Problem One

a.

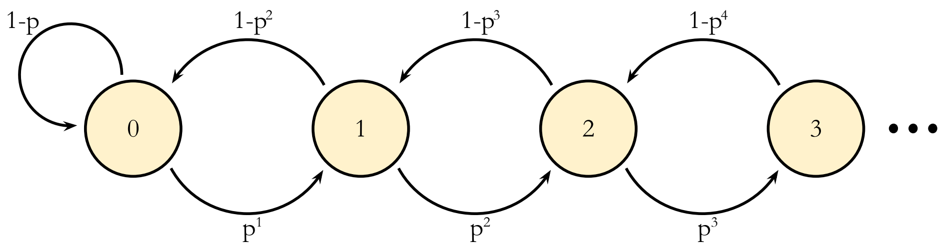

Figure 1: The Markov chain P_{ij} illustrated as a digraph.

The Markov chain P_{ij} in fig. 1 describes a random walk. Starting in any state, you can reach any other state - i.e. all states communicate. Thus, P_{ij} is **strongly connected**, has **one equivalence class** and is **irreducible**. Let X_n and X_{n+s} be two states such that there are $s-1$ intermediate states between them. To get from state i to $j = i + s$ you have to visit all intermediate states at least once[1]. Another trait of P_{ij} is that with growing i, the probability of moving from state i to $j = i + 1$ diminishes exponentially, while the probability of moving back to $j' = i - 1$ grows exponentially. Starting in state i, it is therefore reasonable to claim that it is always possible to return to state i in a finite number of steps. State i is then said to be positive recurrent, and since positive recurrence is a class property, the whole chain is **positive recurrent**.

Starting in state i, let n be the number that allows you to end up in state i after n transitions with the probability $P_{ii}^n > 0$. Let N_i be the set of all such numbers for our chain. The greatest common divisor of the elements in N_i is called the period d of state i. Since $P_{00}^n > 0$ for any n, $N_i = 0, 1, 2, ...$ and so P_{00} has a period of $d = 1$. Since periodicity is a class property, the chain as a whole has period one. A Markov chain with $d = 1$ is called **aperiodic**.

A positive recurrent aperiodic chain is called **ergodic**. Theorem 4.1 in [**?**] states that for an irreducible ergodic Markov chain the limit $\pi_j = \lim_{n\to\infty} P_{ij}^n$, $j \geq 0$ exists,

[1] Equivalently, s is the minimum number of steps you must preform to make the transition from state i to $i + s$

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij} \tag{1}$$

and

$$\sum_{j=0}^{\infty} \pi_j = 1. \tag{2}$$

In this case eq. (1) becomes

$$\pi_i = \begin{cases} \pi_{i-1}P_{i-1,i} + \pi_{i+1}P_{i+1,i} & = \pi_{i-1}p^i + \pi_{i+1}(1-p^{i+2}) \qquad i > 0 \\ \pi_i P_{i,i} + \pi_{i+1}P_{i+1,i} & = \pi_0(1-p) + \pi_1(1-p) \qquad i = 0 \end{cases} \tag{3}$$

The π's are called the limiting probabilities that the process will be in state i at time n, and it can be shown that π_i also equals the long-run proportion of time that the process will be in state i. Below the limiting probabilities are calculated for the first 3 states.

$$\pi_1 = \frac{p}{1-p^2}\pi_0; \quad \pi_2 = \frac{p^2}{1-p^3}\pi_1; \quad \pi_3 = \frac{p^3}{1-p^4}\pi_2$$

By induction this yields the general formula (recurrence relation)

$$\begin{aligned} \pi_{i+1} = \frac{p^{i+1}}{1-p^{i+2}}\pi_i &= \frac{p^{i+1}p^i p^{i-1}\cdots p^2 p^1}{(1-p^{i+2})(1-p^{i+1})(1-p^i)\cdots(1-p^3)(1-p^2)}\pi_0 \\ &= \frac{p^{1+2+\ldots+(i-1)+i+(i+1)}}{\prod_{i=2}^{k=i+2}(1-p^k)}\pi_0 = \frac{p^{\frac{1}{2}(i+1)(i+2)}}{\prod_{i=2}^{k=i+2}(1-p^k)}\pi_0 \end{aligned}$$

or, equivalently,

$$\pi_i = \pi_0 \cdot \prod_{k=1}^{k=i+1} \frac{p^k}{1-p^{k+1}} \tag{4}$$

Now eq. (2) becomes

$$\begin{aligned} \sum_i \pi_i = \pi_0 \left(1 + \sum_i \cdot \prod_{k=1}^{k=i+1} \frac{p^k}{1-p^{k+1}}\right) = 1 \\ \implies \pi_0 = \left(1 + \sum_i \cdot \prod_{k=1}^{k=i+1} \frac{p^k}{1-p^{k+1}}\right)^{-1} \end{aligned} \tag{5}$$

Substituting eq. (5) into eq. (4) gives an expression for the limiting probabilities

$$\pi_i = \frac{\prod_{k=1}^{k=i+1} \frac{p^k}{1-p^{k+1}}}{1 + \sum_i \cdot \prod_{k=1}^{k=i+1} \frac{p^k}{1-p^{k+1}}} \tag{6}$$

The distribution is positive so the limiting probabilities exist and the states are indeed positive recurrent.

b.

Let the set ξ contain the first 10 states, and let T_i be the time spent to reach state 10, starting in state $X_0 = i \in \xi$.

$$\begin{aligned} \mu_i &= E[T_i] \\ &= E[T_i|X_1 = i-1]P(X_1 = i-1) + E[T_i|X_1 = i+1]P(X_1 = i+1) \\ &= (1 + E[T_{i-1}])(1 - p^{i+1}) + (1 + E[T_{i+1}])p^{i+1} \\ &= 1 + \mu_{i-1} + (\mu_{i+1} - \mu_{i-1})p^{i+1} \end{aligned} \tag{7}$$

Note the special cases for $\mu_0 = 1 + \mu_0 + (\mu_1 - \mu_0)p$, and $\mu_9 = 1 + \mu_8(1 - p^{10})$

eq. (7) constitutes a set of 10 equations that can be solved with MATLAB. In table table 1 the solutions are listed for $p = 0.75$ and $p = 0.90$.

	$E[T_i]$	
i	p = 0.75	p = 0.90
0	$3.504 \cdot 10^6$	100.679
1	$3.504 \cdot 10^6$	99.568
2	$3.504 \cdot 10^6$	98.073
3	$3.504 \cdot 10^6$	96.145
4	$3.504 \cdot 10^6$	93.610
5	$3.504 \cdot 10^6$	90.159
6	$3.503 \cdot 10^6$	85.235
7	$3.503 \cdot 10^6$	77.772
8	$3.501 \cdot 10^6$	65.577
9	$3.485 \cdot 10^6$	43.711

Table 1: Mean values of reaching state 10 from any state $i = 0, 1, \cdots, 9$

The exponetial decay for the probabilities for moving on to the next state clearly manifests itself for the case of $p = 0.75$; no matter what state you start in, you'll have to wait for a very long time to reach state 10. The transition from state 9 to 10 is taking very long time compared to the other one-step transitions.

c.

The probability for ever reaching state 100 is 1, no matter what state you start in, since all limiting probabilities are positive. If you start in state $X_0 < 100$, you will have to wait for a very long time to reach state 100, and if $X_0 > 100$ you will only have to wait close to $X_0 - 100$ steps (at least if not $p \approx 1$).

d.

The simulated values in this section were found by running

```
% Obtain both analytical and simulated results
% from the limiting probabilities. \\cref{lim_p}
octave 1a.m

% Find the expected time until the chain reaches
% state 10 through simulations.
octave 1b.m

% Estimate the probability that the time it takes
% visiting state 15 is > 1000 for i=0 and i=10.
octave 1d.m
```

The limiting probabilities discussed in **a** have now been treated in two different ways, and table 2 draws a comparison between the analytical and simulated values obtained for $p = 0.75$. Our estimates are pretty close to the analytical values and comfortably within the confidence intervals in the column to the right. This suggests that our estimation was good.

MATLAB is here used to illustrate some random walks.

```
hold on;
%%% A ten-step illustrated random walks, p=0.90
% Starting in state 0
plot(randwalk(20,0,0.9), "1", \
randwalk(20,0,0.9), "2", randwalk(20,0,0.9), "3" )
% Starting in state 5
```

j	π_j	$\bar{\pi}_j$	C.I.	
0	.1669	0.1677	[.1666,	.1688]
1	.2861	0.2867	[.2856,	.2877]
2	.2789	.2774	[.2767,	.2782]
3	.1718	.1708	[.1699,	.1717]
4	.0713	.0714	[.0706,	.0721]
5	.0206	.0205	[.0201,	.0210]
6	.0042	.0042	[.0040,	.0044]
7	.0006	.0006	[.0005,	.0006]
8	.0001	.0001	[.0000,	.0001]
9	0	0	[0 ,	0]
10	0	0	[0 ,	0]

Table 2: p=0.75: The analytical and simulated limiting probabilities for the first 11 states are shown together with the corresponding 95% confidence interval.

```
plot(randwalk(20,5,0.9), "1", \
randwalk(20,5,0.9), "2", randwalk(20,5,0.9), "3" )
% Starting in state 9
plot(randwalk(20,9,0.9), "1", \
randwalk(20,9,0.9), "2", randwalk(20,9,0.9), "3" )

%%% p=0.75
% Starting in state 0
plot(randwalk(20,0,0.75), "1", \
randwalk(20,0,0.75), "2", randwalk(20,0,0.75), "3" )
% Starting in state 5
plot(randwalk(20,5,0.75), "1", \
randwalk(20,5,0.75), "2", randwalk(20,5,0.75), "3" )
% Starting in state 9
plot(randwalk(20,9,0.75), "1", \
randwalk(20,9,0.75), "2", randwalk(20,9,0.75), "3" )
```

Notice the difference between $p = 0.90$ and $p = 0.75$ in figs. 2 and 3. Walking with $p = 0.75$ does indeed pull you down towards state 0.

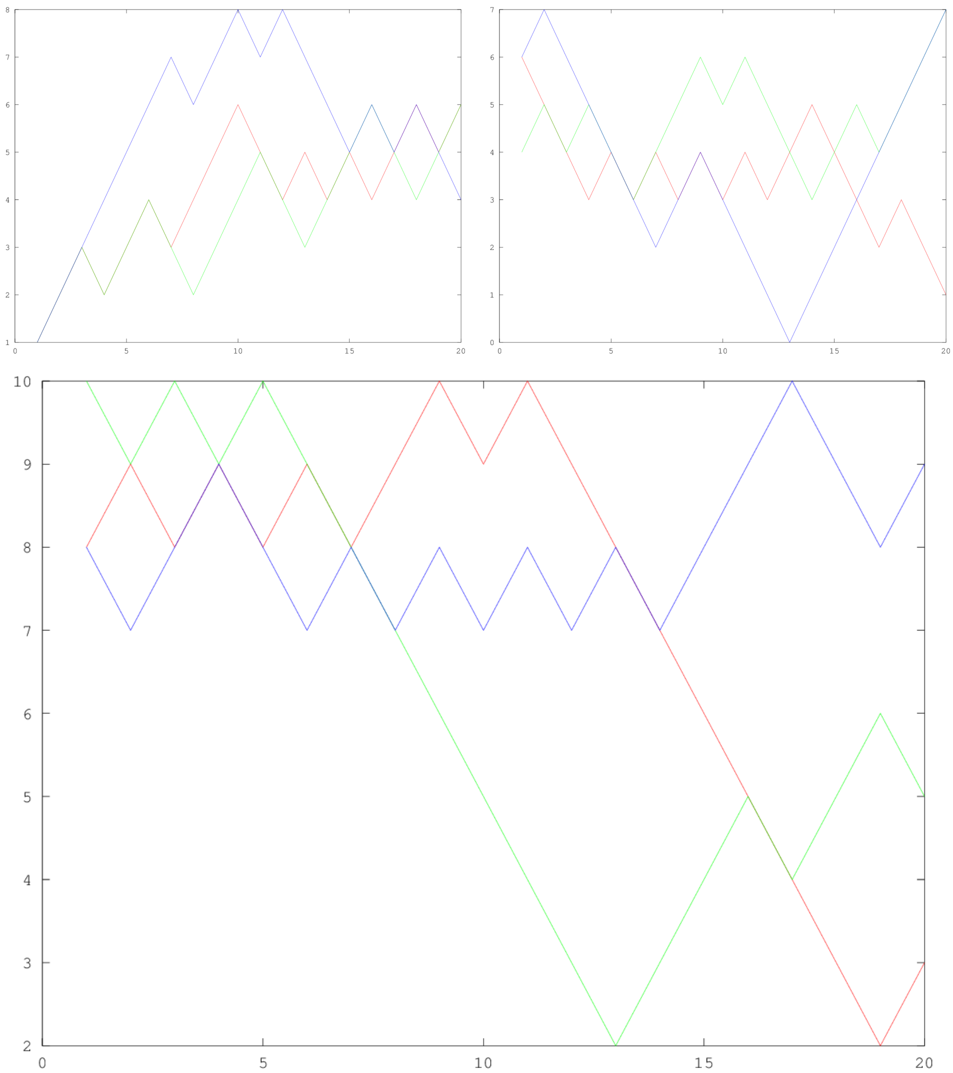

Figure 2: A 20-step random walk w. $p = 0.90$ starting with state 0, 5 and 9, respectively.

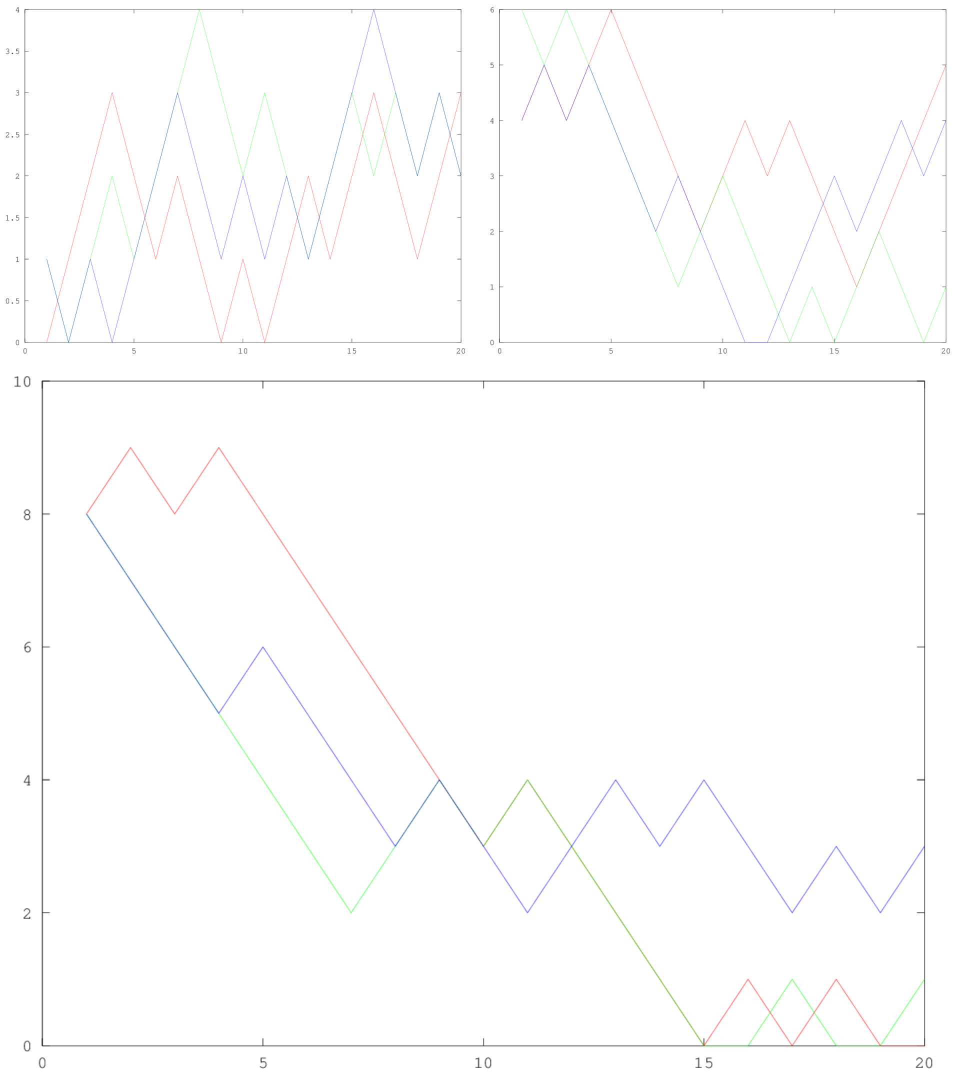

Figure 3: A 20-step random walk w. $p = 75$ starting with state 0, 5 and 9, respectively.