Probability and Measure

by Billingsley

Solutions by

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Section 1

Problems 1, 2, 3, 4, 5, 6, 7

1.1 (a) Show that a *discrete* probability space (see Example 2.8 for the formal definition) cannot contain an infinite sequence $A_1, A_2, \ldots$ of independent events each of probability $\frac{1}{2}$. Since A_n could be identified with heads on the nth toss of a coin, the existence of such a sequence would make this section superfluous.

Proof. Apply 1.1(b). Let $p_n = 0.5$. Therefore $\alpha_n = \min\{0.5, 0.5\} = 0.5$ and $\sum_n \alpha_n$ diverges. Hence a discrete probability space cannot contain independent events $A_1, A_2, \ldots$ each with probability $\frac{1}{2}$. □

(b) Suppse that $0 \leq p_n \leq 1$, and put $\alpha_n = \min\{p_n, 1 - p_n\}$. Show that if $\sum_n \alpha_n$ diverges, then no discrete probability space can contain independent events $A_1, A_2, \ldots$ such that A_n has probability p_n.

Proof. Suppose B_i is A_i or A_i^c. Note $\alpha_n \leq P(B_n) \leq 1 - \alpha_n$. It follows that

$$\begin{aligned} 0 \leq P(B_1 \cap \cdots \cap B_n) &= P(B_1) \cdots P(B_n) && (1) \\ &\leq \prod_{i=1}^{n} (1 - \alpha_i) && (2) \\ &\leq \exp\left[-\sum_{i=1}^{n} \alpha_i\right] && (3) \end{aligned}$$

Inequality (3) comes from combining several $1 + x \leq e^x$ with the added condition that $0 \leq 1 + x$. Since $\sum_n \alpha_n$ diverges, $\exp\{-\sum_{i=1}^{n} \alpha_i\} \downarrow 0$. Taking the limit in n of the inequality above,

$$0 \leq P\left(\cap_{i=1}^{\infty} B_i\right) \leq 0. \tag{4}$$

For all i, each ω in the sample space is in A_i or A_i^c. Therefore, for all n, each ω is in $\cap_{i=1}^n B_i$. Thus, $\omega \in \cap_{i=1}^\infty B_i$ and

$$P(\omega) \leq P(\cap_{i=1}^\infty B_i) = 0 \tag{5}$$

Supposing that Ω were a discrete space leads to the conclusion that

$$1 = P(\Omega) = \sum_{\omega\in\Omega} P(\omega) = \sum_{\omega\in\Omega} 0 = 0, \tag{6}$$

a contradiction. Thus no discrete probability space can contain independent events $A_1, A_2, \ldots$ such that A_n has probability p_n.

□

1.2 Show that N and N^c are dense [A15] in (0,1].

Proof. The definition of *dense*: The set A is *dense* in the set B if for each x in B and each open interval J containing x, J meets A ($J \cap A \neq \emptyset$).

Part 1, N is dense: From Theorem 1.2, N has negligible complement, and therefore $P(N) = 1$. Suppose N is not dense. Then there exists an open interval $J = (a, b) \in (0, 1]$ (with $a \neq b$) such that $J \cap N = \emptyset$. Since this section discusses probabilities of intervals open on the left and closed on the right, take $J' = (a, \frac{a+b}{2}] \subset J$. It follows that $J' \cap N = \emptyset$. Therefore, since $N \subset (0, 1] \setminus J' = (0, a] \cup (\frac{a+b}{2}, 1]$,

$$\begin{aligned} P(N) \leq P((0,1] \setminus J') &= P\left((0, a] \cup \left(\frac{a+b}{2}, 1\right]\right) && (7)\\ &= P((0, a]) + P\left(\left(\frac{a+b}{2}, 1\right]\right) && (8)\\ &= a + \left(1 - \frac{a+b}{2}\right) && (9)\\ &= 1 + \frac{a}{2} - \frac{b}{2} && (10)\\ &< 1, && (11) \end{aligned}$$

a contradiction, since $P(N) = 1$. Therefore N is dense.

Part 2, N^c is dense: Let J be a non-trivial open interval (a, b) in $(0, 1]$. Let $K = \min\{k : 2^{-k} < (b - a)/2,\ k \in \mathbb{Z}^+\}$. By definition, K is constructed such that the length of a dyadic interval of rank K is the largest dyadic interval length less than $\frac{b-a}{2}$. This choice of K guarantees that at least one of the 2^K dyadic intervals of rank K that decompose the unit interval is contained in J. In other words for some integer n with $0 \leq n \leq 2^K - 1$, it follows that $I = (n/2^K, (n + 1)/2^K] \subset J$. Note, $\omega \in I$ means that

$d_i(\omega)$ is fixed for $i \leq K$. Next, consider a finite sequence, S, of 0's and 1's—not all 0's—the mean of which is $\mu \neq \frac{1}{2}$. For the purposes of this proof, let $S_1 = 1, S_2 = 1, S_3 = 0$. Choose the $\omega \in I$ such that $d_{K+n}(\omega) = S_{K+n \mod |S|} = S_{K+n \mod 3}$ for $n \in \mathbb{Z}^+$. Calculating,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} d_i(\omega) = \lim_{n\to\infty} \frac{1}{n|S|} \sum_{i=1}^{n|S|} d_i(\omega) \tag{12}$$

$$= \lim_{n\to\infty} \frac{1}{n|S|} \left(\sum_{i=1}^{K} d_i(\omega) + \sum_{i=K+1}^{n|S|} d_i(\omega) \right) \tag{13}$$

$$= \lim_{n\to\infty} \frac{1}{n|S|} \sum_{i=1}^{K} d_i(\omega) + \lim_{n\to\infty} \frac{1}{n|S|} \sum_{i=K+1}^{n|S|} d_i(\omega) \tag{14}$$

$$= 0 + \mu \tag{15}$$

$$\neq \frac{1}{2} \tag{16}$$

Since the series of partial sums converges to a limit on the left-hand side, all subsequences converge to that same limit, thus justifying (12). By construction, ω is in J, and by (16), $\omega \in N^c$ and thus N^c is dense in $(0, 1]$. □

1.3 ↑ Define a set A to be *trifling* if for each ϵ there exists a *finite* sequence of intervals I_k satisfying (1.22) and (1.23). This definition and the definition of negligibility apply as they stand to all sets on the real line, not just to subsets of $(0, 1]$.

(a) Show that a trifling set is negligible

Proof. Definition of **negligible**: A subset A of Ω is negligible if for each positive ϵ there exists a finite or countable collection $I_1, I_2, \ldots$ of intervals (they may overlap) satisfying

$$A \subset \bigcup_k I_k \tag{17}$$

and

$$\sum_k |I_k| < \epsilon \tag{18}$$

For trifling set A, take the finite sequence of intervals I_k and use them for the finite or countable collection of intervals for the definition of negligible. Therefore A is also negligible. □

(b) Show that the closure of a trifling set is also trifling.

Proof. Suppose A is trifling and let A^- be its closure. Given $\epsilon > 0$, choose intervals $(a_k, b_k], k = 1, \ldots, n$, such that $A \subset \cup_{k=1}^n (a_k, b_k]$ and $\sum_{k=1}^n (b_k - a_k) < \epsilon/2$. Let $x_k = a_k - \epsilon/2$. Then $A^- \subset \cup_{k=1}^n (x_k, b_k]$ and $\sum_{k=1}^n (b_k - x_k) < \epsilon$. □

(c) Find a bounded negligible set that is not trifling.

Proof. □

(d) Show that the closure of a neglibigle set may not be negligible.

(e) Show that finite unions of trifling sets are trifling but that this can fail for countable unions.

1.4 ↑ For $i = 0, \ldots, r-1$, let $A_r(i)$ be the set of numbers in $(0,1]$ whose nonterminating expansions in the base r do not contain digit i.

(a) Show that $A_r(i)$ is trifling.

(b) Find a trifling set A such that every point in the unit interval can be represented in the form $x+y$ with x and y in A.

(c) Let $A_r(i_l, \ldots, i_k)$ consist of the numbers in the unit interval in whose base-r expansion the digits $i_l, \ldots, i_k$ nowhere appear consecutively in that order. Show that it is trifling. What does this imply about the monkey that types at random?

1.5 ↑ The *Cantor set* C can be defined as the closure of $A_3(1)$.

(a) Show that C is uncountable but trifling.

(b) From $[0,1]$ remove the open middle third $\left(\frac{1}{3}, \frac{2}{3}\right)$; from the remainder, a union of two closed intervals, remove the two open middle thirds $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$. Show that C is what remains when this process is continued ad infinitum.

(c) Show that C is perfect [A15].

1.6 Put $M(t) = \int_0^1 e^{ts_n(\omega)} d\omega$, and show by successive differentiation under the integral that

$$M^{(k)}(0) = \int_0^1 s_n^k(\omega) d\omega. \tag{19}$$

Over each dyadic interval of rand n, $s_n(\omega)$ has a constant value of the form $\pm 1 \pm 1 \pm \cdots \pm 1$, and therefore $M(t) = 2^{-n} \sum \exp t(\pm 1 \pm 1 \pm \cdots \pm 1)$, where the sum extends over all 2^n n-long sequences of $+1$'s and -1's. Thus

$$M(t) = \left(\frac{e^t + e^{-t}}{2}\right)^n = (\cosh t)^n. \tag{20}$$

Use this and (1.38) to give new proofs of (1.16), (1.18), and (1.28). (This, the moethod of moment generating functions, will be investigated systematically in Section 9).

1.7 ↑ By an argument similar to that leading to (1.39) show that the Rademacher functions satisfy

$$\int_0^1 \exp\left[i\sum_{k=1}^n a_k r_k(\omega)\right] d\omega = \prod_{k=1}^n \frac{e^{ia_k}+e^{-ia_k}}{2} \tag{21}$$

$$= \prod_{k=1}^n \cos a_k. \tag{22}$$

Take $a_k = t2^{-k}$, and from $\sum_{k=1}^\infty r_k(\omega)2^{-k} = 2\omega - 1$ deduce

$$\frac{\sin t}{t} = \prod_{k=1}^n \cos\frac{t}{2^k} \tag{23}$$

by letting $n \to \infty$ inside the integral above. Derive Vieta's formula

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\frac{\sqrt{2+\sqrt{2}}}{2}\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots. \tag{24}$$