On the Simultaneous Use of Fixed Effects on Cases and Time Points

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Abstract

Time-series cross-sectional (TSCS) data contain a sample of cases observed at repeated time points. Researchers commonly employ fixed effects (FEs) on the cases to remove cross-sectional unobserved heterogeneity from the model. Recently, a great deal of applied work includes FEs on cases and on time points in the same model with the intention of accounting for omitted variables in both the cross-sectional and time dimensions. The properties of the model that includes FEs on both cases and time are not well understood. We derive the formal two-way FE estimator and show that it does not account for unobserved heterogeneity in either the cross-sectional or the time dimension. We further demonstrate that the two-way FE model is sensitive to whether the panels are balanced while a model that includes FEs only on cases or only on time points is not. Using an analysis of the relationship between a country's wealth and level of democracy, we show that the choice of model has a profound influence on the findings. We recommend that researchers avoid the two-way FE model, and instead use a model with FEs only on cases or only on time points, a choice that depends on the research question.

1 Proofs

Lemma 1a. The following equivalence holds in both balanced and unbalanced panels:

$$\sum_{i=1}^{N}\sum_{t=1}^{T_i}(x_{it}-\bar{x}_i)(y_{it}-\bar{y}_i)=\sum_{i=1}^{N}\sum_{t=1}^{T_i}(x_{it}-\bar{x}_i)(y_{it}-\bar{y})=\sum_{i=1}^{N}\sum_{t=1}^{T_i}(x_{it}-\bar{x})(y_{it}-\bar{y}_i). \tag{1}$$

Proof. We prove the lemma for unbalanced panels, but the lemma also holds for balanced panels since balanced panels are the special case in which $T_i = T_j, \forall i, j \in \{1, \ldots, N\}$. We demonstrate that all three expressions are individually equal to $\sum_{i=1}^{N}\sum_{t=1}^{T_i} x_{it}y_{it} - \sum_{i=1}^{N} T_i\bar{x}_i\bar{y}_i$ and are therefore equal to each other:

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{t=1}^{T_i}(x_{it}-\bar{x}_i)(y_{it}-\bar{y}_i) &= \sum_{i=1}^{N}\sum_{t=1}^{T_i}(x_{it}y_{it}-x_{it}\bar{y}_i-\bar{x}_iy_{it}+\bar{x}_i\bar{y}_i)\\
&= \sum_{i=1}^{N}\sum_{t=1}^{T_i}x_{it}y_{it}-\sum_{i=1}^{N}\sum_{t=1}^{T_i}x_{it}\bar{y}_i-\sum_{i=1}^{N}\sum_{t=1}^{T_i}\bar{x}_iy_{it}+\sum_{i=1}^{N}\sum_{t=1}^{T_i}\bar{x}_i\bar{y}_i\\
&= \sum_{i=1}^{N}\sum_{t=1}^{T_i}x_{it}y_{it}-\sum_{i=1}^{N}\bar{y}_i\sum_{t=1}^{T_i}x_{it}-\sum_{i=1}^{N}\bar{x}_i\sum_{t=1}^{T_i}y_{it}+\sum_{i=1}^{N}T_i\bar{x}_i\bar{y}_i\\
&= \sum_{i=1}^{N}\sum_{t=1}^{T_i}x_{it}y_{it}-\sum_{i=1}^{N}\bar{y}_i(T_i\bar{x}_i)-\sum_{i=1}^{N}\bar{x}_i(T_i\bar{y}_i)+\sum_{i=1}^{N}T_i\bar{x}_i\bar{y}_i\\
&= \sum_{i=1}^{N}\sum_{t=1}^{T_i}x_{it}y_{it}-\sum_{i=1}^{N}T_i\bar{x}_i\bar{y}_i.
\end{aligned} \tag{2}$$

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{t=1}^{T_i}(x_{it}-\bar{x}_i)(y_{it}-\bar{y}) &= \sum_{i=1}^{N}\sum_{t=1}^{T_i}(x_{it}y_{it}-x_{it}\bar{y}-\bar{x}_iy_{it}+\bar{x}_i\bar{y})\\
&= \sum_{i=1}^{N}\sum_{t=1}^{T_i}x_{it}y_{it}-\sum_{i=1}^{N}\sum_{t=1}^{T_i}x_{it}\bar{y}-\sum_{i=1}^{N}\sum_{t=1}^{T_i}\bar{x}_iy_{it}+\sum_{i=1}^{N}\sum_{t=1}^{T_i}\bar{x}_i\bar{y}\\
&= \sum_{i=1}^{N}\sum_{t=1}^{T_i}x_{it}y_{it}-\bar{y}\sum_{i=1}^{N}\sum_{t=1}^{T_i}x_{it}-\sum_{i=1}^{N}\bar{x}_i\sum_{t=1}^{T_i}y_{it}+\bar{y}\sum_{i=1}^{N}T_i\bar{x}_i\\
&= \sum_{i=1}^{N}\sum_{t=1}^{T_i}x_{it}y_{it}-\bar{y}\sum_{i=1}^{N}T_i\bar{x}_i-\sum_{i=1}^{N}\bar{x}_i(T_i\bar{y}_i)+\bar{y}\sum_{i=1}^{N}T_i\bar{x}_i\\
&= \sum_{i=1}^{N}\sum_{t=1}^{T_i}x_{it}y_{it}-\sum_{i=1}^{N}T_i\bar{x}_i\bar{y}_i.
\end{aligned} \tag{3}$$

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{t=1}^{T_i}(x_{it}-\bar{x})(y_{it}-\bar{y}_i) &= \sum_{i=1}^{N}\sum_{t=1}^{T_i}(x_{it}y_{it}-x_{it}\bar{y}_i-\bar{x}y_{it}+\bar{x}\bar{y}_i)\\
&= \sum_{i=1}^{N}\sum_{t=1}^{T_i}x_{it}y_{it}-\sum_{i=1}^{N}\sum_{t=1}^{T_i}x_{it}\bar{y}_i-\sum_{i=1}^{N}\sum_{t=1}^{T_i}\bar{x}y_{it}+\sum_{i=1}^{N}\sum_{t=1}^{T_i}\bar{x}\bar{y}_i\\
&= \sum_{i=1}^{N}\sum_{t=1}^{T_i}x_{it}y_{it}-\sum_{i=1}^{N}\bar{y}_i\sum_{t=1}^{T_i}x_{it}-\bar{x}\sum_{i=1}^{N}\sum_{t=1}^{T_i}y_{it}+\bar{x}\sum_{i=1}^{N}T_i\bar{y}_i\\
&= \sum_{i=1}^{N}\sum_{t=1}^{T_i}x_{it}y_{it}-\sum_{i=1}^{N}\bar{y}_i(T_i\bar{x}_i)-\bar{x}\sum_{i=1}^{N}T_i\bar{y}_i+\bar{x}\sum_{i=1}^{N}T_i\bar{y}_i\\
&= \sum_{i=1}^{N}\sum_{t=1}^{T_i}x_{it}y_{it}-\sum_{i=1}^{N}T_i\bar{x}_i\bar{y}_i. \quad \blacksquare
\end{aligned} \tag{4}$$

Lemma 1b. The following equivalence holds in both balanced and unbalanced panels:

$$\sum_{i=1}^{N}\sum_{t=1}^{T_i}(x_{it}-\bar{x}_t)(y_{it}-\bar{y}_t)=\sum_{i=1}^{N}\sum_{t=1}^{T_i}(x_{it}-\bar{x}_t)(y_{it}-\bar{y})=\sum_{i=1}^{N}\sum_{t=1}^{T_i}(x_{it}-\bar{x})(y_{it}-\bar{y}_t). \tag{5}$$

Proof. The proof follows the proof of lemma 1a, substituting $\bar{x}_t$ for $\bar{x}_i$ and $\bar{y}_t$ for $\bar{y}_i$ and rewriting the summation as $\sum_{t=1}^{T}\sum_{i=1}^{N_t}$. ∎

Lemma 2a. The following equivalence holds in both balanced and unbalanced panels:

$$\sum_{i=1}^{N}\sum_{t=1}^{T_i}(x_{it}-\bar{x}_i)^2=\sum_{i=1}^{N}\sum_{t=1}^{T_i}(x_{it}-\bar{x}_i)(x_{it}-\bar{x}). \tag{6}$$

Proof. We prove the lemma for unbalanced panels, but the lemma also holds for balanced panels since balanced panels are the special case in which $T_i = T_j, \forall i, j \in \{1,\ldots,N\}$. We demonstrate that both expressions are individually equal to $\sum_{i=1}^{N}\sum_{t=1}^{T_i}x_{it}^2-\sum_{i=1}^{N}T_i\bar{x}_i^2$ and are therefore equal to each other:

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{t=1}^{T_i}(x_{it}-\bar{x}_i)^2 &= \sum_{i=1}^{N}\sum_{t=1}^{T_i}(x_{it}^2-2x_{it}\bar{x}_i+\bar{x}_i^2)\\
&= \sum_{i=1}^{N}\sum_{t=1}^{T_i}x_{it}^2-2\sum_{i=1}^{N}\sum_{t=1}^{T_i}x_{it}\bar{x}_i+\sum_{i=1}^{N}\sum_{t=1}^{T_i}\bar{x}_i^2\\
&= \sum_{i=1}^{N}\sum_{t=1}^{T_i}x_{it}^2-2\sum_{i=1}^{N}\bar{x}_i\sum_{t=1}^{T_i}x_{it}+\sum_{i=1}^{N}T_i\bar{x}_i^2\\
&= \sum_{i=1}^{N}\sum_{t=1}^{T_i}x_{it}^2-2\sum_{i=1}^{N}T_i\bar{x}_i^2+\sum_{i=1}^{N}T_i\bar{x}_i^2\\
&= \sum_{i=1}^{N}\sum_{t=1}^{T_i}x_{it}^2-\sum_{i=1}^{N}T_i\bar{x}_i^2.
\end{aligned} \tag{7}$$

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{t=1}^{T_i}(x_{it}-\bar{x}_i)(x_{it}-\bar{x}) &= \sum_{i=1}^{N}\sum_{t=1}^{T_i}(x_{it}^2-\bar{x}x_{it}-\bar{x}_ix_{it}+\bar{x}_i\bar{x})\\
&= \sum_{i=1}^{N}\sum_{t=1}^{T_i}x_{it}^2-\sum_{i=1}^{N}\sum_{t=1}^{T_i}\bar{x}x_{it}-\sum_{i=1}^{N}\sum_{t=1}^{T_i}\bar{x}_ix_{it}+\sum_{i=1}^{N}\sum_{t=1}^{T_i}\bar{x}_i\bar{x}\\
&= \sum_{i=1}^{N}\sum_{t=1}^{T_i}x_{it}^2-\bar{x}\sum_{i=1}^{N}\sum_{t=1}^{T_i}x_{it}-\sum_{i=1}^{N}\bar{x}_i\sum_{t=1}^{T_i}x_{it}+\bar{x}\sum_{i=1}^{N}T_i\bar{x}_i\\
&= \sum_{i=1}^{N}\sum_{t=1}^{T_i}x_{it}^2-\bar{x}\sum_{i=1}^{N}T_i\bar{x}_i-\sum_{i=1}^{N}T_i\bar{x}_i^2+\bar{x}\sum_{i=1}^{N}T_i\bar{x}_i\\
&= \sum_{i=1}^{N}\sum_{t=1}^{T_i}x_{it}^2-\sum_{i=1}^{N}T_i\bar{x}_i^2. \quad \blacksquare
\end{aligned} \tag{8}$$

Lemma 2b. The following equivalence holds in both balanced and unbalanced panels:

$$\sum_{i=1}^{N}\sum_{t=1}^{T_i}(x_{it}-\bar{x}_t)^2=\sum_{i=1}^{N}\sum_{t=1}^{T_i}(x_{it}-\bar{x}_t)(x_{it}-\bar{x}). \tag{9}$$

Proof. The proof follows the proof of lemma 2a, substituting $\bar{x}_t$ for $\bar{x}_i$ and rewriting the summation as $\sum_{t=1}^{T}\sum_{i=1}^{N_t}$.
∎

Theorem 1. The two-way fixed effects estimator is a weighted average of five coefficient estimates: (1) the pooled OLS estimator (β_{pool}), (2) the case-level fixed effects estimator (β_{caseFE}), (3) the time-level fixed effects estimator (β_{timeFE}), (4) the OLS estimator applied to the model that removes the case-level means from the outcome and the time-level means from the predictor (β_{casetime}), and (5) the OLS estimator applied to the model that removes the time-level means from the outcome and the case-level means from the predictor (β_{timecase}). Specifically, the two-way fixed effects estimator is

$$\beta_{TW} = \frac{\omega_1\beta_{\text{pool}} - \omega_2\beta_{\text{caseFE}} - \omega_3\beta_{\text{timeFE}} + \omega_4\beta_{\text{casetime}} + \omega_5\beta_{\text{timecase}}}{\omega_1 - \omega_2 - \omega_3 + \omega_4 + \omega_5}, \tag{10}$$

where

$$\omega_1 = \sum_{i=1}^{N}\sum_{t=1}^{T_i}(x_{it}-\bar{x})^2, \quad \omega_2 = -\sum_{i=1}^{N}\sum_{t=1}^{T_i}(x_{it}-\bar{x}_i)^2, \quad \omega_3 = -\sum_{i=1}^{N}\sum_{t=1}^{T_i}(x_{it}-\bar{x}_t)^2, \quad \omega_4 = \omega_5 = \sum_{i=1}^{N}\sum_{t=1}^{T_i}(x_{it}-\bar{x}_i)(x_{it}-\bar{x}_t).$$

Proof. The two-way fixed effects estimator is the OLS estimator applied to the following model,

$$y^*_{it} = \alpha_{TW} + \beta_{TW}x^*_{it} + \varepsilon_{it}, \tag{11}$$

where $*$ denotes the transformation

$$x^*_{it} = x_{it} - \bar{x}_i - \bar{x}_t + \bar{x}. \tag{12}$$

By OLS, the coefficient in equation **??** is given by

$$\begin{aligned}
\beta_{TW} &= \frac{\sum_{i=1}^{N}\sum_{t=1}^{T_i}\Big(x_{it}-\bar{x}_i-\bar{x}_t+\bar{x}\Big)\Big(y_{it}-\bar{y}_i-\bar{y}_t+\bar{y}\Big)}{\sum_{i=1}^{N}\sum_{t=1}^{T_i}\Big(x_{it}-\bar{x}_i-\bar{x}_t+\bar{x}\Big)^2} \\
&= \frac{\sum_{i=1}^{N}\sum_{t=1}^{T_i}\Big(x_{it}+x_{it}-x_{it}-\bar{x}_i-\bar{x}_t+\bar{x}\Big)\Big(y_{it}+y_{it}-y_{it}-\bar{y}_i-\bar{y}_t+\bar{y}\Big)}{\sum_{i=1}^{N}\sum_{t=1}^{T_i}\Big(x_{it}+x_{it}-x_{it}-\bar{x}_i-\bar{x}_t+\bar{x}\Big)^2} \\
&= \frac{\sum_{i=1}^{N}\sum_{t=1}^{T_i}\Big[(x_{it}-\bar{x}_i)+(x_{it}-\bar{x}_t)-(x_{it}-\bar{x})\Big]\Big[(y_{it}-\bar{y}_i)+(y_{it}-\bar{y}_t)-(y_{it}-\bar{y})\Big]}{\sum_{i=1}^{N}\sum_{t=1}^{T_i}\Big[(x_{it}-\bar{x}_i)+(x_{it}-\bar{x}_t)-(x_{it}-\bar{x})\Big]^2}, \\
&= \frac{\sum_{i=1}^{N}\sum_{t=1}^{T_i}A_{it}}{\sum_{i=1}^{N}\sum_{t=1}^{T_i}B_{it}},
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
A_{it} &= (x_{it}-\bar{x}_i)(y_{it}-\bar{y}_i) + (x_{it}-\bar{x}_i)(y_{it}-\bar{y}_t) - (x_{it}-\bar{x}_i)(y_{it}-\bar{y}) \\
&\quad + (x_{it}-\bar{x}_t)(y_{it}-\bar{y}_i) + (x_{it}-\bar{x}_t)(y_{it}-\bar{y}_t) - (x_{it}-\bar{x}_t)(y_{it}-\bar{y}) \\
&\quad - (x_{it}-\bar{x})(y_{it}-\bar{y}_i) - (x_{it}-\bar{x})(y_{it}-\bar{y}_t) + (x_{it}-\bar{x})(y_{it}-\bar{y}),
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
B_{it} &= (x_{it}-\bar{x}_i)^2 + (x_{it}-\bar{x}_t)^2 + (x_{it}-\bar{x})^2 \\
&\quad + 2(x_{it}-\bar{x}_i)(x_{it}-\bar{x}_t) - 2(x_{it}-\bar{x}_i)(x_{it}-\bar{x}) - 2(x_{it}-\bar{x}_t)(x_{it}-\bar{x}).
\end{aligned} \tag{15}$$

From lemmas 1a, 1b, 2a, and 2b, these expressions reduce to

$$\begin{aligned}
A_{it} &= (x_{it}-\bar{x})(y_{it}-\bar{y}) - (x_{it}-\bar{x}_i)(y_{it}-\bar{y}_i) - (x_{it}-\bar{x}_t)(y_{it}-\bar{y}_t) \\
&\quad + (x_{it}-\bar{x}_i)(y_{it}-\bar{y}_t) + (x_{it}-\bar{x}_t)(y_{it}-\bar{y}_i),
\end{aligned} \tag{16}$$

and

$$B_{it} = (x_{it}-\bar{x})^2 - (x_{it}-\bar{x}_i)^2 - (x_{it}-\bar{x}_t)^2 + 2(x_{it}-\bar{x}_i)(x_{it}-\bar{x}_t). \tag{17}$$

Note that the pooled OLS estimator for the coefficient is

$$\beta_{\text{pool}} = \frac{\sum_{i=1}^{N}\sum_{t=1}^{T_i}(x_{it}-\bar{x})(y_{it}-\bar{y})}{\sum_{i=1}^{N}\sum_{t=1}^{T_i}(x_{it}-\bar{x})^2}, \tag{18}$$

the case fixed effects estimator for the coefficient is

$$\beta_{\text{caseFE}} = \frac{\sum_{i=1}^{N} \sum_{t=1}^{T_i} (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_i)}{\sum_{i=1}^{N} \sum_{t=1}^{T_i} (x_{it} - \bar{x}_i)^2}, \tag{19}$$

and the time fixed effects estimator for the coefficient is

$$\beta_{\text{timeFE}} = \frac{\sum_{i=1}^{N} \sum_{t=1}^{T_i} (x_{it} - \bar{x}_t)(y_{it} - \bar{y}_t)}{\sum_{i=1}^{N} \sum_{t=1}^{T_i} (x_{it} - \bar{x}_t)^2}. \tag{20}$$

In addition, define β_{casetime} to be the OLS coefficient obtained by removing the case-level means from the outcome and the time-level means from the predictor, given by

$$\beta_{\text{casetime}} = \frac{\sum_{i=1}^{N} \sum_{t=1}^{T_i} (x_{it} - \bar{x}_t)(y_{it} - \bar{y}_i)}{\sum_{i=1}^{N} \sum_{t=1}^{T_i} (x_{it} - \bar{x}_i)(x_{it} - \bar{x}_t)}. \tag{21}$$

Likewise, define β_{timecase} to be the OLS coefficient obtained by removing the time-level means from the outcome and the case-level means from the predictor, given by

$$\beta_{\text{timecase}} = \frac{\sum_{i=1}^{N} \sum_{t=1}^{T_i} (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_t)}{\sum_{i=1}^{N} \sum_{t=1}^{T_i} (x_{it} - \bar{x}_i)(x_{it} - \bar{x}_t)}. \tag{22}$$

The two-way fixed effects estimator is therefore a weighted average of the preceding five estimates, given by

$$\beta_{TW} = \frac{\omega_1 \beta_{\text{pool}} - \omega_2 \beta_{\text{caseFE}} - \omega_3 \beta_{\text{timeFE}} + \omega_4 \beta_{\text{casetime}} + \omega_5 \beta_{\text{timecase}}}{\omega_1 - \omega_2 - \omega_3 + \omega_4 + \omega_5}, \tag{23}$$

where

$$\omega_1 = \sum_{i=1}^{N} \sum_{t=1}^{T_i} (x_{it} - \bar{x})^2, \quad \omega_2 = -\sum_{i=1}^{N} \sum_{t=1}^{T_i} (x_{it} - \bar{x}_i)^2, \quad \omega_3 = -\sum_{i=1}^{N} \sum_{t=1}^{T_i} (x_{it} - \bar{x}_t)^2, \quad \omega_4 = \omega_5 = \sum_{i=1}^{N} \sum_{t=1}^{T_i} (x_{it} - \bar{x}_i)(x_{it} - \bar{x}_t).$$

Lemma 3. If the panels in the data are balanced, then

$$\sum_{i=1}^{N} \sum_{t=1}^{T} (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_t) = \sum_{i=1}^{N} \sum_{t=1}^{T} (x_{it} - \bar{x}_t)(y_{it} - \bar{y}_i). \tag{24}$$

Proof. This proof depends on the fact that

$$\bar{x} \quad = \frac{\sum_{i=1}^{N} \sum_{t=1}^{T} x_{it}}{NT} \quad = \frac{\sum_{i=1}^{N} \bar{x}_i}{N} \quad = \frac{\sum_{t=1}^{T} \bar{x}_t}{T} \tag{25}$$

if and only if the panels are balanced:

$$
\begin{aligned}
\sum_{i=1}^{N}\sum_{t=1}^{T}(x_{it}-\bar{x}_i)(y_{it}-\bar{y}_t) &= \sum_{i=1}^{N}\sum_{t=1}^{T}(x_{it}y_{it}-x_{it}\bar{y}_t-\bar{x}_i y_{it}+\bar{x}_i\bar{y}_t) \\
&= \sum_{i=1}^{N}\sum_{t=1}^{T}x_{it}y_{it}-\sum_{i=1}^{N}\sum_{t=1}^{T}x_{it}\bar{y}_t-\sum_{i=1}^{N}\sum_{t=1}^{T}\bar{x}_i y_{it}+\sum_{i=1}^{N}\sum_{t=1}^{T}\bar{x}_i\bar{y}_t \\
&= \sum_{i=1}^{N}\sum_{t=1}^{T}x_{it}y_{it}-\sum_{t=1}^{T}\bar{y}_t\sum_{i=1}^{N}x_{it}-\sum_{i=1}^{N}\bar{x}_i\sum_{t=1}^{T}y_{it}+\sum_{i=1}^{N}\bar{x}_i\sum_{t=1}^{T}\bar{y}_t \\
&= \sum_{i=1}^{N}\sum_{t=1}^{T}x_{it}y_{it}-N\sum_{t=1}^{T}\bar{x}_t\bar{y}_t-T\sum_{i=1}^{N}\bar{x}_i\bar{y}_i+\sum_{i=1}^{N}\bar{x}_i\sum_{t=1}^{T}\bar{y}_t \\
&= \sum_{i=1}^{N}\sum_{t=1}^{T}x_{it}y_{it}-N\sum_{t=1}^{T}\bar{x}_t\bar{y}_t-T\sum_{i=1}^{N}\bar{x}_i\bar{y}_i+NT\bar{x}\bar{y} \\
&= \sum_{i=1}^{N}\sum_{t=1}^{T}x_{it}y_{it}-\sum_{t=1}^{T}\bar{x}_t\sum_{i=1}^{N}y_{it}-\sum_{i=1}^{N}\sum_{t=1}^{T}x_{it}\bar{y}_i+\sum_{i=1}^{N}\bar{x}_t\sum_{t=1}^{T}\bar{y}_i \\
&= \sum_{i=1}^{N}\sum_{t=1}^{T}(x_{it}y_{it}-\bar{x}_t y_{it}-x_{it}\bar{y}_i+\bar{x}_t\bar{y}_i) \\
&= \sum_{i=1}^{N}\sum_{t=1}^{T}(x_{it}-\bar{x}_t)(y_{it}-\bar{y}_i). \quad \blacksquare
\end{aligned}
\tag{26}
$$

Lemma 4. If the panels in the data are balanced, then

$$
\sum_{i=1}^{N}\sum_{t=1}^{T}\Big[(x_{it}-\bar{x}_i)(y_{it}-\bar{y}_i)+(x_{it}-\bar{x}_t)(y_{it}-\bar{y}_t)-(x_{it}-\bar{x}_i)(y_{it}-\bar{y}_t)\Big]=\sum_{i=1}^{N}\sum_{t=1}^{T}(x_{it}-\bar{x})(y_{it}-\bar{y}). \tag{27}
$$

Proof.

$$
\begin{aligned}
&\sum_{i=1}^{N}\sum_{t=1}^{T}\Big[(x_{it}-\bar{x}_i)(y_{it}-\bar{y}_i)+(x_{it}-\bar{x}_t)(y_{it}-\bar{y}_t)-(x_{it}-\bar{x}_i)(y_{it}-\bar{y}_t)\Big]\\
&=\sum_{i=1}^{N}\sum_{t=1}^{T}\Big[(x_{it}y_{it}-x_{it}\bar{y}_i-\bar{x}_iy_{it}+\bar{x}_i\bar{y}_i)+(x_{it}y_{it}-x_{it}\bar{y}_t-\bar{x}_ty_{it}+\bar{x}_t\bar{y}_t)-(x_{it}y_{it}-x_{it}\bar{y}_t-\bar{x}_iy_{it}+\bar{x}_i\bar{y}_t)\Big]\\
&=\sum_{i=1}^{N}\sum_{t=1}^{T}\Big[(x_{it}y_{it}+x_{it}y_{it}-x_{it}y_{it})-x_{it}(\bar{y}_i+\bar{y}_t-\bar{y}_t)-(\bar{x}_i+\bar{x}_t-\bar{x}_i)y_{it}+\bar{x}_i\bar{y}_i+\bar{x}_t\bar{y}_t-\bar{x}_i\bar{y}_t\Big]\\
&=\sum_{i=1}^{N}\sum_{t=1}^{T}\Big[x_{it}y_{it}-x_{it}\bar{y}_i-\bar{x}_ty_{it}+\bar{x}_i\bar{y}_i+\bar{x}_t\bar{y}_t-\bar{x}_i\bar{y}_t\Big]\\
&=\sum_{i=1}^{N}\sum_{t=1}^{T}x_{it}y_{it}-\sum_{i=1}^{N}\sum_{t=1}^{T}x_{it}\bar{y}_i-\sum_{i=1}^{N}\sum_{t=1}^{T}\bar{x}_ty_{it}+\sum_{i=1}^{N}\sum_{t=1}^{T}\bar{x}_i\bar{y}_i+\sum_{i=1}^{N}\sum_{t=1}^{T}\bar{x}_t\bar{y}_t-\sum_{i=1}^{N}\sum_{t=1}^{T}\bar{x}_i\bar{y}_t\\
&=\sum_{i=1}^{N}\sum_{t=1}^{T}x_{it}y_{it}-T\sum_{i=1}^{N}\bar{x}_i\bar{y}_i-N\sum_{t=1}^{T}\bar{x}_t\bar{y}_t+T\sum_{i=1}^{N}\bar{x}_i\bar{y}_i+N\sum_{t=1}^{T}\bar{x}_t\bar{y}_t-NT\bar{x}\bar{y}\\
&=\sum_{i=1}^{N}\sum_{t=1}^{T}x_{it}y_{it}-NT\bar{x}\bar{y}\\
&=\sum_{i=1}^{N}\sum_{t=1}^{T}x_{it}y_{it}-NT\bar{x}\bar{y}-NT\bar{x}\bar{y}+NT\bar{x}\bar{y}\\
&=\sum_{i=1}^{N}\sum_{t=1}^{T}x_{it}y_{it}-\sum_{i=1}^{N}\sum_{t=1}^{T}x_{it}\bar{y}-\sum_{i=1}^{N}\sum_{t=1}^{T}\bar{x}y_{it}+\sum_{i=1}^{N}\sum_{t=1}^{T}\bar{x}\bar{y}\\
&=\sum_{i=1}^{N}\sum_{t=1}^{T}(x_{it}y_{it}-x_{it}\bar{y}-\bar{x}y_{it}+\bar{x}\bar{y})\\
&=\sum_{i=1}^{N}\sum_{t=1}^{T}(x_{it}-\bar{x})(y_{it}-\bar{y}). \qquad \blacksquare
\end{aligned}
\tag{28}
$$

Lemma 5. If the panels in the data are balanced, then

$$
\sum_{i=1}^{N}\sum_{t=1}^{T}\Big[(x_{it}-\bar{x}_i)^2+(x_{it}-\bar{x}_t)^2-(x_{it}-\bar{x}_i)(x_{it}-\bar{x}_t)\Big]=\sum_{i=1}^{N}\sum_{t=1}^{T}(x_{it}-\bar{x})^2. \tag{29}
$$

Proof.

$$
\begin{aligned}
&\sum_{i=1}^{N}\sum_{t=1}^{T}\Big[(x_{it}-\bar{x}_i)^2+(x_{it}-\bar{x}_t)^2-(x_{it}-\bar{x}_i)(x_{it}-\bar{x}_t)\Big]\\
&=\sum_{i=1}^{N}\sum_{t=1}^{T}\Big[(x_{it}^2-2x_{it}\bar{x}_i+\bar{x}_i^2)+(x_{it}^2-2x_{it}\bar{x}_t+\bar{x}_t^2)-(x_{it}^2-x_{it}\bar{x}_t-x_{it}\bar{x}_i+\bar{x}_i\bar{x}_t)\Big]\\
&=\sum_{i=1}^{N}\sum_{t=1}^{T}\Big[(x_{it}^2+x_{it}^2-x_{it}^2)-x_{it}(2\bar{x}_i+2\bar{x}_t-\bar{x}_i-\bar{x}_t)+\bar{x}_i^2+\bar{x}_t^2-\bar{x}_i\bar{x}_t\Big]\\
&=\sum_{i=1}^{N}\sum_{t=1}^{T}\Big[x_{it}^2-x_{it}\bar{x}_i+x_{it}\bar{x}_t+\bar{x}_i^2+\bar{x}_t^2-\bar{x}_i\bar{x}_t\Big]\\
&=\sum_{i=1}^{N}\sum_{t=1}^{T}x_{it}^2-\sum_{i=1}^{N}\sum_{t=1}^{T}x_{it}\bar{x}_i+\sum_{i=1}^{N}\sum_{t=1}^{T}x_{it}\bar{x}_t+\sum_{i=1}^{N}\sum_{t=1}^{T}\bar{x}_i^2+\sum_{i=1}^{N}\sum_{t=1}^{T}\bar{x}_t^2-\sum_{i=1}^{N}\sum_{t=1}^{T}\bar{x}_i\bar{x}_t\\
&=\sum_{i=1}^{N}\sum_{t=1}^{T}x_{it}^2-T\sum_{i=1}^{N}\bar{x}_i^2+N\sum_{t=1}^{T}\bar{x}_t^2+T\sum_{i=1}^{N}\bar{x}_i^2+N\sum_{t=1}^{T}\bar{x}_t^2-NT\bar{x}^2\\
&=\sum_{i=1}^{N}\sum_{t=1}^{T}x_{it}^2-NT\bar{x}^2\\
&=\sum_{i=1}^{N}\sum_{t=1}^{T}x_{it}^2-NT\bar{x}^2+NT\bar{x}^2-NT\bar{x}^2\\
&=\sum_{i=1}^{N}\sum_{t=1}^{T}x_{it}^2-2\sum_{i=1}^{N}\sum_{t=1}^{T}x_{it}\bar{x}+\sum_{i=1}^{N}\sum_{t=1}^{T}\bar{x}^2\\
&=\sum_{i=1}^{N}\sum_{t=1}^{T}(x_{it}^2-2x_{it}\bar{x}+\bar{x}^2)\\
&=\sum_{i=1}^{N}\sum_{t=1}^{T}(x_{it}-\bar{x})^2. \quad \blacksquare
\end{aligned}
\tag{30}
$$

Theorem 2. If the panels in the data are balanced, then the two-way fixed effect estimator is given by

$$\beta_{TW}=\frac{\sum_{i=1}^{N}\sum_{t=1}^{T}(x_{it}-\bar{x}_i)(y_{it}-\bar{y}_t)}{\sum_{i=1}^{N}\sum_{t=1}^{T}(x_{it}-\bar{x}_i)(x_{it}-\bar{x}_t)}, \tag{31}$$

or equivalently as

$$\beta_{TW}=\frac{\sum_{i=1}^{N}\sum_{t=1}^{T}(x_{it}-\bar{x}_t)(y_{it}-\bar{y}_i)}{\sum_{i=1}^{N}\sum_{t=1}^{T}(x_{it}-\bar{x}_i)(x_{it}-\bar{x}_t)}. \tag{32}$$

Proof. From the proof of theorem 1, the two-way fixed effects estimator can be written as

$$\beta_{TW}=\frac{\sum_{i=1}^{N}\sum_{t=1}^{T_i}A_{it}}{\sum_{i=1}^{N}\sum_{t=1}^{T_i}B_{it}}, \tag{33}$$

where

$$
\begin{aligned}
A_{it}=&(x_{it}-\bar{x})(y_{it}-\bar{y})-(x_{it}-\bar{x}_i)(y_{it}-\bar{y}_i)-(x_{it}-\bar{x}_t)(y_{it}-\bar{y}_t)\\
&+(x_{it}-\bar{x}_i)(y_{it}-\bar{y}_t)+(x_{it}-\bar{x}_t)(y_{it}-\bar{y}_i),
\end{aligned}
\tag{34}
$$

and

$$B_{it}=(x_{it}-\bar{x})^2-(x_{it}-\bar{x}_i)^2-(x_{it}-\bar{x}_t)^2+2(x_{it}-\bar{x}_i)(x_{it}-\bar{x}_t). \tag{35}$$

By lemma 3, A_{it} simplifies to

$$A_{it}=(x_{it}-\bar{x})(y_{it}-\bar{y})-(x_{it}-\bar{x}_i)(y_{it}-\bar{y}_i)-(x_{it}-\bar{x}_t)(y_{it}-\bar{y}_t)+2(x_{it}-\bar{x}_i)(y_{it}-\bar{y}_t), \tag{36}$$

or equivalently to

$$A_{it} = (x_{it} - \bar{x})(y_{it} - \bar{y}) - (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_i) - (x_{it} - \bar{x}_t)(y_{it} - \bar{y}_t) + 2(x_{it} - \bar{x}_t)(y_{it} - \bar{y}_i). \tag{37}$$

Next, by lemma 4, these expressions further simplify to

$$\begin{aligned} A_{it} = \Big[&(x_{it} - \bar{x}_i)(y_{it} - \bar{y}_i) + (x_{it} - \bar{x}_t)(y_{it} - \bar{y}_t) - (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_t)\Big] \\ &- (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_i) - (x_{it} - \bar{x}_t)(y_{it} - \bar{y}_t) + 2(x_{it} - \bar{x}_i)(y_{it} - \bar{y}_t) \\ \\ &= (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_t), \end{aligned} \tag{38}$$

or equivalently to $(x_{it} - \bar{x}_t)(y_{it} - \bar{y}_i)$ by lemma 3. Applying lemma 5, B_{it} reduces as follows:

$$\begin{aligned} B_{it} = \Big[&(x_{it} - \bar{x}_i)^2 + (x_{it} - \bar{y}_t)^2 - (x_{it} - \bar{x}_i)(x_{it} - \bar{x}_t)\Big] \\ &- (x_{it} - \bar{x}_i)^2 - (x_{it} - \bar{x}_t)^2 + 2(x_{it} - \bar{x}_i)(x_{it} - \bar{x}_t) \\ \\ &= (x_{it} - \bar{x}_i)(x_{it} - \bar{x}_t). \end{aligned} \tag{39}$$

Therefore, the two-way fixed effect estimator is given by

$$\beta_{TW} = \frac{\sum_{i=1}^{N} \sum_{t=1}^{T} (x_{it} - \bar{x}_i)(y_{it} - \bar{y}_t)}{\sum_{i=1}^{N} \sum_{t=1}^{T} (x_{it} - \bar{x}_i)(x_{it} - \bar{x}_t)}, \tag{40}$$

or equivalently as

$$\beta_{TW} = \frac{\sum_{i=1}^{N} \sum_{t=1}^{T} (x_{it} - \bar{x}_t)(y_{it} - \bar{y}_i)}{\sum_{i=1}^{N} \sum_{t=1}^{T} (x_{it} - \bar{x}_i)(x_{it} - \bar{x}_t)}. \quad \blacksquare \tag{41}$$