Numerical Linear Algebra II: Reordering Algorithms

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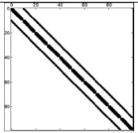
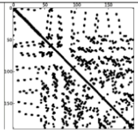
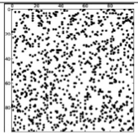
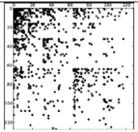
Sparse Matrices Recap - bytes for nonzero entry of double type

	Matrix pattern	Structured		<—>Unstructured		
		DIA	ELL	CSR	HYB	COO
Struct.mesh		8.10	12.16	12.45	12.16	16.00
Unstruct. mesh		328.22	16.60	12.62	13.44	16.00
Random matrix		153.65	21.29	12.42	14.20	16.00
Power-law graph		237.66	74.82	12.73	19.46	16.00

Minimum Degree

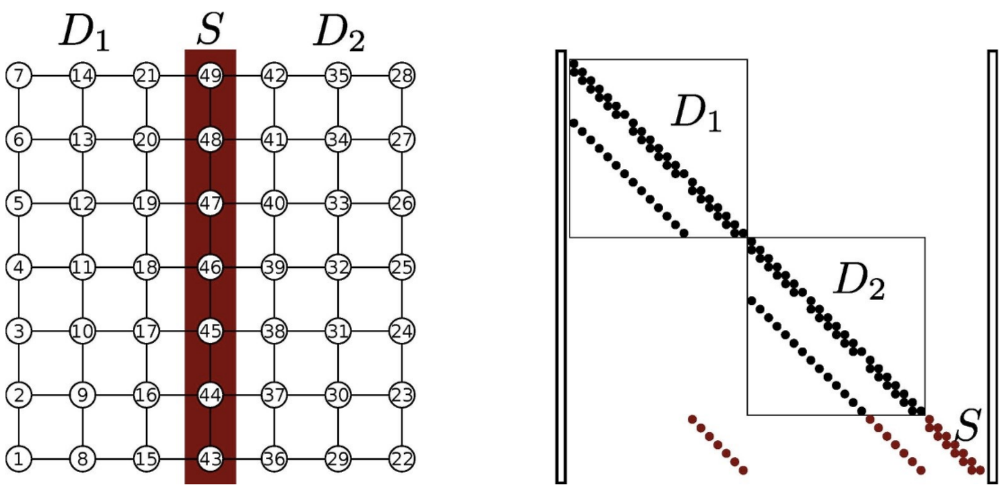
- Eliminate the node with the smallest degree first.
- Fewer neighbors $\Rightarrow$ less fill-in when eliminated.
- Update degrees after each elimination
  - Efficiently using Quotient Minimum Degree (QMD)
- Efficient variants exist like AMD (Approximate Minimum Degree).

Nested Dissection

- Treat the sparse matrix as a graph: nodes = variables, edges = nonzeros.
- Find a small separator that splits the graph into two roughly equal parts.
- Recursively order each part.
- Place separator nodes last.
- Eliminating separated subgraphs independently limits fill-in.
- Small separators mean limited coupling during factorization.
- Very effective for PDE/mesh-derived matrices.

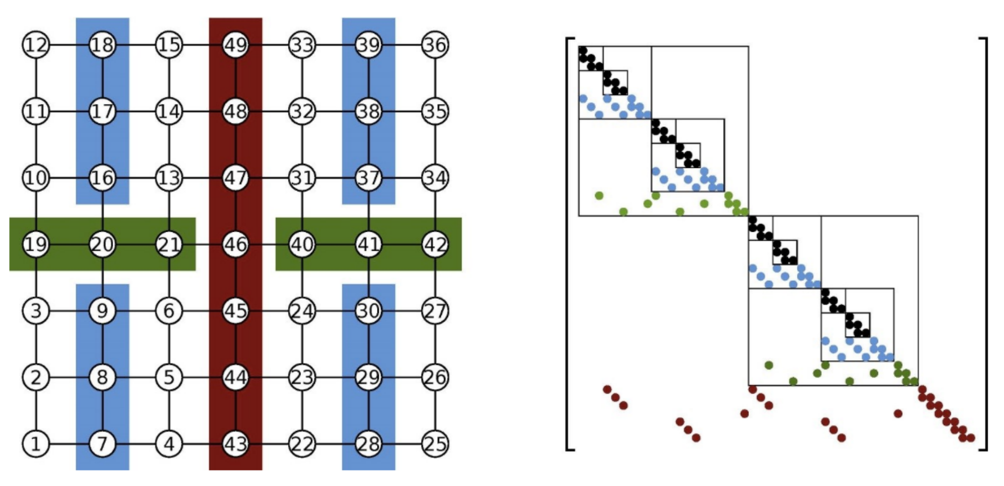
Nested Dissection

[X. S. Li]



Nested Dissection

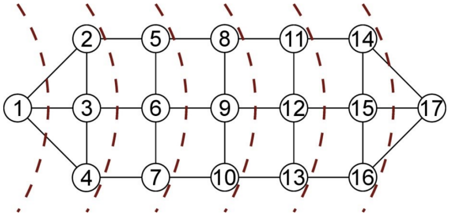
[X. S. Li]



(Reverse) Cuthill–McKee

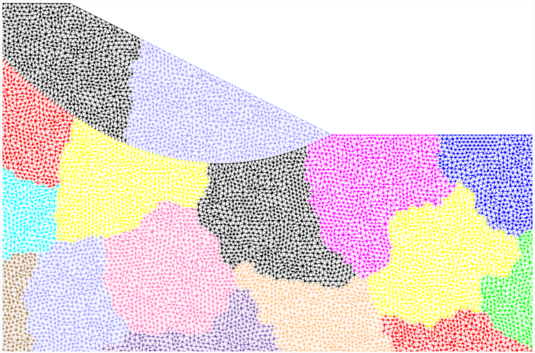
1. Pick a starting node, which has the minimum degree.
2. Perform BFS:
    - Visit nodes level by level.
    - Within each level, sort neighbors by increasing degree.
3. Number nodes in the order they are visited.

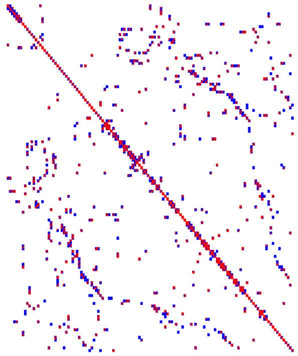
- Produces ordering that reduces matrix bandwidth.
- RCM: Reverse the CM ordering at the end (further bandwidth reduction).



[Duff, Erisman, Reid]

Example – Unstructured Mesh $n = 15,538$, $nnz = 214,380$ [Hrbac ’23]





Reord.	nnz in fact.	Mem. [MiB]	Reord. [s]	Fact. [s]	Solve [s]
NAT	$17,102,083$	198.47	0.00	$9.20\text{e}{+}01$	$5.68\text{e}{-}02$
QMD	$744,283$	11.27	$1.30\text{e}{-}02$	$2.01\text{e}{-}01$	$2.26\text{e}{-}03$
ND	$906,823$	13.13	$8.29\text{e}{-}03$	$2.57\text{e}{-}01$	$2.85\text{e}{-}03$
RCM	$2,156,423$	27.43	$2.09\text{e}{-}03$	$6.37\text{e}{-}01$	$6.63\text{e}{-}03$