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550.621 Probability
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Assignment 1
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Assignment 1

An application of the strong law of large numbers

Suppose $X, X_1, X_2, \dots$ are i.i.d. r.v.'s. Find necessary and sufficient conditions on the distribution of X in order that

$$\frac{S_n}{n/\log n} \to 0 \quad \text{wp1} \tag{1}$$

where $S_n = \sum_{i=1}^n X_i$

Discussion

Notice $x/\log x$ is decreasing on the interval $x \in (0, e]$. To create an increasing function on $[0, \infty)$, define

$$f(x) = \begin{cases} x & \text{for } x \in [0, e) \\ \frac{x}{\log x} & \text{for } x \in [e, \infty) \end{cases}$$

Since $f'(x) = 1$ on $(0, e)$ and $f'(x) = (1/\log x)(1 - 1/\log x)$ on (e, ∞), it is clear f is strictly increasing, except at $x = e$ where the derivative does not exist. On (e, ∞), the function f grows sublinearly ($f'(x) < 1$ for all $x \in (e, \infty)$). Nonetheless, f is injective. Since f tends to infinity as x tends to infinity, f is also surjective. Therefore, f has an inverse f^{-1}, which sends $y \in [0, \infty)$ to $x \in [0, \infty)$ such that $f(x) = y$. A **fact** that will be needed later is that $f(n + 1)/f(n) \leq 2$ for all positive integers n. This follows because $f'(x) \leq 1$ for all $x \in (0, e) \cup (e, \infty)$.

It is proposed that the necessary and sufficient condition for (1) is that

$$E[f^{-1}(|X|)] < \infty \tag{2}$$

and

$$EX = 0 \tag{3}$$

Results given without proof

Chung 3.2.1

$$\sum_{n=1}^{\infty} P(|X| \geq n) \leq E(|X|) \leq 1 + \sum_{n=1}^{\infty} P(|X| \geq n)$$

First Borel-Cantelli Lemma

For arbitrary events $\{E_n\}$,

$$\sum_{n=1}^{\infty} P(E_n) < \infty \quad \Rightarrow \quad P(E_n \text{ i.o.}) = 0$$

Second Borel-Cantelli Lemma

If the events $\{E_n\}$ are independent, then

$$\sum_{n=1}^{\infty} P(E_n) = \infty \quad \Rightarrow \quad P(E_n \text{ i.o.}) = 1$$

Course Notes Lemma 3

Let $\{X_n\}_{n\geq 1}$ be a sequence of independent random variables with zero means (and finite variances). Then

$$\sum_{k=1}^{\infty} \operatorname{var}(X_k) < \infty$$

implies (finite variances and)

$$P\left\{\sum X_k \text{ converges to a finite limit}\right\} = 1$$

Kronecker's Lemma

Let $\{b_n\}_{n\geq 1}$ and $\{x_n\}_{n\geq 1}$ be two real sequences such that

(i) $\sum_{m=1}^{\infty} \frac{x_m}{b_m}$ exists and is finite

(ii) $b_n \uparrow \infty$

Then

(iii) $\frac{S_n}{b_n} \to 0$

where $S_n = \sum_{m=1}^n x_m$

Classical Strong Law of Large Numbers

Let $\{X_n\}$ be a sequence of independent and identically distributed r.v.'s. Then we have

$$E(|X_1|) < \infty \quad \Rightarrow \quad \frac{S_n}{n} \to E(X_1) \quad \text{a.e.}$$

Dominated Convergence Theorem

If $\lim_{n\to\infty} X_n = X$ a.e. or merely in measure on Λ and $\forall n : |X_n| \le Y$ a.e. on Λ, with $\int_\Lambda Y dP < \infty$, then

$$\lim_{n\to\infty} \int_\Lambda X_n dP = \int_\Lambda X dP = \int_\Lambda \lim_{n\to\infty} X_n dP$$

A Generalization of Cesaro's Theorem

If real numbers $x, x_1, x_2, \dots$ satisfy $x_n \to x$, and if $b_n \uparrow \infty$, then, with $b_0 := 0$

$$\frac{1}{b_n} \sum_{m=1}^n (b_m - b_{m-1}) x_m \to x \text{ as } n \to \infty$$

Lemmas

Lemma 1. *Fix $\omega \in \Omega$. Let $\{a_n\}$ be a sequence of positive numbers increasing to infinity. Then*

$$\frac{S_n}{a_n} \to 0 \quad \textit{implies} \quad \frac{X_n}{a_n} \to 0$$

Proof. Let $\epsilon > 0$ and let $S_n/a_n \to 0$. Since $S_n/a_n \to 0$, for $\epsilon/2$, there exists an integer $M(\epsilon/2)$ such that for all $m > M(\epsilon/2)$, it is true that $|S_m/a_m| < \epsilon/2$. Calculating with

$m > M(\epsilon/2)$,

$$\begin{aligned}
\left|\frac{X_{m+1}}{a_{m+1}}\right| &= \left|\frac{S_m}{a_{m+1}} + \frac{X_{m+1}}{a_{m+1}} - \frac{S_m}{a_{m+1}}\right| \\
&= \left|\frac{S_{m+1}}{a_{m+1}} - \frac{S_m}{a_{m+1}}\right| \\
&\leq \left|\frac{S_{m+1}}{a_{m+1}}\right| + \left|\frac{S_m}{a_{m+1}}\right| \text{ by triangle inequality} \\
&= \left|\frac{S_{m+1}}{a_{m+1}}\right| + \left|\frac{S_m}{a_m}\right|\left|\frac{a_m}{a_{m+1}}\right| \\
&\leq \left|\frac{S_{m+1}}{a_{m+1}}\right| + \left|\frac{S_m}{a_m}\right| \text{ since } \{a_n\} \text{ increases} \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Therefore, for all $m > M(\epsilon/2) + 1$ it is true that $|X_m/a_m| < \epsilon$. Since ϵ is arbitrary, it follows $X_n/a_n \to 0$, and the proof is concluded. □

Lemma 2. *Fix $\omega \in \Omega$. Let $\{a_n\}$ be a sequence of positive numbers increasing to infinity. Then*

$$\frac{S_n}{a_n} \to 0 \textit{ only if for every positive integer } N, \textit{ both } \frac{1}{a_n}\sum_{i=1}^{N} X_i \to 0 \textit{ and } \frac{1}{a_n}\sum_{i=N+1}^{n} X_i \to 0 \textit{ as } n \to \infty$$

Proof. Let $\epsilon > 0$ and let $S_n/a_n \to 0$. Since the sequence $\{S_n/a_n\}$ converges, it follows that $\{X_i\} < \infty$. Therefore, $\sum_{i=1}^{N} X_i$ is finite. Hence,

$$\frac{1}{a_n}\sum_{i=1}^{N} X_i \to 0 \text{ as } n \to \infty$$

because $a_n \to \infty$. By definition of limit, there exists N_1 and N_2 such that

$$\left|\frac{1}{a_n}\sum_{i=1}^{N} X_i\right| < \frac{\epsilon}{2}$$

for $n > N_1$, and

$$\left|\frac{1}{a_n}\sum_{i=1}^{n} X_i\right| < \frac{\epsilon}{2}$$

for $n > N_2$. Let $N_3 = \max(N_1, N_2)$. Therefore, for $n > N_3$

$$\begin{aligned}
\left|\frac{1}{a_n}\sum_{i=N+1}^{n} X_i\right| &= \left|\frac{1}{a_n}\sum_{i=1}^{n} X_i - \frac{1}{a_n}\sum_{i=1}^{N} X_i\right| \\
&\leq \left|\frac{1}{a_n}\sum_{i=1}^{n} X_i\right| + \left|\frac{1}{a_n}\sum_{i=1}^{N} X_i\right| \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Since ϵ is arbitrary, it follows $\frac{1}{a_n}\sum_{i=N+1}^{n} \to 0$ as $n \to \infty$, and the proof is concluded. $\square$

Lemma 3. *Suppose r.v.'s $\{X_n\}$ converge to r.v. X almost everywhere and r.v.'s $\{Y_n\}$ conerge to r.v. Y almost everywhere, then r.v.'s*

$$\{X_n + Y_n\} \text{ converge to } X + Y \text{ almost everwhere}$$

Proof. By definition of convergence almost everywhere, there exists a null set N_1 such that for all $\omega \in \Omega \setminus N_1$

$$\lim_{n\to\infty} X_n(\omega) = X(\omega)$$

Similarly, there exists a null set N_2 such that for all $\omega \in \Omega \setminus N_2$

$$\lim_{n\to\infty} Y_n(\omega) = Y(\omega)$$

Let $\epsilon > 0$. Let $\omega \in \Omega \setminus (N_1 \cup N_2)$. Then there exists M_1 such that for all $m > M_1$

$$|X_m(\omega) - X(\omega)| < \frac{\epsilon}{2}$$

and there exists M_2 such that for all $m > M_2$

$$|Y_m(\omega) - Y(\omega)| < \frac{\epsilon}{2}$$

Therefore for all $m > M_3 = \max(M_1, M_2)$

$$\begin{aligned} |X_m(\omega) + Y_m(\omega) - X(\omega) - Y(\omega)| &= |X_m(\omega) - X(\omega) + Y_m(\omega) - Y(\omega)| \\ &\le |X_m(\omega) - X(\omega)| + |Y_m(\omega) - Y(\omega)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Since ϵ is arbitrary and $N_1 \cup N_2$ is a null set, it follows that

$$\lim_{n\to\infty} X_n + Y_n = X + Y$$

almost everywhere. $\square$

Proof of the main result

Proof. ($\Rightarrow$) Let (1) hold. Then for $n \ge 3$

$$\frac{S_n}{n\log n} = \frac{S_n}{f(n)} \quad \text{so} \quad \frac{S_n}{f(n)} \to 0 \quad \text{wp1}$$

By **Lemma 1**,

$$\frac{X_n}{f(n)} \to 0 \tag{4}$$

almost surely. Define

$$E_k := \left\{ \left| \frac{X_k}{f(k)} \right| \geq 1 \right\}$$

By definition of convergence of the limit in (4), for all ω off a null set, there exists an integer $N(\omega)$ such that for $n > N(\omega)$ it is the case that

$$\left| \frac{X_n(\omega)}{f(n)} \right| < 1$$

Therefore, $P(E_k \text{ i.o.}) = 0$. Since X_k are independent by assumption, it follows that E_k are independent. Applying the contrapositive of the **Second Borel-Cantelli Lemma**,

$$\sum_{i=1}^{\infty} P(E_k) < \infty$$

Therefore, by **Chung 3.2.1**

$$\begin{aligned}
\infty &> 1 + \sum_{k=1}^{\infty} P(E_k) \\
&= 1 + \sum_{k=1}^{\infty} P\{|X_k| \geq f(k)\} \\
&= 1 + \sum_{k=1}^{\infty} P\{|X| \geq f(k)\} \text{ since } \{X_i\} \text{ are identically distributed} \\
&= 1 + \sum_{k=1}^{\infty} P\{f^{-1}|X| \geq k\} \text{ because } f^{-1} \text{ is increasing} \\
&\geq E[f^{-1}(|X|)]
\end{aligned}$$

Thus condition (2) is met.

Note $f^{-1}(x) = x$ for $x \in (0, e]$. For $x \in (e, \infty)$, the first derivative of $f(x)$ is strictly between 0 and 1. That means that $f(x) \leq x$, or equivalently $x \leq f^{-1}(x)$, for $x \in (0, \infty)$. Thus $E[|X|]$ is finite, i.e.

$$E|X| \leq E[f^{-1}(|X|)] < \infty \tag{5}$$

Therefore, by the **Classical Strong Law of Large Numbers**,

$$S_n/n \to EX \text{ a.s.} \tag{6}$$

Fix $\epsilon > 0$. Fix ω such that (1) and (6) hold. Then there exists N_1 such that for all $n > N_1$,

$$\left| \frac{S_n}{n} - EX \right| < \frac{\epsilon}{2} \quad \text{or equivalently} \quad \left| EX \log n - \frac{S_n}{n/\log n} \right| < \frac{\epsilon \log n}{2}$$

By (1) there exists N_2 such that for all $n > N_2$,

$$\left|\frac{S_n}{n/\log n}\right| < \frac{\epsilon}{2}$$

Therefore, for all $n > N_3 = \max(N_1, N_2, e)$,

$$\begin{aligned}
|EX \log n| &= \left|EX \log n - \frac{S_n}{n/\log n} + \frac{S_n}{n/\log n}\right| \\
&\leq \left|EX \log n - \frac{S_n}{n/\log n}\right| + \left|\frac{S_n}{n/\log n}\right| \text{ by triangle inequality} \\
&\leq \frac{\epsilon \log n}{2} + \frac{\epsilon}{2}
\end{aligned}$$

Dividing by $\log n$,

$$\begin{aligned}
|EX| &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2 \log n} \text{ and by definition of } N_3 \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Since ϵ is arbitrary, $|EX| = EX = 0$, i.e. (3).

Thus (1) implies (2) and (3). □

Proof. ($\Leftarrow$) Let (2) and (3) hold. Define

$$Y_n = X_n I_{\{|X_n| < f(n)\}}$$

Calculating,

$$
\begin{aligned}
\sum_{n=1}^{\infty} \operatorname{var}(Y_n/f(n)) &\le \sum_{n=1}^{\infty} E\left[\frac{Y_n^2}{f(n)^2}\right] \\
&= \sum_{n=1}^{\infty} E\left[\frac{(X_n I_{\{|X_n|<f(n)\}})^2}{f(n)^2}\right] \\
&= \sum_{n=1}^{\infty} E\left[\frac{(X I_{\{|X|<f(n)\}})^2}{f(n)^2}\right] \text{ since } X_n \text{ are i.i.d.} \\
&= E\left[\sum_{n=1}^{\infty} \frac{X^2 I_{\{|X|<f(n)\}}}{f(n)^2}\right] \text{ by Monotone Convergence Theorem} \\
&\le E\left[\sum_{n=1}^{2} \frac{f(n)^2}{f(n)^2} + \sum_{n=3}^{\infty} \frac{X^2 I_{\{|X|<f(n)\}}}{f(n)^2} \frac{f(n+1)^2}{f(n+1)^2}\right] \\
&= E\left[2 + X^2 \sum_{n=3}^{\infty} \frac{I_{\{f^{-1}(|X|)<n\}}}{f(n+1)^2} \frac{f(n+1)^2}{f(n)^2}\right] \\
&\le E\left[2 + 4X^2 \sum_{n=3}^{\infty} \frac{I_{\{f^{-1}(|X|)<n\}}}{f(n+1)^2}\right] \text{ by the } \textbf{fact} \text{ above} \\
&\le E\left[2 + 4X^2 \left(I_{\{f^{-1}(|X|)\ge 3\}} \int_{f^{-1}(|X|)}^{\infty} \frac{d\xi}{f(\xi)^2} + I_{\{f^{-1}(|X|)<3\}} \int_{3}^{\infty} \frac{d\xi}{f(\xi)^2}\right)\right] \\
&= 2 + E\left[4X^2 I_{\{f^{-1}(|X|)\ge 3\}} \int_{f^{-1}(|X|)}^{\infty} \frac{d\xi}{f(\xi)^2}\right] + E\left[4X^2 I_{\{f^{-1}(|X|)<3\}} \int_{3}^{\infty} \frac{d\xi}{f(\xi)^2}\right]
\end{aligned} \tag{7}
$$

For the first expectation in (7), let $Y = f^{-1}(|X|)$. Then $X^2 = f(Y)^2$. On the set $\{f^{-1}(|X|) \ge 3\} = \{Y \ge 3\}$, it is the case that $f(Y)^2 = Y^2/\log^2 Y$. Calculating,

$$
\begin{aligned}
E\left[4X^2 I_{\{f^{-1}(|X|)\ge 3\}} \int_{f^{-1}(|X|)}^{\infty} \frac{d\xi}{f(\xi)^2}\right] &= E\left[I_{\{Y\ge 3\}} 4 f(Y)^2 \int_{Y}^{\infty} \frac{\log^2 \xi}{\xi^2} d\xi\right] \\
&= E\left[I_{\{Y\ge 3\}} 4 \left(\frac{Y^2}{\log^2 Y}\right) \left(-\frac{\log^2 \xi + 2\log \xi + 2}{\xi}\Bigg|_{\xi=Y}^{\infty}\right)\right] \\
&= E\left[I_{\{Y\ge 3\}} 4 \left(\frac{Y^2}{\log^2 Y}\right) \frac{\log^2 Y + 2\log Y + 2}{Y}\right] \\
&= E\left[I_{\{Y\ge 3\}} \left(4Y + \frac{8Y}{\log Y} + \frac{8Y}{\log^2 Y}\right)\right] \\
&\le E[I_{\{Y\ge 3\}} 20Y] \\
&= 20E[I_{\{f^{-1}(|X|)\ge 3\}} f^{-1}(|X|)] \\
&\le 20E[f^{-1}(|X|)] \qquad \text{since } f^{-1}(|X|) \ge 0 \\
&< \infty
\end{aligned}
$$

For the second expectation in (7), over the set $\{f^{-1}(|X|) < 3\}$, it is the case that $0 \le X^2 \le f(3)^2 = 9/\log^2 3$. Thus

$$\begin{aligned}
E\left[4X^2 I_{\{f^{-1}(|X|)<3\}} \int_3^\infty \frac{d\xi}{f(\xi)^2}\right] &\le E\left[I_{\{f^{-1}(|X|)<3\}} 4\left(\frac{9}{\log^2 3}\right) \int_3^\infty \frac{\log^2 \xi}{\xi^2} d\xi\right] \\
&= E\left[I_{\{f^{-1}(|X|)<3\}} 4\left(\frac{9}{\log^2 3}\right) \left(-\frac{\log^2 \xi + 2\log \xi + 2}{\xi}\bigg|_{\xi=3}^{\infty}\right)\right] \\
&= E\left[I_{\{f^{-1}(|X|)<3\}} 4\left(\frac{9}{\log^2 3}\right) \frac{(\log^2(3/\log 3)) + 2\log(3/\log 3) + 2}{3/\log 3}\right] \\
&< \infty
\end{aligned}$$

Since all three terms in (7) are finite, their sum is finite, and it follows that

$$\sum_{n=1}^\infty \operatorname{var}(Y_n/f(n)) < \infty$$

As shown above, the series of the variances of the truncated $\{X_n/f(n)\}$ converges. By definition of Y_n, it is true that $|Y_n| < f(n)$, thus

$$|EY_n| \le E|Y_n| < f(n) < \infty \tag{8}$$

Hence the series of variances of the centered, truncated $\{X_n/f(n)\}$ also converges, i.e.

$$\sum_{n=1}^\infty \operatorname{var}\left(\frac{Y_n - EY_n}{f(n)}\right) = \sum_{n=1}^\infty \operatorname{var}(Y_n/f(n)) < \infty$$

By **Course Notes Lemma 3**,

$$P\left(\sum_{n=1}^\infty \frac{Y_n - EY_n}{f(n)} \text{ converges to a finite limit}\right) = 1$$

Then by **Kronecker's Lemma**,

$$\lim_{n\to\infty} \frac{1}{f(n)} \sum_{i=1}^n (Y_i - EY_i) = 0$$

almost surely. Calculating,

$$\begin{aligned}
\sum_{n=1}^\infty P\{Y_n \ne X_n\} &= \sum_{n=1}^\infty P\{|X_n| \ge f(n)\} \\
&= \sum_{n=1}^\infty P\{|X| \ge f(n)\} \\
&= \sum_{n=1}^\infty P\{f^{-1}(|X|) \ge n\} \\
&\le E[f^{-1}(|X|)] \\
&< \infty
\end{aligned}$$

By the **First Borel-Cantelli Lemma**, $P(Y_n \neq X_n \text{ i.o.}) = 0$. It follows that for ω off a null set there exists $N(\omega)$ with the property that for all $n > N(\omega)$, one has $Y_n(\omega) = X_n(\omega)$. By **Lemma 2**, as n increases to infinity,

$$\frac{1}{f(n)} \sum_{i=N(\omega)+1}^{n} (X_i(\omega) - EY_i) = \frac{1}{f(n)} \sum_{i=N(\omega)+1}^{n} (Y_i(\omega) - EY_i) \to 0$$

Since $E[f^{-1}(|X|)] < \infty$ by hypothesis, $E|X| < \infty$ by the reasoning in (5). Therefore $\{X_i\}$ is finite off a null set. For ω off the union of the previously two mentioned null sets, it follows that

$$\sum_{i=1}^{N(\omega)} X_i(\omega)$$

is finite. Since $|EY_i|$ is finite by (8), it follows that

$$\sum_{i=1}^{N(\omega)} (X_i(\omega) - EY_i)$$

is finite. Since $f(n)$ diverges to infinity,

$$\lim_{n\to\infty} \frac{1}{f(n)} \sum_{i=1}^{N(\omega)} (X_i(\omega) - EY_i) = 0$$

Combining the previous four statements, it is possible to conclude

$$\lim_{n\to\infty} \frac{1}{f(n)} \sum_{i=1}^{n} (X_i - EY_i) = \lim_{n\to\infty} \frac{1}{n/\log n} \sum_{i=1}^{n} (X_i - EY_i) = 0 \tag{9}$$

almost surely.

Next, the task is to show that

$$\lim_{n\to\infty} \frac{1}{f(n)} \sum_{i=1}^{n} EY_i = \lim_{n\to\infty} \frac{1}{n/\log n} \sum_{i=1}^{n} EY_i = 0 \tag{10}$$

First, it is claimed that

$$f^{-1}(x) = x \log f^{-1}(x)$$

for $x > e$. Calculating,

$$f^{-1}(f(x)) = f(x) \log f^{-1}(f(x)) = \frac{x}{\log x} \log x = x$$

ends the short proof. Notice that

$$(\log n) X I_{\{|X| \geq f(n)\}} = (\log n) X I_{\{f^{-1}(|X|) \geq n\}} \to X$$

pointwise as n tends to ∞. Furthermore, for $\omega \in \{f^{-1}(|X|) \geq \max(e, n)\}$ it is the case that

$$\begin{aligned} f^{-1}(|X|) &\geq n \\ \log f^{-1}(|X|) &\geq \log n \\ |X| \log f^{-1}(|X|) &\geq (\log n)|X| \\ f^{-1}(|X|) &\geq (\log n)|X| I_{\{f^{-1}(|X|) \geq n\}} \text{ by what was claimed above} \end{aligned}$$

For $\omega \in \{e \leq f^{-1}(|X|) < n\}$,

$$\begin{aligned} f^{-1}(|X|) &\geq 0 \\ f^{-1}(|X|) &\geq I_{\{f^{-1}(|X|) \geq n\}} \\ f^{-1}(|X|) &\geq (\log n)|X| I_{\{f^{-1}(|X|) \geq n\}} \end{aligned}$$

For $\omega \in \{f^{-1}(|X|) \leq e\}$,

$$|X| = f^{-1}(|X|) \geq (\log n)|X| I_{\{f^{-1}(|X|) \geq n\}}$$

since $0 \leq (\log n) I_{\{f^{-1}(|X|) \geq n\}} \leq 1$ on this set. Therefore, $f^{-1}(|X|)$ dominates $(\log n)|X| I_{\{f^{-1}(|X|) \geq n\}}$ for all n. Since $f^{-1}(|X|)$ is integrable by (2), by the **Dominated Convergence Theorem**

$$E[(\log n) X I_{\{f^{-1}(|X|) \geq n\}}] \to EX = 0 \quad \text{as} \quad n \to \infty$$

Note $EX = 0$ by (3). Calculating,

$$0 = EX = E[X I_{\{f^{-1}(|X|) \geq n\}} + X I_{\{f^{-1}(|X|) < n\}}]$$

so that by flip-flop

$$E[X I_{\{f^{-1}(|X|) \geq n\}}] = -E[X I_{\{f^{-1}(|X|) < n\}}] = -EY_n$$

By the previous calculations

$$\begin{aligned} -(\log n) EY_n &= (\log n) E[X I_{\{f^{-1}(|X|) \geq n\}}] \\ &= E[(\log n) X I_{\{f^{-1}(|X|) \geq n\}}] \to 0 \text{ as } n \to \infty \end{aligned}$$

Define $a_n = -(\log n) EY_n$. Since $\{a_n\} \to 0$, it follows that the sequence of absolute values $\{|a_n|\} = \{\log(n)|EY_n|\}$ also converges to 0. Let $b_0 = 0, b_1 = 1, b_2 = 2$ and $b_n = n/\log n$ for $n \geq 3$. Applying **A Generalization of Cesaro's Theorem**,

$$\frac{1}{b_n} \sum_{m=1}^{n} (b_m - b_{m-1})|a_m| \to 0 \qquad \text{as } n \to \infty$$

or equivalently (for $n \geq 3$),

$$\frac{1}{n/\log n} \left(\log 2 |EY_2| + (3 - 2\log 3)|EY_3| + \sum_{m=4}^{n} \left(\frac{m}{\log m} - \frac{m-1}{\log(m-1)} \right) (\log m)|EY_m| \right) \to 0 \tag{11}$$

as $n \to \infty$. Let $c_1 = 0, c_2 = \log 2, c_3 = 3 - 2\log 3$ and

$$c_n = \left(\frac{n}{\log n} - \frac{n-1}{\log(n-1)} \right) (\log n)$$

for $n \geq 4$. It can be shown that

$$\lim_{n\to\infty} c_n = 1 \tag{12}$$

by tedious algebra and successive applications of L'Hospital's rule. Also see WolframAlpha: `http://bit.ly/limX621`. Display (11) can be rewritten as

$$\frac{1}{n/\log n} \sum_{m=1}^{n} c_m |EY_m| \to 0 \quad \text{as } n \to \infty \tag{13}$$

for $n \geq 3$. Notice that all the terms in (13) are positive. Fix $1 > \epsilon > 0$. Then since (13) converges to 0, there exists $N_1 = \max(3, N_0)$ such that for all $n > N_1$,

$$\left| \frac{1}{n/\log n} \sum_{m=1}^{n} c_m |EY_m| \right| = \frac{1}{n/\log n} \sum_{m=1}^{n} c_m |EY_m| < \frac{(1-\epsilon)\epsilon}{2}$$

Since (12) converges to 1, there exists N_2 such that for all $n > N_2$,

$$|c_n - 1| < \epsilon \quad \text{or equivalently} \quad 1 - \epsilon < c_n < 1 + \epsilon$$

Thus for all $n > N_3 = \max(N_1, N_2)$,

$$\begin{aligned} \frac{(1-\epsilon)\epsilon}{2} &> \frac{1}{n/\log n} \sum_{m=N_3+1}^{n} c_m |EY_m| \\ &> \frac{1}{n/\log n} \sum_{m=N_3+1}^{n} (1-\epsilon)|EY_m| \\ &= \frac{(1-\epsilon)}{n/\log n} \sum_{m=N_3+1}^{n} |EY_m| \end{aligned}$$

Thus for $n > N_3$

$$\frac{1}{n/\log n} \sum_{m=N_3+1}^{n} |EY_m| < \frac{\epsilon}{2}$$

By (8) it is clear that $|EY_n| < f(n)$. Thus define M to be the sum of $f(m)$ for $1 \leq m \leq N_3$ so that

$$M = \sum_{m=1}^{N_3} f(m) > \sum_{m=1}^{N_3} |EY_m|$$

Clearly, M is finite. Since $n/\log n$ diverges to infinity, there exists N_4 such that for all $n > N_4$,

$$\frac{M}{n/\log n} = \left| \frac{M}{n/\log n} \right| < \frac{\epsilon}{2}$$

Therefore, for $n > N_5 = \max(N_4, N_3)$

$$\begin{aligned}
\left|\frac{1}{n/\log n}\sum_{m=1}^{n} EY_m\right| &\leq \frac{1}{n/\log n}\sum_{m=1}^{n}|EY_m| \quad \text{by triangle inequality}\\
&= \frac{1}{n\log n}\left(\sum_{m=1}^{N_3}|EY_m| + \sum_{m=N_3+1}^{n}|EY_m|\right)\\
&= \frac{1}{n\log n}\sum_{m=1}^{N_3}|EY_m| + \frac{1}{n\log n}\sum_{m=N_3+1}^{n}|EY_m|\\
&< \frac{1}{n\log n}M + \frac{1}{n\log n}\sum_{m=N_3+1}^{n}|EY_m|\\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Since ϵ is arbitrary, (10) holds. Combining (9) with (10) and using **Lemma 3**, it is clear that

$$\lim_{n\to\infty}\frac{1}{f(n)}\sum_{i=1}^{n}X_i = \lim_{n\to\infty}\frac{1}{n/\log n}\sum_{i=1}^{n}X_i = 0$$

almost surely. Or in other words,

$$\frac{S_n}{n/\log n} \to 0 \quad \text{wp1}$$

Thus (2) and (3) imply (1), quod erat demonstrandum. □

Conclusion

It is possible to find equivalent conditions to (2). First a lemma.

Lemma 4. *Suppose for $k \in \{1, 2\}$, the functions $\{f_k\}$ are asymptotically equivalent and real-valued on $[0, \infty)$. Suppose there exists x_0 such that each $|f_k|$ is bounded on $[0, x_0]$ and that each f_k is positive, unbounded, and monotone increasing on (x_0, ∞). Then for all random variables X*

$$E[f_1(|X|)] < \infty \quad \textit{if and only if} \quad E[f_2(|X|)] < \infty$$

Proof. By hypothesis, there exists finite, positive M_k such that $|f_k| < M_k$ on $[0, x_0]$ for $k \in \{1, 2\}$. Thus

$$-M_k < E[f_k(|X|)I_{\{|X|\leq x_0\}}] < M_k$$

Since,

$$E[f_k(|X|)] = E[f_k(|X|)I_{\{|X|\leq x_0\}}] + E[f_k(|X|)I_{\{|X|>x_0\}}],$$

it follows that

$$E[f_k(|X|)] \text{ is finite if and only if } E[f_k(|X|)I_{\{|X|>x_0\}}] \text{ is finite} \tag{14}$$

Fix $\epsilon > 0$. By hypothesis of asymptotic equivalence, there exists positive x_1 such that for all $x > x_1$,

$$\left|\frac{f_1(x)}{f_2(x)} - 1\right| \leq \epsilon$$

or equivalently

$$(1-\epsilon)f_2(x) \leq f_1(x) \leq (1+\epsilon)f_2(x) \tag{15}$$

Let $x_2 = \max(x_1, x_0)$. Then

$$\begin{aligned}
E[f_k(|X|)I_{\{|X|>x_0\}}] &= E[f_k(|X|)I_{\{x_2 \geq |X| > x_0\}}] + E[f_k(|X|)I_{\{|X|>x_2\}}] \\
&\leq E[f_k(x_2)] + E[f_k(|X|)I_{\{|X|>x_2\}}] \text{ by monotonicity} \\
&= f_k(x_2) + E[f_k(|X|)I_{\{|X|>x_2\}}]
\end{aligned}$$

Suppose $E[f_1(|X|)] < \infty$. Then $E[f_1(|X|)I_{\{|X|>x_0\}}] < \infty$ by (14) and

$$\begin{aligned}
E[f_2(|X|)I_{\{|X|>x_0\}}] &\leq f_2(x_2) + E[f_2(|X|)I_{\{|X|>x_2\}}] \\
&\leq f_2(x_2) + E\left[\frac{f_1(|X|)}{1-\epsilon}I_{\{|X|>x_2\}}\right] \text{ by (15)} \\
&\leq f_2(x_2) + \frac{1}{1-\epsilon}E\left[f_1(|X|)I_{\{|X|>x_0\}}\right] \text{ since } f_1 \text{ is positive for } x > x_0 \\
&< \infty
\end{aligned}$$

Therefore $E[f_2(|X|)] < \infty$ by (14).

Similarly, suppose $E[f_2(|X|)] < \infty$. Then $E[f_2(|X|)I_{\{|X|>x_0\}}] < \infty$ by (14) and

$$\begin{aligned}
E[f_1(|X|)I_{\{|X|>x_0\}}] &\leq f_1(x_2) + E[f_1(|X|)I_{\{|X|>x_2\}}] \\
&\leq f_1(x_2) + E\left[f_2(|X|)(1+\epsilon)I_{\{|X|>x_2\}}\right] \text{ by (15)} \\
&\leq f_1(x_2) + (1+\epsilon)E\left[f_2(|X|)I_{\{|X|>x_0\}}\right] \text{ since } f_2 \text{ is positive for } x > x_0 \\
&< \infty
\end{aligned}$$

Therefore, $E[f_1(|X|)] < \infty$ by (14). This concludes the proof of the lemma. □

The result of **Lemma 4** leads to the conclusion if g is asymptotically equivalent to f^{-1}—with g and f^{-1} positive, unbounded, and monotone increasing on (x_0, ∞) and bounded on $[0, x_0]$ for some postive x_0—then (2) is equivalent to $E[g(|X|)] < \infty$.

Let

$$y = f(x) = \frac{x}{\log x}$$

Then

$$\log y = \log x - \log(\log x)$$

It is a fact (that can be shown with L'Hospital's rule) that

$$\lim_{x\to\infty} \frac{\log(\log x)}{\log x} = 0$$

Therefore

$$\lim_{x\to\infty} \frac{\log y}{\log x} = \lim_{x\to\infty} \frac{\log x - \log(\log x)}{\log x} = 1$$

Hence

$$\begin{aligned} \log y &\sim \log x \\ y\log y &\sim y \log x \\ y \log y &\sim x \quad \text{by definition of } y \\ y \log y &\sim f^{-1}(y) \quad \text{by definition of } f \end{aligned}$$

Let $g(x) = x \log x$. It follows from the above that $f^{-1}(x) \sim g(x)$, therefore by **Lemma 4** (let $x_0 = e$ and all the conditions are met), (2) can be replaced by $E[g(|X|)] = E[|X| \log |X|] < \infty$.

Thus necessary and sufficient conditions on the distribution of X in order that

$$\frac{S_n}{n/\log n} \to 0 \quad \text{wp1}$$

are

$$E[|X| \log |X|] < \infty$$

and

$$EX = 0$$