Dear Rex,

I thought that you might like an update on the meshprimp project. Your interest in the project and contributions to the methods are welcome indeed. Currently, I'm struggling with computing the net force on a vertex for two reasons. First, I'm not sure which forces to include in the free-body diagram. Second, my derivation of the angle of an edge includes the cosine of the angle. Since cosine is an even function, I lose information about the sign of the angle.

justin

1 Algorithm overview

The c++ code for moving vertices does so one vertex at a time. However, there are too many vertices in the data set to move them all m times. Instead, we choose to compute the virtual displacement of each vertex during initialization and move the vertex with the largest virtual displacement first. The virtual displacement of a vertex is how far it would move if the spring forces on the vertex were allowed to equilibrate. Our design philosophy is to maintain the state of the spring model by tracking changes in the spring configuration during vertex moves, so that we can also maintain a sorted list of virtual displacements. Futhermore, we are starting with a mesh that has no intersecting faces within the same object or between different objects, and all edge angles are greater than zero radians and less than 2π radians. Since the code monitors edge angles and face intersections, any vertex move that results in an intersection, i.e. a collision, will be undone.

2 Analytical expression for the net force on a vertex

Technically, all forces in the data set are internal forces to the system and the displacement of each vertex should be computed in parallel. But suppose we wanted to discretize the problem by solving an approximate expression for the local forces on a vertex. In other words we would define a fixed boundary around the current vertex. This fixed boundary is equivalent to applying reaction forces to these boundary vertices. **Where do we put the boundary?** Figure 1 illustrates how the boundary is currently implemented.

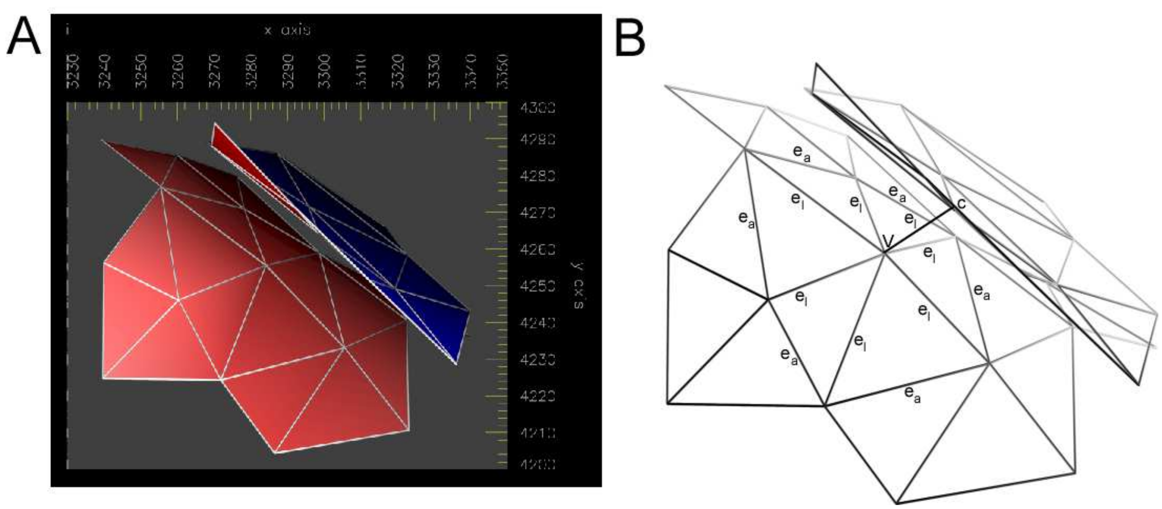

Figure 1: (A) Patch from example mesh object. The red surface is exposed to the extracellular space, while the blue surface is intracellular. (B) V, current vertex. C, closest point to V, if any. Edge e contributes to force on vertex v as either a linear spring, e_l, where the spring force is equal to the product of the spring constant and the spring stretch (current length - resting length), or as an angular spring, e_a, where the spring force is equal to the product of the spring constant and the spring stretch (current angle - resting angle). The vertices defining the angular springs (e_a in figure) constitute the fixed boundary. Angular springs on the e_l edges are NOT contributing to the force at v. **Is this an erroneous ommision?** Note that c is the closest point to v outside of v's neighborhood. c may lie on a face, an edge, or another vertex. Also note that the closest point to other vertices around v may lie on an adjacent face of v, yet NO contribution to the force on v is calculated. **Is this an unacceptable ommision?**

Once the boundary has been decided, can we formulate all forces on the current vertex as a system of linear equations and solve for the equilibrium position? The answer to this question depends on the success of linearizing the edge angle force as seen below.

3 Edge angle force contribution

An edge is defined by two mesh vertices $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$, and two adjacent faces f_1 and f_2 (Figure 2). The faces f_1 and f_2 each have a third vertex in addition to $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$, designated $\overrightarrow{o_1}$ and $\overrightarrow{o_2}$, respectively. Each face also has an outward normal vector computed using the face vertices.

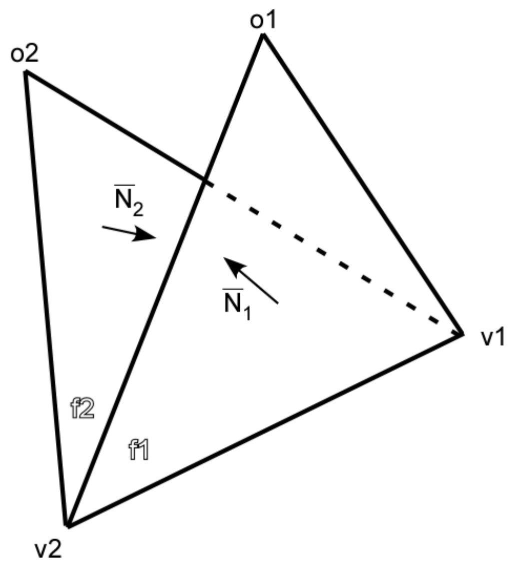

Figure 2: Edge structure.

As shown in Figure A, let $\overrightarrow{I_j}$ $(j = 1, 2)$ be the intersection point of the line segment $\overline{v_1 v_2}$ and a perpendicular line passing through $\overrightarrow{o_j}$. Define $\overrightarrow{r_j} = \overrightarrow{o_j} - \overrightarrow{I_j}$. Let angle γ measure the stretch of the angular spring. Then the spring moment $\overrightarrow{M_j}$ is equivalent to $\overrightarrow{r_j} \times \overrightarrow{F_j}$ (Figure B). Assuming the magnitude of the spring moment varies linearly with spring stretch., then $\left|\overrightarrow{M_j}\right| = k\gamma$. We want $-\pi \leq \gamma \leq \pi$ over the range $0 \leq \theta \leq 2\pi$ so the spring moment is restorative.

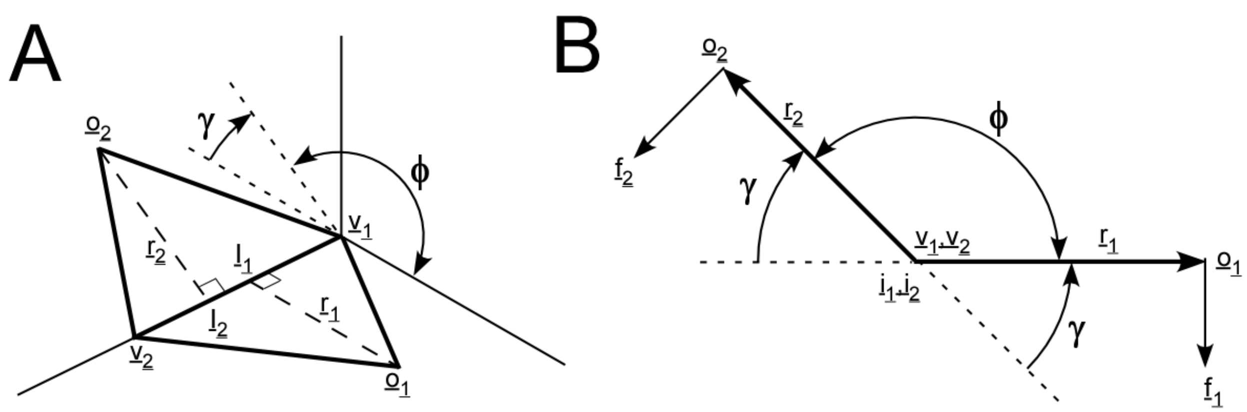

Figure 3: (A) γ measures the stretch of the angular spring. When $\gamma == 0$ the edge is "flat", i.e. the adjacent faces are coplanar and the angular spring is in it's resting position and generates no force. (B) The forces on both $\overrightarrow{o_1}$ and $\overrightarrow{o_2}$ are determined using the same value of γ. Note the underline indicates a vector quantity.

Alternatively, the stretch angle γ can be reformulated as the edge open angle θ. Figure 4 illustrates the calculation of γ, ϕ, and θ for two different edge orientations. For vertex position o_2, $\gamma > 0$, $0 \leq \phi < \pi$ and $0 \leq \theta < \pi$,where as for o_2', $\gamma' < 0$, $\pi < \phi' \leq 2\pi$ and $\pi < \theta' < 2\pi$. It can be shown that $\gamma + \theta = \pi$ for $0 \leq \theta \leq 2\pi$, therefore an alternate expression for the magniatude of the moment is $\left|\overrightarrow{M_j}\right| = k(\pi - \theta)$. Therefore if $\theta < \pi$ then $\gamma > 0$ and $\left|\overrightarrow{M_j}\right| > 0$, and if $\theta > \pi$, then $\gamma < 0$ and $\left|\overrightarrow{M_j}\right| < 0$.

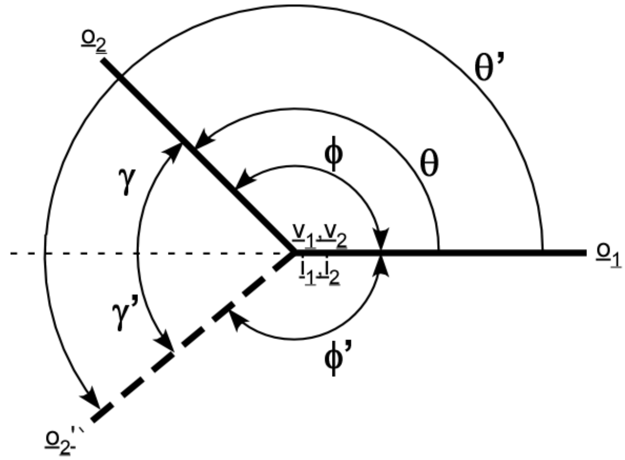

Figure 4: Description of γ, ϕ, and θ for two different edge orientations defined by $\overrightarrow{o_2}$ and $\overrightarrow{o_2'}$. Note that$-\pi < \gamma < \pi$, $0 \leq \phi \leq \pi$, and $0 \leq \theta \leq 2\pi$.

3.1 Relationship between ϕ and γ

From Figure 4 if $\theta < \pi$, then $cos\phi = \frac{\overrightarrow{s_1} \bullet \overrightarrow{s_2}}{|\overrightarrow{s_1}||\overrightarrow{s_2}|} = cos\theta = cos\left(\pi - \gamma\right) = cos\pi cos\gamma + sin\pi sin\gamma = -cos\gamma$. However, because $cos\gamma == cos\left(-\gamma\right)$, the equation effectively returns the absolute value of γ. This is evident for $\theta > \pi$ noting that now $\gamma = \phi - \pi$. The same result above is derived, i.e. $cos\phi = -cos\gamma$. More information is needed to determine the sign of γ. By trial and error it was found that γ is correctly calculated as follows. If $0 \leq \theta \leq \pi$ then $cos\phi = -cos\gamma$, else if $\pi \leq \theta \leq 2\pi$ then $cos\phi = -cos\left(-\gamma\right)$.

How do we unify these two different relations between γ and ϕ? First, we reformulate the expression as $cos\phi = -cos\left(a\gamma\right)$ where $a = 1$ for $0 \leq \theta < \pi$ and $a = -1$ for $\pi < \theta \leq 2\pi$. By trial and error it was noticed that the following expression for a behaves as desired. $a = \frac{\left(\overrightarrow{N_1} \times \overrightarrow{N_2}\right) \bullet \left(\overrightarrow{v_1} - \overrightarrow{v_2}\right)}{\left|\left(\overrightarrow{N_1} \times \overrightarrow{N_2}\right) \bullet \left(\overrightarrow{v_1} - \overrightarrow{v_2}\right)\right|}$ (Figure 5). Note this expression for a is undefined for $\theta == 0, \pi, 2\pi$. Assuming θcan never be 0 or 2π, then the only **remaining problem value is for $\theta == \pi$.**

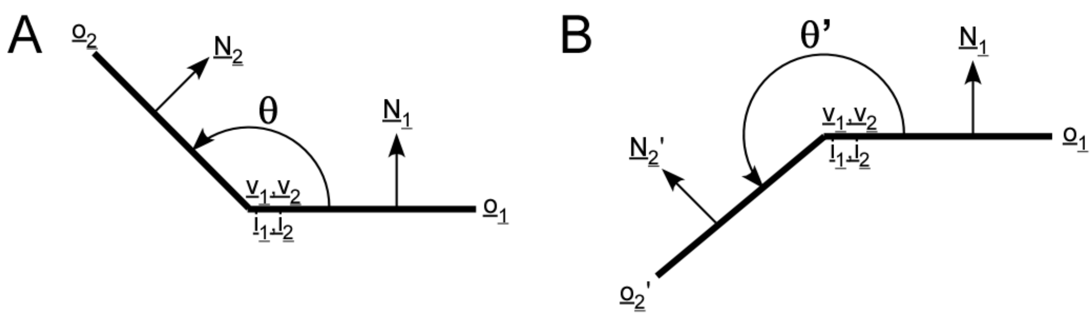

Figure 5: Disambiguating γ and $-\gamma$ using a. (A) Vectors $\overrightarrow{N_1} \times \overrightarrow{N_2}$ and $\overrightarrow{v_1} - \overrightarrow{v_2}$ both point into the page, so their dot product is greater than zero. (B) Vector $\overrightarrow{N_1} \times \overrightarrow{N_2}$ now points out of the page, so the dot product is now less than zero.

3.2 Computing γ and force

Since $cos\phi = -cos\left(a\gamma\right)$ for $0 < \theta < 2\pi$ and $\theta! = \pi$, a Taylor series for cosine can be used to compute γ. $-cos\phi = cos\left(a\gamma\right) = 1 - \frac{a^2\gamma^2}{2} + O\left(\gamma^4\right)$. Therefore $\gamma = \frac{1}{a}\sqrt{2\left(1 + cos\phi\right)}$. Returning to the expression for moment, we see that $\left|\overrightarrow{M_j}\right| = k\gamma = \frac{k}{a}\sqrt{2\left(1 + cos\phi\right)}$. But if $\overrightarrow{M_j} = M_jx\hat{i} + M_jy\hat{j} + M_jz\hat{k}$, then $\sqrt{\left(M_jx\right)^2 + \left(M_jy\right)^2 + \left(M_jz\right)^2} = \frac{k}{a}\sqrt{2\left(1 + cos\phi\right)}$ and then $\left(M_jx\right)^2 + \left(M_jy\right)^2 + \left(M_jz\right)^2 = \left(\frac{k}{a}\right)^2\left(2 + 2os\phi\right)$. **Unfortunately, we see that a is squared which throws away its sign information.**