## Problem Two

In the following $X_n$ is the number of individuals in the $n$'th generation, $X_0$ is the initial population, $\mu = E[\textit{children per individual}]$, $p_j = P\{\textit{an individual has j offspring}\}$. The distributions that are analyzed are listed in table 1

| | $p_0$ | $p_1$ | $p_2$ | $p_3$ |
|---|---|---|---|---|
| **I** | 0.6 | 0.05 | 0.15 | 0.2 |
| **II** | 0.25 | 0.60 | 0.10 | 0.05 |

Table 1: The probability distributions given in Problem Two.

Chapter 4.7 in [**?** ] presents some results that are used in this problem. Below is some properties of branching processes discussed.

Generally, the mean number, and the variance, of offspring of a single individual is

$$\mu = \sum_{j=0}^{\infty} jP_j \qquad \text{and} \qquad \sigma^2 = \sum_{j=0}^{\infty} (j-\mu)^2 P_j \tag{1}$$

respectively.

For both our distributions $\mu = 0.95$, and since $\mu < 1$ the population will eventually die out. Defining $Z_i$ to be the number of offspring of the $i$'th individual for the $(n-1)$th generation, one can find

$$X_n = \sum_{i=1}^{X_{n-1}} Z_i \tag{2}$$

in the edge case where $X_0 = 1$. One can then obtain

$$E[X_n] = E[E[X_n|X_{n-1}]] = E\left[E\left[\sum_{i=1}^{X_{n-1}} Z_i|X_{i-1}\right]\right] = E[X_{n-1}]\mu \tag{3}$$

which leads to the result

$$E[X_1] = \mu, \quad E[X_2] = \mu E[X_1] = \mu^2, \quad \cdots, \quad E[X_n] = \mu^n. \tag{4}$$

It can then be shown that the variance is

$$Var(X_n) = \begin{cases} \sigma^2 \mu^{n-1} \cdot \frac{1-\mu^n}{1-\mu}, & \mu \neq 1 \\ n\sigma^2, & \mu = 1 \end{cases} \tag{5}$$

When doing simulations, the estimated mean value and standard deviation is given by

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i, \qquad S^2 = \frac{1}{n-1} \sum_{i=1}^{n} X_i - \bar{X}, \tag{6}$$

respectively.

**a.**

The mean and standard deviation is found analytically from eq. (1), eq. (4) and eq. (5), and is presented in table 2.

| $n$ | $E[X_n]$ | $SD_I[X_n]$ | $SD_{II}[X_n]$ |
|---|---|---|---|
| 10 | 0.5987 | 2.7977 | 2.7692 |
| 100 | 0.0059 | 0.4379 | 0.0678 |
| 1000 | 5.2912e-23 | 4.1529e-11 | 2.4697e-11 |

Table 2: Analytical values of the mean $E[X_n]$ and the standard deviation, SD ($= Var^{\frac{1}{2}}[X_n]$) for distribution I and II.

The branching process was simulated a hundred thousand times for each of the probability distributions. The values in table 3 were computed by running

```
% No. of simulations
N = 100000;
% Size of the first population
init = 1;
% Distribution I = p1 and II = p2
p1 = [0.60 0.05 0.15 0.20];
p2 = [0.25 0.60 0.10 0.05];
% Run the simulations for I and II
branchtrials(N, init, p1);
branchtrials(N, init, p2);
```

The simulated values is indeed very close to the ones found analytically for both distributions. Given that the MATLAB rand()-function is uniform, the

| $n$ | $E_I[X_n]$ | $E_{II}[X_n]$ | $E_{III}[X_n]$ | $SD_I[X_n]$ | $SD_{II}[X_n]$ | $SD_{III}[X_n]$ |
|---|---|---|---|---|---|---|
| 10 | 0.6007 | 0.5989 | 0.0554 | 2.8099 | 1.6708 | 0.4662 |
| 100 | 0.0060 | 0.0050 | 0 | 0.4583 | 0.2392 | 0 |
| 1000 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 3: Simulated values of the mean $E[X_n]$ and the standard deviation, SD $(= Var^{\frac{1}{2}}[X_n])$ for distribution I and II.

simulated values would probably become even more accurate if the number of simulations is increased. No simulation ever reached $n = 1000$, but this is only natural considering the low probability for that to happen.

Case (i), (ii) and (iii) are presented in table 4 for both distributions.

| | Distribution I | | Distribution II | | Distribution III | |
|---|---|---|---|---|---|---|
| **Case** | $E[\cdot]$ | $SD[\cdot]$ | $E[\cdot]$ | $SD[\cdot]$ | $E[\cdot]$ | $SD[\cdot]$ |
| (i) | 4.5197 | 7.1409 | 8.2460 | 10.8820 | 3.5185 | 2.5587 |
| (ii) | 19.9991 | 109.4967 | 19.9830 | 65.9630 | 3.9787 | 7.0344 |
| (iii) | 2.9699 | 5.4466 | 2.3411 | 2.9429 | 1.5572 | 1.1611 |

Table 4: Simulated values of the mean $E[X_n]$ and the standard deviation, SD $(= Var^{\frac{1}{2}}[X_n])$ for distribution I and II.

**b.**

```
% Defining a third distribution
p3 = [0.50 0.30 0.15 0.05];
% Run the simulations for III
branchtrials(N, init, p3);
```

The three different cases are presented in figs. 1 to 3 as histograms, grouped by probability distribution.

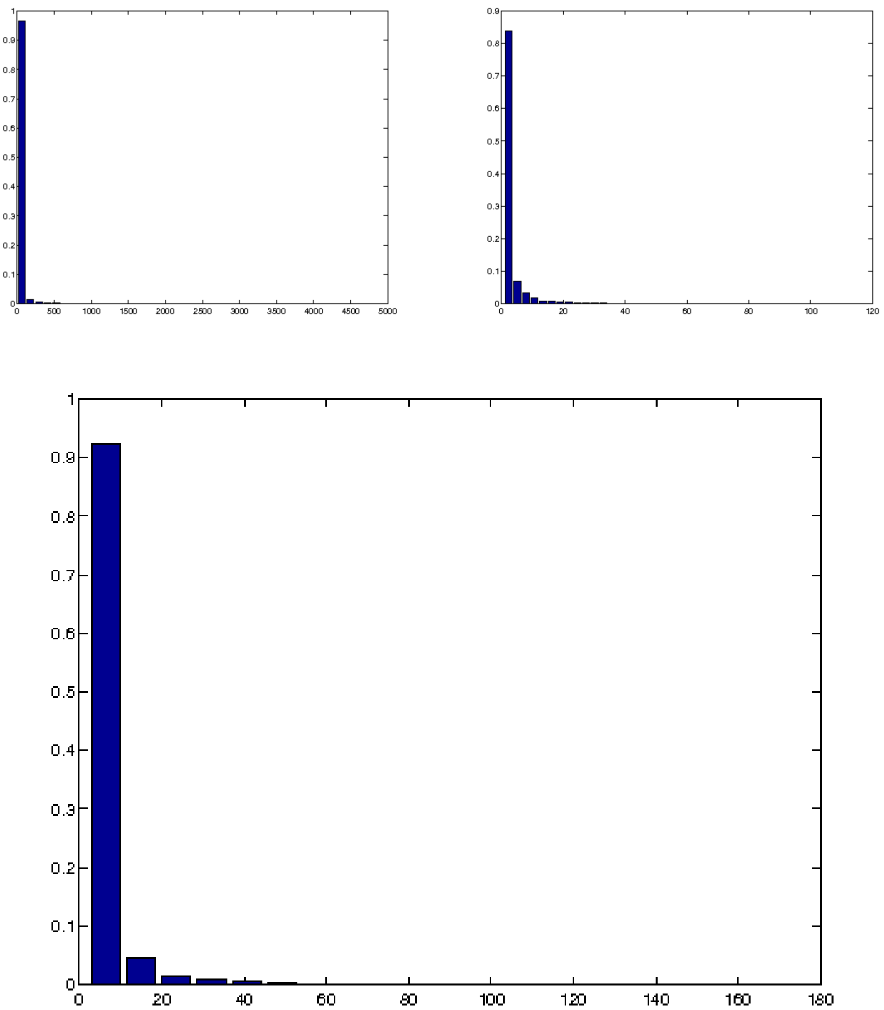


Figure 1: Case (i), (ii) and (iii) for distribution I

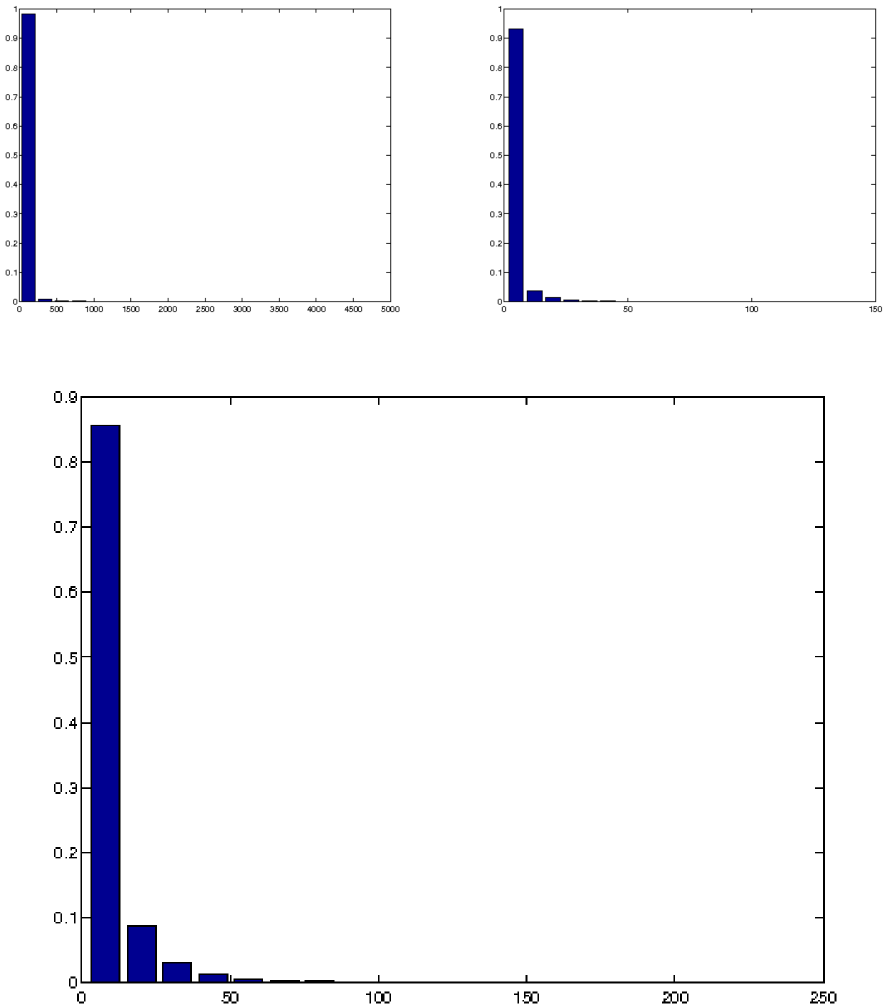


Figure 2: Case (i), (ii) and (iii) for distribution II

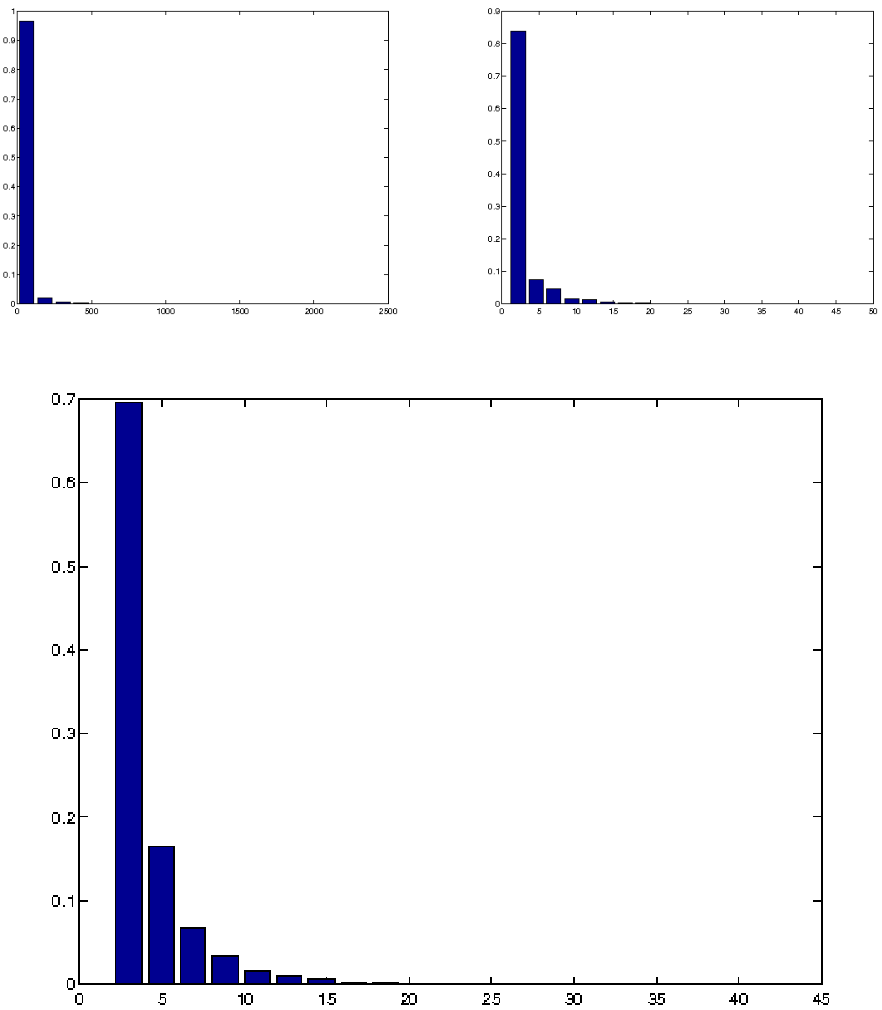


Figure 3: Case (i), (ii) and (iii) for distribution III