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550.621 Probability
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Assignment 5

Radon-Nikodym Theorem and fields

Part 1

Let P and Q be probabilities on a measurable space $(\Omega, \mathcal{A})$. Show that condition 2,

$$\lim_{\eta \to 0} \sup\{P(A) : A \in \mathcal{A} \text{ and } Q(A) < \eta\} = 0$$

is equivalent to $2'$,

$$\lim_{\eta \to 0} \sup\{P(A) : A \in \mathcal{A}_0 \text{ and } Q(A) < \eta\} = 0$$

where $\mathcal{A}_0$ is a field generating $\mathcal{A}$.

Proof. For fixed η, since $\mathcal{A}_0 \subset \mathcal{A}$,

$$\{A \in \mathcal{A}_0 : Q(A) < \eta\} \subset \{A \in \mathcal{A} : Q(A) < \eta\}$$

Hence

$$\{P(A) : A \in \mathcal{A}_0 \text{ and } Q(A) < \eta\} \subset \{P(A) : A \in \mathcal{A} \text{ and } Q(A) < \eta\}$$

Therefore

$$0 \le \sup\{P(A) : A \in \mathcal{A}_0 \text{ and } Q(A) < \eta\} \le \sup\{P(A) : A \in \mathcal{A} \text{ and } Q(A) < \eta\}$$

Since this is true for all η, taking the limit as $\eta \to 0$,

$$0 \le \lim_{\eta \to 0} \sup\{P(A) : A \in \mathcal{A}_0 \text{ and } Q(A) < \eta\} \le \lim_{\eta \to 0} \sup\{P(A) : A \in \mathcal{A} \text{ and } Q(A) < \eta\}$$

By the assumption of condition 2, it follows by squeezing that

$$\lim_{\eta \to 0} \sup\{P(A) : A \in \mathcal{A}_0 \text{ and } Q(A) < \eta\} = 0$$

and condition $2'$ holds. Thus condition 2 implies condition $2'$.

Next suppose that condition $2'$ holds. Note $\emptyset \in \mathcal{A}_0$ since $\Omega \in \mathcal{A}_0$ and $\mathcal{A}_0$ is closed under complementation. Also, $\mathcal{A}_0$ is closed under binary intersection. Finally since $\mathcal{A}_0$ is closed under complementation, $A \subset B$ with $A, B \in \mathcal{A}_0$ implies $B \setminus A = B \cap A^c \in \mathcal{A}_0$. Therefore, $\mathcal{A}_0$ is a semiring. See Billingsley Section 11 for a definition of "semiring."

Fix $\eta > 0$. Let $A \in \mathcal{A}$ such that $Q(A) < \eta$. Thus $0 < \eta - Q(A)$. Note $\sigma(\mathcal{A}_0) = \mathcal{A}$ and Q is a p.m. (hence σ-finite) on $\mathcal{A}$. By Billingsley Theorem 11.4(i), there exists an infinite[1] disjoint sequence $A_1, A_2, \cdots$ of $\mathcal{A}_0$ sets such that

$$A \subset \cup_k A_k$$

and

$$Q((\cup_k A_k) \setminus A) < \eta - Q(A).$$

Thus

$$\begin{aligned} Q((\cup_k A_k) \setminus A) &< \eta - Q(A) \\ Q(\cup_k A_k) - Q(A) &< \eta - Q(A) \text{ by additivity} \\ Q(\cup_k A_k) &< \eta \end{aligned}$$

Let $E_n = \cup_{k=1}^n A_k$, and $E = \cup_k A_k$. From the above, $Q(E) < \eta$. Since $E_n \uparrow E$, for all n

$$Q(E_n) \le Q(E) < \eta$$

By the closure properties of a field, for all n,

$$E_n \in \{A \in \mathcal{A}_0 : Q(A) < \eta\}.$$

Thus for all n

$$P(E_n) \le \sup\{P(A) : A \in \mathcal{A}_0 \text{ and } Q(A) < \eta\}$$

The right-hand side does not depend on n, so passing to the limit

$$\lim_{n\to\infty} P(E_n) \le \sup\{P(A) : A \in \mathcal{A}_0 \text{ and } Q(A) < \eta\}$$

By Monotone Sequential Continuity from Below,

$$\lim_{n\to\infty} P(E_n) = P(E) = P(\cup_k A_k)$$

and since $A \subset \cup_k A_k$

$$P(A) \le P(\cup_k A_k) \le \sup\{P(A) : A \in \mathcal{A}_0 \text{ and } Q(A) < \eta\}$$

[1] The original theorem says "there exists a finite or infinite disjoint sequence..." However, since the empty set is an element of any semiring, then in the finite case with m sets, define $A_n = \emptyset$ for $n > m$.

Since A is arbitrary in $\mathcal{A}$ as long as $Q(A) < \eta$, then for all $A \in \mathcal{A}$ with $Q(A) < \eta$

$$0 \leq P(A) \leq \sup\{P(A) : A \in \mathcal{A}_0 \text{ and } Q(A) < \eta\}$$

Thus

$$0 \leq \sup\{P(A) : A \in \mathcal{A} \text{ and } Q(A) < \eta\} \leq \sup\{P(A) : A \in \mathcal{A}_0 \text{ and } Q(A) < \eta\}$$

Since η is arbitrary, take the limit as $\eta \to 0$,

$$0 \leq \lim_{\eta \to 0} \sup\{P(A) : A \in \mathcal{A} \text{ and } Q(A) < \eta\} \leq \lim_{\eta \to 0} \sup\{P(A) : A \in \mathcal{A}_0 \text{ and } Q(A) < \eta\}$$

By the assumption of condition $2'$, the right-hand limit is 0. Thus by squeezing,

$$\lim_{\eta \to 0} \sup\{P(A) : A \in \mathcal{A} \text{ and } Q(A) < \eta\} = 0$$

and condition 2 holds. Hence condition $2'$ implies condition 2. Since implications in both directions have been proved, conditions 2 and $2'$ are equivalent. □

Part 2

Specialize to $(\Omega, \mathcal{A}, Q) = ([0,1], \text{Borels}, \text{Lebesgue})$ to deduce that a nondecreasing, right-continuous function F on $[0,1]$ with $F(0) = 0$ which is *absolutely continuous* in the classical sense that

> for each $\epsilon > 0$, there exists an $\eta_\epsilon > 0$ such that for every choice of $n \geq 1$ and $a_i < b_i$, $1 \leq i \leq n$, for which $\sum_{i=1}^{n}(b_i - a_i) < \eta_\epsilon$, one has $\sum_{i=1}^{n}(F(b_i) - F(a_i)) < \epsilon$

necessarily is of the form $F(t) = \int_0^t X(s)ds$, $0 \leq t \leq 1$, for some $X \geq 0$.

Proof. First a few definitions. Define $\mathscr{B}_0$ to be the Borel sets on $(0,1]$. Define $\mathscr{B}$ to be the Borel sets on $[0,1]$. Define $\mathscr{B}^1$ to be the Borel sets on $\mathscr{R}^1$, the real line.

The proof is broken into two cases.

Case 1: $F(1) = 0$. Then since F is nondecreasing, $F(t) = 0$ for $0 \leq t \leq 1$. Hence a suitable function $X(s)$ is the constant function 0. For $0 \leq t \leq 1$,

$$0 = F(t) = \int_0^t X(s)ds = \int_0^t 0ds = 0$$

Case 2: $F(1) \neq 0$. Since F is nondecreasing, $F(1) > 0$. Note by the absolute continuity of F, it follows that F is uniformly continuous (take $n = 1$ in $\sum_{i=1}^{n}(b_i - a_i) < \eta_\epsilon$ and $\sum_{i=1}^{n}(F(b_i) - F(a_i)) < \epsilon$). Since $[0,1]$ is an interval, and F is uniformly continuous on that interval, by a result from real analysis, F is bounded on that interval. Hence $F(1)$ is finite. Define

$$G(t) = \begin{cases} 0 & \text{for } t < 0 \\ F(t)/F(1) & \text{for } t \in [0,1] \\ 1 & \text{for } t > 1 \end{cases}$$

Shortly, it is shown that G is a d.f. Note $\lim_{n\to-\infty} G(t) = 0$ and $\lim_{n\to\infty} G(t) = 1$. Also, G is right-continuous on $[0,1)$ because it is a constant $(1/F(1))$ times F. Clearly G is right-continuous on $(-\infty, 0)$. And since $G(1) = F(1)/F(1) = 1$, it follows that G is right-continuous on $[1,\infty)$. Finally, since $F(1)$ is a positive constant, $F(t)/F(1)$ is nondecreasing because F is nondecreasing. Therefore G is nondecreasing. Thus, G is a d.f.

By **Chung Theorem 2.2.4** let μ be the p.m. induced by G on $(\mathscr{R}^1, \mathscr{B}^1)$. Note $0 \le G(0-) \le G(0) = 0$. Thus $0 = G(0-) = \mu((-\infty,0))$. Moreover, $\mu((1,\infty)) = 1 - \mu((-\infty,1]) = 1 - G(1) = 0$. Hence $\mu([0,1]) = 1$.

Define a set function P to be the restriction of μ on $\mathscr{B}$. This is valid since $\mathscr{B} \subset \mathscr{B}^1$. Shortly, it is shown that P is a p.m. As shown above $P([0,1]) = 1$. For any $A \in \mathscr{B}$, it is the case that $A \subset [0,1]$. Thus $0 \le P(A) \le 1$. Finally, let $A_1, A_2, \cdots$ be disjoint sets in $\mathscr{B}$. Then since $\cup_{i=1}^\infty A_i \in \mathscr{B} \subset \mathscr{B}^1$,

$$P(\cup_{i=1}^\infty A_i) = \mu(\cup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mu(A_i) = \sum_{i=1}^\infty P(A_i)$$

Therefore, P is a p.m. on $([0,1], \mathscr{B})$.

Let $\mathcal{A}_0'$ be the class of finite unions of disjoint intervals $(a,b]$ contained in $[0,1]$. By **Chung Example 2.2.2**, $\mathcal{A}_0'$ is a field on $(0,1]$ and generates $\mathscr{B}_0$. However, $\mathcal{A}_0'$ is not a field on $[0,1]$ (because $[0,1]$ is not an element of $\mathcal{A}_0'$). The next task is to describe a field on $[0,1]$ that generates the Borel sets on $[0,1]$.

Let $\mathcal{A}_0$ be the class of sets of $\mathcal{A}_0'$ possibly unioned with $\{0\}$. Hence $A \in \mathcal{A}_0$ can be written as

$$A = A' \quad \text{or} \quad A = \{0\} \cup A'$$

where $A' \in \mathcal{A}_0'$ and

$$A' = \cup_{i=1}^n (a_i, b_i]$$

with $0 \le a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n \le 1$ and $n \ge 0$.

Notice that (using the two possible A definitions above)

$$A^c = \{0\} \cup (0, a_1] \cup (\cup_{i=1}^{n-1} (b_i, a_{i+1}]) \cup (b_n, 1] \quad \text{or} \quad A^c = (0, a_1] \cup (\cup_{i=1}^{n-1} (b_i, a_{i+1}]) \cup (b_n, 1],$$

respectively. Hence $A^c \in \mathcal{A}_0$. Next, suppose $B_1, B_2 \in \mathcal{A}_0$. *Case 1:* $\{0\}$ is not a subset of both B_1 and B_2. Then $B_1 \cap B_2 \in \mathcal{A}_0'$ and hence $B_1 \cap B_2 \in \mathcal{A}_0$. *Case 2:* $\{0\}$ is a subset of both B_1 and B_2. Then $B_1 \cap (B_2 \setminus \{0\}) \in \mathcal{A}_0'$ by case 1, and

$$(B_1 \cap (B_2 \setminus \{0\})) \cup \{0\} = B_1 \cap B_2 \in \mathcal{A}_0$$

by definition of $\mathcal{A}_0$. Hence $\mathcal{A}_0$ is a field on $[0,1]$.

From **Chung Example 2.2.2**, the Borel sets on $[0,1]$ are generated by the Borel sets on $(0,1]$ and the singleton $\{0\}$. In math,

$$\mathscr{B} = \sigma(\mathscr{B}_0, \{0\})$$

Let $A \in \mathcal{A}_0$. Then $A = A'$ or $A = A' \cup \{0\}$ where $A' \in \mathcal{A}_0'$. Note $A' \in \sigma(\mathcal{A}_0') = \mathscr{B}_0$. Therefore, $A' \in \mathscr{B}$. Moreover, $\{0\} \in \mathscr{B}$. Hence by closure of σ-fields, A is an element of $\mathscr{B}$. Since A was arbitrary, $\mathcal{A}_0 \subset \mathscr{B}$. Therefore $\sigma(\mathcal{A}_0) \subset \mathscr{B}$.

Next, let $\mathscr{B}^*$ be the trace of $\sigma(\mathcal{A}_0)$ onto $(0,1]$. In math

$$\mathscr{B}^* = \{A \cap (0,1] : A \in \sigma(\mathcal{A}_0)\}$$

From class notes, 550.620, slide 38, $\mathscr{B}^*$ is a σ-field on $(0,1]$. Since $(0,1] \in \sigma(\mathcal{A}_0)$, it follows that $\mathscr{B}^* \subset \sigma(\mathcal{A}_0)$ by closure of σ-fields. For $A' \in \mathcal{A}_0'$, it is the case that $A' \subset (0,1]$ and $A' \in \mathcal{A}_0 \subset \sigma(\mathcal{A}_0)$. Thus $A' \in \mathscr{B}^*$. Since A' is arbitrary, $\mathcal{A}_0' \subset \mathscr{B}^*$. Hence $\mathscr{B}_0 = \sigma(\mathcal{A}_0') \subset \mathscr{B}^* \subset \sigma(\mathcal{A}_0)$. Finally, since $\{0\} \in \mathcal{A}_0$, it follows that $\{0\} \in \sigma(\mathcal{A}_0)$. The previous two facts show that $\mathscr{B} = \sigma(\mathscr{B}_0, \{0\}) \subset \sigma(\mathcal{A}_0)$. Since containment has been shown in both directions, $\mathscr{B} = \sigma(\mathcal{A}_0)$. In words, the field $\mathcal{A}_0$ on $[0,1]$ generates the σ-field of the Borel sets on $[0,1]$.

Note $Q(\{0\}) = 0$. Hence for $A \in \mathcal{A}_0$, where $A = A'$ or $A = A' \cup \{0\}$ for some $A' \in \mathcal{A}_0'$, it follows that $Q(A) = Q(A')$ or $Q(A') + Q(\{0\}) = Q(A')$. Then by definition of $\mathcal{A}_0$,

$$\{A \in \mathcal{A}_0 : Q(A) < \eta\} = \{A \in \mathcal{A}_0' : Q(A) < \eta\}$$

Note $P(\{0\}) = \mu(\{0\}) = \mu((-\infty, 0]) - \mu((-\infty, 0)) = G(0) - G(0-) = 0$. Hence for $A \in \mathcal{A}_0$, where $A = A'$ or $A = A' \cup \{0\}$ for some $A' \in \mathcal{A}_0'$, it follows that $P(A) = P(A')$ or $P(A') + P(\{0\}) = P(A')$. Therefore, by the two facts above about P and Q, it is the case that

$$\begin{aligned}
&\lim_{\eta \to 0} \sup\{P(A) : A \in \mathcal{A}_0 \text{ and } Q(A) < \eta\} \\
&= \lim_{\eta \to 0} \sup\{P(A) : A \in \mathcal{A}_0' \text{ and } Q(A) < \eta\} \\
&= \lim_{\eta \to 0} \sup\{P(\cup_{i=1}^n (a_i, b_i]) : 0 \le a_1 < b_1 < \cdots < a_n < b_n \le 1, n \ge 0, Q(\cup_{i=1}^n (a_i, b_i]) < \eta\} \\
&= \lim_{\eta \to 0} \sup\{\sum_{i=1}^n \mu((a_i, b_i]) : 0 \le a_1 < b_1 < \cdots < a_n < b_n \le 1, n \ge 0, \sum_{i=1}^n Q((a_i, b_i]) < \eta\} \\
&= \lim_{\eta \to 0} \sup\{\sum_{i=1}^n G(b_i) - G(a_i) : 0 \le a_1 < b_1 < \cdots < a_n < b_n \le 1, n \ge 0, \sum_{i=1}^n b_i - a_i < \eta\} \\
&= \lim_{\eta \to 0} \sup\{\frac{1}{F(1)} \sum_{i=1}^n F(b_i) - F(a_i) : 0 \le a_1 < b_1 < \cdots < a_n < b_n \le 1, n \ge 0, \sum_{i=1}^n b_i - a_i < \eta\}
\end{aligned}$$

Let $\epsilon > 0$. Then by the absolute continuity of F, since $\epsilon F(1) > 0$, there exists a $\delta > 0$, such that if $\sum_{i=1}^n b_i - a_i < \delta$, then $\sum_{i=1}^n F(b_i) - F(a_i) < \epsilon F(1)$. Or in other words,

$$\frac{1}{F(1)} \sum_{i=1}^n F(b_i) - F(a_i) < \epsilon.$$

Notice ϵ is a bound that does not depend on the choice of the a_i's and the b_i's. Therefore, as $\delta \to 0$, i.e. as the bound on $\sum_{i=1}^n b_i - a_i$ tends to 0, it is the case that

$$\frac{1}{F(1)} \sum_{i=1}^n F(b_i) - F(a_i) \to 0$$

uniformly in the choice of $0 \le a_1 < b_1 < \cdots < a_n < b_n \le 1$ and $n \ge 0$. Hence

$$\begin{aligned}
&\lim_{\eta \to 0} \sup\{P(A) : A \in \mathcal{A}_0 \text{ and } Q(A) < \eta\} \\
&= \lim_{\eta \to 0} \sup\{\frac{1}{F(1)} \sum_{i=1}^n F(b_i) - F(a_i) : 0 \le a_1 < b_1 < \cdots < a_n < b_n \le 1 \text{ and } \sum_{i=1}^n b_i - a_i < \eta\} \\
&= 0
\end{aligned}$$

and condition $2'$ holds. By the first part of this homework, condition 2 holds. Thus by the Little Radon-Nykodym theorem, there exists $X \ge 0$ such that $P(A) = \int_A X dQ$ for all $A \in \mathscr{B}$. In particular, take $A = [0, t]$, then

$$F(t)/F(1) = G(t) = \mu((-\infty, t]) = \mu([0, t]) = P([0, t]) = P(A) = \int_A X dQ = \int_0^t X(s) ds$$

Hence there exists a function, $F(1)X$, such that

$$F(t) = \int_0^t F(1)X(s) ds$$

for $0 \le t \le 1$ Furthermore, since $F(1) > 0$, $F(1)X \ge 0$. $\square$