Assignment 6
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Math 427. Partial Differential Equations I
November 28, 2012

5 Sturm-Liouville Eigenvalue Problems

5.3 Sturm-Liouville Eigenvalue Problems

Problem 1. *5.3.8 (5pts)*

Show that $\lambda \geq 0$ for the eigenvalue problem

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + \left(\lambda - x^2\right)\phi = 0 \text{ with } \frac{\mathrm{d}\phi}{\mathrm{d}x}(0) = 0, \quad \frac{\mathrm{d}\phi}{\mathrm{d}x}(1) = 0.$$

Is $\lambda = 0$ an eigenvalue?

Solution.

The Rayleigh quotient is

$$\lambda = \frac{[-p(x)\phi(x)\phi'(x)|_a^b + \int_a^b p(x)[\phi'(x)]^2 - q(x)[\phi(x)]^2 \, \mathrm{d}x}{\int_a^b [\phi(x)]^2 \sigma(x) \, \mathrm{d}x}.$$

Rayleigh Quotient

With $p = 1$, $q = -x^2$, and $\sigma = 1$, the boundary conditions imply $[-\phi\phi'|_0^1 = 0$, so our eigenvalues must satisfy

$$\lambda = \frac{[-\phi\phi'|_0^1 + \int_0^1 [\phi']^2 + x^2\phi^2 \, \mathrm{d}x}{\int_0^1 \phi^2 \, \mathrm{d}x} = \frac{\int_0^1 [\phi']^2 \, \mathrm{d}x + \int_0^1 [x\phi]^2 \, \mathrm{d}x}{\int_0^1 \phi^2 \, \mathrm{d}x} \geq 0.$$

Furthermore, if $\lambda = 0$ then $\int_0^1 [\phi']^2 \, \mathrm{d}x = -\int_0^1 [x\phi]^2 \, \mathrm{d}x$. That is possible only if $\phi'(x) \equiv 0$, which in turn implies that $\phi(x) \equiv c$ for some constant c.

From the boundary conditions, it must be the case that $c = 0$. But this would indicate a trivial solution, and thus $\lambda \neq 0$. So there is no zero eigenvalue.

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5.4 Worked Example: Heat Flow in a Nonuniform Rod without Sources

Problem 2. *5.4.5 (10pts)*

Consider

$$\rho\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} + \alpha u, \text{ subject to } \begin{array}{ll} u(0,t)=0 & u(x,0)=f(x) \\ u(L,t)=0 & u_t(x,0)=g(x) \end{array}$$

where $\rho(x) > 0$, $\alpha(x) < 0$, and T_0 is constant. Assume the appropriate eigenfunctions are known. Solve the initial value problem.

Solution. Since the partial differential equation and its boundary conditions are homogeneous, we may separate variables to yield

$$\frac{1}{h}\frac{\mathrm{d}^2 h}{\mathrm{d}t^2} = \frac{T_0}{\rho\phi}\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + \frac{\alpha}{\rho} = -\lambda.$$

The time problem is thus $h'' + \lambda h = 0$, and the spatial problem is $T_0\phi'' + \alpha\phi + \lambda\rho\phi = 0$ subject to the boundary conditions $\phi(0) = \phi(L) = 0$.

With $p \equiv T_0$, $q = \alpha(x) < 0$, and $\sigma = \rho(x) > 0$, the Rayleigh quotient evaluates to

$$\lambda = \frac{T_0\int_0^L [\phi']^2\,\mathrm{d}x + \int_0^L s\phi^2\,\mathrm{d}x}{\int_0^L \phi^2\rho\,\mathrm{d}x} \geq 0, \text{ where } s = -\alpha > 0.$$

There is a zero eigenvalue $\lambda = 0$ only if $T_0 = -\frac{\int_0^L s\phi^2\,\mathrm{d}x}{\int_0^L [\phi']^2\,\mathrm{d}x}$. Since the physical context of the problem implies $T_0 > 0$, all eigenvalues $\lambda > 0$.

The time-dependent problem thus has a solution of the form

$$h_n(t) = a_n\cos\sqrt{\lambda_n}t + b_n\sin\sqrt{\lambda_n}t$$

We assume the eigenfunctions ϕ_n are known, so our PDE has a product solution

$$u(x,t) = \sum_{n=1}^{\infty}\phi_n h_n = \sum_{n=1}^{\infty}\phi_n\left(a_n\cos\sqrt{\lambda_n}t + b_n\sin\sqrt{\lambda_n}t\right)$$

Applying the initial conditions to $u(x,0) = \sum_{n=1}^{\infty} a_n\phi_n = f(x)$, and to $u_t(x,0) = \sum_{n=1}^{\infty} b_n\sqrt{\lambda_n}\phi_n = g(x)$ yields coefficients determined by

$$\begin{aligned}\int_0^L f(x)\phi_m(x)\rho(x)\,\mathrm{d}x &= \int_0^L \sum_{n=1}^{\infty} a_n\phi_n(x)\phi_m(x)\rho(x)\,\mathrm{d}x \\ \implies a_n &= \frac{\int_0^L f(x)\phi_n(x)\rho(x)\,\mathrm{d}x}{\int_0^L [\phi_n(x)]^2\rho(x)\,\mathrm{d}x}, \qquad \text{[by orthogonality]}\end{aligned}$$

and similarly,

$$b_n\sqrt{\lambda_n} = \frac{\int_0^L g(x)\phi_n(x)\rho(x)\,\mathrm{d}x}{\int_0^L [\phi_n(x)]^2\rho(x)\,\mathrm{d}x}.$$

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5.5 Self-Adjoint Operators and Sturm-Liouville Eigenvalue Problems

Problem 3. *5.5.1g (10pts)*

A Sturm-Liouville eigenvalue problem is called self-adjoint if

$$p\left[u\frac{\mathrm{d}v}{\mathrm{d}x} - v\frac{\mathrm{d}u}{\mathrm{d}x}\right|_a^b = 0$$

because then $\int_a^b uL(v) - vL(u)\,\mathrm{d}x = 0$ for any two functions u and v satisfying the boundary conditions.

Under what conditions is the following self-adjoint (if p is constant)?

$$\phi(L) + \alpha\phi(0) + \beta\phi'(0) = 0$$
$$\phi'(L) + \gamma\phi(0) + \delta\phi'(0) = 0$$

Solution.

Let u and v be any two functions satisfying the above. Since $p = 0$ would imply a trivial solution, $p \neq 0$. Adopting a subscript notation to signify evaluation such that $f_i = f(i)$, we note that the following must hold:

$$\begin{aligned} &[ub' - vu'|_0^L = 0 \qquad & u_L + \alpha u_0 + \beta u_0' = 0 \qquad & v_L + \alpha v_0 + \beta v_0' = 0 \\ \Longrightarrow\ &[uv' - vu'|_L = [uv' - vu'|_0 \qquad & u_L' + \gamma u_0 + \delta u_0' = 0 \qquad & v_L' + \gamma v_0 + \delta v_0' = 0 \end{aligned}$$

It follows that

$$\begin{aligned} u_L &= -\alpha u_0 - \beta u_0' = -(\alpha u_0 + \beta u_0'), \\ u_L' &= -\gamma u_0 - \delta u_0' = -(\gamma u_0 + \delta u_0'), \\ v_L &= -\alpha v_0 - \beta v_0' = -(\alpha v_0 + \beta v_0'), \\ v_L' &= -\gamma v_0 - \delta v_0' = -(\gamma v_0 + \delta v_0'), \end{aligned}$$

and hence

$$\begin{aligned} [uv' - vu'|_L &= u_L v_L' - v_L u_L' \\ &= (\alpha u_0 + \beta u_0')(\gamma v_0 + \delta v_0') - (\alpha v_0 + \beta v_0')(\gamma u_0 + \delta u_0') \\ &= (\alpha\gamma u_0 v_0 + \beta\gamma u_0' v_0 + \alpha\delta u_0 v_0' + \beta\delta u_0' v_0') - (\alpha\gamma u_0 v_0 + \beta\gamma u_0 v_0' + \alpha\delta u_0' v_0 + \beta\delta u_0' v_0') \\ &= \beta\gamma u_0' v_0 + \alpha\delta u_0 v_0' - \beta\gamma u_0 v_0' - \alpha\delta u_0' v_0 \\ &= \alpha\delta u_0 v_0' - \beta\gamma u_0 v_0' - \alpha\delta u_0' v_0 + \beta\gamma u_0' v_0 \\ &= (\alpha\delta - \beta\gamma) u_0 v_0' - (\alpha\delta - \beta\gamma) u_0' v_0 \\ &= (\alpha\delta - \beta\gamma)(u_0 v_0' - u_0' v_0). \end{aligned}$$

So the given conditions are self-adjoint if

$$\begin{aligned} [uv' - vu'|_L &= [uv' - vu'|_0 \\ (\alpha\delta - \beta\gamma)(u_0 v_0' - v_0 u_0') &= u_0 v_0' - v_0 u_0' \\ (\alpha\delta - \beta\gamma)\left[u(0)\frac{\mathrm{d}v}{\mathrm{d}x}(0) - v(0)\frac{\mathrm{d}u}{\mathrm{d}x}(0)\right] &= u(0)\frac{\mathrm{d}v}{\mathrm{d}x}(0) - v(0)\frac{\mathrm{d}u}{\mathrm{d}x}(0). \end{aligned}$$

That is, if $\alpha\delta - \beta\gamma = 1$.

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Problem 4. *5.5.9 (10pts)*

For the eigenvalue problem

$$\frac{\mathrm{d}^4\phi}{\mathrm{d}x^4} + \lambda e^x \phi = 0$$

subject to the boundary conditions $\phi(0) = \phi(1) = \phi'(0) = \phi''(1) = 0$, show that the eigenvalues are less than or equal to zero ($\lambda \leq 0$), as would be expected in a physical context. Is $\lambda = 0$ an eigenvalue?

Solution.

To derive the appropriate quotient expression for λ, we multiply the eigenvalue problem by ϕ and integrate, yielding

$$\lambda = \frac{-\displaystyle\int_0^1 \phi \frac{\mathrm{d}^4\phi}{\mathrm{d}x^4}\,\mathrm{d}x}{\displaystyle\int_0^1 \phi^2 \sigma\,\mathrm{d}x}, \tag{4.1}$$

where $\sigma = e^x$. Manipulating the numerator, we integrate by parts letting $u = \phi$ and $\mathrm{d}v = \dfrac{\mathrm{d}^4\phi}{\mathrm{d}x^4}\mathrm{d}x$.

Then $\mathrm{d}u = \dfrac{\mathrm{d}\phi}{\mathrm{d}x}\mathrm{d}x$ and $v = \dfrac{\mathrm{d}^3\phi}{\mathrm{d}x^3}$, and

$$\int_0^1 \phi \frac{\mathrm{d}^4\phi}{\mathrm{d}x^4}\,\mathrm{d}x = \left[\phi \frac{\mathrm{d}^3\phi}{\mathrm{d}x^3}\right|_0^1 - \int_0^1 \frac{\mathrm{d}\phi}{\mathrm{d}x}\frac{\mathrm{d}^3\phi}{\mathrm{d}x^3}\,\mathrm{d}x = [\phi\phi'''|_0^1 - \int_0^1 \phi'\phi'''\,\mathrm{d}x.$$

Integrating by parts again with $u = \dfrac{\mathrm{d}\phi}{\mathrm{d}x}$ and $\mathrm{d}v = \dfrac{\mathrm{d}^3\phi}{\mathrm{d}x^3}\mathrm{d}x$, we have $\mathrm{d}u = \dfrac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\mathrm{d}x$ and $v = \dfrac{\mathrm{d}^2\phi}{\mathrm{d}x^2}$, and thus

$$\int_0^1 \frac{\mathrm{d}\phi}{\mathrm{d}x}\frac{\mathrm{d}^3\phi}{\mathrm{d}x^3}\,\mathrm{d}x = \left[\frac{\mathrm{d}\phi}{\mathrm{d}x}\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\right|_0^1 - \int_0^1 \left[\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\right]^2\mathrm{d}x = [\phi'\phi''|_0^1 - \int_0^1 [\phi'']^2\,\mathrm{d}x.$$

The numerator of (4.1) is thus

$$\begin{aligned} -\left[[\phi\phi'''|_0^1 - [\phi'\phi''|_0^1 + \int_0^1 [\phi'']^2\,\mathrm{d}x\right] &= -[\phi\phi'''|_0^1 + [\phi'\phi''|_0^1 - \int_0^1 [\phi'']^2\,\mathrm{d}x \\ &= -[0-0] + [0-0] - \int_0^1 [\phi'']^2\,\mathrm{d}x, \qquad \text{[from the boundary conditions]} \end{aligned}$$

leaving us with

$$\lambda = \frac{-\int_0^1 [\phi'']^2\,\mathrm{d}x}{\int_0^1 \phi^2 e^x\,\mathrm{d}x} \leq 0, \text{ as expected.}$$

If $\lambda = 0$, then $\int_0^1 [\phi'']^2\,\mathrm{d}x = 0$, and it would follow that

$$\begin{aligned} &\phi'' \equiv 0 \\ \implies &\phi' \equiv c \quad \text{[a constant]} \\ \implies &\phi' \equiv 0 \quad \text{[from the boundary conditions]} \\ \implies &\phi \equiv c \quad \text{[a constant]} \\ \implies &\phi \equiv 0. \quad \text{[from the boundary conditions]} \end{aligned}$$

Thus $\lambda = 0$ only if $\phi \equiv 0$, which is prohibited by assumption since ϕ is an eigenfunction.

So λ is strictly negative.

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5.8 Boundary Conditions of the Third Kind

Problem 5. *Exercise 5.8.8 (15pts)*

Consider the boundary value problem

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + \lambda\phi = 0 \tag{5.1}$$

with

$$\phi(0) - \frac{\mathrm{d}\phi}{\mathrm{d}x}(0) = 0 \text{ and } \phi(1) + \frac{\mathrm{d}\phi}{\mathrm{d}x}(1) = 0. \tag{5.2}$$

(a) Using the Rayleigh Quotient, show that $\lambda \geq 0$. Why is $\lambda > 0$?

(b) Prove that eigenfunctions corresponding to different eigenvalues are orthogonal.

(c) Show that

$$\tan\sqrt{\lambda} = \frac{2\sqrt{\lambda}}{\lambda - 1}.$$

Determine the eigenvalues graphically. Estimate the large eigenvalues.

(d) Solve

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \tag{5.3}$$

with

$$u(0,t) - \frac{\partial u}{\partial x}(0,t) = 0, \quad u(1,t) + \frac{\partial u}{\partial x}(1,t) = 0, \text{ and } u(x,0) = f(x). \tag{5.4}$$

You may call the relevant eigenfunctions $\phi_n(x)$ and assume they are known.

Solution.

(a) With $p = 1$, $q = 0$, $\sigma = 1$, the Rayleigh quotient for (5.1) becomes

$$\lambda = \frac{[-\phi\phi'|_0^1 + \int_0^1 [\phi']^2 \,\mathrm{d}x}{\int_0^1 \phi^2 \,\mathrm{d}x}.$$

Since the boundary conditions imply that $-\phi_1' = \phi_1$ and $\phi_0' = \phi_0$,

$$[-\phi\phi'|_0^1 = [-\phi_1\phi_1'] - [-\phi_0\phi_0'] = [\phi_1\phi_1] + [\phi_0\phi_0] = \phi_1^2 + \phi_0^2 \geq 0.$$

Thus

$$\lambda = \frac{\phi_1^2 + \phi_0^2 + \int_0^1 [\phi']^2 \,\mathrm{d}x}{\int_0^1 \phi^2 \,\mathrm{d}x} \geq 0.$$

Since $\lambda = 0$ only if $\phi_1^2 + \phi_0^2 = -\int_0^1 [\phi']^2 \,\mathrm{d}x$, which in turn implies the eigenfunction $\phi \equiv 0$, a contradiction. So $\lambda > 0$.

(b) Orthogonality results directly from noticing that (5.1) subject to the boundary conditions in (5.2) is a regular Sturm-Liouville eigenvalue problem. Without relying on other theorems, however, we prove orthogonality by deriving an expression that relates any two of its eigenvalues.

Let λ_m and λ_n be eigenvalues of (5.1) with corresponding eigenfunctions ϕ_m and ϕ_n. Then we have

$$\phi_m'' + \lambda_m\phi_m = 0 \tag{5.5}$$

and

$$\phi_n'' + \lambda_n \phi_n = 0. \tag{5.6}$$

Multiplying (5.5) by ϕ_n and (5.6) by ϕ_m and subtracting yields $\phi_m''\phi_n + \lambda_m\phi_m\phi_n - \phi_n''\phi_m - \lambda_n\phi_n\phi_m = 0$, so

$$\begin{aligned}
\lambda_m\phi_m\phi_n - \lambda_n\phi_n\phi_m &= \phi_n''\phi_m - \phi_n\phi_m'' \\
(\lambda_m - \lambda_n)\phi_n\phi_m &= \phi_n''\phi_m - \phi_n\phi_m''.
\end{aligned}$$

Integrating, we have

$$\begin{aligned}
(\lambda_m - \lambda_n)\int_0^1 \phi_n\phi_m \,\mathrm{d}x &= \int_0^1 \phi_n''\phi_m - \phi_n\phi_m'' \,\mathrm{d}x \\
&= \int_0^1 \phi_n''\phi_m + (\phi_n'\phi_m' - \phi_n'\phi_m') - \phi_n\phi_m'' \,\mathrm{d}x \\
&= \int_0^1 (\phi_n''\phi_m + \phi_n'\phi_m') - (\phi_n'\phi_m' + \phi_n\phi_m'') \,\mathrm{d}x \\
&= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}x}[\phi_n'\phi_m] - \frac{\mathrm{d}}{\mathrm{d}x}[\phi_n\phi_m'] \,\mathrm{d}x \\
&= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}x}[\phi_n'\phi_m] \,\mathrm{d}x - \int_0^1 \frac{\mathrm{d}}{\mathrm{d}x}[\phi_n\phi_m'] \,\mathrm{d}x \\
&= [\phi_n'\phi_m|_0^1 - [\phi_n\phi_m'|_0^1 \\
&= [\phi_n'\phi_m - \phi_n\phi_m'|_0^1 \\
&= \phi_n'\phi_m - \phi_n\phi_m'|_1 - \phi_n'\phi_m - \phi_n\phi_m'|_0 \\
&= -\phi_n\phi_m + \phi_n\phi_m|_1 - \phi_n\phi_m - \phi_n\phi_m|_0 \qquad \text{[from the boundary conditions (5.2)]} \\
&= 0
\end{aligned}$$

Thus if $m \neq n$ then $\int_0^1 \phi_n\phi_m \,\mathrm{d}x = 0$, and for arbitrary m and n the eigenfunctions ϕ_m and ϕ_n corresponding to eigenvalues λ_m and λ_n are orthogonal to each other. □

(c) Keeping in mind that $\lambda > 0$, we impose the boundary conditions to the spatial problem's general solution and its derivative

$$\begin{aligned}
\phi(x) &= c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x \\
\phi'(x) &= -c_1\sqrt{\lambda}\sin\sqrt{\lambda}x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}x
\end{aligned}$$

to yield $c_1 = c_2\sqrt{\lambda}$, implying that the eigenfunction is any multiple of $\phi(x) = \sqrt{\lambda}\cos\sqrt{\lambda}x + \sin\sqrt{\lambda}x$.

Applying the second boundary condition, we have

$$\begin{aligned}
\phi(1) &= \phi'(1) \\
c_2\left[\sqrt{\lambda}\cos\sqrt{\lambda} + \sin\sqrt{\lambda}\right] &= -c_2\left[-\lambda\sin\sqrt{\lambda} + \sqrt{\lambda}\cos\sqrt{\lambda}\right],
\end{aligned}$$

and thus

$$2\sqrt{\lambda}\cos\sqrt{\lambda} = (\lambda - 1)\sin\sqrt{\lambda} \tag{5.7}$$

and since $\cos\sqrt{\lambda} \neq 0$,

$$\tan\sqrt{\lambda} = \frac{2\sqrt{\lambda}}{\lambda - 1}, \text{ as desired.}$$

Division by $\cos\sqrt{\lambda}$ is possible because if $\cos\sqrt{\lambda} = 0$, then $\sin\sqrt{\lambda} = \pm 1 \neq 0$, in which case (5.7) would not be satisfied, since the regularity of (5.1) and (5.2) ensures there are infinite eigenvalues.

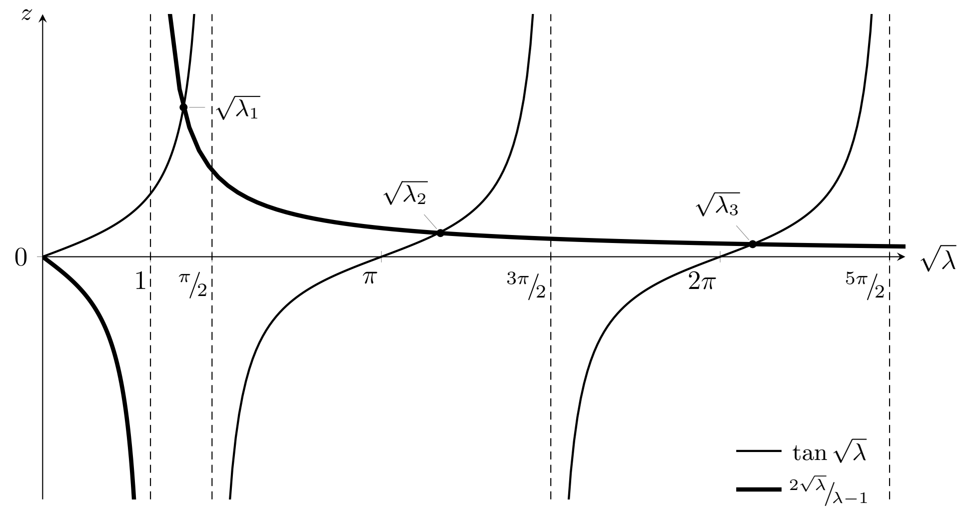

Figure 5.1. Graphical determination of eigenvalues.

To determine the eigenvalues graphically, we note that, letting $s = \sqrt{\lambda} > 0$ and $f(s) = \frac{2s}{s^2-1}$, we have $f'(s) = -\frac{2(s^2+1)}{(s^2-1)^2} < 0$. Thus there is a horizontal asymptote at $f(s)=0$, a vertical asymptote at $s=1$, $f(s) \gtrless 0$ when $s \gtrless 1$, and f is everywhere decreasing, as illustrated in Figure 5.1.

We note from the figure that the graphs of $\tan\sqrt{\lambda}$ and $\frac{2\sqrt{\lambda}}{\lambda-1}$ meet between 1 and $\pi/2$ and that as n grows, solutions approach those of $\tan\sqrt{\lambda} = 0$. That is, as $n \to \infty$, $\lambda_n \to [(n-1)\pi]^2$. Thus for large n we have eigenvalues $\lambda_n \sim (n-1)^2\pi^2$.

(d) From the preceding work, we know the problem in (5.1) and (5.2) has a product solution

$$u(x,t) = \sum_{n=1}^{\infty} a_n \left(\sqrt{\lambda_n}\cos\sqrt{\lambda_n}x + \sin\sqrt{\lambda_n}x\right) e^{-\lambda_n t}$$

such that

$$\tan\sqrt{\lambda_n} = \frac{2\sqrt{\lambda_n}}{\lambda_n - 1}$$

with a_n determined by imposing the initial condition and simplifying by appeal to the orthogonality established in (b), yielding

$$a_n = \frac{\int_0^1 f(x)\phi_n(x)\,\mathrm{d}x}{\int_0^1 \phi_n^2(x)\,\mathrm{d}x}.$$

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