

550.620 Final Examination, Fall 2012

Official Examination Policy

This take-home final examination for 550.620 Probability Theory is due *no later than 12:00:00 noon EST, Monday, December 17, 2012*. There are only three acceptable ways of turning in your exam: (1) Hand it to me in person, or (2) slide it under the door of my office, 306-F Whitehead Hall, or (3) submit it electronically to me at jimfill@jhu.edu. If you choose option (3), please *also* submit an electronic copy to Teaching Assistant Jason Matterer at jmatter4@jhu.edu.

You are not to discuss the exam with anyone except me before the due deadline. Conversation, even on the most casual level, about the exam will be considered a violation of the university's honor code.

Your work should be legibly written in complete English sentences (and paragraphs!). Every solution must be clearly explained. Be sure that your name is on every page of your solutions. Please write on only one side of each page.

You may consult your class notes and the course texts by Billingsley and Chung. You may also consult references on elementary probability and real analysis; these should not, however, make use of measure theory. *You may not use any other references of any kind.*

Point values for each question are shown in parentheses.

You will be graded on a 200-point basis. You MUST indicate which problem parts totaling 200 points you wish to have graded. Your indicated problem parts may not exceed the total of 200 points except due to discreteness; that is, if your total exceeds 200 points, it must be that exclusion of any one of your indicated parts would lower your total below the 200-point threshold.

Examination Problems (400 points total)

1. (70 total)

- (a) **(50)** Let X_n and X be finite real-valued random variables on a measure space $(\Omega, \mathcal{F}, \mu)$, and let $A \in \mathcal{F}$ with $\mu(A) < \infty$. Suppose that $X_n(\omega) \rightarrow X(\omega)$ for each $\omega \in A$. Prove that for each $\epsilon > 0$ there exists a subset $B \in \mathcal{F}$ of A such that $\mu(B) < \epsilon$ and $X_n(\omega) \rightarrow X(\omega)$ uniformly for $\omega \in A - B$. [HINT: Let $B_n^{(k)} := A \cap \cup_{p \geq n} \{|X - X_p| > 1/k\}$. Show (i) that $B_n^{(k)} \downarrow \emptyset$ for each k as $n \uparrow \infty$, (ii) that it is possible to choose a strictly increasing sequence of integers n_k so that $\mu(B_{n_k}^{(k)}) < \epsilon/2^k$ for each k , and (iii) that $B := \cup_{k=1}^{\infty} B_{n_k}^{(k)}$ does the job.]
- (b) **(20)** Show that (a) is false without the hypothesis $\mu(A) < \infty$. [HINT: On the integers with counting measure, consider the indicators of $\{n, n+1, \dots\}$.]

2. (70 total)

- (a) **(10)** On an arbitrary probability space, prove that if an event A is independent of a π -system \mathcal{P} and $A \in \sigma(\mathcal{P})$, then $P(A)$ is either 0 or 1.
- (b) **(15)** The goal of parts (b)–(d) is to prove the following theorem. Let f be a real-valued Borel function on $[0, 1]$ with arbitrarily small periods: For each ϵ there is a p with $0 < p < \epsilon$ such that $f(x) = f(x+p)$ for $0 \leq x \leq 1-p$. Then f is constant almost everywhere. Begin the proof here by showing that it is enough to prove that $P(f^{-1}B)$ is 0 or 1 for every Borel set B , where P is Lebesgue measure on the unit interval.
- (c) **(30)** Show that $f^{-1}B$ is independent of each interval $[0, x]$. [HINT: Let $A := f^{-1}B$ and suppose p is a (small) period of f . Let $m = \lfloor x/p \rfloor$ and $n = \lfloor 1/p \rfloor$. Using periodicity, prove that $P(A \cap [y, y+p])$ is the same for all periods $y \in [0, 1-p]$ and deduce $|P(A \cap [0, x]) - mP(A \cap [0, p])| \leq p$, $|P(A) - nP(A \cap [0, p])| \leq p$, and $|P(A \cap [0, x]) - P(A)x| \leq 2p + |x - m/n| \leq 3p$. Since p can be taken arbitrarily small, $P(A \cap [0, x]) = P(A)x$.]
- (d) **(5)** Conclude that $P(f^{-1}B)$ is 0 or 1.
- (e) **(10)** Show by example that f need not be constant.
3. **(70 total)** Let $Y_{k,n}$ be real-valued random variables, all defined on a common probability space, for $k \in \{1, 2, \dots\}$ and $n \in \{1, 2, \dots\}$. Fix two real numbers p and p' satisfying $1 \leq p < p' < \infty$ and suppose for some nonnegative sequences (b_k) and (b'_k) that
- (i) for each k the sequence $(Y_{k,1}, Y_{k,2}, \dots)$ is a martingale (see below),
 - (ii) for each k we have $\|Y_{k,n}\|_p \leq b_k$ for all $n \in \{1, 2, \dots\}$,
 - (ii') for each k we have $\|Y_{k,n}\|_{p'} \leq b'_k < \infty$ for all $n \in \{1, 2, \dots\}$, and

(iii) $\sum_{k=1}^{\infty} b_k < \infty$.

Prove the following six assertions. In solving a given part, you may use the results of any previous parts but not those of later parts. You don't need to know about martingales—these will be treated thoroughly next semester—but you do need to know, and may use without proof, the *almost sure convergence theorem for martingales*, which asserts that an L^1 -bounded martingale converges almost surely.

- (a) **(10)** For each k there exists a real-valued random variable $Y_{k,0}$ such that $Y_{k,n} \rightarrow Y_{k,0}$ almost surely as $n \rightarrow \infty$.
 - (b) **(10)** With $Y_{k,0}$ as in part (a), hypotheses (ii) and (ii') extend to $n = 0$.
 - (c) **(10)** For each $n \in \{0, 1, 2, \dots\}$ the series $\sum_{k=1}^{\infty} Y_{k,n}$ converges in L^p to some real-valued random variable S_n .
 - (d) **(15)** For each k , the sequences $|Y_{k,n}|^p$ and $|Y_{k,n} - Y_{k,0}|^p$ are both uniformly integrable in $n \in \{1, 2, \dots\}$.
 - (e) **(10)** For each k we have $Y_{k,n} \rightarrow Y_{k,0}$ in L^p as $n \rightarrow \infty$.
 - (f) **(15)** $S_n \rightarrow S_0$ in L^p as $n \rightarrow \infty$.
4. **(60)** Let (X_n) be a sequence of (not necessarily independent) identically distributed strictly positive random variables. For any ϕ such that $\liminf_{n \rightarrow \infty} \phi(n)/n = 0$, show that $P\{S_n > \phi(n) \text{ i.o.}\} = 1$, where $S_n := \sum_{k=1}^n X_k$ for each n . Deduce that $S_n \rightarrow \infty$ almost surely. [HINT: Let N_n denote the number of $k \leq n$ such that $X_k \leq 2\phi(n)/n$. Use Markov's inequality to estimate $P\{N_n \geq n/2\}$ and thereby obtain a lower bound on $P\{S_n > \phi(n)\}$ for each n .]
5. **(130 total) (On Moment Generating Functions)** Let X be a real-valued random variable defined on some probability space. The function Ψ , defined over $[-\infty, \infty]$ (i.e., for $t \in [-\infty, \infty]$) by

$$\Psi(t) := E(e^{tX}),$$

is called the *moment generating function* (mgf) of X . (Use our usual convention that $c \times \infty$ equals ∞ , if $c > 0$; equals 0, if $c = 0$; equals $-\infty$, if $c < 0$.)

- (a) **(15)** Show that $0 \leq \Psi \leq \infty$ over $[-\infty, \infty]$, that $0 < \Psi \leq \infty$ over $(-\infty, \infty)$, and that $\Psi(0) = 1$.
- (b) **(5)** Show that Ψ is convex over $(-\infty, \infty)$.
- (c) **(15)** Show that $T := \{t \in [-\infty, \infty] : \Psi(t) < \infty\}$ is a (possibly degenerate) interval containing 0. This interval T is called the *interval of finiteness* of Ψ .

Let a (resp., b) be the left (resp., right) endpoint of the interval of finiteness of Ψ .

- (d) **(20)** Show that Ψ is left (resp., right) continuous everywhere except possibly at a (resp., b).

- (e) **(25)** Illustrate the various possibilities for the behavior of Ψ at a and/or b using for X random variables with (at least) the following distributions: (i) Cauchy, (ii) positive part of a Cauchy, (iii) exponential, (iv) normal. Explicit computation of the mgfs, though desirable, is not necessary.

Suppose now that the interval (a, b) is nondegenerate.

- (f) **(25)** Show that Ψ is infinitely differentiable on (a, b) , with

$$\Psi^{(n)}(t) = E(X^n e^{tX}), \quad a < t < b,$$

and

$$\Psi^{(n)}(a+) := \lim_{t \downarrow a} \Psi^{(n)}(t) = E(X^n e^{aX}), \quad \Psi^{(n)}(b-) := \lim_{t \uparrow b} \Psi^{(n)}(t) = E(X^n e^{bX})$$

for all n . In particular, X^n is quasi-integrable, with

$$E(X^n) = \Psi^{(n)}(0(\pm))$$

for all n .

- (g) **(25)** For precisely which values of t is it true that

$$\Psi(t) = \sum_{n=0}^{\infty} E(X^n) t^n / n! \quad ? \tag{1}$$

Use (1) to compute the moment generating function of a standard normal random variable.

Problem 1

Part (a) 50 points

Let X_n and X be finite real-valued random variables on a measure space $(\Omega, \mathcal{F}, \mu)$, and let $A \in \mathcal{F}$ with $\mu(A) < \infty$. Suppose that $X_n(\omega) \rightarrow X(\omega)$ for each $\omega \in A$. Prove that for each $\epsilon > 0$ there exists a subset $B \in \mathcal{F}$ of A such that $\mu(B) < \epsilon$ and $X_n(\omega) \rightarrow X(\omega)$ uniformly for $\omega \in A - B$. [HINT: Let $B_n^{(k)} := A \cap \cup_{p \geq n} \{|X - X_p| > 1/k\}$. Show (i) that $B_n^{(k)} \downarrow \emptyset$ for each k as $n \uparrow \infty$, (ii) that it is possible to choose a strictly increasing sequence of integers n_k so that $\mu(B_{n_k}^{(k)}) < \epsilon/2^k$ for each k , and (iii) that $B := \cup_{k=1}^{\infty} B_{n_k}^{(k)}$ does the job.]

Proof. For this proof we will follow the guidelines of the hint. Let

$$B_n^{(k)} := A \cap \cup_{p \geq n} \{|X - X_p| > 1/k\}$$

To show decreasingness for each k , note

$$\begin{aligned} \cup_{p \geq n} \{|X - X_p| > 1/k\} &= \cup_{p=n} \{|X - X_p| > 1/k\} \cup \cup_{p \geq n+1} \{|X - X_p| > 1/k\} \\ &\supset \cup_{p \geq n+1} \{|X - X_p| > 1/k\} \end{aligned}$$

It is clear that all ω in A and in $\cup_{p \geq n+1} \{|X - X_p| > 1/k\}$ are also in A and in $\cup_{p \geq n} \{|X - X_p| > 1/k\}$. Hence,

$$B_n^{(k)} = A \cap \cup_{p \geq n} \{|X - X_p| > 1/k\} \supset A \cap \cup_{p \geq n+1} \{|X - X_p| > 1/k\} = B_{n+1}^{(k)},$$

and it follows that for each positive k , the series $\{B_n^{(k)}\}_n$ is decreasing.

Suppose that the limit of $\{B_n^{(k)}\}_n$ were not the empty set. Then there would be some $\omega \in B_n^{(k)}$ for all n . By hypothesis, $X_n(\omega) \rightarrow X(\omega)$ for each $\omega \in A$. By the definition of convergence, there exists N such that for all $n \geq N$, we have $|X_n(\omega) - X(\omega)| < 1/k$. Therefore $\omega \notin \{B_n^{(k)}\}_{n=N}^{\infty}$. This is a contradiction, and therefore the limit of $\{B_n^{(k)}\}_n$ is the empty set.

Summarizing up to this point, $B_n^{(k)} \downarrow \emptyset$ for each k as $n \uparrow \infty$.

Next we show it is possible to choose a strictly increasing sequence of integers n_k so that $\mu(B_{n_k}^{(k)}) < \epsilon/2^k$ for each k . By the corollary to theorem 3.1.5 in Chung (pg. 39), $|X - X_n|$ is a random variable for all n . Since $(1/k, \infty)$ is in the borel sets on the real line, it follows that the set $\{\omega : |X(\omega) - X_n(\omega)| \in (1/k, \infty)\}$ is a measurable set. Therefore, since $A \in \mathcal{F}$ and \mathcal{F} is closed under countable union and intersection, it is clear $B_n^{(k)} = A \cap \cup_{p \geq n} \{|X - X_p| > 1/k\}$ is measurable. Furthermore, $\emptyset \in \mathcal{F}$ and $\mu(B_1^{(k)}) < \mu(A) < \infty$. Now all the hypotheses of theorem 10.2 in Billingsley (pg 171) are met. Therefore, the conclusion $\mu(B_n^{(k)}) \downarrow \mu(\emptyset) = 0$, the limit in n , is true for all k . By the definition of convergence, given $\epsilon > 0$, there exists N_1 such that for all $n \geq N_1$, it follows $\mu(B_n^{(1)}) < \epsilon/2^1$. Set $n_1 := N_1$ for the sequence $\{n_k\}$. Next, there exists N_2 such that for all $n > N_2$, it follows $\mu(B_n^{(2)}) < \epsilon/2^2$. If $N_2 > N_1$, set $n_2 := N_2$. Else $N_2 \leq N_1$. It is true, then, that for all $n \geq N_1 + 1 > N_2$, it follows $\mu(B_n^{(2)}) < \epsilon/2^2$. Set

$n_2 := N_1 + 1 = n_1 + 1$. In other words, we define n_2 to be $\max(N_2, n_1 + 1)$. We can follow this process as k increases for all sequences $\{B_n^{(k)}\}$. For the next k in the sequence $\{n_k\}$, find N_k such that for all $n \geq N_k$, we have $\mu(B_n^{(k)}) < \epsilon/2^k$. Then set $n_k := \max(N_k, n_{k-1} + 1)$. Hence we constructively determine a strictly increasing sequence $\{n_k\}$ such that $\mu(B_{n_k}^{(k)}) < \epsilon/2^k$ for each k .

Now let $B = \cup_{k=1}^{\infty} B_{n_k}^{(k)}$. Note $B \in \mathcal{F}$ because it is the countable union of $B_{n_k}^{(k)}$, sets in \mathcal{F} . By subadditivity and by construction,

$$\mu(B) = \mu(\cup_{k=1}^{\infty} B_{n_k}^{(k)}) < \cup_{k=1}^{\infty} \mu(B_{n_k}^{(k)}) < \cup_{k=1}^{\infty} \epsilon/2^k = \epsilon$$

Finally we show that $X_n(\omega) \rightarrow X(\omega)$ uniformly for each $\omega \in A - B$. Let $\epsilon > 0$. There exists K such that for all $k \geq K$ we have $1/k < \epsilon$. We claim that for all $n \geq n_K$, the inequality $|X(\omega) - X_n(\omega)| < \epsilon$ holds for all $\omega \in A - B$. From our construction above, the set of all $\omega \in A$ such that $|X(\omega) - X_n(\omega)| \geq \epsilon$ for some $n \geq n_K$ is precisely

$$A \cap \cup_{p > n_K} \{|X - X_p| \geq \epsilon\} \subset A \cap \cup_{p > n_K} \{|X - X_p| > 1/K\} = B_{n_K}^{(K)} \subset B$$

Therefore, all the ‘offending’ sample points are in B . Hence, the claim is verified that $X_n(\omega) \rightarrow X(\omega)$ uniformly for all $\omega \in A - B$, as desired. \square

Problem 1

Part (b) 20 points

Show that (a) is false without the hypothesis $\mu(A) < \infty$. [HINT: On the integers with counting measure, consider the indicators of $\{n, n+1, \dots\}$.]

Proof. Proof by counterexample. Take Ω to be \mathbb{Z}^+ , the positive integers. Let \mathcal{F} to be the total σ -field of Ω . Define μ to be the counting measure. Then $(\Omega, \mathcal{F}, \mu)$ is a measure space (course slides, pg 126). Let $A = \Omega$. Let X_n be the indicator of the set $\{n, n+1, \dots\}$. Hence

$$X_n(\omega) = \begin{cases} 1 & \text{if } \omega \geq n \\ 0 & \text{if } \omega < n \end{cases}$$

Suppose $\omega = k \in \mathbb{Z}^+$. Then for all $n > k$, we have $X_n(\omega) = 0$. Since this is true for all ω , it is clear that the random variable $X(\omega) = 0$ for all $\omega \in \Omega$ is the limiting random variable. Hence $X_n(\omega) \rightarrow X(\omega)$ for each $\omega \in A$. Let $1 > \epsilon > 0$. Since for all sets $S \neq \emptyset$, we have $\mu(S) \geq 1$, it follows that the only set B with measure less than ϵ is the empty set. Hence $A - B = \Omega - \emptyset = \Omega$. It is left to show that $X_n(\omega)$ cannot converge to $X(\omega)$ uniformly on Ω .

We show the negation of the definition of uniform convergence on Ω is true. The negation is “there exists $\epsilon > 0$ such that for all positive N , there exists $n \geq N$ and there exists $\omega \in \Omega$ such that $|X(\omega) - X_n(\omega)| \geq \epsilon$. ”

Let $\epsilon = 0.5$. For all positive N , let $n = N$ and let $\omega = N$. Therefore

$$|X(\omega) - X_N(\omega)| = |X(N) - X_N(N)| = |0 - 1| = 1 \geq 0.5 = \epsilon.$$

This concludes the counterexample and shows that (a) is false without the hypothesis that $\mu(A) < \infty$. \square

Problem 2

Part (a) 10 points

On an arbitrary probability space, prove that if an event A is independent of a π -system \mathcal{P} and $A \in \sigma(\mathcal{P})$, then $P(A)$ is either 0 or 1.

Proof. For reference, we follow closely the proof of theorem 4.2 in Billingsley (pg 58). Let event A be independent of a π -system \mathcal{P} and let $A \in \sigma(\mathcal{P})$.

Let $\mathcal{L} = \{F \in \mathcal{F} : P(F \cap A) = P(F)P(A)\}$. By hypothesis, $\mathcal{P} \subset \mathcal{L}$. We claim \mathcal{L} is a λ -system. Clearly, $\Omega \in \mathcal{L}$, as $P(\Omega \cap A) = P(A) = P(\Omega)P(A)$. Suppose $B \in \mathcal{L}$. It follows

$$\begin{aligned} P(B \cap A) + P(B^c \cap A) &= P(A) \\ P(B \cap A) + P(B^c \cap A) &= P(A)(P(B) + P(B^c)) \\ P(B)P(A) + P(B^c \cap A) &= P(A)P(B) + P(A)P(B^c) \\ P(B^c \cap A) &= P(A)P(B^c), \end{aligned}$$

demonstrating that $B^c \in \mathcal{L}$. Suppose $B_1, B_2, \dots \in \mathcal{L}$ and $B_m \cap B_n = \emptyset$ for $m \neq n$. Then

$$\begin{aligned} P((\cup_n B_n) \cap A) &= P(\cup_n (B_n \cap A)) = \cup_n P(B_n \cap A) \text{ by additivity} \\ &= \cup_n P(B_n)P(A) = (\cup_n P(B_n))P(A) \\ &= P(\cup_n P(B_n))P(A) \text{ again by additivity,} \end{aligned}$$

demonstrating that $\cup_n B_n \in \mathcal{L}$. Thus \mathcal{L} is proven to be a λ -system.

By theorem 3.2 in Billingsley, \mathcal{L} contains $\sigma(\mathcal{P})$. Therefore, A is independent of $\sigma(\mathcal{P})$. In particular, since $A \in \sigma(\mathcal{P})$, we have $P(A) = P(A \cap A) = P(A)P(A)$. Thus, $P(A)$ is either 0 or 1. \square

Problem 2

Part (b) 15 points

The goal of parts (b)–(d) is to prove the following theorem. Let f be a real-valued Borel function on $[0, 1]$ with arbitrarily small periods: For each ϵ there is a p with $0 < p < \epsilon$ such that $f(x) = f(x + p)$ for $0 \leq x \leq 1 - p$. Then f is constant almost everywhere. Begin the proof here by showing that it is enough to prove that $P(f^{-1}B)$ is 0 or 1 for every Borel set B , where P is Lebesgue measure on the unit interval.

Proof. Let $P(f^{-1}B)$ be 0 or 1 for every Borel set B . We show constructively that f is constant almost everywhere. First, if $P(f^{-1}1) = 1$, then we are done, i.e. $f = 1$ almost everywhere. Otherwise, consider the interval $[0, 1)$, call this I . We generate a sequence of 0 and 1, call it $B = \{b_k\}$. Start this “loop.”

1. I is of the form $[a, b)$. Set $I_0 := [a, \frac{a+b}{2})$ and $I_1 := [\frac{a+b}{2}, b)$.
2. Note $P(f^{-1}I) = 1$. Thus $P(f^{-1}I_0) = 1$ or $P(f^{-1}I_1) = 1$ but not both. If $P(f^{-1}I_0) = 1$ and $P(f^{-1}I_1) = 0$, set $b_k := 0$ where this is the k -th iteration of this loop. Set $I := I_0$. Else $P(f^{-1}I_0) = 0$ and $P(f^{-1}I_1) = 1$. Set $b_k := 1$ where this is the k -th iteration of this loop. Set $I := I_1$.
3. Repeat the loop

For example, in the first loop, $I = [0, 1)$. From step one, we have $I_0 = [0, 1/2)$ and $I_1 = [1/2, 1)$. From the reasoning in step two, either $P(f^{-1}I_0) = 1$ or $P(f^{-1}I_1) = 1$, but not both. Whichever one it is (suppose it is I_1), set b_1 appropriately ($b_1 = 1$) and set I to that interval ($I := I_1$).

We claim B is a sequence of binary digits for a number $M \in [0, 1)$. In particular, $M = 0.b_1b_2b_3\cdots$. To prove the claim, first it is clear that $M \in [0, 1]$. We show $M \neq 1 = 0.111\cdots$ in the binary expansion. Suppose by way of contradiction $M = 0.111\cdots$. Then $B = \{1, 1, 1, \cdots\}$. By construction,

$$1 = P(f^{-1}[1/2, 1)) = P(f^{-1}[3/4, 1)) = P(f^{-1}[7/8, 1)) = \cdots$$

In other words, $P(f^{-1}[1 - (1/2)^k, 1)) = 1$ for all k . Notice as $k \uparrow \infty$, we have $[1 - (1/2)^k, 1) \downarrow \emptyset$. By monotone sequential continuity from above, $P(f^{-1}[1 - (1/2)^k, 1)) \downarrow 0$. This is a contradiction, because $P(f^{-1}[1 - (1/2)^k, 1)) = 1$ for all k , so the limit cannot be 0. Therefore, $M \neq 0.111\cdots$, and $M \in [0, 1)$.

Since we are constructively defining M , consider the first j elements of B . Those first j elements define a number by binary representation $0.b_1b_2\cdots b_j = \sum_{i=1}^j b_i(1/2)^i = M_j$. We prove by induction $P(f^{-1}[M_j, M_j + (1/2)^j)) = 1$. The base step is when $j = 1$. Therefore, $0.0 = 0 = M_1$ or $0.1 = 1/2 = M_1$. Case 1: $0.0 = 0 = M_1$. Then on the first iteration of the loop, $I = [0, 1)$, and $I_0 = [0, 0 + 1/2^1) = [0, 1/2)$, and $P(f^{-1}I_0) = 1$, as desired. Case

2: $0.1 = 1/2 = M_1$. Then in the first iteration of the loop $I = [0, 1)$ and $I_1 = [1/2, 1/2 + 1/2^1 = [1/2, 1)$, and $P(f^{-1}I_1) = 1$, as desired. This completes the base step. Now for the inductive step, assume $P(f^{-1}[M_k, M_k + (1/2)^k]) = 1$ for some k . We show the statement is true for $k + 1$. On the $k + 1$ -th iteration of the loop, $I = [M_k, M_k + (1/2)^k)$. Therefore, $I_0 = [M_k, M_k + (1/2)^{k+1})$ and $I_1 = [M_k + (1/2)^{k+1}, M_k + (1/2)^k)$. Case 1: $P(f^{-1}I_0) = 1$. Then $M_{k+1} = 0.b_1b_2 \cdots b_k0 = M_k$. Thus

$$P(f^{-1}[M_k, M_k + (1/2)^{k+1}]) = P(f^{-1}[M_{k+1}, M_{k+1} + (1/2)^{k+1}])$$

as desired. Case 2: $P(f^{-1}I_1) = 1$. Then $M_{k+1} = 0.b_1b_2 \cdots b_k1 = M_k + (1/2)^{k+1}$. Thus

$$P(f^{-1}[M_k + (1/2)^{k+1}, M_k + (1/2)^k]) = P(f^{-1}[M_{k+1}, M_{k+1} + (1/2)^{k+1}])$$

as desired. This completes the inductive step, and it follows that $P(f^{-1}[M_j, M_j + (1/2)^j]) = 1$ for all positive j by induction.

As $j \uparrow \infty$, the sets $[M_j, M_j + (1/2)^j) \downarrow \lim_{j \rightarrow \infty} \{M_j\} = \{M\}$. By monotone sequential continuity from above, $P(f^{-1}[M_j, M_j + (1/2)^j]) \downarrow P(f^{-1}\{M\})$. Since $P(f^{-1}[M_j, M_j + (1/2)^j]) = 1$ for all j , we have $P(f^{-1}\{M\}) = 1$. Therefore, $f = M$ with probability 1, meaning that f is constant almost everywhere. \square

Problem 2

Part (c) 30 points

Show that $f^{-1}B$ is independent of each interval $[0, x]$. [HINT: Let $A := f^{-1}B$ and suppose p is a (small) period of f . Let $m = \lfloor x/p \rfloor$ and $n = \lfloor 1/p \rfloor$. Using periodicity, prove that $P(A \cap [y, y+p])$ is the same for all periods $y \in [0, 1-p]$ and deduce $|P(A \cap [0, x]) - mP(A \cap [0, p])| \leq p$, $|P(A) - nP(A \cap [0, p])| \leq p$, and $|P(A \cap [0, x]) - P(A)x| \leq 2p + |x - m/n| \leq 3p$. Since p can be taken arbitrarily small, $P(A \cap [0, x]) = P(A)x$.]

Proof. We follow the outline in the hint. Let $A = f^{-1}B$. Suppose p , a period of f , is much smaller than x (so that $m = \lfloor x/p \rfloor \neq 0$).

First we prove a lemma. Let $S = A \cap [y, y+q]$ for $y \in [0, 1-q]$ and $0 \leq q \leq p$. We claim $S + np = A \cap [y + np, y + np + q]$ for integer n and $[y + np, y + np + q] \subset [0, 1]$. To show one set inclusion, suppose $x \in S + np$. It follows $x - np \in S$. Since $x - np \in A$, it is clear that $f(x - np) \in B$. By the periodicity of f , we have that $f(x) = f(x - np)$. Therefore, $f(x) \in B$ as well, and thus $x \in A$. Since $x - np \in S$, we have $x - np \in [y, y + q]$. Thus $x \in [y + np, y + np + q]$. It follows $x \in A \cap [y + np, y + np + q]$. Now to show the other set inclusion, suppose $x \in A \cap [y + np, y + np + q]$. Given $x \in A$, by the same periodicity argument above, $x - np \in A$. Furthermore, since $x \in [y + np, y + np + q]$, it follows $x - np \in [y, y + q]$. Thus, $x - np \in A \cap [y, y + q] = S$. Finally, we see that $x \in S + np$. This concludes the lemma that $S + np = A \cap [y + np, y + np + q]$. As an immediate consequence, notice $P(S) = P(S + np)$ because P is Lebesgue measure.

Next we show $P(A \cap [y, y+p]) = P(A \cap [0, p])$ for all $y \in [0, 1-p]$. Let $S = A \cap [y, y+p]$ and let $l = \lfloor y/p \rfloor$. By the consequence of the lemma, $P(S) = P(S - lp) = P(A \cap [y - lp, y - lp + p])$. By another application of the lemma, $P(A \cap [0, y - lp]) = P(A \cap [p, y - lp + p])$. From additivity,

$$\begin{aligned} P(A \cap [y, y+p]) &= P(S) = P(S - lp) \\ &= P(A \cap [y - lp, y - lp + p]) \\ &= P(A \cap [y - lp, p]) + P(A \cap [p, y - lp + p]) \\ &= P(A \cap [0, y - lp]) + P(A \cap [p, y - lp + p]) \\ &= P(A \cap [0, p]), \end{aligned}$$

as desired. For y and y' in $[0, 1-p]$, it is clear that $P(A \cap [y, y+p]) = P(A \cap [0, p]) =$

$P(A \cap [y', y' + p])$. As a consequence of this, by additivity it follows

$$\begin{aligned}
 P(A \cap [y, y + np]) &= P\left(A \cap \left(\bigcup_{k=1}^n [y + (k-1)p, y + kp]\right)\right) \\
 &= \bigcup_{k=1}^n P(A \cap [y + (k-1)p, y + kp]) \\
 &= \bigcup_{k=1}^n P(A \cap [y, y + p]) \\
 &= nP(A \cap [y, y + p]).
 \end{aligned}$$

Stated succinctly,

$$P(A \cap [y, y + np]) = nP(A \cap [y, y + p]) \quad (2)$$

Let $m = \lfloor x/p \rfloor$ and $n = \lfloor 1/p \rfloor$. Calculating,

$$\begin{aligned}
 0 &\leq P(A \cap [0, x]) - mP(A \cap [0, p]) \\
 &= P(A \cap [0, x]) - P(A \cap [0, mp]) \\
 &= P(A \cap [mp, x]) \\
 &\leq P([mp, x]) \\
 &\leq p,
 \end{aligned}$$

and

$$\begin{aligned}
 0 &\leq P(A) - nP(A \cap [0, p]) \\
 &= P(A) - P(A \cap [0, np]) \\
 &= P(A \cap [np, 1]) \\
 &\leq P([np, 1]) \\
 &\leq p.
 \end{aligned}$$

Thus $0 \leq P(A \cap [0, x]) - mP(A \cap [0, p]) \leq p$ and $0 \leq P(A) - nP(A \cap [0, p]) \leq p$. Adding these two inequalities together, we have

$$\begin{aligned}
 0 &\leq P(A \cap [0, x]) - mP(A \cap [0, p]) + P(A) - nP(A \cap [0, p]) \leq 2p \\
 0 &\leq P(A \cap [0, x]) - P(A \cap [0, mp]) + P(A) - P(A \cap [0, np]) \leq 2p \\
 0 &\leq P(A \cap [0, x]) - P(A \cap [0, mp]) + P(A \cap [np, 1]) \leq 2p \\
 0 &\leq P(A \cap [0, x]) - P(A \cap [0, mp]) + P(A \cap [0, 1 - np]) \leq 2p \text{ by (2)} \\
 0 &\leq P(A \cap [0, x]) - P(A \cap [1 - np, mp]) \leq 2p \\
 P(A \cap [1 - np, mp]) &\leq P(A \cap [0, x]) \leq 2p + P(A \cap [1 - np, mp]) \\
 P(A \cap [1 - np, mp]) - P(A)x &\leq P(A \cap [0, x]) - P(A)x \leq 2p + P(A \cap [1 - np, mp]) - P(A)x
 \end{aligned}$$

Consider the right-hand inequality.

$$\begin{aligned}
P(A \cap [0, x]) - P(A)x &\leq 2p + P(A \cap [1 - np, mp]) - P(A)x \\
&\leq 2p + P(A \cap [0, mp]) - P(A)x \\
&= 2p + mP(A \cap [0, p]) - P(A)x \\
&= 2p + m/nP(A \cap [0, np]) - P(A)x \\
&\leq 2p + m/nP(A) - P(A)x \\
&= 2p + (m/n - x)P(A) \\
&\leq 2p + (m/n - x) \\
&\leq 2p + |x - m/n|
\end{aligned}$$

Now consider the left-hand inequality from before.

$$\begin{aligned}
P(A \cap [0, x]) - P(A)x &\geq P(A \cap [1 - np, mp]) - P(A)x \\
&\geq -p + P(A \cap [0, mp]) - P(A)x \\
&= -p + mP(A \cap [0, p]) - P(A)x \\
&= -p + m/nP(A \cap [0, np]) - P(A)x \\
&\geq -2p + m/nP(A) - P(A)x \\
&= -2p + (m/n - x)P(A) \\
&\geq -2p - |m/n - x|P(A) \\
&\geq -2p - |m/n - x|
\end{aligned}$$

And combining the results of both those inequalities, we have

$$|P(A \cap [0, x]) - P(A)x| \leq 2p + |m/n - x|.$$

Now consider $|m/n - x|$. Note $m \in (x/p - 1, x/p]$ and $n \in (1/p - 1, 1/p]$. Calculating,

$$\begin{aligned}
\frac{x/p - 1}{1/p} &< m/n \\
\frac{x/p - p/p}{1/p} - p &< m/n - p \\
\frac{p(x - p)}{p} - p &< m/n - p \\
x - 2p &< m/n - p \\
x - m/n &< p
\end{aligned}$$

Now calculating some more,

$$\begin{aligned}\frac{x/p}{1/p - 1} &> m/n \\ \frac{x}{1 - p} &> m/n \\ x &> m/n - mp/n \\ x - m/n &> -mp/n \geq -p \text{ since } m \leq n\end{aligned}$$

Combining these two most recent inequalities, we have

$$|x - m/n| < p.$$

Now we can extend the inequality

$$|P(A \cap [0, x]) - P(A)x| \leq 2p + |m/n - x| < 3p$$

Since p can be taken arbitrarily small, we have $|P(A \cap [0, x]) - P(A)x| = 0$, so that $P(A \cap [0, x]) = P(A)x$. Therefore $A = f^{-1}B$ is independent of each interval $[0, x]$. \square

Problem 2

Part (d) 5 points

Conclude that $P(f^{-1}B)$ is 0 or 1.

Proof. Let $S = \{[0, x] : x \in [0, 1]\}$. Notice S is a π -system. We show that $\sigma(S)$ is the Borel σ -field on $[0, 1]$. We know from Billingsley, $T = \{(a, b] : a < b \text{ and } a, b \in [0, 1]\}$ generates the Borel σ -field on $(0, 1]$. Therefore, $T \cup \{0\}$ generates the Borel σ -field on $[0, 1]$. Thus it is left to us to show that $T \cup \{0\} \subset \sigma(S)$.

Certainly, $\{0\} = [0, 0] \in S \subset \sigma(S)$. Let $a < b$ and a and b be in $[0, 1]$. Using the set operations that σ -fields are closed under, we see $[0, a]^c = (a, 1] \in \sigma(S)$. Also, $[0, b] \in \sigma(S)$, so $[0, b] \cap (a, 1] = (a, b] \in \sigma(S)$. Therefore, $T \cup \{0\} \subset \sigma(S)$, and it follows that the smallest σ -field containing S is the Borel σ -field. Finally, since B is a Borel set and f is a measurable function, $f^{-1}B$ is also a Borel set, meaning $f^{-1}B \in \sigma(S)$.

By problem 2 part (b), the event $f^{-1}B$ is independent of S , a π -system. Furthermore, $f^{-1}B \in \sigma(S)$. By problem 2 part (a), it follows $P(f^{-1}B)$ is either 0 or 1. \square

Problem 2

Part (e) 10 points

Show by example that f need not be constant.

Proof. Let f be the measurable function $[0, 1] \rightarrow [0, 1]$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

For every $\epsilon > 0$, there exists a rational number $0 < q < \epsilon$. Remember that a rational plus a rational is a rational and an irrational plus a rational is an irrational. Thus

$$f(x + q) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases},$$

and we have $f(x) = f(x + q)$ for $0 \leq x \leq 1 - q$. Hence f is constant on the set of irrational numbers, which is to say almost everywhere. However, f is not constant. \square

Problem 3

Let $Y_{k,n}$ be real-valued random variables, all defined on a common probability space, for $k \in \{1, 2, \dots\}$ and $n \in \{1, 2, \dots\}$. Fix two real numbers p and p' satisfying $1 \leq p < p' < \infty$ and suppose for some nonnegative sequences (b_k) and (b'_k) that

- (i) for each k the sequence $(Y_{k,1}, Y_{k,2}, \dots)$ is a martingale (see below),
- (ii) for each k we have $\|Y_{k,n}\|_p \leq b_k$ for all $n \in \{1, 2, \dots\}$,
- (ii') for each k we have $\|Y_{k,n}\|_{p'} \leq b'_k < \infty$ for all $n \in \{1, 2, \dots\}$, and
- (iii) $\sum_{k=1}^{\infty} b_k < \infty$.

Prove the following six assertions. In solving a given part, you may use the results of any previous parts but not those of later parts. You don't need to know about martingales—these will be treated thoroughly next semester—but you do need to know, and may use without proof, the *almost sure convergence theorem for martingales*, which asserts that an L^1 -bounded martingale converges almost surely.

Part (a) 10 points

For each k there exists a real-valued random variable $Y_{k,0}$ such that $Y_{k,n} \rightarrow Y_{k,0}$ almost surely as $n \rightarrow \infty$.

Proof. From course slides pg. 178, we have $\|X\|_p$ increases with $0 < p \leq \infty$. Therefore, for each k , we have $\|Y_{k,n}\|_1 \leq \|Y_{k,n}\|_p \leq b_k$ (because $1 \leq p$ by hypothesis). By the theorem above, since each sequence $\{Y_{k,n}\}_n$ is a martingale and bounded in L^1 , it converges almost surely. Call the limit $Y_{k,0}$. □

Problem 3

Part (b) 10 points

With $Y_{k,0}$ as in part (a), hypotheses (ii) and (ii') extend to $n = 0$.

Proof. By hypothesis, for each k , we have $\|Y_{k,n}\|_p \leq b_k$ for all n . Hence, $0 \leq (\int Y_{k,n}^p)^{1/p} \leq b_k$, whence $0 \leq \int Y_{k,n}^p \leq b_k^p$. We now use Pratt's theorem, pg. 162 of the course slides, with b_k^p as the upper bound (which is constant in the limiting variable n) and 0 as the lower bound. Thus, applying the limit to the latter inequality we have

$$\begin{array}{ccccc} \lim_n 0 & \leq & \lim_n \int Y_{k,n}^p & \leq & \lim_n b_k^p \\ & & \int Y_{k,0}^p & \leq & b_k^p \end{array}$$

Therefore, $\|Y_{k,0}\|_p = (\int Y_{k,0}^p)^{1/p} \leq b_k$. We can do the exact same thing replacing p with p' and b_k with b'_k . That gives $\|Y_{k,0}\|_{p'} \leq b'_k$. Thus it is demonstrated hypotheses (ii) and (ii') extend to $n = 0$. □

Problem 3

Part (c) 10 points

For each $n \in \{0, 1, 2, \dots\}$ the series $\sum_{k=1}^{\infty} Y_{k,n}$ converges in L^p to some real-valued random variable S_n .

Proof. Define the partial series sequence $\{B_m\} = \sum_{k=1}^m Y_{k,n}$. Since $p \geq 1$, we know L^p is a normed linear space. Let $t \geq M$ and $t' \geq M$, and without loss of generality, assume $t' \geq t$. By the reverse triangle inequality,

$$| \|B_{t'}\|_p - \|B_t\|_p | \leq \|B_{t'} - B_t\|_p = \left\| \sum_{k=t+1}^{t'} Y_{k,n} \right\|_p$$

Let $\epsilon > 0$. Since $\sum_{k=1}^{\infty} b_k < \infty$ and since $b_k \geq 0$, there exists K such that $\sum_{k=K+1}^{\infty} b_k < \epsilon$. Therefore, for all $t \geq K$ and $t' \geq K$, and without loss of generality $t' \geq t$, we have

$$\begin{aligned} | \|B_{t'}\|_p - \|B_t\|_p | &\leq \left\| \sum_{k=t+1}^{t'} Y_{k,n} \right\|_p \\ &\leq \left\| \sum_{k=t+1}^{\infty} Y_{k,n} \right\|_p \\ &\leq \sum_{k=t+1}^{\infty} \|Y_{k,n}\|_p \\ &\leq \sum_{k=t+1}^{\infty} b_k \\ &\leq \epsilon, \end{aligned}$$

demonstrating the Cauchy criterion for convergence for $\{\|B_m\|_p\}$. By pg. 180 of the course slides, we know that Cauchy sequences converge in L^p . Thus $\{\|B_m\|_p\}$ converges to some limit, call it S_n . \square

Problem 3

Part (e) 10 points

For each k we have $Y_{k,n} \rightarrow Y_{k,0}$ in L^p as $n \rightarrow \infty$.

Proof. Since $Y_{k,n} \rightarrow Y_{k,0}$ almost surely for each k , we have $Y_{k,n} \rightarrow Y_{k,0}$ in probability by pg 176 of the course slides. By problem 3 part (d), we have that $|Y_{k,n} - Y_{k,0}|^p$ is uniformly integrable. Therefore,

$$\sup_n E(|Y_{k,n} - Y_{k,0}|^p I_{|Y_{k,n} - Y_{k,0}|^p > A}) \downarrow 0 \text{ as } A \uparrow \infty.$$

Since p is positive, this is the same as

$$\sup_n E(|Y_{k,n} - Y_{k,0}|^p I_{|Y_{k,n} - Y_{k,0}| \geq A}) \downarrow 0 \text{ as } A \uparrow \infty.$$

Uniform integrability implies L^p dominatedness. Our technique is truncation. For each $A > 0$

$$\begin{aligned} \|Y_{k,n} - Y_{k,0}\|_p^p &= E|Y_{k,n} - Y_{k,0}|^p = E(|Y_{k,n} - Y_{k,0}|^p I_{|Y_{k,n} - Y_{k,0}| \geq A}) + E(|Y_{k,n} - Y_{k,0}|^p I_{|Y_{k,n} - Y_{k,0}| < A}) \\ &\leq E(|Y_{k,n} - Y_{k,0}|^p I_{|Y_{k,n} - Y_{k,0}| \geq A}) + A^p \end{aligned}$$

By uniform integrability the first term on the right tends to 0 as $n \rightarrow \infty$. The second term goes to 0 as $A \rightarrow 0$. Thus $Y_{k,n} \rightarrow Y_{k,0}$ in L^p as $n \rightarrow \infty$. \square

Problem 5

Let X be a real-valued random variable defined on some probability space. The function Ψ , defined over $[-\infty, \infty]$ (i.e., for $t \in [-\infty, \infty]$) by

$$\Psi(t) := E(e^{tX}),$$

is called the *moment generating function* (mgf) of X . (Use our usual convention that $c \times \infty$ equals ∞ , if $c > 0$; equals 0, if $c = 0$; equals $-\infty$, if $c < 0$.)

Part (a) 15 points

Show that $0 \leq \Psi \leq \infty$ over $[-\infty, \infty]$, that $0 < \Psi \leq \infty$ over $(-\infty, \infty)$, and that $\Psi(0) = 1$.

Proof. Let X have corresponding probability measure P . Then

$$\Psi(t) = E(e^{tX}) = \int e^{tX} P\{X \in dx\}$$

Since $e^{tx} \geq 0$, for x in the extended reals (using the convention that $e^\infty = \infty$ and $e^{-\infty} = 0$), it follows immediately that $\int e^{tX} P\{X \in dx\} \geq 0$. Hence, $0 \leq \Psi \leq \infty$ over $[-\infty, \infty]$. Similarly, $e^{tx} > 0$ for x in the real line, and $\int e^{tX} P\{X \in dx\} > 0$. And, $0 < \Psi \leq \infty$ over $(-\infty, \infty)$. Finally

$$\Psi(0) = E(e^{0X}) = E(e^0) = E(1) = \int 1 = 1.$$

□

Problem 5

Part (b) 5 points

Show that Ψ is convex over $(-\infty, \infty)$.

Proof. Let $t \in [0, 1]$ and $x_1, x_2 \in (-\infty, \infty)$ and $x_1 \leq x_2$. Note that e^x is convex over $(-\infty, \infty)$. Hence

$$e^{(1-t)x_1X+tx_2X} \leq (1-t)e^{x_1X} + te^{x_2X}.$$

Now taking expectations of both sides (properties of the integral preserve inequality), we have

$$\int e^{(1-t)x_1X+tx_2X} \leq \int (1-t)e^{x_1X} + te^{x_2X},$$

and simplifying gives

$$\int e^{(1-t)x_1X+tx_2X} \leq (1-t) \int e^{x_1X} + t \int e^{x_2X},$$

which is the same as

$$\Psi((1-t)x_1 + tx_2) \leq (1-t)\Psi(x_1) + t\Psi(x_2).$$

Thus Ψ is convex over $(-\infty, \infty)$. □