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Assignment 1
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Homework #1

Suppose $X_1, X_2, \cdots$ are i.i.d. r.v.'s. Find necessary and sufficient conditions on the distribution of X_1 in order that

$$\frac{S_n}{n/\log(n)} \to 0 \text{ wp1} \tag{1}$$

Proof. We claim the condition is that for some $a > 0$.

$$E(|X|^{1+a}) < \infty \tag{2}$$

Now we show the sufficiency of the claim. Assume (2). We claim that for sufficiently large n,

$$\frac{n}{\log n} > n^{\frac{1}{1+a}} \tag{3}$$

To prove that claim, we simplify

$$\begin{aligned} \frac{n}{\log n} &> n^{\frac{1}{1+a}} \\ \frac{n}{n^{\frac{1}{1+a}}} &> \log n \\ n^{\frac{a}{1+a}} &> \log n \\ \log n &> \frac{1+a}{a} \log\log n \end{aligned}$$

which we know to be the case. Since by hyptothesis we have (2), it follows that $E|X| < \infty$. This allows us to assume without loss of generality that $E|X| = 0$. By KMZ SLLN and by (2), we have

$$\frac{S_n}{n^{\frac{1}{1+a}}} \xrightarrow{\text{a.s.}} 0$$

From (3) we have, for sufficiently large n,

$$\frac{S_n}{\frac{n}{\log n}} < \frac{S_n}{n^{\frac{1}{1+a}}} \xrightarrow{\text{a.s.}} 0.$$

Since almost sure convergence implies convergence in probability, we have thus proved (1).

Next we show the necessity of the claim. This, I do not know how to do, so I will leave it here. □

Acknowledgements

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