Using Ulam's method to study the asymptotic behaviour of a dynamical system

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Abstract

The asymptotic behaviour of a dynamical system given by a transformation S can be studied by finding the fixed point of the Frobenius-Perron operator as the fixed point corresponds to the stationary density of the system. This project shall explore under what conditions this stationary density exists and when it can be found using Ulam's method in both one and higher dimensions as well as provide some computational examples. The Lyapunov exponent of the system will then be introduced and this will be used along with the stationary density in considering the 'goodness' of a particular Pseudo-random number generator.

1. Introduction

1.1. Why study Frobenius-Perron operators and absolutely continuous invariant measures?

Let $(X, \mathcal{F}, \mu)$ denote a probability space and $S: X \to X$ be a non-singular transformation such that μ is invariant and ergodic with respect to S (Non-singular means that $u(S^{-1}(A)) = 0$ whenever $u(A) = 0$). Birkhoff's Ergodic theorem implies that the limit

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f\left(S^i(x)\right) = \int_X f d\mu \tag{1.1}$$

exists almost surely, where $f: X \to \mathbb{R}$ is an integrable random variable which corresponds to a physical observable. In practice, how would one calculate this limit and thus the expectation of the observable? Starting at a particular point and iterating the system, whilst calculating the value of the observable at each point and then averaging would be the obvious solution. However, the computer's round-off errors can dominate the calculation, particularly for chaotic mappings sensitive to initial conditions, and make the expected value for the observable very different from theoretical expectations. A simple example of this is shown in [1] where the mapping $x_{n+1} = \frac{x_n}{2}$ has all points at zero when iterated on a computer after a finite number of iterations which should not be the case. Therefore, instead of studying the eventual behaviour of an individual trajectory, it could be beneficial to examine the flow of distributions in the phase space which is given by the Frobenius-Perron operator.

Additionally, if the iteration reaches a fixed point or a periodic point where the invariant measure, μ, is a Dirac measure on these points, the computation would tell us little about the dynamics of the system as a whole. In most practical situations, we are interested in absolutely continuous invariant measures which appear in neural networks, condensed matter physics and large-scale laser arrays to name but a few [2]. The Lebesgue measure corresponds to our intuitive notion of measure with length and volume and thus we want to try and find an absolutely continuous invariant measure with respect to this, which is known as the Sinai-Bowen-Ruelle (SBR) measure of the system.

1.2. The Frobenius-Perron operator and Ulam's method

The Frobenius-Perron operator describes the evolution of the probability density function of the dynamical system.

For a given function $f \in L^1$, define a measure

$$\mu_f(A) = \int_{S^{-1}(A)} f d\mu, \ \ \forall A \in \mathcal{F} \tag{1.2}$$

Since S is nonsingular. The Radon-Nikodym theorem then implies that there exists a unique function, Pf such that

$$\int_A Pf d\mu = \int_{S^{-1}(A)} f d\mu, \ \ \forall A \in \mathcal{F}, \forall f \in L^1(\mu). \tag{1.3}$$

The operator $P: L^1(\mu) \to L^1(\mu)$ is the Frobenius-Perron operator associated with S. Assume that μ_f is S-invariant, that is

$$\mu_f(A) = \mu_f(S^{-1}(A)), \forall A \in \mathcal{F} \tag{1.4}$$

Then from the definition of μ_f and P, we have that

$$\int_A Pf d\mu = \int_A f d\mu, \ \ \forall A \in \mathcal{F}. \tag{1.5}$$

Hence, $Pf = f$. μ_f is S-invariant and is also absolutely continuous with respect to $d\mu$ which we shall define as the Lebesgue measure. Therefore, we have shown that the existence of a Sinai-Bowen-Ruelle measure is equivalent to the existence of a fixed point of the Frobenius-Perron operator.

In many fields of science and engineering, we often need to study the rules under which density functions of physical quantities evolve with respect to time, and their asymptotic and stationary behaviour. Therefore, deriving the Frobenius-Perron operator is extremely useful in areas such as computational molecular dynamics and statistical physics. However, this operator is often impossible to solve exactly and computational methods are required. One way to approximate the Frobenius-Perron operator, P, computationally is by using Ulam's method. This method involves approximating the infinite dimensional Frobenius-Perron operator as a finite dimensional linear operator. For a chosen positive integer n, we divide the space, X, into n subintervals $I_i = [x_{i-1}, x_i]$ for $i = 1, 2, \dots, n$. If we denote

$$1_i = \frac{1}{m(I_i)} \chi_{I_i}, \forall i = 1,2, \dots, n \tag{1.6}$$

where m is the Lebesgue measure on X and each 1_i is a density function. We assume that $h \equiv \max_{i \leq i \leq n}(x_i - x_{i-1})$ is such that $h \to 0$ as $n \to \infty$. The program works by uniformly generating q random numbers in each subinterval, I_i, applying the function S to the random numbers and noting the new subinterval, I_j, of each number. A matrix is then constructed by

$$p_{ij} = \frac{m\left(I_i \cap S^{-1}(I_j)\right)}{m(I_i)} \tag{1.7}$$

From the definition of p_{ij}, one can see that each entry of the matrix at position ij, indicates the probability that a point in the ith subinterval is mapped into the jth subinterval under the mapping S. It is easy to see that p_{ij} is a stochastic matrix and approximates the evolution of the probability density function and thus the Frobenius-Perron operator.

Ulam's conjecture:
If the Frobenius-Perron operator has a stationary density, then there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that

$$\lim_{k \to \infty} f_{n_k} = f^*, \tag{1.8}$$

where f^* is the stationary density of P. Therefore, if there exists an absolutely continuous measure, then the conjecture suggests that the fixed point of the approximated Frobenius-Perron operator will be the stationary density of the dynamical system and will give us insight into the system's limiting behaviour. However, for this to be the case, we need to prove both the existence of an absolutely continuous invariant measure and the validity of Ulam's conjecture.

2. Understanding asymptotic behaviour using Ulam's method in one-dimension

2.1. Existence of Absolutely continuous invariant measures in one-dimension

The existence of absolutely continuous invariant measures in one-dimension has only been proved for certain classes of mappings such as piecewise C^2 expanding maps [3] and piecewise convex mappings with a weak repellor [4]. An outline of the theorem and proof of the existence of an absolutely continuous invariant measure for piecewise C^2 and stretching mappings is shown in section 2.1.2. The difficulty in a generalisation of this proof can be seen by the counterexample of a mapping given by

$$S(x) = \begin{cases} \dfrac{x}{1-x}, & 0 \leq x < \dfrac{1}{2} \\ 2x - 1, & \dfrac{1}{2} \leq x \leq 1 \end{cases} \tag{2.1}$$

For this mapping, the condition $\inf|S'| > 1$ is violated at $x = 0$ and Lasota and Yorke showed that for this mapping the action of the Frobenius-Perron operator converges in measure to zero. Therefore, the equation $P_S f = f$ only has the trivial solution and there is no absolutely continuous measure invariant under S.

2.1.1. Theorem for existence for piecewise C^2 expanding maps

Theorem of Lasota and Yorke:
Let $S: [0,1] \to [0,1]$ be a piecewise C^2 function such that $\inf|S'| > 1$. Then for an $f \in L^1$ that is of bounded variation the sequence

$$\frac{1}{n}\sum_{k=0}^{n-1} P_S^k f \tag{2.2}$$

is convergent in norm to a function $f^* \in L_1$ which has the following properties:

(1) $f \geq 0 \to f^* \geq 0$.
(2) $\int_0^1 f^* dm = \int_0^1 f dm$.
(3) $P_S f^* = f^*$ and therefore, the measure $d\mu^* = f^* dm$ is invariant under S.

2.1.2. Outline of Proof for existence for piecewise C^2 expanding maps

A mapping is piecewise C^2 if there exists a partition $0 = b_0 < b_1 < \cdots < b_n = 1$ of the unit interval such that for each integer $k (k = 1, \dots, n)$ the restriction S_k of S to the open interval (x_{k-1}, x_k) is a C^2-function which can be extended to the closed interval $[b_{k-1}, b_k]$ as a C^2-function.

The space X is given by the interval $X = [0,1]$ and supposing that μ is the Lebsesgue measure m. From equation (1.3), we have that

$$\int_0^x Pf dm = \int_{S^{-1}([0,x])} f dm \,. \tag{2.3}$$

Then, taking derivatives to both sides above with respect to x and using the fundamental theorem of calculus, the Frobenius-Perron operator associated with the interval mapping becomes

$$Pf(x) = \frac{d}{dx}\int_{S^{-1}([0,x])} f dm, \quad x \in [0,1] \;\; a.e. \tag{2.4}$$

Let $\tau = \inf|S'|$. As we are looking at expanding maps where $\inf|S'| > 1$, then we are able to choose a number N such that $\tau^N > 2$. Additionally, given that S is a piecewise C^2 transformation, the function given by $\phi = S^N$ is also piecewise C^2. Denote $b_0, \dots, b_q$ as the corresponding partition for the piecewise function ϕ and denote ϕ_i for the C^2 functions at each interval. As $\tau = \inf|S'|$, we have that $S' \geq \tau$ and thus

$$\left|\phi'_i(x)\right| \leq \tau^{-N}, x \in [b_{i-1}, b_i], i = 1, \dots, q \,. \tag{2.5}$$

Using the discrete version of the Frobenius-Perron operator in equation (2.4) we obtain an operator with our mapping ϕ as

$$P_{\phi} f(x) = \sum_{i=1}^{q} f\big(\psi_i(x)\big)\, |\phi'_i(x)| \chi_i(x) \tag{2.6}$$

where $\psi_i = {\phi_i}^{-1}$ and χ_i is the characteristic function of the interval $I_i = \phi_i([b_{i-1}, b_i])$.

Let $f: [0,1] \to \mathbb{R}$ be a function and let $[a, b]$ be any closed subinterval of $[0,1]$. Define a set as

$$S_f = \left\{ \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| : \{x_i : i \leq i \leq n\}\ is\ a\ partition\ of\ [a, b] \right\}. \tag{2.7}$$

We choose an appropriate f so that the supremum of this set, $V(f, [a, b]) = \sup S_f$ is finite and thus has bounded variation [5]. The symbol $\bigvee_a^b X$ denotes the variation of X in the interval $[a, b]$.

The function ϕ can be discontinuous at the boundaries of the intervals and the variation of $P_\phi f$ over $[0,1]$ must be equal to the sum of the variation for each interval plus the discontinuous jumps at the boundaries. Using this fact and equations (2.5) and (2.6), we get

$$\bigvee_0^1 P_\phi f \leq \sum_{i=1}^{q} \bigvee_{I_i} (f \circ \psi_i)|\phi'_i| + \tau^{-N} \sum_{i=1}^{q} (|f(b_{i-1})| + |f(b_i)|). \tag{2.8}$$

Looking at the first sum, we can write using the definition of variation,

$$\bigvee_{I_i} (f \circ \psi_i)|\phi'_i| = \int_{I_i} |d((f \circ \psi_i)|\phi'_i|)| \tag{2.9}$$

Using the fact that $|xy| \leq |x||y|$ we can then differentiate and obtain,

$$\bigvee_{I_i} (f \circ \psi_i)|\phi'_i| \leq \int_{I_i} |(f \circ \psi_i)||\phi''_i| dm \; + \int_{I_i} |\phi'_i| \, |d(f \circ \psi_i)|$$

$$\leq K \int_{I_i} |(f \circ \psi_i)| \, |\phi'_i| dm + \tau^{-N} \int_{I_i} |d(f \circ \psi_i)| \tag{2.10}$$

where $K = max|\phi''_i|/min|\phi'_i|$. By a change of variables, we then get

$$\bigvee_{I_i}(f \circ \psi_i)|\phi'_i| \leq K \int_{b_{i-1}}^{b_i} |f| dm + \tau^{-N} \int_{b_{i-1}}^{b_i} |df| \tag{2.11}$$

Lasota, and Yorke then used the inequality

$$|f(b_{i-1})| + |f(b_i)| \leq \bigvee_{b_{i-1}}^{b_i} f + 2d_i \tag{2.12}$$

where $d_i = \inf\{|f(x)| : x \in [b_{i-1}, b_i]\}$. This inequality holds in the case of a C^2 expanding transformation. There is also a trivial inequality given by

$$d_i \leq h^{-1} \int_{b_{i-1}}^{b_i} |f| dm\,. \tag{2.13}$$

The second sum in equation (2.8) can now be constrained by $\bigvee_0^1 f + 2h^{-1}\|f\|$ and using this and equation (2.11) we then get

$$\bigvee_0^1 P_\phi f \leq \alpha\|f\| + \beta \bigvee_0^1 f \tag{2.14}$$

where $\alpha = (K + 2h^{-1})$ and $\beta = 2\tau^{-N} > 2$ by our choice of N. Now, if we write $f_k = {P_S}^k f$ and using the fact that ${P_S}^{Nk} f = f_{Nk} = {P_\phi}^k f$ we get

$$\bigvee_0^1 f_{Nk} \leq \alpha\|f_{N(k-1)}\| + \beta \bigvee_0^1 f_{N(k-1)} \tag{2.15}$$

and consequently,

$$\lim_{k\to\infty} \sup \bigvee_0^1 f_{Nk} \leq \alpha(1-\beta)^{-1}\|f\|\,. \tag{2.16}$$

This inequality and the condition $\|f_k\| \leq \|f\|$ which follows from the fact that the Frobenius-Perron operator is positive and preserves integrals (the operator sometimes shrinks the variation of the function) means that the set $C = \{f_{Nk}\}_{k=0}^{\infty}$ is relatively compact in L^1. Since $\{f_k\}_{k=0}^{\infty} \subset \bigcup_{k=0}^{N-1} {P_S}^k C,$ the whole sequence $\{f_k\}_{k=0}^{\infty}$ is also relatively compact. By Mazur's theorem there is also a relatively compact sequence given by

$$\left\{\frac{1}{n}\sum_{k=0}^{n-1} {P_S}^k f\right\}. \tag{2.17}$$

Then the Kakutani-Yosida Theorem can be applied which states that for any $f \in L^1$ the sequence (2.17) converges strongly to a function f^* which is invariant under P_S with the f^* obeying properties (1) and (2) again due to the operator being positive and preserving integrals.

The proof can also be extended to functions that are not of bounded variation by using Helly's theorem. However, the overall proof does heavily use the conditions of the mapping being piecewise C^2 and expanding thus highlighting why a generalisation of the proof has not been established particular the impossibility of one which follows similar arguments.

2.2. Validity of Ulam's conjecture in one-dimension

The validity of Ulam's conjecture in one-dimension is still an open problem and has only been proved for some classes of transformations S such as for the class of piecewise C^2 and expanding mappings by Li [1], for piecewise convex mapping with a strong repellor by Miller, or for Anosov Maps [6]. Below, I have outlined Li's theorem and proof for the class of piecewise C^2 expanding mappings.

2.2.1. Theorem for Ulam's Conjecture for piecewise C^2 expanding maps

Theorem of Li:
Let $S: [0,1] \to [0,1]$ be a piecewise C^2-function with $M = \inf|S'| > 1$. Suppose P_S has a unique fixed point, f^*. Then, for any positive integer number of subintervals n, P_n has a fixed point f_n with $\|f_n\| = 1$ and $\{f_n\}$ converges to f^*. Additionally, even if f^* is not unique then there exists a relatively compact set C of fixed points of P_S such that $d(f_n, \bar{C}) \to 0$ in L^1. So, for large enough n, every f_n approximates an absolutely continuous invariant measure of S.

2.2.2. Outline of Proof for Ulam's conjecture for piecewise C^2 expanding maps

Let $\Delta_n^1 = \{\sum_{i=1}^n a_i\chi_i : \ a_i \geq 0 \ and \ \sum_{i=1}^n a_i = 1\}$ then P_n is invariant on the compact subset Δ_n^1.

Proof:
Let $f = \sum_{i=1}^n a_i\chi_i$. Then $f \in \Delta_n^1$ and

$$P_n f = P_n\left(\sum_{i=1}^n a_i\chi_i\right) = \sum_{i=1}^n a_i(P_n\chi_i) = \sum_{i=1}^n\left(\sum_{i=1}^n a_i\, P_{ij}\right)\chi_i \tag{2.18}$$

where P_{ij} is the approximated Frobenius-Perron operator found using Ulam's method. As this is a transition matrix and thus the sum of the rows is equal to one, we get

$$\sum_{i=1}^n\left(\sum_{i=1}^n a_i\, P_{ij}\right) = \sum_{i=1}^n a_i\left(\sum_{i=1}^n P_{ij}\right) = \sum_{i=1}^n a_i = 1. \tag{2.19}$$

Therefore, we have that $P_n f \in \Delta_n^1$.
Since $P_n(\Delta_n^1) \subset \Delta_n^1$ is a compact convex set there exists, by the Brouwer fixed point theorem, a point $g_n \in \Delta_n^1$ for which $P_n g_n = g_n$. Let $f_n = n g_n$, then $f_n \in \Delta_n$ and $\|f_n\| = 1$ for all n.
Now, we need to prove that $\{f_n\}$ converges. This was done by introducing an operator Q_n defined by

$$Q_n f = \sum_{i=1}^{n} c_i \chi_i \tag{2.20}$$

where

$$c_i = \frac{1}{m(I_i)} \int_{I_i} f(s) ds.$$

Lemma 2.1: For $f \in L^1$, the sequence $Q_n f$ converges in L^1 to f as $n \to \infty$.

Proof: Since $f \in L^1$, for any $\varepsilon > 0$ there exists a continuous function g such that $\|g - f\| < \frac{\varepsilon}{3}$. g must be uniformly continuous as it is continuous in [0,1]. We may choose N large enough such that for $n > N$, we have $|g(x_1) - g(x_2)| < \frac{\varepsilon}{3}$ for all x_1, x_2 in $I_i, i = 1, \dots, n$. It follows that,

$$\int_{I_i} |(Q_n g)(s) - g(s)| ds = \int_{I_i} \left| \left(\frac{1}{m(I_i)}\right) \int_{I_i} g(s') ds' - g(s) \right| ds$$

$$\leq \int_{I_i} \left(\frac{1}{m(I_i)}\right) \left(\int_{I_i} |g(s') - g(s)| ds' \right) ds$$

$$\leq m(I_i) \times \frac{\varepsilon}{2}. \tag{2.21}$$

Hence,

$$\|Q_n g - g\| = \int_0^1 |Q_n g - g| = \sum_{i=1}^{n} \int_{I_i} |Q_n g - g| < \frac{\varepsilon}{3}. \tag{2.22}$$

We also have that for $f \geq 0$,

$$\int_0^1 Q_n f = \int_0^1 \left(\frac{1}{m(I_i)} \int_{I_i} f(s) ds \right) \chi_i(s') ds'$$

$$= \sum_{i=1}^{n} \int_{I_i} f(s)ds = \int_0^1 f. \tag{2.23}$$

Therefore, $\|Q_n\| = 1$ and hence,

$$\|Q_n f - f\| \leq \|Q_n f - Q_n g\| + \|Q_n g - g\| + \|g - f\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \tag{2.24}$$

Thus, we have proved Lemma 2.1.

Lemma 2.2: For f in Δ_n we have $P_n f = Q_n P_S f$.

Proof: We only need to show that $P_n \chi_i = Q_n P_S \chi_i$ for $1 \leq i \leq n$. From the definition of the Frobenius-Perron operator in equation X we have

$$Q_n(P_S \chi_i) = \sum_{i=1}^{n} \left[\frac{1}{m(I_i)} \int_{I_j} \left(\frac{d}{dx} \int_{S^{-1}([0,x])} \chi_j(s) \right) dx \right] \chi_i$$

$$= \sum_{i=1}^{n} \left[\frac{1}{m(I_i)} \int_{S^{-1}(I_i)} \chi_j(s) ds \right] \chi_i$$

$$= \sum_{i=1}^{n} P_{ij} \chi_i = P_n \chi_i. \tag{2.25}$$

Li then uses this relation between Q_n, P_n and P_S and the same relatively compact argument as I outlined in the proof by Lasoka and Yorke, to show that the set $C = \{f_n\}_{n=0}^{\infty}$ is relatively compact which follows from the bounded variation of f_n and the fact that P_n shares the same property as P_S in that it can shrink the variation of the function. Then let $\{f_{n_i}\}$ be any convergent subsequence of C and let $f = \lim_{i \to \infty} f_{n_i}$. We have from the triangle inequality that

$$\begin{aligned} \|f - P_S f\| \leq & \left\|f - f_{n_i}\right\| + \left\|f_{n_i} - Q_{n_i} P_S f_{n_i}\right\| + \left\|Q_{n_i} P_S f_{n_i} - Q_{n_i} P_S f\right\| \\ & + \left\|Q_{n_i} P_S f - P_S f\right\|. \end{aligned} \tag{2.26}$$

Given that f_{n_i} is a fixed point of P_{n_i}, Lemma 2.1 implies that the right-hand side of the above inequality tends to zero when n_i tends to infinity and thus $f = P_S f$. Therefore, any convergent subsequence of C converges to a fixed point of P_S.

From both the existence of an SBR measure and Ulam's conjecture proofs, if S is a piecewise C^2 function such that $inf|S'| > 1$ then Ulam's method will lead to the approximation of an absolutely continuous invariant measure of the dynamical system. An example of the bent Baker's tent map is studied to show what can be done with the theory discussed as well as to look at the speed of convergence of Ulam's method and whether it is practically possible by computing on a finite number of subintervals.

2.3. Computational example for piecewise C^2 expanding maps in one-dimension

The stationary density was calculated using Ulam's method for the bent Baker's map which is defined by,

$$S(x) = \frac{4\sqrt{6}}{3}x^3 - 2\sqrt{6}x^2 + \left(2 + \frac{2\sqrt{6}}{3}\right)x \ (mod1). \qquad (2.27)$$

$inf|S'| = 2 - \frac{\sqrt{6}}{3} \approx 1.18$ meaning an approximation of an absolutely continuous invariant measure can be found for the transformation using Ulam's method. The Baker's map can be understood as a bilateral shift operator of a bi-infinite two-state lattice and has been used in the study of species co-existence [7]. Therefore, it is important to understand the asymptotic behaviour of the mapping and thus to find an absolutely continuous invariant measure. The code to do this is shown in the Appendix. 10,000 random numbers were generated in each subinterval and the subinterval of the iterate found. An approximated Frobenius-Perron operator was then constructed from this. To find the fixed point of this operator the first left eigenvector was found using the discreteMarkovChain package in Python and this was then plotted in the interval [0,1]. Additionally, the number of subintervals was changed in order to see the convergence of the approximated stationary density.

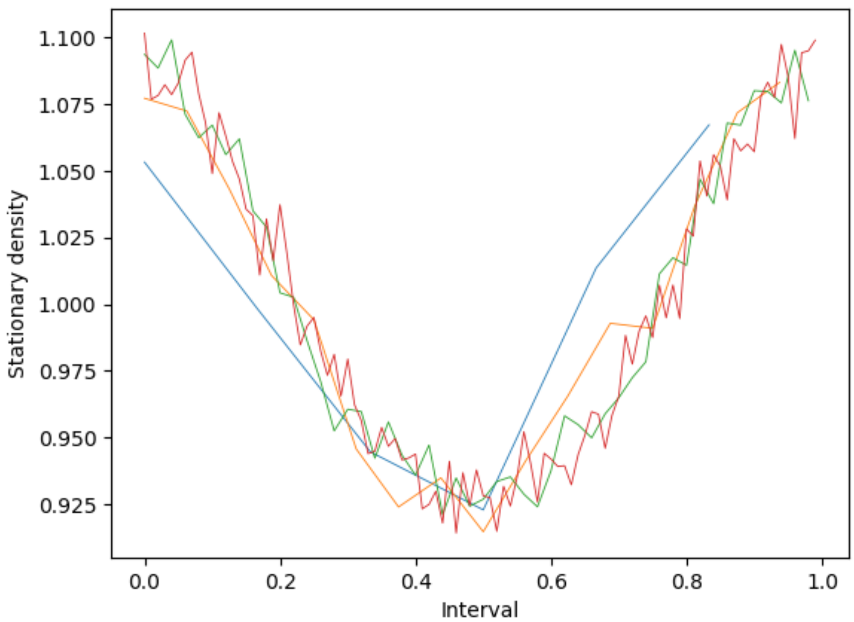

Figure 1: Graph of the stationary density of the bent Baker's map over the interval [0,1] found using Ulam's method. The number of subintervals is given by: Blue = 6, Orange = 16, Green = 50, Red = 100.

Figure 1 shows the approximation of the SBR measure. As can be seen the stationary density seems to converge after just 16 iterations showing that a large number of subintervals is not necessarily required for the approximation. Moreover, the density found is equivalent to that found by Froyland in [8] thus confirming the validity of the program.

3. Understanding asymptotic behaviour using Ulam's method in higher dimensions

3.1. Existence of absolutely continuous invariant measures in higher dimensions

Though it is natural to expect the existence of absolutely continuous invariant measures to be simply extended to higher dimensions, this is not the case. The main difficulty in higher dimensions is the fact that the partition of the domain into the regions where an iteration of the map is smooth can be very complicated. Keller treated piecewise C^2 expanding maps in his thesis [9] and gave some criterion for the existence of absolutely continuous invariant measures. Tsujii's proved the existence of absolutely continuous invariant measures for piecewise C^2 real-analytic expanding maps on the plane [10]. The real-analytic property somewhat relaxes the difficulty in suitably partitioning the domain. The most general result in this area was given by Gora and Boyarski, where they gave a lower bound for the expansion rate that insures the existence of an absolutely continuous invariant measure and is valid in arbitrary dimension [11]. Their approach is based on the modern notion of functions of bounded variation in several variables using Schwartz's distribution theory.

3.2. Validity of Ulam's conjecture in higher dimensions

Additionally, Ulam's method can be extended to multi-dimensions [12]. Let $\Omega \in \mathbb{R}^N$ be a bounded open region, m be the Lebesgue measure on $\mathbb{R}^N$ and $S\colon \Omega \to \Omega$ be a non-singular mapping. The closed region $\overline{\Omega}$ is partitioned into l subregions $\mathcal{P}_h = \{\Omega_i : i = 1,2,\dots,l\}$ such that the diameter, $diam\ \Omega_i \equiv sup\{\|\boldsymbol{x}-\boldsymbol{y}\|_2 : \boldsymbol{x},\boldsymbol{y} \in \Omega_i\}$ is bounded by a positive discretisation parameter h. We assume that the partition is quasi-uniform in the sense that there exists a constant $\gamma > 0$ such that

$$h \leq \gamma \min_{i=1\to l} diam\ B_{\Omega_i} \tag{3.1}$$

where B_{Ω_i} is the closed balled in the region $\overline{\Omega_\iota}$. In a similar manner to the one-dimensional case, we can produce random numbers in each subinterval and monitor which subintervals the mappings are sent to. We then approximate the Frobenius-Perron operator with

$$p_{ij} = \frac{m\left(\Omega_i \cap S^{-1}(\Omega_j)\right)}{m(\Omega_i)}, \forall i,j = 1,2,\dots,l. \tag{3.2}$$

Ulam's conjecture then implies that $\lim_{k\to\infty} f_{n_k} = f^*$. The proof of this conjecture is difficult to extend to a higher dimensional setting. The main difficulty is due to the definition of bounded variation in n dimensions which is complicated and does not possess the same intuitive properties as one-dimensional bounded variation [13]. Boyarsky and Lou proved an n-dimensional version of Ulam's conjecture for a special class of higher-dimensional transformations called Jablonski transformations [14]. Such transformations are defined on rectangular partitions and on each partition S depends only on one variable. Recently, these transformations have found an interesting application to cellular automata [15].

3.3. Computational example in two dimensions

Ulam's method was then applied to the two-dimensional mapping S: $[0,1] \times [0,1] \to [0,1] \times [0,1]$, where $S(x,y) = \big(T(x), S(y)\big)$ with

$$T(x) = \begin{cases} 2x, & 0 \le x < \frac{1}{2} \\ 2(1-x), & \frac{1}{2} \le x \le 1 \end{cases}$$

and

$$S(y) = \begin{cases} \frac{2y}{1-y}, & 0 \le y \le \frac{1}{3} \\ \frac{1-y}{2y}, & \frac{1}{3} \le y \le 1 \end{cases}$$

This is an example of a Jablonski transformation and thus an absolutely continuous invariant measure exists and Ulam's method will converge to the stationary density. The two-dimensional space is split into $n \times n$ equal squares. In each square, 10000 random numbers are produced with a uniform distribution throughout the area of the square and the square in which the mapping sends each random number is noted. The Frobenius-Perron operator is then approximated using equation (3.2). This operator is of the size $n^2 \times n^2$ in order to completely specify all possible transitions amongst the squares. In the same way as the one-dimensional case, the stationary density is then found from this approximated operator by using the discreteMarkovChain package in python. This is a method to find the eigenvector for which the eigenvalue is equal to one. The code to carry out this example and plot the invariant probability distribution is shown in the appendix. *Figure 2*, shows a plot of the stationary distribution of the mapping. As can be seen, a grid of at least 30×30 is required in order for Ulam's method to converge to the stationary density in this case as the invariant distribution is different to that with less squares. Additionally, it takes longer for the stationary density to be found from the operator due to the difficulty in finding the appropriate eigenvector for a matrix of size $n^d \times n^d$, where d is the dimension of the mapping as the size of the matrix required can become very large. Therefore, it may become impractical to use Ulam's method for mappings of high dimension, at least when using the Monte Carlo approach to construct the Frobenius-Perron operator.

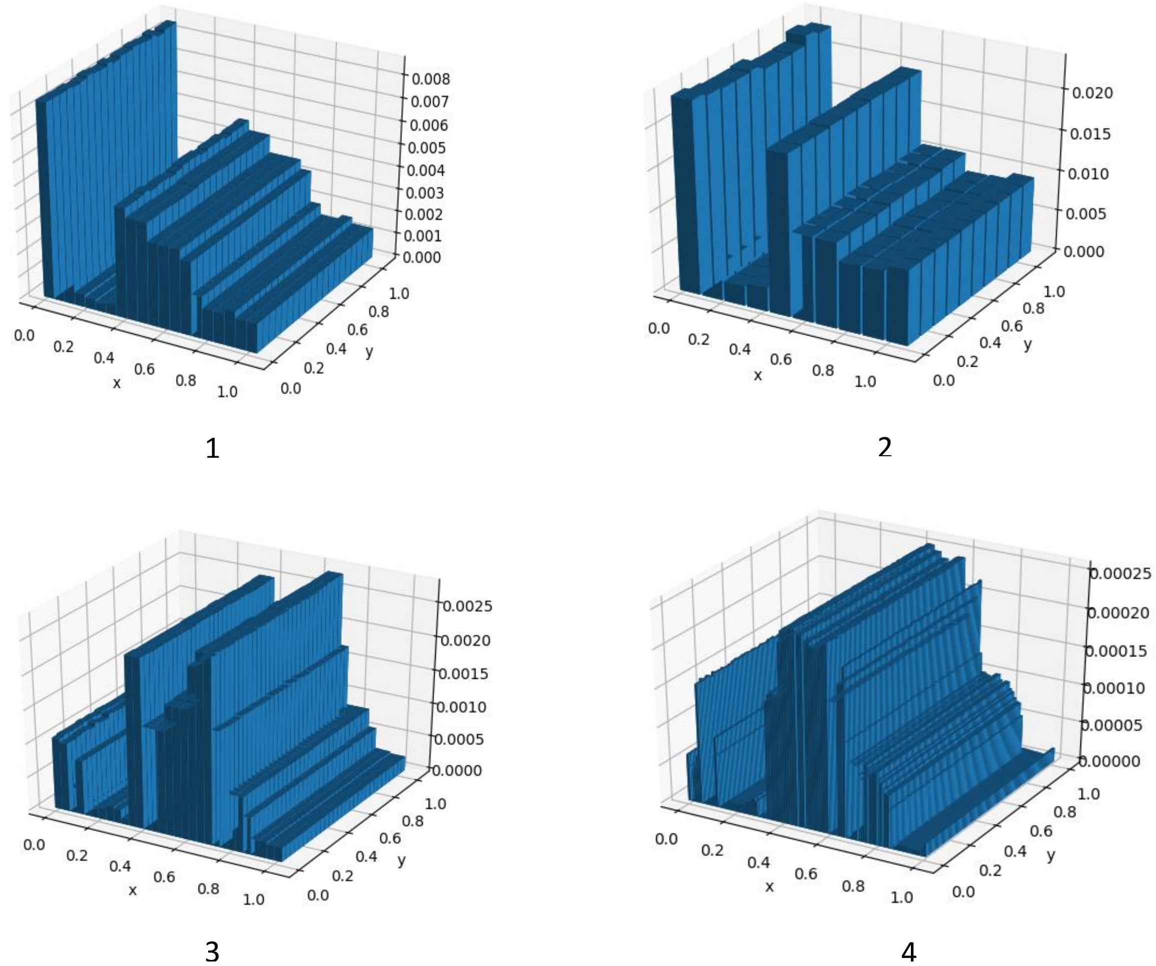

Figure 2: Graph of the stationary density of the 2D map over the space $[0,1] \times [0,1]$ found using Ulam's method. The number of subintervals is given by: 1) = 10×10. 2) = 20×20. 3) = 30×30. 4) = 100×100.

4. Using Ulam's method to Calculate the Lyapunov exponent

Once we have found the SBR measure, μ by approximating the Frobenius-Perron operator, we can also calculate the Lyapunov exponent of S with respect to μ, which is also intricately linked to the metric entropy of the system. Approximating the Lyapunov exponent is useful as it is one of the few unambiguous indications of chaos in the dynamical system. Other estimates that may be found using Ulam's method are for the decay of correlations or for the rate of escape for repellors [8], however, I shall not explore these in this project.

The Lyapunov exponent quantifies the average rate of expansion or contraction for a local interval under iteration. If there exists an SBR measure and Ulam's conjecture is valid, then when we partition the interval into n subintervals $I_i = [x_{i-1}, x_i]$ for $i = 1, 2, \dots, n$, the Lyapunov exponent Λ is exactly computed as

$$\Lambda = \int_0^1 \ln|S'(x)| f^* dx \tag{4.1}$$

By splitting the integral over [0,1] into the sum of the integrals for each subinterval we get

$$\Lambda = \int_{x_0}^{x_1} \ln|S'(x)| f_1^* dx + \cdots + \int_{x_{n-1}}^{x_n} \ln|S'(x)|\, f_n^* dx$$

$$= \ln[\, S'(x_{0.5})] \int_{x_0}^{x_1} f_1^* dx + \cdots + [S'(x_{n-0.5})] \int_{x_{n-1}}^{x_n} \ln f_n^* dx$$

$$= \sum_{i=1}^{n} \ln|S'(\, x_{i-0.5})|(x_i - x_{i-1}) f_i^* \tag{4.2}$$

Again, using the bent Baker's map as an example, I have calculated the Lyapunov exponent of the system and compared my solution to the results of Froyland in order to confirm the validity of the program. Having found the stationary density previously, I iterated through the intervals and calculated the sum given by equation (4.2). The code I used to do this is shown in the appendix. I did this calculation for different numbers of subintervals to see if the Lyapunov exponent converged with a finite number of subintervals.

Number of Subintervals	Lyapunov Exponent
2	0.585493
4	0.630425
8	0.645789
16	0.648360
32	0.648438
64	0.649542
128	0.649467
256	0.649223
512	0.649486
1024	0.649380

Table 1: Estimates for the Lyapunov exponents for the bent Baker's map for different number of subintervals. The estimates were found by using Ulam's method to approximate the absolutely continuous invariant measure.

Table 1 suggests that the value of the Lyapunov exponent converges rapidly and gives the value 0.65 to 2 significant figures after only 8 iterations. However, there does seem to be some error fluctuations with the Lyapunov exponent as it is never obtained to 4 significant figures. This error is shown in the graphs of the stationary density which are not smooth functions of position. As the Lyapunov exponent is positive, the map is chaotic and thus has a sensitive dependence on initial conditions. This may explain the lack of smoothness in the approximation of the absolutely continuous invariant measure as small changes in position within a subinterval will have differing dynamics. My value of the lyapunov exponent corresponds with that of Froyland who gets a value of 0.64946, confirming my computational method.

5. Using both the stationary density and Lyapunov exponent to assess a Pseudo-random number generator.

Random number generators are important in cryptographic systems, communication systems, statistical simulation systems, image encryption and any area involving Monte Carlo methods [16]. We can generate numbers using the deterministic mapping S. In order to be a good pseudo-random number generator, the probability of generating any number in the interval [0,1] should be a constant and the generated number should be unpredictable. To test the 'goodness' of the random number generator, we can use the ideas of stationary density and Lyapunov exponents discussed earlier. The Stationary density of the mapping should be as close to a constant as possible as each number should be picked equally as $t \to \infty$ and thus have equal measure. Moreover, the Lyapunov exponent should be positive i.e. the mapping is chaotic. Therefore, the generated number would be highly-sensitive to initial conditions and would therefore be more unpredictable. This is a very interesting consequence, as a purely deterministic mapping which we know completely can produce uniformly pseudo-random numbers which are statistically unpredictable.

Ulam's method can be used if the mapping obeys suitable conditions, to find the stationary density and the Lyapunov exponent of the transformation and thus give an indication of the suitability of the mapping as a Pseudo-random number generator. However, other factors which contribute to a good pseudo-random number generator such as high-dimensionality of the parent dynamical system or the very large period of the generated sequences [4], are not guaranteed or explored.

The stationary density and the Lyapunov exponent was found for the Hewlett-Packard company's mapping which is given by

$$S(x) = (x+\pi)^5 \ (mod 1). \tag{5.1}$$

$inf|S'| = 5\pi^4 > 1$ meaning an approximation of an absolutely continuous invariant measure can be found for the transformation using Ulam's method and from this the Lyapunov exponent. The number of subintervals was set at 100 and the discreteMarkovChain package was again used to find the fixed point of the approximated Frobenius-Perron operator. The code to do this is outlined in the appendix.

The Lyapunov exponent is found to be 6.765719, and given the large positive value suggests that the Hewlett-Packard company's mapping is highly chaotic and thus makes a good pseudo-random number generator. Additionally, from *Figure 3* it can be seen that there are only small variations to the stationary distribution over the interval and thus the generated numbers will be approximately uniform.

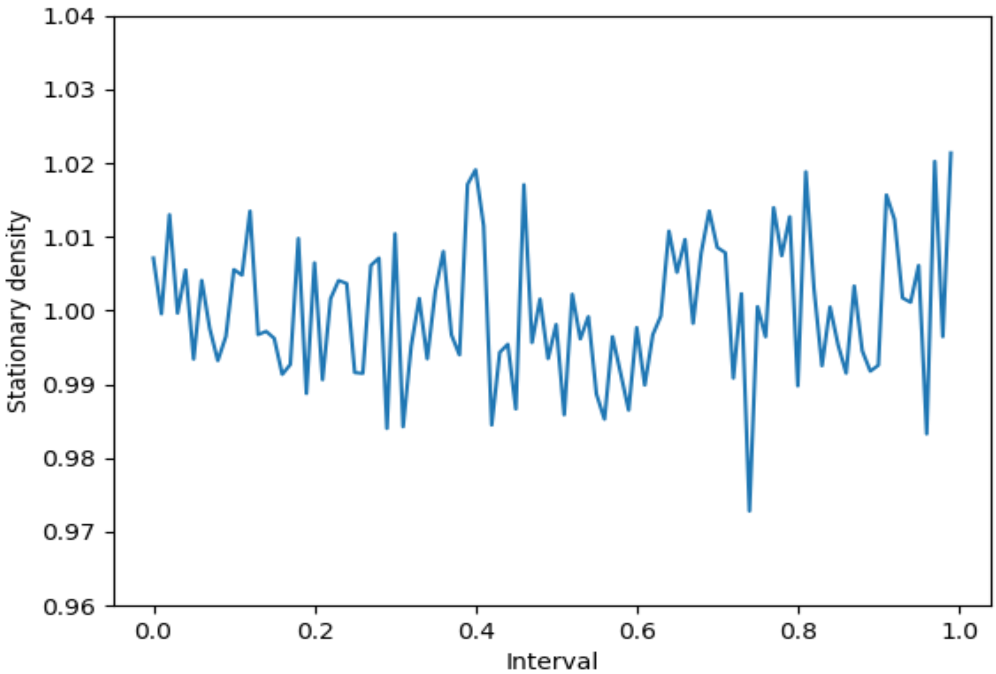

Figure 3: Graph of the stationary density of the Hewlett-Packard company map over the interval [0,1] found using Ulam's method. The number of subintervals is 100.

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