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550.621 Probability
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Assignment 3
March 26, 2014

Assignment 3

Chung 6.4.19

Problem

The problem is to provide a solution to Chung Exercise 6.4.19, which is:

> Reformulate Exercise 18 in terms of d.f.'s and deduce the following consequence. Let F_n be a sequence of d.f.'s a_n, a'_n real constants, $b_n > 0$, $b'_n > 0$. If
>
> $$F_n(b_n x + a_n) \xrightarrow{v} F(x) \quad \text{and} \quad F_n(b'_n x + a'_n) \xrightarrow{v} F(x),$$
>
> where F is a nondegenerate d.f., then
>
> $$\frac{b_n}{b'_n} \to 1 \quad \text{and} \quad \frac{a_n - a'_n}{b_n} \to 0$$

First, **Chung Exercise 6.4.18** is stated and the reformulation is presented. Following that, the consequence is deduced.

Chung Exercise 6.4.18

Let f_n be ch.f.'s. Let f and g be two nondegenerate ch.f.'s. Suppose that there exist real constants a_n and $b_n > 0$ such that for every t:

$$f_n(t) \to f(t) \quad \text{and} \quad e^{ita_n/b_n} f_n\left(\frac{t}{b_n}\right) \to g(t) \tag{1}$$

Then $a_n \to a$, $b_n \to b$, where a is finite, $0 < b < \infty$, and $g(t) = e^{ita/b} f(t/b)$.

Reformulation of 6.4.18 in terms of d.f.'s

Let Φ_n be d.f.'s with ch.f.'s f_n. Let Φ and G be two nondegenerate d.f.'s with characteristic functions f and g, respectively. Suppose that there exist real constants α_n and $\beta_n > 0$ such that:

$$\Phi_n(x) \xrightarrow{v} \Phi(x) \quad \text{and} \quad \Phi_n(\beta_n x - \alpha_n) \xrightarrow{v} G(x) \tag{2}$$

Then $\alpha_n \to \alpha$, $\beta_n \to \beta$, where α is finite, $0 < \beta < \infty$, and $G(x) = \Phi(\beta x - \alpha)$.

Proof. Let $\Phi_n(x) \xrightarrow{v} \Phi(x)$. Then by definition, the corresponding p.m.'s on $\mathbb{R}$ converge weakly, and by the convergence theorem (**Chung Theoremm 6.3.1**), $f_n(t) \to f(t)$ for every t.

Let $\Phi_n(\beta_n x - \alpha_n) \xrightarrow{v} G(x)$. Since $\beta_n > 0$, it follows that for all n, the function $T_n(x) = \beta_n x - \alpha_n$ is a continuous, increasing bijection. Therefore, $\Phi_n(T_n(x)) = \Phi_n(\beta_n x - \alpha_n)$ is a distribution function. The characteristic function of $\Phi_n(\beta_n x - \alpha_n)$ is

$$\begin{aligned} \int e^{itx} d\Phi_n(\beta_n x - \alpha_n) &= \int e^{it\frac{x+\alpha_n}{\beta_n}} d\Phi_n(x) \\ &= e^{it\alpha_n/\beta_n} \int e^{i\frac{t}{\beta_n}x} d\Phi_n(x) \\ &= e^{it\alpha_n/\beta_n} f_n\left(\frac{t}{\beta_n}\right) \end{aligned}$$

Since $\Phi_n(\beta_n x - \alpha_n) \xrightarrow{v} G(x)$, the corresponding p.m.'s on $\mathbb{R}$ converge weakly. Thus by the convergence theorem (**Chung Theorem 6.3.1**), the ch.f.'s converge for every t, i.e.

$$e^{it\alpha_n/\beta_n} f_n\left(\frac{t}{\beta_n}\right) \to g(t)$$

for every t.

The above shows that the set of assumptions in (2) imply the set of assumptions in (1). Therefore, given (2), it follows by **Chung Exercise 6.4.18** that $\alpha_n \to \alpha$, $\beta_n \to \beta$, where α is finite, $0 < \beta < \infty$, and $g(t) = e^{it\alpha/\beta} f(t/\beta)$. Additionally, by **Chung Theorem 6.3.2**, the sequence of d.f.'s $\{\Phi_n(\beta_n x - \alpha_n)\}_{n \geq 1}$ converges weakly to a d.f. G with ch.f. $e^{it\alpha/\beta} f(t/\beta)$. Notice that $\Phi(\beta x - \alpha)$ has ch.f. $e^{it\alpha/\beta} f(t/\beta)$. Since any two d.f.'s with the same ch.f. are the same d.f. by **Chung Theorem 6.2.2**, it follows that $G(x) = \Phi(\beta x - \alpha)$. □

Deduction of the Consequence

Proof. Let F_n be a sequence of d.f.'s a_n, a_n' real constants $b_n > 0$, $b_n' > 0$. Let

$$F_n(b_n x + a_n) \xrightarrow{v} F(x) \quad \text{and} \quad F_n(b_n' x + a_n') \xrightarrow{v} F(x)$$

where F is a nondegenerate d.f. Since $b_n > 0$, for all n it follows that $T_n(x) = b_n x + a_n$ is a continuous, increasing bijection. Therefore $\Phi_n(x) = F_n(T_n(x)) = F_n(b_n x + a_n)$ is a d.f. Using this new notation, $\Phi_n(x) \xrightarrow{v} F(x)$. Define

$$\begin{aligned}\beta_n &:= \frac{b_n'}{b_n}\\ \alpha_n &:= -\frac{a_n' - a_n}{b_n}\end{aligned}$$

Notice β_n is positive and α_n is some real number. Calculating,

$$\begin{aligned}\Phi_n(\beta_n x - \alpha_n) &= F_n(T_n(\beta_n x - \alpha_n))\\ &= F_n(b_n(\beta_n x - \alpha_n) + a_n)\\ &= F_n\left(b_n\left(\frac{b_n'}{b_n}x + \frac{a_n' - a_n}{b_n}\right) + a_n\right)\\ &= F(b_n' x + a_n')\end{aligned}$$

By hypothesis then, $\Phi_n(\beta_n x - \alpha_n) = F(b_n' x + a_n') \xrightarrow{v} F(x)$. By the **Reformulation of 6.4.18**,

$$\lim_{n\to\infty} \alpha_n = \alpha$$

where α is finite, and

$$\lim_{n\to\infty} \beta_n = \beta$$

where $0 < \beta < \infty$. Furthermore, as a result of the **Reformulation of 6.4.18**, the weak limit of $\{\Phi_n(\beta_n x - \alpha_n)\}_{n\geq 1}$ is $F(x) = F(\beta x - \alpha)$. Since this equality is true for all x and since F is nondegenerate, by **Billingsley Section 14, Lemma 5** it follows that $\beta = 1$ and $\alpha = 0$.

Therefore,

$$\lim_{n\to\infty} -\frac{a_n' - a_n}{b_n} = \lim_{n\to\infty} \frac{a_n - a_n'}{b_n} = \lim_{n\to\infty} \alpha_n = \alpha = 0$$

and

$$\lim_{n\to\infty} \frac{b_n'}{b_n} = \lim_{n\to\infty} \beta_n = \beta = 1,$$

whence

$$\frac{b_n}{b_n'} \to 1$$

as desired. $\square$