Applications of Fractional Linear Transformations

In this document (adapted from a **Math 555: Complex Analysis** homework set created by Professor Divakar Vinswanath), fractional linear transformations are utilized to prove results in complex analysis and geometry.

0.1 Generalized circles and rays about arbitrary fixed points

Consider the fractional linear transformation $w = \frac{z-\alpha}{z-\beta}$. Circles about the origin with $r = |w|$ are the image of $r = \frac{z-\alpha}{z-\beta}$. Rays through the origin with $\arg(w) = \theta$ are the image of $\arg(\frac{z-\alpha}{z-\beta}) = \theta$. We want to investigate what the geometries of these z inputs look like.

Let $\alpha = a_1 + b_1 i$, $\beta = a_2 + b_2 i$, and $z = x + yi$. Let $r > 0, r \neq 1$. We observe:

$$
\begin{aligned}
|\frac{z-\alpha}{z-\beta}| = r \Rightarrow |(x-a_1)+(y-b_1)i| &= |(x-a_2)+(y-b_2)i|r \\
&\Rightarrow (x-a_1)^2 + (y-b_1)^2 = ((x-a_2)^2 + (y-b_2)^2)r^2 \\
&\Rightarrow (1-r^2)x^2 + 2(r^2a_2 - a_1)x + (1-r^2)y^2 + 2(r^2b_2 - b_1)y + (a_1^2 + b_1^2 - r^2a_2^2 - r^2b_2^2) = 0 \\
&\Rightarrow \big(x + \frac{r^2a_2 - a_1}{1-r^2}\big)^2 + \big(y + \frac{r^2b_2 - b_1}{1-r^2}\big)^2 \\
&\qquad = -\big(\frac{a_1^2 + b_1^2 - r^2a_2^2 - r^2b_2^2}{1-r^2}\big) + \big(\frac{r^2a_2 - a_1}{1-r^2}\big)^2 + \big(\frac{r^2b_2 - b_1}{1-r^2}\big)^2
\end{aligned}
$$

Which is the equation for a circle centered at $\big(\frac{r^2a_2-a_1}{1-r^2}, \frac{r^2b_2-b_1}{1-r^2}\big)$.

Let $r = 1$. Then we observe:

$$
\begin{aligned}
|\frac{z-\alpha}{z-\beta}| = r \Rightarrow |(x-a_1)+(y-b_1)i| &= |(x-a_2)+(y-b_2)i|r \\
&\Rightarrow (x-a_1)^2 + (y-b_1)^2 = ((x-a_2)^2 + (y-b_2)^2)r^2 \\
&\Rightarrow x^2 - 2a_1x + a_1^2 + y^2 - 2b_1y + b_1^2 = x^2 - 2a_2x + a_2^2 + y^2 - 2b_2x + b_2^2
\end{aligned}
$$

Which is the equation of a line since our second order terms cancel.

An upsetting fact: the technical term for these circles and lines is *circlines*. They are depicted on the Riemann Sphere in Figure 1.

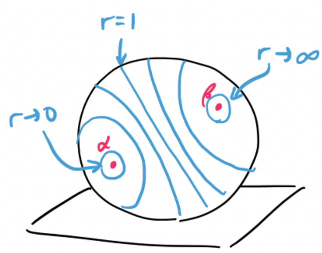

Figure 1: Circlines on the Riemann Sphere

Now let's consider $\arg(\frac{z-\alpha}{z-\beta}) = \theta$. Define $\theta_1 = \arg(z-\alpha), \theta_2 = \arg(z-\beta)$. So $\theta_1 + \theta_2 = \theta$ The point C is defined such that $\triangle zC\beta$ and $\triangle zC\alpha$ are isosceles, as shown in Figure 2. Then the angle $\phi = 360 - (180 - 2\theta_2) - (180 - 2\theta_1) = 2(\theta_1 + \theta_2) = 2\theta$ is constant. So we must have that $\triangle \alpha C\beta$ is fixed by SAS identity, hence C is a fixed point and z travels in an arc centered at C from α to β, shown on the Riemann Sphere in Figure 2.

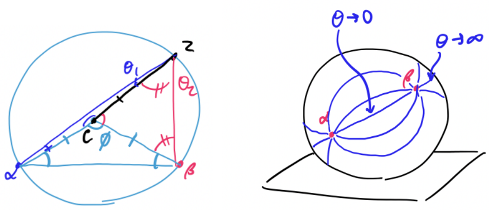

Figure 2: Geometric definitions and arcs on the Riemann Sphere

Since these circlines around α and β and arcs through α and β are mapped to circles and rays through the origin by a fractional linear transformation, they must be orthogonal by preservation of angles. This orthogonality and mapping is illustrated below.

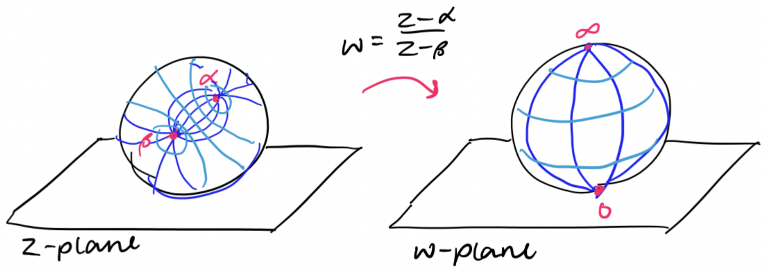

Figure 3: Orthogonality of the circlines and arcs

0.2 Mapping the upper half plane to itself

Suppose we want to map the upper half plane (aka $\{z \in \mathbb{C} \mid \Im(z) > 0\}$) to itself. Equivalently, we are mapping the upper half plane to the unit circle, rotating it, then mapping back to to upper half plane. This is demonstrated below in Figure 4.

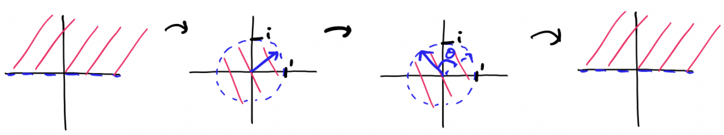

Figure 4: The vision for mapping the upper half plane to itself

We first note that $f_\alpha(z) = \frac{z-\alpha}{z-\bar{\alpha}}$ sends the upper half plane to the unit disk (ensuring this map is holomorphic in the upper half plane region by specifying $\Im(\alpha) > 0$). We make a second note that $g(z) = e^{i\theta} z$ rotates the unit disk by θ. So by composing these transformations as $w = f_i^{-1}(g(f_\alpha(z)))$, we map the upper half-plane to itself. By rearranging, we attain $f_i(w) = g(f_\alpha(z))$, or equivalently,

$$\frac{w-i}{w+i} = e^{i\theta} \frac{z-\alpha}{z-\bar{\alpha}} \tag{1}$$

Now write $\frac{w-i}{w+i} = e^{i\theta} \frac{z-\alpha}{z-\bar{\alpha}}$ in the form $\left[\begin{smallmatrix} 1 & -i \\ 1 & i \end{smallmatrix}\right]\left[\begin{smallmatrix} w \\ 1 \end{smallmatrix}\right] = e^{i\theta}\left[\begin{smallmatrix} 1 & -\alpha \\ 1 & -\bar{\alpha} \end{smallmatrix}\right]\left[\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right]$. Observe then:

$$\begin{aligned}
\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} w \\ 1 \end{smallmatrix}\right] &= \frac{1}{2}\left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & -i \\ 1 & i \end{smallmatrix}\right]\left[\begin{smallmatrix} w \\ 1 \end{smallmatrix}\right] \\
&= \frac{1}{2} e^{i\theta}\left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & -\alpha \\ 1 & -\bar{\alpha} \end{smallmatrix}\right]\left[\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right] \\
&= \frac{1}{2} e^{i\theta}\left[\begin{smallmatrix} 2 & -2\Re(\alpha) \\ 0 & 2\Im(\alpha) \end{smallmatrix}\right]\left[\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right]
\end{aligned}$$

which has real entries and a positive determinant, hence

$$w = \frac{az+b}{cz+d}, \quad \text{with} \quad a, b, c, d \in \mathbb{R} \tag{2}$$

Both (1) and (2) will equivalently map the upper hemisphere to the upper hemisphere.

0.3 Fractional linear transformations for a target mapping

Suppose we want to map the region $\Im(z) > 0 \ \cap \ |z| > 0$ to the first quadrant, being $\Im(w) > 0 \ \cap \ \Re(w) > 0$ We will map $z \to w^* \to w$ as depicted in Figure 5.

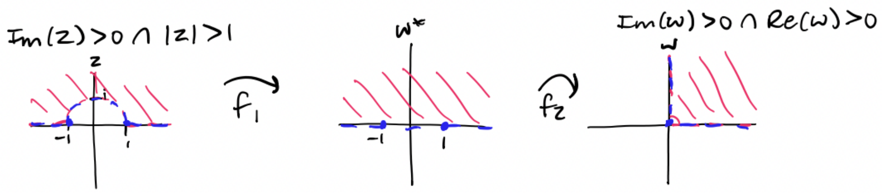

Figure 5: The vision for mapping to the first quadrant

We first map $\Im(z) > 0 \ \cap \ |z| > 0$ to $\Re(w^*) > 0$. Let 1 and -1 be fixed and let i be mapped to 0. Then we have $\frac{w^*-1}{w^*+1} = a\frac{z-1}{z+1}$. By solving for a using the mapping from $z = i$ to $w = 0$, we attain: $w^* + 1 = i\frac{z-1}{z+1}$, or equivalently, $w^* = \frac{i(z-1)+(z+1)}{(z+1)-i(z-1)}$.

We now map to the first quadrant. Let 1 and 0 be fixed and let -1 be mapped to i. We have $\frac{w}{w-1} = b\frac{w^*}{w^*-1}$. By solving for b using the mapping from $z = -1$ to $w = i$, we attain: $\frac{w}{w-1} = (1-i)\frac{w^*}{w^*-1}$.

Now by combining our transformations $z \to w^* \to w$ and doing a bit of algebra that I am not willing to type out, we attain the mapping:

$$\tfrac{w+(w-1)(1-i)}{(w-1)(1-i)-w} = -2\left(\tfrac{i(z-1)+(z+1)}{(z+1)-i(z-1)}\right) - 1$$

0.4 Generalized Pythagorean's Theorem

The generalized version of Pythagorean's Theorem can be proved rather elegantly by applying some complex analysis. What an exciting use of complex numbers!

Define the vertices of a quadrilateral A, B, C, D cyclically as in the below figure. The Generalized Pythagorean Theorem states that

$$\overline{AC} \cdot \overline{BD} = \overline{AB} \cdot \overline{CD} + \overline{BC} \cdot \overline{DA}$$

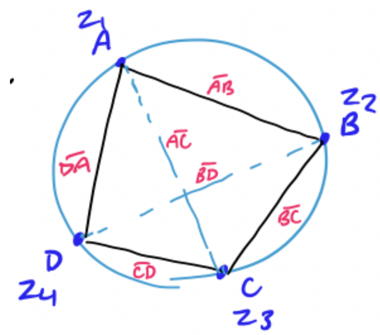

Figure 6: Defining a cyclic quadrilateral

Lemma. Define z_1, z_2, z_3, z_4 cyclically. We will show that $\frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z_3}, \frac{1}{z_4}$ lie on a line.

Consider the fractional linear transformation $w = \frac{1}{z}$, and consider a circle through the origin defined by $|z - a|^2 = |a|^2$. Equivalently, $(z - a)(\overline{z - a}) = a\overline{a}$ and equivalently again: $z\overline{z} - z\overline{a} - \overline{z}a = 0$.

Note that $\overline{z} = \frac{1}{\overline{w}}$. So by substituting this and letting $w = x + yi$, we have:

$$\begin{aligned} 0 &= \frac{1}{w\overline{w}} - \frac{\overline{a}}{w} - \frac{a}{\overline{w}} \\ &= 1 - \overline{a}\overline{w} - aw \\ &= 1 - \overline{a}x + \overline{a}iy - ax - aiy \\ &= 1 - (\overline{a} + a)x + (\overline{a} - a)iy \\ &= 1 - 2\Re(a)x - 2\Im(a)y \end{aligned}$$

which is the equation of a line. So $w = \frac{1}{z}$ maps circles through the origin to lines, and hence $\frac{1}{z_2}, \frac{1}{z_3}, \frac{1}{z_4}$ lie on a line.

Proof of Generalized Pythagorean's Theorem. We want to show that $|z_3 - z_1||z_4 - z_2| = |z_4 - z_1||z_3 - z_2| + |z_2 - z_1||z_4 - z_3|$. Observe that

$$\begin{aligned} |z_3 - z_1||z_4 - z_2| &= |z_3 z_4 - z_1 z_4 - z_2 z_3 + z_1 z_2| \\ &= |(z_3 - z_4)(z_1 - z_2) + (z_1 - z_4)(z_2 - z_3)| \\ &\leq |z_4 - z_1||z_3 - z_2| + |z_2 - z_1||z_4 - z_3| \text{ by triangle inequality.} \end{aligned}$$

If we can show that $(z_4 - z_1)(z_3 - z_2)$ and $(z_2 - z_1)(z_4 - z_3)$ lie on the same line, the last inequality will become an equality which completes our proof. Following from the Lemma, note that $0 = \arg\left(\frac{1}{z_j} - \frac{1}{z_i}\right) = \arg(\frac{z_j - z_i}{z_i z_j}) = \arg(z_j - z_i) - \arg(z_i z_j)$.

$$
\begin{aligned}
\text{Hence: } 0 &= \arg\left(\frac{1}{z_1} - \frac{1}{z_4}\right) + \arg\left(\frac{1}{z_2} - \frac{1}{z_3}\right) - \arg\left(\frac{1}{z_1} - \frac{1}{z_2}\right) - \arg\left(\frac{1}{z_3} - \frac{1}{z_4}\right) \\
&= \arg(z_4 - z_1) - \arg(z_4 z_1) + \arg(z_3 - z_2) - \arg(z_3 z_2) \\
&\qquad - \arg(z_2 - z_1) + \arg(z_2 z_1) - \arg(z_4 - z_3) + \arg(z_4 z_3) \\
&= \arg\big((z_4 - z_1)(z_3 - z_2)\big) - \arg(z_4 z_1 z_3 z_2) - \arg\big((z_2 - z_1)(z_4 - z_3)\big) + \arg(z_2 z_1 z_4 z_3) \\
&= \arg\big((z_4 - z_1)(z_3 - z_2)\big) - \arg\big((z_2 - z_1)(z_4 - z_3)\big)
\end{aligned}
$$

So $(z_4 - z_1)(z_3 - z_2)$ and $(z_2 - z_1)(z_4 - z_3)$ must lie on the same line, which completes our proof of the Generalized Pythagorean Theorem. ■

Let's observe why this theorem generalizes the Pythagorean theorem. The Pythagorean Theorem is a more specific case in which $\angle A = \angle B = \angle C = \angle D = 90$. Here, $\overline{AC} = \overline{BD}$ $\overline{DA} = \overline{BC}$, and $\overline{AB} = \overline{DC}$. So from the generalized Pythagorean theorem, we obtain $\overline{AC} \cdot \overline{AC} = \overline{AB} \cdot \overline{AB} + \overline{BC} \cdot \overline{BC}$ (Pythagorean's theorem).

0.5 Rotations about two antipodal points

Suppose we would like to find the transformation that maps the Riemann Sphere about arbitrary antipodal points α and α'.

We first determine α' in terms of α. Let $\alpha = a + bi$. On the Riemann Sphere, its coordinates are:

$$\xi = \frac{a}{1 + a^2 + b^2}, \quad \eta = \frac{b}{1 + a^2 + b^2}, \quad \zeta = \frac{a^2 + b^2}{1 + a^2 + b^2}.$$

With some geometric intuition, we can see that its antipodal point of has coordinates:

$$\xi = \frac{-a}{1 + a^2 + b^2}, \quad \eta = \frac{-b}{1 + a^2 + b^2}, \quad \zeta = 1 - \frac{a^2 + b^2}{1 + a^2 + b^2}$$

And now by transforming these coordinates to the complex plane (we use **stereographic projection** to do this), we have:

$$
\begin{aligned}
x &= \frac{-a}{1 + a^2 + b^2}\left(\frac{a^2 + b^2}{1 + a^2 + b^2}\right)^{-1} = \frac{-a}{a^2 + b^2} \\
y &= \frac{-b}{1 + a^2 + b^2}\left(\frac{a^2 + b^2}{1 + a^2 + b^2}\right)^{-1} = \frac{-b}{a^2 + b^2}
\end{aligned}
$$

Hence the antipodal point of α is $\alpha' = \frac{1}{|\alpha|^2}(-a - bi) = -\frac{\alpha}{\alpha\overline{\alpha}} = -\frac{1}{\overline{\alpha}}$

Now we can easily find that a general fractional linear transformation with a rotation of θ about fixed points α and $-\frac{1}{\overline{\alpha}}$ is of the form

$$\frac{w - \alpha}{w + \frac{1}{\overline{\alpha}}} = e^{i\theta} \frac{z - \alpha}{z + \frac{1}{\overline{\alpha}}}$$