2 are elements of $\mathscr{G}$ with $B_1 \subset B_2$. Since B_1 and B_2 are elements of $\mathcal{A}_2$, their difference $B_2 \setminus B_1$ is an element of $\mathcal{A}_2$ because $\mathcal{A}_2$ is a σ-field. By (1), it follows $T(\omega_1; B_2 \setminus B_1) = T(\omega_1; B_2) - T(\omega_1; B_1)$ for all $\omega_1 \in \Omega_1$. Since, $T(\,\cdot\,; B_1)$ and $T(\,\cdot\,; B_2)$ are $\mathcal{A}_1$-measurable, then, by the corollary to Theorem 3.1.5 in Chung (the closure theorem), their difference, $T(\,\cdot\,; B_2 \setminus B_1)$, is $\mathcal{A}_1$-measurable and is therefore a member of $\mathscr{G}$.

Closure under increasing union: Finally, let $B_1, B_2, \cdots$ be a sequence of sets in $\mathscr{G}$ with $B_n \subset B_{n+1}$ for all $n \geq 1$. Since $\mathcal{A}_2$ is a σ-field, it contains the countable union of the B_n. By (1) and the properties of a probability measure (i.e. monotone sequential continuity from below), for all $\omega_1 \in \Omega_1$ it is true that

$$\lim_{n\to\infty} T(\omega_1; B_n) = T(\omega_1; \cup_n B_n)$$

and the limit exists because it is the limit of a bounded monotone increasing. By Theorem 13.4 in Billingsley, since the limit of $\mathcal{A}_1$-measurable $T(\,\cdot\,; B_n)$ exists everywhere (on Ω_1), then that limit, $T(\,\cdot\,; \cup_n B_n)$, is $\mathcal{A}_1$-measurable. Thus it is in $\mathscr{G}$. We can conclude that $\mathscr{G}$ is a λ-system.

By hypothesis, $\mathscr{P} \subset \mathscr{G}$, whence by the π-λ theorem we have $\mathcal{A}_2 = \sigma(\mathscr{P}) \subset \mathscr{G}$. By the definition of $\mathscr{G}$, it is clear $\mathscr{G} \subset \mathcal{A}_2$. Therefore $\mathscr{G} = \mathcal{A}_2$, which demonstrates (2). $\square$

6. Let $(\Omega_1, \mathcal{A}_1), (\Omega_2, \mathcal{A}_2)$ be measurable spaces, $\Omega = \Omega_1 \times \Omega_2$, $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$. Let M be a probability on $(\Omega_1, \mathcal{A}_1)$, and let $(T_{\omega_1})_{\omega_1 \in \Omega_1}$ be a transition probability from Ω_1 to Ω_2. Let $\mathscr{G} = \{A \in \mathcal{A} : \omega_1 \mapsto T_{\omega_1}(A_{\omega_1}) \text{ is } \mathcal{A}_1\text{-measurable}\}$. Show that

$$T_{\omega_1}((A_1 \times A_2)_{\omega_1}) = I_{A_1}(\omega_1) T_{\omega_1}(A_2). \tag{3}$$

Also show $\mathscr{G}$ contains the π-system $\mathcal{R}$.

Proof. Calculating,

$$T_{\omega_1}((A_1 \times A_2)_{\omega_1}) = T(\omega_1; (A_1 \times A_2)_{\omega_1}) = \begin{cases} T(\omega_1; \emptyset) = 0 & \text{if } \omega_1 \notin A_1 \\ T(\omega_1; A_2) = T_{\omega_1}(A_2) & \text{if } \omega_1 \in A_1 \end{cases}.$$

Since,

$$I_{A_1}(\omega_1) T_{\omega_1}(A_2) = \begin{cases} 0 & \text{if } \omega_1 \notin A_1 \\ T_{\omega_1}(A_2) & \text{if } \omega_1 \in A_1 \end{cases},$$

we have (3), as desired.

To show $\mathscr{G}$ contains $\mathcal{R}$, let $R = A_1 \times A_2 \in \mathcal{R}$ with $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$. Our task is to show that the function on Ω_1 that sends $\omega_1 \mapsto T_{\omega_1}(R_{\omega_1})$ is an $\mathcal{A}_1$-measurable function. By (3), we have

$$T_{\omega_1}(R_{\omega_1}) = T_{\omega_1}((A_1 \times A_2)_{\omega_1}) = I_{A_1}(\omega_1)T_{\omega_1}(A_2).$$

Since T_{ω_1} is a transition probability, condition (2) holds that $T(\,\cdot\,; A_2)$ is $\mathcal{A}_1$-measurable for each $A_2 \in \mathcal{A}_2$. In particular, for the A_2 in $R = A_1 \times A_2$, we have $T_{\omega_1}(A_2) = T(\,\cdot\,; A_2)$ is $\mathcal{A}_1$-measurable. Also since $A_1 \in \mathcal{A}_1$, the indicator I_{A_1} is $\mathcal{A}_1$-measurable. Note the product of $\mathcal{A}_1$-measurable functions is measurable by the corollary to Theorem 3.1.5 in Chung (the closure theorem). Hence $T_{\omega_1}(R_{\omega_1})$ is $\mathcal{A}_1$-measurable and $R \in \mathscr{G}$. Therefore, $\mathcal{R} \subset \mathscr{G}$. □

7. Notations are the same as Problem 6. Show that $\mathscr{G}$ is a λ-system.

Proof. **Contains Ω:** Since for all $\omega_1 \in \Omega_1$, we have $T_{\omega_1}(\Omega_{\omega_1}) = T_{\omega_1}(\Omega_2) = 1$, then $\omega_1 \mapsto T_{\omega_1}(\Omega_{\omega_1})$ is a constant function. Thus it is $\mathcal{A}_1$-measurable, and $\Omega \in \mathscr{G}$.

Closure under complementation: Let $A \in \mathscr{G}$. Thus $\omega_1 \mapsto T_{\omega_1}(A_{\omega_1})$ is $\mathcal{A}_1$-measurable. Then, since sectioning commutes with set operations,

$$\omega_1 \mapsto T_{\omega_1}((A^c)_{\omega_1}) = T_{\omega_1}((A_{\omega_1})^c) = 1 - T_{\omega_1}(A_{\omega_1})$$

is A_1-measurable by the closure theorem.

Closure under countable union of disjoint sets: Let $B_1, B_2, \cdots$ be mutually disjoint elements of $\mathscr{G}$. Then for all $\omega_1 \in \Omega_1$ and for all integers $I \geq 1$ we have

$$\begin{aligned} T_{\omega_1}\left(\left(\bigcup_{i=1}^{I} B_i\right)_{\omega_1}\right) &= T_{\omega_1}\left(\bigcup_{i=1}^{I} (B_i)_{\omega_1}\right) \quad \text{since sectioning commutes} \\ &= \sum_{i=1}^{I} T_{\omega_1}\left((B_i)_{\omega_1}\right) \end{aligned}$$

since $T(\omega_1;\,\cdot\,) = T_{\omega_1}(\,\cdot\,)$ is a probability measure on $(\Omega_2, \mathcal{A}_2)$ and the B_i are disjoint. For all ω_1,

$$\lim_{I\to\infty} T_{\omega_1}\left(\left(\bigcup_{i=1}^{I} B_i\right)_{\omega_1}\right) = \lim_{I\to\infty} \sum_{i=1}^{I} T_{\omega_1}\left((B_i)_{\omega_1}\right) \tag{4}$$

The series converges since it is the limit of a bounded (by 1), monotone increasing sequence of partial sums.

Calculating, for all ω_1

$$\begin{aligned} T_{\omega_1}\left(\left(\lim_{I\to\infty}\bigcup_{i=1}^{I} B_i\right)_{\omega_1}\right) &= T_{\omega_1}\left(\lim_{I\to\infty}\left(\bigcup_{i=1}^{I} B_i\right)_{\omega_1}\right) \quad \text{since sectioning commutes} \\ &= \lim_{I\to\infty} T_{\omega_1}\left(\left(\bigcup_{i=1}^{I} B_i\right)_{\omega_1}\right) \qquad (5) \end{aligned}$$

by monotone sequential continuity from below.

Therefore by (4) and (5) and Theorem 13.4 (ii) in Billingsley, we have

$$\omega_1 \mapsto T_{\omega_1}\left(\left(\lim_{I\to\infty}\bigcup_{i=1}^{I} B_i\right)_{\omega_1}\right)$$

is measurable. Hence, $\lim_{I\to\infty}\cup_{i=1}^{I} B_i \in \mathscr{G}$. We conclude $\mathscr{G}$ is a λ-system. □

8. Notations are the same as Problem 6. Let M be a probability on $(\Omega_1, \mathcal{A}_1)$. The set function MT, defined on $\mathcal{A}$ is

$$MT(A) := \int_{\Omega_1} T_{\omega_1}(A_{\omega_1})M(d\omega_1).$$

Furthermore, for any nonnegative $\mathcal{A}$-random variable X on Ω, the map TX on Ω_1 is defined as

$$\omega_1 \mapsto \int_{\Omega_2} X_{\omega_1}(\omega_2)T_{\omega_1}(d\omega_2)$$

Let $\mathcal{G}$ denote the collection of $\mathcal{A}$-measurable nonnegative random variables X such that $\omega_1 \mapsto \int_{\Omega_2} X_{\omega_1}(\omega_2)T_{\omega_1}(\mathrm{d}\omega_2)$ is A_1-measurable and $\langle MT, X\rangle = \langle M, TX\rangle$. Show that $I_A \in \mathcal{G}$ for every $A \in \mathcal{A}$.

Proof. Let $A \in \mathcal{A}$. By problem 2 of this homework, we have $(I_A)_{\omega_1} = I_{A_{\omega_1}}$. Thus

$$\begin{aligned} \omega_1 \mapsto \int_{\Omega_2} (I_A)_{\omega_1}(\omega_2)T_{\omega_1}(d\omega_2) &= \int_{\Omega_2} I_{A_{\omega_1}}(\omega_2)T_{\omega_1}(d\omega_2) \qquad (6) \\ &= \int_{A_{\omega_1}} T_{\omega_1}(d\omega_2) \qquad (7) \\ &= T_{\omega_1}(A_{\omega_1}) \quad \text{since } T_{\omega_1} \text{ is a probability measure} \qquad (8) \end{aligned}$$

is $\mathcal{A}_1$-measurable by the reasoning that follows after problem 7 in the course slides (that $\mathscr{G} = \mathcal{A}$).

For the second part, calculating gives

$$\begin{aligned}
\langle MT, I_A \rangle &= \int_\Omega I_A(\omega) MT(d\omega) \\
&= \int_A MT(d\omega) \\
&= MT(A) \quad \text{since } MT \text{ is a probability measure} \\
&= \int_{\Omega_1} T_{\omega_1}(A_{\omega_1}) M(d\omega_1) \quad \text{by the definition of } MT \\
&= \int_{\Omega_1} \left(\int_{\Omega_2} I_{A_{\omega_1}}(\omega_2) T_{\omega_1}(d\omega_2) \right) M(d\omega_1) \quad \text{by (6) = (8)} \\
&= \int_{\Omega_1} (TI_A)(\omega_1) M(d\omega_1) \quad \text{by the definition of } TI_A\text{, see bottom pg. 51} \\
&= \langle M, TI_A \rangle
\end{aligned}$$

Thus $I_A \in \mathcal{G}$. □

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Billingsley 2.9

Show that, if $B \in \sigma(\mathscr{A})$, then there exists a countable subclass $\mathscr{A}_B$ of $\mathscr{A}$ such that $B \in \sigma(\mathscr{A}_B)$.

Proof. If $\mathscr{A}$ is at most countable, then take $\mathscr{A}_B = \mathscr{A}$. So suppose $\mathscr{A}$ is uncountable. Define

$$\mathcal{Z} = \{\zeta : \zeta \subset \mathscr{A} \text{ and } \zeta \text{ is at most countable}\}$$

Here "at most countable" means empty, finite, or countable. Note $\mathcal{Z}$ is nonempty. Let

$$\mathcal{F} = \bigcup_{\zeta \in \mathcal{Z}} \sigma(\zeta)$$

Then $\mathcal{F}$ is union of all σ-fields generated by an element of $\mathcal{Z}$. The goal is to show that $\mathcal{F} = \sigma(\mathscr{A})$.

Given $\zeta \in \mathcal{Z}$, it follows that $\Omega \in \sigma(\zeta)$ by properties of σ-field. Therefore $\Omega \in \mathcal{F}$ because $\sigma(\zeta) \subset \mathcal{F}$.

Let $A \in \mathcal{F}$. Then exists a $\zeta_0 \in \mathcal{Z}$ such that $A \in \sigma(\zeta_0)$. By properties of σ-field, $A^c \in \sigma(\zeta_0)$. Thus

$$A^c \in \sigma(\zeta_0) \subset \mathcal{F}$$

Suppose $A_1, A_2, \cdots$ is a sequence of elements in $\mathcal{F}$. Then there exist $\zeta_1, \zeta_2, \cdots$ in $\mathcal{Z}$ such that $A_i \in \sigma(\zeta_i)$ for all i. Since the countable union of countable sets is itself countable, it follows that $\cup_{j=1}^{\infty} \zeta_j \in \mathcal{Z}$. For all i,

$$\zeta_i \subset \bigcup_{j=1}^{\infty} \zeta_j \subset \sigma\left(\bigcup_{j=1}^{\infty} \zeta_j\right) \subset \mathcal{F}$$

Therefore, since $\sigma(\zeta_i)$ is a subset of all σ-fields that contain ζ_i, it follows that

$$\sigma(\zeta_i) \subset \sigma\left(\bigcup_{j=1}^{\infty} \zeta_j\right)$$

for all i. Hence,

$$A_i \in \sigma(\zeta_i) \subset \sigma\left(\bigcup_{j=1}^{\infty} \zeta_j\right)$$

for all i. By properties of σ-field,

$$\bigcup_{i=1}^{\infty} A_i \in \sigma\left(\bigcup_{j=1}^{\infty} \zeta_j\right) \subset \mathcal{F}$$

Thus $\mathcal{F}$ is closed under countable union, and it has been demonstrated that $\mathcal{F}$ is a σ-field.

Suppose $A \in \mathscr{A}$. Then $\{A\} \in \mathcal{Z}$ and it follows that $A \in \sigma(\{A\}) \subset \mathcal{F}$. Thus $\mathcal{F}$ is a σ-field that contains $\mathscr{A}$. Hence $\sigma(\mathscr{A}) \subset \mathcal{F}$. Suppose $F \in \mathcal{F}$. Then there exists $\zeta \in \mathcal{Z}$ such that $F \in \sigma(\zeta)$. Since $\zeta \subset \mathscr{A} \subset \sigma(\mathscr{A})$, it follows that $\sigma(\zeta) \subset \sigma(\mathscr{A})$. Therefore, $F \in \sigma(\zeta) \subset \sigma(\mathscr{A})$ and $\mathcal{F} \subset \sigma(\mathscr{A})$. Since the set inclusion has been shown in both directions,

$$\mathcal{F} = \sigma(\mathscr{A})$$

As shown in the paragraph above, given any $B \in \mathcal{F} = \sigma(\mathscr{A})$, there exists $\zeta \in \mathcal{Z}$, some subset of $\mathscr{A}$, such that $B \in \sigma(\zeta)$. Define $\mathscr{A}_B := \zeta$, and it is clear that $\mathscr{A}_B$ is a countable subclass of $\mathscr{A}$. $\square$