James K. Pringle
550.621 Probability
Dr. Jim Fill
Assignment 4
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Chung Exercise 7.2.4

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Prove the sufficiency part of Theorem 7.2.1 without using Theorem 7.1.2, but by elaborating the proof of the latter.

Chung Theorem 7.2.1

Assume $\sigma_{nj}^2 < \infty$ for each n and j and the reduction hypotheses

$$\sum_{j=1}^{k_n} \sigma^2(X_{nj}) = 1 \tag{1}$$

$$E(X_{nj}) = 0 \tag{2}$$

of Sec 7.1. In order that as $n \to \infty$ the two conclusions below both hold

(I) S_n converges in dist. to Φ (standard normal)

(II) the double array (2) of Sec. 7.1 is holospoudic,

it is necessary and sufficient that for each $\eta > 0$, we have

$$\sum_{j=1}^{k_n} \int_{|x|>\eta} x^2 dF_{nj}(x) \to 0 \tag{3}$$

Assumptions

By way of notation, let $F_{nj}(x)$ be the d.f. of X_{nj}. Assume the hypotheses of **Chung Theorem 7.2.1** and assume (3). Note by (1)

$$1 = \sum_{j=1}^{k_n} \sigma^2(X_{nj}) = \sum_{j=1}^{k_n} \int x^2 dF_{nj}(x) = \sum_{j=1}^{k_n} \left(\int_{|x|\leq\eta} + \int_{|x|>\eta} \right) x^2 dF_{nj}(x)$$

so that

$$0 = \lim_{n\to\infty} \sum_{j=1}^{k_n} \int_{|x|>\eta} x^2 dF_{nj}(x) = 1 - \lim_{n\to\infty} \sum_{j=1}^{k_n} \int_{|x|\leq\eta} x^2 dF_{nj}(x)$$

Thus

$$\lim_{n\to\infty} \sum_{j=1}^{k_n} \int_{|x|\leq\eta} x^2 dF_{nj}(x) = 1 \tag{4}$$

Discussion

The proof examines the convergence of ch.f.'s of the rows of the double array. By hypothesis, S_n is the sum of the k_n independent random variables in the n-th row. By independence, the ch.f. of S_n is the product of the ch.f.'s of the random variables of that row. Define f_n to be the ch.f. of S_n. Define f_{nj} to be the ch.f. of X_{nj}

$$f_n(t) = f_{S_n}(t) = \prod_{j=1}^{k_n} f_{X_{nj}}(t) = \prod_{j=1}^{k_n} f_{nj}(t)$$

By **Chung Theorem 6.3.2** it is sufficient to show for all $t \in \mathbb{R}$ that $f_n(t)$ converges to the ch.f. of standard normal, $e^{-\frac{t^2}{2}}$ in order that S_n converges in distribution to Φ.

Next follow some lemmas that are required for the proof.

Lemma 1

Let $\{\theta_{nj}, 1 \leq j \leq k_n, 1 \leq n\}$ be a double array of complex numbers satisfying the following conditions as $n \to \infty$

(i) $\max_{1\leq j\leq k_n} |\theta_{nj}| \to 0$

(ii) $\sum_{j=1}^{k_n} |\theta_{nj}| \leq M < \infty$, where M does not depend on n

(iii) $\sum_{j=1}^{k_n} \theta_{nj} \to \theta$, where θ is a (finite) complex number.

Then we have

$$\prod_{j=1}^{k_n}(1+\theta_{nj}) \to e^{\theta}$$

Lemma 2

According to Billingsley 26.4_1

$$|e^{ix}-(1+ix)| \le \min\{\frac{1}{2}x^2, 2|x|\}$$

Lemma 3

According to Billingsley 26.4_2

$$|e^{ix}-(1+ix-\frac{1}{2}x^2)| \le \min\{\frac{1}{6}|x|^3, x^2\}$$

Proof of the Main Result

Fix $t \in \mathbb{R}$. Notice

$$f_n(t) = \prod_{j=1}^{k_n} f_{nj}(t) = \prod_{j=1}^{k_n} 1 + (f_{nj}(t) - 1)$$

To apply **Lemma 1**, let $\theta_{nj} = f_{nj}(t) - 1$.

Condition (i)

Let $\eta > 0$. Calculating,

$$\begin{aligned}
\max_{1\le j\le k_n} |\theta_{nj}| &= \max_{1\le j\le k_n} |f_{nj}(t) - 1| \\
&= \max_{1\le j\le k_n} |f_{nj}(t) - 1 - itE(X_{nj})| \quad \text{by (2)} \\
&= \max_{1\le j\le k_n} |\int (e^{itx} - 1 - itx) dF_{nj}(x)| \\
&\le \max_{1\le j\le k_n} \int |e^{itx} - 1 - itx| dF_{nj}(x) \quad \text{by modulus inequality} \\
&\le \max_{1\le j\le k_n} \int \frac{1}{2}(tx)^2 dF_{nj}(x) \quad \text{by } \textbf{Lemma 2} \\
&= \max_{1\le j\le k_n} \frac{1}{2}t^2 \left(\int_{|x|\le\eta} x^2 dF_{nj}(x) + \int_{|x|>\eta} x^2 dF_{nj}(x) \right) \\
&\le \frac{1}{2}t^2\eta^2 + \frac{1}{2}t^2 \max_{1\le j\le k_n} \int_{|x|>\eta} x^2 dF_{nj}(x) \\
&\le \frac{1}{2}t^2\eta^2 + \frac{1}{2}t^2 \sum_{j=1}^{k_n} \int_{|x|>\eta} x^2 dF_{nj}(x)
\end{aligned}$$

since the max of a set of positive numbers is less than or equal to the sum of all of them. Thus, by (3),

$$0 \le \lim_{n\to\infty} \left(\max_{1\le j\le k_n} |\theta_{nj}| \right) \le \frac{1}{2}t^2\eta^2 + \frac{1}{2}t^2 \lim_{n\to\infty} \left(\sum_{j=1}^{k_n} \int_{|x|>\eta} x^2 dF_{nj}(x) \right) = \frac{1}{2}t^2\eta^2$$

Since there is no dependence on η in the second term above, and since η is arbitrary and positive,

$$0 \le \lim_{n\to\infty} \left(\max_{1\le j\le k_n} |\theta_{nj}| \right) \le \lim_{\eta\to 0} \left(\frac{1}{2}t^2\eta^2 \right) = 0$$

Hence **Condition (i)** holds.

Condition (ii)

Calculating,

$$\begin{aligned}
\sum_{j=1}^{k_n} |\theta_{nj}| &= \sum_{j=1}^{k_n} |f_{nj}(t) - 1| \\
&\leq \sum_{j=1}^{k_n} \int \frac{1}{2}(tx)^2 dF_{nj}(x) \quad \text{by same calculations as for } \textbf{Condition (i)} \\
&= \frac{1}{2}t^2 \sum_{j=1}^{k_n} \int x^2 dF_{nj}(x) \\
&= \frac{1}{2}t^2 \sum_{j=1}^{k_n} \sigma^2(X_{nj}) \\
&= \frac{1}{2}t^2 \quad \text{by (1).}
\end{aligned}$$

Since this bound does not depend on n and it is finite, **Condition (ii)** holds.

Condition (iii)

Let $\eta > 0$. It is claimed that $\theta = -\frac{t^2}{2}$, so that

$$\sum_{j=1}^{k_n} \theta_{nj} \to \theta.$$

Calculating,

$$
\begin{aligned}
\left|\left(\sum_{j=1}^{k_n}\theta_{nj}\right)-\theta\right| &= \left|\sum_{j=1}^{k_n}(f_{nj}(t)-1)+\frac{t^2}{2}\right| \\
&= \left|\sum_{j=1}^{k_n}(f_{nj}(t)-1)-\frac{(it)^2}{2}\left(\sum_{j=1}^{k_n}\sigma^2(X_{nj})\right)\right| \quad \text{by (1)} \\
&= \left|\sum_{j=1}^{k_n}(f_{nj}(t)-1-itE(X_{nj}))-\frac{(it)^2}{2}\left(\sum_{j=1}^{k_n}E(X_{nj}^2)\right)\right| \quad \text{by (2)} \\
&= \left|\sum_{j=1}^{k_n}\left(f_{nj}(t)-1-itE(X_{nj})-\frac{(it)^2}{2}E(X_{nj}^2)\right)\right| \\
&= \left|\sum_{j=1}^{k_n}\left(\int e^{itx}-1-itx+\frac{(tx)^2}{2}dF_{nj}(x)\right)\right| \\
&\leq \sum_{j=1}^{k_n}\int\left|e^{itx}-1-itx+\frac{(tx)^2}{2}\right|dF_{nj}(x) \quad \text{by triangle and modulus inequalities} \\
&= \sum_{j=1}^{k_n}\left(\int_{|x|\leq\eta}\left|e^{itx}-1-itx+\frac{(tx)^2}{2}\right|dF_{nj}(x)+\int_{|x|>\eta}\left|e^{itx}-1-itx+\frac{(tx)^2}{2}\right|dF_{nj}(x)\right) \\
&\leq \sum_{j=1}^{k_n}\left(\int_{|x|\leq\eta}\frac{|x|^3}{6}dF_{nj}(x)+\int_{|x|>\eta}x^2dF_{nj}(x)\right) \quad \text{by } \textbf{Lemma 3} \\
&\leq \frac{\eta}{6}\sum_{j=1}^{k_n}\int_{|x|\leq\eta}x^2dF_{nj}(x)+\sum_{j=1}^{k_n}\int_{|x|>\eta}x^2dF_{nj}(x)
\end{aligned}
$$

Taking the limit of this nonegative quantity as n tends to infinity,

$$
\begin{aligned}
0 &\leq \lim_{n\to\infty}\left|\left(\sum_{j=1}^{k_n}\theta_{nj}\right)-\theta\right| \\
&\leq \lim_{n\to\infty}\left(\frac{\eta}{6}\sum_{j=1}^{k_n}\int_{|x|\leq\eta}x^2dF_{nj}(x)+\sum_{j=1}^{k_n}\int_{|x|>\eta}x^2dF_{nj}(x)\right) \\
&= \frac{\eta}{6}\lim_{n\to\infty}\left(\sum_{j=1}^{k_n}\int_{|x|\leq\eta}x^2dF_{nj}(x)\right)+\lim_{n\to\infty}\left(\sum_{j=1}^{k_n}\int_{|x|>\eta}x^2dF_{nj}(x)\right) \\
&= \frac{\eta}{6} \quad \text{by (4) and (3)}
\end{aligned}
$$

Since η is arbitrary and positive, by squeezing,

$$
0 \leq \lim_{n\to\infty}\left|\left(\sum_{j=1}^{k_n}\theta_{nj}\right)-\theta\right| \leq \lim_{\eta\to 0}\frac{\eta}{6} = 0
$$

and it follows that

$$\lim_{n\to\infty}\left|\left(\sum_{j=1}^{k_n}\theta_{nj}\right)-\theta\right|=0$$

Hence, $\sum_{j=1}^{k_n}\theta_{nj}$ converges to $\theta=-\frac{t^2}{2}$ as n tends to infinity. **Condition (iii)** holds. Therefore by **Lemma 1**,

$$\lim_{n\to\infty}f_{S_n}(t)=\lim_{n\to\infty}f_n(t)=\prod_{j=1}^{k_n}1+(f_{nj}(t)-1)=\prod_{j=1}^{k_n}1+\theta_{nj}=e^{\theta}=e^{-\frac{t^2}{2}}$$

Note $t\in\mathbb{R}$ is arbitrary. Thus for all $t\in\mathbb{R}$ it is the case that $f_{S_n}(t)$ converges to the ch.f. of standard normal, $e^{-\frac{t^2}{2}}$, and by **Chung Theorem 6.3.2** S_n converges in distribution to Φ. Thus (I).

Condition (II), holospoudicity, follows from **Chung Theorem 7.1.1** since **Condition (i)** holds. Alternatively, the proof of holospoudicity copies over from Chung since it does not rely on **Chung Theorem 7.1.2**. Therefore, (3) is sufficient for (I) and (II).

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