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Assignment 9
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Homework #9
Chung 4.4.6

Let the r.v.'s $\{X_\alpha\}$ have the p.m.'s $\{\mu_\alpha\}$. If for some real $r > 0$, $\mathscr{E}\{|X_\alpha|^r\}$ is bounded in α, then $\{\mu_\alpha\}$ is tight.

Proof. Let the r.v.'s $\{X_\alpha\}$ have the p.m.'s $\{\mu_\alpha\}$. Let $\mathscr{E}\{|X_\alpha|^r\}$ be bounded in α for some real $r > 0$. Hence, for all α, from their index set A, we have

$$0 \leq \mathscr{E}\{|X_\alpha|^r\} = \int |X|^r d\mu_\alpha < M \tag{1}$$

for some finite, positive M.

Suppose by way of contradiction that $\{\mu_\alpha\}$ is not tight. It follows from negating the definition of tightness (Chung 94) that there exists some $\epsilon > 0$ such that for all finite intervals I we have

$$\inf_{\alpha \in A} \mu_\alpha(I) \leq 1 - \epsilon. \tag{2}$$

Since $\{\mu_\alpha\}$ are probability measures, $\mu_\alpha(I) = 1 - \mu_\alpha(I^c)$. Remember that for real-valued sets S, it is true that $\inf -S = -\sup S$. Hence (2) becomes

$$\begin{aligned}
\inf_{\alpha \in A} \mu_\alpha(I) &\leq 1 - \epsilon \\
\inf_{\alpha \in A} (1 - \mu_\alpha(I^c)) &\leq 1 - \epsilon \\
1 + \inf_{\alpha \in A} -\mu_\alpha(I^c) &\leq 1 - \epsilon \\
1 - \sup_{\alpha \in A} \mu_\alpha(I^c) &\leq 1 - \epsilon \\
\sup_{\alpha \in A} \mu_\alpha(I^c) &\geq \epsilon.
\end{aligned}$$

Therefore, there must be some $\alpha' \in A$, such that

$$\mu_{\alpha'}(I^c) > \epsilon/2 \tag{3}$$

for all finite intervals I.

Let I be the finite interval $I_b = (-b, b)$ for a positive real b. Thus $I_b^c = (-\infty, -b] \cup [b, \infty)$. Notice that $\mathscr{P}\{|X_{\alpha'}| \geq b\} = \mu_{\alpha'}(I_b^c)$. Using inequality (3) and applying Chebyshev's inequality (Chung 51),

$$\frac{\epsilon}{2} < \mu_{\alpha'}(I_b^c) = \mathscr{P}\{|X_{\alpha'}| \geq b\} \leq \frac{\mathscr{E}\{|X_{\alpha'}|^r\}}{b^r}.$$

After rearranging,

$$\frac{\epsilon b^r}{2} < \mathscr{E}\{|X_{\alpha'}|^r\}. \tag{4}$$

Since b is restricted to be positive real choose $b = (2M/\epsilon)^{1/r}$. Now (4) becomes

$$M = \frac{\epsilon}{2}\frac{2M}{\epsilon} = \frac{\epsilon}{2}\left(\left(\frac{2M}{\epsilon}\right)^{1/r}\right)^r = \frac{\epsilon b^r}{2} < \mathscr{E}\{|X_{\alpha'}|^r\},$$

a contradiction of inequality (1).

Therefore, given that for some real $r > 0$, $\mathscr{E}\{|X_\alpha|^r\}$ is bounded in α, we conclude that $\{\mu_\alpha\}$ is tight. $\square$

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