James K. Pringle
140.673 Stat Theory
Dr. Constantine Frangakis
Problem set 1
February 6, 2014

Problem set 1

Decision theory and loss functions

Problem 1

Among the population, P, of women who visit phyisicians for screening for a disease, assume that the screening test has specificity and sensitivity as the one discussed in lecture one, where, here, probability statements mean fractions of women in the population P (for example, specificity of 98% means that, of all true negative women in P, 98% would test negative.). For a woman i, denote $\theta(i) = 1$ if the woman is truly positive, and 2 if truly negative; and denote $(l_i(a_1), l_i(a_2))$ to be the loss to that woman if treated, or if not treated, respectively (in the last expressions, her true status is captured already in the notation "i"). Suppose that the averages of the losses in the diseased and non-diseased women, if treated and if not treated, are:

$$\begin{aligned} l(1, a_1) &:= E\{l_i(a_1)|\theta(i) = 1\} = 2 \\ l(1, a_2) &:= E\{l_i(a_2)|\theta(i) = 1\} = 5 \\ l(2, a_1) &:= E\{l_i(a_1)|\theta(i) = 2\} = 1 \\ l(2, a_2) &:= E\{l_i(a_2)|\theta(i) = 2\} = 0 \end{aligned}$$

but that l_i may generally vary from woman to woman, and that among women of a particular status (diseased, or not diseased), the losses l_i may be correlated with the value X_i that the diagnostic test would show for woman i. For the strategy s defined as $s(X_i) = a_1$ if X_i is positive, and $s(X_i) = a_2$ if X_i is negative, which of the following conditions 1.-3. would make the losses

$$E\{l_i(s(X_i))|\theta(i)\} \text{ and } E\{l(\theta(i), s(X_i))|\theta(i)\} \tag{1}$$

equal and why?

1. For a fixed action a, $l_i(a)$ is constant within women of common disease status $\theta(i)$.
2. For a fixed action a, X_i is independent of $l_i(a)$.
3. For a fixed action a, X_i is independent of $l_i(a)$ given $\theta(i)$.

Proof. If $X_i \geq 0$, then $s(X_i) = a_1$ and if $X_i < 0$, then $s(X_i) = a_2$. Thus,

$$l(\theta(i), s(X_i)|\theta_i) = l(\theta(i), a_1)I_{X_i \geq 0} + l(\theta(i), a_2)I_{X_i<0}$$

Hence, the right-hand side becomes

$$\begin{aligned} E\{l(\theta(i), s(X_i)|\theta_i) \mid \theta(i)\} &= E\{l(\theta(i), a_1)I_{X_i \geq 0} + l(\theta(i), a_2)I_{X_i<0} \mid \theta(i)\} && (2)\\ &= E\{l(\theta(i), a_1)I_{X_i \geq 0} \mid \theta(i)\} + E\{l(\theta(i), a_2)I_{X_i<0} \mid \theta(i)\} && (3) \end{aligned}$$

Since the expression above is conditioned on $\theta(i)$, the expressions $l(\theta(i), a_1)$ and $l(\theta(i), a_2)$ are constant by hypothesis. The right-hand side can be further simplified to

$$l(\theta(i), a_1)E\{I_{X_i \geq 0} \mid \theta(i)\} + l(\theta(i), a_2)E\{I_{X_i<0} \mid \theta(i)\} \quad (4)$$

Similarly, the left-hand side can be written as

$$\begin{aligned} E\{l_i(s(X_i)) \mid \theta(i)\} &= E\{l_i(a_1)I_{X_i \geq 0} + l_i(a_2)I_{X_i<0} \mid \theta(i)\} && (5)\\ &= E\{l_i(a_1)I_{X_i \geq 0} \mid \theta(i)\} + E\{l_i(a_2)I_{X_i<0} \mid \theta(i)\} && (6) \end{aligned}$$

Examine condition 1. For fixed a, Let $l_i(a)$ be constant within women of common disease status $\theta(i)$. The expectation of a constant function is that same constant, hence

$$E\{l_i(a) \mid \theta(i) = j\} = l_i(a)|_{\theta(i)=j} \quad (7)$$

Combining (6) with (7)

$$\begin{aligned} E\{l_i(s(X_i)) \mid \theta(i)\} &= E\{l_i(a_1)I_{X_i \geq 0} \mid \theta(i)\} + E\{l_i(a_2)I_{X_i<0} \mid \theta(i)\} && (8)\\ &= E\{E\{l_i(a_1) \mid \theta(i)\}I_{X_i \geq 0} \mid \theta(i)\} + E\{E\{l_i(a_2) \mid \theta(i)\}I_{X_i<0} \mid \theta(i)\} && (9)\\ &= E\{l_i(a_1) \mid \theta(i)\}E\{I_{X_i \geq 0} \mid \theta(i)\} + E\{l_i(a_2) \mid \theta(i)\}E\{I_{X_i<0} \mid \theta(i)\} && (10)\\ &= l(\theta(i), a_1)E\{I_{X_i \geq 0} \mid \theta(i)\} + l(\theta(i), a_2)E\{I_{X_i<0} \mid \theta(i)\} && (11) \end{aligned}$$

Clearly, (4) and (11) are the same, and therefore the right- and left-hand sides of (1) are equal given condition one.

Given condition three, for a fixed action a, X_i is independent of $l_i(a)$ given $\theta(i)$. Therefore the exents in $\sigma(X_i)$ and $\sigma(l_i(a))$ are independent given $\theta(i)$. Since the expectation of independent events is the production of the expectation of those events, from (6) it follows that

$$\begin{aligned} E\{l_i(s(X_i)) \mid \theta(i)\} &= E\{l_i(a_1)I_{X_i \geq 0} \mid \theta(i)\} + E\{l_i(a_2)I_{X_i<0} \mid \theta(i)\} && (12)\\ &= E\{l_i(a_1) \mid \theta(i)\}E\{I_{X_i \geq 0} \mid \theta(i)\} + E\{l_i(a_2) \mid \theta(i)\}E\{I_{X_i<0} \mid \theta(i)\} && (13)\\ &= l(\theta(i), a_1)E\{I_{X_i \geq 0} \mid \theta(i)\} + l(\theta(i), a_2)E\{I_{X_i<0} \mid \theta(i)\} && (14) \end{aligned}$$

Clearly, (4) and (14) are the same, and therefore the right- and left-hand sides of (1) are equal given condition three.

Finally, condition 2 says that for a fixed action a, X_i is independent of $l_i(a)$. This however does not imply that conditional on the event $\theta(i)$ that X_i is independent of $l_i(a)$. For a counterexample, let $\cup_{i=1}^4 A_i = \Omega$ and let the A_i be disjoint.

Event	P	X	Y	Z
A_1	$P(A_1) = 0.25$	$X = 0$	$Y = 0$	$Z = 0$
A_2	$P(A_2) = 0.25$	$X = 1$	$Y = 0$	$Z = 0$
A_3	$P(A_3) = 0.25$	$X = 0$	$Y = 1$	$Z = 0$
A_4	$P(A_4) = 0.25$	$X = 1$	$Y = 1$	$Z = 1$

Clearly, X and Y are independent. However, conditional on $Z = 0$,

$$0 = P(X = 1 \cap Y = 1 \mid Z = 0) \neq P(X = 1 \mid Z = 0)P(Y = 1 \mid Z = 0) = 1/3^2 = 1/9$$

Therefore, X and Y are not conditionally independent (on Z), and it is shown that independence does not in general imply conditional independence. Hence,

$$\begin{aligned} E\{l_i(s(X_i)) \mid \theta(i)\} &= E\{l_i(a_1)I_{X_i \geq 0} \mid \theta(i)\} + E\{l_i(a_2)I_{X_i<0} \mid \theta(i)\} && (15) \\ &\neq E\{l_i(a_1) \mid \theta(i)\}E\{I_{X_i \geq 0} \mid \theta(i)\} + E\{l_i(a_2) \mid \theta(i)\}E\{I_{X_i<0} \mid \theta(i)\} && (16) \\ &= l(\theta(i), a_1)E\{I_{X_i \geq 0} \mid \theta(i)\} + l(\theta(i), a_2)E\{I_{X_i<0} \mid \theta(i)\} && (17) \end{aligned}$$

Therefore the right- and left-hand sides of (1) are not in general equal given condition two. □

Problem 2

Now assume that the way the test X_i is determined is by measuring a continuous variable X_i^*, and calling X_i positive if $X^* > 0$, otherwise calling X_i negative. Assuming that $\mathrm{pr}(X_i^*|\theta(i))$ is normal with variance 1, find $E(X_i^*|\theta(i))$ for the two disease conditions. Also, assume that, for a fixed action a, $\mathrm{pr}(l_i(a)|\theta(i))$ is normal with the means given above, variance 10, and that $cor(l_i(a), X_i^*|\theta(i)) = 0.7$. Using simulation of 1000 diseased and 1000 non-diseased women, or otherwise, estimate the two average losses in (1).

Proof. From the lecture notes, the specificiy of the test is 0.98 and the sensitivity is 0.94. Thus $P(X_i^* > 0|\theta(i) = 1) = 0.94$. Since we assume a normal distribution, with variance $\sigma^2 = 1$, we have

$$\begin{aligned} 0.94 &= P(X^* > 0|\theta(i) = 1) \\ 0.94 &= P((X^* - \mu_1)/\sigma > -\mu_1/\sigma|\theta(i) = 1) \\ 0.94 &= P(Z > -\mu_1|\theta(i) = 1) \end{aligned}$$

Thus we solve for $\mu_1 = E(X_i^*|\theta(i) = 1)$ in $1 - \Phi(-\mu_1) = 0.94$ where Φ is the distribution function for the standard normal distribution. It follows that $\mu_1 = -\Phi^{-1}(0.06) = -\texttt{qnorm}(0.06) = 1.5548$.

Similarly solving for $\mu_2 = E(X_i^*|\theta(i) = 2)$, one finds that

$$\begin{aligned} 0.98 &= P(X^* < 0|\theta(i) = 2) \\ 0.98 &= P((X^* - \mu_2)\ \sigma < -\mu_2/\sigma|\theta(i) = 2) \\ 0.98 &= P(Z < -\mu_2|\theta(i) = 2) \end{aligned}$$

So $\mu_2 = -\Phi^{-1}(0.98) = -\texttt{qnorm}(0.98) = -2.0537$.

Simulation

The algorithm is to

(1) Choose $\theta = 1$ and n, the number of simulated scores

(2) Generate n of X_i^*, the distribution of which depends on θ.

(3) Generate $l_i(s(X_i))$, the distribution of which depends on X_i^* and θ, using bivariate normal conditional distribution. Average these results.

(4) Generate $l(\theta(i), s(X_i^*))$, the distribution of which has two values given θ and the value depends on X_i. Average these results.

(5) Set $\theta = 2$ and $n = n$, go to (2)

```
library(xtable)

set.seed(2014 - 2 - 5)

# Generate a random l(a) | X* = x
r_cond <- function(x, theta = 1) {
    rho <- 0.7
    sigma_l <- sqrt(10)
    sigma_x <- 1

    # Mean of l(a1) and l(a2) given a theta value
    mu_a1 <- 2
    mu_a2 <- 5
    if (theta == 2) {
        mu_a1 <- 1
        mu_a2 <- 0
    }
```

```
    mu_l <- ifelse(x >= 0, mu_a1, mu_a2)
    # Mean of X* given a theta value
    mu_x <- ifelse(theta == 1, -qnorm(0.06), -qnorm(0.98))

    new_mean <- mu_l + sigma_l/sigma_x * rho * (x - mu_x)
    new_var <- (1 - rho^2) * sigma_l^2
    new_sd <- sqrt(new_var)
    val <- rnorm(length(x), new_mean, new_sd)
    return(val)
}

r_cond_ave <- function(x, theta = 1) {
    mean(r_cond(x, theta))
}

mean_loss <- function(x, theta = 1) {
    mu_a1 <- 2
    mu_a2 <- 5
    if (theta == 2) {
        mu_a1 <- 1
        mu_a2 <- 0
    }
    return(ifelse(x >= 0, mu_a1, mu_a2))
}

mean_loss_ave <- function(x, theta = 1) {
    mean(mean_loss(x, theta))
}

compare_eq_1 <- function(n_sample, theta = 1) {
    mu_x <- ifelse(theta == 1, -qnorm(0.06), -qnorm(0.98))
    x_star <- rnorm(n_sample, mu_x, 1)
    lhs <- r_cond_ave(x_star, theta)
    rhs <- mean_loss_ave(x_star, theta)
    return(c(lhs, rhs))
}

get_table <- function(n_sample) {
    theta_1 <- compare_eq_1(n_sample, 1)
    theta_2 <- compare_eq_1(n_sample, 2)
```

```
    df <- as.data.frame(rbind(theta_1, theta_2))
    colnames(df) <- c("$E\\{l_i(s(X_i))| \\theta(i)\\}$", "$E\\{l(\\theta(i), s(X_i))|\\t
    rownames(df) <- c("$\\theta(i)=1$", "$\\theta(i)=2$")
    return(df)
}

results <- get_table(1000)
```

The simulation results are summarized in the following table.

	$E\{l_i(s(X_i))\|\theta(i)\}$	$E\{l(\theta(i), s(X_i))\|\theta(i)\}$
$\theta(i) = 1$	2.23	2.16
$\theta(i) = 2$	0.07	0.02

□

Problem 3

Assume that the random variable X has finite $E|X|$ and is continuous (has a density). Show that $E|X - a|$ is minimun at $a = \text{median}(X)$.

Proof. Let $f(x)$ be the density function of X. Calculating,

$$\begin{aligned}
\frac{d}{da}E|X-a| &= \frac{d}{da}\int_{-\infty}^{\infty}|x-a|f(x)dx \\
&= \int_{-\infty}^{\infty}\frac{d}{da}|x-a|f(x)dx \qquad \text{by Liebniz's Rule} \\
&= \int_{-\infty}^{a}f(x)dx + \int_{a}^{\infty}-f(x)dx \\
&= \int_{-\infty}^{a}f(x)dx - \int_{a}^{\infty}f(x)dx \\
&= F(a) - (1 - F(a)) \\
&= 2F(a) - 1
\end{aligned}$$

Setting this derivative to 0, we have equality for the value of a for which $F(a) = 0.5$, or when $a = \text{median}(X)$. The second derivative with respect to a is $\frac{d}{da}(2F(a) - 1) = 2f(a) > 0$. Thus the a that we found is a minimum for the original function. □

Problem 4

We want to estimate the true value of the scalar θ, and we have a loss function $l(\theta, a) = |\theta - a|$. Based on previous similar studies, we believe that, a priori, $P(\theta) = N(\mu_0, \tau_0^2)$, where μ_0 and τ_0 are known values. To help us estimate θ, we design a study that gives us data X where $P(X|\theta) = N(\theta, \sigma_0^2)$, and where σ_0^2 is assumed known.

Part 1

Find the posterior distribution $P(\theta|X)$.

Proof. As a lemma, let the likelihood $\mathrm{pr}(X \mid \theta) = N_k(\theta, \Sigma)$ be multivariate normal with prior $\mathrm{pr}(\theta) = N_k(\mu, V)$ multivariate normal. The posterior $\mathrm{pr}(\theta \mid X)$ is proportional to the product of the likelihood and the prior. Hence

$$\mathrm{pr}(\theta \mid X) \propto \exp\left[-\frac{1}{2}(x-\theta)'\Sigma^{-1}(x-\theta) - \frac{1}{2}(\theta-\mu)'V^{-1}(\theta-\mu)\right] \tag{18}$$

$$\propto \exp\left[-\frac{1}{2}(\theta'\Sigma^{-1}\theta + \theta'V^{-1}\theta - 2\theta'\Sigma^{-1}x - 2\theta'V^{-1}\mu)\right] \tag{19}$$

$$= \exp\left[-\frac{1}{2}(\theta'(\Sigma^{-1} + V^{-1})\theta - 2\theta'(\Sigma^{-1}x + V^{-1}\mu))\right] \tag{20}$$

Therefore, the posterior is proportional to an unnormalized multivariate normal distribution with

$$N_k((\Sigma^{-1} + V^{-1})^{-1}(\Sigma^{-1}x + V^{-1}\mu), (\Sigma^{-1} + V^{-1})^{-1}) \tag{21}$$

For this problem, there are multiple observations X and a common mean θ. Notice

$$\mathrm{pr}(x_1, x_2, \ldots, x_n \mid \theta) \propto_\theta \exp\left[-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n}(x_i - \theta)^2\right] \tag{22}$$

$$\propto_\theta \exp\left[-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n}(x_i^2 - 2x_i\theta + \theta^2)\right] \tag{23}$$

$$\propto_\theta \exp\left[-\frac{n}{2\sigma_0^2}\sum_{i=1}^{n}(2\bar{x_n}\theta + \theta^2)\right] \tag{24}$$

$$\propto_\theta \exp\left[-\frac{n}{2\sigma_0^2}\sum_{i=1}^{n}(\bar{x_n} - \theta)^2\right] \tag{25}$$

which is recognized as the normal distribution $\mathrm{pr}(\bar{x_n}|\theta) = N(\theta, \frac{\sigma_0^2}{n})$. Calculating more,

$$\mathrm{pr}(\theta \mid x_1, x_2, \ldots, x_n) \propto_\theta \mathrm{pr}(x_1, x_2, \ldots, x_n \mid \theta)\mathrm{pr}(\theta) \tag{26}$$

$$\propto_\theta \mathrm{pr}(\bar{x_n}|\theta)\mathrm{pr}(\theta) \tag{27}$$

$$\propto_\theta \mathrm{pr}(\theta \mid \bar{x_n}) \tag{28}$$

which according to the lemma, follows the

$$N\left(\left(\frac{n}{\sigma_0^2}+\frac{1}{\tau_0^2}\right)^{-1}\left(\frac{n}{\sigma_0^2}\bar{x_n}+\frac{1}{\tau_0^2}\mu_0\right),\left(\frac{n}{\sigma_0^2}+\frac{1}{\tau_0^2}\right)^{-1}\right)=N\left(\frac{\tau_0^2}{\frac{\sigma_0^2}{n}+\tau_0^2}\bar{x_n}+\frac{\sigma_0^2}{\sigma_0^2+n\tau_0^2}\mu_0,\left(\frac{n}{\sigma_0^2}+\frac{1}{\tau_0^2}\right)^{-1}\right) \tag{29}$$

Source: Berkeley course notes □

Part 2

Using Exercise 3, find the Bayes estimator for this problem, i.e. the estimate $s(X)$ that minimizes $E\{E(l(\theta, s(X)) \mid \theta)\}$, where the outer expectation is with respect to the prior distribution for θ.

Proof. According to the class notes, slide notes (Chapter 1, pg. 20),

$$E_\theta[E_{X\mid\theta}\{l(\theta,s(X))\mid\theta\}]=E_X[E_{\theta\mid X}\{l(\theta,s(X))\mid X\}] \tag{30}$$

The $s(X)$ is chosen that minimizes the posterior loss

$$E_{\theta\mid X}\{l(\theta,s(x))\mid X=x\}=\int_\theta l(\theta,s(x))\mathrm{pr}(\theta\mid X=x)d\theta \tag{31}$$

To do that, first the “no-data problem” is solved. Assume a working prior $\pi(\theta)$. The s^* that minimizes

$$\int_\theta l(\theta,s^*)\pi(\theta)d\theta=E_\pi[|\theta-s^*|] \tag{32}$$

is $s^*_{(\pi)}=\mathrm{median}(\theta)$ by problem 3. Therefore, by slide notes (Chapter 1, pg. 21), the Bayes estimator is

$$s^*(X)=s^*_{\mathrm{pr}(\theta\mid X)}=\mathrm{median}(\theta)=\frac{\tau_0^2}{\frac{\sigma_0^2}{n}+\tau_0^2}\bar{x_n}+\frac{\sigma_0^2}{\sigma_0^2+n\tau_0^2}\mu_0$$

□

Problem 5

Refer to Problem 4, and suppose we have iid observations from the likelihood. By considering a sequence of priors, each as in problem 4, but with mean 0 and τ_0 increasing with the sequence, show that the sample average is a minimax estimator. (Hint: note that the sample average is equalizer).

Proof. Let $\{\tau_j\}\uparrow\infty$ be a sequence increasing to infinity. Let θ_j be distributed as $N(0,\tau_j^2)$. Then the posterior distribution $\mathrm{pr}(\theta_j|x_1,x_2,\ldots,x_n)$ is

$$N\left(\frac{\tau_j^2}{\frac{\sigma_0^2}{n}+\tau_j^2}\bar{x_n},\left(\frac{n}{\sigma_0^2}+\frac{1}{\tau_j^2}\right)^{-1}\right)$$

It is easy to see the limit of the posterior is

$$\lim_{j\to\infty} \mathrm{pr}(\theta_j \mid x_1, x_2, \ldots, x_n) = N\left(\bar{x_n}, \frac{\sigma_0^2}{n}\right)$$

We will use the following result to prove the sample average is minimax.

Result 1: If s_j is a Bayes rule with respect to π_j, with $L(\theta, s_j) \to c \in \mathbb{R}$ and if there exists a strategy s_0 such that $L(\theta, s_0) \leq c$ for every $\theta \in \Theta$, then s_0 is minimax with "value"

$$\inf_s \sup_\theta L(\theta, s) = c = \sup_\theta L(\theta, s_0)$$

Suppose $s|\theta \sim N(\mu, \sigma^2)$. Calculating,

$$\begin{aligned}
E_{s|\theta}[l(\theta, s)] &= \int_{-\infty}^{\infty} |\theta - s| f_{s|\theta}(s) ds && (33)\\
&= \int_{-\infty}^{\theta} (\theta - s) f_{s|\theta}(s) ds + \int_{\theta}^{\infty} (s - \theta) f_{s|\theta}(s) ds && (34)\\
&= \int_{-\infty}^{\theta} (\theta - \mu + \mu - s) f_{s|\theta}(s) ds + \int_{\theta}^{\infty} (s - \mu + \mu - \theta) f_{s|\theta}(s) ds && (35)\\
&= (\theta - \mu) F_{s|\theta}(\theta) + (\mu - \theta)(1 - F_{s|\theta}(\theta)) + \int_{-\infty}^{\theta} (\mu - s) f_{s|\theta}(s) ds + \int_{\theta}^{\infty} (s - \mu) f_{s|\theta}(s) ds && (36)\\
&= (\theta - \mu)(2F_{s|\theta}(\theta) - 1) + \int_{-\theta}^{\infty} (s - \mu) f_{s|\theta}(s) ds + \int_{\theta}^{\infty} (s - \mu) f_{s|\theta}(s) ds && (37)
\end{aligned}$$

And doing a u substitution with $u = \frac{s-\mu}{\sigma}$

$$\begin{aligned}
\int_{-\theta}^{\infty} (s - \mu) f_{s|\theta}(s) ds + \int_{\theta}^{\infty} (s - \mu) f_{s|\theta}(s) ds &= \frac{\sigma}{\sqrt{2\pi}} \left(\int_{-\frac{\theta-\mu}{\sigma}}^{\infty} u \exp\{-0.5u^2\} + \int_{\frac{\theta-\mu}{\sigma}}^{\infty} u \exp\{-0.5u^2\} \right) && (38)\\
&= \frac{\sigma}{\sqrt{2\pi}} \left(2 \exp\left\{ -\frac{1}{2\sigma^2}(\theta - \mu)^2 \right\} \right) && (39)
\end{aligned}$$

Hence

$$E_{s|\theta}[l(\theta, s)] = (\theta - \mu)(2F_{s|\theta}(\theta) - 1) + \frac{2\sigma}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2\sigma^2}(\theta - \mu)^2 \right\} \tag{40}$$

Note that $E_{s|\theta}[l(\theta, s)] > 0$ when $|\theta - \mu| > 0$.

Since

$$s_{\pi_j}(X) = \frac{\tau_j^2}{\frac{\sigma_0^2}{n} + \tau_j^2} \bar{x_n}$$

we have that

$$s_{\pi_j}(X)|\theta \sim N\left(\frac{\tau_j^2}{\frac{\sigma_0^2}{n}+\tau_j^2}\theta, \frac{\sigma_0^2}{n}\left(\frac{\tau_j^2}{\frac{\sigma_0^2}{n}+\tau_j^2}\right)^2\right)$$

Using the result above,

$$L(\theta, s_{\pi_j}) = (\theta - \frac{\tau_j^2}{\frac{\sigma_0^2}{n}+\tau_j^2}\theta)(2F_{s_{\pi_j}|\theta}(\theta)-1) + \frac{2\sigma_0}{\sqrt{2\pi n}}\left(\frac{\tau_j^2}{\frac{\sigma_0^2}{n}+\tau_j^2}\right)\exp\left\{-\frac{n}{2\sigma_0^2}\left(\frac{\frac{\sigma_0^2}{n}+\tau_j^2}{\tau_j^2}\right)\left(\theta - \frac{\tau_j^2}{\frac{\sigma_0^2}{n}+\tau_j^2}\theta\right)^2\right\} \tag{41}$$

Taking the limit,

$$\lim_{j\to\infty} L(\theta, s_{\pi_j}) = \frac{2\sigma_0}{\sqrt{2\pi n}}$$

Now $L(\theta, \bar{x_n}) = E_{\bar{x_n}|\theta}[l(\theta, \bar{x_n})]$ and since

$$\bar{x_n}|\theta \sim N\left(\theta, \frac{\sigma_0^2}{n}\right)$$

it follows

$$L(\theta, \bar{x_n}) = \frac{2\sigma_0}{\sqrt{2\pi n}}$$

That is $\bar{x_n}$ is equalizer. Therefore, we have the conclusion of the **Result 1** above, i.e. that the sample average is equalizer. □