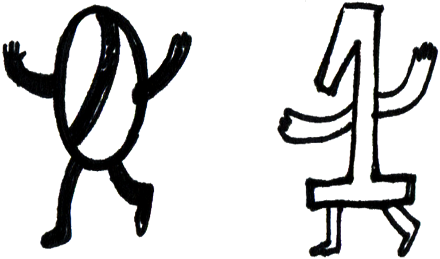
How to Build a Logit Model from Scratch



Jon Kropko
University of Virginia
June 11, 2019

Before we get started

You can access the slides, data, and code for today's class at

github.com/jkropko/ds_sampleclass

I imagine this class to be in the middle of a semester on probability modeling. I assume students have prior exposure to the following topics:

- Probability and distributions
- Linear regression and OLS
- R programming, including writing functions
- Pre-calculus, derivatives, and basic matrix operations

Please feel free to ask questions at any time!

Logistic Regression (logit)

Logistic regression (also called **logit**) is a tool for modeling the variance of a binary (two values, usually coded 0 and 1) outcome variable.

Logistic Regression (logit)

Logistic regression (also called **logit**) is a tool for modeling the variance of a binary (two values, usually coded 0 and 1) outcome variable.

There are two important ways we use logit models:

Logistic Regression (logit)

Logistic regression (also called **logit**) is a tool for modeling the variance of a binary (two values, usually coded 0 and 1) outcome variable.

There are two important ways we use logit models:

1. Calculating the probability that the outcome is 1 (vs. 0) given values of a set of X variables

Logistic Regression (logit)

Logistic regression (also called **logit**) is a tool for modeling the variance of a binary (two values, usually coded 0 and 1) outcome variable.

There are two important ways we use logit models:

1. Calculating the probability that the outcome is 1 (vs. 0) given values of a set of X variables

2. Assessing whether each X has an effect on this probability (and how big is the effect)

Logistic Regression (logit)

Logistic regression (also called **logit**) is a tool for modeling the variance of a binary (two values, usually coded 0 and 1) outcome variable.

There are two important ways we use logit models:

1. Calculating the probability that the outcome is 1 (vs. 0) given values of a set of X variables

2. Assessing whether each X has an effect on this probability (and how big is the effect)

In practice, the code and results of a logit model look a lot like linear regression.

R Code

Let's use data from the American National Election Study. There are 2,795 observations collected from surveys just before and after the 2016 U.S. presidential election.

R Code

Let's use data from the American National Election Study. There are 2,795 observations collected from surveys just before and after the 2016 U.S. presidential election.

The outcome variable is **vote**, where 0 indicates a vote for Donald Trump and 1 indicates a vote for Hillary Clinton.

R Code

Let's use data from the American National Election Study. There are 2,795 observations collected from surveys just before and after the 2016 U.S. presidential election.

The outcome variable is **vote**, where 0 indicates a vote for Donald Trump and 1 indicates a vote for Hillary Clinton.

I regress the vote on a voter's:

- age – years
- marital status – single, married, no longer married
- education – less than a HS diploma, HS diploma, some college, college degree, graduate degree
- union membership status – member (1) or non-member (0)
- race – white, black, Hispanic, other
- gender – male, female

R Code

The code to run a logit model is

```
logit <- glm(vote ~ age + marital + education + union +
                   race + gender,
             data = anes,
             family=binomial(link="logit"))
```

R Code

The code to run a logit model is

```
logit <- glm(vote ~ age + marital + education + union +
                    race + gender,
               data = anes,
               family=binomial(link="logit"))
```

There are two differences between this code and the code to run a linear regression:

R Code

The code to run a logit model is

```
logit <- glm(vote ~ age + marital + education + union +
                    race + gender,
               data = anes,
               family=binomial(link="logit"))
```

There are two differences between this code and the code to run a linear regression:

1. The function is `glm()` instead of `lm()`, because logit is a special example of a **generalized linear model** (GLM)

R Code

The code to run a logit model is

```
logit <- glm(vote ~ age + marital + education + union +
                    race + gender,
             data = anes,
             family=binomial(link="logit"))
```

There are two differences between this code and the code to run a linear regression:

1. The function is `glm()` instead of `lm()`, because logit is a special example of a **generalized linear model** (GLM)

2. The argument `family=binomial(link="logit")` tells R that the particular GLM we want to run is logit. We will discuss the logic of this syntax more detail in a few minutes.

R Code

The code to run a logit model is

```
logit <- glm(vote ~ age + marital + education + union +
                  race + gender,
             data = anes,
             family=binomial(link="logit"))
```

There are two differences between this code and the code to run a linear regression:

1. The function is `glm()` instead of `lm()`, because logit is a special example of a **generalized linear model** (GLM)

2. The argument `family=binomial(link="logit")` tells R that the particular GLM we want to run is logit. We will discuss the logic of this syntax more detail in a few minutes.

Try running the model yourself.

Climbing out of the Black Box

Great! We've used the right code to run a logit model. If all we care about is getting coefficient estimates, we can stop here.

Climbing out of the Black Box

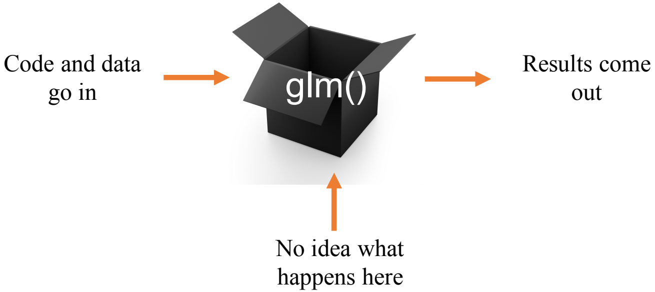
Great! We've used the right code to run a logit model. If all we care about is getting coefficient estimates, we can stop here.

That's an example of something called a black box approach to statistics:

Climbing out of the Black Box

Great! We've used the right code to run a logit model. If all we care about is getting coefficient estimates, we can stop here.

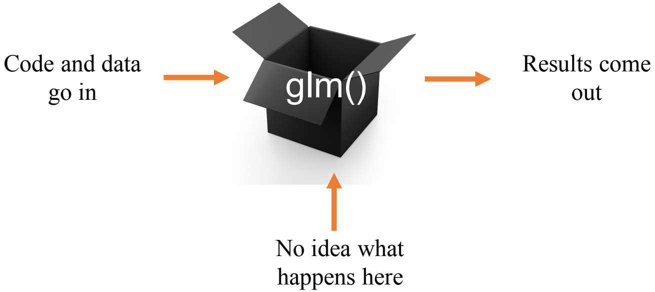
That's an example of something called a black box approach to statistics:



Climbing out of the Black Box

Great! We've used the right code to run a logit model. If all we care about is getting coefficient estimates, we can stop here.

That's an example of something called a black box approach to statistics:



Many systems in real life work this way (the human brain for example!) but logit doesn't have to be a black box.

Climbing out of the Black Box

Logit is a mathematical model of probability. If we delve into the underlying math, we will *really* understand how logit works.

Climbing out of the Black Box

Logit is a mathematical model of probability. If we delve into the underlying math, we will *really* understand how logit works.

Then we can adapt it to our own problems. We can even **invent new methods** by changing parts of this construction.

Climbing out of the Black Box

Logit is a mathematical model of probability. If we delve into the underlying math, we will *really* understand how logit works.

Then we can adapt it to our own problems. We can even **invent new methods** by changing parts of this construction.

Our plan today:

Climbing out of the Black Box

Logit is a mathematical model of probability. If we delve into the underlying math, we will *really* understand how logit works.

Then we can adapt it to our own problems. We can even **invent new methods** by changing parts of this construction.

Our plan today:

1. Go over the math, and build the log-likelihood function: a blueprint for a logit model

Climbing out of the Black Box

Logit is a mathematical model of probability. If we delve into the underlying math, we will *really* understand how logit works.

Then we can adapt it to our own problems. We can even **invent new methods** by changing parts of this construction.

Our plan today:

1. Go over the math, and build the log-likelihood function: a blueprint for a logit model

2. Program our own logit models **from scratch** in R, without using any pre-programmed logit function like `glm()`.

Climbing out of the Black Box

Logit is a mathematical model of probability. If we delve into the underlying math, we will *really* understand how logit works.

Then we can adapt it to our own problems. We can even **invent new methods** by changing parts of this construction.

Our plan today:

1. Go over the math, and build the log-likelihood function: a blueprint for a logit model

2. Program our own logit models **from scratch** in R, without using any pre-programmed logit function like `glm()`.

Also, there are better ways than reporting coefficients to express the results of a logit model. For details, see the appendix to these slides.

Generalized Linear Models (GLMs)

The purpose of a GLM is to map values of X variables onto probabilities of values of Y.

Generalized Linear Models (GLMs)

The purpose of a GLM is to map values of X variables onto probabilities of values of Y.

A GLM has three parts:

Generalized Linear Models (GLMs)

The purpose of a GLM is to map values of X variables onto probabilities of values of Y.

A GLM has three parts:

The family: a **probability density/mass function** for the outcome

Generalized Linear Models (GLMs)

The purpose of a GLM is to map values of X variables onto probabilities of values of Y.

A GLM has three parts:

The family: a **probability density/mass function** for the outcome

The linear model: a linear combination of the X variables that will be substituted for a parameter of the family

Generalized Linear Models (GLMs)

The purpose of a GLM is to map values of X variables onto probabilities of values of Y.

A GLM has three parts:

The family: a **probability density/mass function** for the outcome

The linear model: a linear combination of the X variables that will be substituted for a parameter of the family

A link function: a mathematical function that transforms the linear model so that it has the same support as the parameter it's being plugged into

Logit as a GLM

Logit is just one example of a GLM. Other famous models (probit, ordered logit, Poisson, etc.) are also GLMs, and differ from logit only in the choice of family and link function.

Logit as a GLM

Logit is just one example of a GLM. Other famous models (probit, ordered logit, Poisson, etc.) are also GLMs, and differ from logit only in the choice of family and link function.

Logit's family is the **Bernoulli distribution** (a special case of the binomial distribution with one trial):

$$f(y_i|p_i) = {p_i}^{y_i}(1-p_i)^{1-y_i}$$

Logit as a GLM

Logit is just one example of a GLM. Other famous models (probit, ordered logit, Poisson, etc.) are also GLMs, and differ from logit only in the choice of family and link function.

Logit's family is the **Bernoulli distribution** (a special case of the binomial distribution with one trial):

$$f(y_i|p_i) = {p_i}^{y_i}(1 - p_i)^{1-y_i}$$

There's one parameter p_i, which represents the probability that $y_i = 1$.

Logit as a GLM

Logit is just one example of a GLM. Other famous models (probit, ordered logit, Poisson, etc.) are also GLMs, and differ from logit only in the choice of family and link function.

Logit's family is the **Bernoulli distribution** (a special case of the binomial distribution with one trial):

$$f(y_i|p_i) = {p_i}^{y_i}(1-p_i)^{1-y_i}$$

There's one parameter p_i, which represents the probability that $y_i = 1$.

The linear model is an equation containing all of the Xs. We've seen equations like this before with regression models. I denote the linear model as y_i^*:

$$y_i^* = \alpha + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_k x_{ik}.$$

Logit as a GLM

Family: $f(y_i|p_i) = p_i^{\,y_i}(1-p_i)^{1-y_i}$

Linear model: $y_i^* = \alpha + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_k x_{ik}$

Logit as a GLM

$$\text{Family:} \quad f(y_i|p_i) = {p_i}^{y_i}(1-p_i)^{1-y_i}$$
$$\text{Linear model:} \quad y_i^* = \alpha + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_k x_{ik}$$

The linear model can take on all real numbers. And we want to substitute it for the probability parameter p_i, which can only be between 0 and 1.

Logit as a GLM

$$\begin{aligned} \text{Family:} \quad & f(y_i|p_i) = p_i^{\,y_i}(1-p_i)^{1-y_i} \\ \text{Linear model:} \quad & y_i^* = \alpha + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_k x_{ik} \end{aligned}$$

The linear model can take on all real numbers. And we want to substitute it for the probability parameter p_i, which can only be between 0 and 1.

We need a math function that **takes all reals** as inputs, but outputs **numbers between 0 and 1**. Can you think of one?

Logit as a GLM

$$\begin{aligned} \text{Family:} \quad & f(y_i|p_i) = {p_i}^{y_i}(1-p_i)^{1-y_i} \\ \text{Linear model:} \quad & y_i^* = \alpha + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_k x_{ik} \end{aligned}$$

The linear model can take on all real numbers. And we want to substitute it for the probability parameter p_i, which can only be between 0 and 1.

We need a math function that **takes all reals** as inputs, but outputs **numbers between 0 and 1**. Can you think of one?

For logit, we use this particular link function:

$$p_i = \frac{1}{1+e^{-y_i^*}},$$

which is CDF of the standard logistic function – hence the name "logit".

Logit as a GLM

$$\begin{aligned}
\text{Family:} \quad & f(y_i|p_i) = {p_i}^{y_i}(1-p_i)^{1-y_i} \\
\text{Linear model:} \quad & y_i^* = \alpha + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_k x_{ik} \\
\text{Link function:} \quad & p_i = \frac{1}{1+e^{-y_i^*}}
\end{aligned}$$

Logit as a GLM

Family: $f(y_i|p_i) = p_i{}^{y_i}(1-p_i)^{1-y_i}$

Linear model: $y_i^* = \alpha + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_k x_{ik}$

Link function: $p_i = \dfrac{1}{1+e^{-y_i^*}}$

One way to write the complete GLM for logit is

$$f(y_i|y_i^*) = \left(\frac{1}{1+e^{-y_i^*}}\right)^{y_i}\left(1-\frac{1}{1+e^{-y_i^*}}\right)^{1-y_i}$$

where $y_i^* = \alpha + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_k x_{ik}$.

Logit as a GLM

$$\begin{aligned}
\text{Family:} \quad & f(y_i|p_i) = p_i^{\,y_i}(1-p_i)^{1-y_i} \\
\text{Linear model:} \quad & y_i^* = \alpha + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_k x_{ik} \\
\text{Link function:} \quad & p_i = \frac{1}{1+e^{-y_i^*}}
\end{aligned}$$

One way to write the complete GLM for logit is

$$f(y_i|y_i^*) = \left(\frac{1}{1+e^{-y_i^*}}\right)^{y_i}\left(1 - \frac{1}{1+e^{-y_i^*}}\right)^{1-y_i}$$

where $y_i^* = \alpha + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_k x_{ik}$.

This GLM expresses the probabilities for one particular observation's outcome. Next we need to write the *joint distribution* for ALL the observations in the data.

Logit's Likelihood Function

$$f(y_i|y_i^*) = \left(\frac{1}{1+e^{-y_i^*}}\right)^{y_i}\left(1-\frac{1}{1+e^{-y_i^*}}\right)^{1-y_i}$$

We need to assume that all the observations are **independent** and **identically distributed** (they all share this GLM).

Logit's Likelihood Function

$$f(y_i|y_i^*) = \left(\frac{1}{1+e^{-y_i^*}}\right)^{y_i}\left(1-\frac{1}{1+e^{-y_i^*}}\right)^{1-y_i}$$

We need to assume that all the observations are **independent** and **identically distributed** (they all share this GLM).

Remember, if random variables are independent, then their joint distribution is the product of their marginal distributions.

Logit's Likelihood Function

$$f(y_i|y_i^*) = \left(\frac{1}{1+e^{-y_i^*}}\right)^{y_i}\left(1-\frac{1}{1+e^{-y_i^*}}\right)^{1-y_i}$$

We need to assume that all the observations are **independent** and **identically distributed** (they all share this GLM).

Remember, if random variables are independent, then their joint distribution is the product of their marginal distributions.

So the joint distribution of all observations in the data is the **product of all observations' GLMs**:

$$f(Y|Y^*) = \prod_{i=1}^{N}\left(\frac{1}{1+e^{-y_i^*}}\right)^{y_i}\left(1-\frac{1}{1+e^{-y_i^*}}\right)^{1-y_i}$$

Logit's Likelihood Function

$$f(Y|Y^*) = \prod_{i=1}^{N} \left(\frac{1}{1+e^{-y_i^*}}\right)^{y_i} \left(1 - \frac{1}{1+e^{-y_i^*}}\right)^{1-y_i}$$

Notice that this function contains both data (in the X and Y variables) and coefficients (in the linear model).

Logit's Likelihood Function

$$f(Y|Y^*) = \prod_{i=1}^{N} \left(\frac{1}{1+e^{-y_i^*}} \right)^{y_i} \left(1 - \frac{1}{1+e^{-y_i^*}} \right)^{1-y_i}$$

Notice that this function contains both data (in the X and Y variables) and coefficients (in the linear model).

If we think of the **coefficients as the unknowns** and the data as known, then we call this function a likelihood function, with this notation:

$$L(\alpha, \beta_1, \dots, \beta_k | X, Y) = \prod_{i=1}^{N} \left(\frac{1}{1+e^{-y_i^*}} \right)^{y_i} \left(1 - \frac{1}{1+e^{-y_i^*}} \right)^{1-y_i}$$

Logit's Likelihood Function

$$f(Y|Y^*) = \prod_{i=1}^{N} \left(\frac{1}{1+e^{-y_i^*}}\right)^{y_i} \left(1 - \frac{1}{1+e^{-y_i^*}}\right)^{1-y_i}$$

Notice that this function contains both data (in the X and Y variables) and coefficients (in the linear model).

If we think of the **coefficients as the unknowns** and the data as known, then we call this function a likelihood function, with this notation:

$$L(\alpha, \beta_1, \ldots, \beta_k|X, Y) = \prod_{i=1}^{N} \left(\frac{1}{1+e^{-y_i^*}}\right)^{y_i} \left(1 - \frac{1}{1+e^{-y_i^*}}\right)^{1-y_i}$$

A likelihood function is the DNA of a statistical model. It's a complete representation of all information in the model, and we need the likelihood function to get coefficient estimates.

Logit's Likelihood Function

$$L(\alpha, \beta_1, \ldots, \beta_k | X, Y) = \prod_{i=1}^{N} \left(\frac{1}{1 + e^{-y_i^*}} \right)^{y_i} \left(1 - \frac{1}{1 + e^{-y_i^*}} \right)^{1 - y_i}$$

Here's how likelihood functions work:

1. Plug candidate values for every coefficient into the likelihood function

Logit's Likelihood Function

$$L(\alpha, \beta_1, \ldots, \beta_k | X, Y) = \prod_{i=1}^{N} \left(\frac{1}{1 + e^{-y_i^*}} \right)^{y_i} \left(1 - \frac{1}{1 + e^{-y_i^*}} \right)^{1-y_i}$$

Here's how likelihood functions work:

1. Plug candidate values for every coefficient into the likelihood function
2. The function outputs a value called a likelihood

Logit's Likelihood Function

$$L(\alpha, \beta_1, \ldots, \beta_k | X, Y) = \prod_{i=1}^{N} \left(\frac{1}{1 + e^{-y_i^*}} \right)^{y_i} \left(1 - \frac{1}{1 + e^{-y_i^*}} \right)^{1 - y_i}$$

Here's how likelihood functions work:

1. Plug candidate values for every coefficient into the likelihood function
2. The function outputs a value called a likelihood
3. The higher the likelihood, the more likely it is that the data could have been generated by a model with the coefficients you've inputted

Logit's Likelihood Function

$$L(\alpha, \beta_1, \ldots, \beta_k | X, Y) = \prod_{i=1}^{N} \left(\frac{1}{1 + e^{-y_i^*}} \right)^{y_i} \left(1 - \frac{1}{1 + e^{-y_i^*}} \right)^{1-y_i}$$

Here's how likelihood functions work:

1. Plug candidate values for every coefficient into the likelihood function
2. The function outputs a value called a likelihood
3. The higher the likelihood, the more likely it is that the data could have been generated by a model with the coefficients you've inputted

Our goal: find the coefficient values that MAXIMIZE the likelihood function.

Logit's Log-Likelihood Function

To solve this optimization problem, we will use a hill-climber algorithm, which is built into R.

Logit's Log-Likelihood Function

To solve this optimization problem, we will use a hill-climber algorithm, which is built into R.

We will get faster and more accurate results, however, if we maximize the natural logarithm of the likelihood function. The log has several really useful properties:

Logit's Log-Likelihood Function

To solve this optimization problem, we will use a hill-climber algorithm, which is built into R.

We will get faster and more accurate results, however, if we maximize the natural logarithm of the likelihood function. The log has several really useful properties:

- Taking the log of a function does not change the location of maximum and minimum points

Logit's Log-Likelihood Function

To solve this optimization problem, we will use a hill-climber algorithm, which is built into R.

We will get faster and more accurate results, however, if we maximize the natural logarithm of the likelihood function. The log has several really useful properties:

- Taking the log of a function does not change the location of maximum and minimum points
- Exponents inside a log become factors outside the log

Logit's Log-Likelihood Function

To solve this optimization problem, we will use a hill-climber algorithm, which is built into R.

We will get faster and more accurate results, however, if we maximize the natural logarithm of the likelihood function. The log has several really useful properties:

- Taking the log of a function does not change the location of maximum and minimum points
- Exponents inside a log become factors outside the log
- Multiplication inside a log becomes addition outside the log

Logit's Log-Likelihood Function

To solve this optimization problem, we will use a hill-climber algorithm, which is built into R.

We will get faster and more accurate results, however, if we maximize the natural logarithm of the likelihood function. The log has several really useful properties:

- Taking the log of a function does not change the location of maximum and minimum points
- Exponents inside a log become factors outside the log
- Multiplication inside a log becomes addition outside the log
- Division inside a log becomes subtraction outside the log

Logit's Log-Likelihood Function

To solve this optimization problem, we will use a hill-climber algorithm, which is built into R.

We will get faster and more accurate results, however, if we maximize the natural logarithm of the likelihood function. The log has several really useful properties:

- Taking the log of a function does not change the location of maximum and minimum points
- Exponents inside a log become factors outside the log
- Multiplication inside a log becomes addition outside the log
- Division inside a log becomes subtraction outside the log

These properties mean that the hill-climber will have a much easier time maximizing the **log-likelihood function**.

Logit's Log-Likelihood Function

Likelihood function:

$$L(\alpha, \beta_1, \ldots, \beta_k | X, Y) = \prod_{i=1}^{N} \left(\frac{1}{1 + e^{-y_i^*}} \right)^{y_i} \left(1 - \frac{1}{1 + e^{-y_i^*}} \right)^{1-y_i},$$

Logit's Log-Likelihood Function

Likelihood function:

$$L(\alpha, \beta_1, \ldots, \beta_k | X, Y) = \prod_{i=1}^{N} \left(\frac{1}{1 + e^{-y_i^*}} \right)^{y_i} \left(1 - \frac{1}{1 + e^{-y_i^*}} \right)^{1-y_i},$$

Log-likelihood function (denoted with ℓ instead of L):

$$\ell(\alpha, \beta_1, \ldots, \beta_k | X, Y) = \ln \left[\prod_{i=1}^{N} \left(\frac{1}{1 + e^{-y_i^*}} \right)^{y_i} \left(1 - \frac{1}{1 + e^{-y_i^*}} \right)^{1-y_i} \right]$$

Logit's Log-Likelihood Function

Likelihood function:

$$L(\alpha, \beta_1, \ldots, \beta_k | X, Y) = \prod_{i=1}^{N} \left(\frac{1}{1 + e^{-y_i^*}} \right)^{y_i} \left(1 - \frac{1}{1 + e^{-y_i^*}} \right)^{1-y_i},$$

Log-likelihood function (denoted with ℓ instead of L):

$$\ell(\alpha, \beta_1, \ldots, \beta_k | X, Y) = \ln \left[\prod_{i=1}^{N} \left(\frac{1}{1 + e^{-y_i^*}} \right)^{y_i} \left(1 - \frac{1}{1 + e^{-y_i^*}} \right)^{1-y_i} \right]$$

The log-likelihood function algebraically reduces to:

$$\ell(\alpha, \beta_1, \ldots, \beta_k | X, Y) = \sum_{i=1}^{N} -y_i^*(1 - y_i) - \ln(1 + e^{-y_i^*}).$$

(To see the algebra, see the appendix.)

Coding the log-likelihood as an R function

To estimate any model, all you need to do is program the model's log-likelihood function into R.

Coding the log-likelihood as an R function

To estimate any model, all you need to do is program the model's log-likelihood function into R.

We'll do that by writing a function whose inputs are possible values of every coefficient, and whose output is a single number representing the value of the log-likelihood.

Coding the log-likelihood as an R function

To estimate any model, all you need to do is program the model's log-likelihood function into R.

We'll do that by writing a function whose inputs are possible values of every coefficient, and whose output is a single number representing the value of the log-likelihood.

We will also use our **ANES data** inside this function.

Coding the log-likelihood as an R function

To estimate any model, all you need to do is program the model's log-likelihood function into R.

We'll do that by writing a function whose inputs are possible values of every coefficient, and whose output is a single number representing the value of the log-likelihood.

We will also use our **ANES data** inside this function.

Here's the code to program the logit log-likelihood into R. We'll go over this code line by line:

```
logit.ll <- function(parameters, formula, data, outcome){
      Xmat <- model.matrix(terms(formula), data=data)
      ystar <- Xmat %*% parameters
      ll <- -ystar*(1 - outcome) - log(1 + exp(-ystar))
      return(sum(ll))
}
```

Coding the log-likelihood as an R function

```
logit.ll <- function(parameters, formula, data, outcome){
      Xmat <- model.matrix(terms(formula), data=data)
      ystar <- Xmat %*% parameters
      ll <- -ystar*(1 - outcome) - log(1 + exp(-ystar))
      return(sum(ll))
}
```

The function has four arguments:

Coding the log-likelihood as an R function

```
logit.ll <- function(parameters, formula, data, outcome){
      Xmat <- model.matrix(terms(formula), data=data)
      ystar <- Xmat %*% parameters
      ll <- -ystar*(1 - outcome) - log(1 + exp(-ystar))
      return(sum(ll))
}
```

The function has four arguments:

- `parameters` – a vector of possible values for the intercept and each coefficient

Coding the log-likelihood as an R function

```
logit.ll <- function(parameters, formula, data, outcome){
      Xmat <- model.matrix(terms(formula), data=data)
      ystar <- Xmat %*% parameters
      ll <- -ystar*(1 - outcome) - log(1 + exp(-ystar))
      return(sum(ll))
}
```

The function has four arguments:

- `parameters` – a vector of possible values for the intercept and each coefficient
- `formula` – the linear model, using the same syntax we use with the `lm()` of `glm()` functions

Coding the log-likelihood as an R function

```
logit.ll <- function(parameters, formula, data, outcome){
      Xmat <- model.matrix(terms(formula), data=data)
      ystar <- Xmat %*% parameters
      ll <- -ystar*(1 - outcome) - log(1 + exp(-ystar))
      return(sum(ll))
}
```

The function has four arguments:

- `parameters` – a vector of possible values for the intercept and each coefficient
- `formula` – the linear model, using the same syntax we use with the `lm()` of `glm()` functions
- `data` – the data frame to analyze

Coding the log-likelihood as an R function

```
logit.ll <- function(parameters, formula, data, outcome){
      Xmat <- model.matrix(terms(formula), data=data)
      ystar <- Xmat %*% parameters
      ll <- -ystar*(1 - outcome) - log(1 + exp(-ystar))
      return(sum(ll))
}
```

The function has four arguments:

- `parameters` – a vector of possible values for the intercept and each coefficient
- `formula` – the linear model, using the same syntax we use with the `lm()` of `glm()` functions
- `data` – the data frame to analyze
- `outcome` – the dependent variable

Coding the log-likelihood as an R function

```
logit.ll <- function(parameters, formula, data, outcome){
      Xmat <- model.matrix(terms(formula), data=data)
      ystar <- Xmat %*% parameters
      ll <- -ystar*(1 - outcome) - log(1 + exp(-ystar))
      return(sum(ll))
}
```

The `model.matrix()` and `terms()` functions together restrict the data to only the variables we need. They also break categorical variables into binary ones, leaving out the reference category.

Coding the log-likelihood as an R function

```
logit.ll <- function(parameters, formula, data, outcome){
      Xmat <- model.matrix(terms(formula), data=data)
      ystar <- Xmat %*% parameters
      ll <- -ystar*(1 - outcome) - log(1 + exp(-ystar))
      return(sum(ll))
}
```

The `model.matrix()` and `terms()` functions together restrict the data to only the variables we need. They also break categorical variables into binary ones, leaving out the reference category.

The %*% syntax is **matrix multiplication**. We multiply the X matrix by the coefficients to get the linear model.

Coding the log-likelihood as an R function

```
logit.ll <- function(parameters, formula, data, outcome){
      Xmat <- model.matrix(terms(formula), data=data)
      ystar <- Xmat %*% parameters
      ll <- -ystar*(1 - outcome) - log(1 + exp(-ystar))
      return(sum(ll))
}
```

On these two lines of code, we type the logit log-likelihood directly into R. Take a look at how the code matches up with:

$$\ell(\alpha, \beta_1, \ldots, \beta_k | X, Y) = \sum_{i=1}^{N} -y_i^*(1 - y_i) - \ln(1 + e^{-y_i^*}).$$

Coding the log-likelihood as an R function

```
logit.ll <- function(parameters, formula, data, outcome){
      Xmat <- model.matrix(terms(formula), data=data)
      ystar <- Xmat %*% parameters
      ll <- -ystar*(1 - outcome) - log(1 + exp(-ystar))
      return(sum(ll))
}
```

On these two lines of code, we type the logit log-likelihood directly into R. Take a look at how the code matches up with:

$$\ell(\alpha, \beta_1, \dots, \beta_k | X, Y) = \sum_{i=1}^{N} -y_i^*(1 - y_i) - \ln(1 + e^{-y_i^*}).$$

The `sum()` function represents the summation.

Coding the log-likelihood as an R function

```
logit.ll <- function(parameters, formula, data, outcome){
      Xmat <- model.matrix(terms(formula), data=data)
      ystar <- Xmat %*% parameters
      ll <- -ystar*(1 - outcome) - log(1 + exp(-ystar))
      return(sum(ll))
}
```

On these two lines of code, we type the logit log-likelihood directly into R. Take a look at how the code matches up with:

$$\ell(\alpha, \beta_1, \ldots, \beta_k | X, Y) = \sum_{i=1}^{N} -y_i^*(1 - y_i) - \ln(1 + e^{-y_i^*}).$$

The `sum()` function represents the summation.

The `return()` function sets the value of the log-likelihood to be the output of the function.

Using optim()

Now that we've programmed the log-likelihood, we call a hill-climbing algorithm by calling the optim() function:

```
our.logit <- optim(par = c(3,rep(0, 12)),
                   fn = logit.ll,
                   formula = as.formula(~ age + marital + education +
                                          union + race + gender),
                   data = anes,
                   outcome = anes$vote,
                   method="BFGS",
                   hessian = TRUE,
                   control=list(fnscale=-1, maxit=5000))
```

We'll go over this code line by line as well.

Using optim()

```
our.logit <- optim(par = c(3,rep(0, 12)),
                   fn = logit.ll,
                   formula = as.formula(~ age + marital + education +
                                          union + race + gender),
                   data = anes,
                   outcome = anes$vote,
                   method="BFGS",
                   hessian = TRUE,
                   control=list(fnscale=-1, maxit=5000))
```

Hill-climbing algorithms have to start somewhere. The pars argument sets the starting values of the parameters to be estimated.

Using optim()

```
our.logit <- optim(par = c(3,rep(0, 12)),
                   fn = logit.ll,
                   formula = as.formula(~ age + marital + education +
                                          union + race + gender),
                   data = anes,
                   outcome = anes$vote,
                   method="BFGS",
                   hessian = TRUE,
                   control=list(fnscale=-1, maxit=5000))
```

Hill-climbing algorithms have to start somewhere. The pars argument sets the starting values of the parameters to be estimated.

Our logit model has an intercept and 12 coefficients. I set the starting value of the intercept to 3, and to 0 for the 12 coefficients.

Using `optim()`

```
our.logit <- optim(par = c(3,rep(0, 12)),
                   fn = logit.ll,
                   formula = as.formula(~ age + marital + education +
                                        union + race + gender),
                   data = anes,
                   outcome = anes$vote,
                   method="BFGS",
                   hessian = TRUE,
                   control=list(fnscale=-1, maxit=5000))
```

`fn` is the function we created to express the log-likelihood function.

Using optim()

```
our.logit <- optim(par = c(3,rep(0, 12)),
                   fn = logit.ll,
                   formula = as.formula(~ age + marital + education +
                                          union + race + gender),
                   data = anes,
                   outcome = anes$vote,
                   method="BFGS",
                   hessian = TRUE,
                   control=list(fnscale=-1, maxit=5000))
```

`fn` is the function we created to express the log-likelihood function.

`formula`, `data`, and `outcome` are the same as defined in the `logit.ll()` function.

Using `optim()`

```
our.logit <- optim(par = c(3,rep(0, 12)),
                   fn = logit.ll,
                   formula = as.formula(~ age + marital + education +
                                          union + race + gender),
                   data = anes,
                   outcome = anes$vote,
                   method="BFGS",
                   hessian = TRUE,
                   control=list(fnscale=-1, maxit=5000))
```

`method` allows us to choose between different hill-climbing algorithms. BFGS is the "Broyden-Fletcher-Goldfarb-Shanno" algorithm. (To see other options, type `?optim` and read under Details.)

Using optim()

```
our.logit <- optim(par = c(3,rep(0, 12)),
                   fn = logit.ll,
                   formula = as.formula(~ age + marital + education +
                                          union + race + gender),
                   data = anes,
                   outcome = anes$vote,
                   method="BFGS",
                   hessian = TRUE,
                   control=list(fnscale=-1, maxit=5000))
```

`method` allows us to choose between different hill-climbing algorithms. BFGS is the "Broyden-Fletcher-Goldfarb-Shanno" algorithm. (To see other options, type `?optim` and read under Details.)

`hessian=TRUE` estimates variances and covariances for our estimates, and allows us to extract standard errors.

Using `optim()`

```
our.logit <- optim(par = c(3,rep(0, 12)),
                   fn = logit.ll,
                   formula = as.formula(~ age + marital + education +
                                          union + race + gender),
                   data = anes,
                   outcome = anes$vote,
                   method="BFGS",
                   hessian = TRUE,
                   control=list(fnscale=-1, maxit=5000))
```

By default, `optim()` **minimizes** functions instead of maximizing. Setting `fnscale=-1` tells `optim()` to maximize the log-likelihood function.

Using optim()

```
our.logit <- optim(par = c(3,rep(0, 12)),
                   fn = logit.ll,
                   formula = as.formula(~ age + marital + education +
                                        union + race + gender),
                   data = anes,
                   outcome = anes$vote,
                   method="BFGS",
                   hessian = TRUE,
                   control=list(fnscale=-1, maxit=5000))
```

By default, `optim()` **minimizes** functions instead of maximizing. Setting `fnscale=-1` tells `optim()` to maximize the log-likelihood function.

`maxit=5000` sets a maximum of 5000 iterations to arrive at the maximum of the log-likelihood function. Most of the time, however, we only need a couple dozen iterations.

Using optim()

```
our.logit <- optim(par = c(3,rep(0, 12)),
                   fn = logit.ll,
                   formula = as.formula(~ age + marital + education +
                                            union + race + gender),
                   data = anes,
                   outcome = anes$vote,
                   method="BFGS",
                   hessian = TRUE,
                   control=list(fnscale=-1, maxit=5000))
```

Try running the code!

- `our.logit$par` shows our coefficients.
- `sqrt(diag(solve(-our.logit$hessian)))` provides the standard errors.

How do these compare to the results we get by using the canned `glm()` function?

You are a Craftsman, not just a User

To recap: we rebuilt logit from a theoretically chosen family, linear model, and link function. You can change these if you want!

You are a Craftsman, not just a User

To recap: we rebuilt logit from a theoretically chosen family, linear model, and link function. You can change these if you want!

We used these elements to construct logit's **log-likelihood function**. And we passed the log-likelihood to a hill-climber in R to estimate the model.

You are a Craftsman, not just a User

To recap: we rebuilt logit from a theoretically chosen family, linear model, and link function. You can change these if you want!

We used these elements to construct logit's **log-likelihood function**. And we passed the log-likelihood to a hill-climber in R to estimate the model.

These steps are important for getting a complete understanding of an important method like logit. But more than that:

You are a Craftsman, not just a User

To recap: we rebuilt logit from a theoretically chosen family, linear model, and link function. You can change these if you want!

We used these elements to construct logit's **log-likelihood function**. And we passed the log-likelihood to a hill-climber in R to estimate the model.

These steps are important for getting a complete understanding of an important method like logit. But more than that:

You have the tools to build and estimate your own original probability models.

As such, you should think of yourselves as people who can craft methods tailored to a problem at hand.

Thank you!

Interpreting results

For logit models, the interpretation of β you learned in regression class no longer applies. My personal rule is:

If all you can do is interpret the sign and significance of an estimate, you've done it wrong.

Interpreting results

For logit models, the interpretation of β you learned in regression class no longer applies. My personal rule is:

If all you can do is interpret the sign and significance of an estimate, you've done it wrong.

We need to be able to interpret the magnitude of an effect in order to make results substantively meaningful.

Interpreting results

For logit models, the interpretation of β you learned in regression class no longer applies. My personal rule is:

If all you can do is interpret the sign and significance of an estimate, you've done it wrong.

We need to be able to interpret the magnitude of an effect in order to make results substantively meaningful.

There are three ways to interpret binary regression results:

Interpreting results

For logit models, the interpretation of β you learned in regression class no longer applies. My personal rule is:

If all you can do is interpret the sign and significance of an estimate, you've done it wrong.

We need to be able to interpret the magnitude of an effect in order to make results substantively meaningful.

There are three ways to interpret binary regression results:

1. Odds ratios (going out of style fast)

Interpreting results

For logit models, the interpretation of β you learned in regression class no longer applies. My personal rule is:

If all you can do is interpret the sign and significance of an estimate, you've done it wrong.

We need to be able to interpret the magnitude of an effect in order to make results substantively meaningful.

There are three ways to interpret binary regression results:

1. Odds ratios (going out of style fast)
2. Predicted probabilities (and differences in probability)

Interpreting results

For logit models, the interpretation of β you learned in regression class no longer applies. My personal rule is:

If all you can do is interpret the sign and significance of an estimate, you've done it wrong.

We need to be able to interpret the magnitude of an effect in order to make results substantively meaningful.

There are three ways to interpret binary regression results:

1. Odds ratios (going out of style fast)
2. Predicted probabilities (and differences in probability)
3. Marginal changes in probability (the first derivative of probability)

Odds

Odds: The probability **for** an event divided by the probability **against** the event.

Example: probability of $y = 1$ is .6. Odds are

Odds

Odds: The probability **for** an event divided by the probability **against** the event.

Example: probability of $y = 1$ is .6. Odds are

$$\frac{.6}{.4} = \frac{6}{4} = \frac{3}{2} = \textbf{``3 to 2 for''}$$

Odds

Odds: The probability **for** an event divided by the probability **against** the event.

Example: probability of $y = 1$ is .6. Odds are

$$\frac{.6}{.4} = \frac{6}{4} = \frac{3}{2} = \textbf{“3 to 2 for”}$$

If we flip the fraction, it's called **“2 to 3 against”**.

Odds

<u>Odds</u>: The probability **for** an event divided by the probability **against** the event.

Example: probability of $y = 1$ is .6. Odds are

$$\frac{.6}{.4} = \frac{6}{4} = \frac{3}{2} = \textbf{“3 to 2 for”}$$

If we flip the fraction, it's called **“2 to 3 against”**.

Usually, when a gambler talks about odds, it's the odds against. Odds are useful in gambling because they are interpreted as payouts. **The second number is the bet required to get the first number in profit.**

Odds

Odds: The probability **for** an event divided by the probability **against** the event.

Example: probability of $y = 1$ is .6. Odds are

$$\frac{.6}{.4} = \frac{6}{4} = \frac{3}{2} = \textbf{“3 to 2 for”}$$

If we flip the fraction, it's called **“2 to 3 against”**.

Usually, when a gambler talks about odds, it's the odds against. Odds are useful in gambling because they are interpreted as payouts. **The second number is the bet required to get the first number in profit.**

So in the above example, if I bet \$3 on an event and it happens, I get \$5: my \$3 returned and \$2 in profit.

Odds ratios from a logit model

The sizes of the coefficients for probit and c-log-log mean nothing.
But logit coefficients can be interpreted as "**log-odds**":

Odds ratios from a logit model

The sizes of the coefficients for probit and c-log-log mean nothing.
But logit coefficients can be interpreted as "**log-odds**":

$$p_i = \frac{1}{1 + e^{-y_i^*}},$$

Odds ratios from a logit model

The sizes of the coefficients for probit and c-log-log mean nothing.
But logit coefficients can be interpreted as "**log-odds**":

$$p_i = \frac{1}{1 + e^{-y_i^*}}, \qquad (1 + e^{-y_i^*})p_i = 1,$$

Odds ratios from a logit model

The sizes of the coefficients for probit and c-log-log mean nothing. But logit coefficients can be interpreted as "**log-odds**":

$$p_i = \frac{1}{1+e^{-y_i^*}}, \qquad (1+e^{-y_i^*})p_i = 1,$$

$$p_i + p_i e^{-y_i^*} = 1,$$

Odds ratios from a logit model

The sizes of the coefficients for probit and c-log-log mean nothing. But logit coefficients can be interpreted as "**log-odds**":

$$p_i = \frac{1}{1 + e^{-y_i^*}}, \qquad (1 + e^{-y_i^*})p_i = 1,$$

$$p_i + p_i e^{-y_i^*} = 1, \qquad p_i e^{-y_i^*} = 1 - p_i,$$

Odds ratios from a logit model

The sizes of the coefficients for probit and c-log-log mean nothing. But logit coefficients can be interpreted as "**log-odds**":

$$p_i = \frac{1}{1+e^{-y_i^*}}, \qquad (1+e^{-y_i^*})p_i = 1,$$

$$p_i + p_i e^{-y_i^*} = 1, \qquad p_i e^{-y_i^*} = 1 - p_i,$$

$$e^{-y_i^*} = \frac{1-p_i}{p_i},$$

Odds ratios from a logit model

The sizes of the coefficients for probit and c-log-log mean nothing.
But logit coefficients can be interpreted as "**log-odds**":

$$p_i = \frac{1}{1 + e^{-y_i^*}}, \qquad (1 + e^{-y_i^*})p_i = 1,$$

$$p_i + p_i e^{-y_i^*} = 1, \qquad p_i e^{-y_i^*} = 1 - p_i,$$

$$e^{-y_i^*} = \frac{1 - p_i}{p_i}, \qquad e^{y_i^*} = \frac{p_i}{1 - p_i},$$

Odds ratios from a logit model

The sizes of the coefficients for probit and c-log-log mean nothing.
But logit coefficients can be interpreted as "**log-odds**":

$$p_i = \frac{1}{1 + e^{-y_i^*}}, \qquad (1 + e^{-y_i^*})p_i = 1,$$

$$p_i + p_i e^{-y_i^*} = 1, \qquad p_i e^{-y_i^*} = 1 - p_i,$$

$$e^{-y_i^*} = \frac{1 - p_i}{p_i}, \qquad e^{y_i^*} = \frac{p_i}{1 - p_i},$$

$$y_i^* = \ln\left(\frac{p_i}{1 - p_i}\right),$$

Odds ratios from a logit model

The sizes of the coefficients for probit and c-log-log mean nothing. But logit coefficients can be interpreted as "**log-odds**":

$$p_i = \frac{1}{1+e^{-y_i^*}}, \qquad (1+e^{-y_i^*})p_i = 1,$$

$$p_i + p_i e^{-y_i^*} = 1, \qquad p_i e^{-y_i^*} = 1 - p_i,$$

$$e^{-y_i^*} = \frac{1-p_i}{p_i}, \qquad e^{y_i^*} = \frac{p_i}{1-p_i},$$

$$y_i^* = \ln\left(\frac{p_i}{1-p_i}\right),$$

$$\alpha + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_k x_{ik} = \ln\left(\frac{p_i}{1-p_i}\right)$$

Odds ratios from a logit model

$$\alpha + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_k x_{ik} = \ln\left(\frac{p_i}{1-p_i}\right)$$

Since we take the log, $\ln\left(\frac{p_i}{1-p_i}\right)$ called the "log-odds".

Odds ratios from a logit model

$$\alpha + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_k x_{ik} = \ln\left(\frac{p_i}{1-p_i}\right)$$

Since we take the log, $\ln\left(\frac{p_i}{1-p_i}\right)$ called the "log-odds".

Direct interpretation of a logit coefficient: one-unit increase in x_{ik} is associated with a β_k change in the log-odds, on average, after controlling for the other x variables.

Odds ratios from a logit model

$$\alpha + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_k x_{ik} = \ln\left(\frac{p_i}{1 - p_i}\right)$$

Since we take the log, $\ln\left(\frac{p_i}{1-p_i}\right)$ called the "log-odds".

Direct interpretation of a logit coefficient: one-unit increase in x_{ik} is associated with a β_k change in the log-odds, on average, after controlling for the other x variables.

But this statement doesn't really tell us anything about the substance of an effect.

Odds ratios from a logit model

$$\alpha + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_k x_{ik} = \ln\left(\frac{p_i}{1-p_i}\right)$$

Since we take the log, $\ln\left(\frac{p_i}{1-p_i}\right)$ called the "log-odds".

Direct interpretation of a logit coefficient: one-unit increase in x_{ik} is associated with a β_k change in the log-odds, on average, after controlling for the other x variables.

But this statement doesn't really tell us anything about the substance of an effect.

To express the odds, exponentiate both sides:

$$e^{\alpha+\beta_1 x_{i1}+\beta_2 x_{i2}+\ldots+\beta_k x_{ik}} = \frac{p_i}{1-p_i}.$$

Odds ratios from a logit model

Odds Ratio: How do the odds change (**multiplicatively**) for a one-unit increase in x?

Odds ratios from a logit model

Odds Ratio: How do the odds change (**multiplicatively**) for a one-unit increase in x?

For logit, the odds ratio is

$$\frac{\text{Odds with } (x_{i1}+1)}{\text{Odds with } (x_{i1})} = \frac{e^{\alpha+\beta_1(x_{i1}+1)+\beta_2 x_{i2}+\ldots+\beta_k x_{ik}}}{e^{\alpha+\beta_1 x_{i1}+\beta_2 x_{i2}+\ldots+\beta_k x_{ik}}}$$

Odds ratios from a logit model

Odds Ratio: How do the odds change (**multiplicatively**) for a one-unit increase in x?

For logit, the odds ratio is

$$\frac{\text{Odds with } (x_{i1}+1)}{\text{Odds with } (x_{i1})} = \frac{e^{\alpha+\beta_1(x_{i1}+1)+\beta_2 x_{i2}+\ldots+\beta_k x_{ik}}}{e^{\alpha+\beta_1 x_{i1}+\beta_2 x_{i2}+\ldots+\beta_k x_{ik}}}$$

$$= \frac{e^{\alpha+\beta_1 x_{i1}+\beta_1+\beta_2 x_{i2}+\ldots+\beta_k x_{ik}}}{e^{\alpha+\beta_1 x_{i1}+\beta_2 x_{i2}+\ldots+\beta_k x_{ik}}}$$

Odds ratios from a logit model

<u>Odds Ratio</u>: How do the odds change (**multiplicatively**) for a one-unit increase in x?

For logit, the odds ratio is

$$\frac{\text{Odds with } (x_{i1}+1)}{\text{Odds with } (x_{i1})} = \frac{e^{\alpha+\beta_1(x_{i1}+1)+\beta_2 x_{i2}+\ldots+\beta_k x_{ik}}}{e^{\alpha+\beta_1 x_{i1}+\beta_2 x_{i2}+\ldots+\beta_k x_{ik}}}$$

$$= \frac{e^{\alpha+\beta_1 x_{i1}+\beta_1+\beta_2 x_{i2}+\ldots+\beta_k x_{ik}}}{e^{\alpha+\beta_1 x_{i1}+\beta_2 x_{i2}+\ldots+\beta_k x_{ik}}}$$

$$= \frac{e^{\beta_1}\, e^{\alpha+\beta_1 x_{i1}+\beta_2 x_{i2}+\ldots+\beta_k x_{ik}}}{e^{\alpha+\beta_1 x_{i1}+\beta_2 x_{i2}+\ldots+\beta_k x_{ik}}}$$

Odds ratios from a logit model

Odds Ratio: How do the odds change (**multiplicatively**) for a one-unit increase in x?

For logit, the odds ratio is

$$\frac{\text{Odds with } (x_{i1}+1)}{\text{Odds with } (x_{i1})} = \frac{e^{\alpha+\beta_1(x_{i1}+1)+\beta_2 x_{i2}+\ldots+\beta_k x_{ik}}}{e^{\alpha+\beta_1 x_{i1}+\beta_2 x_{i2}+\ldots+\beta_k x_{ik}}}$$

$$= \frac{e^{\alpha+\beta_1 x_{i1}+\beta_1+\beta_2 x_{i2}+\ldots+\beta_k x_{ik}}}{e^{\alpha+\beta_1 x_{i1}+\beta_2 x_{i2}+\ldots+\beta_k x_{ik}}}$$

$$= \frac{e^{\beta_1}\, e^{\alpha+\beta_1 x_{i1}+\beta_2 x_{i2}+\ldots+\beta_k x_{ik}}}{e^{\alpha+\beta_1 x_{i1}+\beta_2 x_{i2}+\ldots+\beta_k x_{ik}}} = e^{\beta_1}.$$

Odds ratios from a logit model

Odds Ratio: How do the odds change (**multiplicatively**) for a one-unit increase in x?

For logit, the odds ratio is

$$\frac{\text{Odds with } (x_{i1}+1)}{\text{Odds with } (x_{i1})} = \frac{e^{\alpha+\beta_1(x_{i1}+1)+\beta_2 x_{i2}+\ldots+\beta_k x_{ik}}}{e^{\alpha+\beta_1 x_{i1}+\beta_2 x_{i2}+\ldots+\beta_k x_{ik}}}$$

$$= \frac{e^{\alpha+\beta_1 x_{i1}+\beta_1+\beta_2 x_{i2}+\ldots+\beta_k x_{ik}}}{e^{\alpha+\beta_1 x_{i1}+\beta_2 x_{i2}+\ldots+\beta_k x_{ik}}}$$

$$= \frac{e^{\beta_1} e^{\alpha+\beta_1 x_{i1}+\beta_2 x_{i2}+\ldots+\beta_k x_{ik}}}{e^{\alpha+\beta_1 x_{i1}+\beta_2 x_{i2}+\ldots+\beta_k x_{ik}}} = e^{\beta_1}.$$

Odds ratio interpretation of a logit coefficient: "a one-unit increase in x_{ik} is associated with **multiplying the odds** by e^{β_k}, on average, after controlling for the other x variables."

Odds ratios from a logit model

Example: logit coefficient is $\beta = 0.25$.

Odds ratios from a logit model

Example: logit coefficient is $\beta = 0.25$.

A one-unit increase in x_i is associated with multiplying the odds by $e^{.25} = 1.28$, on average, after controlling for the other x variables.

Odds ratios from a logit model

Example: logit coefficient is $\beta = 0.25$.

A one-unit increase in x_i is associated with multiplying the odds by $e^{.25} = 1.28$, on average, after controlling for the other x variables.

A one-unit increase in x_i is associated with the outcome becoming 28% "more likely", on average, after controlling for the other x variables.

Odds ratios from a logit model

Example: logit coefficient is $\beta = 0.25$.

A one-unit increase in x_i is associated with multiplying the odds by $e^{.25} = 1.28$, on average, after controlling for the other x variables.

A one-unit increase in x_i is associated with the outcome becoming 28% "more likely", on average, after controlling for the other x variables.

Example: logit coefficient is $\beta = -0.5$.

Odds ratios from a logit model

Example: logit coefficient is $\beta = 0.25$.

A one-unit increase in x_i is associated with multiplying the odds by $e^{.25} = 1.28$, on average, after controlling for the other x variables.

A one-unit increase in x_i is associated with the outcome becoming 28% "more likely", on average, after controlling for the other x variables.

Example: logit coefficient is $\beta = -0.5$.

A one-unit increase in x_i is associated with multiplying the odds by $e^{-.5} = .61$, on average, after controlling for the other x variables.

Odds ratios from a logit model

Example: logit coefficient is $\beta = 0.25$.

A one-unit increase in x_i is associated with multiplying the odds by $e^{.25} = 1.28$, on average, after controlling for the other x variables.

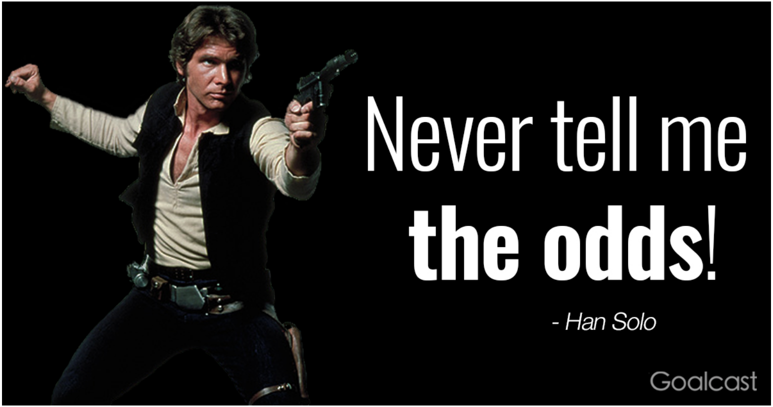
A one-unit increase in x_i is associated with the outcome becoming 28% "more likely", on average, after controlling for the other x variables.

Example: logit coefficient is $\beta = -0.5$.

A one-unit increase in x_i is associated with multiplying the odds by $e^{-.5} = .61$, on average, after controlling for the other x variables.

A one-unit increase in x_i is associated with the outcome becoming 39% "less likely", on average, after controlling for the other x variables.

Why odds ratios are going out of style



Why odds ratios are going out of style

The odds are odd!

Why odds ratios are going out of style

The odds are odd!

Odds are used in things like horse racing, but social scientists NEVER talk about probability in this weird way.

Why odds ratios are going out of style

The odds are odd!

Odds are used in things like horse racing, but social scientists NEVER talk about probability in this weird way.

Which makes more sense?

Why odds ratios are going out of style

The odds are odd!

Odds are used in things like horse racing, but social scientists NEVER talk about probability in this weird way.

Which makes more sense?

.25 probability

Why odds ratios are going out of style

The odds are odd!

Odds are used in things like horse racing, but social scientists NEVER talk about probability in this weird way.

Which makes more sense?

.25 probability or

Why odds ratios are going out of style

The odds are odd!

Odds are used in things like horse racing, but social scientists NEVER talk about probability in this weird way.

Which makes more sense?

.25 probability or $\frac{p}{1-p} = \frac{.25}{.75} =$ **3 to 1 odds against**

Why odds ratios are going out of style

The odds are odd!

Odds are used in things like horse racing, but social scientists NEVER talk about probability in this weird way.

Which makes more sense?

.25 probability or $\frac{p}{1-p} = \frac{.25}{.75} =$ **3 to 1 odds against**

.8 probability

Why odds ratios are going out of style

The odds are odd!

Odds are used in things like horse racing, but social scientists NEVER talk about probability in this weird way.

Which makes more sense?

.25 probability or $\frac{p}{1-p} = \frac{.25}{.75} =$ **3 to 1 odds against**

.8 probability or

Why odds ratios are going out of style

The odds are odd!

Odds are used in things like horse racing, but social scientists NEVER talk about probability in this weird way.

Which makes more sense?

.25 probability or $\frac{p}{1-p} = \frac{.25}{.75} =$ **3 to 1 odds against**

.8 probability or $\frac{p}{1-p} = \frac{.8}{.2} =$ **4 to 1 odds for**

Why odds ratios are going out of style

The odds can be misleading!

Why odds ratios are going out of style

The odds can be misleading!

Example: Suppose when $X = c$, $p = .4$, and the odds are

$$\frac{.4}{.6} = .667.$$

Why odds ratios are going out of style

The odds can be misleading!

Example: Suppose when $X = c$, $p = .4$, and the odds are

$$\frac{.4}{.6} = .667.$$

Also suppose when $X = c + 1$, $p = .95$, then the odds are

$$\frac{.95}{.05} = 19.$$

Why odds ratios are going out of style

The odds can be misleading!

Example: Suppose when $X = c$, $p = .4$, and the odds are

$$\frac{.4}{.6} = .667.$$

Also suppose when $X = c + 1$, $p = .95$, then the odds are

$$\frac{.95}{.05} = 19.$$

Then the odds ratio (multiplicative change in the odds) is

$$\frac{19}{.667} = 28.5.$$

Why odds ratios are going out of style

The odds can be misleading!

Example: Suppose when $X = c$, $p = .4$, and the odds are

$$\frac{.4}{.6} = .667.$$

Also suppose when $X = c + 1$, $p = .95$, then the odds are

$$\frac{.95}{.05} = 19.$$

Then the odds ratio (multiplicative change in the odds) is

$$\frac{19}{.667} = 28.5.$$

So a one-unit increase in X is associated with a 2750% increase in the odds that $y = 1$. That's not wrong, *but it is a misleading way* to communicate this finding. It's much better to report that we go from a .4 probability to a .95 probability.

Why odds ratios are going out of style

Odds are easy to misunderstand and incorrectly interpret!

Why odds ratios are going out of style

Odds are easy to misunderstand and incorrectly interpret!

Odds ratios are the multiplicative change in the odds for an event.

Why odds ratios are going out of style

Odds are easy to misunderstand and incorrectly interpret!

Odds ratios are the multiplicative change in the odds for an event.

They are **NOT** multiplicative changes in probability.

Why odds ratios are going out of style

Odds are easy to misunderstand and incorrectly interpret!

Odds ratios are the multiplicative change in the odds for an event.

They are **NOT** multiplicative changes in probability.

$$\text{Mult. change in odds} = \frac{.95/.05}{.4/.6} = 28.5$$

Why odds ratios are going out of style

Odds are easy to misunderstand and incorrectly interpret!

Odds ratios are the multiplicative change in the odds for an event.

They are **NOT** multiplicative changes in probability.

$$\text{Mult. change in odds} = \frac{.95/.05}{.4/.6} = 28.5$$

$$\text{Mult. change in probability} = \frac{.95}{.4} = 2.38$$

Why odds ratios are going out of style

Odds are easy to misunderstand and incorrectly interpret!

Odds ratios are the multiplicative change in the odds for an event.

They are **NOT** multiplicative changes in probability.

$$\text{Mult. change in odds} = \frac{.95/.05}{.4/.6} = 28.5$$

$$\text{Mult. change in probability} = \frac{.95}{.4} = 2.38$$

When a researcher says "more likely" it's not clear if that refers to odds or probability. *Is the event 28.5 times more likely, or just 2.38 times more likely?*

Why odds ratios are going out of style

Odds are easy to misunderstand and incorrectly interpret!

Odds ratios are the multiplicative change in the odds for an event.

They are **NOT** multiplicative changes in probability.

$$\text{Mult. change in odds} = \frac{.95/.05}{.4/.6} = 28.5$$

$$\text{Mult. change in probability} = \frac{.95}{.4} = 2.38$$

When a researcher says "more likely" it's not clear if that refers to odds or probability. *Is the event 28.5 times more likely, or just 2.38 times more likely?*

Instead, let's compute predicted probability and marginal changes in probability.

Predicted Probability

For logit, predicted probability is

$$P(y_i = 1) = \pi_i = \frac{1}{1 + e^{-(\hat{\alpha} + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2} + \ldots + \hat{\beta}_k x_{ik})}}$$

where ^ indicates the ML estimate for the parameter.

Predicted Probability

For logit, predicted probability is

$$P(y_i = 1) = \pi_i = \frac{1}{1 + e^{-(\hat{\alpha}+\hat{\beta}_1 x_{i1}+\hat{\beta}_2 x_{i2}+\ldots+\hat{\beta}_k x_{ik})}}$$

where ˆ indicates the ML estimate for the parameter.

For probit, predicted probability is

$$P(y_i = 1) = \pi_i = \Phi(\hat{\alpha} + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2} + \ldots + \hat{\beta}_k x_{ik})$$

These probabilities are **different** for every observation, depending on the covariates.

Predicted Probability

For logit, predicted probability is

$$P(y_i = 1) = \pi_i = \frac{1}{1 + e^{-(\hat{\alpha} + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2} + \ldots + \hat{\beta}_k x_{ik})}}$$

where ˆ indicates the ML estimate for the parameter.

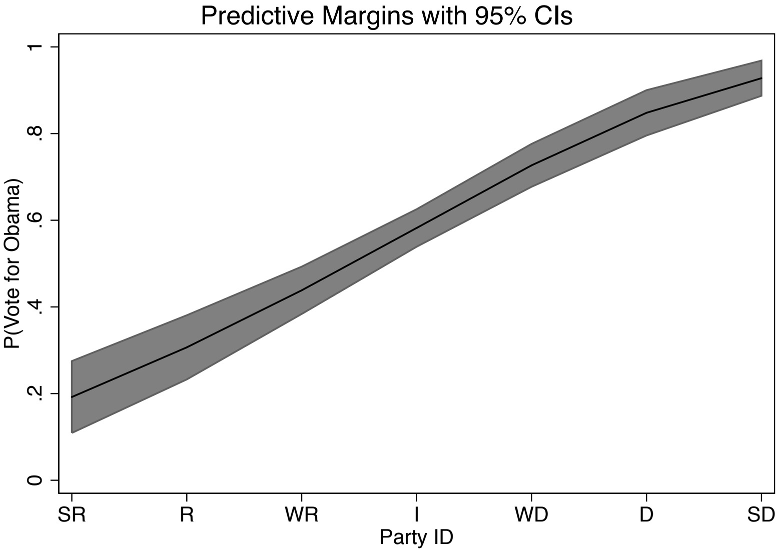
For probit, predicted probability is

$$P(y_i = 1) = \pi_i = \Phi(\hat{\alpha} + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2} + \ldots + \hat{\beta}_k x_{ik})$$

These probabilities are **different** for every observation, depending on the covariates.

Strategy: plot the predicted probabilities over the x of interest to show what's really going on with the model.

Predicted Probability



Predicted Probability

Plotting predicted probabilities is a wonderful illustration of what's really going on with the model results. But they don't directly **test hypotheses**.

Predicted Probability

Plotting predicted probabilities is a wonderful illustration of what's really going on with the model results. But they don't directly **test hypotheses**.

For a direct hypothesis test, try one of these approaches:

Predicted Probability

Plotting predicted probabilities is a wonderful illustration of what's really going on with the model results. But they don't directly **test hypotheses**.

For a direct hypothesis test, try one of these approaches:

1. Calculate the predicted probability at two meaningful and distinct values of the x of interest. Take the difference in the probabilities.

Predicted Probability

Plotting predicted probabilities is a wonderful illustration of what's really going on with the model results. But they don't directly **test hypotheses**.

For a direct hypothesis test, try one of these approaches:

1. Calculate the predicted probability at two meaningful and distinct values of the x of interest. Take the difference in the probabilities.

2. Compute the marginal change in probability, which is the first derivative of probability with respect to the x of interest.

Predicted Probability

Plotting predicted probabilities is a wonderful illustration of what's really going on with the model results. But they don't directly **test hypotheses**.

For a direct hypothesis test, try one of these approaches:

1. Calculate the predicted probability at two meaningful and distinct values of the x of interest. Take the difference in the probabilities.

2. Compute the marginal change in probability, which is the first derivative of probability with respect to the x of interest.

Then use the **delta method**, simulation, or another method to compute standard errors. Use the standard errors to compute p-values and confidence intervals to test whether the difference/derivative is different from 0.

Marginal Change in Probability

For logit, the marginal change in probability (or just **marginal effect**) of x is the derivative of probability:

$$\frac{\partial \pi_i}{\partial x_{ik}} = \frac{\partial}{\partial x_{ik}} \left(\frac{1}{1 + e^{-(\hat{\alpha} + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2} + \ldots + \hat{\beta}_k x_{ik})}} \right)$$

Marginal Change in Probability

For logit, the marginal change in probability (or just **marginal effect**) of x is the derivative of probability:

$$\frac{\partial \pi_i}{\partial x_{ik}} = \frac{\partial}{\partial x_{ik}} \left(\frac{1}{1 + e^{-(\hat{\alpha} + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2} + \ldots + \hat{\beta}_k x_{ik})}} \right)$$

$$= \frac{\hat{\beta}_k e^{-(\hat{\alpha} + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2} + \ldots + \hat{\beta}_k x_{ik})}}{(1 + e^{-(\hat{\alpha} + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2} + \ldots + \hat{\beta}_k x_{ik})})^2}.$$

Marginal Change in Probability

For logit, the marginal change in probability (or just **marginal effect**) of x is the derivative of probability:

$$\frac{\partial \pi_i}{\partial x_{ik}} = \frac{\partial}{\partial x_{ik}} \left(\frac{1}{1 + e^{-(\hat{\alpha} + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2} + \ldots + \hat{\beta}_k x_{ik})}} \right)$$

$$= \frac{\hat{\beta}_k e^{-(\hat{\alpha} + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2} + \ldots + \hat{\beta}_k x_{ik})}}{(1 + e^{-(\hat{\alpha} + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2} + \ldots + \hat{\beta}_k x_{ik})})^2}.$$

For probit, the marginal effect is

$$\frac{\partial}{\partial x_{ik}} P(y_i = 1) = \frac{\partial}{\partial x_{ik}} \Phi(\hat{\alpha} + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2} + \ldots + \hat{\beta}_k x_{ik})$$

$$= \hat{\beta}_k \phi(\hat{\alpha} + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2} + \ldots + \hat{\beta}_k x_{ik})$$

where $\phi()$ denotes the standard normal PDF instead of the CDF.

Logit's Log-Likelihood Function

Here's the algebraic simplification of the log-likelihood function:

$$\ell(\alpha, \beta_1, \ldots, \beta_k | X, Y) = \ln \left[\prod_{i=1}^{N} \left(\frac{1}{1 + e^{-y_i^*}} \right)^{y_i} \left(1 - \frac{1}{1 + e^{-y_i^*}} \right)^{1-y_i} \right]$$

Logit's Log-Likelihood Function

Here's the algebraic simplification of the log-likelihood function:

$$\ell(\alpha, \beta_1, \ldots, \beta_k | X, Y) = \ln \left[\prod_{i=1}^{N} \left(\frac{1}{1 + e^{-y_i^*}} \right)^{y_i} \left(1 - \frac{1}{1 + e^{-y_i^*}} \right)^{1-y_i} \right]$$

$$= \sum_{i=1}^{N} \ln \left[\left(\frac{1}{1 + e^{-y_i^*}} \right)^{y_i} \left(1 - \frac{1}{1 + e^{-y_i^*}} \right)^{1-y_i} \right]$$

Logit's Log-Likelihood Function

Here's the algebraic simplification of the log-likelihood function:

$$\ell(\alpha, \beta_1, \ldots, \beta_k | X, Y) = \ln\left[\prod_{i=1}^{N}\left(\frac{1}{1+e^{-y_i^*}}\right)^{y_i}\left(1-\frac{1}{1+e^{-y_i^*}}\right)^{1-y_i}\right]$$

$$= \sum_{i=1}^{N} \ln\left[\left(\frac{1}{1+e^{-y_i^*}}\right)^{y_i}\left(1-\frac{1}{1+e^{-y_i^*}}\right)^{1-y_i}\right]$$

$$= \sum_{i=1}^{N} \ln\left[\left(\frac{1}{1+e^{-y_i^*}}\right)^{y_i}\right] + \ln\left[\left(1-\frac{1}{1+e^{-y_i^*}}\right)^{1-y_i}\right]$$

Logit's Log-Likelihood Function

Here's the algebraic simplification of the log-likelihood function:

$$\ell(\alpha, \beta_1, \ldots, \beta_k | X, Y) = \ln\left[\prod_{i=1}^{N}\left(\frac{1}{1+e^{-y_i^*}}\right)^{y_i}\left(1-\frac{1}{1+e^{-y_i^*}}\right)^{1-y_i}\right]$$

$$= \sum_{i=1}^{N} \ln\left[\left(\frac{1}{1+e^{-y_i^*}}\right)^{y_i}\left(1-\frac{1}{1+e^{-y_i^*}}\right)^{1-y_i}\right]$$

$$= \sum_{i=1}^{N} \ln\left[\left(\frac{1}{1+e^{-y_i^*}}\right)^{y_i}\right] + \ln\left[\left(1-\frac{1}{1+e^{-y_i^*}}\right)^{1-y_i}\right]$$

$$= \sum_{i=1}^{N} y_i \ln\left[\frac{1}{1+e^{-y_i^*}}\right] + (1-y_i)\ln\left[\left(1-\frac{1}{1+e^{-y_i^*}}\right)\right]$$

Logit's Log-Likelihood Function

$$= \sum_{i=1}^{N} y_i \ln \left[\frac{1}{1+e^{-y_i^*}} \right] + (1-y_i) \ln \left[\left(\frac{e^{-y_i^*}}{1+e^{-y_i^*}} \right) \right]$$

Logit's Log-Likelihood Function

$$= \sum_{i=1}^{N} y_i \ln \left[\frac{1}{1+e^{-y_i^*}} \right] + (1-y_i) \ln \left[\left(\frac{e^{-y_i^*}}{1+e^{-y_i^*}} \right) \right]$$

$$= \sum_{i=1}^{N} -y_i \ln(1+e^{-y_i^*}) + (1-y_i) \ln(e^{-y_i^*}) - (1-y_i) \ln(1+e^{-y_i^*})$$

Logit's Log-Likelihood Function

$$= \sum_{i=1}^{N} y_i \ln\left[\frac{1}{1+e^{-y_i^*}}\right] + (1-y_i)\ln\left[\left(\frac{e^{-y_i^*}}{1+e^{-y_i^*}}\right)\right]$$

$$= \sum_{i=1}^{N} -y_i \ln(1+e^{-y_i^*}) + (1-y_i)\ln(e^{-y_i^*}) - (1-y_i)\ln(1+e^{-y_i^*})$$

$$= \sum_{i=1}^{N} -y_i \ln(1+e^{-y_i^*}) - y_i^*(1-y_i) - \ln(1+e^{-y_i^*}) + y_i \ln(1+e^{-y_i^*})$$

Logit's Log-Likelihood Function

$$= \sum_{i=1}^{N} y_i \ln\left[\frac{1}{1+e^{-y_i^*}}\right] + (1-y_i)\ln\left[\left(\frac{e^{-y_i^*}}{1+e^{-y_i^*}}\right)\right]$$

$$= \sum_{i=1}^{N} -y_i \ln(1+e^{-y_i^*}) + (1-y_i)\ln(e^{-y_i^*}) - (1-y_i)\ln(1+e^{-y_i^*})$$

$$= \sum_{i=1}^{N} -y_i \ln(1+e^{-y_i^*}) - y_i^*(1-y_i) - \ln(1+e^{-y_i^*}) + y_i \ln(1+e^{-y_i^*})$$

$$\ell(\alpha, \beta_1, \ldots, \beta_k | X, Y) = \sum_{i=1}^{N} -y_i^*(1-y_i) - \ln(1+e^{-y_i^*})$$