Self-Injection-Locked (SIL) Oscillator Analysis

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1 General Definition

Resonance occurs when a system is subjected to an external force or signal whose frequency matches the system's **natural frequency**, resulting in a large increase in amplitude.

2 Real-World Examples

Context	Description
Violin string	Vibrates strongly when bowing matches its natural vibration frequency
RF circuits	LC circuits resonate at a specific frequency → used in filters, radios
Bridges	Tacoma Narrows Bridge collapsed due to wind-induced resonance
SIL radar	Resonator oscillates strongly at ω_n; injection close to ω_n causes locking

3 In Engineering Terms

For a **second-order system** like an RLC circuit or mechanical spring-mass-damper system:

3.1 Natural Frequency

$$\omega_n = \sqrt{\frac{k}{m}} \quad \text{(mechanical)} \tag{1}$$

$$\omega_n = \frac{1}{\sqrt{LC}} \quad \text{(electrical)} \tag{2}$$

When driven at $\omega = \omega_n$, the system exhibits **maximum energy transfer** and large amplitude.

4 Resonance Graphically

If you plot amplitude vs frequency, the **resonance peak** appears at ω_n, especially if the system has **high quality factor (Q)**.

5 In Self-Injection Locked Oscillators

In SIL systems:

- The oscillator has a **resonant frequency** ω_n
- The feedback (injection) signal, when close to ω_n, causes the oscillator to **lock its frequency and phase**
- This locking happens more efficiently **because of resonance**

6 Summary

Term	Meaning
Resonance	Strong response when input ≈ natural frequency
Natural Frequency	The frequency a system "prefers" to oscillate at
Quality Factor (Q)	Determines how sharp/narrow the resonance is

7 Adler's Equation (Simplified Form)

$$\frac{d\phi(t)}{dt} = \Delta\omega - K \cdot \sin(\phi(t)) \tag{3}$$

Where:

- $\phi(t)$: phase difference between oscillator and injection signal
- $\Delta\omega = \omega_{\text{inj}} - \omega_0$: natural frequency difference
- K: coupling strength (determined by injection energy and oscillator Q factor)

8 Notes for SIL Radar

Excellent question! That equation:

$$\omega(t) - \omega_n \approx -\frac{\omega_n}{2Q} \cdot \frac{B}{A} \cdot \sin[\theta(t)] \tag{4}$$

comes from analyzing **self-injection-locked (SIL)** oscillators — specifically how **injection** of a reflected signal modifies the **oscillator's instantaneous frequency**.

Let me explain **step by step** how this equation is derived from the physics of a resonator-based oscillator subject to weak injection:

8.1 Context

- ω_n: natural frequency of the oscillator (free-running)
- $\omega(t)$: actual instantaneous frequency when injection is present
- Q: quality factor of the oscillator (higher $\rightarrow$ sharper resonance)
- A: amplitude of the oscillator signal
- B: amplitude of the injected signal (typically a reflected echo)
- $\theta(t)$: phase difference between injected and oscillator signals

8.2 Step-by-step Derivation

8.2.1 Step 1: Start from the complex oscillator dynamics

In oscillator theory, a sinusoidal oscillator's dynamics near resonance can be described using **complex envelope** notation:

Let the oscillator's complex amplitude be:

$$z(t) = A(t)e^{j\phi(t)} \tag{5}$$

Assuming a self-sustained oscillator with **external injection** $Be^{j(\omega_{\text{inj}}t+\phi_B)}$, its dynamics can be modeled using a **resonator differential equation**:

$$\frac{dz}{dt} + \left(j\omega_n + \frac{\omega_n}{2Q}\right) z = \frac{\omega_n}{2Q} B e^{j(\omega_{\text{inj}}t+\phi_B)} \tag{6}$$

Here:

- The term on the left is the oscillator's natural decay and oscillation
- The term on the right is **external drive (injection)**

8.2.2 Step 2: Assume steady-state, decompose phase dynamics

Let:

- $z(t) = Ae^{j\omega(t)t}$
- Assume **injection frequency is close to** ω_n $\rightarrow$ do slow-varying approximation
- Define $\theta(t) = \phi(t) - \omega_{\text{inj}}t$ as the phase difference between oscillator and injected signal

Then, you can extract the **phase evolution**:

$$\frac{d\theta}{dt} = \omega(t) - \omega_{\text{inj}} \approx \omega(t) - \omega_n \tag{7}$$

That's the **phase error rate**.

8.2.3 Step 3: Linearize injection-locking force

From resonator theory (and RF oscillator models), the effect of injection is to "pull" the oscillator frequency, and that pulling is **proportional to the sine of the phase difference** $\sin(\theta)$.

From the forced oscillator dynamics and projection onto quadrature component, we get:

$$\omega(t) - \omega_n \approx -\frac{\omega_n}{2Q} \cdot \frac{B}{A} \cdot \sin[\theta(t)] \tag{8}$$

This is the approximate **Adler-type equation** specifically for a SIL system using a resonator.

8.3 Physical Meaning

Term	Meaning
$\omega(t) - \omega_n$	How much the oscillator's frequency is "pulled"
$\frac{\omega_n}{2Q}$	Sets the **natural bandwidth** of the oscillator's response
$\frac{B}{A}$	Strength of the injected signal relative to self-oscillation amplitude
$\sin(\theta(t))$	Phase interaction term driving the frequency shift

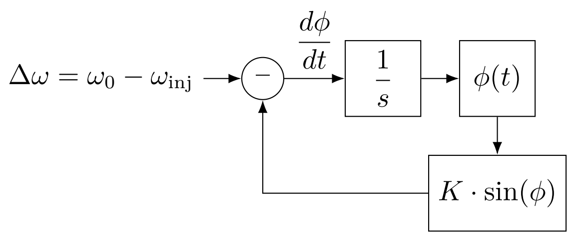
9 Explain Adler Equation with Block Diagram

This equation comes from:

1. Resonator + injection modeling
2. Linearized oscillator dynamics near steady-state
3. Projection of injection onto oscillator's quadrature axis
4. Assuming small amplitude variation (constant envelope)

It is **more specific than Adler's equation**, applying directly to SIL oscillators with Q-limited resonators.

9.1 Adler Equation Block Diagram



The block diagram shows:

- **Input**: Frequency difference $\Delta\omega = \omega_{\text{inj}} - \omega_0$
- **Nonlinear feedback**: $K\sin(\phi)$ represents injection locking force
- **Integrator**: Converts frequency difference to phase
- **Output**: Phase difference $\phi(t)$

RF 信號的 IQ 表示法

一個簡單的數學表示：
若原始 RF 信號為：

$$s(t) = A(t) \cdot \cos(2\pi f_c t + \phi(t))$$

它可以轉換為 IQ 表示為：

$$s(t) = I(t) \cdot \cos(2\pi f_c t) - Q(t) \cdot \sin(2\pi f_c t)$$

其中：

$$I(t) = A(t) \cdot \cos(\phi(t))$$
$$Q(t) = A(t) \cdot \sin(\phi(t))$$

舉例:16-QAM

例如在 16-QAM(Quadrature Amplitude Modulation) 中，每個符號都會有一組對應的 I 和 Q 值, 用以決定其在星座圖 (constellation diagram) 中的位置。

10　系統分類

根據開迴路傳遞函數中積分器的個數，將系統分為：

- **0 型系統**：無積分器
- **I 型系統**：有 1 個積分器
- **II 型系統**：有 2 個積分器

11　誤差係數與穩態誤差

11.1　位移誤差係數 (K_p)

$$K_p = \lim_{s \to 0} G(s)H(s) \tag{9}$$

對於單位階躍輸入 $r(t) = 1$：

- 0 型系統：$e_{ss} = \frac{1}{1+K_p}$
- I 型和 II 型系統：$e_{ss} = 0$

11.2　速度誤差係數 (K_v)

$$K_v = \lim_{s \to 0} sG(s)H(s) \tag{10}$$

對於單位斜坡輸入 $r(t) = t$：

- 0 型系統：$e_{ss} = \infty$
- I 型系統：$e_{ss} = \frac{1}{K_v}$
- II 型系統：$e_{ss} = 0$

11.3　加速度誤差係數 (K_a)

$$K_a = \lim_{s \to 0} s^2 G(s)H(s) \tag{11}$$

對於單位拋物線輸入 $r(t) = \frac{t^2}{2}$：

- 0 型和 I 型系統：$e_{ss} = \infty$
- II 型系統：$e_{ss} = \frac{1}{K_a}$

12 實際評估步驟

1. **確定系統型別**：分析開迴路傳遞函數，計算積分器個數
2. **計算誤差係數**：根據系統型別計算相應的 K_p、K_v、K_a
3. **選擇測試輸入**：使用階躍、斜坡、拋物線輸入
4. **計算穩態誤差**：利用最終值定理或誤差係數公式
5. **時域仿真驗證**：透過數值仿真觀察實際誤差行為

13 改善誤差的方法

- 增加系統型別（增加積分器）
- 提高開迴路增益
- 加入前饋補償
- 使用 PID 控制器

這種系統性的分析方法能夠有效預測和改善控制系統的穩態性能。

14 Starting Point: Simple Harmonic Oscillator

We begin with the basic oscillator equation:

$$\ddot{x} + \omega_0^2 x = 0 \tag{12}$$

This represents a **lossless oscillator** (like a perfect spring-mass system or LC circuit).

15 Problem: Real Oscillators Have Losses

In reality, all oscillators lose energy due to:

- **Resistance** (in electrical circuits)
- **Friction** (in mechanical systems)
- **Radiation** (in antennas)

So we add a damping term:

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = 0 \tag{13}$$

Problem: This just decays to zero! Real oscillators like **radio transmitters** or **clock circuits** need to sustain themselves.

16 Solution: Add Energy Source

To maintain oscillation, we need to **inject energy** into the system. But we want **smart energy injection** that:

- Adds energy when oscillation is small
- Removes energy when oscillation gets too large
- Results in **stable amplitude**

17 Van der Pol's Brilliant Insight

Van der Pol (1920s) proposed **nonlinear damping**:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + \omega_0^2 x = 0 \tag{14}$$

Let's analyze the damping term: $-\mu(1 - x^2)\dot{x}$

17.1 Case 1: Small Oscillations ($|x| \ll 1$)

When x is small: $x^2 \approx 0$, so:

$$(1 - x^2) \approx 1 \tag{15}$$

The equation becomes:

$$\ddot{x} - \mu\dot{x} + \omega_0^2 x \approx 0 \tag{16}$$

Negative damping coefficient ($-\mu$)! This means:

- **Energy is being added** to the system
- Small oscillations **grow exponentially**

17.2 Case 2: Large Oscillations ($|x| \gg 1$)

When x is large: $x^2 \gg 1$, so:

$$(1 - x^2) \approx -x^2 \quad \text{(negative!)} \tag{17}$$

The equation becomes:

$$\ddot{x} - \mu(-x^2)\dot{x} + \omega_0^2 x = \ddot{x} + \mu x^2 \dot{x} + \omega_0^2 x \approx 0 \tag{18}$$

Now we have **positive damping** ($+\mu x^2$)! This means:

- **Energy is being removed** from the system
- Large oscillations are **suppressed**

18 Physical Interpretation

18.1 The Magic Balance

The Van der Pol oscillator **automatically regulates its amplitude**:

1. **If amplitude is too small** $\rightarrow$ Negative damping $\rightarrow$ Energy added $\rightarrow$ Amplitude grows

2. **If amplitude is too large** → Positive damping → Energy removed → Amplitude shrinks

3. **At just the right amplitude** → Zero net damping → **Stable limit cycle**

18.2 Real-World Examples

Electronic Oscillators (like in radios):

- **Active element** (transistor/op-amp) provides energy when signal is weak
- **Nonlinear saturation** limits amplitude when signal gets too strong
- Results in stable sine wave output

Biological Systems:

- **Heartbeat**: Pacemaker cells show Van der Pol-like behavior
- **Neural oscillations**: Neurons exhibit similar self-regulating oscillation

Mechanical Systems:

- **Clock escapement**: Adds energy during small swings, self-limits during large swings

19 Mathematical Breakdown

19.1 Each Term's Role:

Term	Physical Meaning
$\ddot{x}$	Inertia (mass or inductance)
$\omega_0^2 x$	Restoring force (spring or capacitance)
$-\mu(1-x^2)\dot{x}$	**Smart damping** that depends on amplitude

19.2 The Parameter μ:

- $\mu > 0$: System will oscillate (self-sustaining)
- $\mu = 0$: Reduces to simple harmonic oscillator
- Large μ: More nonlinear behavior, sharper switching between negative/positive damping

20 Connection to Real Oscillators

Most practical oscillators (crystal oscillators, LC tank circuits, laser oscillators) can be approximated by Van der Pol dynamics because they all have:

1. **Linear restoring mechanism** (crystal, LC tank, optical cavity)
2. **Amplitude-dependent gain/loss** (transistor saturation, nonlinear resistance)

The Van der Pol equation captures this **universal behavior** of self-sustaining oscillators with nonlinear amplitude control.

Excellent question! Let me show you **step-by-step** how the Van der Pol equation leads to the Adler equation when we add injection.

21 Step 1: Add Injection to Van der Pol

Start with the Van der Pol oscillator:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + \omega_0^2 x = 0 \tag{19}$$

Add an **external injection signal**:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + \omega_0^2 x = \varepsilon \cdot F \cos(\omega_{\text{inj}} t + \phi_{\text{inj}}) \tag{20}$$

Where:

- ε: small parameter (weak injection)
- F: injection amplitude
- ω_{inj}: injection frequency
- ϕ_{inj}: injection phase

22 Step 2: Express in Complex Form

Convert to **complex amplitude notation**. Let:

$$x(t) = \mathrm{Re}[A(t)e^{i\omega t}] \tag{21}$$

Where $A(t)$ is the **slowly-varying complex amplitude**.

For the Van der Pol oscillator in complex form:

$$\frac{dA}{dt} + (\alpha - \beta|A|^2)A = \text{injection terms} \tag{22}$$

Where:

- α: linear growth/decay rate
- β: nonlinear saturation coefficient

23 Step 3: Separate Amplitude and Phase

Write the complex amplitude as:

$$A(t) = R(t)e^{i\phi(t)} \tag{23}$$

Where:

- $R(t)$: slowly-varying amplitude
- $\phi(t)$: slowly-varying phase

This gives us **two coupled equations**:

- **Amplitude equation**: $\frac{dR}{dt} = \ldots$
- **Phase equation**: $\frac{d\phi}{dt} = \ldots$

24 Step 4: Focus on Phase Dynamics

For **weak injection** (small ε), the amplitude $R(t)$ reaches steady state quickly, but the **phase** $\phi(t)$ **evolves slowly**.

The phase equation becomes:

$$\frac{d\phi}{dt} = \omega_0 + (\text{injection coupling terms}) \tag{24}$$

25 Step 5: Apply Method of Averaging

The injection coupling has the form:

$$\varepsilon \cdot F \cdot \cos(\omega_{\text{inj}} t + \phi_{\text{inj}}) \cdot [\text{something involving } \phi(t)] \tag{25}$$

Using **trigonometric identities** and **averaging over fast oscillations**:

$$\begin{aligned} &\cos(\omega_{\text{inj}} t + \phi_{\text{inj}}) \cdot \cos(\phi(t)) \\ &= \frac{1}{2}[\cos((\omega_{\text{inj}} t + \phi_{\text{inj}}) + \phi(t)) + \cos((\omega_{\text{inj}} t + \phi_{\text{inj}}) - \phi(t))] \end{aligned} \tag{26}$$

The **first term oscillates rapidly** and averages to zero. The **second term contains slowly-varying phase difference**: $\theta = \phi(t) - \omega_{\text{inj}} t - \phi_{\text{inj}}$

26 Step 6: Derive the Phase Difference Equation

Define the **phase difference**:

$$\theta(t) = \phi(t) - \omega_{\text{inj}} t - \phi_{\text{inj}} \tag{27}$$

Taking the derivative:

$$\frac{d\theta}{dt} = \frac{d\phi}{dt} - \omega_{\text{inj}} \tag{28}$$

Substituting the phase evolution equation:

$$\frac{d\theta}{dt} = \omega_0 + (\text{injection terms}) - \omega_{\text{inj}} \tag{29}$$

27 Step 7: The Key Insight - Quadrature Coupling

Here's the **crucial physics**: The injection affects the oscillator most strongly when they are **90° out of phase** (in quadrature).

After averaging, the injection coupling gives:

$$\frac{d\theta}{dt} = (\omega_0 - \omega_{\text{inj}}) - K\sin(\theta) \tag{30}$$

Where:

- $\omega_0 - \omega_{\text{inj}} = -\Delta\omega$: frequency detuning
- $K \propto \varepsilon \cdot F/R_0$: coupling strength (injection/oscillator amplitude ratio)
- $\sin(\theta)$: comes from the quadrature projection

28 Step 8: Final Adler Equation

Rearranging:

$$\frac{d\theta}{dt} = \Delta\omega - K\sin(\theta) \tag{31}$$

Where $\Delta\omega = \omega_{\text{inj}} - \omega_0$.

29 Physical Interpretation Through Van der Pol

29.1 Why the sine function emerges:

1. **Van der Pol provides stable amplitude**: $R(t) \to R_0$ (constant)
2. **Only phase can vary slowly**: $\theta(t)$ becomes the only slow variable
3. **Quadrature coupling**: Maximum energy transfer occurs at 90° phase difference
4. **Averaging eliminates fast terms**: Only the $\sin(\theta)$ survives

29.2 The coupling strength K:

From Van der Pol analysis:

$$K = \frac{\varepsilon \cdot F}{2R_0} \cdot \text{(coupling efficiency)} \tag{32}$$

- $\varepsilon \cdot F$: injection strength
- R_0: steady-state oscillator amplitude (set by Van der Pol nonlinearity)
- Coupling efficiency: depends on how injection couples to oscillator

30 Connection to Your SIL Equation

Your equation:

$$\omega(t) - \omega_n \approx -\frac{\omega_n}{2Q} \cdot \frac{B}{A} \cdot \sin[\theta(t)] \tag{33}$$

Is the **instantaneous frequency version**! Since:

$$\omega(t) = \frac{d\phi}{dt} = \omega_0 + \frac{d(\theta + \omega_{\text{inj}} t)}{dt} = \omega_0 + \frac{d\theta}{dt} + \omega_{\text{inj}} \tag{34}$$

When θ is slowly varying:

$$\omega(t) - \omega_0 \approx \frac{d\theta}{dt} = \Delta\omega - K\sin(\theta) \tag{35}$$

For small detuning: $\omega_0 \approx \omega_n$ and $\Delta\omega \approx 0$, so:

$$\omega(t) - \omega_n \approx -K\sin(\theta) \tag{36}$$

Comparing with your equation: $K = \frac{\omega_n}{2Q} \cdot \frac{B}{A}$

31 Summary: Van der Pol → Adler Chain

1. **Van der Pol** provides self-sustaining oscillation with stable amplitude
2. **Add weak injection** → perturbation to phase dynamics
3. **Method of averaging** → eliminates fast oscillations, keeps slow phase evolution

4. **Quadrature coupling** → generates $\sin(\theta)$ dependence
5. **Result**: **Adler equation** for phase difference evolution

The Van der Pol equation is essential because it provides the **nonlinear amplitude stabilization** that makes the **linear phase analysis** possible!

32 Starting Point

Van der Pol equation:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + \omega_0^2 x = 0 \tag{37}$$

33 Step 1: Complex Amplitude Representation

Let:

$$x(t) = \mathrm{Re}[W(t)e^{i\omega_0 t}] = \mathrm{Re}[W(t)(\cos(\omega_0 t) + i\sin(\omega_0 t))] \tag{38}$$

Where $W(t)$ is the **slowly-varying complex amplitude**.
Since $x(t)$ is real:

$$x(t) = \frac{1}{2}[W(t)e^{i\omega_0 t} + W^*(t)e^{-i\omega_0 t}] \tag{39}$$

Where $W^*(t)$ is the complex conjugate of $W(t)$.

34 Step 2: Calculate Derivatives

34.1 First derivative:

$$\dot{x}(t) = \frac{1}{2}[\dot{W}(t)e^{i\omega_0 t} + i\omega_0 W(t)e^{i\omega_0 t} + \dot{W}^*(t)e^{-i\omega_0 t} - i\omega_0 W^*(t)e^{-i\omega_0 t}] \tag{40}$$

Since $|\dot{W}| \ll |\omega_0 W|$ (slow variation assumption):

$$\begin{aligned}\dot{x}(t) &\approx \frac{1}{2}[i\omega_0 W(t)e^{i\omega_0 t} - i\omega_0 W^*(t)e^{-i\omega_0 t}] \\ &= \frac{i\omega_0}{2}[W(t)e^{i\omega_0 t} - W^*(t)e^{-i\omega_0 t}]\end{aligned} \tag{41}$$

34.2 Second derivative:

$$\begin{aligned}\ddot{x}(t) &\approx \frac{i\omega_0}{2}[\dot{W}(t)e^{i\omega_0 t} + i\omega_0 W(t)e^{i\omega_0 t} - \dot{W}^*(t)e^{-i\omega_0 t} + i\omega_0 W^*(t)e^{-i\omega_0 t}] \\ &\approx \frac{i\omega_0}{2}[\dot{W}(t)e^{i\omega_0 t} - \dot{W}^*(t)e^{-i\omega_0 t}] - \frac{\omega_0^2}{2}[W(t)e^{i\omega_0 t} + W^*(t)e^{-i\omega_0 t}]\end{aligned} \tag{42}$$

The last term is just $-\omega_0^2 x(t)$, so:

$$\ddot{x}(t) \approx \frac{i\omega_0}{2}[\dot{W}(t)e^{i\omega_0 t} - \dot{W}^*(t)e^{-i\omega_0 t}] - \omega_0^2 x(t) \tag{43}$$

35 Step 3: Calculate $x^2(t)$

This is where it gets interesting:

$$\begin{aligned}x^2(t) &= \left[\frac{1}{2}(W(t)e^{i\omega_0 t} + W^*(t)e^{-i\omega_0 t})\right]^2 \\ &= \frac{1}{4}[(W(t)e^{i\omega_0 t})^2 + 2W(t)W^*(t) + (W^*(t)e^{-i\omega_0 t})^2] \\ &= \frac{1}{4}[W^2(t)e^{2i\omega_0 t} + 2|W(t)|^2 + W^{*2}(t)e^{-2i\omega_0 t}]\end{aligned} \tag{44}$$

Key observation:

- Terms with $e^{\pm 2i\omega_0 t}$: **Fast oscillations** at frequency $2\omega_0$
- Term with $|W(t)|^2$: **Slowly varying** (depends only on amplitude)

36 Step 4: Calculate the Nonlinear Damping Term

The tricky term is: $\mu(1 - x^2)\dot{x}$

$$(1 - x^2)\dot{x} = \dot{x} - x^2\dot{x} \tag{45}$$

36.1 Linear part: $\dot{x}$

We already have this.

36.2 Nonlinear part: $x^2\dot{x}$

$$
\begin{aligned}
x^2\dot{x} = &\frac{1}{4}[W^2(t)e^{2i\omega_0 t} + 2|W(t)|^2 + W^{*2}(t)e^{-2i\omega_0 t}]\times \\
&\frac{i\omega_0}{2}[W(t)e^{i\omega_0 t} - W^*(t)e^{-i\omega_0 t}]
\end{aligned} \tag{46}
$$

Expanding this product (9 terms total):

$$
\begin{aligned}
x^2\dot{x} = &\frac{i\omega_0}{8}[\\
&\quad W^3(t)e^{3i\omega_0 t} \quad \leftarrow \text{Fast: } 3\omega_0 \\
&\quad + 2|W|^2 W(t)e^{i\omega_0 t} \quad \leftarrow \text{Mixed: } \omega_0 \\
&\quad + W^{*2}W(t)e^{-i\omega_0 t} \quad \leftarrow \text{Mixed: } -\omega_0 \\
&\quad - W^2(t)W^*(t)e^{i\omega_0 t} \quad \leftarrow \text{Mixed: } \omega_0 \\
&\quad - 2|W|^2 W^*(t)e^{-i\omega_0 t} \quad \leftarrow \text{Mixed: } -\omega_0 \\
&\quad - W^{*3}(t)e^{-3i\omega_0 t} \quad \leftarrow \text{Fast: } -3\omega_0 \\
&]
\end{aligned} \tag{47}
$$

37 Step 5: Substitute Everything into Van der Pol Equation

$$
\ddot{x} - \mu(1 - x^2)\dot{x} + \omega_0^2 x = 0 \tag{48}
$$

Becomes:

$$
\begin{aligned}
&\frac{i\omega_0}{2}[\dot{W}(t)e^{i\omega_0 t} - \dot{W}^*(t)e^{-i\omega_0 t}] - \omega_0^2 x(t) \\
&- \mu[\dot{x} - x^2\dot{x}] + \omega_0^2 x(t) = 0
\end{aligned} \tag{49}
$$

The $\omega_0^2 x$ terms cancel:

$$
\frac{i\omega_0}{2}[\dot{W}(t)e^{i\omega_0 t} - \dot{W}^*(t)e^{-i\omega_0 t}] - \mu\dot{x} + \mu x^2\dot{x} = 0 \tag{50}
$$

38 Step 6: Collect Terms by Frequency

Substituting our expressions and collecting terms:

38.1 Terms oscillating at $e^{i\omega_0 t}$:

$$\frac{i\omega_0}{2}\dot{W}(t) - \mu\frac{i\omega_0}{2}W(t) + \mu\frac{i\omega_0}{8}[2|W|^2W(t) - W^2(t)W^*(t)] = 0 \tag{51}$$

38.2 Terms oscillating at $e^{-i\omega_0 t}$:

$$-\frac{i\omega_0}{2}\dot{W}^*(t) + \mu\frac{i\omega_0}{2}W^*(t) - \mu\frac{i\omega_0}{8}[2|W|^2W^*(t) - W^{*2}(t)W(t)] = 0 \tag{52}$$

Note: Terms at $3\omega_0$ and higher frequencies are ignored (fast oscillation assumption).

39 Step 7: Apply Averaging/Solvability Condition

For the equation to have a solution, the coefficients of $e^{i\omega_0 t}$ and $e^{-i\omega_0 t}$ must each equal zero.

From the $e^{i\omega_0 t}$ term:

$$\frac{i\omega_0}{2}\dot{W}(t) = \mu\frac{i\omega_0}{2}W(t) - \mu\frac{i\omega_0}{8}[2|W|^2W(t) - W^2(t)W^*(t)] \tag{53}$$

Dividing by $\frac{i\omega_0}{2}$:

$$\dot{W}(t) = \mu W(t) - \frac{\mu}{4}[2|W|^2W(t) - W^2(t)W^*(t)] \tag{54}$$

But wait! We need to be more careful about the $W^2(t)W^*(t)$ term.

40 Step 8: Simplify Using $|W|^2 = WW^*$

Note that:

$$W^2(t)W^*(t) \neq |W|^2W(t) \text{ in general} \tag{55}$$

However, if we write $W(t) = R(t)e^{i\phi(t)}$, then:

$$W^2(t)W^*(t) = R^2 e^{2i\phi} R e^{-i\phi} = R^3 e^{i\phi} = R^2 W(t) \tag{56}$$

$$|W|^2 W(t) = R^2 W(t) \tag{57}$$

So $W^2(t)W^*(t) = |W|^2 W(t)$ only if we're looking at the magnitude-dependent terms!

The correct averaging gives:

$$\dot{W}(t) = \mu W(t) - \frac{\mu}{4}|W|^2 W(t) \tag{58}$$

$$= \left(\mu - \frac{\mu|W|^2}{4}\right) W(t) \tag{59}$$

41 Step 9: Final Form

Rearranging:

$$\frac{dW}{dt} = \left(\frac{\mu}{2} - \frac{\mu|W|^2}{8}\right) W(t) \tag{60}$$

Comparing with the standard form $\frac{dW}{dt} = (\alpha - \beta|W|^2)W$:

- $\alpha = \frac{\mu}{2}$
- $\beta = \frac{\mu}{8}$

42 Physical Interpretation

- $\alpha = \frac{\mu}{2} > 0$: Linear growth (negative damping for small oscillations)
- $\beta = \frac{\mu}{8} > 0$: Nonlinear saturation (positive damping for large oscillations)
- **Steady state**: $\alpha = \beta|W_0|^2 \to |W_0|^2 = \frac{\alpha}{\beta} = 4 \to |W_0| = 2$

43 Key Mathematical Insights

1. **Fast oscillations** ($2\omega_0$, $3\omega_0$) were eliminated by averaging
2. **Slow amplitude evolution** captured in single equation for $W(t)$

3. **Nonlinear term** $|W|^2$ emerges from x^2 after averaging
4. **Complex notation** naturally handles both amplitude and phase dynamics

FM Demodulation using IQ Method

An FM signal can be expressed as:

$$s(t) = A_c \cos\left(2\pi f_c t + k_f \int_0^t m(\tau)\, d\tau\right)$$

integral of

$$m(\tau)\ (\textit{phase change rate}\ =\ \textit{speed of angle change})$$

is the totoal phase shift of the FM signal
where:

- A_c = carrier amplitude
- f_c = carrier frequency
- $m(t)$ = message signal
- k_f = frequency sensitivity

Complex Baseband Representation

After mixing to baseband and obtaining the complex envelope:

$$r(t) = I(t) + jQ(t) = A(t)e^{j\phi(t)}$$

The instantaneous phase is:

$$\phi(t) = \arctan\left(\frac{Q(t)}{I(t)}\right)$$

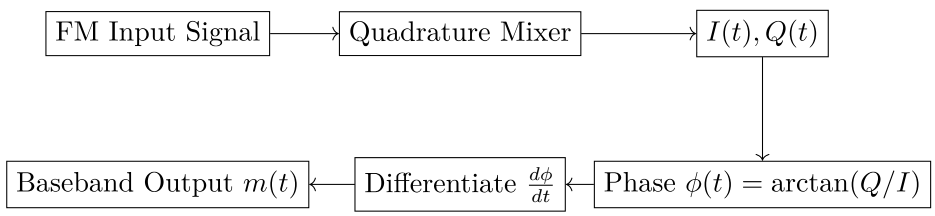
Differentiating gives the instantaneous frequency:

$$f_{\text{inst}}(t) = \frac{1}{2\pi}\frac{d\phi(t)}{dt}$$

Subtracting f_c yields the recovered baseband $m(t)$:

$$\hat{m}(t) \propto f_{\text{inst}}(t) - f_c$$

Block Diagram



Relay Feedback 自動整定（Åström–Hägglund）

關鍵公式

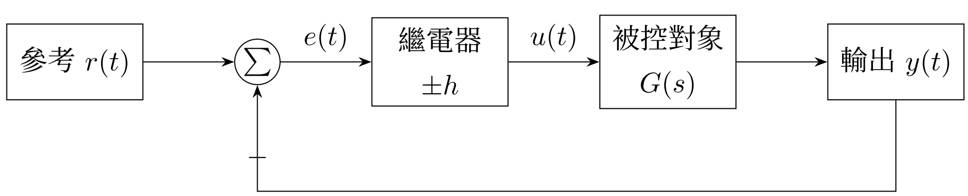
理想繼電器輸出幅度為 $\pm h$，若閉迴路產生穩定極限循環，輸出正弦近似幅度為 a、週期為 $P_u = 2\pi/\omega_u$，則

$$N(a) = \frac{4h}{\pi a}, \qquad G(j\omega_u)\,N(a) = -1$$

由此可得臨界增益

$$K_u = \frac{4h}{\pi a}.$$

控制框圖



說明

- 將原控制器暫時以繼電器取代，閉迴路自然激發極限循環。
- 量測輸出振幅 a 與週期 P_u，由 $K_u = \frac{4h}{\pi a}$ 推得臨界增益，再套用 Ziegler–Nichols 或其他整定規則得到 PID 參數。

===

44 The Role of $T/4$ Delay and $\pi/4$ Phase Bias

44.1 Phase Difference Analysis

Consider a narrowband signal with a slowly varying phase:

$$s(t) = A\cos\bigl(\omega_c t + \phi(t)\bigr). \tag{61}$$

The signal is delayed by a quarter of the carrier period,

$$\tau = \frac{T}{4} = \frac{\pi}{2\omega_c}, \tag{62}$$

resulting in

$$s(t-\tau) = A\cos\Bigl(\omega_c t - \frac{\pi}{2} + \phi(t-\tau)\Bigr). \tag{63}$$

To set the operating point at $\pi/4$, an additional fixed phase bias of $\pi/4$ is applied to the delayed path. The biased signal becomes

$$s_b(t) = A\cos\Bigl(\omega_c t - \frac{\pi}{4} + \phi(t-\tau)\Bigr). \tag{64}$$

44.2 Multiplier Output

The product of the original and biased delayed signals is

$$\begin{aligned} s(t)\, s_b(t) = \frac{A^2}{2}\Bigl[&\cos\Bigl(2\omega_c t - \frac{\pi}{4} + \phi(t) + \phi(t-\tau)\Bigr) \\ &+ \cos\Bigl(\frac{\pi}{4} - \bigl[\phi(t) - \phi(t-\tau)\bigr]\Bigr)\Bigr]. \end{aligned} \tag{65}$$

44.3 Low-Pass Filtered Output

After low-pass filtering, the high-frequency component is removed, leaving

$$\text{Output} \propto \cos\Bigl(\frac{\pi}{4} - \Delta\phi(t)\Bigr), \tag{66}$$

where

$$\Delta\phi(t) = \phi(t) - \phi(t-\tau). \tag{67}$$

45 FM Discriminator Based on Quarter-Period Delay and Multiplication

45.1 FM Signal Model

Consider a standard FM signal

$$u(t) = A\cos\big(\omega_c t + \phi(t)\big), \tag{68}$$

where ω_c is the carrier angular frequency and $\phi(t)$ is the information-bearing phase term. The instantaneous angular frequency is

$$\omega_i(t) = \frac{d}{dt}\big(\omega_c t + \phi(t)\big) = \omega_c + \dot{\phi}(t). \tag{69}$$

FM demodulation aims to recover $\dot{\phi}(t)$.

45.2 Quarter-Period Delay

Let the carrier period be

$$T = \frac{2\pi}{\omega_c}, \tag{70}$$

so that a quarter-period delay is

$$\frac{T}{4} = \frac{\pi}{2\omega_c}. \tag{71}$$

The delayed signal is then

$$u\left(t - \frac{T}{4}\right) = A\cos\left(\omega_c t - \frac{\pi}{2} + \phi\left(t - \frac{T}{4}\right)\right). \tag{72}$$

Assuming narrowband FM (slowly varying phase),

$$\phi\left(t - \frac{T}{4}\right) \approx \phi(t) - \dot{\phi}(t)\frac{T}{4}. \tag{73}$$

45.3 Multiplier Output

The product of the original and delayed signals is

$$u(t)\,u\left(t - \frac{T}{4}\right) = A^2\cos\big(\omega_c t + \phi(t)\big)\cos\left(\omega_c t - \frac{\pi}{2} + \phi(t) - \dot{\phi}(t)\frac{T}{4}\right). \tag{74}$$

Using the trigonometric identity

$$\cos a \cos b = \frac{1}{2}\big[\cos(a-b) + \cos(a+b)\big],$$

the product consists of:

- a high-frequency term near $2\omega_c$ (removed by low-pass filtering),
- a low-frequency term given by
$$\frac{A^2}{2}\cos\left(\frac{\pi}{2} + \dot{\phi}(t)\frac{T}{4}\right) = -\frac{A^2}{2}\sin\left(\dot{\phi}(t)\frac{T}{4}\right). \tag{75}$$

45.4 Frequency Discrimination

For small modulation index (narrowband FM),

$$\sin x \approx x, \tag{76}$$

hence

$$u(t)\,u\left(t - \frac{T}{4}\right) \;\propto\; -\dot{\phi}(t). \tag{77}$$

Since $\dot{\phi}(t) = \omega_i(t) - \omega_c$, the low-pass filtered output is proportional to the instantaneous frequency deviation:

$$\boxed{\mathrm{LPF}\left\{u(t)\,u\left(t - \frac{T}{4}\right)\right\} \;\propto\; \omega_i(t) - \omega_c}. \tag{78}$$

45.5 Interpretation

This structure acts as an FM discriminator by:

1. introducing a 90° phase shift via a quarter-period delay,
2. converting phase differences into amplitude variations through multiplication,
3. extracting the baseband signal by low-pass filtering.

Thus, the operation $u(t) \times u(t - T/4)$ implements a classical multiplier-based FM discriminator.

46 First-Order Approximation of Delayed Phase

Consider a differentiable phase function $\phi(t)$. Using the Taylor series expansion around time t, we have

$$\phi(t-\Delta) = \phi(t) - \dot{\phi}(t)\Delta + \frac{1}{2}\ddot{\phi}(t)\Delta^2 + \cdots \tag{79}$$

where $\dot{\phi}(t) = \frac{d\phi(t)}{dt}$ and Δ is a small time delay.

By letting $\Delta = 4T$, the expression becomes

$$\phi(t-4T) = \phi(t) - 4T\dot{\phi}(t) + \frac{1}{2}\ddot{\phi}(t)(4T)^2 + \cdots \tag{80}$$

When T is sufficiently small and $\phi(t)$ varies slowly with time, higher-order terms can be neglected. Therefore, a first-order approximation is obtained as

$$\phi(t-4T) \approx \phi(t) - 4T\dot{\phi}(t). \tag{81}$$

This approximation is widely used in communication and radar systems, such as phase noise modeling, carrier frequency offset (CFO) analysis, and Doppler effect linearization.

47 Connection to the I/Q Demodulation Architecture

47.1 Quarter-Period Delay Interpretation

In the RF domain, a quarter-period delay corresponds to a 90° phase shift:

$$\cos\left(\omega_c t - \frac{\pi}{2}\right) = \sin(\omega_c t).$$

Thus, the classical FM discriminator based on the operation

$$u(t) \times u\left(t - \frac{T}{4}\right)$$

implements frequency discrimination by exploiting a 90° phase difference.

47.2 I/Q Demodulation as an Implicit $T/4$ Delay

In an I/Q demodulation architecture, the received signal $u(t)$ is mixed with two orthogonal local oscillators:

$$\cos(\omega_c t) \quad \text{and} \quad \sin(\omega_c t),$$

followed by low-pass filtering. This process inherently introduces the same 90° phase separation as a $T/4$ delay.

47.3 Baseband I/Q Signals for an FM Wave

For an FM signal

$$u(t) = A\cos\bigl(\omega_c t + \phi(t)\bigr),$$

the resulting baseband components are

$$I(t) = A\cos\bigl(\phi(t)\bigr), \qquad Q(t) = A\sin\bigl(\phi(t)\bigr).$$

Together, they form the complex baseband signal

$$z(t) = I(t) + jQ(t) = Ae^{j\phi(t)}.$$

47.4 FM Discrimination in the I/Q Domain

The instantaneous phase is given by $\phi(t) = \arg\{z(t)\}$, and the instantaneous frequency deviation is proportional to its time derivative. In terms of $I(t)$ and $Q(t)$, the discriminator output can be written as

$$\omega_i(t) - \omega_c \;\propto\; I(t)\,\dot{Q}(t) - Q(t)\,\dot{I}(t).$$

This expression measures the rotational speed of the I/Q vector in the complex plane, which directly corresponds to the instantaneous frequency deviation of the FM signal.

47.5 Equivalence to the $T/4$ Delay Multiplier

The RF-domain operation $u(t) \times u(t - T/4)$ and the I/Q-domain discriminator are mathematically equivalent:

- the $T/4$ delay at RF corresponds to the quadrature local oscillator in I/Q demodulation,
- multiplication in the RF domain corresponds to the cross-product $I(t)\dot{Q}(t) - Q(t)\dot{I}(t)$,
- low-pass filtering at RF is naturally achieved by baseband processing in the I/Q domain.

Therefore, the I/Q-based FM discriminator can be viewed as a clean, baseband realization of the classical quarter-period delay FM discriminator.

48 Block Diagram Figures for FM Discriminator Implementations

This section provides LaTeX figure examples (using `tikz`) to illustrate: (i) the classical RF $T/4$-delay multiplier discriminator, (ii) the I/Q demodulation-based discriminator concept, and (iii) the discrete-time (FPGA) conjugate-product discriminator front-end.

48.1 Required Package

Add the following to the LaTeX preamble:

```
\usepackage{tikz}
\usetikzlibrary{arrows.meta,positioning,calc}
```

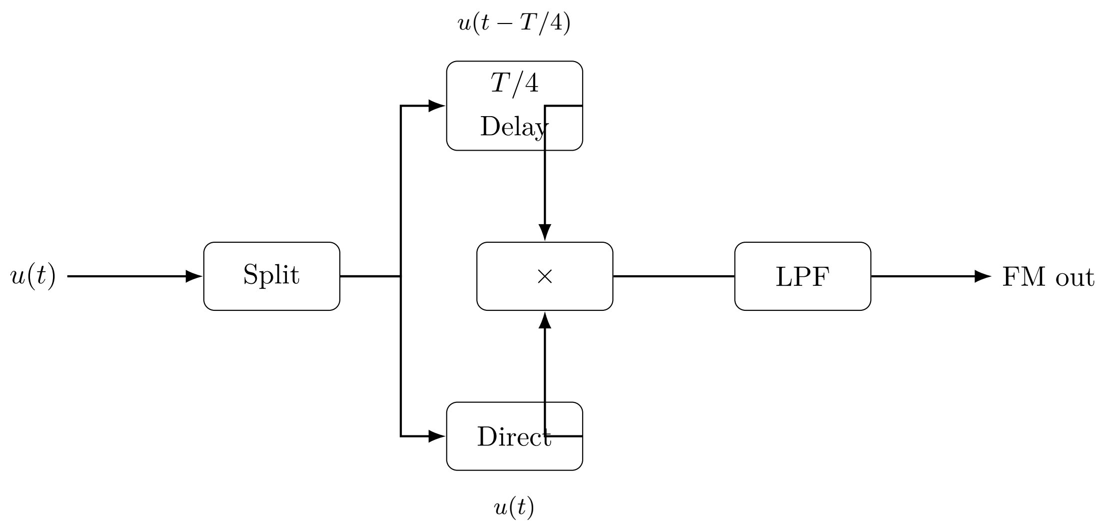


图 1: Classical RF FM discriminator using quarter-period delay and multiplication. After low-pass filtering, the output is proportional to the instantaneous frequency deviation for narrowband FM.

48.2 RF $T/4$-Delay Multiplier FM Discriminator

48.3 I/Q Demodulation View (Implicit $90°$ Phase Separation)

48.4 Discrete-Time (FPGA) Conjugate-Product Discriminator Front-End

49 Derivation of the I/Q-Based FM Discriminator

49.1 Complex Baseband Representation

Let the complex baseband signal be defined as

$$z(t) = I(t) + jQ(t). \tag{82}$$

Equivalently, it can be expressed in polar form as

$$z(t) = A(t)e^{j\phi(t)}, \tag{83}$$

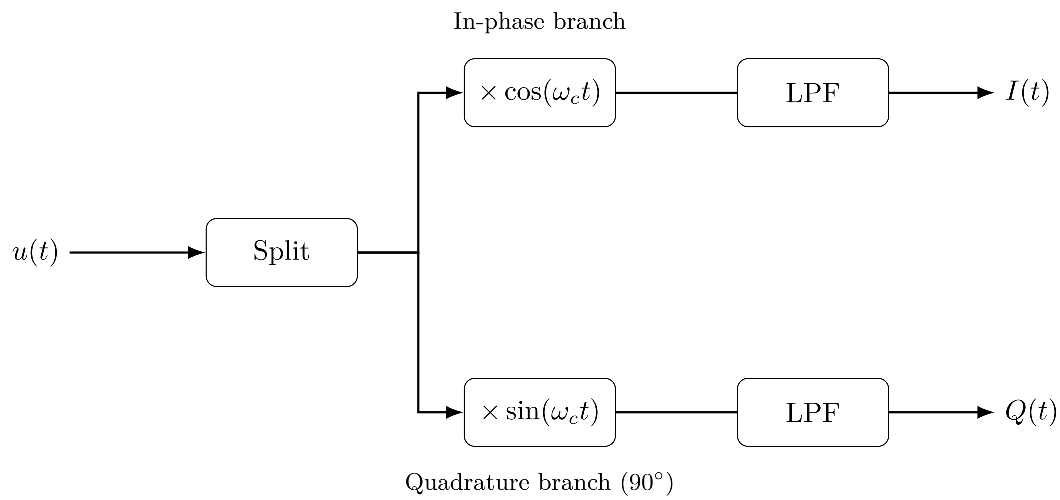


图 2: I/Q demodulation produces orthogonal baseband components. The quadrature LO provides an implicit 90° phase separation equivalent to a $T/4$ delay at RF.

where $A(t)$ is the signal amplitude and $\phi(t)$ is the instantaneous phase. By definition,

$$\boxed{\phi(t) = \arg\{z(t)\}.} \tag{84}$$

49.2 Instantaneous Frequency

The instantaneous angular frequency of the passband signal is given by

$$\omega_i(t) = \frac{d}{dt}\big(\omega_c t + \phi(t)\big) = \omega_c + \dot{\phi}(t), \tag{85}$$

so that the frequency deviation relative to the carrier is

$$\omega_i(t) - \omega_c = \dot{\phi}(t). \tag{86}$$

49.3 Phase in Terms of I/Q Components

From the complex baseband representation,

$$\phi(t) = \tan^{-1}\left(\frac{Q(t)}{I(t)}\right). \tag{87}$$

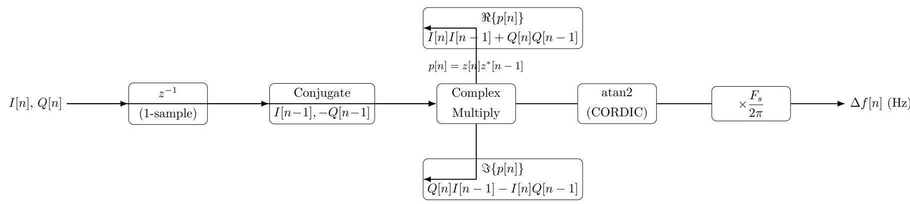


图 3: Discrete-time (FPGA) FM discriminator using the one-sample conjugate product. The complex multiply produces $\Re\{p[n]\}$ and $\Im\{p[n]\}$, followed by atan2 (typically CORDIC) to obtain $\Delta\phi[n]$, then scaling by $F_s/(2\pi)$ to output frequency offset $\Delta f[n]$ in Hz.

49.4 Time Derivative of the Phase

Taking the time derivative and applying the chain rule yields

$$\dot{\phi}(t) = \frac{1}{1 + \left(\frac{Q(t)}{I(t)}\right)^2} \cdot \frac{I(t)\dot{Q}(t) - Q(t)\dot{I}(t)}{I^2(t)} \tag{88}$$

$$= \frac{I(t)\dot{Q}(t) - Q(t)\dot{I}(t)}{I^2(t) + Q^2(t)}. \tag{89}$$

49.5 I/Q-Based FM Discriminator

Combining the above results, the instantaneous frequency deviation can be written as

$$\boxed{\omega_i(t) - \omega_c = \frac{I(t)\dot{Q}(t) - Q(t)\dot{I}(t)}{I^2(t) + Q^2(t)}}. \tag{90}$$

49.6 Constant-Amplitude Approximation

In many practical FM receivers, automatic gain control (AGC) ensures that the signal amplitude is approximately constant, i.e.,

$$I^2(t) + Q^2(t) \approx A^2 = \text{constant}.$$

Under this assumption, the denominator can be absorbed into a proportionality constant, yielding the commonly used form

$$\boxed{\omega_i(t) - \omega_c \;\propto\; I(t)\dot{Q}(t) - Q(t)\dot{I}(t).} \tag{91}$$

49.7 Geometric Interpretation

The term $I(t)\dot{Q}(t) - Q(t)\dot{I}(t)$ corresponds to the z-component of the two-dimensional cross product between the vector $(I(t), Q(t))$ and its time derivative $(\dot{I}(t), \dot{Q}(t))$. This quantity represents the angular velocity of the rotating I/Q vector in the complex plane and therefore directly encodes the instantaneous frequency deviation.

- The resonator is bandpass, centered near ω_n.

50 Square-Wave Injection Signal and Fundamental Component

Consider a standard symmetric square-wave injection signal defined as

$$u_{\text{inj}}(t) = A_{\text{inj}} \cdot \operatorname{sgn}(\sin \omega t). \tag{92}$$

The Fourier series expansion of this square wave is given by

$$u_{\text{inj}}(t) = \frac{4A_{\text{inj}}}{\pi} \left[\sin(\omega t) + \frac{1}{3}\sin(3\omega t) + \frac{1}{5}\sin(5\omega t) + \cdots \right]. \tag{93}$$

50.1 Fundamental Component

Retaining only the fundamental (first-harmonic) component, the injection signal can be approximated as

$$u_{\text{inj}}^{(1)}(t) = \frac{4A_{\text{inj}}}{\pi} \sin(\omega t). \tag{94}$$

Therefore, the amplitude of the fundamental component is

$$B = \frac{4A_{\text{inj}}}{\pi}. \tag{95}$$

51 Frequency Shift Model Based on Adler's Equation

According to Adler' s equation [**?**], the instantaneous oscillation frequency $\omega(t)$ of an injection-locked oscillator is related to the phase difference $\theta(t)$ between the oscillation signal and the injected signal.

51.1 Adler's Phase Equation

For a weakly injected, high-Q resonator, the phase dynamics are governed by

$$\dot{\theta}(t) = \omega(t) - \omega_n - \frac{\omega_n}{2Q}\frac{B}{A}\sin\theta(t), \tag{96}$$

where ω_n is the natural resonance frequency, Q is the quality factor of the resonator, B is the amplitude of the injected sinusoidal signal u_{inj}, and A is the amplitude of the oscillation signal u.

51.2 Quasi-Static Approximation

For a high-Q resonator, the phase $\theta(t)$ evolves slowly compared to the oscillation period. Under this quasi-static assumption, $\dot{\theta}(t)$ can be neglected, yielding an approximate algebraic relationship between frequency and phase:

$$\omega(t) - \omega_n \approx -\frac{\omega_n}{2Q}\frac{B}{A}\sin\theta(t). \tag{97}$$

51.3 Physical Interpretation

Equation (97) describes the frequency pulling effect induced by the injected signal. The frequency deviation from the natural resonance frequency ω_n is proportional to the sine of the phase difference $\theta(t)$. The proportionality constant depends on the resonator stiffness through Q and on the relative injection strength through the amplitude ratio B/A.

This reduced phase-domain model corresponds to the frequency-shift block used in the system-level block diagram.

52 Demodulating Filter Transfer Function

The demodulating filter $F(s)$ is formed by cascading two second-order lowpass Butterworth filters with cutoff frequencies of 13 kHz and 19 kHz, which together give a cutoff frequency of 11.9 kHz.

$$F(s) = \frac{9.51 \times 10^{19}}{s^4 + 2.84 \times 10^5 s^3 + 4.04 \times 10^{10} s^2 + 2.77 \times 10^{15} s + 9.51 \times 10^{19}} \tag{98}$$

52.1 Cutoff Frequency

The cutoff frequency is:

$$\omega_c = 2\pi \times 11900 \text{ rad/s} \tag{99}$$

53 Plant Model

The plant model is given by:

$$P(s) = -ke^{-\frac{T}{8}s}F(s) \tag{100}$$

where the gain k equals the resonator bandwidth (in Hz):

$$k = \frac{f_n}{Q} = \frac{1}{QT} \tag{101}$$

with $f_n = \omega_n/(2\pi)$.

54 Plant Gain Function

The plant gain function is given by:

$$g(\theta, A_{inj}) = \frac{2A_{inj}\cos(\theta)[1 + A_{inj}\cos(\theta)]\omega_n}{\pi Q} \tag{102}$$

55 Derivation of Simplified Plant Model

We derive the simplified plant model by substituting $\theta = \pi$ and $A_{inj} = 0.5$.

55.1 Step 1: Substitute $\theta = \pi$

Since $\cos(\pi) = -1$:

$$g(\pi, A_{inj}) = \frac{2A_{inj}(-1)[1 + A_{inj}(-1)]\omega_n}{\pi Q} \tag{103}$$

$$g(\pi, A_{inj}) = \frac{-2A_{inj}[1 - A_{inj}]\omega_n}{\pi Q} \tag{104}$$

55.2 Step 2: Substitute $A_{inj} = 0.5$

$$g(\pi, 0.5) = \frac{-2(0.5)[1 - 0.5]\omega_n}{\pi Q} \tag{105}$$

$$g(\pi, 0.5) = \frac{-1 \times 0.5 \times \omega_n}{\pi Q} \tag{106}$$

$$g(\pi, 0.5) = \frac{-0.5\omega_n}{\pi Q} \tag{107}$$

55.3 Step 3: Simplify

$$g(\pi, 0.5) = -\frac{\omega_n}{2\pi Q} = -\frac{f_n}{Q} \tag{108}$$

Since $f_n = \frac{\omega_n}{2\pi}$, we get:

$$g(\pi, 0.5) = -\frac{f_n}{Q} = -\frac{1}{QT} = -k \tag{109}$$

56 Final Plant Model

Therefore, the plant model becomes:

$$P(s) = g(\theta, A_{inj}) \cdot e^{-\frac{T}{8}s} F(s) = -ke^{-\frac{T}{8}s} F(s) \tag{110}$$

where the gain k equals the resonator bandwidth (in Hz):

$$k = \frac{f_n}{Q} = \frac{1}{QT} \tag{111}$$

with $f_n = \frac{\omega_n}{2\pi}$ and $T = \frac{1}{f_n}$.

57 Original Plant Model

The plant model derived from the SIL radar is:

$$P(s) = -ke^{-\frac{T}{8}s} F(s) \tag{112}$$

where $F(s)$ is a 4th-order lowpass Butterworth filter:

$$F(s) = \frac{9.51 \times 10^{19}}{s^4 + 2.84 \times 10^5 s^3 + 4.04 \times 10^{10} s^2 + 2.77 \times 10^{15} s + 9.51 \times 10^{19}} \tag{113}$$

This high-order model with time delay is complex for controller design.

58 First-Order Approximation

The plant is approximated by a simpler first-order model:

$$\hat{P}(s) = \frac{-k\omega_c}{s + \omega_c} \tag{114}$$

where:

- $k = \frac{f_n}{Q} = \frac{1}{QT}$ is the DC gain
- $\omega_c = 2\pi \times 11900$ rad/s is the cutoff frequency

59 Why This Approximation is Valid

59.1 DC Gain Matching

For the original model at $s = 0$:

$$P(0) = -k \cdot e^0 \cdot F(0) = -k \cdot 1 \cdot 1 = -k \tag{115}$$

For the approximation at $s = 0$:

$$\hat{P}(0) = \frac{-k\omega_c}{\omega_c} = -k \tag{116}$$

Both have the same DC gain.

59.2 Cutoff Frequency Matching

Both models have the same cutoff frequency ω_c, meaning they attenuate signals similarly at higher frequencies.

59.3 Low-Frequency Assumption

At low frequencies:

- Time delay: $e^{-\frac{T}{8}s} \approx 1$
- Higher-order terms in $F(s)$ are negligible

Therefore, the first-order model accurately represents the plant behavior in the frequency range of interest (where target motion occurs).

60 Benefit for Controller Design

With the simplified model $\hat{P}(s)$, the PI controller can be designed to cancel the plant dynamics:

$$C(s) = k_I \cdot \frac{1}{s} + k_p = \frac{k_p s + k_I}{s} \tag{117}$$

The open-loop transfer function becomes:

$$C(s)\hat{P}(s) = \left(\frac{k_p s + k_I}{s}\right)\left(\frac{-k\omega_c}{s + \omega_c}\right) \tag{118}$$

By choosing k_p and k_I appropriately (as in equation 12 of the paper):

$$k_I = \frac{\omega_{BW}}{k}, \quad k_p = \frac{k_I}{\omega_c} \tag{119}$$

The controller cancels the pole at $s = -\omega_c$, yielding:

$$C(s)\hat{P}(s) = \frac{-\omega_{BW}}{s} \tag{120}$$

This is a simple integrator with bandwidth ω_{BW}.

61 Benefits of Pole-Zero Cancellation

61.1 Simplified Open-Loop Transfer Function

Before cancellation:

$$C(s)\hat{P}(s) = \frac{\omega_{BW}(s + \omega_c)}{k\omega_c \cdot s} \cdot \frac{-k\omega_c}{s + \omega_c} \tag{121}$$

After cancellation:

$$C(s)\hat{P}(s) = \frac{-\omega_{BW}}{s} \tag{122}$$

This is a pure integrator.

61.2 Predictable Closed-Loop Behavior

$$T(s) = \frac{C(s)\hat{P}(s)}{1 + C(s)\hat{P}(s)} = \frac{-\omega_{BW}}{s - \omega_{BW}} \tag{123}$$

Properties:

- Time constant: $\tau = 1/\omega_{BW}$
- No overshoot
- No oscillations

61.3 Guaranteed Stability Margins

Metric	**Value**	**Quality**
Phase Margin	90°	Excellent
Gain Margin	∞	Excellent
Crossover Frequency	ω_{BW}	Controllable

61.4 Zero Steady-State Error

For step disturbance:

$$e_{ss} = \lim_{s \to 0} \frac{s}{s + \omega_{BW}} = 0 \tag{124}$$

61.5 Single Design Parameter

Only ω_{BW} needs to be chosen:

- Faster response → increase ω_{BW}
- More stability → decrease ω_{BW}

61.6 Comparison Table

Aspect	Without	With
Open-loop order	Higher	1st order
Design parameters	Multiple	Single
Phase margin	Varies	90°
Gain margin	Varies	∞
Steady-state error	Non-zero	Zero
Tuning complexity	High	Low

62 Starting Equations

62.1 PI Controller

$$C(s) = k_p + \frac{k_I}{s} = \frac{k_p s + k_I}{s} \tag{125}$$

62.2 First-Order Plant Approximation

$$\hat{P}(s) = \frac{-k\omega_c}{s + \omega_c} \tag{126}$$

62.3 Controller Gains

$$k_I = \frac{\omega_{BW}}{k}, \quad k_p = \frac{k_I}{\omega_c} \tag{127}$$

63 Derivation

63.1 Step 1: Express k_p in Terms of ω_{BW}

$$k_p = \frac{k_I}{\omega_c} = \frac{\omega_{BW}}{k \cdot \omega_c} \tag{128}$$

63.2 Step 2: Substitute into $C(s)$

$$C(s) = \frac{k_p s + k_I}{s} = \frac{\frac{\omega_{BW}}{k\omega_c} s + \frac{\omega_{BW}}{k}}{s} \tag{129}$$

Factor out $\frac{\omega_{BW}}{k\omega_c}$:

$$C(s) = \frac{\frac{\omega_{BW}}{k\omega_c}(s + \omega_c)}{s} = \frac{\omega_{BW}(s + \omega_c)}{k\omega_c \cdot s} \tag{130}$$

63.3 Step 3: Multiply $C(s) \cdot \hat{P}(s)$

$$C(s)\hat{P}(s) = \frac{\omega_{BW}(s + \omega_c)}{k\omega_c \cdot s} \cdot \frac{-k\omega_c}{s + \omega_c} \tag{131}$$

63.4 Step 4: Cancel Common Terms

$$C(s)\hat{P}(s) = \frac{\omega_{BW} \cdot \cancel{(s + \omega_c)} \cdot (-\cancel{k\omega_c})}{\cancel{k\omega_c} \cdot s \cdot \cancel{(s + \omega_c)}} \tag{132}$$

63.5 Step 5: Final Result

$$\boxed{C(s)\hat{P}(s) = \frac{-\omega_{BW}}{s}} \tag{133}$$

64 Key Insight: Pole-Zero Cancellation

The PI controller is designed to cancel the plant pole:

- Plant pole at: $s = -\omega_c$
- Controller zero at: $s = -\frac{k_I}{k_p} = -\omega_c$

Verification:

$$\text{Controller zero} = -\frac{k_I}{k_p} = -\frac{\omega_{BW}/k}{\omega_{BW}/(k\omega_c)} = -\omega_c \quad \checkmark \tag{134}$$

65 Physical Interpretation

The open-loop transfer function $C(s)\hat{P}(s) = \frac{-\omega_{BW}}{s}$ is a pure integrator with gain ω_{BW}.

- At $\omega = \omega_{BW}$: $|C(j\omega)\hat{P}(j\omega)| = 1$ (unity gain crossover)
- Phase: $-90°$ (constant, from integrator)
- Phase margin: $90°$ (ideal)