Triangle inequality, Euler's formula, complex maps

In this document (adapted from a **Math 555: Complex Analysis** homework set created by Professor Divakar Vinswanath), basic results of complex analysis are derived.

0.1 Triangle inequality: $|z_1 + z_2| \leq |z_1| + |z_2|$

Proof of triangle inequality.

Let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ where $a_1, a_2, b_1, b_2 \in \mathbb{R}$.

We will first show $2a_1a_2 + 2b_1b_2 \leq 2(a_1^2a_2^2 + b_1^2b_2^2 + a_1^2b_2^2 + a_2^2b_1^2)^{1/2}$ by considering the difference of their squares. Observe:

$$\begin{aligned}
&4(a_1^2a_2^2 + b_1^2b_2^2 + a_1^2b_2^2 + a_2^2b_1^2) - (2a_1a_2 + 2b_1b_2)^2 \\
&\qquad = 4(a_1^2a_2^2 + b_1^2b_2^2 + a_1^2b_2^2 + a_2^2b_1^2) - 4(a_1^2a_2^2 + b_1^2b_2^2 + 2a_1a_2b_1b_2) \\
&\qquad = a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2 \\
&\qquad = (a_1b_2 - a^2b_1)^2 \geq 0
\end{aligned}$$

It now follows that:

$$\begin{aligned}
(a_1 + a_2)^2 + (b_1 + b_2)^2 &= a_1^2 + a_2^2 + b_1^2 + b_2^2 + 2a_1a_2 + 2b_1b_2 \\
&\leq a_1^2 + a_2^2 + b_1^2 + b_2^2 + 2(a_1^2a_2^2 + b_1^2b_2^2 + a_1^2b_2^2 + a_2^2b_1^2)^{1/2} \\
&= (a_1^2 + b_1^2) + 2\big((a_1^2 + b_1^2)(a_2^2 + b_2^2)\big)^{1/2} + (a_2^2 + b_2^2) \\
&= \big((a_1^2 + b_1^2)^{1/2} + (a_2^2 + b_2^2)^{1/2}\big)^2
\end{aligned}$$

This proves the triangle identity. ■

Geometric interpretation. One side length of a triangle is no greater than the sum of the other two side lengths.

Generalization. $|z_1 + ... + z_n| \leq |z_1| + ... + |z_n|$

We carry out inductively on n. Note that for $n = 1$, $|z_1| \leq |z_1|$ is trivial. Let $j \in \mathbb{N}$, and assume $|z_1 + ... + z_j| \leq |z_1| + ... + |z_j|$. By applying triangle inequality, we have

$$|z_1 + ... + z_j + z_{j+1}| \leq |z_1 + ... + z_j| + |z_j + 1| \leq |z_1| + ..._{|}z_j| + |z_{j+1}|$$

So the generalization of triangle inequality holds via induction. ■

Useful corollaries. These relationships follow from triangle inequality:

a. $|z_1 - z_2| = |z_1 + (-z_2)| \leq |z_1| + |-z_2| = |z_1| + |z_2|$

b. $|z_1| = |z_1 + z_2 - z_2| \leq |z_1 + z_2| + |z_2|$

c. $|z_1| - |z_2| = |z_1 - z_2 + z_2| - |z_2| \leq |z_1 - z_2|$ and $|z_2| - |z_1| = |z_2 - z_1 + z_1| - |z_1| \leq |z_2 - z_1|$

An example application. Consider $z^8 + z^3 + 7 = 0$. We can use triangle inequality to show that $|z| \geq 1$. Note that $|z^8 + z^3| = 7$, and suppose for contradiction that $|z| < 1$. Then we have

$$|z^8 + z^3| \leq |z^8| + |z^3| = |z|^8 + |z|^3 < 2 \neq 7$$

So by contradiction, $|z| \geq 1$ ■

0.2 Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$

Alternate interpretation. We observe that $\cos\theta$ and $\sin\theta$ can be rewritten as follows.

$$\begin{aligned}
\cos\theta &= \frac{1}{2}(\cos\theta + i\sin\theta + \cos\theta - i\sin\theta) \\
&= \frac{1}{2}(\cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta)) \\
&= \frac{e^{i\theta} + e^{-i\theta}}{2}
\end{aligned}$$

$$\begin{aligned}
\sin\theta &= \frac{1}{2i}(\cos\theta + i\sin\theta - \cos\theta + i\sin\theta) \\
&= \frac{1}{2i}(\cos\theta + i\sin\theta - \cos(-\theta) - i\sin(-\theta)) \\
&= \frac{e^{i\theta} - e^{-i\theta}}{2i}
\end{aligned}$$

Applying Euler's formula. Euler's formula provides a clean proof of our classic trigonometric identities. As examples, we show versions of sum to product and product to sum.

$$\begin{aligned}
\cos\alpha\cos\beta - \sin\alpha\sin\beta &= \left(\frac{e^{i\alpha} + e^{-i\alpha}}{2}\right)\left(\frac{e^{i\beta} + e^{-i\beta}}{2}\right) - \left(\frac{e^{i\alpha} - e^{-i\alpha}}{2i}\right)\left(\frac{e^{i\beta} - e^{-i\beta}}{2i}\right) \\
&= \left(\frac{e^{i(\alpha+\beta)} + e^{i(\alpha-\beta)} + e^{-i(\alpha-\beta)} + e^{-i(\alpha+\beta)}}{4}\right) \\
&\qquad - \left(\frac{e^{i(\alpha+\beta)} - e^{i(\alpha-\beta)} - e^{-i(\alpha-\beta)} + e^{-i(\alpha+\beta)}}{4}\right) \\
&= \frac{e^{i(\alpha+\beta)} + e^{-i(\alpha+\beta)}}{2} \\
&= \cos(\alpha+\beta)
\end{aligned}$$

$$\begin{aligned}
2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) &= 2\left(\frac{e^{i\frac{\alpha+\beta}{2}} + e^{-i\frac{\alpha+\beta}{2}}}{2}\right)\left(\frac{e^{i\frac{\alpha-\beta}{2}} + e^{-i\frac{\alpha-\beta}{2}}}{2}\right) \\
&= \frac{1}{2}\left(e^{i(\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2})} + e^{i(\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2})} + e^{i(\frac{-\alpha-\beta}{2}+\frac{\alpha-\beta}{2})} + e^{i(\frac{-\alpha-\beta}{2}-\frac{\alpha-\beta}{2})}\right) \\
&= \frac{1}{2}\left(e^{i\alpha} + e^{i\beta} + e^{-i\beta} + e^{-i\alpha}\right) \\
&= \frac{e^{i\alpha} + e^{-i\alpha}}{2} + \frac{e^{i\beta} + e^{-i\beta}}{2} \\
&= \cos\alpha + \cos\beta
\end{aligned}$$

0.3 Maps in the complex plane

Mapping the real line to the unit circle:
Consider $w = \frac{z-i}{z+i}$ where $z \in \mathbb{R}$. Then

$$|w| = \left|\frac{z-i}{z+i}\right| = \frac{|z-i|}{|z+i|} = \frac{\sqrt{z^2+1^2}}{\sqrt{z^2+1^2}} = 1$$

So the real line in the z plane maps to the unit circle in the w plane.

Mapping the unit circle to the imaginary line:
Now consider $z = \frac{w-1}{w+1}$ where $w = x + yi$ and $|w|^2 = x^2 + y^2 = 1$. Then

$$z = \frac{w-1}{w+1}\frac{(\bar{w}+1)}{(\bar{w}+1)} = \frac{|w|^2 + w - \bar{w} - 1}{|w|^2 + w + \bar{w} - 1} = \frac{x + yi - x + yi}{z + x + yi + x - yi} = \frac{2yi}{2(1+x)} = \frac{y}{1+x}i$$

Note that $\frac{y}{1+x} \subseteq \mathrm{Im}$. We next show that $\frac{y}{1+x}$ spans Im. We have $x = \cos\theta$ and $y = \sin\theta$ for $\theta \in (0, 2\pi]$. Notice $\frac{\sin\theta}{1+\cos\theta} = \tan\frac{\theta}{2}$ which spans $\mathbb{R}$. So z indeed maps the unit circle to the entire imaginary line.