In a general nonlinear state-space system

$$\dot{x} = f(t, x, u), \qquad y = h(t, x, u),$$

an *equilibrium point* (x_e, u_e) is a state–input pair such that, if the system starts there, it remains there forever. That is,

$$\dot{x} = 0 \quad \text{when } x = x_e,\ u = u_e,$$

or equivalently,

$$f(t, x_e, u_e) = 0.$$

The corresponding output at equilibrium is

$$y_e = h(t, x_e, u_e).$$

Isolated vs. continuum of equilibrium points.

- **Isolated equilibrium:** There is only one equilibrium point in its neighborhood. Example:

  $$\dot{x} = -x,$$

  which has a single equilibrium at $x = 0$.

- **Continuum of equilibria:** There exists an infinite (continuous) set of equilibrium points forming a line, plane, or manifold. Example:

  $$\dot{x}_1 = 0, \qquad \dot{x}_2 = -x_2,$$

  where any state of the form $(x_1, 0)$ is an equilibrium. This is called a *continuum of equilibrium points* (平衡點連續).

Linear systems. For a linear time-varying system,

$$\dot{x} = A(t)x + B(t)u, \qquad y = C(t)x + D(t)u,$$

which is a special case of the nonlinear model above where f and h are linear in x and u.

Equilibria in linear systems. For the time-invariant case,

$$0 = Ax_e + Bu_e.$$

- If A is invertible, there is a single (isolated) equilibrium:

$$x_e = -A^{-1}Bu_e.$$

- If A is singular, there exists a *continuum of equilibria.*

Multiple isolated equilibria. A linear system can have only one isolated equilibrium point; thus, it has only one steady-state operating point that attracts all trajectories (if stable).

For a nonlinear system,

$$\dot{x} = f(x, u),$$

there may exist more than one isolated equilibrium point, i.e. multiple distinct solutions of

$$f(x_e, u_e) = 0.$$

The state of the system may converge to one of several steady-state operating points, depending on the *initial state of the system.* Each equilibrium has its own *basin of attraction.* **多個孤立平衡點（Multiple isolated equilibria）**

對於**線性系統**而言，它只能有一個孤立的平衡點，因此也只有一個穩態工作點（steady-state operating point）。若系統是穩定的，所有狀態最終都會收斂到該唯一的平衡點，與初始狀態無關。

對於**非線性系統**

$$\dot{x} = f(x, u),$$

可能存在多組滿足

$$f(x_e, u_e) = 0$$

的解 (x_e, u_e)，也就是**多個孤立平衡點**。

系統的狀態可能會根據**初始狀態**不同，而收斂到不同的平衡點。每個平衡點附近的區域稱為其**吸引域（basin of attraction）**。

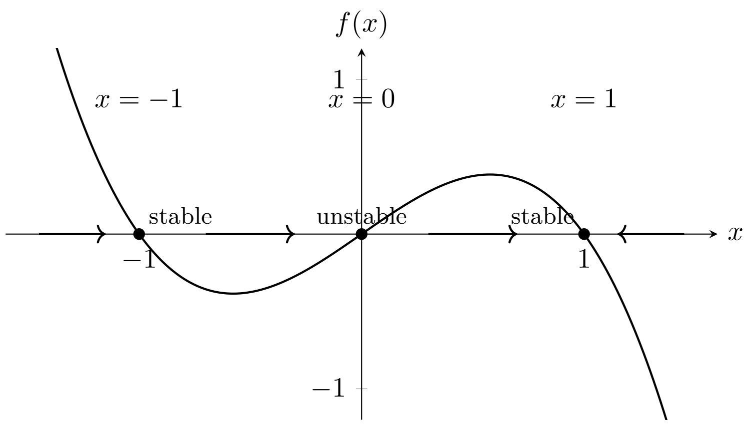
舉例：

$$\dot{x} = x - x^3$$

滿足平衡條件 $0 = x - x^3$，得到三個孤立平衡點：

$$x_e = 0,\ \pm 1.$$

此系統中，初始狀態決定了最終收斂到哪一個平衡點。



不穩定平衡（Unstable Equilibrium）

考慮一個一維自主系統：

$$\dot{x} = f(x).$$

若存在某一點 $x = x_e$ 使得

$$f(x_e) = 0,$$

則 x_e 為系統的**平衡點（equilibrium point）**。當系統狀態位於該點時，$\dot{x} = 0$，狀態將保持不變。

穩定與不穩定的差異如下：

類型	微小擾動後的行為	幾何意義	物理比喻
穩定平衡 (Stable)	狀態回到原平衡點	周圍箭頭指向該點	小球在碗底
不穩定平衡 (Unstable)	狀態遠離原平衡點	周圍箭頭遠離該點	小球在山頂

數學上（Lyapunov 意義）：

若對任意 $\varepsilon > 0$，存在 $\delta > 0$，使得當 $|x(0) - x_e| < \delta$ 時，系統軌跡滿足 $|x(t) - x_e| < \varepsilon$ 對所有 $t > 0$ 皆成立，則稱 x_e **穩定**。反之，若存在再小的擾動都會使軌跡離開該點附近，則 x_e 為**不穩定平衡點**。

範例：

$$\dot{x} = x - x^3$$

其平衡條件為：

$$x - x^3 = 0 \quad \Rightarrow \quad x_e = -1, 0, 1.$$

取導數：

$$f'(x) = 1 - 3x^2.$$

代入：

$f'(-1) = -2 < 0 \Rightarrow$ 穩定, $\quad f'(0) = 1 > 0 \Rightarrow$ 不穩定, $\quad f'(1) = -2 < 0 \Rightarrow$ 穩定.

因此，$x = 0$ 為**不穩定平衡點**，而 $x = \pm 1$ 為**穩定平衡點**。

圖像化理解：

當$f(x) > 0 \Rightarrow \dot{x} > 0$(狀態向右移動），$\quad f(x) < 0 \Rightarrow \dot{x} < 0$(狀態向左移動）.

若在平衡點兩側，箭頭「遠離」該點，則該點為**不穩定平衡**。

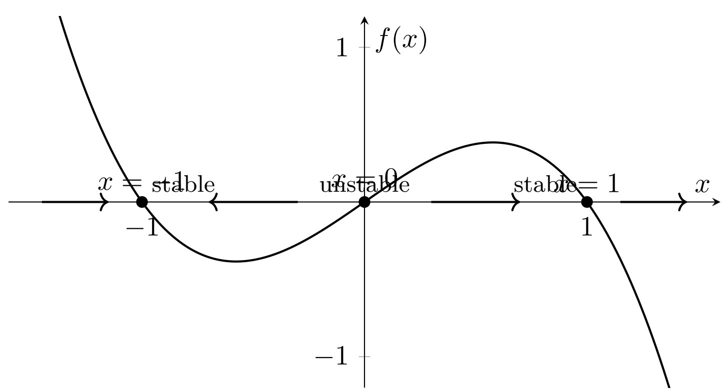


图 1: 相圖示意：$f(x) = x - x^3$，其中 $x = 0$ 為不穩定平衡，$x = \pm 1$ 為穩定平衡。

Tunnel Diode

A **tunnel diode** (also called an *Esaki diode*) is a special type of semiconductor diode that exhibits **negative resistance** due to the phenomenon of **quantum tunneling** inside its p–n junction.

1. Construction

A tunnel diode is fabricated similarly to a normal p–n junction diode, but with **extremely heavy doping** on both sides — about 10^3 times higher than that of an ordinary diode.

Because of the very high doping concentration:

- The **depletion layer** becomes extremely narrow (a few nanometers).
- Electrons can **tunnel** through the potential barrier even when they do not have enough classical energy to overcome it.

This tunneling mechanism gives rise to a unique current–voltage characteristic.

2. Current–Voltage Characteristic

The *i*–*v* curve of a tunnel diode has three distinct regions:

Region	Voltage range	Behavior / Description
1.Tunneling region	Small forward bias	Current rises rapidly due to tunneling.
2.Negative resistance region	Medium bias	Current decreases as voltage increases.
3.Forward conduction region	High bias	Current rises again (normal diode behavior).

Graphically, it appears as an "N" or "S" shaped curve:

$$\frac{di}{dv} < 0 \quad \text{in the negative resistance region.}$$

3. Negative Resistance

In the middle region, as the voltage v increases, the current i actually decreases, giving

$$\frac{di}{dv} < 0$$

This property is known as **negative differential resistance**, which allows the tunnel diode to:

- Amplify small signals,
- Generate oscillations,
- Operate at extremely high switching speeds.

4. Typical Parameters

$$\begin{aligned}
V_p &: \text{ Peak voltage (typically 0.1–0.3 V)} \\
I_p &: \text{ Peak current (a few mA)} \\
V_v &: \text{ Valley voltage (0.3–0.6 V)} \\
I_v &: \text{ Valley current, smaller than } I_p
\end{aligned}$$

5. Applications

Thanks to its negative resistance and high-speed behavior, tunnel diodes are widely used in:

- High-frequency oscillators,
- Microwave amplifiers,
- Fast-switching circuits,
- Bistable multivibrators (memory elements).

6. Historical Note

The tunnel diode was invented by **Leo Esaki** in 1957. He was awarded the **Nobel Prize in Physics** in 1973 for discovering the tunneling phenomenon in semiconductors.

Summary:

A tunnel diode is a heavily doped p–n junction device that exhibits negative resistance due to quantum

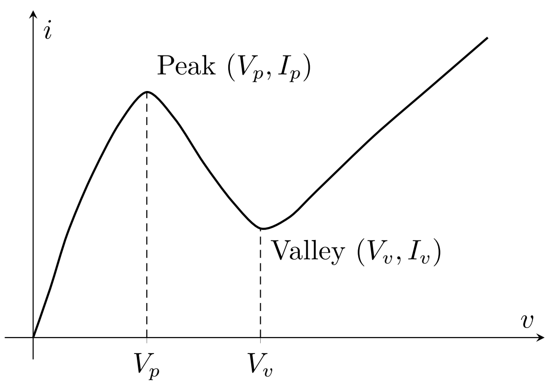


图 2: Tunnel diode i–v characteristic showing the negative-resistance region between V_p and V_v.

Symmetric Tunnel-Diode Characteristic

Consider a circuit consisting of two **tunnel diodes** connected in opposite directions, each in series with a 0.3 V bias source:

$$\text{Positive branch: Diode conducts for } v > +0.3\,\text{V}, \quad \text{Negative branch: Diode conducts for } v < -0.3\,\text{V}.$$

Because of the opposite orientations, the overall element exhibits a **symmetric nonlinear current–voltage characteristic** described by

$$i = h(v), \qquad \text{with } h(-v) = -h(v).$$

1. Right-hand side of the curve $(v > 0)$

When the applied voltage v exceeds approximately +0.3 V:

- The tunnel diode oriented for forward conduction becomes active.

- The current i first increases rapidly (tunneling region), then decreases (negative-resistance region), and finally rises again (normal forward conduction).

This produces the **right-hand lobe** of the $i-v$ curve.

2. Left-hand side of the curve ($v < 0$)

When v becomes less than approximately -0.3 V:

- The oppositely oriented tunnel diode is forward biased.
- It exhibits the same behavior as the first diode but with reversed polarity.

Thus, the left-hand side of the curve is the **mirror image** of the right-hand side.

3. Around $v = 0$

For small voltages $|v| < 0.3$ V, both diodes are reverse biased, and the current is negligible ($i \approx 0$). The overall curve passes smoothly through the origin.

4. Combined characteristic

The complete $i = h(v)$ characteristic is shown below.

Summary:
Each half of the curve corresponds to one tunnel diode's forward-bias behavior. The right side arises from the diode conducting for positive voltages, while the left side is its mirror image for negative voltages. Together they form a symmetric nonlinear element with regions of **negative differential**

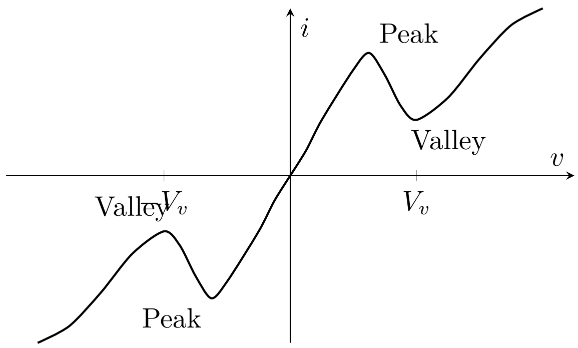
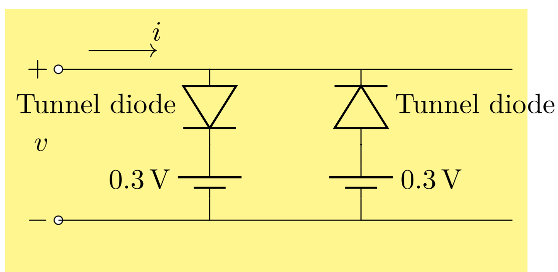


图 3: Symmetric tunnel-diode characteristic $i = h(v)$ produced by two oppositely oriented tunnel diodes.



resistance.

Relation between the Circuit and its i–v Characteristic

The circuit below consists of two **tunnel diodes** connected in opposite directions, each in series with a small bias source of 0.3 V. It realizes a symmetric nonlinear current–voltage characteristic described by

$$i = h(v), \qquad h(-v) = -h(v).$$

1. Circuit Operation

- When $v > +0.3$ V, the **left-hand tunnel diode** conducts. Its current rises rapidly (tunneling region), then falls (negative resistance), and finally rises again.

- When $v < -0.3$ V, the **right-hand tunnel diode** conducts with the same shape, but mirrored.
- Around $v = 0$, both diodes are reverse-biased and the current is nearly zero.

Thus, each diode produces one lobe of the i–v curve, and their combination yields a symmetric nonlinear function.

2. Circuit Realization

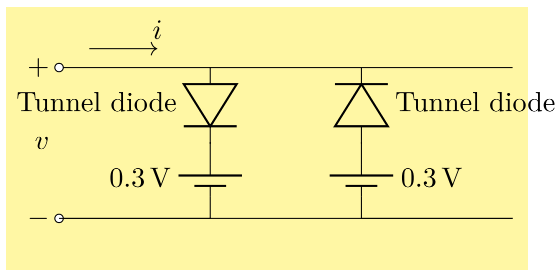


图 4: Circuit producing the nonlinear characteristic $i = h(v)$ using two oppositely oriented tunnel diodes.

3. Nonlinear Characteristic $i = h(v)$

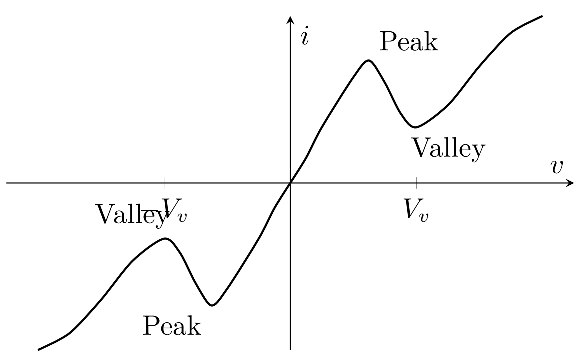


图 5: Symmetric i–v characteristic $i = h(v)$ resulting from the circuit. Each half corresponds to one tunnel diode.

4. Explanation

- The **right-hand branch** ($v > 0$) arises from the left diode' s forward characteristic.
- The **left-hand branch** ($v < 0$) arises from the right diode' s mirrored characteristic.
- Around $v = 0$, both diodes are reverse-biased, producing nearly zero current.

Thus, the overall element behaves as a **symmetric nonlinear resistor** with regions of **negative differential resistance**, as shown in the $i = h(v)$ plot.

Single Tunnel-Diode Circuit and its i–v Characteristic

The circuit shown below consists of a **tunnel diode** connected in series with a 0.3 V bias source. The external terminals are labeled by voltage v and current i.

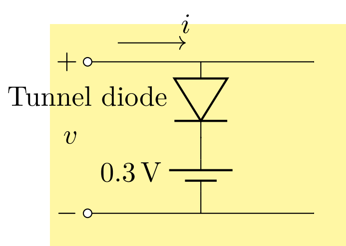


图 6: Single tunnel-diode circuit with a 0.3 V series bias source.

1. Voltage relationship

Let v_d denote the voltage across the tunnel diode itself. Because the 0.3 V battery is in series, the external voltage v is

$$v = v_d + 0.3, \quad \text{or equivalently} \quad v_d = v - 0.3.$$

Thus, the external current–voltage relationship becomes

$$i = f(v - 0.3),$$

where $f(v_d)$ is the intrinsic diode characteristic.

—

2. Effect on the i–v curve

The series 0.3 V bias source shifts the entire tunnel-diode characteristic **leftward by 0.3 V** on the voltage axis.

- When $v < 0$: the diode is reverse biased ($v_d < -0.3$ V), and $i \approx 0$.
- Around $v \approx -0.3$ V: the diode just begins to conduct (tunneling region).
- For larger v: the diode enters its negative-resistance region and then normal forward conduction.

—

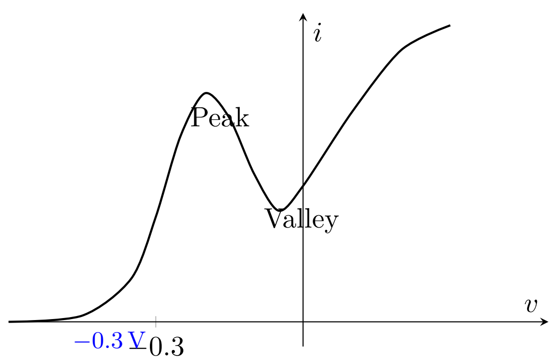


图 7: i–v characteristic of the single tunnel-diode circuit. The 0.3 V source shifts the curve left by 0.3 V.

3. Interpretation

The 0.3 V bias source provides a **DC offset** that moves the conduction region of the tunnel diode away from the origin. From the external terminals:

- For $0 \le v < 0.29$ V, the current is nearly zero.
- When $v \approx 0.3$ V, tunneling conduction begins.
- The external peak appears near $v = -0.3$ V, showing the leftward shift caused by the bias source.

$$\boxed{\text{The 0.3 V bias source shifts the tunnel diode's } i\text{–}v \text{ curve leftward by 0.3 V.}}$$

為什麼在 $v < 0.3\,\mathrm{V}$ 時出現負電流？

考慮下圖的電路：一個 **隧道二極體（tunnel diode）**串聯一個 $0.3\,\mathrm{V}$ 的偏壓電池。外部端點的電壓為 v，電流為 i。

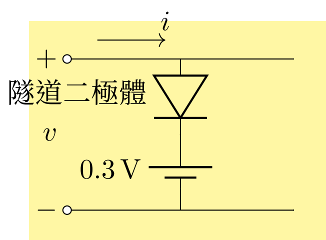


图 8: 含 0.3V 偏壓的隧道二極體電路。

—

1. 電壓關係

設隧道二極體兩端的實際電壓為 v_d。由於串聯一個 $0.3\,\mathrm{V}$ 的電池，有：

$$v = v_d + 0.3 \quad \Rightarrow \quad v_d = v - 0.3.$$

因此，當 $v < 0.3\,\mathrm{V}$ 時，二極體端電壓 $v_d < 0$。

即二極體處於**反向偏壓（reverse bias）**狀態。

—

2. 隧穿效應與反向電流

隧道二極體的耗盡層極薄（數奈米），因此即使在反向偏壓下，電子仍能利用 **量子隧穿效應（quantum tunneling）**穿越能障。

這種反向穿隧電流方向與定義的正電流相反，因此在外部量測上呈現為**負電流**：

$$i < 0 \quad (\text{在 } v < 0.3\,\mathrm{V} \text{ 時}).$$

當 $v > 0.3\,\mathrm{V}$ 時，二極體轉為正向偏壓，開始產生正向隧穿電流，電流變為正值並迅速上升。

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3. 外部 i–v 特性

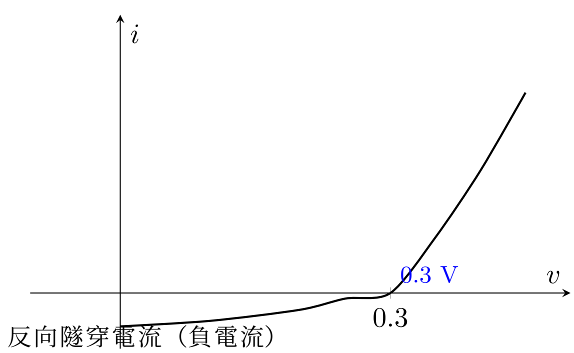

图 9: 在 $v < 0.3\,\mathrm{V}$ 時的反向隧穿電流導致 $i < 0$。

—

4. 物理解釋（能帶觀點）

- 當 $v < 0.3\,\mathrm{V}$ 時，二極體內部電壓 $v_d < 0$，其 n 區填滿的能態與 p 區空能態對齊，電子可穿隧從 p 區至 n 區，電流方向與外部定義的正電流相反，故 $i < 0$。
- 當 $v \approx 0.3\,\mathrm{V}$ 時，反向偏壓被抵消，隧穿電流趨近於零。

- 當 $v > 0.3\,\mathrm{V}$ 時，二極體正向偏壓，電子由 n 區注入 p 區，出現正向隧穿電流，$i > 0$。

—

5. 總結

結論

在 $v < 0.3\,\mathrm{V}$ 時，由於隧道二極體受到反向偏壓，產生方向相反的反向隧穿電流，因此量測到的外部電流為負值。

參考資料

$$M = [v_1, v_2], \qquad \overline{M^{-1}} = (M^{-1})^*.$$
$$M = [v_1, v_2], \qquad \overline{M^{-1}}.$$

推導

$$\dot{z}_1 = \lambda_1 z_1, \qquad \dot{z}_2 = \lambda_2 z_2.$$

其解為

$$z_1(t) = z_{10} e^{\lambda_1 t}, \qquad z_2(t) = z_{20} e^{\lambda_2 t}.$$

消去時間變數 t，由第一式得

$$t = \frac{1}{\lambda_1} \ln \frac{z_1}{z_{10}}.$$

代入第二式得

$$z_2 = z_{20} e^{\lambda_2 \left(\frac{1}{\lambda_1} \ln \frac{z_1}{z_{10}}\right)} = z_{20} \left(\frac{z_1}{z_{10}}\right)^{\lambda_2/\lambda_1} = c\, z_1^{\lambda_2/\lambda_1},$$

其中

$$c = z_{20} z_{10}^{-\lambda_2/\lambda_1}.$$