# Perturbations in Celestial Mechanics

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*These are notes I have compiled through numerous textbooks and published articles which can be found cited through the text and in the bibliography. The intention of these notse is to be as explicit as possible with any derivations such that there are no "gaps" in understanding. If you find any errors, please email me.*

## 1 Introduction

TODO: Rewrite this and cite The foundations of perturbation theory in celestial mechanics were laid in the 18th and 19th centuries, primarily to address the complexities of multi-body gravitational interactions in the solar system. Joseph-Louis Lagrange was a pivotal figure, developing the method of variation of parameters and the celebrated Lagrange planetary equations, which formalized how orbital elements evolve under perturbative forces. In the early twentieth century, E.T. Whittaker's analytical dynamics provided a rigorous and modern framework for Hamiltonian mechanics and perturbative expansions, influencing generations of celestial mechanicians. Meanwhile, Ernest William Brown advanced the application of these methods to the Moon's motion, culminating in his monumental lunar theory that remained the basis for ephemerides until the space age. Together, these contributions formed the backbone of modern analytical perturbation theory and remain deeply embedded in the theory and practice of orbital mechanics.

### 1.1 Equations of Motion: N-Body

TODO: figure Newton's law: two point masses attract each other with a force that acts along a line "joining" them, and the force is proportional to the product of their masses, inversely proportional to the square of the distance between them:

$$\vec{F} = G\frac{m_1 m_2}{r^2}\,\hat{r}, \quad \vec{r} = \hat{r}r, \quad \hat{r} = \frac{\vec{r}}{r}. \tag{1}$$

The position of the $i$-th mass is given by

$$\vec{r}_i = x_i\hat{\mathbf{i}} + y_i\hat{\mathbf{j}} + z_i\hat{\mathbf{k}}, \tag{2}$$

with velocity

$$\vec{v}_i = \frac{d\vec{r}_i}{dt} = \frac{dx_i}{dt}\hat{\mathbf{i}} + \frac{dy_i}{dt}\hat{\mathbf{j}} + \frac{dz_i}{dt}\hat{\mathbf{k}}. \tag{3}$$

The magnitude of the total force on the $i$-th point mass $m_i$ from all other $n-1$ masses is

$$\vec{F}_i = G\,m_i \sum_{\substack{j=1 \\ j\neq i}}^{n} m_j \frac{\vec{r}_j - \vec{r}_i}{r_{ij}^3}. \tag{4}$$

From Newton's second law:

$$\vec{F}_i = m_i\frac{d^2\vec{r}_i}{dt^2} = m_i\frac{d\vec{v}_i}{dt}. \tag{5}$$

Thus, the equations of motion become:

$$\boxed{\frac{d^2\vec{r}_i}{dt^2} = G\sum_{\substack{j=1 \\ j\neq i}}^{n} m_j \frac{\vec{r}_j - \vec{r}_i}{r_{ij}^3}} \tag{6}$$

With initial conditions on position and velocity, we can describe the motion of the system of $n$ point masses.

### 1.2 Equation of Motion: Two Bodies

In this chapter, we develop the vector equations of two-body motion and solve by power series.

For two bodies, the motion is described by:

$$m_1 \frac{d^2\vec{r}_1}{dt^2} = \frac{Gm_1m_2}{r_{12}^3}(\vec{r}_2 - \vec{r}_1), \tag{7}$$

$$m_2 \frac{d^2\vec{r}_2}{dt^2} = \frac{Gm_2m_1}{r_{12}^3}(\vec{r}_1 - \vec{r}_2) = -\frac{Gm_2m_1}{r_{12}^3}(\vec{r}_2 - \vec{r}_1). \tag{8}$$

Given initial conditions $\vec{r}_1(t_0)$, $\vec{r}_2(t_0)$, $\vec{v}_1(t_0)$, $\vec{v}_2(t_0)$, finding the positions and velocities at future times is the **two-body problem**.

Subtracting equation **??** - **??**

$$\frac{d^2}{dt^2}(\vec{r}_2 - \vec{r}_1) = -G\frac{(\vec{r}_2 - \vec{r}_1)(m_1 + m_2)}{r_{12}^3}. \tag{9}$$

Using $\vec{r} = \vec{r}_2 - \vec{r}_1$ and $\mu = G(m_1 + m_2)$, we get:

$$\frac{d^2\vec{r}}{dt^2} + \frac{\mu}{r^3}\vec{r} = 0. \tag{10}$$

This is really three simultaneous, second-order, non-linear differential equations in components:

$$\frac{d^2x}{dt^2} + \frac{\mu}{r^3}x = 0, \tag{11}$$

$$\frac{d^2y}{dt^2} + \frac{\mu}{r^3}y = 0, \tag{12}$$

$$\frac{d^2z}{dt^2} + \frac{\mu}{r^3}z = 0, \tag{13}$$

where

$$r = \sqrt{x^2 + y^2 + z^2}. \tag{14}$$

## 2 Perturbations

In the case where a body only moves under the influence of a central gravitational force, the equation of motion gives us Keplerian motion. However, real celestial bodies are influenced by other effects such as additional gravitating bodies, oblateness etc. which create additional perturbing accelerations.

In [5], Equation 2.1 lists the general form of the equation of motion with perturbations which can be expressed in Earth-Centered Inertial (ECI) Cartesian coordinates as:

$$\frac{d^2\vec{r}}{dt^2} = \vec{a}_{gravity} + \vec{a}_{3rd} + \vec{a}_{SRP} + \vec{a}_D + \vec{a}_{sf} \tag{15}$$

where $\vec{a}_{gravity}$ is the acceleration (per unit mass) resulting from gravity of the central body. The components of $\vec{a}_{gravity}$ are $\{\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y,}, \frac{\partial\Phi}{\partial z}\}$. The perturbing accelerations are the result of the following:

- $\vec{a}_{3rd}$ - gravity as a result of the third body
- $\vec{a}_{SRP}$ - solar radiation pressure from the momentum of photons from the sun's radiation
- $\vec{a}_D$ - atmospheric drag effects that oppose the motion of an object moving through the atmosphere

- $\vec{a}_{sf}$ - sum of other small forces causing acceleration

The last value $\vec{a}_{sf}$ is

$$\vec{a}_{gf} = (\text{solid tides}) + (\text{ocean tides}) + \vec{a}_{rel} + \vec{a}_{ir} + \vec{a}_{op} + \vec{a}_e + \vec{a}_s \tag{16}$$

where

- $\vec{a}_{rel}$ results from the relativistic effects

- $\vec{a}_{ir}$ results from Earth radiation (infrared)

- $\vec{a}_{op}$ results from Earth albedo (optical)

- $\vec{a}_e$ and $\vec{a}_s$ result from Earth and solar Yarkovsky forces, respectively.

In the absence of perturbing forces, the solution to equation 10 describes the motion of a body by six constants of integration:

$$c_1, c_2, \ldots, c_6$$

These parameters are typically chosen to be a set of orbital elements, such as $\{a, e, i, \omega, \Omega, f\}$ as shown in figure 1 or an alternative representation such as equinoctial elements, depending on the formulation.

TODO: rewrite these

- $a$ — **Semi-major axis:** Describes the size of the orbit; it is half the longest diameter of the elliptical orbit.
- $e$ — **Eccentricity:** Measures the shape of the orbit, indicating how elongated it is; $e = 0$ is circular, $0 < e < 1$ is elliptical.
- $i$ — **Inclination:** The angle between the orbital plane and the reference plane (usually the equatorial or ecliptic plane), measured at the ascending node.
- $\omega$ — **Argument of periapsis:** The angle from the ascending node to the closest approach point of the orbit (periapsis), measured within the orbital plane.
- $\Omega$ — **Right ascension of the ascending node (RAAN):** The angle from the reference direction (typically the vernal equinox) to the ascending node, measured in the reference plane.
- $f$ — **True anomaly:** The angle between the periapsis and the current position of the satellite, measured in the direction of motion along the orbit.

## 2.1 Variation of Parameters

This section on the variation of parameters method, which was developed by Lagrange in 1788[14]. An in-depth review of this method can be found in chapter 3.5 of Longuski, Hoots, Pollock IV[15], chapter 11 in Brouwer, Clemence[2], TODO: more citations.

The position and velocity of the body at any time $t$ are then given by smooth functions of these constants and time:

$$\begin{aligned} x &= f_1(c_1, c_2, \ldots, c_6, t), & \dot{x} &= g_1(c_1, c_2, \ldots, c_6, t), \\ y &= f_2(c_1, c_2, \ldots, c_6, t), & \dot{y} &= g_2(c_1, c_2, \ldots, c_6, t), \\ z &= f_3(c_1, c_2, \ldots, c_6, t), & \dot{z} &= g_3(c_1, c_2, \ldots, c_6, t) \end{aligned} \tag{17}$$

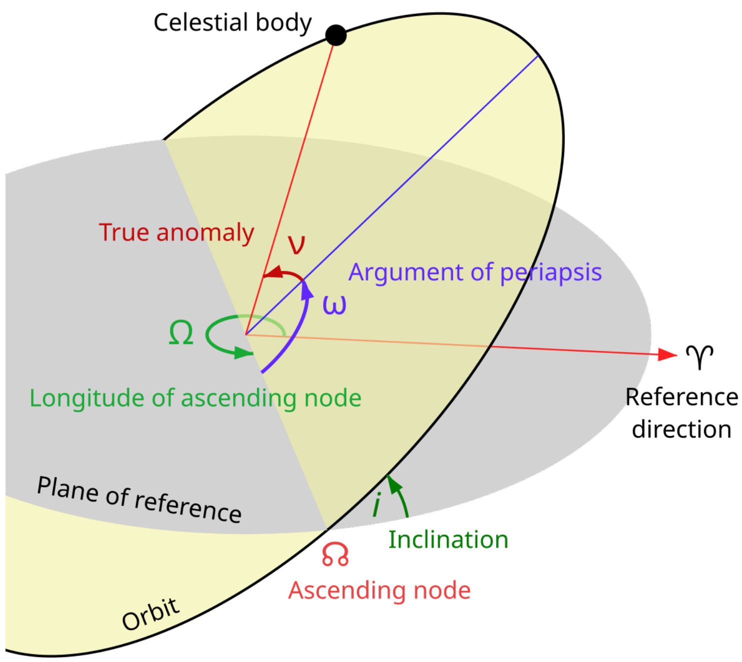


Figure 1: TODO: caption this

which describe the Keplerian (unperturbed) motion. Since the elements $c_k$ are constant in the unperturbed problem, the velocity components are simply the partial derivatives of the position functions with respect to time:

$$g_k = \frac{\partial f_k}{\partial t}, \quad \text{for } k = 1, 2, 3 \tag{18}$$

In the presence of some perturbing force, the full equations of motion are:

$$\ddot{x} + \frac{\mu x}{r^3} = X, \quad \ddot{y} + \frac{\mu y}{r^3} = Y, \quad \ddot{z} + \frac{\mu z}{r^3} = Z \tag{19}$$

where $(X, Y, Z)$ are the "perturbing" accelerations per unit mass.

Assuming the perturbing accelerations are conservative,

$$\vec{F} = \vec{\nabla} R \tag{20}$$

they can be written as the gradient of a scalar potential, which is called the disturbing function $R$:

$$X = \frac{\partial R}{\partial x}, \quad Y = \frac{\partial R}{\partial y}, \quad Z = \frac{\partial R}{\partial z}$$

In the case of perturbed motion, we're trying to satisfy equations 5 by the values of equation 3, but obviously the set $\{c_1, ..., c_6\}$ are not constant. So now, we should derive differential equations for this variable element set. Starting with

$$x = f_1(c_1, c_2, ..., c_6, t) \tag{21}$$

Notice the time dependence. To compute how $x$ changes with time under time dependence, we use the chain rule. The total time derivative of $x$ is:

$$\frac{dx}{dt} = \frac{\partial f_1}{\partial t} + \sum_{j=1}^{6} \frac{\partial f_1}{\partial c_j} \cdot \frac{dc_j}{dt} \tag{22}$$

The same applies for $\frac{dy}{dt}$ and $\frac{dz}{dt}$. To obtain the equations of motion, we now differentiate again to get the second derivatives.

Applying the product rule:

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{\partial f_1}{\partial t} + \sum_{j=1}^{6} \frac{\partial f_1}{\partial c_j}\frac{dc_j}{dt}\right) \tag{23}$$

$$= \frac{\partial^2 f_1}{\partial t^2} + \sum_{j=1}^{6} \frac{\partial^2 f_1}{\partial c_j \partial t}\frac{dc_j}{dt} + \sum_{j=1}^{6} \frac{\partial f_1}{\partial c_j}\frac{d^2 c_j}{dt^2} + \sum_{j=1}^{6}\sum_{k=1}^{6} \frac{\partial^2 f_1}{\partial c_j \partial c_k}\frac{dc_j}{dt}\frac{dc_k}{dt} \tag{24}$$

TODO: write out product rule Similar expressions hold for $\ddot{y}$ and $\ddot{z}$. These are then substituted into the equations of motion, TODO: actually do this

which take the general form:

$$\ddot{x} - \frac{\mu x}{r^3} = X = \frac{\partial R}{\partial x} \tag{25}$$

$$\ddot{y} - \frac{\mu y}{r^3} = Y = \frac{\partial R}{\partial y} \tag{26}$$

$$\ddot{z} - \frac{\mu z}{r^3} = Z = \frac{\partial R}{\partial z} \tag{27}$$

Substituting the full expressions for $\{\ddot{x}, \ddot{y}, \ddot{z}\}$ into these equations yields a system of 3 second-order differential equations involving the quantities $\frac{dc_j}{dt}$ and $\frac{d^2 c_j}{dt^2}$. These equations can be symbolically written as:

$$F_x\left(c_j, \frac{dc_j}{dt}, \frac{d^2 c_j}{dt^2}\right) = \frac{\partial R}{\partial x} \tag{28}$$

and similarly for $y$ and $z$.
At this point, we are faced with a system of only 3 equations but 6 unknowns: $\frac{dc_1}{dt}, \ldots, \frac{dc_6}{dt}$. Therefore, the system is **underdetermined**. There are infinitely many ways to choose the six functions $c_j(t)$ to satisfy the 3 equations of motion.
TODO: flesh this part out To make the system uniquely solvable, we must impose 3 additional gauge conditions [9].

$$\sum_j \frac{\partial f_1}{\partial c_j} \cdot \frac{dc_j}{dt} = 0 \tag{29}$$

$$\sum_j \frac{\partial f_2}{\partial c_j} \cdot \frac{dc_j}{dt} = 0 \tag{30}$$

$$\sum_j \frac{\partial f_3}{\partial c_j} \cdot \frac{dc_j}{dt} = 0 \tag{31}$$

$$\tag{32}$$

These choices eliminate the second terms in equation 22, so that the first derivatives of the coordinates are simply the same as the first derivative of the unperturbed orbit

$$\frac{dx}{dt} = \frac{\partial f_1}{\partial t} = g_1 \tag{33}$$

$$\frac{dy}{dt} = \frac{\partial f_2}{\partial t} = g_2 \tag{34}$$

$$\frac{dz}{dt} = \frac{\partial f_3}{\partial t} = g_3 \tag{35}$$

These expressions define the **osculating elements**. At each moment, the position and velocity vectors are consistent with Keplerian motion using the instantaneous orbital elements.

Continuing on, differentiate these expressions with respect to time again using the chain rule:

$$\ddot{x} = \frac{d}{dt}\left(\frac{\partial f_1}{\partial t}\right) = \frac{\partial^2 f_1}{\partial t^2} + \sum_{j=1}^{6} \frac{\partial g_1}{\partial c_j} \cdot \frac{dc_j}{dt} \tag{36}$$

$$\ddot{y} = \frac{\partial^2 f_2}{\partial t^2} + \sum_{j=1}^{6} \frac{\partial g_2}{\partial c_j} \cdot \frac{dc_j}{dt} \tag{37}$$

$$\ddot{z} = \frac{\partial^2 f_3}{\partial t^2} + \sum_{j=1}^{6} \frac{\partial g_3}{\partial c_j} \cdot \frac{dc_j}{dt} \tag{38}$$

$$\tag{39}$$

Substituting the expressions for $\{\ddot{x}, \ddot{y}, \ddot{z}\}$ from above into the perturbed equation of motion in equation 25

$$\frac{\partial^2 f_1}{\partial t^2} - \frac{f_1}{r^3}\mu + \sum_{j=1}^{6} \frac{\partial g_1}{\partial c_j} \cdot \frac{dc_j}{dt} = \frac{\partial R}{\partial x} \tag{40}$$

$$\frac{\partial^2 f_2}{\partial t^2} - \frac{f_2}{r^3}\mu + \sum_{j=1}^{6} \frac{\partial g_2}{\partial c_j} \cdot \frac{dc_j}{dt} = \frac{\partial R}{\partial y} \tag{41}$$

$$\frac{\partial^2 f_3}{\partial t^2} - \frac{f_3}{r^3}\mu + \sum_{j=1}^{6} \frac{\partial g_3}{\partial c_j} \cdot \frac{dc_j}{dt} = \frac{\partial R}{\partial z} \tag{42}$$

Since $f_1$ is a solution of the unperturbed problem, it satisfies the equation:

$$\frac{\partial^2 f_1}{\partial t^2} + \frac{f_1}{r^3}\mu = 0 \tag{43}$$

So the first two terms in equation 40 are zero, and we are left with:

$$\sum_{j=1}^{6} \frac{\partial g_1}{\partial c_j} \frac{dc_j}{dt} = \frac{\partial R}{\partial x} \tag{44}$$

$$\sum_{j=1}^{6} \frac{\partial g_2}{\partial c_j} \frac{dc_j}{dt} = \frac{\partial R}{\partial y} \tag{45}$$

$$\sum_{j=1}^{6} \frac{\partial g_3}{\partial c_j} \frac{dc_j}{dt} = \frac{\partial R}{\partial z} \tag{46}$$

Switching notation from $f_k, g_k$ to explicitly writing out the coordinates $\{x, y, z, \dot{x}, \dot{y}, \dot{z}\}$, we now collect the six first-order equations for the time derivatives $\frac{dc_j}{dt}$:

TODO: remove the cdots here and add more explanation

$$
\begin{align}
\frac{\partial x}{\partial c_1}\frac{dc_1}{dt} + \frac{\partial x}{\partial c_2}\frac{dc_2}{dt} + \cdots + \frac{\partial x}{\partial c_6}\frac{dc_6}{dt} &= 0 \tag{47}\\
\frac{\partial y}{\partial c_1}\frac{dc_1}{dt} + \frac{\partial y}{\partial c_2}\frac{dc_2}{dt} + \cdots + \frac{\partial y}{\partial c_6}\frac{dc_6}{dt} &= 0 \tag{48}\\
\frac{\partial z}{\partial c_1}\frac{dc_1}{dt} + \frac{\partial z}{\partial c_2}\frac{dc_2}{dt} + \cdots + \frac{\partial z}{\partial c_6}\frac{dc_6}{dt} &= 0 \tag{49}\\
\frac{\partial \dot{x}}{\partial c_1}\frac{dc_1}{dt} + \frac{\partial \dot{x}}{\partial c_2}\frac{dc_2}{dt} + \cdots + \frac{\partial \dot{x}}{\partial c_6}\frac{dc_6}{dt} &= \frac{\partial R}{\partial x} \tag{50}\\
\frac{\partial \dot{y}}{\partial c_1}\frac{dc_1}{dt} + \frac{\partial \dot{y}}{\partial c_2}\frac{dc_2}{dt} + \cdots + \frac{\partial \dot{y}}{\partial c_6}\frac{dc_6}{dt} &= \frac{\partial R}{\partial y} \tag{51}\\
\frac{\partial \dot{z}}{\partial c_1}\frac{dc_1}{dt} + \frac{\partial \dot{z}}{\partial c_2}\frac{dc_2}{dt} + \cdots + \frac{\partial \dot{z}}{\partial c_6}\frac{dc_6}{dt} &= \frac{\partial R}{\partial z} \tag{52}\\
& \tag{53}
\end{align}
$$

These six equations, each first-order in $\frac{dc_j}{dt}$, form the complete system known as **Equation (9)** in Brouwer & Clemence[2]. They describe how the osculating elements evolve in time under a perturbing potential $R$. These six first order equations are exactly equivalent to the original second order three equations. What has been accomplished is a transformation from the old variables $\{x, y, z\}$ to the variables $\{c_1, ..., c_6\}$. This form is not convenient

## 2.2 Definition of Lagrange's Brackets

TODO: More explanation To simplify equation 47, we multiply:

- the first equation (47) by $-\dfrac{\partial \dot{x}}{\partial c_k}$,
- the second (48) by $-\dfrac{\partial \dot{y}}{\partial c_k}$,
- the third (49) by $-\dfrac{\partial \dot{z}}{\partial c_k}$,
- the fourth (50) by $\dfrac{\partial x}{\partial c_k}$,
- the fifth (51) by $\dfrac{\partial y}{\partial c_k}$,
- the sixth (52) by $\dfrac{\partial z}{\partial c_k}$,

and add the resulting equations. TODO: actually do this multiplication
The left-hand side becomes:

$$\sum_{j=1}^{6}\left[-\frac{\partial \dot{x}}{\partial c_k}\frac{\partial x}{\partial c_j}-\frac{\partial \dot{y}}{\partial c_k}\frac{\partial y}{\partial c_j}-\frac{\partial \dot{z}}{\partial c_k}\frac{\partial z}{\partial c_j}+\frac{\partial x}{\partial c_k}\frac{\partial \dot{x}}{\partial c_j}+\frac{\partial y}{\partial c_k}\frac{\partial \dot{y}}{\partial c_j}+\frac{\partial z}{\partial c_k}\frac{\partial \dot{z}}{\partial c_j}\right]\frac{dc_j}{dt}$$

Group terms:

$$\sum_{j=1}^{6}\left[\left(\frac{\partial x}{\partial c_k}\frac{\partial \dot{x}}{\partial c_j}-\frac{\partial x}{\partial c_j}\frac{\partial \dot{x}}{\partial c_k}\right)+\left(\frac{\partial y}{\partial c_k}\frac{\partial \dot{y}}{\partial c_j}-\frac{\partial y}{\partial c_j}\frac{\partial \dot{y}}{\partial c_k}\right)+\left(\frac{\partial z}{\partial c_k}\frac{\partial \dot{z}}{\partial c_j}-\frac{\partial z}{\partial c_j}\frac{\partial \dot{z}}{\partial c_k}\right)\right]\frac{dc_j}{dt}$$

Define the **Lagrange bracket**:

$$[c_j, c_k] = \sum_{\alpha=x,y,z}\left(\frac{\partial x^\alpha}{\partial c_k}\frac{\partial \dot{x}^\alpha}{\partial c_j}-\frac{\partial x^\alpha}{\partial c_j}\frac{\partial \dot{x}^\alpha}{\partial c_k}\right)$$

So the left-hand side becomes:

$$\sum_{j=1}^{6}[c_j, c_k]\frac{dc_j}{dt}$$

### Right-Hand Side

The right-hand side becomes:

$$\frac{\partial x}{\partial c_k}\frac{\partial R}{\partial x}+\frac{\partial y}{\partial c_k}\frac{\partial R}{\partial y}+\frac{\partial z}{\partial c_k}\frac{\partial R}{\partial z}=\frac{\partial R}{\partial c_k}$$

(using the chain rule, since $R$ depends on $x, y, z$ which in turn depend on the $c_j$).

### Final Form: Equation (10)

Putting it all together, we arrive at:

$$\sum_{j=1}^{6}[c_j, c_k]\frac{dc_j}{dt}=\frac{\partial R}{\partial c_k}, \quad \text{for } k = 1, \ldots, 6$$

This is the compact form of **Equation (10)** in Brouwer & Clemence, giving the evolution of the osculating elements governed by the Lagrange brackets and the disturbing function.

### Properties of the Lagrange Brackets

From the definition:

$$[c_j, c_j] = 0, \tag{54}$$
$$[c_j, c_k] = -[c_k, c_j] \tag{55}$$

TODO: proof for this explicitly. TODO: add to this explanation from 278 brouwer Thus, the matrix $[c_j, c_k]$ is antisymmetric with 15 distinct nonzero entries (since it's $6 \times 6$).

# 3 Whitaker's Method

TODO: words Still working on the left-hand side of the system of equations. We need a way to valuate Lagrange brackets for a given set of orbital elements. Whittaker's method for evaluating Lagrange brackets depends on how the brackets change with successive rotations. Begin by rotating around the $z$ axis through an angle $\Omega$, which will bring the ascending node to the x-axis TODO: make diagram for this rotation

$$x = x^{'} \cos\Omega - y^{'} \sin\Omega \tag{56}$$
$$y = x^{'} \sin\Omega + y^{'} \cos\Omega \tag{57}$$
$$z = z^{'} \tag{58}$$

The derivatives with respect to $p$ will look like TODO: actually do this differentiation and then write these after the coefficients

$$\frac{\partial x}{\partial p} = A_1 \cos\Omega - B_1 \sin\Omega \qquad \frac{\partial \dot{x}}{\partial p} = C_1 \cos\Omega - D_1 \sin\Omega$$
$$\frac{\partial y}{\partial p} = B_1 \cos\Omega + A_1 \sin\Omega \qquad \frac{\partial \dot{y}}{\partial p} = D_1 \cos\Omega + C_1 \sin\Omega$$

and with respect to $q$

$$\frac{\partial x}{\partial q} = A_1 \cos\Omega - B_1 \sin\Omega \qquad \frac{\partial \dot{x}}{\partial q} = C_1 \cos\Omega - D_1 \sin\Omega$$
$$\frac{\partial y}{\partial q} = B_1 \cos\Omega + A_1 \sin\Omega \qquad \frac{\partial \dot{y}}{\partial q} = D_1 \cos\Omega + C_1 \sin\Omega$$

we can define these coefficients

$$A_1 = \frac{\partial x^{'}}{\partial p} - y^{'} \frac{\partial \Omega}{\partial p} \qquad B_1 = \frac{\partial y^{'}}{\partial p} + x^{'} \frac{\partial \Omega}{\partial p}$$
$$C_1 = \frac{\partial x^{'}}{\partial p} - y^{'} \frac{\partial \Omega}{\partial p} \qquad D_1 = \frac{\partial y^{'}}{\partial p} + x^{'} \frac{\partial \Omega}{\partial p}$$

and we can do the same for $A_2, B_2, C_2, D_2$ by replacing $p$ with $q$ in these expressions. It is then found that

Which means that we can write the equations like TODO: the ones we wrote previously

Now that we have the derivatives of the coordinates with respect to $\{p, q\}$, we can write the Lagrange bracket after the rotation around the z axis through the angle $\Omega$

$$\frac{\partial\left(x,\dot{x}\right)}{\partial\left(p,q\right)} = \left(A_1C_2 - A_2C_1\right)\cos^2\Omega + \left(B_1D_2 - B_2D_1\right)\sin^2\Omega \tag{59}$$
$$+\left(-A_1D_2 - B_1C_2 + A_2D_1 + B_2C_1\right)\sin\Omega\cos\Omega$$
$$\frac{\partial\left(y,\dot{y}\right)}{\partial\left(p,q\right)} = \left(B_1D_2 - B_2D_1\right)\cos^2\Omega + \left(A_1C_2 - A_2C_1\right)\sin^2\Omega \tag{60}$$
$$+\left(A_1D_2 + B_1C_2 - A_2D_1 - B_2C_1\right)\sin\Omega\cos\Omega$$

Thus, the lagrange bracket is

$$[p,q] = \frac{\partial\left(x,\dot{x}\right)}{\partial\left(p,q\right)} + \frac{\partial\left(y,\dot{y}\right)}{\partial\left(p,q\right)} + \frac{\partial\left(z,\dot{z}\right)}{\partial\left(p,q\right)} \tag{61}$$

$$[p,q] = A_1C_2 - A_2C_1 + B_1D_2 - B_2D_1 + \frac{\partial\left(z,\dot{z}\right)}{\partial\left(p,q\right)} \tag{62}$$

Now, we can look at the individual terms on the right hand side. The first two

$$\begin{aligned} A_1C_2 - A_2C_1 &= \left(\frac{\partial x^{'}}{\partial p} - y^{'}\frac{\partial\Omega}{\partial p}\right)\left(\frac{\partial y^{'}}{\partial q} - \dot{y}^{'}\frac{\partial\Omega}{\partial q}\right) \\ &\quad - \left(\frac{\partial x^{'}}{\partial q} - \dot{y}^{'}\frac{\partial\Omega}{\partial q}\right)\left(\frac{\partial\dot{x}^{'}}{\partial p} - \dot{y}^{'}\frac{\partial\Omega}{\partial p}\right) \\ &= \frac{\partial\left(x^{'},\dot{x^{'}}\right)}{\partial\left(p,q\right)} + \left(-y^{'}\frac{\partial\dot{x}^{'}}{\partial q} + \dot{y}^{'}\frac{\partial x}{\partial q}\right)\frac{\partial\Omega}{\partial p} + \left(-\dot{y}^{'}\frac{\partial x^{'}}{\partial p} + y^{'}\frac{\partial\dot{x}^{'}}{\partial p}\right)\frac{\partial\Omega}{\partial q} \end{aligned} \tag{63}$$

The next terms

$$\begin{aligned} B_1D_2 - B_2D_1 &= \left(\frac{\partial y^{'}}{\partial p} + x^{'}\frac{\partial\Omega}{\partial p}\right)\left(\frac{\partial y^{'}}{\partial q} + \dot{x}^{'}\frac{\partial\Omega}{\partial q}\right) \\ &\quad - \left(\frac{\partial y^{'}}{\partial q} + \dot{x}^{'}\frac{\partial\Omega}{\partial q}\right)\left(\frac{\partial\dot{y}^{'}}{\partial p} + \dot{x}^{'}\frac{\partial\Omega}{\partial p}\right) \\ &= \frac{\partial\left(y^{'},\dot{y^{'}}\right)}{\partial\left(p,q\right)} + \left(x^{'}\frac{\partial\dot{y}^{'}}{\partial q} - \dot{x}^{'}\frac{\partial y^{'}}{\partial q}\right)\frac{\partial\Omega}{\partial p} + \left(\dot{x}^{'}\frac{\partial y^{'}}{\partial p} - x^{'}\frac{\partial\dot{y}^{'}}{\partial p}\right)\frac{\partial\Omega}{\partial q} \end{aligned} \tag{64}$$

The Lagrange Bracket in the prime systems (after rotation) is

$$[p,q]^{'} = \frac{\partial\left(x^{'},\dot{x^{'}}\right)}{\partial\left(p,q\right)} + \frac{\partial\left(y^{'},\dot{y^{'}}\right)}{\partial\left(p,q\right)} + \frac{\partial\left(z^{'},\dot{z^{'}}\right)}{\partial\left(p,q\right)} \tag{65}$$

Because the rotation is around the $z$ axis, the $z^{'}$ axis is the $z$ axis, meaning the coordinates in the unprimed coordinate system are the same as in the primed coordinate system: $z = z^{'}$, $\dot{z} = \dot{z}^{'}$. The lagrange bracket in the unprimed system can be written in terms of the lagrange bracket in the primed system

TODO: fix this huge equation and add more steps

$$[p,q] = \frac{\partial\left(x^{'},\dot{x^{'}}\right)}{\partial\left(p,q\right)} + \left(-y^{'}\frac{\partial\dot{x}^{'}}{\partial q} + \dot{y}^{'}\frac{\partial x}{\partial q}\right)\frac{\partial\Omega}{\partial p} + \left(-\dot{y}^{'}\frac{\partial x^{'}}{\partial p} + y^{'}\frac{\partial\dot{x}^{'}}{\partial p}\right)\frac{\partial\Omega}{\partial q} + \frac{\partial\left(y^{'},\dot{y^{'}}\right)}{\partial\left(p,q\right)} + \left(x^{'}\frac{\partial\dot{y}^{'}}{\partial q} - \dot{x}^{'}\frac{\partial y^{'}}{\partial q}\right)\frac{\partial\Omega}{\partial p} + \left(\dot{x}^{'}\frac{\partial y^{'}}{\partial p} - x^{'} \tag{66}$$

This simplifies to

$$[p,q] = [p,q]^{'} + \frac{\partial\left(\Omega, x^{'}\dot{y}^{'} - y^{'}\dot{x}^{'}\right)}{\partial(p,q)} \tag{67}$$

since

$$x^{'}\dot{y}^{'} - y^{'}\dot{x}^{'} = H \tag{68}$$

$$[p,q] = [p,q]^{'} + \frac{\partial(\Omega, H)}{\partial(p,q)} \tag{69}$$

now do another rotation about $x^{'}$ through an angle $I$

$$x^{'} = x^{''} \tag{70}$$

$$y^{'} = y^{''}\cos I - z^{''}\sin I \tag{71}$$

$$z^{'} = y^{''}\sin I + z^{''}\cos I \tag{72}$$

which follows from the previous calculation

$$[p,q]^{'} = [p,q]^{''} + \frac{\partial\left(\Omega, y^{''}\dot{z}^{''} - z^{''}\dot{y}^{''}\right)}{\partial(p,q)} \tag{73}$$

because $z^{''}$ and $\dot{z}^{''}$

$$[p,q]^{'} = [p,q]^{''} \tag{74}$$

Finally, a rotation about the $z^{''}$ axis

$$X^{'}\dot{Y}^{'} - Y^{'}\dot{X}^{'} = G \tag{75}$$

$$[p,q]^{''} = [p,q]^{'''} + \frac{\partial\left(\tilde{\omega} - \Omega, X\dot{Y} - Y\dot{X}\right)}{\partial(p,q)} \tag{76}$$

$$= [p,q]^{'''} + \frac{\partial(\tilde{\omega} - \Omega, G)}{\partial(p,q)} \tag{77}$$

$$\tag{78}$$

if we define

$$H = G\cos I \tag{79}$$

adding the equations, gives us the lagrange bracket in terms of the the triple prime lagrange bracket

$$[p,q] = [p,q]^{'''} + \frac{\partial(\tilde{\omega} - \Omega, G)}{\partial(p,q)} + \frac{\partial(\Omega, H)}{\partial(p,q)} \tag{80}$$

but we still must evaluate that first term $[p,q]^{'''}$, which is

$$[p,q]^{'''} = \frac{\partial X}{\partial p}\frac{\partial \dot{X}}{\partial q} - \frac{\partial X}{\partial p}\frac{\partial \dot{X}}{\partial q} + \frac{\partial Y}{\partial p}\frac{\partial \dot{Y}}{\partial q} - \frac{\partial Y}{\partial p}\frac{\partial \dot{Y}}{\partial q} \tag{81}$$

The coordinates $X$ and $Y$ are on the ellipse meaning we can write them as

$$X = ... \tag{82}$$
$$Y = ... \tag{83}$$

Expanding these as a maclauren series

$$X = X_0 + \dot{X}_0(t-T) + \frac{1}{2}\ddot{X}_0(t-T)^2 \tag{84}$$
$$Y = Y_0 + \dot{Y}_0(t-T) + \frac{1}{2}\ddot{Y}_0(t-T)^2 \tag{85}$$

$$\begin{aligned} X &= ... & \dot{X} &= ... \\ Y &= .. & \dot{Y} &= D_1 \cos\Omega + C_1 \sin\Omega \end{aligned}$$

evaluating these

# 4 The Disturbing Function

In the previous section, we've arrived at a set of differential equations describing the motion of a body in the presence of some undefined perturbing/disturbing force/potential. We have also looked at how to evaluate the left-hand side of the system of equations in terms of a set of orbital elements. In this section, we'll focus on the right hand side of the equations and derive some specific functions for the disturbing (perturbing) force in terms of the orbital elements so that we may have a system of equations capable of being solved.

## 4.1 Third Body Perturbations

From Fitzpatrick 9.2 The scenario:

A central mass (the Sun), M, at $\vec{R}_s$, a secondary smaller mass body (planet), m, located at $\vec{R}$. A third mass

$$M\ddot{\mathbf{R}}_s = GMm\frac{\mathbf{r}}{r^3} + GMm'\frac{\mathbf{r}'}{r'^3} \tag{9.1}$$
$$m\ddot{\mathbf{R}} = Gmm'\frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}'-\mathbf{r}|^3} - GmM\frac{\mathbf{r}}{r^3} \tag{9.2}$$
$$m'\ddot{\mathbf{R}}' = Gm'm\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} - Gm'M\frac{\mathbf{r}'}{r'^3} \tag{9.3}$$

Let $\mu = G(M+m)$, $\mu' = G(M+m')$, $\tilde{\mu} = Gm$, and $\tilde{\mu}' = Gm'$.

$$\ddot{\mathbf{r}} + \mu\frac{\mathbf{r}}{r^3} = Gm'\left(\frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}'-\mathbf{r}|^3} - \frac{\mathbf{r}'}{r'^3}\right) \tag{9.4}$$
$$\ddot{\mathbf{r}}' + \mu'\frac{\mathbf{r}'}{r'^3} = Gm\left(\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} - \frac{\mathbf{r}}{r^3}\right) \tag{9.5}$$

These right-hand sides can be written as gradients of disturbing functions:

$$\ddot{\mathbf{r}} + \mu\frac{\mathbf{r}}{r^3} = \nabla\mathcal{R} \tag{9.6}$$
$$\ddot{\mathbf{r}}' + \mu'\frac{\mathbf{r}'}{r'^3} = \nabla'\mathcal{R}' \tag{9.7}$$

with disturbing functions:

$$\mathcal{R}(\mathbf{r}, \mathbf{r}') = \tilde{\mu}' \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{\mathbf{r} \cdot \mathbf{r}'}{r'^3} \right) \tag{9.8}$$

$$\mathcal{R}'(\mathbf{r}, \mathbf{r}') = \tilde{\mu} \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{\mathbf{r} \cdot \mathbf{r}'}{r^3} \right) \tag{9.9}$$

1.5 Longuski/1.4.1 Vallado
We start with the form of the disturbing function from equation (9.8) in the book:

$$\mathcal{R}(\mathbf{r}, \mathbf{r}') = \tilde{\mu}' \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{\mathbf{r} \cdot \mathbf{r}'}{r'^3} \right)$$

Note: in equations 9.8 and 9.9,

$$\tilde{\mu}' = Gm', \quad \tilde{\mu} = Gm$$

## Gradient of Scalar Potential

This can be conveniently expressed as the gradient of a scalar potential

## Gradient of the Second term Term

## Gradient of the First Term

We want to compute:

$$\nabla \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right)$$

Let:

$$s = |\mathbf{r} - \mathbf{r}'| = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

Then:

$$\nabla \left( \frac{1}{s} \right) = \left( \frac{\partial}{\partial x} \hat{\mathbf{i}} + \frac{\partial}{\partial y} \hat{\mathbf{j}} + \frac{\partial}{\partial z} \hat{\mathbf{k}} \right) \left( s^{-1} \right)$$

Use the identity:

$$\nabla \left( s^{-1} \right) = -\frac{1}{s^2} \nabla s \quad \text{and} \quad \nabla s = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

So:

$$\nabla \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}$$

## Component-wise Gradient Example

Let:

$$s = \sqrt{x^2 + y^2 + z^2} \quad \Rightarrow \quad \frac{1}{s} = (x^2 + y^2 + z^2)^{-1/2}$$

Then:

$$\nabla \left( \frac{1}{s} \right) = \nabla \left( (x^2 + y^2 + z^2)^{-1/2} \right)$$

By the chain rule:

$$= -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2} \cdot 2x\hat{\mathbf{i}} - \frac{1}{2}(x^2 + y^2 + z^2)^{-3/2} \cdot 2y\hat{\mathbf{j}} - \frac{1}{2}(x^2 + y^2 + z^2)^{-3/2} \cdot 2z\hat{\mathbf{k}}$$

$$= -(x^2 + y^2 + z^2)^{-3/2}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) = -\frac{\mathbf{r}}{r^3}$$

This is consistent with the known identity for the gradient of a Newtonian potential.
Let the disturbing function be defined as:

$$\mathcal{R} = Gm' \left[ \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\phi}} - \frac{r\cos\phi}{r'^2} \right]$$

Let:

$$\zeta^2 = |\mathbf{r} - \mathbf{r}'|^2 = r^2 + r'^2 - 2rr'\cos\phi$$

## Factoring and Expansion

Factor out $\frac{1}{r'}$:

$$\mathcal{R} = \frac{Gm'}{r'} \left[ \left( 1 + \left(\frac{r}{r'}\right)^2 - 2\left(\frac{r}{r'}\right)\cos\phi \right)^{-1/2} - \left(\frac{r}{r'}\right)\cos\phi \right]$$

Let:

$$\omega = \left(\frac{r}{r'}\right)^2 - 2\left(\frac{r}{r'}\right)\cos\phi$$

Then:

$$(1+\omega)^{-1/2} = 1 - \frac{1}{2}\omega + \frac{3}{8}\omega^2 - \frac{5}{16}\omega^3 + \dots$$

## Binomial Expansion Reference

For any $(1+x)^n$, we have:

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

In our case, $n = -\frac{1}{2}$, and $x = \omega$.

## Explicit Terms for $\omega$

$$\omega = \left(\frac{r}{r'}\right)^2 - 2\left(\frac{r}{r'}\right)\cos\phi$$

$$\omega^2 = \left( \left(\frac{r}{r'}\right)^2 - 2\left(\frac{r}{r'}\right)\cos\phi \right)^2 = \left(\frac{r}{r'}\right)^4 - 4\left(\frac{r}{r'}\right)^3\cos\phi + 4\left(\frac{r}{r'}\right)^2\cos^2\phi$$

$$\omega^3 = \left( \left(\frac{r}{r'}\right)^2 - 2\left(\frac{r}{r'}\right)\cos\phi \right)^3 = \left(\frac{r}{r'}\right)^6 - 6\left(\frac{r}{r'}\right)^5\cos\phi + 12\left(\frac{r}{r'}\right)^4\cos^2\phi - 8\left(\frac{r}{r'}\right)^3\cos^3\phi$$

$$\omega^4 = \left( \left(\frac{r}{r'}\right)^2 - 2\left(\frac{r}{r'}\right)\cos\phi \right)^4 = \left(\frac{r}{r'}\right)^8 - 8\left(\frac{r}{r'}\right)^7\cos\phi + 2\cdot\left(\frac{r}{r'}\right)^6\cos^2\phi - 32\left(\frac{r}{r'}\right)^5\cos^3\phi + 16\left(\frac{r}{r'}\right)^4\cos^4\phi$$

## Putting It All Together

$$\begin{aligned}\mathcal{R} = \frac{Gm'}{r'}\Bigg[&1 + \left(\frac{r}{r'}\right)\cos\phi + \left(\frac{r}{r'}\right)^2\left(\frac{1}{2} + \frac{3}{2}\cos^2\phi\right)\\ &+ \left(\frac{r}{r'}\right)^3\left(-\frac{3}{2}\cos\phi + \frac{5}{2}\cos^3\phi\right)\\ &+ \left(\frac{r}{r'}\right)^4\left(\frac{3}{8} - \frac{15}{4}\cos^2\phi + \frac{35}{8}\cos^4\phi\right) + \cdots\Bigg]\end{aligned}$$

These coefficients match the form of Legendre polynomials:

$$P_0(\cos\phi),\ P_1(\cos\phi),\ P_2(\cos\phi),\ \ldots$$

$$\begin{aligned}P_0(\cos\phi) &= 1\\ P_1(\cos\phi) &= \cos\phi\\ P_2(\cos\phi) &= \frac{1}{2}\left(3\cos^2\phi - 1\right)\\ P_3(\cos\phi) &= \frac{1}{2}\left(5\cos^3\phi - 3\cos\phi\right)\\ P_4(\cos\phi) &= \frac{1}{8}\left(35\cos^4\phi - 30\cos^2\phi + 3\right)\end{aligned} \tag{6.13}$$

## Trigonometric Power Relations

$$\cos^2\phi = \frac{1}{2}(1 + \cos 2\phi) \tag{a}$$

$$\cos^3\phi = \frac{1}{4}(3\cos\phi + \cos 3\phi) \tag{b}$$

$$\cos^4\phi = \frac{1}{8}(3 + 4\cos 2\phi + \cos 4\phi) \tag{c}$$

## Trig Expansion of $P_2$

$$P_2 = \frac{1}{2}\left(3\cos^2\phi - 1\right)$$

Substitute from (a):

$$= \frac{1}{2}\left(3\cdot\frac{1}{2}(1+\cos 2\phi) - 1\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{3}{2}\cos 2\phi - 1\right) = \frac{1}{4}(1 + 3\cos 2\phi) \tag{6.15}$$

## Trig Expansion of $P_3$

$$P_3 = \frac{1}{2}\left(5\cos^3\phi - 3\cos\phi\right)$$

Substitute from (b):

$$= \frac{1}{2}\left(5\cdot\frac{1}{4}(3\cos\phi + \cos 3\phi) - 3\cos\phi\right) = \frac{1}{2}\left(\frac{15}{4}\cos\phi + \frac{5}{4}\cos 3\phi - 3\cos\phi\right) = \frac{1}{8}(5\cos 3\phi + 3\cos\phi) \tag{6.16}$$

## Trig Expansion of $P_4$

$$P_4 = \frac{1}{8}\left(35\cos^4\phi - 30\cos^2\phi + 3\right)$$

Substitute from (a) and (c):

$$= \frac{1}{8}\left[35\cdot\frac{1}{8}(3 + 4\cos 2\phi + \cos 4\phi) - 30\cdot\frac{1}{2}(1+\cos 2\phi) + 3\right]$$

$$= \frac{1}{8}\left(\frac{105}{8} + \frac{140}{8}\cos 2\phi + \frac{35}{8}\cos 4\phi - 15 - 15\cos 2\phi + 3\right)$$

$$= \frac{1}{64}(105 + 140\cos 2\phi + 35\cos 4\phi - 120 - 120\cos 2\phi + 24) = \frac{1}{64}(35\cos 4\phi + 20\cos 2\phi + 9) \tag{6.17}$$

## Final Expressions

$$\begin{aligned}
P_0(\cos\phi) &= 1\\
P_1(\cos\phi) &= \cos\phi\\
P_2(\cos\phi) &= \frac{1}{4}(3\cos 2\phi + 1)\\
P_3(\cos\phi) &= \frac{1}{8}(5\cos 3\phi + 3\cos\phi)\\
P_4(\cos\phi) &= \frac{1}{64}(35\cos 4\phi + 20\cos 2\phi + 9)
\end{aligned}$$

## Legendre Series Expansion of the Disturbing Function

The series:

$$\sum_{n=0}^{\infty}\left(\frac{r}{r'}\right)^n P_n(\cos\phi)$$

is convergent when:

$$\frac{r}{r'} < 1 \tag{6.20}$$

Substituting into the disturbing function:

$$\mathcal{R} = \frac{Gm'}{r'}\left[1 + \left(\frac{r}{r'}\right)^2 P_2 + \left(\frac{r}{r'}\right)^3 P_3 + \left(\frac{r}{r'}\right)^4 P_4 + \cdots\right] \tag{6.21}$$

(Note: The $P_1$ terms cancel due to symmetry.)
Now we need to look at two terms in this disturbing function (in red)
To expand $\cos E$ in terms of the mean anomaly $M$, to order $e^3$, we use:

$$\cos E = \frac{1}{e}\left[1 - \frac{1}{2}e^2 - \sqrt{1-e^2}\cos M\right] \tag{6.28}$$

$$= \cos M + \frac{e}{2}(\cos 2M - 1) + \frac{e^2}{8}(3\cos 3M - \cos M)$$

$$= \cos M + \frac{e}{2}\cos 2M - \frac{e}{2} + \frac{3e^2}{8}\cos 3M - \frac{e^2}{8}\cos M \tag{6.29}$$

# Rewriting $(r/r')^2$ using the expansion of $\cos E$

From Kepler's equation:

$$E - e\sin E = M$$

We also have:

$$\left(\frac{r}{r'}\right)^2 = \left(\frac{a(1-e\cos E)}{r'}\right)^2 \tag{6.30}$$

Substitute $\cos E$ from Eq. 6.28:

$$= \left(\frac{a}{r'}\right)^2 \left[1 + e^2 - 2e\cos M + 2e^2\cos^2 M - 2e^2\cos M + e^2\right]$$

Collecting terms:

$$\begin{aligned}\left(\frac{r}{r'}\right)^2 &= \left(\frac{a}{r'}\right)^2 \left[1 + e^2 + 2e^2\cos^2 M - 4e\cos M - 2e^2\cos M\right] \\ &= \left(\frac{a}{r'}\right)^2 \left[1 + e^2 + 2e^2\cos^2 M - 2e(2+e)\cos M\right] \end{aligned}\tag{6.37}$$

# Further Simplification with Trig Identities

Using:

$$\cos^2 M = \frac{1}{2}(1 + \cos 2M) \tag{6.38}$$

$$\cos M \cos M = \frac{1}{2}[\cos(M+M) + \cos(M-M)] = \frac{1}{2}(\cos 2M + 1) \tag{6.39}$$

Then Eq. 6.37 becomes:

$$\left(\frac{r}{r'}\right)^2 = \left(\frac{a}{r'}\right)^2 \left[1 + e^2 + e^2\cos 2M - 2e(2+e)\cos M\right] \tag{6.40}$$

# Structure of the Expansion

We observe from Eq. 6.40 that $(r/r')^2$ is the sum of terms of the form:

$$A_{pq}\cos(pM + qM)$$

where $p$ and $q$ are integers (positive, negative, or zero), and the coefficients $A_{pq} \equiv A_{pq}(a, e, i)$.
From Eq. 6.40:

$$\left(\frac{r}{r'}\right)^2 = \sum A_{pq}\cos(pM + qM')$$

where $p, q \in \mathbb{Z}$ and the coefficients depend on:

$$A_{pq} = A_{pq}(a, a', e, e') \tag{6.41}$$

### 6.1.2 The Factor $P_2(\cos\phi)$

From Eq. 6.42:

$$P_2(\cos\phi) = -\frac{1}{2} + \frac{3}{2}\cos^2\phi \tag{6.42}$$

# Position Vector and Unit Vectors

The position vector in the perifocal frame:

$$\mathbf{r} = \xi\mathbf{u}_P + \eta\mathbf{u}_Q = a\left[(\cos E - e)\mathbf{u}_P + \sqrt{1-e^2}\sin E\,\mathbf{u}_Q\right] \tag{6.43}$$

### Perifocal Unit Vectors

$$\mathbf{u}_P = P_1\hat{\imath} + P_2\hat{\jmath} + P_3\hat{k}, \quad \mathbf{u}_Q = Q_1\hat{\imath} + Q_2\hat{\jmath} + Q_3\hat{k}, \quad \mathbf{u}_W = W_1\hat{\imath} + W_2\hat{\jmath} + W_3\hat{k} \tag{2.50}$$

With:

$$\begin{aligned}
P_1 &= \cos\Omega\cos\omega - \cos i \sin\Omega \sin\omega \\
P_2 &= \sin\Omega\cos\omega + \cos i \cos\Omega \sin\omega \\
P_3 &= \sin i \sin\omega \\
Q_1 &= -\sin\omega\cos\Omega - \cos\omega\cos i \sin\Omega \\
Q_2 &= -\sin\omega\sin\Omega + \cos\omega\cos i \cos\Omega \\
Q_3 &= \cos\omega \sin i \\
W_1 &= \sin i \sin\Omega, \quad W_2 = -\sin i \cos\Omega, \quad W_3 = \cos i
\end{aligned} \tag{2.51}$$

### Angle Between Radius Vectors

From Eq. 6.44:

$$\cos\phi = \frac{\mathbf{r}\cdot\mathbf{r}'}{rr'} = \frac{\xi\xi'\mathbf{u}_P\cdot\mathbf{u}'_P + \xi\eta'\mathbf{u}_P\cdot\mathbf{u}'_Q + \eta\xi'\mathbf{u}_Q\cdot\mathbf{u}'_P + \eta\eta'\mathbf{u}_Q\cdot\mathbf{u}'_Q}{rr'} \tag{6.44}$$

$$\xi = a(\cos E - e), \quad \xi' = a'(\cos E' - e') \tag{6.45}$$

Use the expansion from Eq. 6.29:
A typical product is of the form:

$$\cos pM \cos qM' = \frac{1}{2}\left[\cos(pM + qM') + \cos(pM - qM')\right] \tag{6.47}$$

### Dot Product Expansion

We also have from Eq. 6.48:

$$\mathbf{u}_P\cdot\mathbf{u}'_P = P_1P'_1 + P_2P'_2 + P_3P'_3 \tag{6.48}$$

And:

$$\cos i = 1 - 2\sin^2\left(\frac{i}{2}\right) \tag{6.52}$$

## Summary of Terms

Thus, the disturbing function term:

$$\left(\frac{r}{r'}\right)^2 P_2(\cos\phi)$$

is ultimately composed of cosine sums of the form:

$$B_{pq}\cos(pM + qM') \tag{6.47}$$

with coefficients depending on orbital parameters:

$$B_{pq} = B_{pq}(a, a', e, e')$$

### 6.1.3 The $\left(\frac{r}{r'}\right)^2 P_2(\cos\phi)$ Term

We have been considering (as an illustration) what terms are involved in the disturbing function, $\mathcal{R}$, as expressed in Eq. (6.21), when we expand the Legendre term $\left(\frac{r}{r'}\right)^2 P_2(\cos\phi)$ as a function of the mean anomalies, $M$ and $M'$. In particular, we have been examining the factor $P_2(\cos\phi)$. We must still consider the product of $\mathbf{u}_P \cdot \mathbf{u}'_P$ in Eq. (6.44), which can be written as

$$\mathbf{u}_P \cdot \mathbf{u}'_P = P_1 P'_1 + P_2 P'_2 + P_3 P'_3, \tag{6.48}$$

where, from Eq. (2.51),

$$P_1 = \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i, \tag{6.49}$$
$$P_2 = \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i, \tag{6.50}$$
$$P_3 = \sin\omega\sin i, \tag{6.51}$$

with analogous expressions for $P'_1, P'_2, P'_3$.

For convenience, let

$$\cos i = 1 - 2\sin^2\!\left(\tfrac{i}{2}\right). \tag{6.52}$$

In many applications, the inclination $i$ is a small quantity so that

$$\sin\!\left(\tfrac{i}{2}\right) \approx \frac{i}{2} = \gamma, \tag{6.53}$$
$$\cos i \approx 1 - 2\gamma^2, \tag{6.54}$$

where $\gamma$ is used as the new inclination variable. Thus, for small $i$,

$$P_1 \approx \cos\Omega\cos\omega - \sin\Omega\sin\omega + 2\gamma^2\sin\Omega\sin\omega. \tag{6.55a}$$

Using the identity

$$2\sin\omega\sin\Omega = \cos(\omega-\Omega) - \cos(\omega+\Omega), \tag{6.56}$$

we may write

$$P_1 \approx \cos(\Omega+\omega) + 2\gamma^2\sin\Omega\sin\omega. \tag{6.55b}$$

From Eq. (6.55) we can show that

$$P_1 \approx (1-\gamma^2)\,\cos(\Omega+\omega) + \gamma^2\cos(\omega-\Omega), \tag{6.57}$$

and similarly

$$P'_1 \approx (1-\gamma'^2)\,\cos(\Omega'+\omega') + \gamma'^2\cos(\omega'-\Omega'). \tag{6.58}$$

We are interested only in the general form of the product $\mathbf{u}_P \cdot \mathbf{u}'_P$ as typified by the term $P_1 P'_1$. From Eqs. (6.57) and (6.58), we see that the term $P_1 P'_1$ consists of terms of the form

$$\cos(\Omega+\omega)\,\cos(\Omega'+\omega'),$$

which can be reduced to sums such as

$$\frac{1}{2}\Big[\cos\big(\Omega+\omega+\Omega'+\omega'\big) + \cos\big(\Omega+\omega-\Omega'-\omega'\big)\Big].$$

Thus, all products arising from $\mathbf{u}_P \cdot \mathbf{u}'_P$ and the other scalar products in Eq. (6.44) take the form

$$C_j\cos\big(j_1\Omega + j_2\Omega' + j_3\omega + j_4\omega'\big),$$

where the $j_i$ ($i = 1,2,3,4$) are integers (positive, negative or zero) and the coefficients $C_j$ are functions of the inclination variables $\gamma$ and $\gamma'$. In general, $\cos^2\phi$ (which appears in the Legendre polynomial $P_2(\cos\phi) = -\frac{1}{2} + \frac{3}{2}\cos^2\phi$) and higher powers of $\cos\phi$ can be expressed as products of such functions and ultimately reduced by trigonometric identities to sums of cosines of multiple angles.

## 6.2 Form of the Perturbing Function

In the final analysis, the perturbation function $\mathcal{R}$ takes the form

$$\mathcal{R} = Gm' \sum_p C_p\big(a, a', e, e', \gamma, \gamma'\big)\ \cos\!\big(p_1 M + p_2 M' + p_3 \Omega + p_4 \Omega' + p_5 \omega + p_6 \omega'\big), \tag{6.59}$$

where the integers $p_i$ $(i = 1, 2, 3, 4, 5, 6)$ may be positive, negative, or zero. Equation (6.59) can now be used in Lagrange's planetary equations (Eqs. 5.27 or 5.37) to obtain the perturbations in the orbital elements.

Let

$$M = nt + \sigma, \qquad M' = n't + \sigma', \tag{6.60}$$

so that

$$p_1 M + p_2 M' = (p_1 n + p_2 n')\, t + p_1 \sigma + p_2 \sigma'. \tag{6.61}$$

Denote the argument of the cosine in $\mathcal{R}$ by $\theta$:

$$\theta = (p_1 n + p_2 n')\, t + p_1 \sigma + p_2 \sigma' + p_3 \Omega + p_4 \Omega' + p_5 \omega + p_6 \omega'. \tag{6.62}$$

We make the additional assumption that the orbital elements of the perturbing body $m'$ can be considered constant. That is, the body of interest is assumed to have no significant effect on the motion of the perturbing body. Thus, Eq. (6.59) can be written as

$$\mathcal{R} = Gm' \sum_p C_p \cos\Big[(p_1 n + p_2 n')\, t + p_1 \sigma + p_3 \Omega + p_4 \Omega' + p_5 \omega + \theta_0\Big], \tag{6.64}$$

where $\theta_0$ contains all the contributions due to combinations of $p_2, p_4, p_6$ with $\sigma', \Omega', \omega'$ and the summation refers to all $p_i$ $(i = 1, 2, 3, 4, 5, 6)$. Taking partial derivatives of $\mathcal{R}$ with respect to the orbital elements yields

$$\frac{\partial \mathcal{R}}{\partial \sigma} = -\, Gm' \sum_p C_p\, p_1 \sin\theta, \qquad (p_1 \neq 0), \tag{6.65}$$

$$\frac{\partial \mathcal{R}}{\partial \sigma} = 0, \qquad (p_1 = 0), \tag{6.66}$$

$$\frac{\partial \mathcal{R}}{\partial \Omega} = -\, Gm' \sum_p C_p\, p_3 \sin\theta, \tag{6.67}$$

$$\frac{\partial \mathcal{R}}{\partial \omega} = -\, Gm' \sum_p C_p\, p_5 \sin\theta. \tag{6.68}$$

### Partial Derivatives with respect to $e$, $\gamma$ and $a$

We also have

$$\frac{\partial \mathcal{R}}{\partial e} = Gm' \sum_p \frac{\partial C_p}{\partial e}\ \cos\theta, \tag{6.69}$$

$$\frac{\partial \mathcal{R}}{\partial \gamma} = Gm' \sum_p \frac{\partial C_p}{\partial \gamma}\ \cos\theta. \tag{6.70}$$

Since $\gamma = \sin\!\big(\frac{i}{2}\big)$, the relation $\partial\gamma/\partial i = \frac{1}{2}\cos\!\big(\frac{i}{2}\big)$ allows us to write

$$\frac{\partial \mathcal{R}}{\partial i} = \frac{1}{2}\, Gm' \cos\!\Big(\frac{i}{2}\Big)\ \sum_p \frac{\partial C_p}{\partial \gamma}\ \cos\theta. \tag{6.71}$$

Differentiation with respect to $a$ gives

$$\frac{\partial \mathcal{R}}{\partial a} = Gm' \sum_p \frac{\partial C_p}{\partial a}\ \cos\theta\ -\ Gm' \sum_p C_p\, (p_1 t)\, \frac{\partial n}{\partial a}\ \sin\theta, \tag{6.72}$$

where, in Eq. (6.72), we use the relation

$$\frac{\partial n}{\partial a} = -\,\frac{3\,n}{2a}.$$

As an example, consider the $\dot{\Omega}$ equation of Lagrange's planetary equations (Eqs. 5.27 or 5.37):

$$\dot{\Omega} = \frac{1}{a^2 n\nu\sqrt{1-e^2}\sin i}\,\frac{\partial \mathcal{R}}{\partial i}. \tag{6.73}$$

Substituting Eq. (6.71) into Eq. (6.73) yields

$$\dot{\Omega} = \frac{Gm' \cos\!\left(\frac{i}{2}\right)}{2\,a^2 n\nu\sqrt{1-e^2}\,\sin i}\sum_p \frac{\partial C_p}{\partial \gamma}\,\cos\theta, \tag{6.74}$$

where

$$\theta = (p_1 n + p_2 n')\,t + p_1\sigma + p_3\Omega + p_5\omega + \theta_0. \tag{6.75}$$

## 6.2 Short- and Long-Period Terms

If $p_1 n + p_2 n'$ is large, then perturbations to the orbital element have small amplitudes and short periods (high frequencies). Such perturbations are referred to as *short-period inequalities.* Conversely, if $p_1 n + p_2 n'$ is small, perturbations have large amplitudes and long periods (low frequencies); these are called *long-period inequalities.* From Lagrange's planetary equations (Eqs. 5.27 or 5.37) and Eqs. (6.65)–(6.72), all of the elements except the semimajor axis $a$ exhibit secular as well as periodic changes (to first order in $m'$).

### Separation of $\dot{\Omega}$ into Secular and Periodic Parts

We can separate $\dot{\Omega}$ into two pieces as

$$\dot{\Omega} = A + \sum_p B_p \cos\!\big[(p_1 n + p_2 n')\,t + \theta_1\big], \tag{6.76}$$

where the $p_1$ and $p_2$ are not zero simultaneously. Integrating Eq. (6.76) yields

$$\Omega = \Omega_0 + At + \sum_p \frac{B_p}{(p_1 n + p_2 n')}\,\sin\!\big[(p_1 n + p_2 n')\,t + \theta_1\big], \tag{6.77}$$

where the subscript zero denotes fixed elements. The term $At$ is a secular perturbation term. If $p_1 n + p_2 n' = 0$, we still have secular terms arising from Eq. (6.76). In this case we have a commensurability in the periods $P$ and $P'$ of the perturbed and perturbing bodies. If $P$ and $P'$ are the periods then

$$\frac{P'}{P} = \frac{n}{n'} = -\,\frac{p_2}{p_1}, \tag{6.80}$$

where $p_1$ and $p_2$ are integers.

## 6.4 Stability of the Semimajor Axis

For the semimajor axis, Lagrange's equations yield

$$\dot{a} = \frac{2}{an}\,\frac{\partial \mathcal{R}}{\partial \sigma} - \frac{2\,Gm'}{an}\sum_p C_p\, p_1 \sin\theta, \qquad (p_1 \neq 0), \tag{6.81}$$

$$\dot{a} = 0, \qquad (p_1 = 0). \tag{6.82}$$

Integrating Eq. (6.81) gives

$$\delta(a) = \frac{2\,Gm'}{a_0 n}\sum_p C_p\,\frac{p_1}{(p_1 n + p_2 n')}\,\cos\theta, \tag{6.83}$$

where

$$a = a_0 + \delta(a). \tag{6.84}$$

Thus, the semimajor axis oscillates about the mean value $a_0$ with period

$$P = \frac{2\pi}{p_1 n + p_2 n'}. \tag{6.85}$$

We have shown that the semimajor axis exhibits no secular change in the first-order theory; if it did, then the orbit would expand or contract indefinitely and the orbit would be unstable. In another situation, if the eccentricity were to increase secularly, a close approach with another planet could occur and disrupt the system. Lagrange showed that when all powers of $e$ are included to first order in $m'$ that the semimajor axis undergoes no secular change. Subsequent work by Poisson and Haretu extended this result to higher orders.

# 5 Other Disturbing Functions

[1] [2] [3] [4] [5] [6] [7] [8] [9] [10] [11] [12] [13] [15] [16] [17] [18] [19] [20] [21] [22] [23]

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