Problem One

a.

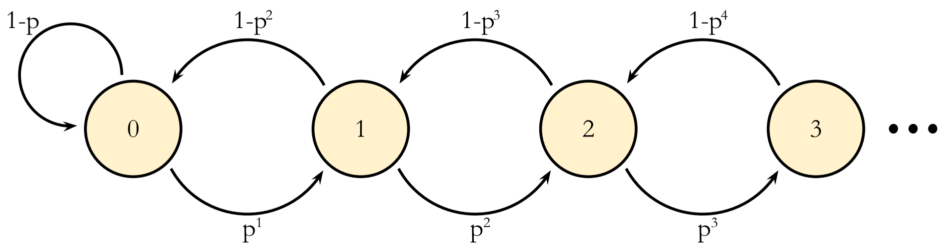

Figure 1: The Markov chain P_{ij} illustrated as a digraph.

The Markov chain P_{ij} in fig. 1 describes a random walk. Starting in any state, you can reach any other state - i.e. all states communicate. Thus, P_{ij} is **strongly connected**, has **one equivalence class** and is **irreducible**. Let X_n and X_{n+s} be two states such that there are $s-1$ intermediate states between them. To get from state i to $j = i+s$ you have to visit all intermediate states at least once[1]. Another trait of P_{ij} is that with growing i, the probability of moving from state i to $j = i+1$ diminishes exponentially, while the probability of moving back to $j' = i-1$ grows exponentially. Starting in state i, it is therefore reasonable to claim that it is always possible to return to state i in a finite number of steps. State i is then said to be positive recurrent, and since positive recurrence is a class property, the whole chain is **positive recurrent**.

Starting in state i, let n be the number that allows you to end up in state i after n transitions with the probability $P_{ii}^n > 0$. Let N_i be the set of all such numbers for our chain. The greatest common divisor of the elements in N_i is called the period d of state i. Since $P_{00}^n > 0$ for any n, $N_i = 0, 1, 2, ...$ and so P_{00} has a period of $d = 1$. Since periodicity is a class property, the chain as a whole has period one. A Markov chain with $d = 1$ is called **aperiodic**.

A positive recurrent aperiodic chain is called **ergodic**. Theorem 4.1 in [**?**] states that for an irreducible ergodic Markov chain the limit $\pi_j = \lim_{n\to\infty} P_{ij}^n$, $j \geq 0$ exists,

[1]Equivalently, s is the minimum number of steps you must preform to make the transition from state i to $i+s$

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij} \tag{1}$$

and

$$\sum_{j=0}^{\infty} \pi_j = 1. \tag{2}$$

In this case eq. (1) becomes

$$\pi_i = \begin{cases} \pi_{i-1}P_{i-1,i} + \pi_{i+1}P_{i+1,i} & = \pi_{i-1}p^i + \pi_{i+1}(1-p^{i+2}) \qquad i > 0 \\ \pi_i P_{i,i} + \pi_{i+1}P_{i+1,i} & = \pi_0(1-p) + \pi_1(1-p) \qquad i = 0 \end{cases} \tag{3}$$

The π's are called the limiting probabilities that the process will be in state i at time n, and it can be shown that π_i also equals the long-run proportion of time that the process will be in state i. Below the limiting probabilities are calculated for the first 3 states.

$$\pi_1 = \frac{p}{1-p^2}\pi_0; \quad \pi_2 = \frac{p^2}{1-p^3}\pi_1; \quad \pi_3 = \frac{p^3}{1-p^4}\pi_2$$

By induction this yields the general formula (recurrence relation)

$$\begin{aligned} \pi_{i+1} = \frac{p^{i+1}}{1-p^{i+2}}\pi_i &= \frac{p^{i+1}p^i p^{i-1}\cdots p^2 p^1}{(1-p^{i+2})(1-p^{i+1})(1-p^i)\cdots(1-p^3)(1-p^2)}\pi_0 \\ &= \frac{p^{1+2+\ldots+(i-1)+i+(i+1)}}{\prod_{i=2}^{k=i+2}(1-p^k)}\pi_0 = \frac{p^{\frac{1}{2}(i+1)(i+2)}}{\prod_{i=2}^{k=i+2}(1-p^k)}\pi_0 \end{aligned}$$

or, equivalently,

$$\pi_i = \pi_0 \cdot \prod_{k=1}^{k=i+1} \frac{p^k}{1-p^{k+1}} \tag{4}$$

Now eq. (2) becomes

$$\begin{aligned} \sum_i \pi_i = \pi_0 \left(1 + \sum_i \cdot \prod_{k=1}^{k=i+1} \frac{p^k}{1-p^{k+1}}\right) = 1 \\ \Longrightarrow \pi_0 = \left(1 + \sum_i \cdot \prod_{k=1}^{k=i+1} \frac{p^k}{1-p^{k+1}}\right)^{-1} \end{aligned} \tag{5}$$

Substituting eq. (5) into eq. (4) gives an expression for the limiting probabilities

$$\pi_i = \frac{\prod_{k=1}^{k=i+1} \frac{p^k}{1-p^{k+1}}}{1 + \sum_i \cdot \prod_{k=1}^{k=i+1} \frac{p^k}{1-p^{k+1}}} \tag{6}$$

The distribution is positive so the limiting probabilities exist and the states are indeed positive recurrent.

b.

Let the set ξ contain the first 10 states, and let T_i be the time spent to reach state 10, starting in state $X_0 = i \in \xi$.

$$\begin{aligned}
\mu_i &= E[T_i] \\
&= E[T_i|X_1 = i-1]P(X_1 = i-1) + E[T_i|X_1 = i+1]P(X_1 = i+1) \\
&= (1 + E[T_{i-1}])(1 - p^{i+1}) + (1 + E[T_{i+1}])p^{i+1} \\
&= 1 + \mu_{i-1} + (\mu_{i+1} - \mu_{i-1})p^{i+1}
\end{aligned} \tag{7}$$

Note the special cases for $\mu_0 = 1 + \mu_0 + (\mu_1 - \mu_0)p$, and $\mu_9 = 1 + \mu_8(1 - p^{10})$

eq. (7) constitutes a set of 10 equations that can be solved with MATLAB. In table ¡TABLE¿ the solutions are listed for $p = 0.75$ and $p = 0.90$.

The exponetial decay for the probabilities for moving on to the next state clearly manifests itself for the case of $p = 0.75$; no matter what state you start in, you'll have to wait for a very long time to reach state 10. The transition from state 9 to 10 is taking very long time compared to the other one-step transitions.

c.

The probability for ever reaching state 100 is 1, no matter what state you start in, since all limiting probabilities are positive. If you start in state $X_0 < 100$, you will have to wait for a very long time to reach state 100, and if $X_0 > 100$ you will only have to wait close to $X_0 - 100$ steps (at least if not $p \approx 1$).

d.

```
hold on;
%%% A ten-step illustrated random walks, p=0.90
% Starting in state 0
plot(randwalk(20,0,0.9), "1", \
randwalk(20,0,0.9), "2", randwalk(20,0,0.9), "3" )
% Starting in state 5
plot(randwalk(20,5,0.9), "1", \
randwalk(20,5,0.9), "2", randwalk(20,5,0.9), "3" )
% Starting in state 9
plot(randwalk(20,9,0.9), "1", \
randwalk(20,9,0.9), "2", randwalk(20,9,0.9), "3" )

%%% p=0.75
% Starting in state 0
plot(randwalk(20,0,0.75), "1", \
randwalk(20,0,0.75), "2", randwalk(20,0,0.75), "3" )
% Starting in state 5
plot(randwalk(20,5,0.75), "1", \
randwalk(20,5,0.75), "2", randwalk(20,5,0.75), "3" )
% Starting in state 9
plot(randwalk(20,9,0.75), "1", \
randwalk(20,9,0.75), "2", randwalk(20,9,0.75), "3" )
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Notice the difference between $p = 0.90$ and $p = 0.75$ in figs. 2 and 3.

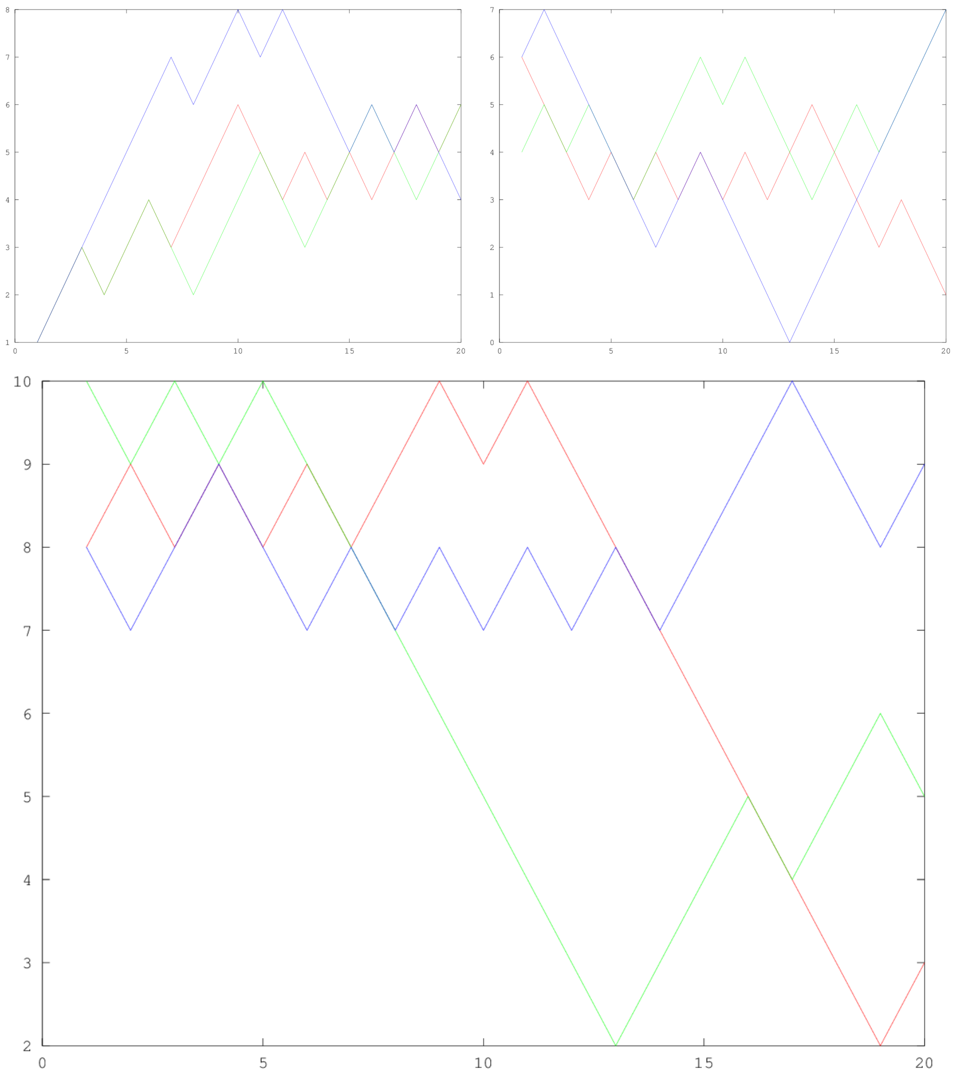

Figure 2: A 20-step random walk w. $p = 0.90$ starting with state 0, 5 and 9, respectively.

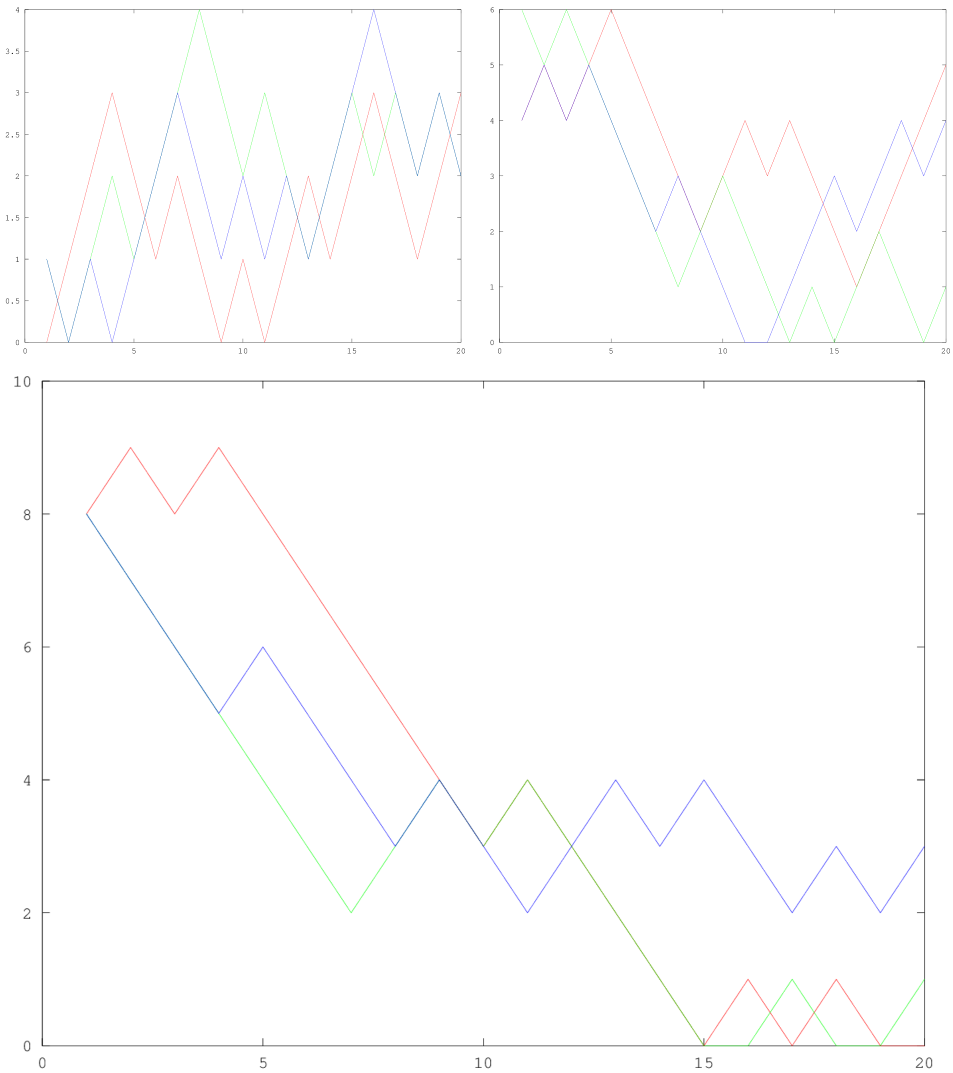

Figure 3: A 20-step random walk w. $p = 75$ starting with state 0, 5 and 9, respectively.