Self-Injection-Locked (SIL) Oscillator Analysis

carlos ma

2025 年 8 月 11 日

1 General Definition

Resonance occurs when a system is subjected to an external force or signal whose frequency matches the system's **natural frequency**, resulting in a large increase in amplitude.

2 Real-World Examples

Context	Description
Violin string	Vibrates strongly when bowing matches its natural vibration frequency
RF circuits	LC circuits resonate at a specific frequency → used in filters, radios
Bridges	Tacoma Narrows Bridge collapsed due to wind-induced resonance
SIL radar	Resonator oscillates strongly at ω_n; injection close to ω_n causes locking

3 In Engineering Terms

For a **second-order system** like an RLC circuit or mechanical spring-mass-damper system:

3.1 Natural Frequency

$$\omega_n = \sqrt{\frac{k}{m}} \quad \text{(mechanical)} \tag{1}$$

$$\omega_n = \frac{1}{\sqrt{LC}} \quad \text{(electrical)} \tag{2}$$

When driven at $\omega = \omega_n$, the system exhibits **maximum energy transfer** and large amplitude.

4 Resonance Graphically

If you plot amplitude vs frequency, the **resonance peak** appears at ω_n, especially if the system has **high quality factor (Q)**.

5 In Self-Injection Locked Oscillators

In SIL systems:

- The oscillator has a **resonant frequency** ω_n
- The feedback (injection) signal, when close to ω_n, causes the oscillator to **lock its frequency and phase**
- This locking happens more efficiently **because of resonance**

6 Summary

Term	Meaning
Resonance	Strong response when input $\approx$ natural frequency
Natural Frequency	The frequency a system "prefers" to oscillate at
Quality Factor (Q)	Determines how sharp/narrow the resonance is

7 Adler's Equation (Simplified Form)

$$\frac{d\phi(t)}{dt} = \Delta\omega - K \cdot \sin(\phi(t)) \tag{3}$$

Where:

- $\phi(t)$: phase difference between oscillator and injection signal
- $\Delta\omega = \omega_{\text{inj}} - \omega_0$: natural frequency difference
- K: coupling strength (determined by injection energy and oscillator Q factor)

8 Notes for SIL Radar

Excellent question! That equation:

$$\omega(t) - \omega_n \approx -\frac{\omega_n}{2Q} \cdot \frac{B}{A} \cdot \sin[\theta(t)] \tag{4}$$

comes from analyzing **self-injection-locked (SIL)** oscillators — specifically how **injection** of a reflected signal modifies the **oscillator's instantaneous frequency**.

Let me explain **step by step** how this equation is derived from the physics of a resonator-based oscillator subject to weak injection:

8.1 Context

- ω_n: natural frequency of the oscillator (free-running)
- $\omega(t)$: actual instantaneous frequency when injection is present
- Q: quality factor of the oscillator (higher $\rightarrow$ sharper resonance)
- A: amplitude of the oscillator signal
- B: amplitude of the injected signal (typically a reflected echo)
- $\theta(t)$: phase difference between injected and oscillator signals

8.2 Step-by-step Derivation

8.2.1 Step 1: Start from the complex oscillator dynamics

In oscillator theory, a sinusoidal oscillator's dynamics near resonance can be described using **complex envelope** notation:

Let the oscillator's complex amplitude be:

$$z(t) = A(t)e^{j\phi(t)} \tag{5}$$

Assuming a self-sustained oscillator with **external injection** $Be^{j(\omega_{\text{inj}}t+\phi_B)}$, its dynamics can be modeled using a **resonator differential equation**:

$$\frac{dz}{dt} + \left(j\omega_n + \frac{\omega_n}{2Q}\right)z = \frac{\omega_n}{2Q}Be^{j(\omega_{\text{inj}}t+\phi_B)} \tag{6}$$

Here:

- The term on the left is the oscillator's natural decay and oscillation
- The term on the right is **external drive (injection)**

8.2.2 Step 2: Assume steady-state, decompose phase dynamics

Let:

- $z(t) = Ae^{j\omega(t)t}$
- Assume **injection frequency is close to** ω_n $\rightarrow$ do slow-varying approximation
- Define $\theta(t) = \phi(t) - \omega_{\text{inj}}t$ as the phase difference between oscillator and injected signal

Then, you can extract the **phase evolution**:

$$\frac{d\theta}{dt} = \omega(t) - \omega_{\text{inj}} \approx \omega(t) - \omega_n \tag{7}$$

That's the **phase error rate**.

8.2.3 Step 3: Linearize injection-locking force

From resonator theory (and RF oscillator models), the effect of injection is to "pull" the oscillator frequency, and that pulling is **proportional to the sine of the phase difference** $\sin(\theta)$.

From the forced oscillator dynamics and projection onto quadrature component, we get:

$$\omega(t) - \omega_n \approx -\frac{\omega_n}{2Q} \cdot \frac{B}{A} \cdot \sin[\theta(t)] \tag{8}$$

This is the approximate **Adler-type equation** specifically for a SIL system using a resonator.

8.3 Physical Meaning

Term	Meaning
$\omega(t) - \omega_n$	How much the oscillator's frequency is "pulled"
$\frac{\omega_n}{2Q}$	Sets the **natural bandwidth** of the oscillator's response
$\frac{B}{A}$	Strength of the injected signal relative to self-oscillation amplitude
$\sin(\theta(t))$	Phase interaction term driving the frequency shift

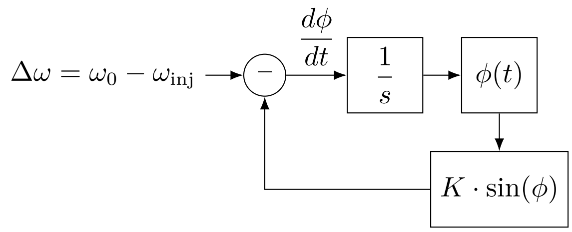
9 Explain Adler Equation with Block Diagram

This equation comes from:

1. Resonator + injection modeling
2. Linearized oscillator dynamics near steady-state
3. Projection of injection onto oscillator's quadrature axis
4. Assuming small amplitude variation (constant envelope)

It is **more specific than Adler's equation**, applying directly to SIL oscillators with Q-limited resonators.

9.1 Adler Equation Block Diagram



The block diagram shows:

- **Input**: Frequency difference $\Delta\omega = \omega_{\text{inj}} - \omega_0$
- **Nonlinear feedback**: $K \sin(\phi)$ represents injection locking force
- **Integrator**: Converts frequency difference to phase
- **Output**: Phase difference $\phi(t)$

RF 信號的 IQ 表示法

一個簡單的數學表示：
若原始 RF 信號為：

$$s(t) = A(t) \cdot \cos(2\pi f_c t + \phi(t))$$

它可以轉換為 IQ 表示為：

$$s(t) = I(t) \cdot \cos(2\pi f_c t) - Q(t) \cdot \sin(2\pi f_c t)$$

其中：

$$I(t) = A(t) \cdot \cos(\phi(t))$$
$$Q(t) = A(t) \cdot \sin(\phi(t))$$

舉例:16-QAM

例如在 16-QAM(Quadrature Amplitude Modulation) 中，每個符號都會有一組對應的 I 和 Q 值, 用以決定其在星座圖 (constellation diagram) 中的位置。

10 系統分類

根據開迴路傳遞函數中積分器的個數，將系統分為：

- **0 型系統**：無積分器
- **I 型系統**：有 1 個積分器
- **II 型系統**：有 2 個積分器

11 誤差係數與穩態誤差

11.1 位移誤差係數 (K_p)

$$K_p = \lim_{s \to 0} G(s)H(s) \tag{9}$$

對於單位階躍輸入 $r(t) = 1$：

- 0 型系統：$e_{ss} = \frac{1}{1+K_p}$
- I 型和 II 型系統：$e_{ss} = 0$

11.2 速度誤差係數 (K_v)

$$K_v = \lim_{s \to 0} sG(s)H(s) \tag{10}$$

對於單位斜坡輸入 $r(t) = t$：

- 0 型系統：$e_{ss} = \infty$
- I 型系統：$e_{ss} = \frac{1}{K_v}$
- II 型系統：$e_{ss} = 0$

11.3 加速度誤差係數 (K_a)

$$K_a = \lim_{s \to 0} s^2 G(s)H(s) \tag{11}$$

對於單位拋物線輸入 $r(t) = \frac{t^2}{2}$：

- 0 型和 I 型系統：$e_{ss} = \infty$
- II 型系統：$e_{ss} = \frac{1}{K_a}$

12 實際評估步驟

1. **確定系統型別**：分析開迴路傳遞函數，計算積分器個數
2. **計算誤差係數**：根據系統型別計算相應的 K_p、K_v、K_a
3. **選擇測試輸入**：使用階躍、斜坡、拋物線輸入
4. **計算穩態誤差**：利用最終值定理或誤差係數公式
5. **時域仿真驗證**：透過數值仿真觀察實際誤差行為

13 改善誤差的方法

- 增加系統型別（增加積分器）
- 提高開迴路增益
- 加入前饋補償
- 使用 PID 控制器

這種系統性的分析方法能夠有效預測和改善控制系統的穩態性能。

14 Starting Point: Simple Harmonic Oscillator

We begin with the basic oscillator equation:

$$\ddot{x} + \omega_0^2 x = 0 \tag{12}$$

This represents a **lossless oscillator** (like a perfect spring-mass system or LC circuit).

15 Problem: Real Oscillators Have Losses

In reality, all oscillators lose energy due to:

- **Resistance** (in electrical circuits)
- **Friction** (in mechanical systems)
- **Radiation** (in antennas)

So we add a damping term:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0 \tag{13}$$

Problem: This just decays to zero! Real oscillators like **radio transmitters** or **clock circuits** need to sustain themselves.

16 Solution: Add Energy Source

To maintain oscillation, we need to **inject energy** into the system. But we want **smart energy injection** that:

- Adds energy when oscillation is small
- Removes energy when oscillation gets too large
- Results in **stable amplitude**

17 Van der Pol's Brilliant Insight

Van der Pol (1920s) proposed **nonlinear damping**:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + \omega_0^2 x = 0 \tag{14}$$

Let's analyze the damping term: $-\mu(1 - x^2)\dot{x}$

17.1 Case 1: Small Oscillations ($|x| \ll 1$)

When x is small: $x^2 \approx 0$, so:

$$(1 - x^2) \approx 1 \tag{15}$$

The equation becomes:

$$\ddot{x} - \mu\dot{x} + \omega_0^2 x \approx 0 \tag{16}$$

Negative damping coefficient $(-\mu)$! This means:

- **Energy is being added** to the system
- Small oscillations **grow exponentially**

17.2 Case 2: Large Oscillations ($|x| \gg 1$)

When x is large: $x^2 \gg 1$, so:

$$(1 - x^2) \approx -x^2 \quad \text{(negative!)} \tag{17}$$

The equation becomes:

$$\ddot{x} - \mu(-x^2)\dot{x} + \omega_0^2 x = \ddot{x} + \mu x^2 \dot{x} + \omega_0^2 x \approx 0 \tag{18}$$

Now we have **positive damping** $(+\mu x^2)$! This means:

- **Energy is being removed** from the system
- Large oscillations are **suppressed**

18 Physical Interpretation

18.1 The Magic Balance

The Van der Pol oscillator **automatically regulates its amplitude**:

1. **If amplitude is too small** → Negative damping → Energy added → Amplitude grows

2. **If amplitude is too large** → Positive damping → Energy removed → Amplitude shrinks

3. **At just the right amplitude** → Zero net damping → **Stable limit cycle**

18.2 Real-World Examples

Electronic Oscillators (like in radios):

- **Active element** (transistor/op-amp) provides energy when signal is weak
- **Nonlinear saturation** limits amplitude when signal gets too strong
- Results in stable sine wave output

Biological Systems:

- **Heartbeat**: Pacemaker cells show Van der Pol-like behavior
- **Neural oscillations**: Neurons exhibit similar self-regulating oscillation

Mechanical Systems:

- **Clock escapement**: Adds energy during small swings, self-limits during large swings

19 Mathematical Breakdown

19.1 Each Term's Role:

Term	Physical Meaning
$\ddot{x}$	Inertia (mass or inductance)
$\omega_0^2 x$	Restoring force (spring or capacitance)
$-\mu(1-x^2)\dot{x}$	**Smart damping** that depends on amplitude

19.2 The Parameter μ:

- $\mu > 0$: System will oscillate (self-sustaining)
- $\mu = 0$: Reduces to simple harmonic oscillator
- Large μ: More nonlinear behavior, sharper switching between negative/positive damping

20 Connection to Real Oscillators

Most practical oscillators (crystal oscillators, LC tank circuits, laser oscillators) can be approximated by Van der Pol dynamics because they all have:

1. **Linear restoring mechanism** (crystal, LC tank, optical cavity)
2. **Amplitude-dependent gain/loss** (transistor saturation, nonlinear resistance)

The Van der Pol equation captures this **universal behavior** of self-sustaining oscillators with nonlinear amplitude control.

Excellent question! Let me show you **step-by-step** how the Van der Pol equation leads to the Adler equation when we add injection.

21 Step 1: Add Injection to Van der Pol

Start with the Van der Pol oscillator:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + \omega_0^2 x = 0 \tag{19}$$

Add an **external injection signal**:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + \omega_0^2 x = \varepsilon \cdot F \cos(\omega_{\text{inj}} t + \phi_{\text{inj}}) \tag{20}$$

Where:

- ε: small parameter (weak injection)
- F: injection amplitude
- ω_{inj}: injection frequency
- ϕ_{inj}: injection phase

22 Step 2: Express in Complex Form

Convert to **complex amplitude notation**. Let:

$$x(t) = \text{Re}[A(t)e^{i\omega t}] \tag{21}$$

Where $A(t)$ is the **slowly-varying complex amplitude**.

For the Van der Pol oscillator in complex form:

$$\frac{dA}{dt} + (\alpha - \beta|A|^2)A = \text{injection terms} \tag{22}$$

Where:

- α: linear growth/decay rate
- β: nonlinear saturation coefficient

23 Step 3: Separate Amplitude and Phase

Write the complex amplitude as:

$$A(t) = R(t)e^{i\phi(t)} \tag{23}$$

Where:

- $R(t)$: slowly-varying amplitude
- $\phi(t)$: slowly-varying phase

This gives us **two coupled equations**:

- **Amplitude equation**: $\frac{dR}{dt} = \dots$
- **Phase equation**: $\frac{d\phi}{dt} = \dots$

24 Step 4: Focus on Phase Dynamics

For **weak injection** (small ε), the amplitude $R(t)$ reaches steady state quickly, but the **phase** $\phi(t)$ **evolves slowly**.

The phase equation becomes:

$$\frac{d\phi}{dt} = \omega_0 + (\text{injection coupling terms}) \tag{24}$$

25 Step 5: Apply Method of Averaging

The injection coupling has the form:

$$\varepsilon \cdot F \cdot \cos(\omega_{\text{inj}} t + \phi_{\text{inj}}) \cdot [\text{something involving } \phi(t)] \tag{25}$$

Using **trigonometric identities** and **averaging over fast oscillations**:

$$\begin{aligned} &\cos(\omega_{\text{inj}} t + \phi_{\text{inj}}) \cdot \cos(\phi(t)) \\ &= \frac{1}{2}[\cos((\omega_{\text{inj}} t + \phi_{\text{inj}}) + \phi(t)) + \cos((\omega_{\text{inj}} t + \phi_{\text{inj}}) - \phi(t))] \end{aligned} \tag{26}$$

The **first term oscillates rapidly** and averages to zero. The **second term contains slowly-varying phase difference**: $\theta = \phi(t) - \omega_{\text{inj}} t - \phi_{\text{inj}}$

26 Step 6: Derive the Phase Difference Equation

Define the **phase difference**:

$$\theta(t) = \phi(t) - \omega_{\text{inj}} t - \phi_{\text{inj}} \tag{27}$$

Taking the derivative:

$$\frac{d\theta}{dt} = \frac{d\phi}{dt} - \omega_{\text{inj}} \tag{28}$$

Substituting the phase evolution equation:

$$\frac{d\theta}{dt} = \omega_0 + (\text{injection terms}) - \omega_{\text{inj}} \tag{29}$$

27 Step 7: The Key Insight - Quadrature Coupling

Here's the **crucial physics**: The injection affects the oscillator most strongly when they are **90° out of phase** (in quadrature).

After averaging, the injection coupling gives:

$$\frac{d\theta}{dt} = (\omega_0 - \omega_{\text{inj}}) - K \sin(\theta) \tag{30}$$

Where:

- $\omega_0 - \omega_{\text{inj}} = -\Delta\omega$: frequency detuning
- $K \propto \varepsilon \cdot F/R_0$: coupling strength (injection/oscillator amplitude ratio)
- $\sin(\theta)$: comes from the quadrature projection

28 Step 8: Final Adler Equation

Rearranging:

$$\frac{d\theta}{dt} = \Delta\omega - K \sin(\theta) \tag{31}$$

Where $\Delta\omega = \omega_{\text{inj}} - \omega_0$.

29 Physical Interpretation Through Van der Pol

29.1 Why the sine function emerges:

1. **Van der Pol provides stable amplitude**: $R(t) \to R_0$ (constant)
2. **Only phase can vary slowly**: $\theta(t)$ becomes the only slow variable
3. **Quadrature coupling**: Maximum energy transfer occurs at 90° phase difference
4. **Averaging eliminates fast terms**: Only the $\sin(\theta)$ survives

29.2 The coupling strength K:

From Van der Pol analysis:

$$K = \frac{\varepsilon \cdot F}{2R_0} \cdot (\text{coupling efficiency}) \tag{32}$$

- $\varepsilon \cdot F$: injection strength
- R_0: steady-state oscillator amplitude (set by Van der Pol nonlinearity)
- Coupling efficiency: depends on how injection couples to oscillator

30 Connection to Your SIL Equation

Your equation:

$$\omega(t) - \omega_n \approx -\frac{\omega_n}{2Q} \cdot \frac{B}{A} \cdot \sin[\theta(t)] \tag{33}$$

Is the **instantaneous frequency version**! Since:

$$\omega(t) = \frac{d\phi}{dt} = \omega_0 + \frac{d(\theta + \omega_{\text{inj}} t)}{dt} = \omega_0 + \frac{d\theta}{dt} + \omega_{\text{inj}} \tag{34}$$

When θ is slowly varying:

$$\omega(t) - \omega_0 \approx \frac{d\theta}{dt} = \Delta\omega - K \sin(\theta) \tag{35}$$

For small detuning: $\omega_0 \approx \omega_n$ and $\Delta\omega \approx 0$, so:

$$\omega(t) - \omega_n \approx -K \sin(\theta) \tag{36}$$

Comparing with your equation: $K = \frac{\omega_n}{2Q} \cdot \frac{B}{A}$

31 Summary: Van der Pol → Adler Chain

1. **Van der Pol** provides self-sustaining oscillation with stable amplitude
2. **Add weak injection** → perturbation to phase dynamics
3. **Method of averaging** → eliminates fast oscillations, keeps slow phase evolution

4. **Quadrature coupling** → generates $\sin(\theta)$ dependence
5. **Result**: **Adler equation** for phase difference evolution

The Van der Pol equation is essential because it provides the **nonlinear amplitude stabilization** that makes the **linear phase analysis** possible!

32 Starting Point

Van der Pol equation:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + \omega_0^2 x = 0 \tag{37}$$

33 Step 1: Complex Amplitude Representation

Let:

$$x(t) = \mathrm{Re}[W(t)e^{i\omega_0 t}] = \mathrm{Re}[W(t)(\cos(\omega_0 t) + i\sin(\omega_0 t))] \tag{38}$$

Where $W(t)$ is the **slowly-varying complex amplitude**.
Since $x(t)$ is real:

$$x(t) = \frac{1}{2}[W(t)e^{i\omega_0 t} + W^*(t)e^{-i\omega_0 t}] \tag{39}$$

Where $W^*(t)$ is the complex conjugate of $W(t)$.

34 Step 2: Calculate Derivatives

34.1 First derivative:

$$\dot{x}(t) = \frac{1}{2}[\dot{W}(t)e^{i\omega_0 t} + i\omega_0 W(t)e^{i\omega_0 t} + \dot{W}^*(t)e^{-i\omega_0 t} - i\omega_0 W^*(t)e^{-i\omega_0 t}] \tag{40}$$

Since $|\dot{W}| \ll |\omega_0 W|$ (slow variation assumption):

$$\begin{aligned}\dot{x}(t) &\approx \frac{1}{2}[i\omega_0 W(t)e^{i\omega_0 t} - i\omega_0 W^*(t)e^{-i\omega_0 t}] \\ &= \frac{i\omega_0}{2}[W(t)e^{i\omega_0 t} - W^*(t)e^{-i\omega_0 t}] \qquad (41)\end{aligned}$$

34.2 Second derivative:

$$\begin{aligned}\ddot{x}(t) &\approx \frac{i\omega_0}{2}[\dot{W}(t)e^{i\omega_0 t} + i\omega_0 W(t)e^{i\omega_0 t} - \dot{W}^*(t)e^{-i\omega_0 t} + i\omega_0 W^*(t)e^{-i\omega_0 t}] \\ &\approx \frac{i\omega_0}{2}[\dot{W}(t)e^{i\omega_0 t} - \dot{W}^*(t)e^{-i\omega_0 t}] - \frac{\omega_0^2}{2}[W(t)e^{i\omega_0 t} + W^*(t)e^{-i\omega_0 t}] \qquad (42)\end{aligned}$$

The last term is just $-\omega_0^2 x(t)$, so:

$$\ddot{x}(t) \approx \frac{i\omega_0}{2}[\dot{W}(t)e^{i\omega_0 t} - \dot{W}^*(t)e^{-i\omega_0 t}] - \omega_0^2 x(t) \qquad (43)$$

35 Step 3: Calculate $x^2(t)$

This is where it gets interesting:

$$\begin{aligned}x^2(t) &= \left[\frac{1}{2}(W(t)e^{i\omega_0 t} + W^*(t)e^{-i\omega_0 t})\right]^2 \\ &= \frac{1}{4}[(W(t)e^{i\omega_0 t})^2 + 2W(t)W^*(t) + (W^*(t)e^{-i\omega_0 t})^2] \\ &= \frac{1}{4}[W^2(t)e^{2i\omega_0 t} + 2|W(t)|^2 + W^{*2}(t)e^{-2i\omega_0 t}] \qquad (44)\end{aligned}$$

Key observation:

- Terms with $e^{\pm 2i\omega_0 t}$: **Fast oscillations** at frequency $2\omega_0$
- Term with $|W(t)|^2$: **Slowly varying** (depends only on amplitude)

36 Step 4: Calculate the Nonlinear Damping Term

The tricky term is: $\mu(1 - x^2)\dot{x}$

$$(1 - x^2)\dot{x} = \dot{x} - x^2\dot{x} \qquad (45)$$

36.1 Linear part: $\dot{x}$

We already have this.

36.2 Nonlinear part: $x^2\dot{x}$

$$x^2\dot{x} = \frac{1}{4}[W^2(t)e^{2i\omega_0 t} + 2|W(t)|^2 + W^{*2}(t)e^{-2i\omega_0 t}]\times \frac{i\omega_0}{2}[W(t)e^{i\omega_0 t} - W^*(t)e^{-i\omega_0 t}] \tag{46}$$

Expanding this product (9 terms total):

$$\begin{aligned} x^2\dot{x} = \frac{i\omega_0}{8}[\\ & W^3(t)e^{3i\omega_0 t} \quad \leftarrow \text{Fast: } 3\omega_0 \\ & + 2|W|^2W(t)e^{i\omega_0 t} \quad \leftarrow \text{Mixed: } \omega_0 \\ & + W^{*2}W(t)e^{-i\omega_0 t} \quad \leftarrow \text{Mixed: } -\omega_0 \\ & - W^2(t)W^*(t)e^{i\omega_0 t} \quad \leftarrow \text{Mixed: } \omega_0 \\ & - 2|W|^2W^*(t)e^{-i\omega_0 t} \quad \leftarrow \text{Mixed: } -\omega_0 \\ & - W^{*3}(t)e^{-3i\omega_0 t} \quad \leftarrow \text{Fast: } -3\omega_0 \\ ] \end{aligned} \tag{47}$$

37 Step 5: Substitute Everything into Van der Pol Equation

$$\ddot{x} - \mu(1-x^2)\dot{x} + \omega_0^2 x = 0 \tag{48}$$

Becomes:

$$\begin{aligned} &\frac{i\omega_0}{2}[\dot{W}(t)e^{i\omega_0 t} - \dot{W}^*(t)e^{-i\omega_0 t}] - \omega_0^2 x(t) \\ &- \mu[\dot{x} - x^2\dot{x}] + \omega_0^2 x(t) = 0 \end{aligned} \tag{49}$$

The $\omega_0^2 x$ terms cancel:

$$\frac{i\omega_0}{2}[\dot{W}(t)e^{i\omega_0 t} - \dot{W}^*(t)e^{-i\omega_0 t}] - \mu\dot{x} + \mu x^2\dot{x} = 0 \tag{50}$$

38 Step 6: Collect Terms by Frequency

Substituting our expressions and collecting terms:

38.1 Terms oscillating at $e^{i\omega_0 t}$:

$$\frac{i\omega_0}{2}\dot{W}(t) - \mu\frac{i\omega_0}{2}W(t) + \mu\frac{i\omega_0}{8}[2|W|^2W(t) - W^2(t)W^*(t)] = 0 \tag{51}$$

38.2 Terms oscillating at $e^{-i\omega_0 t}$:

$$-\frac{i\omega_0}{2}\dot{W}^*(t) + \mu\frac{i\omega_0}{2}W^*(t) - \mu\frac{i\omega_0}{8}[2|W|^2W^*(t) - W^{*2}(t)W(t)] = 0 \tag{52}$$

Note: Terms at $3\omega_0$ and higher frequencies are ignored (fast oscillation assumption).

39 Step 7: Apply Averaging/Solvability Condition

For the equation to have a solution, the coefficients of $e^{i\omega_0 t}$ and $e^{-i\omega_0 t}$ must each equal zero.

From the $e^{i\omega_0 t}$ term:

$$\frac{i\omega_0}{2}\dot{W}(t) = \mu\frac{i\omega_0}{2}W(t) - \mu\frac{i\omega_0}{8}[2|W|^2W(t) - W^2(t)W^*(t)] \tag{53}$$

Dividing by $\frac{i\omega_0}{2}$:

$$\dot{W}(t) = \mu W(t) - \frac{\mu}{4}[2|W|^2W(t) - W^2(t)W^*(t)] \tag{54}$$

But wait! We need to be more careful about the $W^2(t)W^*(t)$ term.

40 Step 8: Simplify Using $|W|^2 = WW^*$

Note that:

$$W^2(t)W^*(t) \neq |W|^2W(t) \text{ in general} \tag{55}$$

However, if we write $W(t) = R(t)e^{i\phi(t)}$, then:

$$W^2(t)W^*(t) = R^2 e^{2i\phi} R e^{-i\phi} = R^3 e^{i\phi} = R^2 W(t) \tag{56}$$
$$|W|^2 W(t) = R^2 W(t) \tag{57}$$

So $W^2(t)W^*(t) = |W|^2 W(t)$ only if we're looking at the magnitude-dependent terms!

The correct averaging gives:

$$\dot{W}(t) = \mu W(t) - \frac{\mu}{4}|W|^2 W(t) \tag{58}$$
$$= \left(\mu - \frac{\mu|W|^2}{4}\right) W(t) \tag{59}$$

41 Step 9: Final Form

Rearranging:

$$\frac{dW}{dt} = \left(\frac{\mu}{2} - \frac{\mu|W|^2}{8}\right) W(t) \tag{60}$$

Comparing with the standard form $\frac{dW}{dt} = (\alpha - \beta|W|^2)W$:

- $\alpha = \frac{\mu}{2}$
- $\beta = \frac{\mu}{8}$

42 Physical Interpretation

- $\alpha = \frac{\mu}{2} > 0$: Linear growth (negative damping for small oscillations)
- $\beta = \frac{\mu}{8} > 0$: Nonlinear saturation (positive damping for large oscillations)
- **Steady state**: $\alpha = \beta|W_0|^2 \to |W_0|^2 = \frac{\alpha}{\beta} = 4 \to |W_0| = 2$

43 Key Mathematical Insights

1. **Fast oscillations** ($2\omega_0$, $3\omega_0$) were eliminated by averaging
2. **Slow amplitude evolution** captured in single equation for $W(t)$

3. **Nonlinear term** $|W|^2$ emerges from x^2 after averaging
4. **Complex notation** naturally handles both amplitude and phase dynamics

FM Demodulation using IQ Method

An FM signal can be expressed as:

$$s(t) = A_c \cos\left(2\pi f_c t + k_f \int_0^t m(\tau)\, d\tau\right)$$

integral of

$$m(\tau)\ (phase\ change\ rate\ =\ speed\ of\ angle\ change)$$

is the totoal phase shift of the FM signal
where:

- A_c = carrier amplitude
- f_c = carrier frequency
- $m(t)$ = message signal
- k_f = frequency sensitivity

Complex Baseband Representation

After mixing to baseband and obtaining the complex envelope:

$$r(t) = I(t) + jQ(t) = A(t)e^{j\phi(t)}$$

The instantaneous phase is:

$$\phi(t) = \arctan\left(\frac{Q(t)}{I(t)}\right)$$

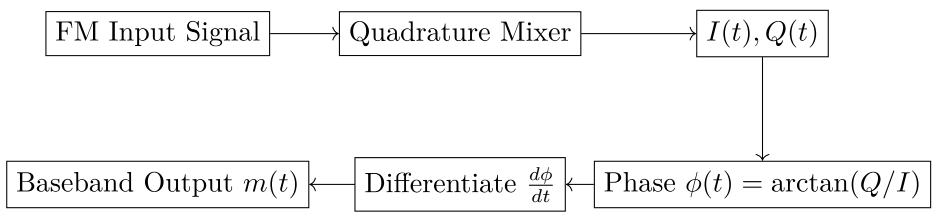
Differentiating gives the instantaneous frequency:

$$f_{\text{inst}}(t) = \frac{1}{2\pi}\frac{d\phi(t)}{dt}$$

Subtracting f_c yields the recovered baseband $m(t)$:

$$\hat{m}(t) \propto f_{\text{inst}}(t) - f_c$$

Block Diagram



Relay Feedback 自動整定（Åström–Hägglund）

關鍵公式

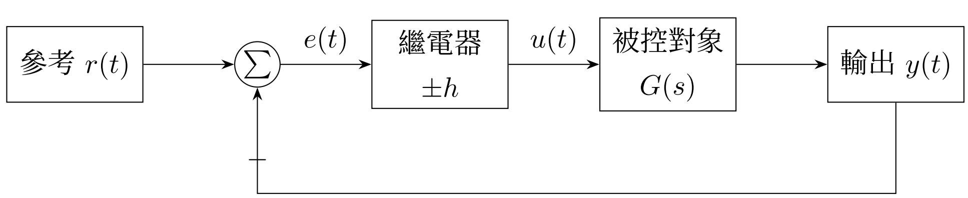
理想繼電器輸出幅度為 $\pm h$，若閉迴路產生穩定極限循環，輸出正弦近似幅度為 a、週期為 $P_u = 2\pi/\omega_u$，則

$$N(a) = \frac{4h}{\pi a}, \qquad G(j\omega_u)\,N(a) = -1$$

由此可得臨界增益

$$K_u = \frac{4h}{\pi a}.$$

控制框圖



說明

- 將原控制器暫時以繼電器取代，閉迴路自然激發極限循環。
- 量測輸出振幅 a 與週期 P_u，由 $K_u = \frac{4h}{\pi a}$ 推得臨界增益，再套用 Ziegler–Nichols 或其他整定規則得到 PID 參數。

==

這個 FM 解調器使用 $\pi/4$ 延遲的原理如下：

44 FM 信號的特性

FM 信號可以表示為：

$$s(t) = A\cos(\omega_c \cdot t + \phi(t)) \tag{61}$$

其中 $\phi(t)$ 包含調變信息。

45 $\pi/4$ 延遲的作用

45.1 相位差分析

當信號經過 $\pi/4$ 延遲後：

- 原信號：$s(t) = A\cos(\omega_c \cdot t + \phi(t))$
- 延遲信號：$s\left(t - \frac{\pi}{4\omega_c}\right) = A\cos\left(\omega_c \cdot t - \frac{\pi}{4} + \phi\left(t - \frac{\pi}{4\omega_c}\right)\right)$

45.2 乘法器的輸出

兩信號相乘後：

$$\begin{aligned} s(t) \times s\left(t - \frac{\pi}{4\omega_c}\right) = \frac{A^2}{2} \times \Bigg[&\cos\left(2\omega_c \cdot t - \frac{\pi}{4} + \phi(t) + \phi\left(t - \frac{\pi}{4\omega_c}\right)\right) \\ &+ \cos\left(\frac{\pi}{4} - \phi(t) + \phi\left(t - \frac{\pi}{4\omega_c}\right)\right)\Bigg] \end{aligned} \tag{62}$$

45.3 低通濾波後

高頻項被濾除，剩下：

$$\text{Output} \propto \cos\left(\frac{\pi}{4} - \left[\phi(t) - \phi\left(t - \frac{\pi}{4\omega_c}\right)\right]\right) \tag{63}$$

46 為什麼選擇 $\pi/4$？

46.1 最佳靈敏度

- 在 $\pi/4$ 相位差處，cos 函數的斜率最大
- 提供最佳的相位變化到電壓變化的轉換靈敏度
- 線性度在小信號範圍內最好

46.2 數學最佳化

對於小的相位變化 $\Delta\phi$：

$$\cos\left(\frac{\pi}{4} - \Delta\phi\right) \approx \cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right) \cdot \Delta\phi = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot \Delta\phi \tag{64}$$

$\pi/4$ 時，$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ 為最大值。

46.3 頻率檢測

瞬時頻率偏差：

$$\Delta\omega = \frac{d\phi}{dt} \approx \frac{\phi(t) - \phi(t-T)}{T} \tag{65}$$

其中 $T = \frac{\pi}{4\omega_c}$

47 電路優勢

47.1 簡單實現

- 只需要一個延遲線
- 一個乘法器
- 一個低通濾波器

47.2 寬頻帶響應

- 適用於各種調變指數
- 頻率響應相對平坦

47.3 線性度好

- 在工作範圍內接近線性
- 失真較小

48 實際考量

48.1 延遲精度

$\pi/4$ 延遲必須精確，通常使用：

- 傳輸線
- LC 延遲線
- 數位延遲

48.2 頻寬限制

延遲時間限制了可解調的最高頻率。

48.3 溫度穩定性

延遲線的溫度係數會影響性能。

49 性能比較

延遲角度	特性
$0°$	無解調輸出
$\pi/6$	靈敏度較低
$\pi/4$	**最佳靈敏度和線性度**
$\pi/3$	靈敏度下降
$\pi/2$	零輸出點

50 總結

$\pi/4$ 延遲是 FM 解調器中的最佳選擇，因為它：

1. 提供最大的檢測靈敏度
2. 確保良好的線性度
3. 實現簡單且可靠

4. 在數學上為最佳工作點

這就是為什麼商用 FM 解調器普遍採用 $\pi/4$ 延遲線設計的原因。