

ASSIGNMENT I: THE AIYGARI MODEL

Vision: This project teaches you to solve for the *stationary equilibrium* in a neoclassical-style heterogeneous agent model and analyze the economic results.

- **Problem:** The problem consists of
 1. A number of questions (page 2)
 2. A model (page 3 onward, incl. solution tricks)
- **Code:** The problem is designed to be solved with the *GEModelTools* package.
- **Structure:** Your project should consist of
 1. A single self-contained pdf-file with all results
 2. A single Jupyter notebook showing how the results are produced
 3. Well-documented *.py* files
- **Hand-in:** Upload a single zip-file on Absalon (and nothing else)
- **Deadline:** 14th of October 2022
- **Exam:** Your Aiygari-project will be a part of your exam portfolio.
You can incorporate feedback before handing in the final version.

Questions

1. **Define the stationary equilibrium for the model on the next page**

2. **Solve for the stationary equilibrium**

Show aggregate quantities and prices

Illustrate household behavior

Note: *You can restrict attention to equilibria with a positive real interest rate*

3. **Illustrate how changes in the tax rates affect the stationary equilibrium**

4. **Discuss the social optimal level of taxation**

Begin with average household utility as a social welfare criterion

Other aspects of social welfare can also be introduced

Note: *Searching over a fixed grid of tax rates is fine*

5. **Suggest and implement an extension which improves the tax system**

The definition of »improves« is up to you

1. Model

Households. The model has a continuum of infinitely lived households indexed by $i \in [0, 1]$. Households are *ex ante* heterogeneous in terms of their dis-utility of labor, φ_i , and their time-invariant productivity, ζ_i . Households are *ex post* heterogeneous in terms of their time-varying stochastic productivity, z_t , and their (end-of-period) savings, a_{t-1} . The distribution of households over idiosyncratic states is denoted \underline{D}_t before shocks are realized and \mathbf{D}_t afterwards. Households choose to supply labor, ℓ_t , and consumption, c_t . Households are not allowed to borrow. The real interest rate is r_t , the real wage is w_t , and real-profits are Π_t . Interest-rate income is taxed with the rate $\tau_t^a \in [0, 1]$ and labor income is taxes with the rate $\tau_t^\ell \in [0, 1]$.

The household problem is

$$\begin{aligned} v_t(z_t, a_{t-1}) &= \max_{c_t, \ell_t} \frac{c_t^{1-\sigma}}{1-\sigma} - \varphi_i \frac{\ell_t^{1+\nu}}{1+\nu} + \beta \mathbb{E} [v_{t+1}(z_{t+1}, a_t) \mid z_t, a_t] \\ \text{s.t. } a_t + c_t &= (1 + \tilde{r}_t) a_{t-1} + \tilde{w}_t \ell_t \zeta_i z_t + \Pi_t \\ \log z_{t+1} &= \rho_z \log z_t + \psi_{t+1}, \psi_t \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \mathbb{E}[z_t] = 1 \\ a_t &\geq 0 \end{aligned}$$

where $\tilde{r}_t = (1 - \tau_t^a) r_t$ and $\tilde{w}_t = (1 - \tau_t^\ell) w_t$. Aggregate quantities are

$$\begin{aligned} L_t^{hh} &= \int \ell_t \zeta_i z_t d\mathbf{D}_t \\ C_t^{hh} &= \int c_t d\mathbf{D}_t \\ A_t^{hh} &= \int a_t d\mathbf{D}_t \end{aligned}$$

Firms. A representative firm rents capital, K_{t-1} , and hire labor, L_t , to produce goods, with the production functio

$$Y_t = \Gamma K_{t-1}^\alpha L_t^{1-\alpha} \quad (1)$$

where Γ is technology. Capital depreciates with the rate $\delta \in (0, 1)$. The real rental price of capital is r_t^K and the real wage is w_t . Profits are

$$\Pi_t = Y_t - w_t L_t - r_t^K K_{t-1} \quad (2)$$

The law-of-motion for capital is

$$K_t = (1 - \delta)K_{t-1} + I_t \quad (3)$$

The households own the representative firm in equal shares.

Government. The budget constraint for the government is

$$\begin{aligned} B_t &= (1 + r_t^B)B_{t-1} + G_t - \int \left[\tau_t^a r_t a_{t-1} + \tau_t^\ell w_t \ell_t \zeta_i z_t \right] d\mathbf{D}_t \\ &= (1 + r_t^B)B_{t-1} + G_t - \tau_t^a r_t A_t^{hh} - \tau_t^\ell w_t L_t^{hh} \end{aligned} \quad (4)$$

where G_t is exogenous government spending not entering household utility, B_t is (end-of-period) government bonds, and r_t^B is the real interest rate on government bonds.

Market clearing. Arbitrage implies that all assets must give the same rate of return

$$r_t = r_t^B = r_t^K - \delta \quad (5)$$

Market clearing implies

1. Labor market: $L_t = L_t^{hh}$
2. Goods market: $Y_t = C_t^{hh} + I_t + G_t$
3. Asset market: $K_t + B_t = A_t^{hh}$

2. Calibration

The parameters and steady state government behavior are as follows:

1. **Preferences and abilities:** $\beta = 0.96$, $\sigma = 2$, $\varphi_i \in \{0.9, 1.1\}$, $\nu = 1.0$, $\zeta_i \in \{0.9, 1.1\}$

$$\Pr[\varphi_i = 0.9, \zeta_i = 0.9] = 0.25$$

$$\Pr[\varphi_i = 1.1, \zeta_i = 0.9] = 0.25$$

$$\Pr[\varphi_i = 0.9, \zeta_i = 1.1] = 0.25$$

$$\Pr[\varphi_i = 1.1, \zeta_i = 1.1] = 0.25$$

2. **Income:** $\rho_z = 0.96, \sigma_\psi = 0.15$
3. **Production:** $\Gamma = 1, \alpha = 0.3, \delta = 0.1$
4. **Government:** $G_{ss} = 0.30, \tau_{ss}^a = 0.1, \tau_{ss}^\ell = 0.30$

3. Finding the steady state

Let $\mathcal{K}_{ss} = K_{ss}/L_{ss}$ denote the steady state capital-labor ratio.

From a guess on \mathcal{K}_{ss} we can derive:

1. Calculate $r_{ss} = \alpha\Gamma(\mathcal{K}_{ss})^{\alpha-1}$
2. Calculate $w_{ss} = (1-\alpha)\Gamma(\mathcal{K}_{ss})^\alpha$
3. Calculate $r_{ss} = r_{ss}^B = r_{ss}^K - \delta$
4. Solve and simulate household problem to obtain A_{ss}^{hh} and L_{ss}^{hh}
5. Calculate $B_{ss} = \frac{\tau_{ss}^a r_{ss} A_{ss}^{hh} + \tau_{ss}^\ell w_{ss} L_{ss}^{hh} - G_{ss}}{r_{ss}^B}$
6. Calculate $L_{ss} = L_{ss}^{hh}$
7. Calculate $K_{ss} = \mathcal{K}_{ss} L_{ss}$

We then only need to check the asset market clearing condition:

$$K_{ss} + B_{ss} - A_{ss}^{hh} = 0$$

We can also derive a *lower* and an *upper* bound on \mathcal{K}_{ss} . From ensuring a positive real interest rate, we get

$$r_{ss} > 0 \Leftrightarrow r_{ss}^K - \delta > 0 \Leftrightarrow \mathcal{K}_{ss} > \left(\frac{\delta}{\alpha\Gamma} \right)^{\frac{1}{\alpha-1}}$$

From ensuring the real interest rate is not too high relative to the household discount factor, we get

$$1 + r_{ss} < \frac{1}{\beta} \Leftrightarrow r_{ss}^K - \delta < \frac{1}{\beta} - 1 \Leftrightarrow \mathcal{K}_{ss} < \left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha\Gamma} \right)^{\frac{1}{\alpha-1}}$$

4. Solving the household problem

The following provides a recipe for solving the household problem for fixed $\varphi_i = \varphi$ and $\zeta_i = \zeta$.

The envelope condition implies

$$\underline{v}_{a,t+1}(z_{t-1}, a_{t-1}) = \mathbb{E} \left[(1 + \tilde{r}_t) c_t^{-\rho} \mid z_{t-1}, a_{t-1} \right] \quad (6)$$

The first order conditions imply

$$c_t = (\beta \underline{v}_{a,t+1}(z_t, a_t))^{-\frac{1}{\sigma}} \quad (7)$$

$$\ell_t = \left(\frac{\tilde{w}_t \zeta_i z_t}{\varphi_i} \right)^{\frac{1}{\nu}} c_t^{-\sigma/\nu} \quad (8)$$

The household problem can be solved with an extended EGM:

1. Calculate c_t and ℓ_t over end-of-period states from FOCs
2. Construct endogenous grid $m_t = c_t + a_t - \tilde{w}_t \ell_t \zeta_i z_t$
3. Use linear interpolation to find consumption $c^*(z_t, a_{t-1})$ and labor supply $\ell^*(z_t, a_{t-1})$ with $m_t = (1 + \tilde{r}_t) a_{t-1}$
4. Calculate savings $a^*(z_t, a_{t-1}) = (1 + \tilde{r}_t) a_{t-1} + \tilde{w}_t \ell_t^* \zeta_i z_t - c_t^*$
5. If $a^*(z_t, a_{t-1}) < 0$ set $a^*(z_t, a_{t-1}) = 0$ and search for ℓ_t such that $f(\ell_t) \equiv \ell_t - \left(\frac{\tilde{w}_t z_t}{\varphi_i} \right)^{\frac{1}{\nu}} c_t^{-\sigma/\nu} = 0$ holds and $c_t = (1 + \tilde{r}_t) a_{t-1} + \tilde{w}_t \ell_t \zeta_i z_t$. This can be done with a Newton solver with an update from step j to step $j + 1$ by

$$\begin{aligned} \ell_t^{j+1} &= \ell_t^j - \frac{f(\ell_t)}{f'(\ell_t)} \\ &= \ell_t^j - \frac{\ell_t - \left(\frac{\tilde{w}_t z_t}{\varphi_i} \right)^{\frac{1}{\nu}} c_t^{-\sigma/\nu}}{1 - \left(\frac{\tilde{w}_t z_t}{\varphi_i} \right)^{\frac{1}{\nu}} (-\sigma/\nu) \frac{\partial c_t}{\partial \ell_t}} \\ &= \ell_t^j - \frac{\ell_t - \left(\frac{\tilde{w}_t z_t}{\varphi_i} \right)^{\frac{1}{\nu}} c_t^{-\sigma/\nu}}{1 - \left(\frac{\tilde{w}_t z_t}{\varphi_i} \right)^{\frac{1}{\nu}} (-\sigma/\nu) c_t^{-\sigma/\nu-1} \tilde{w}_t \zeta_i z_t} \end{aligned}$$

The next page contains a code snippet with $\zeta_i z_t = 1$ you can base your code on.

```

1 # a. prepare
2 fac = (wt/varphi)**(1/nu)
3
4 # b. use FOCs
5 c_endo = (beta*vbeg_a_plus)**(-1/sigma)
6 ell_endo = fac*(c_endo)**(-sigma/nu)
7
8 # c. interpolation
9 m_endo = c_endo + a_grid - wt*ell_endo
10 m_exo = (1+rt)*a_grid
11 c = np.zeros(Na)
12 interp_1d_vec(m_endo, c_endo, m_exo, c)
13 ell = np.zeros(Na)
14 interp_1d_vec(m_endo, ell_endo, m_exo, ell)
15
16 a = m_exo + wt*ell - c
17
18 # d. refinement at borrowing constraint
19 for i_a in range(Na):
20
21     if a[i_a] < 0.0:
22
23         # i. binding constraint for a
24         a[i_a] = 0.0
25
26         # ii. solve FOC for ell
27         elli = ell[i_a]
28
29         it = 0
30         while True:
31
32             ci = (1+rt)*a_grid[i_a] + wt*elli
33
34             error = elli - fac*ci**(-sigma/nu)
35             if np.abs(error) < tol_ell:
36                 break
37             else:
38                 derror = 1 - fac*(-sigma/nu)*ci**(-sigma/nu-1)*wt
39                 elli = elli - error/derror
40
41             it += 1
42             if it > max_iter_ell: raise ValueError('too many iterations')
43
44         # iii. save
45         c[i_a] = ci
46         ell[i_a] = elli

```

Listing 1: Extended EGM