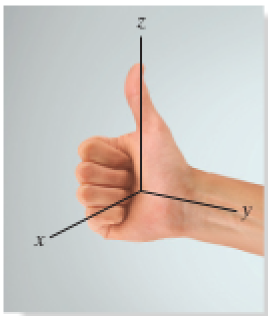
Lesson 3: Cartesian Vectors

The operations of vector algebra, when applied to solving problems in *three dimensions*, are greatly simplified if the vectors are represented in Cartesian vector form.

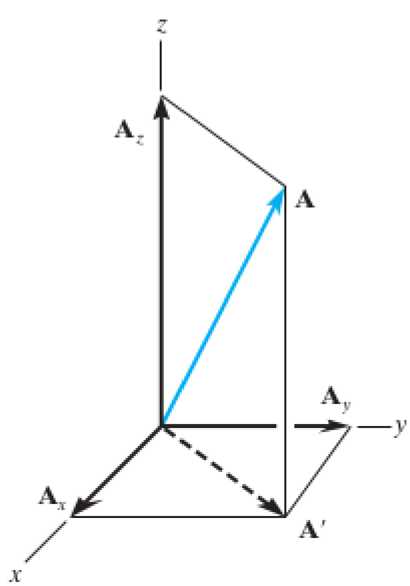
Right-Handed Coordinate System

A rectangular coordinate system is said to be **right-handed** if the thumb of the right hand points in the direction of the positive z axis when the right-hand fingers are curled about this axis and directed from the positive x towards the positive Y axis.



Rectangular Components of a Vector

A vector $\mathbf{A}$ will have three rectangular components along the x, y, z coordinate axes and the set of Cartesian unit vectors **i**, **j**, and **k**, is used to designate the directions of x, y, z axes.



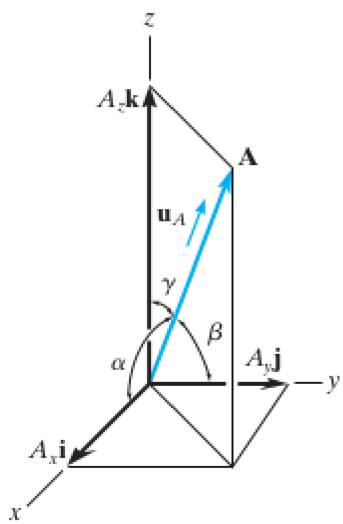
$$\mathbf{A} = \mathbf{A_x} + \mathbf{A_y} + \mathbf{A_z}$$
$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

The magnitude of $\mathbf{A}$ is

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

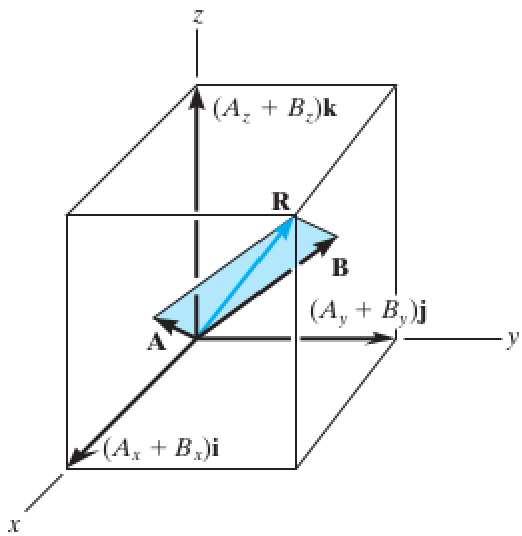
The direction of $\mathbf{A}$ can be defined the coordinate direction angles α, β, γ, measured between the tail of $\mathbf{A}$ and the *positive* x, y, z axes. Each of these angles will be between 0^o and 180^o. These are also called **direction cosines**.

$$\cos\alpha = \frac{A_x}{A}$$
$$\cos\beta = \frac{A_y}{A}$$
$$\cos\gamma = \frac{A_z}{A}$$



Addition of Cartesian Vectors

The addition or subtraction of two or more vectors is greatly simplified if the vectors are expressed in terms of Cartesian components.



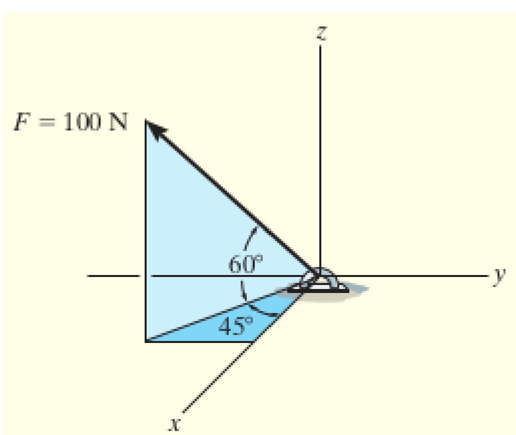
$$\mathbf{R} = \mathbf{A} + \mathbf{B}$$
$$\mathbf{R} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k}$$

In general, the force resultant is the vector sum of all forces and can be written as

$$\mathbf{R} = \sum \mathbf{F} = \sum F_x\mathbf{i} + \sum F_y\mathbf{j} + \sum F_z\mathbf{k}$$

Example 1

Express force $\mathbf{F}$ shown in the figure below as a Cartesian vector.



Solution

$$F = 100 \ \ (\mathrm{N})$$

Calculation of xy component of F

$$F_{xy} = F \cdot \cos\left(60 \cdot \frac{\pi}{180}\right) = 100 \cdot \cos\left(60 \cdot \frac{3.142}{180}\right) \qquad = 50.0 \ \ (\mathrm{N})$$

Calculation of F_x

$$F_x = F_{xy} \cdot \cos\left(45 \cdot \frac{\pi}{180}\right) = 50.0 \cdot \cos\left(45 \cdot \frac{3.142}{180}\right) \qquad = 35.355 \ \ (\mathrm{N})$$

Calculation of F_y

$$F_y = F_{xy} \cdot \sin\left(45 \cdot \frac{\pi}{180}\right) = 50.0 \cdot \sin\left(45 \cdot \frac{3.142}{180}\right) \qquad = 35.355 \ \ (\mathrm{N})$$

Calculation of F_z

$$F_z = F \cdot \sin\left(60 \cdot \frac{\pi}{180}\right) = 100 \cdot \sin\left(60 \cdot \frac{3.142}{180}\right) \qquad = 86.603 \ \ (\mathrm{N})$$

$$\begin{aligned} F &= F_x \cdot i - F_y \cdot j + F_z \cdot k \\ &= 35.355 \cdot i - 35.355 \cdot j + 86.603 \cdot k \\ &= 35.355i - 35.355j + 86.603k \end{aligned}$$

Calculation of Directions Cosines

$$F = 100$$

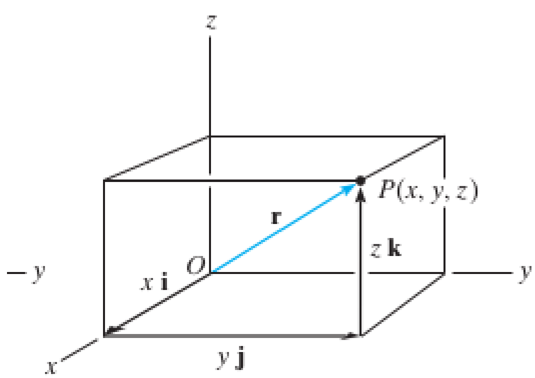
$$\alpha = \arccos\left(\frac{F_x}{F}\right) \cdot \frac{180}{\pi} = \arccos\left(\frac{35.355}{100}\right) \cdot \frac{180}{3.142} = 69.295$$

$$\beta = \arccos\left(\frac{(-F_y)}{F}\right) \cdot \frac{180}{\pi} = \arccos\left(\frac{(-35.355)}{100}\right) \cdot \frac{180}{3.142} = 110.705$$

$$\gamma = \arccos\left(\frac{F_z}{F}\right) \cdot \frac{180}{\pi} = \arccos\left(\frac{86.603}{100}\right) \cdot \frac{180}{3.142} = 30.0$$

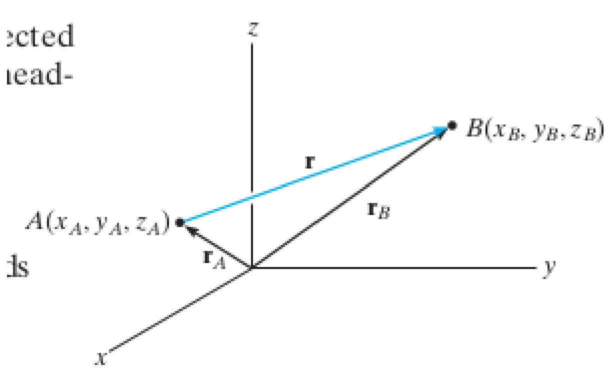
Position Vectors

A *position vector* of **r** is defined as a fixed vector which locates a point in space relative to another point.



$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

In general, the position vector may be directed from point *A* to *B* in space. By the head-to-tail vector addition:



$$\mathbf{r_A} + \mathbf{r} = \mathbf{r_B}$$

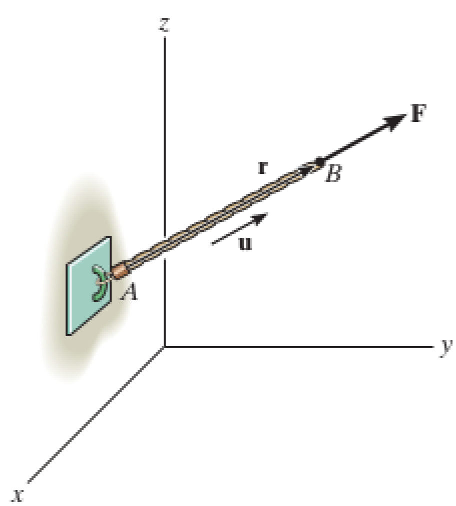
Solving for $\mathbf{r}$

$$\mathbf{r} = \mathbf{r_B} - \mathbf{r_A}$$
$$\mathbf{r} = (x_B\mathbf{i} + y_B\mathbf{j} + z_B\mathbf{k}) - (x_A\mathbf{i} + y_A\mathbf{j} + z_A\mathbf{k})$$
$$\mathbf{r} = (x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k}$$

Force Directed along a Line

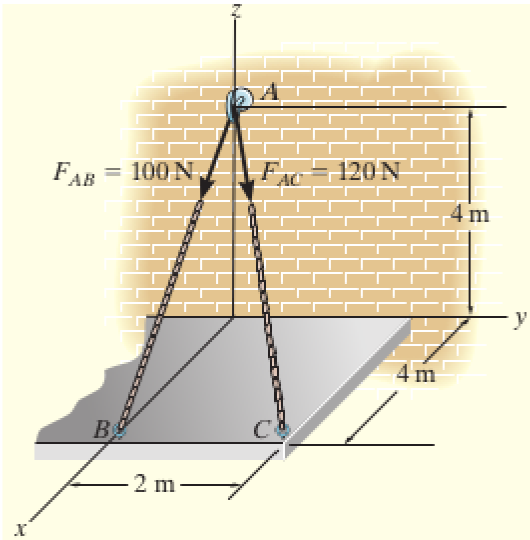
A force $\mathbf{F}$ acting in the direction of a position vector $\mathbf{r}$ can be represented in Cartesian form if the unit vector $\mathbf{u}$ of the position vector is determined and it is multiplied by the magnitude of the force.

$$\mathbf{F} = F\mathbf{u} = F\frac{\mathbf{r}}{r}$$
$$\mathbf{F} = F\frac{(x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k}}{sqrt(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$$



Example 2

The roof is supported by two cables as shown in the figure below. If the cables exert forces $F_{AB} = 100$ N and $F_{AC} = 120$ N on the wall hook at *A*, determine the resultant force acting at *A*. Express the resultant as a Cartesian vector.



Solution

Considering $\mathbf{F_{AB}}$

$$x_A = 0 \qquad y_A = 0 \qquad z_A = 4$$

$$x_B = 4 \qquad y_B = 0 \qquad z_B = 0$$

$$x_{AB} = x_B - x_A = 4 - 0$$

$$y_{AB} = y_B - y_A = 0 - 0$$

$$z_{AB} = z_B - z_A = 0 - 4$$

$$r_{AB} = x_{AB} \cdot i + y_{AB} \cdot j + z_{AB} \cdot k = 4 \cdot i + 0 \cdot j + -4 \cdot k \qquad =$$

$$u_{AB} = \frac{r_{AB}}{\sqrt{(x_{AB})^2 + (y_{AB})^2 + (z_{AB})^2}} = \frac{4i - 4k}{\sqrt{(4)^2 + (0)^2 + (-4)^2}} \qquad = 0.7071i -$$

$$F_{AB} = 100 \cdot u_{AB} = 100 \cdot 0.7071i - 0.7071k \qquad = 70.711i -$$

Considering $\mathbf{F_{AC}}$

$$x_C = 4 \qquad y_C = 2 \qquad z_C = 0$$

$$x_{AC} = x_C - x_A = 4 - 0 \qquad =$$

$$y_{AC} = y_C - y_A = 2 - 0 \qquad =$$

$$z_{AC} = z_C - z_A = 0 - 4 \qquad = -$$

$$\begin{aligned} r_{AC} &= x_{AC} \cdot i + y_{AC} \cdot j + z_{AC} \cdot k \\ &= 4 \cdot i + 2 \cdot j + -4 \cdot k \\ &= 4i + 2j - 4k \end{aligned}$$

$$\begin{aligned} u_{AC} &= \frac{r_{AC}}{\sqrt{(x_{AC})^2 + (y_{AC})^2 + (z_{AC})^2}} \\ &= \frac{4i + 2j - 4k}{\sqrt{(4)^2 + (2)^2 + (-4)^2}} \\ &= 0.6667i + 0.3333j - 0.6667k \end{aligned}$$

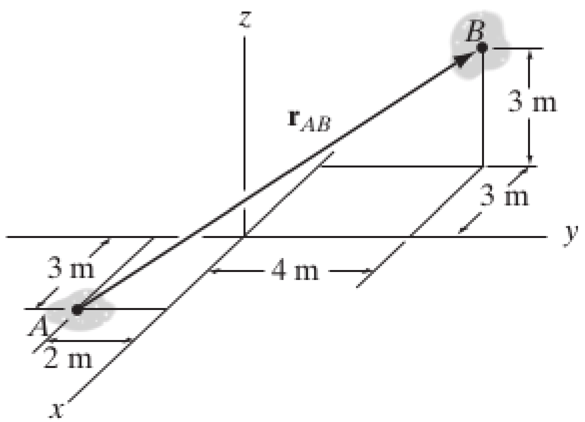
$$F_{AC} = 120 \cdot u_{AC} = 120 \cdot 0.6667i + 0.3333j - 0.6667k \qquad = 80.0i + 40.0j - 80.$$

Calculation of the resultant, R

$$R = F_{AB} + F_{AC} = 70.711i - 70.711k + 80.0i + 40.0j - 80.0k \qquad = 150.711i +$$

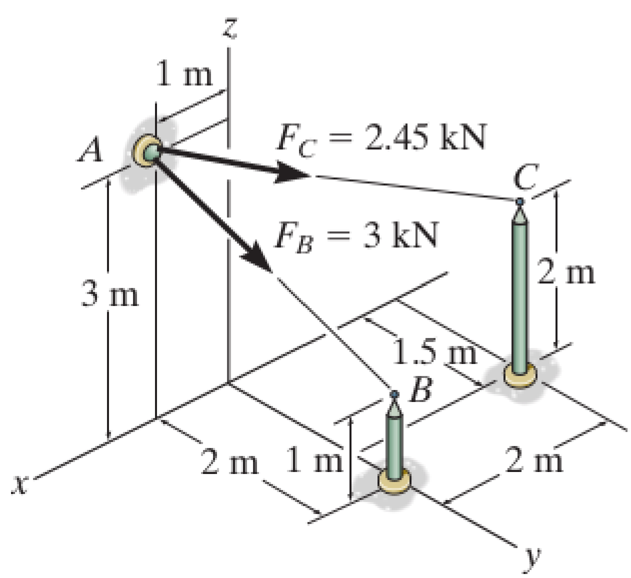
Problem 1

Express $\mathbf{r_{AB}}$ as a Cartesian vector, then determine its magnitude and coordinate direction angles.



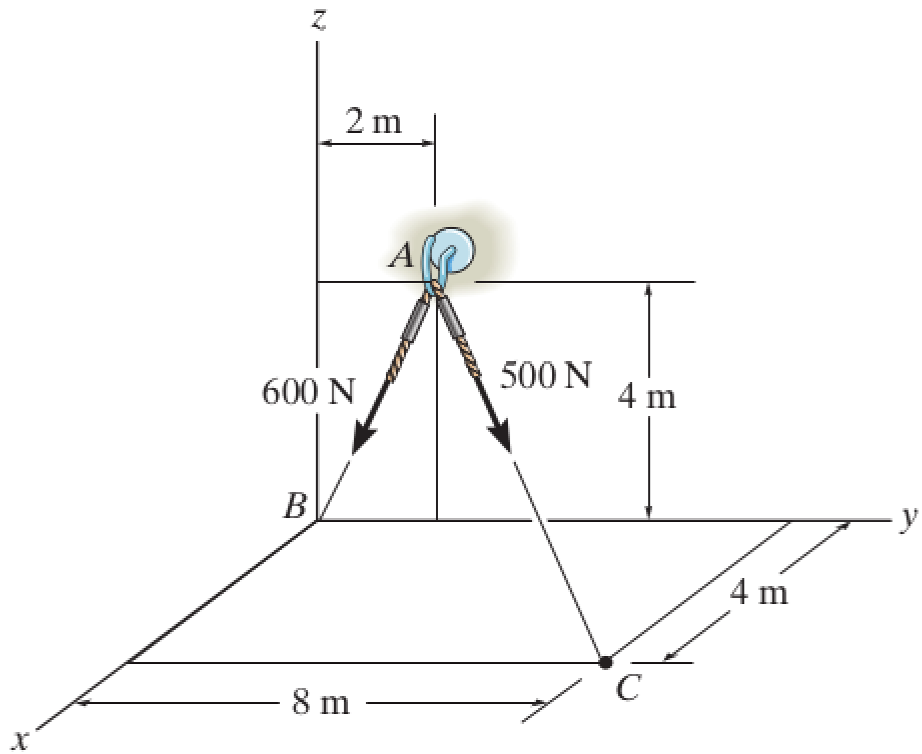
Problem 2

Determine the resultant force at *A*.



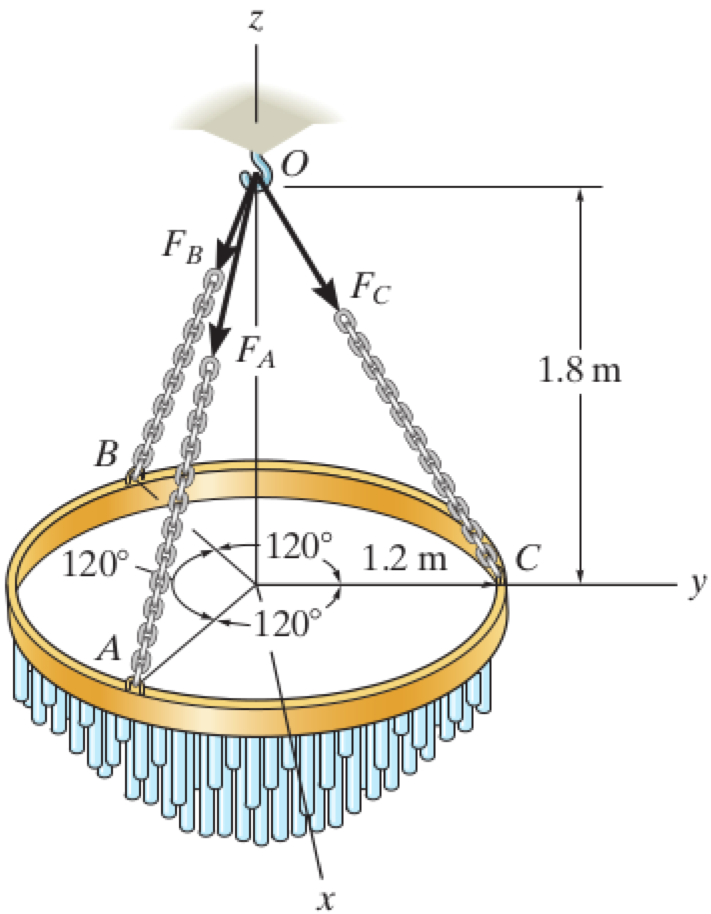
Problem 3

Determine the magnitude and coordinate direction angles of the resultant force.



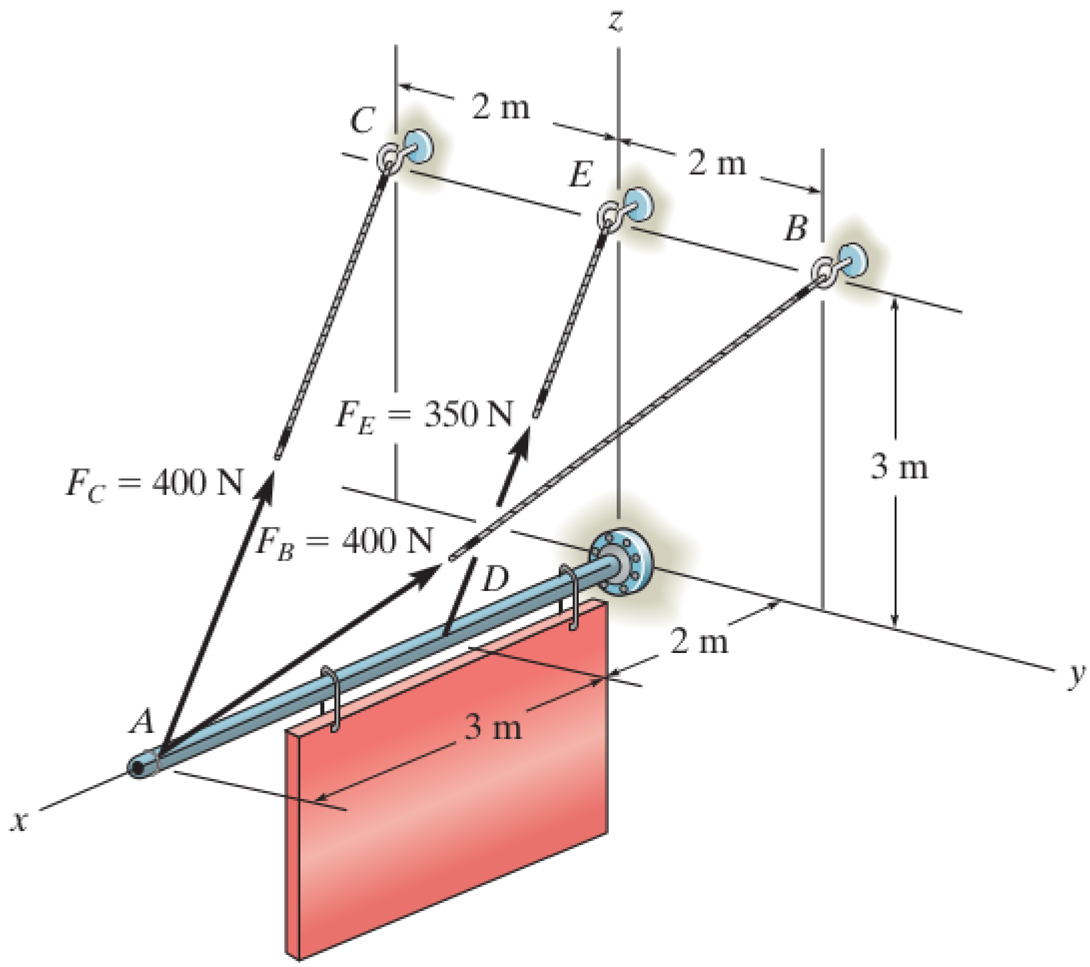
Problem 4

The chandelier is supported by three chains which are concurrent at point O. If the resultant force at O has a magnitude of 650 **N** and is directed along the negative z axis, determine the force in each chain.



Problem 5

Determine the magnitude and coordinate direction angles of the resultant force of the two forces acting at point *A*.


z
2 m
C
E
2 m
B
$F_E = 350$ N
3 m
$F_C = 400$ N
$F_B = 400$ N
D
2 m
y
A
3 m
x