

# Self-Injection-Locked (SIL) Oscillator Analysis

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## 1 General Definition

**Resonance** occurs when a system is subjected to an external force or signal whose frequency matches the system’s **natural frequency**, resulting in a large increase in amplitude.

## 2 Real-World Examples

Context	Description
Violin string	Vibrates strongly when bowing matches its natural vibration frequency
RF circuits	LC circuits resonate at a specific frequency → used in filters, radios
Bridges	Tacoma Narrows Bridge collapsed due to wind-induced resonance
SIL radar	Resonator oscillates strongly at $\omega_n$ ; injection close to $\omega_n$ causes locking

## 3 In Engineering Terms

For a **second-order system** like an RLC circuit or mechanical spring-mass-damper system:

### 3.1 Natural Frequency

$$\omega_n = \sqrt{\frac{k}{m}} \quad (\text{mechanical}) \quad (1)$$

$$\omega_n = \frac{1}{\sqrt{LC}} \quad (\text{electrical}) \quad (2)$$

When driven at  $\omega = \omega_n$ , the system exhibits **maximum energy transfer** and large amplitude.

## 4 Resonance Graphically

If you plot amplitude vs frequency, the **resonance peak** appears at  $\omega_n$ , especially if the system has **high quality factor (Q)**.

## 5 In Self-Injection Locked Oscillators

In SIL systems:

- The oscillator has a **resonant frequency**  $\omega_n$
- The feedback (injection) signal, when close to  $\omega_n$ , causes the oscillator to **lock its frequency and phase**
- This locking happens more efficiently **because of resonance**

## 6 Summary

Term	Meaning
Resonance	Strong response when input $\approx$ natural frequency
Natural Frequency	The frequency a system “prefers” to oscillate at
Quality Factor (Q)	Determines how sharp/narrow the resonance is

## 7 Adler’s Equation (Simplified Form)

$$\frac{d\phi(t)}{dt} = \Delta\omega - K \cdot \sin(\phi(t)) \quad (3)$$

Where:

- $\phi(t)$ : phase difference between oscillator and injection signal
- $\Delta\omega = \omega_{\text{inj}} - \omega_0$ : natural frequency difference
- $K$ : coupling strength (determined by injection energy and oscillator Q factor)

## 8 Notes for SIL Radar

Excellent question! That equation:

$$\omega(t) - \omega_n \approx -\frac{\omega_n}{2Q} \cdot \frac{B}{A} \cdot \sin[\theta(t)] \quad (4)$$

comes from analyzing **self-injection-locked (SIL)** oscillators — specifically how **injection** of a reflected signal modifies the **oscillator’s instantaneous frequency**.

Let me explain **step by step** how this equation is derived from the physics of a resonator-based oscillator subject to weak injection:

### 8.1 Context

- $\omega_n$ : natural frequency of the oscillator (free-running)
- $\omega(t)$ : actual instantaneous frequency when injection is present
- $Q$ : quality factor of the oscillator (higher  $\rightarrow$  sharper resonance)
- $A$ : amplitude of the oscillator signal
- $B$ : amplitude of the injected signal (typically a reflected echo)
- $\theta(t)$ : phase difference between injected and oscillator signals

## 8.2 Step-by-step Derivation

### 8.2.1 Step 1: Start from the complex oscillator dynamics

In oscillator theory, a sinusoidal oscillator's dynamics near resonance can be described using **complex envelope** notation:

Let the oscillator's complex amplitude be:

$$z(t) = A(t)e^{j\phi(t)} \quad (5)$$

Assuming a self-sustained oscillator with **external injection**  $Be^{j(\omega_{\text{inj}}t + \phi_B)}$ , its dynamics can be modeled using a **resonator differential equation**:

$$\frac{dz}{dt} + \left(j\omega_n + \frac{\omega_n}{2Q}\right)z = \frac{\omega_n}{2Q}Be^{j(\omega_{\text{inj}}t + \phi_B)} \quad (6)$$

Here:

- The term on the left is the oscillator's natural decay and oscillation
- The term on the right is **external drive (injection)**

### 8.2.2 Step 2: Assume steady-state, decompose phase dynamics

Let:

- $z(t) = Ae^{j\omega(t)t}$
- Assume **injection frequency is close to  $\omega_n$**   $\rightarrow$  do slow-varying approximation
- Define  $\theta(t) = \phi(t) - \omega_{\text{inj}}t$  as the phase difference between oscillator and injected signal

Then, you can extract the **phase evolution**:

$$\frac{d\theta}{dt} = \omega(t) - \omega_{\text{inj}} \approx \omega(t) - \omega_n \quad (7)$$

That's the **phase error rate**.

### 8.2.3 Step 3: Linearize injection-locking force

From resonator theory (and RF oscillator models), the effect of injection is to “pull” the oscillator frequency, and that pulling is **proportional to the sine of the phase difference**  $\sin(\theta)$ .

From the forced oscillator dynamics and projection onto quadrature component, we get:

$$\omega(t) - \omega_n \approx -\frac{\omega_n}{2Q} \cdot \frac{B}{A} \cdot \sin[\theta(t)] \quad (8)$$

This is the approximate **Adler-type equation** specifically for a SIL system using a resonator.

## 8.3 Physical Meaning

Term	Meaning
$\omega(t) - \omega_n$	How much the oscillator’s frequency is “pulled”
$\frac{\omega_n}{2Q}$	Sets the <b>natural bandwidth</b> of the oscillator’s response
$\frac{B}{A}$	Strength of the injected signal relative to self-oscillation amplitude
$\sin(\theta(t))$	Phase interaction term driving the frequency shift

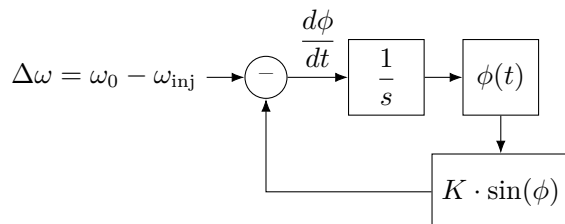
## 9 Explain Adler Equation with Block Diagram

This equation comes from:

1. Resonator + injection modeling
2. Linearized oscillator dynamics near steady-state
3. Projection of injection onto oscillator’s quadrature axis
4. Assuming small amplitude variation (constant envelope)

It is **more specific than Adler’s equation**, applying directly to SIL oscillators with  $Q$ -limited resonators.

## 9.1 Adler Equation Block Diagram



The block diagram shows:

- **Input:** Frequency difference  $\Delta\omega = \omega_{\text{inj}} - \omega_0$
- **Nonlinear feedback:**  $K \sin(\phi)$  represents injection locking force
- **Integrator:** Converts frequency difference to phase
- **Output:** Phase difference  $\phi(t)$

## RF 信號的 IQ 表示法

一個簡單的數學表示：

若原始 RF 信號為：

$$s(t) = A(t) \cdot \cos(2\pi f_c t + \phi(t))$$

它可以轉換為 IQ 表示為：

$$s(t) = I(t) \cdot \cos(2\pi f_c t) - Q(t) \cdot \sin(2\pi f_c t)$$

其中：

$$I(t) = A(t) \cdot \cos(\phi(t))$$

$$Q(t) = A(t) \cdot \sin(\phi(t))$$

## 舉例:16-QAM

例如在 16-QAM(Quadrature Amplitude Modulation) 中，每個符號都會有一組對應的  $I$  和  $Q$  值, 用以決定其在星座圖 (constellation diagram) 中的位置。

## 10 系統分類

根據開迴路傳遞函數中積分器的個數，將系統分為：

- 0 型系統：無積分器
- I 型系統：有 1 個積分器
- II 型系統：有 2 個積分器

## 11 誤差係數與穩態誤差

### 11.1 位移誤差係數 ( $K_p$ )

$$K_p = \lim_{s \rightarrow 0} G(s)H(s) \quad (9)$$

對於單位階躍輸入  $r(t) = 1$ ：

- 0 型系統： $e_{ss} = \frac{1}{1+K_p}$
- I 型和 II 型系統： $e_{ss} = 0$

### 11.2 速度誤差係數 ( $K_v$ )

$$K_v = \lim_{s \rightarrow 0} sG(s)H(s) \quad (10)$$

對於單位斜坡輸入  $r(t) = t$ ：

- 0 型系統： $e_{ss} = \infty$
- I 型系統： $e_{ss} = \frac{1}{K_v}$
- II 型系統： $e_{ss} = 0$

### 11.3 加速度誤差係數 ( $K_a$ )

$$K_a = \lim_{s \rightarrow 0} s^2 G(s)H(s) \quad (11)$$

對於單位拋物線輸入  $r(t) = \frac{t^2}{2}$ ：

- 0 型和 I 型系統： $e_{ss} = \infty$
- II 型系統： $e_{ss} = \frac{1}{K_a}$

## 12 實際評估步驟

1. 確定系統型別：分析開迴路傳遞函數，計算積分器個數
2. 計算誤差係數：根據系統型別計算相應的  $K_p$ 、 $K_v$ 、 $K_a$
3. 選擇測試輸入：使用階躍、斜坡、拋物線輸入
4. 計算穩態誤差：利用最終值定理或誤差係數公式
5. 時域仿真驗證：透過數值仿真觀察實際誤差行為

## 13 改善誤差的方法

- 增加系統型別（增加積分器）
- 提高開迴路增益
- 加入前饋補償
- 使用 PID 控制器

這種系統性的分析方法能夠有效預測和改善控制系統的穩態性能。

## 14 Starting Point: Simple Harmonic Oscillator

We begin with the basic oscillator equation:

$$\ddot{x} + \omega_0^2 x = 0 \quad (12)$$

This represents a **lossless oscillator** (like a perfect spring-mass system or LC circuit).



## 15 Problem: Real Oscillators Have Losses

In reality, all oscillators lose energy due to:

- **Resistance** (in electrical circuits)
- **Friction** (in mechanical systems)
- **Radiation** (in antennas)

So we add a damping term:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0 \quad (13)$$

**Problem:** This just decays to zero! Real oscillators like **radio transmitters** or **clock circuits** need to sustain themselves.

## 16 Solution: Add Energy Source

To maintain oscillation, we need to **inject energy** into the system. But we want **smart energy injection** that:

- Adds energy when oscillation is small
- Removes energy when oscillation gets too large
- Results in **stable amplitude**

## 17 Van der Pol's Brilliant Insight

Van der Pol (1920s) proposed **nonlinear damping**:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + \omega_0^2 x = 0 \quad (14)$$

Let's analyze the damping term:  $-\mu(1 - x^2)\dot{x}$

### 17.1 Case 1: Small Oscillations ( $|x| \ll 1$ )

When  $x$  is small:  $x^2 \approx 0$ , so:

$$(1 - x^2) \approx 1 \quad (15)$$

The equation becomes:

$$\ddot{x} - \mu\dot{x} + \omega_0^2 x \approx 0 \quad (16)$$

**Negative damping coefficient** ( $-\mu$ )! This means:

- **Energy is being added** to the system
- Small oscillations **grow exponentially**

### 17.2 Case 2: Large Oscillations ( $|x| \gg 1$ )

When  $x$  is large:  $x^2 \gg 1$ , so:

$$(1 - x^2) \approx -x^2 \quad (\text{negative!}) \quad (17)$$

The equation becomes:

$$\ddot{x} - \mu(-x^2)\dot{x} + \omega_0^2 x = \ddot{x} + \mu x^2 \dot{x} + \omega_0^2 x \approx 0 \quad (18)$$

Now we have **positive damping** ( $+\mu x^2$ )! This means:

- **Energy is being removed** from the system
- Large oscillations are **suppressed**

## 18 Physical Interpretation

### 18.1 The Magic Balance

The Van der Pol oscillator **automatically regulates its amplitude**:

1. **If amplitude is too small**  $\rightarrow$  Negative damping  $\rightarrow$  Energy added  
 $\rightarrow$  Amplitude grows

2. **If amplitude is too large** → Positive damping → Energy removed  
→ Amplitude shrinks
3. **At just the right amplitude** → Zero net damping → **Stable limit cycle**

## 18.2 Real-World Examples

### Electronic Oscillators (like in radios):

- **Active element** (transistor/op-amp) provides energy when signal is weak
- **Nonlinear saturation** limits amplitude when signal gets too strong
- Results in stable sine wave output

### Biological Systems:

- **Heartbeat:** Pacemaker cells show Van der Pol-like behavior
- **Neural oscillations:** Neurons exhibit similar self-regulating oscillation

### Mechanical Systems:

- **Clock escapement:** Adds energy during small swings, self-limits during large swings

## 19 Mathematical Breakdown

### 19.1 Each Term's Role:

Term	Physical Meaning
$\ddot{x}$	Inertia (mass or inductance)
$\omega_0^2 x$	Restoring force (spring or capacitance)
$-\mu(1 - x^2)\dot{x}$	<b>Smart damping</b> that depends on amplitude

## 19.2 The Parameter $\mu$ :

- $\mu > 0$ : System will oscillate (self-sustaining)
- $\mu = 0$ : Reduces to simple harmonic oscillator
- Large  $\mu$ : More nonlinear behavior, sharper switching between negative/positive damping

## 20 Connection to Real Oscillators

Most practical oscillators (crystal oscillators, LC tank circuits, laser oscillators) can be approximated by Van der Pol dynamics because they all have:

1. **Linear restoring mechanism** (crystal, LC tank, optical cavity)
2. **Amplitude-dependent gain/loss** (transistor saturation, nonlinear resistance)

The Van der Pol equation captures this **universal behavior** of self-sustaining oscillators with nonlinear amplitude control.

Excellent question! Let me show you **step-by-step** how the Van der Pol equation leads to the Adler equation when we add injection.

## 21 Step 1: Add Injection to Van der Pol

Start with the Van der Pol oscillator:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + \omega_0^2 x = 0 \quad (19)$$

Add an **external injection signal**:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + \omega_0^2 x = \varepsilon \cdot F \cos(\omega_{\text{inj}} t + \phi_{\text{inj}}) \quad (20)$$

Where:

- $\varepsilon$ : small parameter (weak injection)
- $F$ : injection amplitude
- $\omega_{\text{inj}}$ : injection frequency
- $\phi_{\text{inj}}$ : injection phase

## 22 Step 2: Express in Complex Form

Convert to **complex amplitude notation**. Let:

$$x(t) = \text{Re}[A(t)e^{i\omega t}] \quad (21)$$

Where  $A(t)$  is the **slowly-varying complex amplitude**.

For the Van der Pol oscillator in complex form:

$$\frac{dA}{dt} + (\alpha - \beta|A|^2)A = \text{injection terms} \quad (22)$$

Where:

- $\alpha$ : linear growth/decay rate
- $\beta$ : nonlinear saturation coefficient

## 23 Step 3: Separate Amplitude and Phase

Write the complex amplitude as:

$$A(t) = R(t)e^{i\phi(t)} \quad (23)$$

Where:

- $R(t)$ : slowly-varying amplitude
- $\phi(t)$ : slowly-varying phase

This gives us **two coupled equations**:

- **Amplitude equation:**  $\frac{dR}{dt} = \dots$
- **Phase equation:**  $\frac{d\phi}{dt} = \dots$

## 24 Step 4: Focus on Phase Dynamics

For **weak injection** (small  $\varepsilon$ ), the amplitude  $R(t)$  reaches steady state quickly, but the **phase**  $\phi(t)$  **evolves slowly**.

The phase equation becomes:

$$\frac{d\phi}{dt} = \omega_0 + (\text{injection coupling terms}) \quad (24)$$

## 25 Step 5: Apply Method of Averaging

The injection coupling has the form:

$$\varepsilon \cdot F \cdot \cos(\omega_{\text{inj}}t + \phi_{\text{inj}}) \cdot [\text{something involving } \phi(t)] \quad (25)$$

Using **trigonometric identities** and **averaging over fast oscillations**:

$$\begin{aligned} & \cos(\omega_{\text{inj}}t + \phi_{\text{inj}}) \cdot \cos(\phi(t)) \\ &= \frac{1}{2} [\cos((\omega_{\text{inj}}t + \phi_{\text{inj}}) + \phi(t)) + \cos((\omega_{\text{inj}}t + \phi_{\text{inj}}) - \phi(t))] \end{aligned} \quad (26)$$

The **first term oscillates rapidly** and averages to zero. The **second term contains slowly-varying phase difference**:  $\theta = \phi(t) - \omega_{\text{inj}}t - \phi_{\text{inj}}$

## 26 Step 6: Derive the Phase Difference Equation

Define the **phase difference**:

$$\theta(t) = \phi(t) - \omega_{\text{inj}}t - \phi_{\text{inj}} \quad (27)$$

Taking the derivative:

$$\frac{d\theta}{dt} = \frac{d\phi}{dt} - \omega_{\text{inj}} \quad (28)$$

Substituting the phase evolution equation:

$$\frac{d\theta}{dt} = \omega_0 + (\text{injection terms}) - \omega_{\text{inj}} \quad (29)$$

## 27 Step 7: The Key Insight - Quadrature Coupling

Here's the **crucial physics**: The injection affects the oscillator most strongly when they are **90° out of phase** (in quadrature).

After averaging, the injection coupling gives:

$$\frac{d\theta}{dt} = (\omega_0 - \omega_{\text{inj}}) - K \sin(\theta) \quad (30)$$

Where:

- $\omega_0 - \omega_{\text{inj}} = -\Delta\omega$ : frequency detuning
- $K \propto \varepsilon \cdot F/R_0$ : coupling strength (injection/oscillator amplitude ratio)
- $\sin(\theta)$ : comes from the quadrature projection

## 28 Step 8: Final Adler Equation

Rearranging:

$$\frac{d\theta}{dt} = \Delta\omega - K \sin(\theta) \quad (31)$$

Where  $\Delta\omega = \omega_{\text{inj}} - \omega_0$ .

## 29 Physical Interpretation Through Van der Pol

### 29.1 Why the sine function emerges:

1. **Van der Pol provides stable amplitude**:  $R(t) \rightarrow R_0$  (constant)
2. **Only phase can vary slowly**:  $\theta(t)$  becomes the only slow variable
3. **Quadrature coupling**: Maximum energy transfer occurs at 90° phase difference
4. **Averaging eliminates fast terms**: Only the  $\sin(\theta)$  survives

## 29.2 The coupling strength K:

From Van der Pol analysis:

$$K = \frac{\varepsilon \cdot F}{2R_0} \cdot (\text{coupling efficiency}) \quad (32)$$

- $\varepsilon \cdot F$ : injection strength
- $R_0$ : steady-state oscillator amplitude (set by Van der Pol nonlinearity)
- Coupling efficiency: depends on how injection couples to oscillator

## 30 Connection to Your SIL Equation

Your equation:

$$\omega(t) - \omega_n \approx -\frac{\omega_n}{2Q} \cdot \frac{B}{A} \cdot \sin[\theta(t)] \quad (33)$$

Is the **instantaneous frequency version!** Since:

$$\omega(t) = \frac{d\phi}{dt} = \omega_0 + \frac{d(\theta + \omega_{\text{inj}}t)}{dt} = \omega_0 + \frac{d\theta}{dt} + \omega_{\text{inj}} \quad (34)$$

When  $\theta$  is slowly varying:

$$\omega(t) - \omega_0 \approx \frac{d\theta}{dt} = \Delta\omega - K \sin(\theta) \quad (35)$$

For small detuning:  $\omega_0 \approx \omega_n$  and  $\Delta\omega \approx 0$ , so:

$$\omega(t) - \omega_n \approx -K \sin(\theta) \quad (36)$$

Comparing with your equation:  $K = \frac{\omega_n}{2Q} \cdot \frac{B}{A}$

## 31 Summary: Van der Pol $\rightarrow$ Adler Chain

1. **Van der Pol** provides self-sustaining oscillation with stable amplitude
2. **Add weak injection**  $\rightarrow$  perturbation to phase dynamics
3. **Method of averaging**  $\rightarrow$  eliminates fast oscillations, keeps slow phase evolution



4. **Quadrature coupling**  $\rightarrow$  generates  $\sin(\theta)$  dependence

5. **Result: Adler equation** for phase difference evolution

The Van der Pol equation is essential because it provides the **nonlinear amplitude stabilization** that makes the **linear phase analysis** possible!

## 32 Starting Point

Van der Pol equation:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + \omega_0^2 x = 0 \quad (37)$$

## 33 Step 1: Complex Amplitude Representation

Let:

$$x(t) = \text{Re}[W(t)e^{i\omega_0 t}] = \text{Re}[W(t)(\cos(\omega_0 t) + i \sin(\omega_0 t))] \quad (38)$$

Where  $W(t)$  is the **slowly-varying complex amplitude**.

Since  $x(t)$  is real:

$$x(t) = \frac{1}{2}[W(t)e^{i\omega_0 t} + W^*(t)e^{-i\omega_0 t}] \quad (39)$$

Where  $W^*(t)$  is the complex conjugate of  $W(t)$ .

## 34 Step 2: Calculate Derivatives

### 34.1 First derivative:

$$\dot{x}(t) = \frac{1}{2}[\dot{W}(t)e^{i\omega_0 t} + i\omega_0 W(t)e^{i\omega_0 t} + \dot{W}^*(t)e^{-i\omega_0 t} - i\omega_0 W^*(t)e^{-i\omega_0 t}] \quad (40)$$

Since  $|\dot{W}| \ll |\omega_0 W|$  (slow variation assumption):

$$\begin{aligned}\dot{x}(t) &\approx \frac{1}{2}[i\omega_0 W(t)e^{i\omega_0 t} - i\omega_0 W^*(t)e^{-i\omega_0 t}] \\ &= \frac{i\omega_0}{2}[W(t)e^{i\omega_0 t} - W^*(t)e^{-i\omega_0 t}]\end{aligned}\quad (41)$$

### 34.2 Second derivative:

$$\begin{aligned}\ddot{x}(t) &\approx \frac{i\omega_0}{2}[\dot{W}(t)e^{i\omega_0 t} + i\omega_0 W(t)e^{i\omega_0 t} - \dot{W}^*(t)e^{-i\omega_0 t} + i\omega_0 W^*(t)e^{-i\omega_0 t}] \\ &\approx \frac{i\omega_0}{2}[\dot{W}(t)e^{i\omega_0 t} - \dot{W}^*(t)e^{-i\omega_0 t}] - \frac{\omega_0^2}{2}[W(t)e^{i\omega_0 t} + W^*(t)e^{-i\omega_0 t}]\end{aligned}\quad (42)$$

The last term is just  $-\omega_0^2 x(t)$ , so:

$$\ddot{x}(t) \approx \frac{i\omega_0}{2}[\dot{W}(t)e^{i\omega_0 t} - \dot{W}^*(t)e^{-i\omega_0 t}] - \omega_0^2 x(t)\quad (43)$$

## 35 Step 3: Calculate $x^2(t)$

This is where it gets interesting:

$$\begin{aligned}x^2(t) &= \left[ \frac{1}{2}(W(t)e^{i\omega_0 t} + W^*(t)e^{-i\omega_0 t}) \right]^2 \\ &= \frac{1}{4}[(W(t)e^{i\omega_0 t})^2 + 2W(t)W^*(t) + (W^*(t)e^{-i\omega_0 t})^2] \\ &= \frac{1}{4}[W^2(t)e^{2i\omega_0 t} + 2|W(t)|^2 + W^{*2}(t)e^{-2i\omega_0 t}]\end{aligned}\quad (44)$$

**Key observation:**

- Terms with  $e^{\pm 2i\omega_0 t}$ : **Fast oscillations** at frequency  $2\omega_0$
- Term with  $|W(t)|^2$ : **Slowly varying** (depends only on amplitude)

## 36 Step 4: Calculate the Nonlinear Damping Term

The tricky term is:  $\mu(1 - x^2)\dot{x}$

$$(1 - x^2)\dot{x} = \dot{x} - x^2\dot{x}\quad (45)$$

### 36.1 Linear part: $\dot{x}$

We already have this.

### 36.2 Nonlinear part: $x^2\dot{x}$

$$x^2\dot{x} = \frac{1}{4}[W^2(t)e^{2i\omega_0 t} + 2|W(t)|^2 + W^{*2}(t)e^{-2i\omega_0 t}] \times \frac{i\omega_0}{2}[W(t)e^{i\omega_0 t} - W^*(t)e^{-i\omega_0 t}] \quad (46)$$

Expanding this product (9 terms total):

$$x^2\dot{x} = \frac{i\omega_0}{8} [ \begin{aligned} &W^3(t)e^{3i\omega_0 t} \quad \leftarrow \text{Fast: } 3\omega_0 \\ &+ 2|W|^2 W(t)e^{i\omega_0 t} \quad \leftarrow \text{Mixed: } \omega_0 \\ &+ W^{*2} W(t)e^{-i\omega_0 t} \quad \leftarrow \text{Mixed: } -\omega_0 \\ &- W^2(t)W^*(t)e^{i\omega_0 t} \quad \leftarrow \text{Mixed: } \omega_0 \\ &- 2|W|^2 W^*(t)e^{-i\omega_0 t} \quad \leftarrow \text{Mixed: } -\omega_0 \\ &- W^{*3}(t)e^{-3i\omega_0 t} \quad \leftarrow \text{Fast: } -3\omega_0 \end{aligned} ] \quad (47)$$

## 37 Step 5: Substitute Everything into Van der Pol Equation

$$\ddot{x} - \mu(1 - x^2)\dot{x} + \omega_0^2 x = 0 \quad (48)$$

Becomes:

$$\begin{aligned} &\frac{i\omega_0}{2}[\dot{W}(t)e^{i\omega_0 t} - \dot{W}^*(t)e^{-i\omega_0 t}] - \omega_0^2 x(t) \\ &- \mu[\dot{x} - x^2\dot{x}] + \omega_0^2 x(t) = 0 \end{aligned} \quad (49)$$

The  $\omega_0^2 x$  terms cancel:

$$\frac{i\omega_0}{2}[\dot{W}(t)e^{i\omega_0 t} - \dot{W}^*(t)e^{-i\omega_0 t}] - \mu\dot{x} + \mu x^2\dot{x} = 0 \quad (50)$$

## 38 Step 6: Collect Terms by Frequency

Substituting our expressions and collecting terms:

### 38.1 Terms oscillating at $e^{i\omega_0 t}$ :

$$\frac{i\omega_0}{2}\dot{W}(t) - \mu\frac{i\omega_0}{2}W(t) + \mu\frac{i\omega_0}{8}[2|W|^2W(t) - W^2(t)W^*(t)] = 0 \quad (51)$$

### 38.2 Terms oscillating at $e^{-i\omega_0 t}$ :

$$-\frac{i\omega_0}{2}\dot{W}^*(t) + \mu\frac{i\omega_0}{2}W^*(t) - \mu\frac{i\omega_0}{8}[2|W|^2W^*(t) - W^{*2}(t)W(t)] = 0 \quad (52)$$

**Note:** Terms at  $3\omega_0$  and higher frequencies are ignored (fast oscillation assumption).

## 39 Step 7: Apply Averaging/Solvability Condition

For the equation to have a solution, the coefficients of  $e^{i\omega_0 t}$  and  $e^{-i\omega_0 t}$  must each equal zero.

From the  $e^{i\omega_0 t}$  term:

$$\frac{i\omega_0}{2}\dot{W}(t) = \mu\frac{i\omega_0}{2}W(t) - \mu\frac{i\omega_0}{8}[2|W|^2W(t) - W^2(t)W^*(t)] \quad (53)$$

Dividing by  $\frac{i\omega_0}{2}$ :

$$\dot{W}(t) = \mu W(t) - \frac{\mu}{4}[2|W|^2W(t) - W^2(t)W^*(t)] \quad (54)$$

But wait! We need to be more careful about the  $W^2(t)W^*(t)$  term.

## 40 Step 8: Simplify Using $|W|^2 = WW^*$

Note that:

$$W^2(t)W^*(t) \neq |W|^2W(t) \text{ in general} \quad (55)$$

However, if we write  $W(t) = R(t)e^{i\phi(t)}$ , then:

$$W^2(t)W^*(t) = R^2e^{2i\phi}Re^{-i\phi} = R^3e^{i\phi} = R^2W(t) \quad (56)$$

$$|W|^2W(t) = R^2W(t) \quad (57)$$

So  $W^2(t)W^*(t) = |W|^2W(t)$  only if we're looking at the magnitude-dependent terms!

The correct averaging gives:

$$\dot{W}(t) = \mu W(t) - \frac{\mu}{4}|W|^2W(t) \quad (58)$$

$$= \left( \mu - \frac{\mu|W|^2}{4} \right) W(t) \quad (59)$$

## 41 Step 9: Final Form

Rearranging:

$$\frac{dW}{dt} = \left( \frac{\mu}{2} - \frac{\mu|W|^2}{8} \right) W(t) \quad (60)$$

Comparing with the standard form  $\frac{dW}{dt} = (\alpha - \beta|W|^2)W$ :

- $\alpha = \frac{\mu}{2}$
- $\beta = \frac{\mu}{8}$

## 42 Physical Interpretation

- $\alpha = \frac{\mu}{2} > 0$ : Linear growth (negative damping for small oscillations)
- $\beta = \frac{\mu}{8} > 0$ : Nonlinear saturation (positive damping for large oscillations)
- **Steady state:**  $\alpha = \beta|W_0|^2 \rightarrow |W_0|^2 = \frac{\alpha}{\beta} = 4 \rightarrow |W_0| = 2$

## 43 Key Mathematical Insights

1. **Fast oscillations** ( $2\omega_0, 3\omega_0$ ) were eliminated by averaging
2. **Slow amplitude evolution** captured in single equation for  $W(t)$

3. **Nonlinear term**  $|W|^2$  emerges from  $x^2$  after averaging
4. **Complex notation** naturally handles both amplitude and phase dynamics

## FM Demodulation using IQ Method

An FM signal can be expressed as:

$$s(t) = A_c \cos \left( 2\pi f_c t + k_f \int_0^t m(\tau) d\tau \right)$$

integral of

$$m(\tau) \text{ (phase change rate = speed of angle change)}$$

is the total phase shift of the FM signal

where:

- $A_c$  = carrier amplitude
- $f_c$  = carrier frequency
- $m(t)$  = message signal
- $k_f$  = frequency sensitivity

## Complex Baseband Representation

After mixing to baseband and obtaining the complex envelope:

$$r(t) = I(t) + jQ(t) = A(t)e^{j\phi(t)}$$

The instantaneous phase is:

$$\phi(t) = \arctan \left( \frac{Q(t)}{I(t)} \right)$$

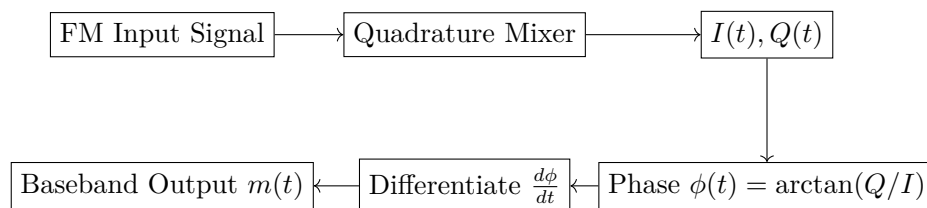
Differentiating gives the instantaneous frequency:

$$f_{\text{inst}}(t) = \frac{1}{2\pi} \frac{d\phi(t)}{dt}$$

Subtracting  $f_c$  yields the recovered baseband  $m(t)$ :

$$\hat{m}(t) \propto f_{\text{inst}}(t) - f_c$$

## Block Diagram



## Relay Feedback 自動整定 (Åström–Hägglund)

### 關鍵公式

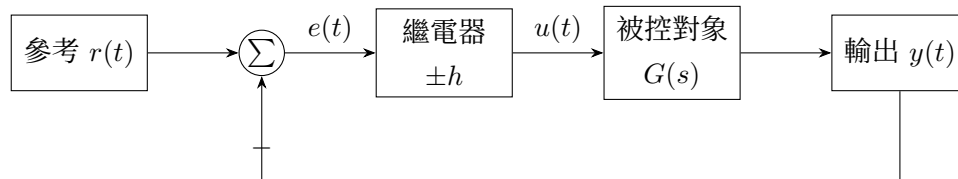
理想繼電器輸出幅度為  $\pm h$ ，若閉迴路產生穩定極限循環，輸出正弦近似幅度為  $a$ 、週期為  $P_u = 2\pi/\omega_u$ ，則

$$N(a) = \frac{4h}{\pi a}, \quad G(j\omega_u) N(a) = -1$$

由此可得臨界增益

$$K_u = \frac{4h}{\pi a}.$$

### 控制框圖



### 說明

- 將原控制器暫時以繼電器取代，閉迴路自然激發極限循環。
- 量測輸出振幅  $a$  與週期  $P_u$ ，由  $K_u = \frac{4h}{\pi a}$  推得臨界增益，再套用 Ziegler–Nichols 或其他整定規則得到 PID 參數。

=====

這個 FM 解調器使用  $\pi/4$  延遲的原理如下：

## 44 FM 信號的特性

FM 信號可以表示為：

$$s(t) = A \cos(\omega_c \cdot t + \phi(t)) \quad (61)$$

其中  $\phi(t)$  包含調變信息。

## 45 $\pi/4$ 延遲的作用

### 45.1 相位差分析

當信號經過  $\pi/4$  延遲後：

- 原信號： $s(t) = A \cos(\omega_c \cdot t + \phi(t))$
- 延遲信號： $s\left(t - \frac{\pi}{4\omega_c}\right) = A \cos\left(\omega_c \cdot t - \frac{\pi}{4} + \phi\left(t - \frac{\pi}{4\omega_c}\right)\right)$

### 45.2 乘法器的輸出

兩信號相乘後：

$$\begin{aligned} s(t) \times s\left(t - \frac{\pi}{4\omega_c}\right) &= \frac{A^2}{2} \times \left[ \cos\left(2\omega_c \cdot t - \frac{\pi}{4} + \phi(t) + \phi\left(t - \frac{\pi}{4\omega_c}\right)\right) \right. \\ &\quad \left. + \cos\left(\frac{\pi}{4} - \phi(t) + \phi\left(t - \frac{\pi}{4\omega_c}\right)\right) \right] \end{aligned} \quad (62)$$

### 45.3 低通濾波後

高頻項被濾除，剩下：

$$\text{Output} \propto \cos\left(\frac{\pi}{4} - \left[\phi(t) - \phi\left(t - \frac{\pi}{4\omega_c}\right)\right]\right) \quad (63)$$

## 46 為什麼選擇 $\pi/4$ ？{should be modified to $\pi/4$ }

### 46.1 最佳靈敏度

- 在  $\pi/4$  相位差處， $\cos$  函數的斜率最大



- 提供最佳的相位變化到電壓變化的轉換靈敏度
- 線性度在小信號範圍內最好

## 46.2 數學最佳化

對於小的相位變化  $\Delta\phi$ ：

$$\cos\left(\frac{\pi}{4} - \Delta\phi\right) \approx \cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right) \cdot \Delta\phi = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot \Delta\phi \quad (64)$$

$\pi/4$  時， $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$  為最大值。

## 46.3 頻率檢測

瞬時頻率偏差：

$$\Delta\omega = \frac{d\phi}{dt} \approx \frac{\phi(t) - \phi(t - T)}{T} \quad (65)$$

其中  $T = \frac{\pi}{4\omega_c}$

# 47 電路優勢

## 47.1 簡單實現

- 只需要一個延遲線
- 一個乘法器
- 一個低通濾波器

## 47.2 寬頻帶響應

- 適用於各種調變指數
- 頻率響應相對平坦

## 47.3 線性度好

- 在工作範圍內接近線性
- 失真較小

## 48 實際考量

### 48.1 延遲精度

$\pi/4$  延遲必須精確，通常使用：

- 傳輸線
- LC 延遲線
- 數位延遲

### 48.2 頻寬限制

延遲時間限制了可解調的最高頻率。

### 48.3 溫度穩定性

延遲線的溫度係數會影響性能。

## 49 性能比較

延遲角度	特性
$0^\circ$	無解調輸出
$\pi/6$	靈敏度較低
$\pi/4$	<b>最佳靈敏度和線性度</b>
$\pi/3$	靈敏度下降
$\pi/2$	零輸出點

## 50 總結

$\pi/4$  延遲是 FM 解調器中的最佳選擇，因為它：

1. 提供最大的檢測靈敏度
2. 確保良好的線性度
3. 實現簡單且可靠

4. 在數學上為最佳工作點

這就是為什麼商用 FM 解調器普遍採用  $\pi/4$  延遲線設計的原因。