Topological quantum computation

using Fibonacci anyons

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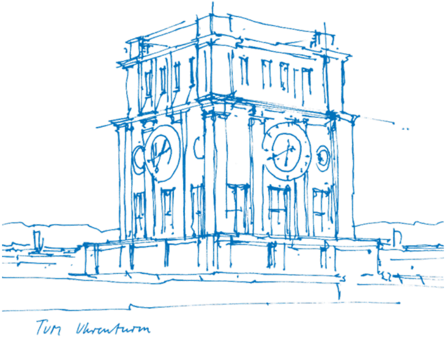

Outline

1 Introduction

2 Background on particles

3 Braid Group

4 Fibonacci anyons

5 Computing using braids

6 examples of quantum gates

Idea

- Classical quantum computing suffers from decoherence caused by interactions with the environment
- Idea: local pertubations do not affect topological structures
- First mentioned by Kitaev in 1997[1]

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[1] A. Yu Kitaev. *Fault-tolerant quantum computation by anyons.* arXiv:quant-ph/9707021. July 1997. DOI: 10.48550/arXiv.quant-ph/9707021. URL: http://arxiv.org/abs/quant-ph/9707021 (visited on 10/26/2024).

[2] Quantum Matter Theory Research Group. *Quantum Matter Theory Research Group | RIKEN Center for Emergent Matter Science (CEMS).* https://cems.riken.jp/en/laboratory/qmtrt. Accessed: 2024-12-20.

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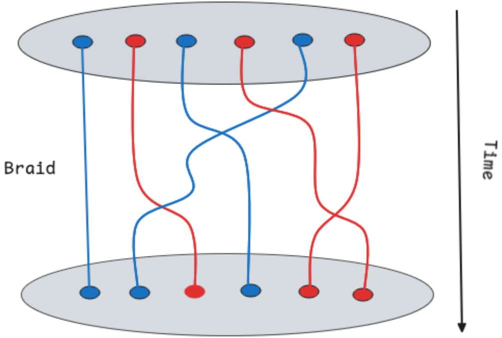
Topological quantum field theories

Topological quantum field theories depend on the topology (not the geometry!) of the spacetime-history of the particles' locations in the space.

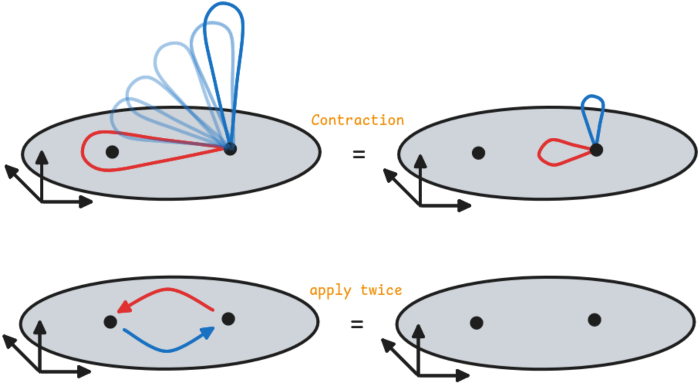


Particles

Traditional statistics

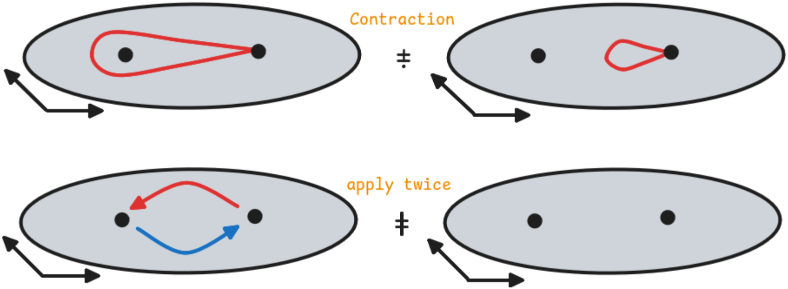
All particles are identical (up to position).

- In 3D systems, exchanging twice does not change the wavefunction
- Exchange operation is square root of identity (1 or -1)
- Topologically, this is allowed because of:



Exchange in 2D

- In 2 dimensions, we cannot contract the loop to the identity



- Naive approach: Fractional particle exchange with

$$\psi(x_1, x_2) = e^{i\theta}\psi(x_2, x_1)$$

Particles
Fractional statistics

- Particle exchange: $\psi(x_1, x_2) = e^{i\theta}\psi(x_2, x_1)$
- 1-Dimensional representation because of nondegenerate groundstate
- This type of quasiparticle is called an abelian "Anyon"

- Anabelian anyons: $\psi(x_1, x_2) = U\psi(x_2, x_1)$ for some $U \in U(k), k \geq 2$
- 2 or higher dimensional representation with a degenerate groundstate
- Anabelian anyons allow universal computation

- This interpretation is consistent with quantum mechanics[3]
- Anyons arent really particles, but rather defects in the ground state.

[3] J. M. Leinaas and J. Myrheim. “On the theory of identical particles”. en. In: *Il Nuovo Cimento B (1971-1996)* 37.1 (Jan. 1977), pp. 1–23. ISSN: 1826-9877. DOI: 10.1007/BF02727953. URL: https://doi.org/10.1007/BF02727953 (visited on 12/19/2024).

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Braid diagrams

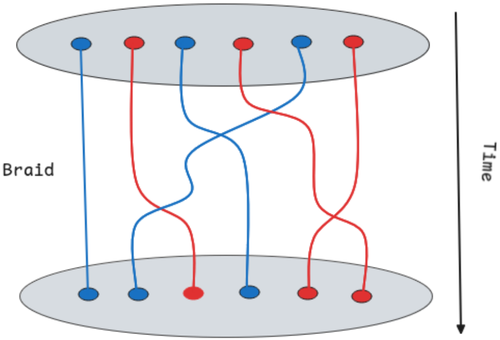


Figure 1 Time slices of 6 particles in $\mathbb{R}^2$

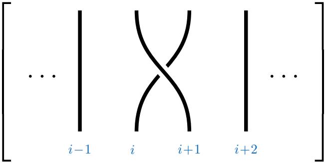
Braid Group

- The braid group on n points is generated by the Artin relations:

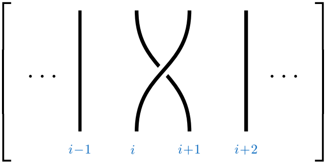
$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j| > 1 \tag{1}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad ; i \in [0, n-1] \tag{2}$$

$\sigma_i =$

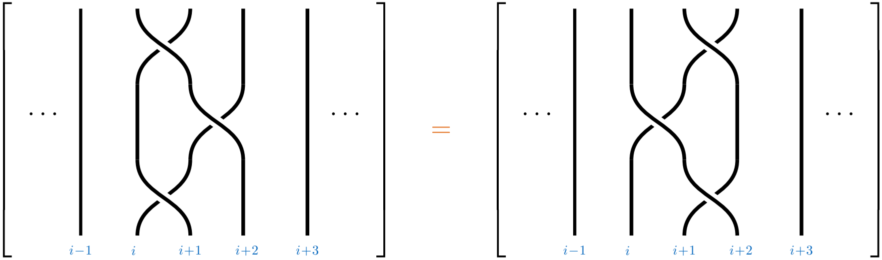


$\sigma_i^{-1} =$



Yang Baxter equation

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$



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Fusion and splitting

- Anyons can be created from a vacuuum in pairs
- All anyons carry a charge q_i, classified into n types
- Every class of anyons have a trivial "vacuum" type 1
- Anyons can be fused together to create a new anyon with a superposition of types

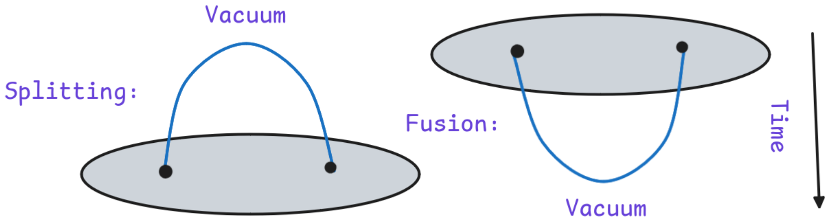

Figure 2 Splitting and Fusion of anyons

Fibonacci anyons
Fusion rules

- Two types: $\{1, \tau\}$
- Fusion rules:

$$1 \otimes 1 = 1$$
$$1 \otimes \tau = \tau$$
$$\tau \otimes 1 = \tau$$
$$\tau \otimes \tau = 1 \oplus \tau$$

- $\otimes$: fusion operator tensor product
- $\oplus$: superposition operator
- Probabilites of fusing to 1 or τ are $\frac{1}{\phi^2}$ and $\frac{1}{\phi}$

Fibonacci anyons

$$
\begin{aligned}
\tau \otimes \tau &= 1 \oplus \tau \\
(\tau \otimes \tau) \otimes \tau &= (1 \oplus \tau) \otimes \tau \\
&= \tau \oplus (\tau \otimes \tau) \\
&= \tau \oplus \tau \oplus 1 \\
&= 1 \oplus 2\tau \\
\tau^{\otimes 4} &= 2 \cdot 1 \oplus 3\tau \\
\tau^{\otimes 5} &= 3 \cdot 1 \oplus 5\tau
\end{aligned}
$$

Fusion diagrams

We can denote the fusion rules in graphical calculus as:

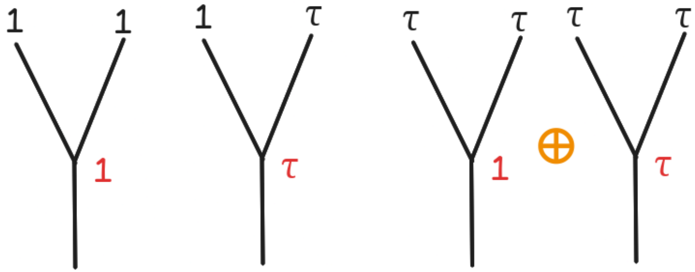

Figure 3 Fusion rules for Fib Anyons in graphical calculus

Fusion diagrams (cont.)

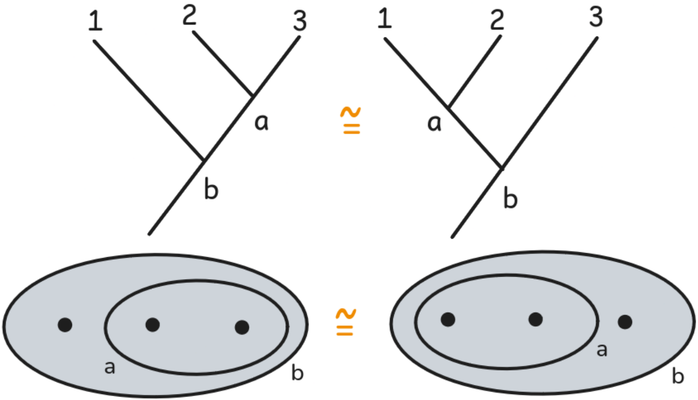

Figure 4 Top: Fusion trees are isomorphic
Bottom: "qubit" notation for fusion trees

Qubits
Fusion basis

There are 3 unique fusion trees with 3τ anyons:

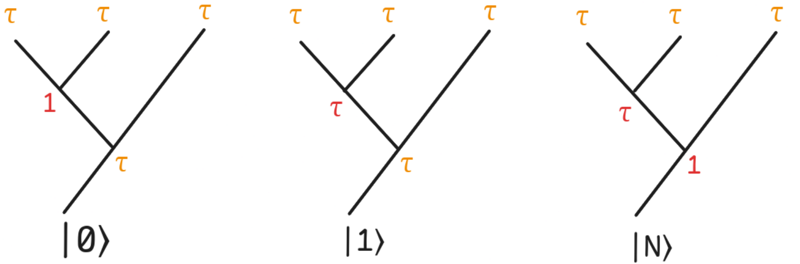

Figure 5 Basis trees for 3 anyons

$$|\psi\rangle_{Fib} = \frac{1}{\phi}|0\rangle + \frac{1}{\sqrt{\phi}}|1\rangle$$

$$\left|\frac{1}{\phi^2} + \frac{1}{\phi}\right| = 1$$

Qubits
representation

Fusion state has to result in same charge

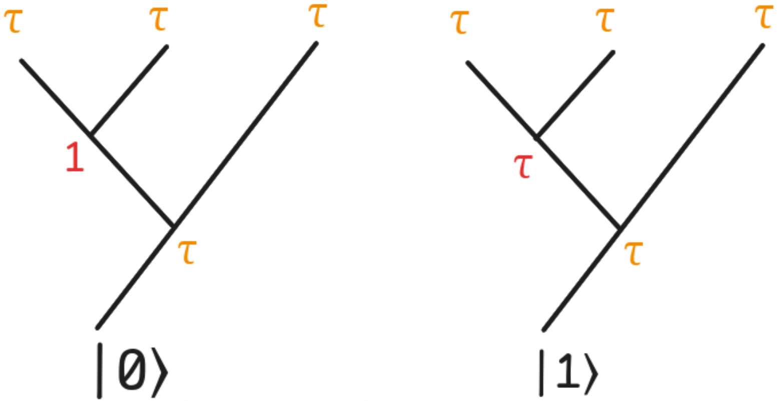

Figure 6 Orthogonal fusion basis

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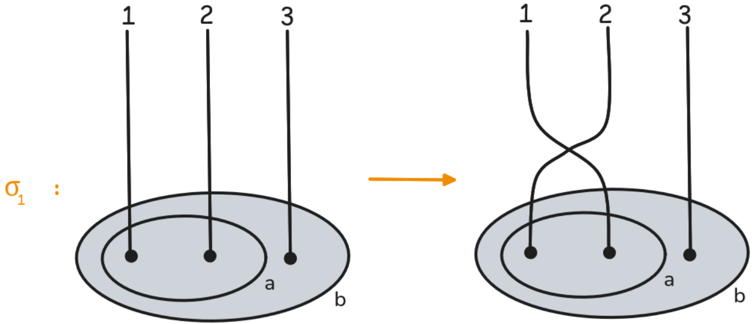
Representation

- We want to find a representation of B_n, specifically its generators σ_i, in some unitary group $U(k)$.
- Operations on single topological qubit $\in B_3$: $\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}$
- Operation on a single classical qubit $\in U(2)$
- to find:

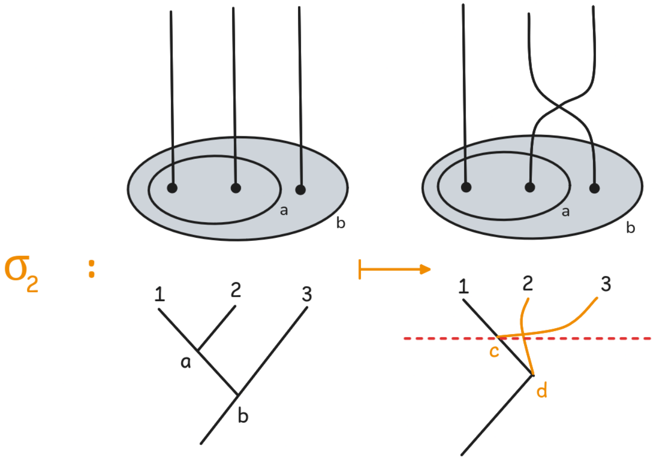
$$\rho(B_3) \to U(2)$$

- Example:



Braiding on fusion trees

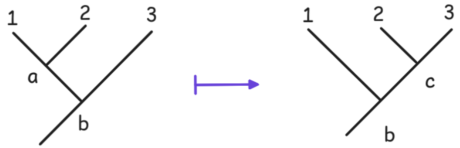
- Braiding in the designated fusion channel is trivial.
- Braiding 2 and 3 "violates" the fusion order / channel, so we have to do a change of base.



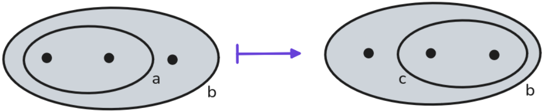
Braiding
F-moves

The F-move:

$$\left[F_b^{123}\right]_a^c : ((1 \otimes 2) \otimes 3, b) \to (1 \otimes (2 \otimes 3), b)$$

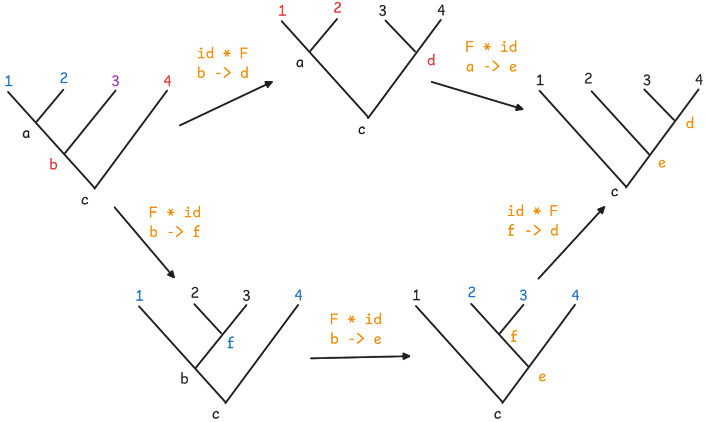


$\left[F_b^{123}\right]_a^c$:



Braiding
F-move : pentagon

There are many ways to describe the same space over fusion channels. In fact, for 4 anyons, we can derive the following identity:



Braiding
F-move : pentagon 2

This is the strongest constraint on F-moves there is. Guaranteed by the *MacLane Coherence theorem*[4].
Algebraically, we can write it as

$$F_c^{a34} F_c^{12e} = \sum_f F_e^{234} F_c^{afd} F_b^{123}$$

Given

$$F_1^{\tau\tau\tau} = F_\tau^{1\tau\tau} = F_\tau^{1\tau 1} = F_\tau^{\tau\tau 1} = 1$$

Using the pentagon we can derive the braid matrix for any theory of anyons, here for Fibonacci anyons:

$$F_\tau^{\tau\tau\tau} := \begin{pmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & \phi^{-1} \end{pmatrix}.$$

[4] Saunders Mac Lane. *Categories for the Working Mathematician*. Vol. 5. Graduate Texts in Mathematics. New York, NY: Springer, 1978. ISBN: 978-1-4419-3123-8 978-1-4757-4721-8. DOI: 10.1007/978-1-4757-4721-8. URL: http://link.springer.com/10.1007/978-1-4757-4721-8 (visited on 12/19/2024).

R-move

- The R-move exchanges two particles
- This is defined if they are in the same fusion channel

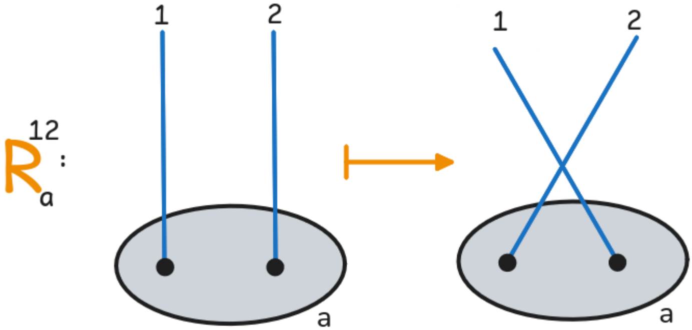


Figure 7 The R-move on two anyons

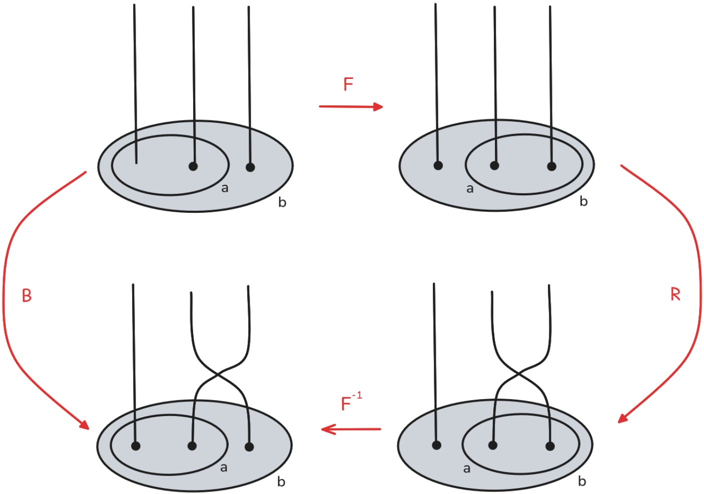


Figure 8 Braiding of the second fusion channel

Hexagon relation

We can derive Coherence relations between R and F using the Yang-Baxter equation 8:

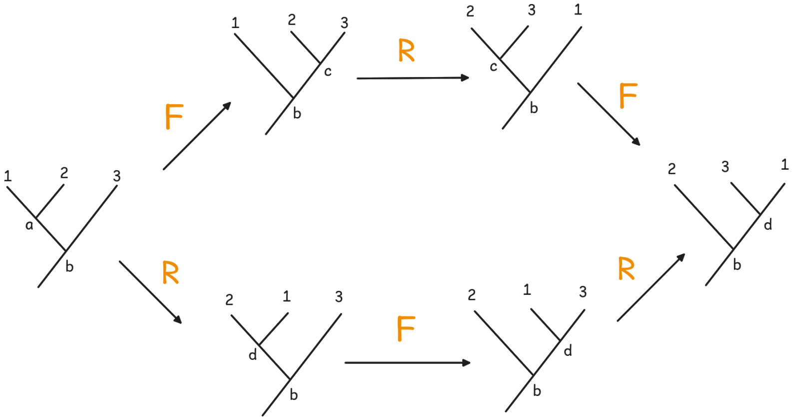


Figure 9 Hexagon relation

B-move

Using the hexagon relation and Yang-Baxter gives:

$$R_{\tau}^{\tau 1} = R_{\tau}^{1\tau} = 1$$

and therefore

$$R^{\tau\tau} = \begin{pmatrix} e^{-4\pi i/5} & 0 \\ 0 & e^{3\pi i/5} \end{pmatrix}$$

Giving

$$B = FRF^{-1} = \begin{pmatrix} \phi^{-1} e^{4i\pi/5} & \phi^{-1/2} e^{-i3\pi/5} \\ \phi^{-1/2} e^{-i3\pi/5} & -\phi^{-1} \end{pmatrix}$$

- Representation acting on the states $|N\rangle, |0\rangle, |1\rangle$

$$\rho(\sigma_1) = \begin{pmatrix} e^{3\pi i/5} & 0 & 0 \\ 0 & e^{-4\pi i/5} & 0 \\ 0 & 0 & e^{3\pi i/5} \end{pmatrix}$$

$$\rho(\sigma_2) = \begin{pmatrix} e^{3\pi i/5} & 0 & 0 \\ 0 & \phi^{-1} e^{4\pi i/5} & \phi^{-1/2} e^{-3\pi i/5} \\ 0 & \phi^{-1/2} e^{-3\pi i/5} & -\phi^{-1} \end{pmatrix}$$

- Lower 2x2 matrix braids computational states.

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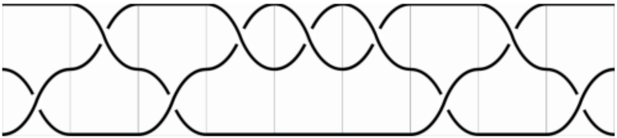
Single qubit gates

- $\overline{\rho(B_3)} \supset U(2)$ (B_3 is dense in $U(2)$) proven by Freedman et al.[5]
- We can represent every $U \in U(2)$ with arbitrary accuracy $\epsilon > 0$
- Example:

$$U_{target} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$U_{target} = \sigma_2 \sigma_1^3 \sigma_2 \quad , \epsilon \approx 0.18$$

- Braid for the Hadamard gate of length 9 with error $\epsilon = 0.079$:



[5] Michael H. Freedman et al. *Topological Quantum Computation*. en. arXiv:quant-ph/0101025. Sept. 2002. URL: `http://arxiv.org/abs/quant-ph/0101025` (visited on 11/01/2024).

Entangling gates

- "Weaving" scheme devised by Bonesteel et al.[6].
- Universal computation using only one moving particle
- recall that braiding with the 1 type anyon does not affect the system

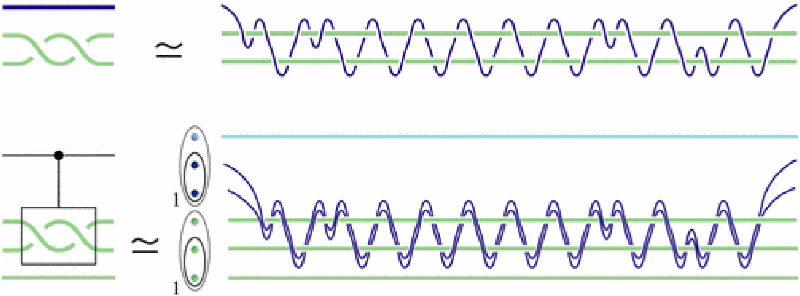
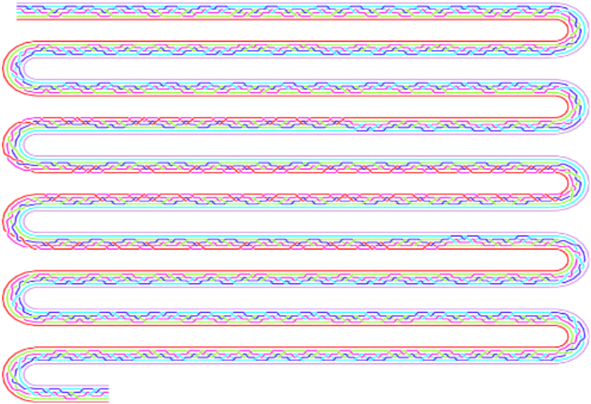


Figure 10 An entangling weave

[6] N. E. Bonesteel et al. "Braid Topologies for Quantum Computation". en. In: *Physical Review Letters* 95.14 (Sept. 2005). arXiv:quant-ph/0505065, p. 140503. ISSN: 0031-9007, 1079-7114. DOI: 10.1103/PhysRevLett.95.140503. URL: http://arxiv.org/abs/quant-ph/0505065 (visited on 10/27/2024).

Solovay-Kitaev Improved CNOT



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