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Billiard Trajectories : The Mathematics in Billiards (First Draft)

Billiards has existed for several hundred years, and throughout its lifetime it has evolved into several different variations of the game, such as English snooker, 8 ball pool, 3 cushion, and many more. Although there are many different forms of the game, almost all of them involve a rectangular table with walls that serve as a boundary for balls to bounce off of after they are hit in a certain direction by a long wooden object used to apply a force to the ball, known as a cue. The trajectory that a pool ball follows after it is hit by the cue is far from random, and geometry can be used to model the game of billiards mathematically. In this mathematical analysis, we will take a billiards table of dimensions $m \times n$, and we will assume conservation of energy on a frictionless surface to model various unique situations in which the ball travels. In particular, we will use mathematics to predict which trajectories never "close up," with the ball never returning to its starting position, as well as trajectories that do close up and with what angle.

The first step in analyzing the trajectories of billiards is figuring out how the ball interacts with the walls of the table. In an ideal setting, billiards follows the physics law of optics, which is shown in figure 1. After applying a force to the ball, it follows a straight path until it hits one of the walls, where it will bounce off with an angle of reflection that is equal to its angle of incidence, as shown in the diagram below. Since we are assuming that our table is frictionless and void of any other possible obstructions which may alter the direction of the ball other than the walls and the cue, then this law of physics will hold true. We will also assume that our ball has no volume, so that it will not get "stuck" near the corners, meaning that it can only hit one wall or an intersection of two walls at any given time. Finally, we will assume that the force applied by the cue always causes the ball to move in a straight line, even though the game often involves techniques to add spin to the ball, which can cause it to move in a more complicated trajectory.

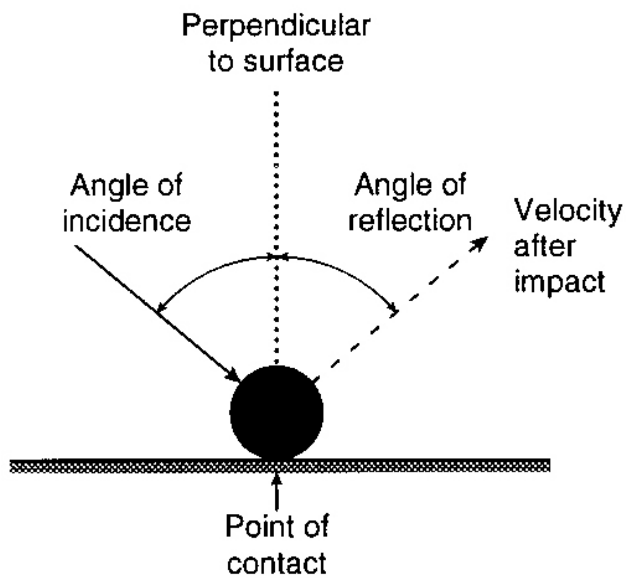
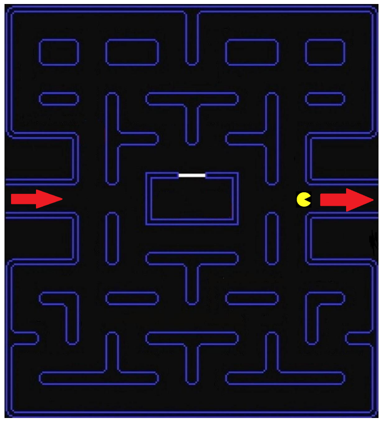
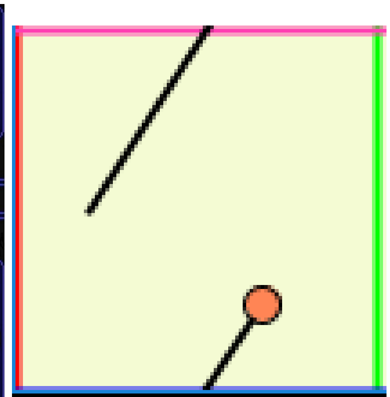


Figure 1: Angle of Incidence (Medici 1)

Now that we have some basic guidelines set on how the cue, ball, and table interact, we can begin to look at the different paths that the ball can take within our billiards table. We can start with a table of $m \times n$ dimensions with $m, n \in \mathbb{Z}^+$, and then create an equation for the travel of the ball using a vector to represent the horizontal and vertical speed of the ball in standard motion. Right away it is obvious that we have too many variables to work with. Instead, we can represent the vector with an angle, θ, since the length of the vector is irrelevant. Another problem arises due to the "bouncing" aspect of the game of billiards. Since this makes our projectile path much more complicated, we can look at a similar situation, in which the billiards table has the same characteristics of an old arcade game, such as Pacman. When Pacman leaves the side of the screen, he end up on the other side, traveling at the same angle. An example of this, as well as what this effect looks like in billiards are shown below. Just like in the game of Pacman, if the ball hits the top of the billiards table (the pink side in the figure below), it will continue traveling at the same angle from the bottom of the table (the blue side).



(a) Pacman Example



(b) Billiards Example

From this perspective, some important patterns start to emerge. Since some of these patterns are easier to see with a square billiards table, we will continue to use this shape as an example. If we continue to follow the initial path of the projectile and duplicate the pool table, we can use the intersections of this path with the new sides to figure out the next path that the ball will take.

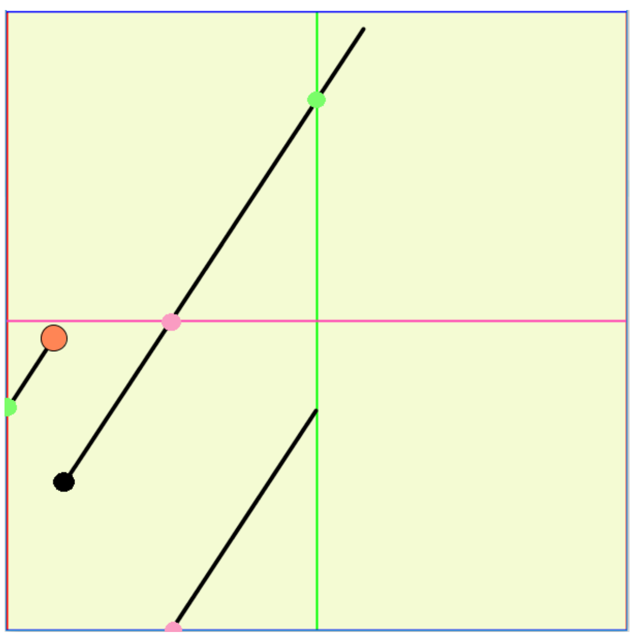


Figure 3: Following the Initial Path of the Ball

In the figure above, the ball starts at the black circle, and continues to move and reflect using the rules we set earlier, until it reaches the orange circle. From the diagram, we can see how the original straight line of the ball's path can be used to find every point that the ball will pass through. The ball is reflected to the bottom of the table when it hits the top side, but if we add another table on top of the original table, then we can easily contain the next path of the ball. The extended path will pass through the new table as if the ball was starting its trajectory at the bottom of the table, like when it performs one of the Pacman reflections. This method of finding the next path of the ball is not limited to just one duplicated table. As in the diagram, the next path of the ball can be modeled in yet another replicated table. This is an effective way to model the path of the billiards ball, since we cut down all of our potential variables to only the angle θ and the starting position of the ball.

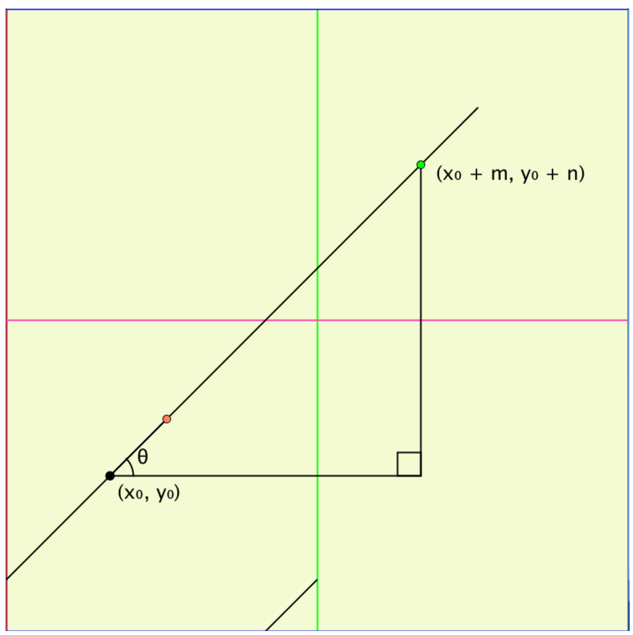


Figure 4: Tracing a Billiards Trajectory with (x_0, y_0) and θ

Now that we have developed an effective model of the modified "Pacman" version of billiards, we can find the closed paths of the ball. We know that the model predicts which points the ball will pass through from the initial trajectory, so it is crucial that the initial trajectory eventually passes through the same starting point in one of the duplicated tables. If the ball starts at some point (x_0, y_0), we can hit it at an angle θ so that it passes through a point in one of the duplicated tables with the same (x_0, y_0) relative to that table, which we will define as the

”equivalent starting coordinates”. If the ball passes through its equivalent starting coordinates at 1 table to the right and 1 table up, then the resulting billiards trajectory could look like the figure above.The ball starts at the point represented by the black circle at (x_0, y_0), and it travels at angle θ, with its original trajectory intersecting the equivalent starting coordinates at $(x_0 + m, y_0 + n)$, represented by the green point. The ball is represented by the orange circle, and at this point in time it has just looped passed the initial starting point. From this representation, the angle θ of the trajectory can be easily calculated using trigonometry. If the base of the triangle is b, and the height is h, then we can calculate these variables using our coordinate system, and take the tangent to find θ.

$$w = x_0 + m - x_0 = m \qquad h = y_0 + n - y_0 = n$$
$$tan(\theta) = \tfrac{h}{w} = \tfrac{n}{m}$$
$$\theta = tan^{-1}(\tfrac{n}{m})$$

From this calculation, we discover another useful property that these trajectories have. Even though the trajectory starts at the point (x_0, y_0), since we are looking for closed loops which end up at this same point (x_0, y_0) in a duplicated table, then these coordinates will cancel out, indicating that the angle θ for closed loops is independent of the starting point. In the case of this trajectory, we found that $\theta = tan^{-1}(\frac{n}{m})$, but that does not mean that this value is unique. For example, as shown in the figure below, we could use a different trajectory that passes through 3 duplicated tables before intersecting with its equivalent starting coordinates in a duplicate.

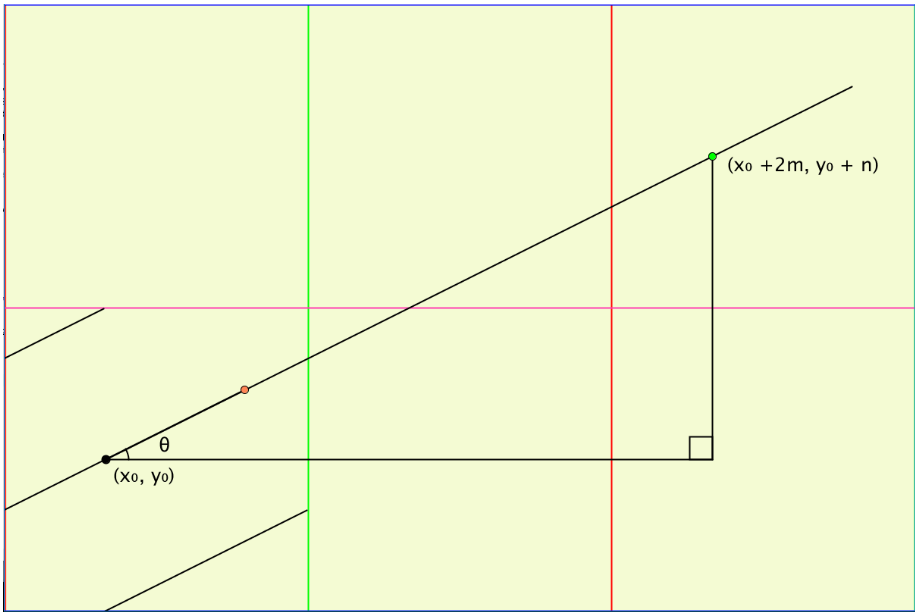


Figure 5: Alternate θ for a perfect loop

We can calculate the angle θ for this new scenario:

$$w = x_0 + 2m - x_0 = 2m \qquad h = y_0 + n - y_0 = n$$

$$tan(\theta) = \frac{h}{w} = \frac{n}{2m}$$

$$\theta = tan^{-1}(\frac{n}{2m})$$

Even though these two angles are different, they both follow an important pattern. The angle θ of the trajectory will always be the inverse tangent of $\{\frac{n}{m} \times \lambda : \lambda \in \mathbb{Q}\}$. If we further analyze λ], it is clear that for $\lambda = \frac{a}{b}$, where $\frac{a}{b}$ is in its simplest form, a is the number of duplicate table rows that the initial trajectory must pass through other than the initial row before intersecting the equivalent starting coordinates, and b is the number of duplicate table columns other than the initial column that the trajectory must pass through. Also, for each new column or row that the trajectory passes through before intersecting the equivalent starting coordinates in one of the duplicates, the ball does another "Pacman reflection," so we can find the total number of unique reflections, r, that the ball undergoes for a closed loop to be $r = \{a + b : \lambda = \frac{a}{b}\}$. If a loop is not closed, then that means λ is irrational. We just figured out that the number of unique reflections, or paths, that the ball takes is equal to $a + b$, so how many unique reflections does an open path have with an irrational λ? A simple way to think about this is to try to convert an irrational number into a fraction. We can start by using a rational number as an example, $q = 1.912$. To convert q to a fraction, the process is as follows:

$$q = \frac{a}{b} = \frac{1}{1} + \frac{9}{10} + \frac{1}{100} + \frac{2}{1000}$$

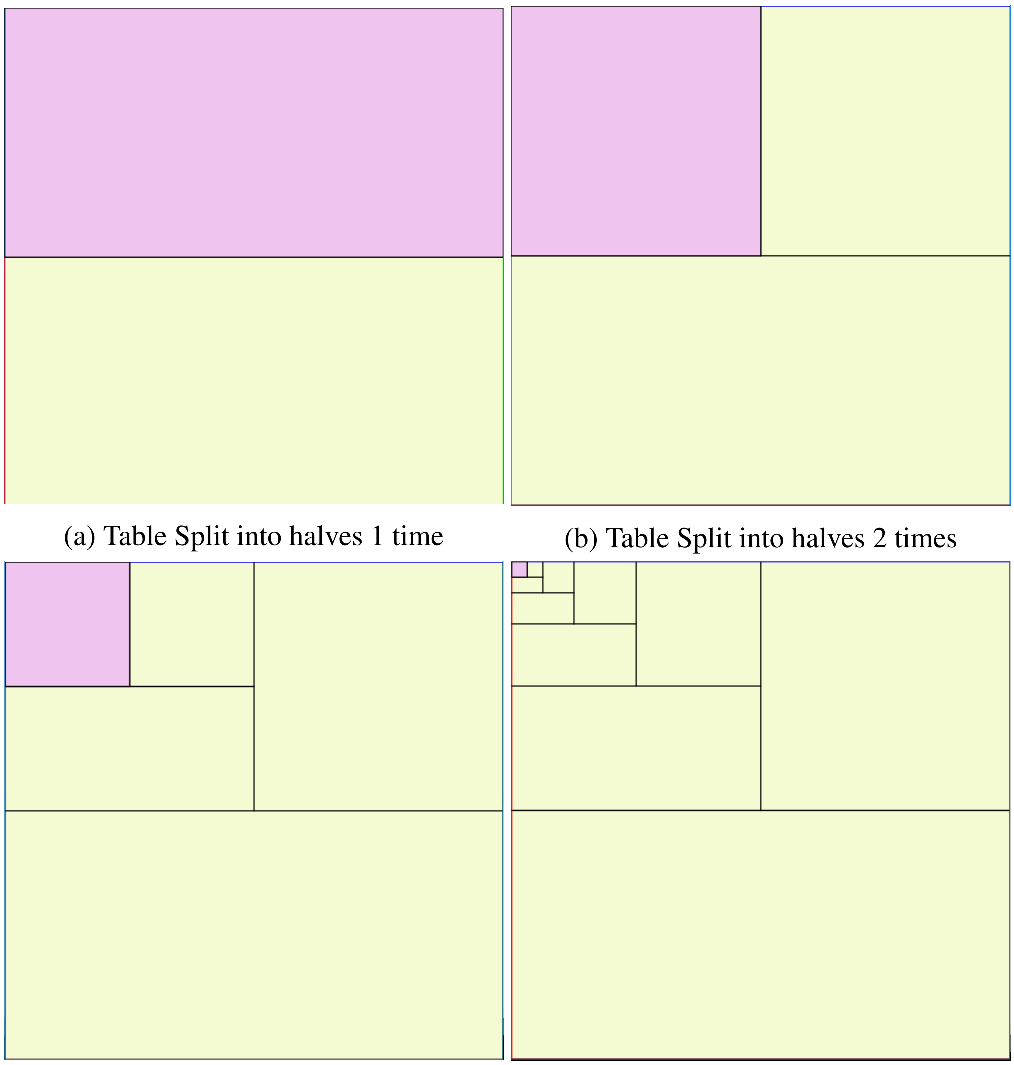
If we do the same process for an irrational, such as $p = \pi \approx 3.14159\ldots$

$$p = \frac{a}{b} = \frac{3}{1} + \frac{1}{10} + \frac{4}{100} + \frac{1}{1000} + \frac{5}{10000} + \frac{9}{100000} + \ldots$$

No matter how many combinations of fractions we add together, there will always be another smaller fraction to add. For every new fraction that we add to the sum of fractions, the denominator will continue to increase. Even if the fraction could be simplified, such as for $\frac{4}{100}$ in the case above, the denominator of the fraction that represents the digit which holds the place of 10^{-n} will always be greater than the previous fraction which holds the place of 10^{-n+1}. This can be validated by showing that for some number represented by a digit d from 0 to 9, $d \times 10^{-n} < 1 \times 10^{-n+1}$, where $1 \times 10^{-n+1}$ is the smallest possible value that the previous digit can represent. Therefore, since we know that the denominator is infinitely increasing, we can conclude that a path which does not close up will have infinitely many unique reflections.

Despite this conclusion, the ball still does not go through every point on the table. For example, with a table of rational dimensions. If the trajectory has a $tan(\theta)$ that is irrational, then the path will only go through irrational points for all generic starting points (the ball can go through one rational point if the starting point has irrational parts that are canceled out by the irrational parts of the ball's trajectory, but this is not relevant to analyzing the paths that the ball takes). Knowing this information, we can conclude that even though the ball has infinitely many unique paths and therefore passes through infinitely many points, it still misses infinitely many points since it does not go through any of the rationals. This analysis concludes that the ball misses some points, but it does not yield enough information to conclude that the ball misses all regions. In order to find out if the ball passes through every region, we can apply the pigeon hole principle. The pigeon hole principle states that for finite sets, given n pigeons and $m < n$

pigeonholes, at least one pigeonhole must contain more than one pigeon. (Wolfram 1) This principle can be applied to infinite sets as well. If infinitely many pigeons are stuffed into a finite number of pigeonholes, there will exist at least one pigeonhole having infinitely many pigeons stuffed into it. In the case of a billiards table, the pigeons are the unique paths of the ball after every reflection, and the pigeonholes are the smaller regions of some larger region.



(a) Table Split into halves 1 time

(b) Table Split into halves 2 times

(c) Table Split into halves 4 times

(d) Table Split into halves 10 times

We can start off by analyzing the entire region of the table split in half (a). By the pigeonhole principle, since a trajectory that is not closed has infinitely many unique paths, then this implies that at least one of these halves must contain infinitely many unique paths. If we decide that the top half, filled in pink, contains infinitely many unique paths, then we can split this region again (b), and conclude that one of the halves has infinitely many unique paths. We can keep on splitting our region (c, d), and we will always come to the same conclusion; one of the halves of the region contains infinitely many unique paths, no matter how small the region is. Therefore, we can conclude that there exists some region, which we can refer to as c, which contains infinitely many unique paths, as long as c has some real dimensions (it is not just a point). This is still not enough to claim that the ball does not miss any regions, since all the pigeonhole principle helps to conclude is that there exists some c that the ball passes through, no matter what its size, within the larger region. We are able to finally conclude that the ball does not miss any region by realizing that *every* region has infinitely many unique paths. If we define $\{r_1 < r_2 : r_1, r_2 \in \mathbb{R}\}$, then it is easy to see that there are infinitely many real numbers between these two reals, since for some real scalar, $\mu < 1$, we can find another real $\{r_3 : r_1 < r_3 < r_2\}$, where $r_3 = r_1 + \mu(r_2 - r_1)$. Therefore, we also know that there are infinitely many irrationals between every two rationals. The ball *must* pass through infinitely many of these irrationals, since for our region c that we found earlier with the pigeonhole principle, there must be some other region which has the same cardinality of paths as c, since every path that passes through c is reflected by the same rules. Since we know that the path never closes up, then c cannot be equal to d. Thus, for every set of paths A, $\{\exists! B : |A| = |B| \implies A \longleftrightarrow B\}$. Thus, any larger region can be shrunk down with the pigeonhole principle to some smaller region for which we are unsure that the ball hits.

Up until this point, we have assumed that for our billiards table of dimensions $m \times n$, both m and n are integers. Now that we have analyzed some of the closed paths on this specific type of table, we can analyze some other cases, such as a table with dimensions $1 \times k$. One side can be left with a size of 1, since in analyzing paths we only take into account the tangent of the angle. We found earlier that for a path to close up it needs to have a $tan(\theta)$ that is equal to some scalar of the ratio of the table height over the table width, so if the tangent of the angle of some unclosed path is $\frac{a}{k}$, then scaling this quantity by $\frac{1}{a}$ will still yield another unclosed path, but with $tan(\theta) = \frac{1}{ak}$. Therefore, we can leave a side as 1, since one of the sides can always be scaled down to 1. So, if we have a table of dimensions $1 \times k$, then instead of all closed paths having a rational $tan(\theta)$ and all unclosed paths having an irrational $tan(\theta)$, all closed paths now have a $tan(\theta)$ that is equal to any rational multiplied by the height of the table, k, since for $\{\frac{n}{m} \times c = \frac{k}{1} \times c = k \times c : c \in \mathbb{Q}\}$. It is important to note this small difference, since $\{k \times c \notin \mathbb{Q} : c \in \mathbb{Q}, k \notin \mathbb{Q}\}$, so if $k \notin \mathbb{Q}$, then the $tan(\theta)$ of any closed path will also be irrational.

The ”Pacman” version of billiards is a good starting point, but there is still more to investigate with normal billiards, which involves reflecting off of the walls rather than reappearing across the width or height in a ”Pacman reflection.” We can use the same duplicate table model to analyze billiards, but there are some important differences. The first major difference lies in the geometry of the path compared to the initial trajectory. In the Pacman billiards, every path line was parallel to the initial trajectory, but now only half of the path lines are.

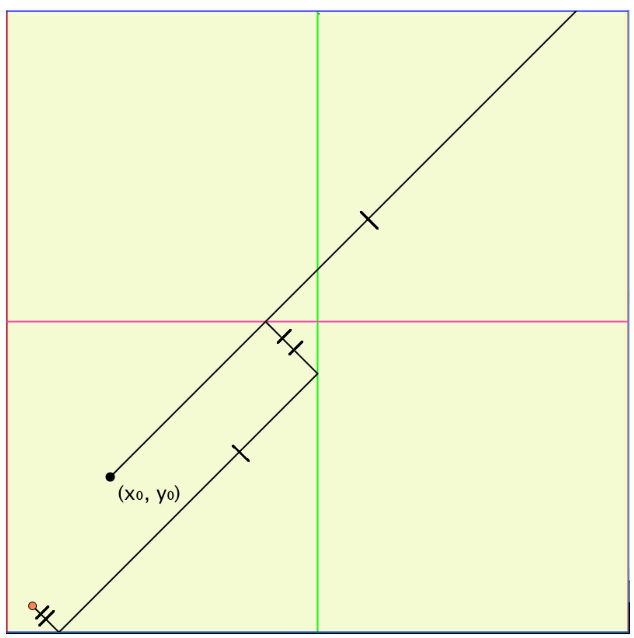
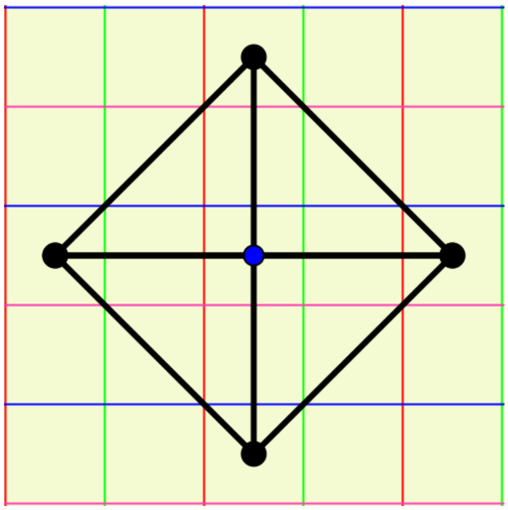


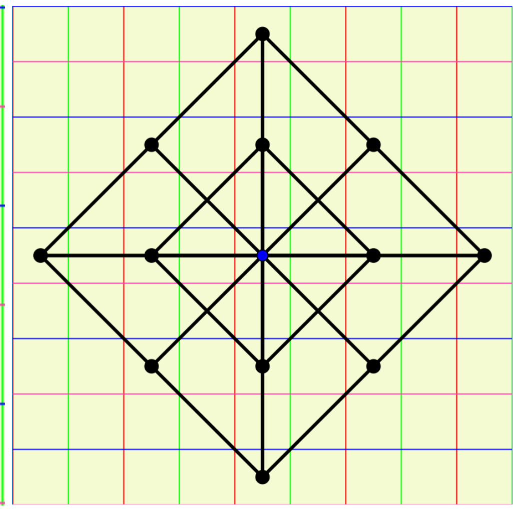
Figure 7: Billiards with Normal Reflections

Also, the orientation of the initial trajectory does not always line up with the orientation of the path trajectories, even if they are parallel. From the diagram, the first duplicate board that the table must go through has to be flipped along the x-axis to match up with the direction of the real path. The second duplicate board must be flipped across the x-axis *and* the y-axis to match up with the orientation. It is clear that for every horizontal duplicate table that the ball passes through the table must be flipped across the y-axis, and for every vertical duplicate table that the ball passes through the table must flip across the x-axis. Since two flips compose to the identity function, we can conclude that for every z duplicated horizontal tables that the ball passes through, the table must be flipped once across the y-axis if z is odd, and if z is even then no flip is necessary. Similarly, for every w duplicated vertical tables that the ball passes through, the table must be flipped once across the x-axis if w is odd, and no times if w is even. Therefore, it is clear that for a path to close up, it must pass through an even number of tables, otherwise it is impossible for the the angle of the ball's travel to be the same as the angle of the starting trajectory, since even if it does happen to go through the starting point, it will be perpendicular to the starting angle. To conclude all of these statements, if we are analyzing closed paths of a table with billiards trajectories, then we only have to look at the paths for which $w + z$ is even, if w represents the number of duplicated vertical tables that the ball passes through before hitting our point of analysis, and z the number duplicated horizontal tables.

We can model what all of these closed paths look like by plotting the initial starting point in the duplicated tables, while also taking into account our extra rules for billiards. The graphing of these paths resolves quite neatly by drawing them in sets, where for a complete set of paths, each path passes through the same number of duplicated tables. For example, a path that ends up intersecting with its duplicated starting point at 3 horizontal duplicated tables and one vertical duplicated table away from its original table would be graphed in the same set as the path that intersects with its duplicated starting point at 1 horizontal duplicated table and 3 vertical duplicated tables away. We can define a set P^n to include all of the paths which intersect with their duplicate starting point exactly n tables away, and n will be even for any closed path, so $n \in 2\mathbb{Z}$ for our closed path analysis. The simplest closed paths are those which intersect with their duplicate starting point in a table that has no flip, which is any table that has both an even w and an even z. We will also consider these simple paths to be a special case and give them their own set, denoted by S^n, where $S^n \subset P^n$. If we start off by just graphing S^2 and S^4, we can already start to see a pattern emerge.



(a) Graph of S^2



(b) Graph of S^2 and S^4

Clearly, the shape of $\{S^n : n \in 2\mathbb{Z}\}$ is a rhombus. Since the duplicated starting point is never reflected, then the angle of each trajectory stays constant for all starting points. Even if the starting point in this image is translated, the intersection point will be translated by an equal amount since the table that it lies in is a direct copy of the starting table. It is also interesting to note that in increasing n from 2 to 4, only 4 more unique paths were added. The horizontal and vertical paths will always remain the same, but every time we increase n by 2, the number of duplicated tables that can be chosen from increases. For S^2, the only tables with $z + w = 2$ are $(z, w) = (2, 0), (0, 2)$. For every unique pair (z, w) that does not contain 0, there are 4 more paths, and 2 more paths if z or w is equal to 0. We can confirm this by thinking of z and w as having direction, so that z represents the quantity of duplicated tables to the right if positive, and to the left if negative. Similarly, w represents the amount of duplicated tables up if positive, and down if negative. To be in S^n, all that matters is the magnitude of z and w, but their direction is useful in determining the total amount of paths. Therefore, if we know that the path goes through a certain number of tables horizontally or vertically, there must be a path that travels through those tables positively, and a path that travels through them negatively. So it makes sense that S^2 has 4 paths.

For S^4, the pairings of z and w that exist are $(z, w) = (4, 0), (0, 4), (2, 2)$. Out of these pairings, 2 are scalars of the pairings from S^2 since $2 * (2, 0) = (4, 0)$ and $2 * (0, 2) = (0, 4)$. Therefore, we expect to only get 4 more different paths for S^4 from $(2, 2)$, and 8 paths in total. If we keep going on to analyze S^6, we can see that $(z, w) = (6, 0), (0, 6), (2, 4), (4, 2)$, from which $(2, 4)$ and $(4, 2)$ are not scalars of any of the previous paths from S^2 or S^4. Therefore, S^6 has 8 more unique paths added to it, and 12 paths total.

We can find the total number of combinations of z and w for any n by utilizing multisets. If we find the number of multisets of cardinality 1 taken from a set of size $n + 1$, then this is equivalent to finding the number of pairs (z, w) for which $z + w = n$. This can be verified with the stars and bars graphical aid (Brilliant 1). If we view n as a row of n stars, then it is possible to figure out all of the ways to split this row of stars in two by putting a bar somewhere between them. For example, if $n = 4$, then our configuration might look like the following.

$$\star \quad \star \quad \star \quad | \quad \star$$

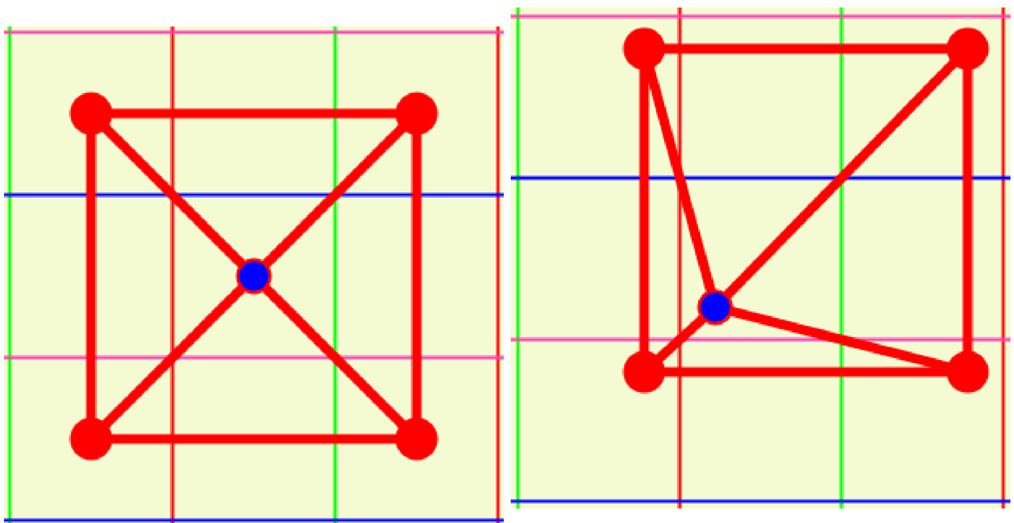
Since we are finding pairs, then we are splitting the stars in half, so let $k = 2$. It is clear that there are $n + k - 1$ total objects, and if we place all $k - 1$ bars, then that leaves only n available places for the stars. Thus, we can find the number of pairs (z, w) for which $z + w = n$ by computing $\binom{n+k-1}{n}$. Now that we know the number of total pairs, we need to find the total number of pairs for which z and w are even so that their coordinate produces a path in S. Two of the configurations of stars and bars for $n = 4$ will look like the following:

$$| \quad \star \quad \star \quad \star \quad \star$$
$$\star \quad \star \quad \star \quad \star \quad |$$

As we determined earlier, every unique pair will yield 4 paths if both z and w are not equal to 0, and 2 paths otherwise. Since we that there are 2 star and bar configurations yield a pair with 0 (as shown above), then we know that there are 2 pairs which only represent 2 paths instead of 4. Thus, we can finally find the total number of paths produced by the pairs with the expression

$$4 * \binom{n+k-1}{n} - 2 * 2 = 4 * \frac{(n+2-1)!}{(n)!(n+2-1-n)!} - 4 = \frac{4(n+1)!}{n!} - 4 = 4(n+1) - 4 = 4n$$

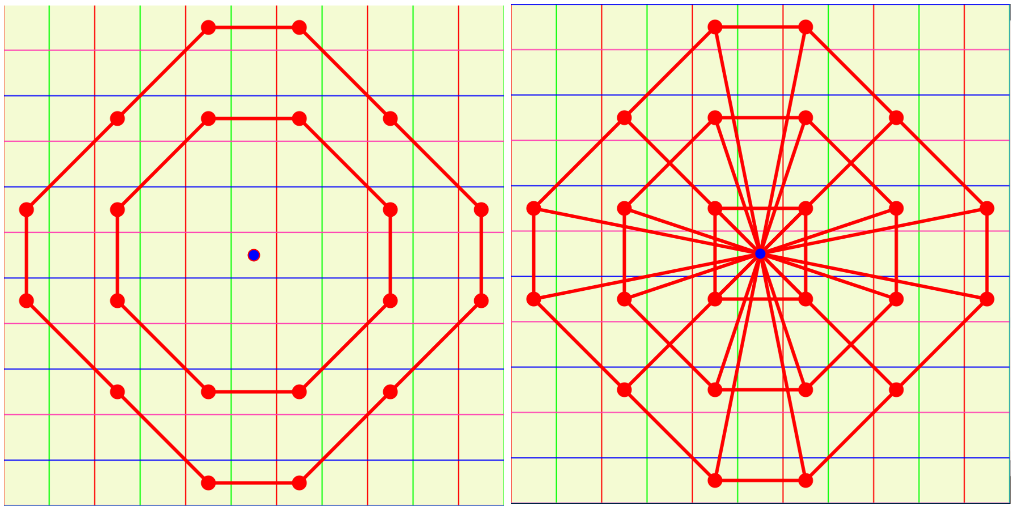
For S, we are only looking at paths for which z and w are both even, so we can divide the total number of paths by 2. Finally, we can conclude that for S^n, there are $2n$ paths. We can also conclude that since S^n always contains 4 paths that are not unique (4 paths are always associated with some pair that has either $z = 0$ or $w = 0$), then as n increases by 2, there are $2(n + 2) - 4$ more unique paths.



(a) Graph of F^2

(b) Graph of F^2 with translated starting point

Next, we can analyze the paths that intersect their duplicated starting point in a table that is flipped. Since we are only analyzing closed paths, then for P^n, n must still be even, but now z and w are both odd. Therefore, the table with the intersection point will be flipped along both the y-axis and the x-axis. We can also consider these paths that end up in double flipped tables to be a special case and give them their own set, denoted by F^n, where $F^n \subset P^n$ and $|F^n| + |S^n| = |P^n|$. If we start off by just graphing F^2, we can see that we also get a rhombus shape (figure b), but now when the starting point is translated, the angles of the paths change. The previous angles will still result in closed paths, but they will need to pass through more than 2 tables to close up. This major difference between the path angles of F^2 and S^2 is due to the table with the intersection being flipped. Translating the starting point by some (x,y) now translates the intersection point by (-x,-y), so as we translate the starting point within the table (figure a), the angle of the path between the starting point and the intersection point (of distance n tables away) will not always be constant as it was in S^2.



(a) Hexagon Shapes for $n = 4$ and $n = 6$ (b) Graph of F^2, F^4, and F^6

It is also interesting to note that as we increase n, the shape of F^n forms a hexagon rather than a rhombus as in F^2. This is not entirely surprising, since if we just connect the points of the middle of all tables with a certain n, where w and z are both odd, then we can see that a hexagon shape is always formed. Unlike our analysis of S, we need to take non-generic staring points into account for a unique closed path analysis of F. A non-generic point is any point that lies on the very edge of the table, or also on the diagonal axes of the table. These starting points are not typical, because they allow some of the paths to have the same angles. For a path on the edge of the table, many of the closed paths will have an angle of either $\frac{\pi}{2}$ for the vertical sides or 0 for the horizontal sides. If the starting point lies on the diagonal of the table, then any paths with some (z, q) that is a scalar multiple of some other path's (z, q) will have the same angle. If the starting point lies on the horizontal or vertical axes, then some paths will be reflections of each other over the x-axis and the y-axis ($\theta_1 = -\pi - \theta_2$), however we will still consider these paths to be different since the two angles are not equal. For any generic point, *none* of the paths will be the same, since every path's $tan(\theta)$ will be different. Since we determined earlier that there are $4n$ total paths for P^n, and $2n$ total paths for S^n, then since $|F^n| + |S^n| = |P^n|$, there will be $2n$ total paths in F^n. Lastly, because every path is unique, then for any generic starting point, as n increases by 2 there will be an additional $|F^{n+2}|$ unique paths.

In conclusion, the in depth analysis of billiards trajectories can yield some interesting and beautiful results. By using the duplicate table method to find closed trajectory paths, many billiards trajectory problems become much more accessible. The ”bouncing” aspect of billiards does make working in the Pacman version of the game slightly easier, but much of the analysis in one game carries over to the other. It is possible to determine if a path is closed in both games by knowing only the dimensions of the table and the angle, θ, of the ball’s trajectory, since for a table of dimensions $m \times n$, a closed path must have $\{tan(\theta) = \mu \frac{n}{m} : \mu \in \mathbb{Q}\}$. It is also clear, after creating a bijection of paths that go through a region determined by the pigeonhole principle to any another region of the same size, that every path which is *not* closed must not miss any region on the table, even if it misses infinitely many points. Lastly, we examined an interesting quality that is unique to the game with standard billiards rules, which involves graphing the closed path intersections in duplicate tables as their own sets, for which some are in flipped tables and some are not. Each have their own unique properties and graphical images. Billiards may just be a table game, but the mathematics behind it can be fascinating.

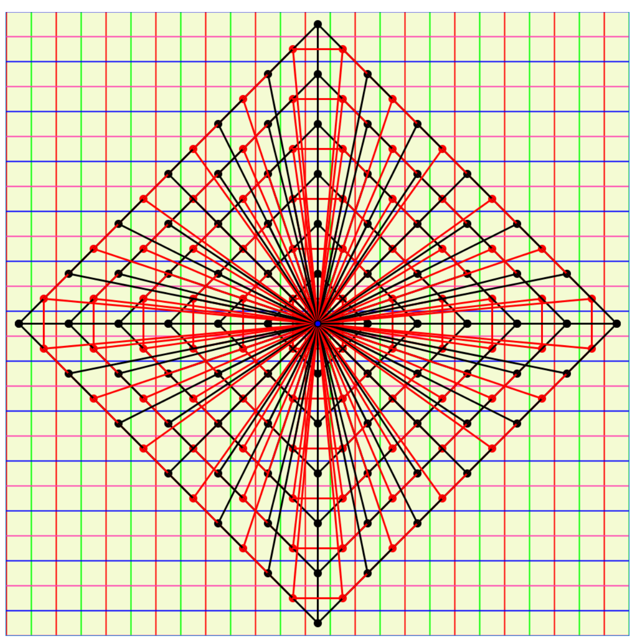


Figure 11: Billiards Trajectories can be beautiful

Works Cited

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