JAMES K. PRINGLE
550.621 Probability
Dr. Jim Fill
Assignment 2
February 28, 2014

Assignment 2

Billingsley 2.9

Show that, if $B \in \sigma(\mathscr{A})$, then there exists a countable subclass $\mathscr{A}_B$ of $\mathscr{A}$ such that $B \in \sigma(\mathscr{A}_B)$.

Proof. Implicit in the statement of the problem is that $\mathscr{A}$ is a subset of the power set of Ω.

If $\mathscr{A}$ is countable, then take $\mathscr{A}_B = \mathscr{A}$. So suppose $\mathscr{A}$ is uncountable. Define

$$\mathcal{Z} = \{\zeta : \zeta \subset \mathscr{A} \text{ and } \zeta \text{ is countable}\}$$

Note $\mathcal{Z}$ is nonempty since $\mathscr{A}$ is nonempty. Let

$$\mathcal{F} = \bigcup_{\zeta \in \mathcal{Z}} \sigma(\zeta)$$

Then $\mathcal{F}$ is union of all σ-algebras generated by an element of $\mathcal{Z}$. The goal is to show that $\mathcal{F} = \sigma(\mathscr{A})$.

Given $\zeta \in \mathcal{Z}$, it follows that $\Omega \in \sigma(\zeta)$ by properties of σ-algebra. Therefore $\Omega \in \mathcal{F}$ because $\sigma(\zeta) \subset \mathcal{F}$.

Let $A \in \mathcal{F}$. Then exists a $\zeta_0 \in \mathcal{Z}$ such that $A \in \sigma(\zeta_0)$. By properties of σ-algebra, $A^c \in \sigma(\zeta_0)$. Thus

$$A^c \in \sigma(\zeta_0) \subset \mathcal{F}$$

Suppose $A_1, A_2, \cdots$ is a sequence of elements in $\mathcal{F}$. Then there exist $\zeta_1, \zeta_2, \cdots$ in $\mathcal{Z}$ such that $A_i \in \sigma(\zeta_i)$ for all i. Since the countable union of countable sets is itself countable, it follows that $\cup_{j=1}^{\infty} \zeta_j \in \mathcal{Z}$. For all i,

$$\zeta_i \subset \bigcup_{j=1}^{\infty} \zeta_j \subset \sigma\left(\bigcup_{j=1}^{\infty} \zeta_j\right) \subset \mathcal{F}$$

Therefore, since $\sigma(\zeta_i)$ is a subset of all σ-algebras that contain ζ_i, it follows that

$$\sigma(\zeta_i) \subset \sigma\left(\bigcup_{j=1}^{\infty} \zeta_j\right)$$

for all i. Hence,

$$A_i \in \sigma(\zeta_i) \subset \sigma\left(\bigcup_{j=1}^{\infty} \zeta_j\right)$$

for all i. By properties of σ-field,

$$\bigcup_{i=1}^{\infty} A_i \in \sigma\left(\bigcup_{j=1}^{\infty} \zeta_j\right) \subset \mathcal{F}$$

Thus $\mathcal{F}$ is closed under countable union, and it has been demonstrated that $\mathcal{F}$ is a σ-algebra.

Suppose $A \in \mathscr{A}$. Then $\{A\} \in \mathcal{Z}$ and it follows that $A \in \sigma(\{A\}) \subset \mathcal{F}$. Thus $\mathcal{F}$ is a σ-field that contains $\mathscr{A}$. Hence $\sigma(\mathscr{A}) \subset \mathcal{F}$. Suppose $F \in \mathcal{F}$. Then there exists $\zeta \in \mathcal{Z}$ such that $F \in \sigma(\zeta)$. Since $\zeta \subset \mathscr{A} \subset \sigma(\mathscr{A})$, it follows that $\sigma(\zeta) \subset \sigma(\mathscr{A})$. Therefore, $F \in \sigma(\zeta) \subset \sigma(\mathscr{A})$ and $\mathcal{F} \subset \sigma(\mathscr{A})$. Since the set inclusion has been shown in both directions,

$$\mathcal{F} = \sigma(\mathscr{A})$$

As shown in the paragraph above, given any $B \in \mathcal{F} = \sigma(\mathscr{A})$, there exists $\zeta \in \mathcal{Z}$, some subset of $\mathscr{A}$, such that $B \in \sigma(\zeta)$. Define $\mathscr{A}_B := \zeta$, and it is clear that $\mathscr{A}_B$ is a countable subclass of $\mathscr{A}$. $\square$

References

- Notes on the definition of countable `http://en.wikipedia.org/wiki/Countable_set`. In this problem, "countable" means "has the same cardinality as a subset of the set of natural numbers."
- `http://math.stackexchange.com/questions/297942/reference-request-set-theory-of-sigm`
- `http://math.stackexchange.com/questions/344784/show-that-for-any-a-in-sigma-varthet`
- `http://math.stackexchange.com/questions/61617/sigma-algebras`
- `http://math.stackexchange.com/questions/496837/you-only-need-countable-many-sets-ea`