James K. Pringle
550.621 Probability
Dr. Jim Fill
Assignment 2
February 27, 2013

Assignment 2

All the exercises on the transition probabilities handout

1. Let $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ be two measurable spaces and let $\pi_i : \Omega_1 \times \Omega_2 \to \Omega_i$ be the ith projection, $i = 1, 2$. Set

$$\mathcal{C} := \pi_1^{-1}(\mathcal{A}_1) \cup \pi_2^{-1}(\mathcal{A}_2), \text{ the class of } \textit{measurable cylinders,}$$
$$\mathcal{R} := \{A_1 \times A_2 : A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2\}, \text{ the class of } \textit{measurable rectangles,}$$
$$\mathcal{U} := \left\{ \sum_{j \in J} R_j : J \text{ finite, } R_j \in \mathcal{R} \text{ for each } j \right\}.$$

(a) Show $\mathcal{C}$ is closed under complementation.

Proof. Given an element of C of $\mathcal{C}$, we have $C = A_1 \times \Omega_2$ or $C = \Omega_1 \times A_2$ for some $A_1 \in \mathcal{A}_1$ or $A_2 \in \mathcal{A}_2$. The complement of C is $C^c = A_1^c \times \Omega_2$ or $C^c = \Omega_1 \times A_2^c$. Since σ-fields are closed under complementation, $A_1^c \in \mathcal{A}_1$ and $A_2^c \in \mathcal{A}_2$. We have covered all possible cases to show that $C^c \in \mathcal{C}$. □

(b) Show $\mathcal{R}$ is a π-system.

Proof. Let $A_1 \times A_2 \in \mathcal{R}$ and $B_1 \times B_2 \in \mathcal{R}$. Taking the intersection of both sets, we have

$$(A_1 \times A_2) \cap (B_1 \times B_2) = (A_1 \cap B_1) \times (A_2 \times B_2) \tag{1}$$

Since $A_i \cap B_i \in \mathcal{A}_i$ for $i = 1, 2$ by the closure under countable intersections of σ-fields, $(A_1 \times A_2) \cap (B_1 \times B_2) \in \mathcal{R}$. □

(c) Show $\mathcal{U}$ is the field generated by $\mathcal{C}$ (and by $\mathcal{R}$).

Proof. We already have $\mathcal{C} \subset \mathcal{R} \subset \mathcal{U}$. Let $\langle \mathcal{C} \rangle$ denote the field generated by $\mathcal{C}$. Let $U \in \mathcal{U}$. By definition, $U = \sum R_j$ (a finite sum). Each R_i in the sum can be written as $A_{i1} \times A_{i2}$. By (1), $A_{i1} \times A_{i2} = (A_{i1} \times \Omega_2) \cap (\Omega_1 \times A_{i2})$, the intersection of elements of $\mathcal{C}$. Since $\langle \mathcal{C} \rangle$ is generated from finite complementation, union, and intersection of sets in $\mathcal{C}$, it is clear that each $R_i \in \langle \mathcal{C} \rangle$ (by finite intersection). By this same reasoning, every rectangle is in $\langle\mathcal{C}\rangle$. Thus $\langle\mathcal{R}\rangle \subset \langle\mathcal{C}\rangle$. Furthermore, $U = \sum R_j \in \langle\mathcal{R}\rangle \subset \langle\mathcal{C}\rangle$ by finite union. This shows $\mathcal{U} \subset \langle\mathcal{R}\rangle \subset \langle\mathcal{C}\rangle$.

Now we show $\mathcal{U}$ is field. Once we have shown that, it will be clear that the smallest field containing $\mathcal{C}$, which is $\langle\mathcal{C}\rangle$, is a subset of $\mathcal{U}$. That, combined with $\mathcal{U} \subset \langle\mathcal{C}\rangle$, will show $\mathcal{U} = \langle\mathcal{C}\rangle$. Likewise, that will show that the smallest field containing $\mathcal{R}$, which is $\langle\mathcal{R}\rangle$, is a subset of $\mathcal{U}$. That combined with $\mathcal{U} \subset \langle\mathcal{R}\rangle$ will give $\mathcal{U} = \langle\mathcal{R}\rangle$.

Closure under binary union: Let $U_1, U_2 \in \mathcal{U}$ with $U_i = \sum_j^{J_i} R_{ij}$, a finite sum over j with $i = 1, 2$. Notice elements of $\mathcal{U}$ are finite unions of disjoint rectangles. We now show that the union of U_1 and U_2 is a union of disjoint rectangles. Define

$$B_{1k} = R_{1k} \setminus (\sum_j R_{2j}) = R_{1k} \setminus R_{21} \setminus R_{22} \setminus \cdots \setminus R_{2J_2} \tag{2}$$

where here we are evaluating the binary set difference operators from left to right (written that way to avoid copious amounts of parentheses), and $k \in \{1, 2, \cdots, K\}$. Since

$$(A_1 \times A_2) \setminus (B_1 \times B_2) = (A_1 \cap B_1^c) \times (A_2 \times B_2^c) + (A_1 \cap B_1) \times (A_2 \times B_2^c) + (A_1 \cap B_1^c) \times (A_2 \times B_2),$$

a sum of rectangles, we have by "quick" induction (quick because the obvious inductive step—a sum of rectangles minus a last rectangle is the sum of each rectangle in the sum minus the last rectangle—is skipped) that (2) is a sum of rectangles. Therefore, each B_{1j} is a sum of rectangles. Since $B_{1j} \subset R_{1j}$ the B_{1j} are mutually disjoint. Then by construction, B_{1j} and R_{2j} are mutually disjoint and their sum is the union is the finite sum of disjoint rectangles equal to the union of U_1 and U_2. Hence $\mathcal{U}$ is closed under binary union.

Closure under complementation: First a little lemma. Let $R = A_1 \times A_2$ be a rectangle in $\mathcal{R}$. Then

$$R^c = (A_1^c \times A_2) \cup (A_1 \times A_2^c) \cup (A_1^c \times A_2^c)$$

is a finite union of rectangles in $\mathcal{R}$. Let $U \in \mathcal{U}$. Then $U = \sum_{j=1}^J R_j$ for rectangles R_j. We have

$$U^c = \left(\sum_{j=1}^J R_j\right)^c = \bigcap_{j=1}^J R_j^c = \bigcap_{j=1}^J (\cup_{i=1}^{I_j} S_i)$$

where S_i are rectangles by the little lemma. By the distributive law for sets, $\cap_{j=1}^J (\cup_{i=1}^{I_j} S_i)$ is the union of intersections of rectangles (it is hard to write a closed form since I_j is variable). By "quick" induction, the intersection of any finite number of rectangles is a rectangle (the base case is (1), and we skip the obvious inductive step). Therefore, the union of intersections of rectangles is the union of rectangles, and we have

$$U^c = \bigcap_{j=1}^{J} (\cup_{i=1}^{I_j} S_i) = \bigcup_{i=1}^{I} T_i$$

where T_i are rectangles and I is finite. Now we show U^c can be reduced to the union of disjoint rectangles. We do the classic trick where we let $B_1 = T_1$, then $B_i = T_i \backslash (\cup_{j=1}^{i-1} T_j)$. Then we have $\cup_{i=1}^{I} B_i = \cup_{i=1}^{I} T_i$ and the B_i are mutually disjoint. Furthermore, by the "quick" induction based on (2), each B_i is a sum of dijoint rectangles. This shows $U^c \in \mathcal{U}$ and we conclude $\mathcal{U}$ is a field. From our argument above, we can finally conclude $\mathcal{U} = \langle \mathcal{C} \rangle$ and $\mathcal{U} = \langle \mathcal{R} \rangle$. □

2. Let Ω_1 and Ω_2 be two spaces; set $\Omega = \Omega_1 \times \Omega_2$. Let $X : \Omega \to \Psi$ (respectively, $A \subset \Omega$). The section of X (resp., of A) at $\omega_1 \in \Omega_1$ is defined to be the function $X_{\omega_1} : \Omega_2 \to \Psi$ (resp., the set $A_{\omega_1} \subset \Omega_2$) given by $X_{\omega_1}(\omega_2) = X(\omega_1, \omega_2)$ (resp., by $A_{\omega_1} = \{\omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in A\}$).

   Show that $(I_A)_{\omega_1} = I_{A_{\omega_1}}$ for $A \subset \Omega$ and $(X^{-1}(B))_{\omega_1} = X_{\omega_1}^{-1}(B)$ for $B \subset \Psi$.

   Proof. Let $A \subset \Omega$. Suppose $\omega_2 \in A_{\omega_1}$. Thus $(\omega_1, \omega_2) \in A$. Therefore, $I_{A_{\omega_1}}(\omega_2) = 1$ and $(I_A)_{\omega_1}(\omega_2) = I_A(\omega_1, \omega_2) = 1$. And we see both functions agree. Now suppose $\omega_2 \notin A_{\omega_1}$. Then $(\omega_1, \omega_2) \notin A$. Thus $I_{A_{\omega_1}}(\omega_2) = 0$ and $(I_A)_{\omega_1}(\omega_1) = I_A(\omega_1, \omega_2) = 0$. And we conclude $(I_A)_{\omega_1} = I_{A_{\omega_1}}$ for $A \subset \Omega$.

   For the second part, note for all $B \subset \Psi$

$$\begin{aligned}
(X^{-1}(B))_{\omega_1} &= \{\omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in X^{-1}(B)\} \\
&= \{\omega_2 \in \Omega_2 : X(\omega_1, \omega_2) \in B\} \\
&= \{\omega_2 \in \Omega_2 : X_{\omega_1}(\omega_2) \in B\} \\
&= (X_{\omega_1})^{-1}(B) \\
&= X_{\omega_1}^{-1}(B)
\end{aligned}$$

   as desired. □

3. Notations are the same as Problem 2. Let $i_{\omega_1} : \Omega_2 \to \Omega$ be the injection mapping defined by

$$i_{\omega_1}(\omega_2) = (\omega_1, \omega_2).$$

   Show that

$$A_{\omega_1} = i_{\omega_1}^{-1}(A), \qquad X_{\omega_1} = X \circ i_{\omega_1}.$$

Proof. For the first part, notice

$$\begin{aligned}
A_{\omega_1} &= \{\omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in A\} \\
&= \{\omega_2 \in \Omega_2 : i_{\omega_1}(\omega_2) \in A\} \\
&= (i_{\omega_1})^{-1}(A) \\
&= i_{\omega_1}^{-1}(A)
\end{aligned}$$

as desired. Also, for all $\omega_2 \in \Omega_2$ it is the case that

$$X_{\omega_1}(\omega_2) = X(\omega_1, \omega_2) = X(i_{\omega_1}(\omega_2)) = X \circ i_{\omega_1}(\omega_2).$$

Hence $X_{\omega_1} = X \circ i_{\omega_1}$ as desired. □

4. Let Ψ be an uncountable set, and let $\mathcal{B}$ be the σ-field in Ψ generated by the singletons. ($\mathcal{B}$ consists of the countable and co-countable subsets of Ψ.) Take $(\Omega_1, \mathcal{A}_1) = (\Psi, \mathcal{B}) = (\Omega_2, \mathcal{A}_2)$. Consider the diagonal $\Delta := \{(\psi, \psi) : \psi \in \Psi\}$ of $\mathcal{A} = \mathcal{B} \otimes \mathcal{B}$. Show that every section of Δ is in $\mathcal{B}$, but $\Delta \notin \mathcal{A}$.

Proof. Since $\mathcal{B}$ is generated by the singletons of Ψ, it follows that $\mathcal{A} = \mathcal{B} \otimes \mathcal{B}$ is generated by sets of the form $\{\psi\} \times \Psi$ and $\Psi \times \{\psi\}$ for all $\psi \in \Psi$. Let

$$\mathcal{S} = \{\{\psi\} \times \Psi \mid \psi \in \Psi\} \cup \{\Psi \times \{\psi\} \mid \psi \in \Psi\},$$

and with the new notation, $\mathcal{A} = \sigma(\mathcal{S})$.

Suppose, by way of contradiction, $\Delta \in \sigma(\mathcal{S})$. Problem 2.9 in Billingsley states

Theorem. *If $B \in \sigma(\mathscr{A})$, then there exists a countable subclass $\mathscr{A}_B$ of $\mathscr{A}$ such that $B \in \sigma(\mathscr{A}_B)$.*

In the current solution, by problem 2.9, there exists some countable sublass $\mathcal{S}_\Delta$ of $\mathcal{S}$ such that $\Delta \in \sigma(\mathcal{S}_\Delta)$. Now it is possible to write out the members of $\mathcal{S}_\Delta$. We conclude

$$\mathcal{S}_\Delta = \{\{\psi_i\} \times \Psi \mid i \in I_1\} \cup \{\Psi \times \{\psi_i\} \mid i \in I_2\},$$

Where I_1 and I_2 are at most countable indexing sets and could possibly be empty. Let $Q = \{\psi_i \mid i \in I_1 \cup I_2\}$ so that Q is the set of all elements of Ψ that form a singleton cylinder in $\mathcal{S}_\Delta$.

Let $\mathcal{P}$ be the class containing Q^c and the singletons $\{\psi_i\}$ where $i \in I_1 \cup I_2$. Let $\mathcal{P}'$ be the class of countable unions of sets (or blocks) in the partition $[P_1 \times P_2 : P_1, P_2 \in \mathcal{P}]$ (partition of $\Psi \times \Psi$) and let $\mathcal{P}'$ include the empty set. (An example, for demonstration, of an element in $\mathcal{P}'$ is the set $\{\psi_1\} \times \{\psi_1\} \cup \{\psi_2\} \times Q^c$). We claim (i) $\mathcal{P}'$ is a σ-field and (ii) $\sigma(\mathcal{S}_\Delta) \subset \mathcal{P}'$.

(i) By definition, $\mathcal{P}'$ includes the empty set. Since $\mathcal{P}$ is countable, then the blocks of the partition $[P_1 \times P_2 : P_1, P_2 \in \mathcal{P}]$ are countable. The union of all those blocks is $\Psi \times \Psi$, which is therefore contained in $\mathcal{P}'$. A set of $\mathcal{P}'$ is the countable union of blocks of the partition, and that set's complement is the countable union of all the other blocks. Thus $\mathcal{P}'$ is closed under complementation. Finally, $\mathcal{P}'$ is closed under countable union by definition. Hence $\mathcal{P}'$ has the three properties of a σ-field.

(ii) The elements of $\mathcal{S}_\Delta$ are contained in $\mathcal{P}'$ since

$$\{\psi_i\} \times \Psi = \left(\bigcup_{P \in \mathcal{P}} \{\psi_i\} \times P \right) \in \mathcal{P}' \quad \text{and similarly}$$
$$\Psi \times \{\psi_i\} = \left(\bigcup_{P \in \mathcal{P}} P \times \{\psi_i\} \right) \in \mathcal{P}'.$$

By (i), $\mathcal{P}'$ is a σ-field containing $\mathcal{S}_\Delta$. Thus $\sigma(\mathcal{S}_\Delta) \subset \mathcal{P}'$ and $\Delta \in \mathcal{P}'$.

Now we seek the contradiction. Set

$$D = \Delta \setminus (\bigcup_{\psi \in Q} \{\psi\} \times \{\psi\}) \subset Q^c \times Q^c.$$

We have $D \in \mathcal{P}'$ by the closure properties of $\mathcal{P}'$. Since Q^c is uncountable, there exist ψ_1 and ψ_2 in Q^c with $\psi_1 \neq \psi_2$. It follows that (ψ_1, ψ_2) is an element of $Q^c \times Q^c$ but it is not an element of D. Thus D is a strict subset of $Q^c \times Q^c$. Since $D \in \mathcal{P}'$ it is equal to U, some countable union of sets in the partition from above. One of those sets in the union that is U must be $Q^c \times Q^c$ (if not, then D is not in U). However, D is a strict subset $Q^c \times Q^c \subset U$. Thus $D \neq U$. This is our contradiction. Therefore $\Delta \neq \sigma(\mathcal{S}) = \mathcal{A}$ as desired. However, for all $\psi_0 \in \Psi$, the section

$$\Delta_{\psi_0} = \{\psi : (\psi_0, \psi) \in \Psi \times \Psi\} = \{\psi : (\psi, \psi_0) \in \Psi \times \Psi\} = \{\psi_0\}$$

is an element of $\mathcal{B}$. □

5. A transition probability is defined as follows:

A mapping $T : \Omega_1 \times \mathcal{A}_2 \to [0, 1]$ is called a transition probability from $(\Omega_1, \mathcal{A}_1)$ to $(\Omega_2, \mathcal{A}_2)$ (briefly, from Ω_1 to Ω_2) if

(1) $T(\omega_1; \cdot\,)$ is a probability measure on $(\Omega_2, \mathcal{A}_2)$ for each $\omega_1 \in \Omega_1$, and

(2) $T(\,\cdot\,; A_2)$ is $\mathcal{A}_1$-measurable for each $A_2 \in \mathcal{A}_2$.

Show that given (1), then (2) holds if $T(\,\cdot\,; A_2)$ is $\mathcal{A}_1$-measurable for all A_2 in some π-system generating $\mathcal{A}_2$.

Proof. Let (1) be true. Let $T(\,\cdot\,;A_2)$ be $\mathcal{A}_1$-measurable for all A_2 in some π-system $\mathscr{P}$ generating $\mathcal{A}_2$. Define

$$\mathscr{G} = \{A_2 \in \mathcal{A}_2 : T(\,\cdot\,;A_2) \text{ is } \mathcal{A}_1\text{-measurable }\}$$

We now show that $\mathscr{G}$ is a λ-system. **Contains Ω_2:** By (1), we have $T(\omega_1;\Omega_2) = 1$ for all $\omega_1 \in \Omega_1$. Thus $T(\,\cdot\,;\Omega_2)$ is a constant function. Hence it is $\mathcal{A}_1$-measurable, and $\Omega_2 \in \mathscr{G}$. **Closure under proper difference:** Now suppose B_1 and B_2 are elements of $\mathscr{G}$ with $B_1 \subset B_2$. By (1), it follows $T(\omega_1; B_2 \setminus B_1) = T(\omega_1;B_2) - T(\omega_1;B_1)$ for all $\omega_1 \in \Omega_1$. Since, $T(\,\cdot\,;B_1)$ and $T(\,\cdot\,;B_2)$ are $\mathcal{A}_1$-measurable, then, by the corollary to Theorem 3.1.5 in Chung (the closure theorem), their difference, $T(\,\cdot\,;B_2 \setminus B_1)$, is $\mathcal{A}_1$-measurable and is therefore a member of $\mathscr{G}$. **Closure under increasing union:** Finally, let $B_1, B_2, \cdots$ be a sequence of sets in $\mathscr{G}$ with $B_n \subset B_{n+1}$ for all $n \geq 1$. By (1) and the properties of a probability measure (i.e. monotone sequential continuity from below), for all $\omega_1 \in \Omega_1$ it is true that

$$\lim_{n\to\infty} T(\omega_1;B_n) = T(\omega_1;\cup_n B_n)$$

and the limit exists because it is the limit of a bounded monotone increasing. By Theorem 13.4 in Billingsley, since the limit of $\mathcal{A}_1$-measurable $T(\,\cdot\,;B_n)$ exists everywhere (on Ω_1), then that limit, $T(\,\cdot\,;\cup_n B_n)$, is $\mathcal{A}_1$-measurable. Thus it is in $\mathscr{G}$. We can conclude that $\mathscr{G}$ is a λ-system.

By hypothesis, $\mathscr{P} \subset \mathscr{G}$, whence by the π-λ theorem we have $\sigma(\mathscr{P}) = \mathcal{A}_2 \subset \mathscr{G}$. By the definition of $\mathscr{G}$, it is clear $\mathscr{G} \subset \mathcal{A}_2$. Therefore $\mathscr{G} = \mathcal{A}_2$, which demonstrates (2). $\square$

6. Let $(\Omega_1, \mathcal{A}_1), (\Omega_2, \mathcal{A}_2)$ be measurable spaces, $\Omega = \Omega_1 \times \Omega_2$, $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$. Let M be a probability on $(\Omega_1, \mathcal{A}_1)$, and let $(T_{\omega_1})_{\omega_1 \in \Omega_1}$ be a transition probability from Ω_1 to Ω_2. Let $\mathscr{G} = \{A \in \mathcal{A} : \omega_1 \mapsto T_{\omega_1}(A_{\omega_1}) \text{ is } \mathcal{A}_1\text{-measurable}\}$. Show that

$$T_{\omega_1}((A_1 \times A_2)_{\omega_1}) = I_{A_1}(\omega_1) T_{\omega_1}(A_2). \tag{3}$$

Also show $\mathscr{G}$ contains the π-system $\mathcal{R}$.

Proof. Calculating,

$$T_{\omega_1}((A_1 \times A_2)_{\omega_1}) = T(\omega_1;(A_1 \times A_2)_{\omega_1}) = \begin{cases} T(\omega_1;\emptyset) = 0 & \text{if } \omega_1 \notin A_1 \\ T(\omega_1;A_2) = T_{\omega_1}(A_2) & \text{if } \omega_1 \in A_1 \end{cases}.$$

Since,

$$I_{A_1}(\omega_1)T_{\omega_1}(A_2) = \begin{cases} 0 & \text{if } \omega_1 \notin A_1 \\ T_{\omega_1}(A_2) & \text{if } \omega_1 \in A_1 \end{cases},$$

we have (3), as desired.

To show $\mathscr{G}$ contains $\mathcal{R}$, let $R = A_1 \times A_2 \in \mathcal{R}$. Our task is to show that the function on Ω_1 that sends $\omega_1 \mapsto T_{\omega_1}(R_{\omega_1})$ is a $\mathcal{A}_1$-measurable function. By the (3), we have

$$T_{\omega_1}(R_{\omega_1}) = T_{\omega_1}((A_1 \times A_2)_{\omega_1}) = I_{A_1}(\omega_1)T_{\omega_1}(A_2).$$

Since T_{ω_1} is a transition probability, for all $\omega_1 \in \Omega_1$, we have $T_{\omega_1}(A_2)$ is constant. The product of $\mathcal{A}_1$-measurable functions (an indicator and a constant function) is measurable by the corollary to Theorem 3.1.5 in Chung (the closure theorem). Hence $T_{\omega_1}(R_{\omega_1})$ is measurable and $R \in \mathscr{G}$. Therefore, $\mathcal{R} \subset \mathscr{G}$. □

7. Notations are the same as Problem 6. Show that $\mathscr{G}$ is a λ-system.

Proof. **Contains Ω:** Since for all $\omega_1 \in \Omega_1$, we have $T_{\omega_1}(\Omega_{\omega_1}) = T_{\omega_1}(\Omega_2) = 1$ is constant, then $\omega_1 \mapsto T_{\omega_1}(R_{\omega_1})$ is $\mathcal{A}_1$-measurable. **Closure under complementation:** Let $A \in \mathscr{G}$. Thus $\omega_1 \mapsto T_{\omega_1}(A_{\omega_1})$ is $\mathcal{A}_1$-measurable. Then, since sectioning commutes with set operations,

$$\omega_1 \mapsto T_{\omega_1}((A^c)_{\omega_1}) = T_{\omega_1}((A_{\omega_1})^c) = 1 - T_{\omega_1}(A_{\omega_1})$$

is A_1-measurable by the closure theorem. **Closure under countable union of disjoint sets:** Let $B_1, B_2, \cdots$ be mutually disjoint elements of $\mathscr{G}$. Then for all $\omega_1 \in \Omega_1$ we have

$$\begin{aligned} \omega_1 \mapsto T_{\omega_1}\left(\left(\lim_{I\to\infty}\bigcup_{i=1}^{I} B_i\right)_{\omega_1}\right) &= T_{\omega_1}\left(\lim_{I\to\infty}\bigcup_{i=1}^{I}(B_i)_{\omega_1}\right) \quad \text{since sectioning commutes} \\ &= \lim_{I\to\infty} T_{\omega_1}\left(\bigcup_{i=1}^{I}(B_i)_{\omega_1}\right) \end{aligned}$$

by monotone sequential continuity from below, and that limit exists because it is the limit of a bounded, monotone increasing sequence. Thus by Theorem 13.4 in Billingsley, we have $\omega_1 \mapsto T_{\omega_1}\left(\left(\lim_{I\to\infty}\cup_{i=1}^{I} B_i\right)_{\omega_1}\right)$ is measurable. Hence, $\lim_{I\to\infty}\cup_{i=1}^{I} B_i \in \mathscr{G}$. We conclude $\mathscr{G}$ is a λ-system. □

8. Notations are the same as Problem 6. Let M be a probability on $(\Omega_1, \mathcal{A}_1)$. The set function MT, defined on $\mathcal{A}$ is

$$MT(A) := \int_{\Omega_1} T_{\omega_1}(A_{\omega_1})M(d\omega_1).$$

Furthermore, for any nonnegative $\mathcal{A}$-random variable X on Ω, the map TX on Ω_1 is defined as

$$\omega_1 \mapsto \int_{\Omega_2} X_{\omega_1}(\omega_2)T_{\omega_1}(d\omega_2)$$

Let $\mathcal{G}$ denote the collection of $\mathcal{A}$-measurable nonnegative random variables X such that $\omega_1 \mapsto \int_{\Omega_2} X_{\omega_1}(\omega_2)T_{\omega_1}(\mathrm{d}\omega_2)$ is A_1-measurable and $\langle MT, X\rangle = \langle M, TX\rangle$. Show that $I_A \in \mathcal{G}$ for every $A \in \mathcal{A}$.

Proof. Let $A \in \mathcal{A}$. By problem 2 of this homework, we have $(I_A)_{\omega_1} = I_{A_{\omega_1}}$. Thus

$$\begin{aligned}
\omega_1 \mapsto \int_{\Omega_2} (I_A)_{\omega_1}(\omega_2) T_{\omega_1}(d\omega_2) &= \int_{\Omega_2} I_{A_{\omega_1}}(\omega_2) T_{\omega_1}(d\omega_2) && (4)\\
&= \int_{A_{\omega_1}} T_{\omega_1}(d\omega_2) && (5)\\
&= T_{\omega_1}(A_{\omega_1}) && (6)
\end{aligned}$$

is $\mathcal{A}_1$-measurable by the reasoning that follows after problem 7 in the course slides (that $\mathscr{G} = \mathcal{A}$).

For the second part, calculating gives

$$\begin{aligned}
\langle MT, I_A \rangle &= \int_{\Omega} I_A(\omega) MT(d\omega)\\
&= \int_A MT(d\omega)\\
&= MT(A)\\
&= \int_{\Omega_1} T_{\omega_1}(A_{\omega_1}) M(d\omega_1) \quad \text{by the definition of } MT\\
&= \int_{\Omega_1} \left(\int_{\Omega_2} I_{A_{\omega_1}}(\omega_2) T_{\omega_1}(d\omega_2) \right) M(d\omega_1) \quad \text{by (4) = (6)}\\
&= \int (TI_A) dM \quad \text{by the definition of } TI_A\\
&= \langle M, TI_A \rangle
\end{aligned}$$

Thus $I_A \in \mathcal{G}$. ∎

Acknowledgements

I would like to thank Tianchen for TeXing some of the problems. That saved time. I am grateful to Jason for spending time with me to explain problems 4 and 5, and for proving Problem 2.9 in Billingsley in Fall 2012. Also, I explained problem 4 to most of the biostats people, so they may have similar notation and methods to mine.