Perturbations in Celestial Mechanics

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These are notes I have compiled through numerous textbooks and published articles which can be found cited throughout the text and in the bibliography. The intention of these notes is to be as explicit as possible with any derivations such that there are no "gaps" in understanding. If you find any errors, please email me.

Contents

1 Introduction

Perturbation theory in celestial mechanics originated in the 18th and 19th centuries as mathematicians sought methods to account for the deviations of planetary and lunar motions from ideal Keplerian orbits. Joseph-Louis Lagrange made foundational contributions through his *Mécanique Analytique* (1788) [14], where he introduced the method of variation of parameters and the Lagrange planetary equations to study the long-term evolution of orbital elements. Pierre-Simon Laplace's *Mécanique Céleste* (1799–1825) expanded Lagrange's techniques to explain the stability of the Solar System and resonant interactions, particularly in the Jupiter–Saturn system. Later, Sir Edmund Whittaker formalized the Hamiltonian approach in *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (1904) [23], demonstrating how canonical transformations and the theory of invariants refined the perturbative treatment of orbits. By the mid-20th century, work by Ernest W. Brown on lunar theory and Dirk Brouwer and Gerald Clemence's Methods of Celestial Mechanics (1961) synthesized these classical ideas into rigorous computational procedures, laying the groundwork for modern perturbation methods used in astrodynamics and space mission design.

These notes will start with the general N-Body equations of motion, and then focus on the specific case of a two-body system. The second section introduces perturbations, detailing the method of variation of parameters and the role of Lagrange brackets in transforming orbital equations. Building on this, a full treatment of Whittaker's method is given, organized into three successive canonical rotations that simplify the dynamical system. The notes then turn to the disturbing function, including detailed consideration of third-body effects and other perturbing influences, connecting classical derivations with modern interpretations. The final section provides references to additional topics drawn from textbooks and published literature. Throughout, the emphasis is on explicit derivations with minimal gaps, offering a clear pathway from first principles to advanced perturbation methods in celestial mechanics.

2 Two-Particle System and Newtonian Attraction

Consider the system of two particles in (Fig. 1), where P_1 and P_2, observed from an inertial frame with orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$. The only internal forces are due to mutual gravitation. Denote the internal force on P_1 due to P_2 by $\boldsymbol{f}_{12}$, and let $\hat{\boldsymbol{u}}_{12}$ be the *unit* vector directed from P_1 toward P_2. Newton's law of gravitation gives the force on P_1 due to P_2 as

$$\vec{\boldsymbol{f}}_{12} = \frac{Gm_1m_2}{r_{12}^2}\,\hat{\boldsymbol{u}}_{12} \tag{1}$$

where m_1 and m_2 are the particle masses, G is the gravitational constant, and r_{12} is the distance between the particles.

Being explicit in labeling the distance vectors, let $\vec{\boldsymbol{r}}^{\,OP_1}$ and $\vec{\boldsymbol{r}}^{\,OP_2}$ denote the position vectors of P_1 and P_2 measured from the origin O of the inertial frame. The distance between particles r_{12} satisfies

$$r_{12} = |\vec{\boldsymbol{r}}^{\,OP_1} - \vec{\boldsymbol{r}}^{\,OP_2}| = |-\vec{\boldsymbol{r}}^{\,P_1P_2}| = |\vec{\boldsymbol{r}}^{\,P_2P_1}| \tag{2}$$

Writing the position vectors with individual components gives the following.

$$\begin{aligned}
\vec{\boldsymbol{r}}^{\,OP_1} &= x_1\,\hat{\boldsymbol{e}}_1 + y_1\,\hat{\boldsymbol{e}}_2 + z_1\,\hat{\boldsymbol{e}}_3 \\
\vec{\boldsymbol{r}}^{\,OP_2} &= x_2\,\hat{\boldsymbol{e}}_1 + y_2\,\hat{\boldsymbol{e}}_2 + z_2\,\hat{\boldsymbol{e}}_3
\end{aligned}$$

The superscripts indicate the origin and endpoint of each position vector (ex: $\boldsymbol{r}^{\,OP_1}$ is the position vector from O to P_1). We may write r_{12} as

$$\begin{aligned}
r_{12} &= \left|\vec{\boldsymbol{r}}^{\,OP_1} - \vec{\boldsymbol{r}}^{\,OP_2}\right| \\
&= \left|(x_1 - x_2)\,\hat{\boldsymbol{e}}_1 + (y_1 - y_2)\,\hat{\boldsymbol{e}}_2 + (z_1 - z_2)\,\hat{\boldsymbol{e}}_3\right| \\
&= \left[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\right]^{1/2}
\end{aligned} \tag{3}$$

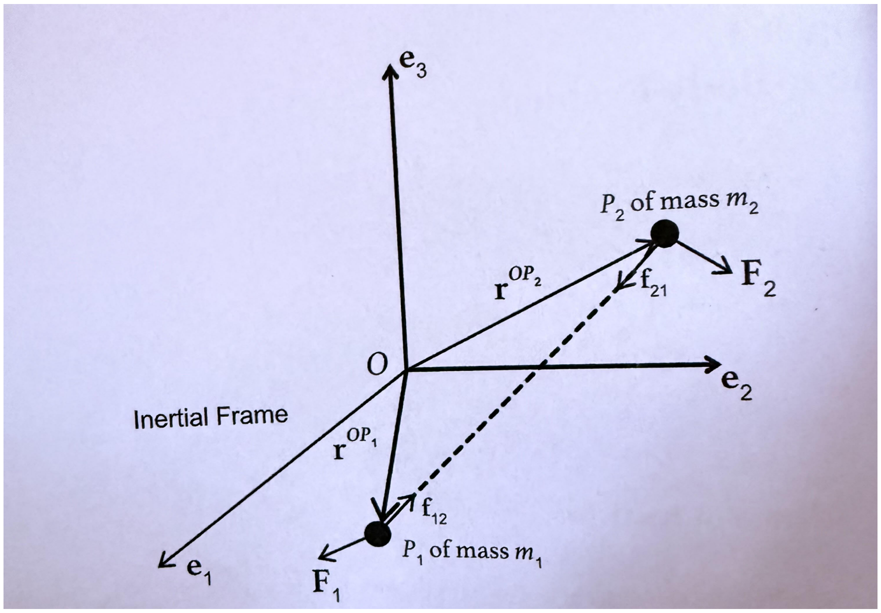

Figure 1: TODO: Remove background A system of point masses $\{m_1, m_2\}$ in an inertial frame with a fixed origin O

We want to write out $\hat{\boldsymbol{u}}_{12}$ in (Eq. 1)

$$\hat{\boldsymbol{u}}_{12} = \frac{\vec{\boldsymbol{r}}^{P_1P_2}}{|\vec{\boldsymbol{r}}^{P_1P_2}|} \tag{4}$$
$$= \frac{\vec{\boldsymbol{r}}^{P_1O} + \vec{\boldsymbol{r}}^{OP_2}}{|\vec{\boldsymbol{r}}^{P_1O} + \vec{\boldsymbol{r}}^{OP_2}|}$$

Using vector subtractiion, which is apparent from figure systemOfParticles

$$\hat{\boldsymbol{u}}_{12} = \frac{\vec{\boldsymbol{r}}^{OP_2} - \vec{\boldsymbol{r}}^{OP_1}}{|\vec{\boldsymbol{r}}^{OP_2} - \vec{\boldsymbol{r}}^{OP_1}|} = \frac{(x_2 - x_1)\hat{\boldsymbol{e}}_1 + (y_2 - y_1)\hat{\boldsymbol{e}}_2 + (z_2 - z_1)\hat{\boldsymbol{e}}_3}{\left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{1/2}} \tag{5}$$

Writing out the internal force $\boldsymbol{f}_{12}$, we have can use r_{12} from equation (Eq. 3) and $\hat{\boldsymbol{u}}_{12}$ from equation (Eq. 5)

$$\vec{\boldsymbol{f}}_{12} = \frac{Gm_1m_2}{r_{12}^2}\hat{\boldsymbol{u}}_{12}$$
$$= \frac{Gm_1m_2\left[(x_2 - x_1)\hat{\boldsymbol{e}}_1 + (y_2 - y_1)\hat{\boldsymbol{e}}_2 + (z_2 - z_1)\hat{\boldsymbol{e}}_3\right]}{\left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{3/2}}$$

Or by using equation (Eq. 2) for r_{12} and equation (Eq. 4) for $\hat{\boldsymbol{u}}_{12}$

$$\vec{\boldsymbol{f}}_{12} = \frac{Gm_1m_2}{|\vec{\boldsymbol{r}}^{P_1P_2}|^2}\frac{\vec{\boldsymbol{r}}^{P_1P_2}}{|\vec{\boldsymbol{r}}^{P_1P_2}|} = \frac{Gm_1m_2\,\vec{\boldsymbol{r}}^{P_1P_2}}{|\vec{\boldsymbol{r}}^{P_1P_2}|^3}$$

This expression can be generalized for an arbitrary choice of 2 particles i and j in a system of n masses:

$$\vec{\boldsymbol{f}}_{ij} = \frac{Gm_im_j\,\vec{\boldsymbol{r}}^{P_iP_j}}{|\vec{\boldsymbol{r}}^{P_iP_j}|^3}$$

Newton's first law gives us the equation of motion for P_1 as the sum of all forces from each of the masses in the system $\vec{f}_{1j}$ along with some arbitrary external force $\vec{\boldsymbol{F}}_1$:

$$m_1 \ddot{\vec{\boldsymbol{r}}}^{OP_1} = \sum_{j=1}^{n} \vec{\boldsymbol{f}}_{1j} + \vec{\boldsymbol{F}}_1 \tag{6}$$

where $\boldsymbol{f}_{11} \equiv \boldsymbol{0}$, as a particle cannot exert a force upon itself. The single dot above a quantity denotes the first derivative with respect to time, and two dots indicates the second derivative. Since the frame is inertial, we can directly write the acceleration of P_1 with respect to O as

$$\ddot{\vec{\boldsymbol{r}}}^{OP_1} = \ddot{x}_1 \hat{\boldsymbol{e}}_1 + \ddot{y}_1 \hat{\boldsymbol{e}}_2 + \ddot{z}_1 \hat{\boldsymbol{e}}_3$$

Using this as the acceleration in the equation of motion (eq. (Eq. 6))

$$m_1\big(\ddot{x}_1 \hat{\boldsymbol{e}}_1 + \ddot{y}_1 \hat{\boldsymbol{e}}_2 + \ddot{z}_1 \hat{\boldsymbol{e}}_3\big) = \sum_{j=2}^{n} \frac{G m_1 m_j \, \vec{\boldsymbol{r}}^{P_1 P_j}}{|\vec{\boldsymbol{r}}^{P_1 P_j}|^3} + \vec{\boldsymbol{F}}_1$$

Where we start the summation at $j = 2$ because $\boldsymbol{f}_{11} \equiv 0$. Thus, for the ith particle the equation of motion is

$$m_i\big(\ddot{x}_i \hat{\boldsymbol{e}}_1 + \ddot{y}_i \hat{\boldsymbol{e}}_2 + \ddot{z}_i \hat{\boldsymbol{e}}_3\big) = \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{G m_i m_j \, \vec{\boldsymbol{r}}^{P_i P_j}}{|\vec{\boldsymbol{r}}^{P_i P_j}|^3} + \vec{\boldsymbol{F}}_i$$

or equivalently

$$m_i \ddot{\vec{\boldsymbol{r}}}^{OP_i} = \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{G m_i m_j \, \vec{\boldsymbol{r}}^{P_i P_j}}{|\vec{\boldsymbol{r}}^{P_i P_j}|^3} + \vec{\boldsymbol{F}}_i \tag{7}$$

We note that the gravitational forces are internal to the system of particles and obey the relation

$$\vec{\boldsymbol{f}}_{ij} = -\vec{\boldsymbol{f}}_{ji},$$

according to Newton's third law.

Furthermore, we can write out the summation over all masses

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} \vec{\boldsymbol{f}}_{ij} = {} & \vec{\boldsymbol{f}}_{11} + \vec{\boldsymbol{f}}_{12} + \vec{\boldsymbol{f}}_{13} + \cdots + \vec{\boldsymbol{f}}_{1n} \\
& + \vec{\boldsymbol{f}}_{21} + \vec{\boldsymbol{f}}_{22} + \vec{\boldsymbol{f}}_{23} + \cdots + \vec{\boldsymbol{f}}_{2n} \\
& + \cdots \\
& + \vec{\boldsymbol{f}}_{n1} + \vec{\boldsymbol{f}}_{n2} + \vec{\boldsymbol{f}}_{n3} + \cdots + \vec{\boldsymbol{f}}_{nn} \\
= {} & \vec{\boldsymbol{0}}
\end{aligned} \tag{8}$$

Since

$$\vec{\boldsymbol{f}}_{ii} \equiv \vec{\boldsymbol{0}}$$

the diagonal terms vanish in equation (Eq. 8), and the off-diagonal terms are equal and opposite pairs. For example,

$$\vec{\boldsymbol{f}}_{12} + \vec{\boldsymbol{f}}_{21} = 0$$

The center of mass C, or barycenter of the system of particles is located, with respect to O, by the weighted average of the mass locations in the system:

$$\vec{\boldsymbol{r}}^{OC} = \frac{1}{m} \sum_{i=1}^{n} m_i \, \vec{\boldsymbol{r}}^{OP_i} \tag{9}$$

where the total system mass is

$$m \equiv \sum_{i=1}^{n} m_i$$

Differentiating equation (Eq. 9) twice and multiplying through by the total mass m, we arrive at the following expressions:

$$m\,\vec{\boldsymbol{r}}^{\,OC} = \sum_{i=1}^{n} m_i\,\vec{\boldsymbol{r}}^{\,OP_i}$$

$$m\,\dot{\vec{\boldsymbol{r}}}^{\,OC} = \sum_{i=1}^{n} m_i\,\dot{\vec{\boldsymbol{r}}}^{\,OP_i}$$

$$m\,\ddot{\vec{\boldsymbol{r}}}^{\,OC} = \sum_{i=1}^{n} m_i\,\ddot{\vec{\boldsymbol{r}}}^{\,OP_i} \tag{10}$$

Summing over i in equation (Eq. 7), we obtain

$$\sum_{i=1}^{n} m_i \ddot{\vec{\boldsymbol{r}}}^{\,OP_i} = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{G m_i m_j\,\vec{\boldsymbol{r}}^{\,P_iP_j}}{|\vec{\boldsymbol{r}}^{\,P_iP_j}|^3} + \sum_{i=1}^{n} \vec{\boldsymbol{F}}_i \qquad j \neq i \tag{11}$$

We now use equation (Eq. 10) to rewrite the left-hand side of equation (Eq. 11), and from (Eq. 8) the first term on the right-hand side vanishes. The result is

$$m\ddot{\vec{\boldsymbol{r}}}^{\,OC} = \vec{\boldsymbol{F}} \tag{12}$$

where the total force is just the sum of all external forces

$$\vec{\boldsymbol{F}} = \sum_{i=1}^{n} \vec{\boldsymbol{F}}_i \tag{13}$$

(Eq. 12) is Newton's second law for a system of particles i.e. the motion of the center of mass is the same as if the entire mass of the system were concentrated there and acted upon by the resultant of external forces.

Next, we consider changing the reference point in (Eq. 7) from the origin O to the barycenter C, using the following vector relationships:

$$\vec{\boldsymbol{r}}^{\,OP_i} = \vec{\boldsymbol{r}}^{\,OC} + \vec{\boldsymbol{r}}^{\,CP_i}$$

$$\dot{\vec{\boldsymbol{r}}}^{\,OP_i} = \dot{\vec{\boldsymbol{r}}}^{\,OC} + \dot{\vec{\boldsymbol{r}}}^{\,CP_i}$$

$$\ddot{\vec{\boldsymbol{r}}}^{\,OP_i} = \ddot{\vec{\boldsymbol{r}}}^{\,OC} + \ddot{\vec{\boldsymbol{r}}}^{\,CP_i} \tag{14}$$

Substituting (Eq. 14) into (Eq. 7) provides

$$m_i\ddot{\vec{\boldsymbol{r}}}^{\,OC} + m_i\ddot{\vec{\boldsymbol{r}}}^{\,CP_i} = \sum_{j=1}^{n} \frac{G m_i m_j}{|\vec{\boldsymbol{r}}^{\,P_iC} + \vec{\boldsymbol{r}}^{\,CP_j}|^3}\left(\vec{\boldsymbol{r}}^{\,P_iC} + \vec{\boldsymbol{r}}^{\,CP_j}\right) + \vec{\boldsymbol{F}}_i \qquad j \neq i \tag{15}$$

From (Eq. 12), we note that

$$m_i\ddot{\vec{\boldsymbol{r}}}^{\,OC} = \frac{m_i}{m}\,\vec{\boldsymbol{F}}$$

which we substitute for the first term on the left-hand side of (Eq. 15), then rearrange to write

$$m_i\ddot{\vec{\boldsymbol{r}}}^{\,CP_i} = \sum_{j=1}^{n} \frac{G m_i m_j}{|\vec{\boldsymbol{r}}^{\,P_iC} + \vec{\boldsymbol{r}}^{\,CP_j}|^3}\left(\vec{\boldsymbol{r}}^{\,P_iC} + \vec{\boldsymbol{r}}^{\,CP_j}\right) + \vec{\boldsymbol{F}}_i - \frac{m_i}{m}\vec{\boldsymbol{F}} \qquad j \neq i \tag{16}$$

(Eq. 16) gives the accelerations of the particles P_i relative to the center of mass (or barycenter) of the system.

1.2 Equations of Relative Motion

Another possibility is to use P_1 as the reference point. Similarly to (Eq. 14)

$$\begin{aligned}
\vec{\boldsymbol{r}}^{\,OP_i} &= \vec{\boldsymbol{r}}^{\,OP_1} + \vec{\boldsymbol{r}}^{\,P_1P_i} \\
\dot{\vec{\boldsymbol{r}}}^{\,OP_i} &= \dot{\vec{\boldsymbol{r}}}^{\,OP_1} + \dot{\vec{\boldsymbol{r}}}^{\,P_1P_i} \\
\ddot{\vec{\boldsymbol{r}}}^{\,OP_i} &= \ddot{\vec{\boldsymbol{r}}}^{\,OP_1} + \ddot{\vec{\boldsymbol{r}}}^{\,P_1P_i}
\end{aligned} \tag{17}$$

Substituting (Eq. 17) into (Eq. 7) gives the equation of motion for the ith particle

$$m_i\big(\ddot{\vec{\boldsymbol{r}}}^{\,OP_1} + \ddot{\vec{\boldsymbol{r}}}^{\,P_1P_i}\big) = \sum_{j=1}^{n} \frac{Gm_im_j}{|\vec{\boldsymbol{r}}^{\,P_iP_j}|^3}\,\vec{\boldsymbol{r}}^{\,P_iP_j} + \vec{\boldsymbol{F}}_i \tag{18}$$

where the vector addition is

$$\vec{\boldsymbol{r}}^{\,P_iP_j} = \vec{\boldsymbol{r}}^{\,P_1P_j} - \vec{\boldsymbol{r}}^{\,P_1P_i}$$

whenever $\vec{\boldsymbol{r}}^{\,P_iP_j}$ appears in (Eq. 18).Also from (Eq. 18), setting $i = 1$, we obtain the following.

$$m_1\ddot{\vec{\boldsymbol{r}}}^{\,OP_1} = \sum_{j=2}^{n} \frac{Gm_1m_j}{|\vec{\boldsymbol{r}}^{\,P_1P_j}|^3}\,\vec{\boldsymbol{r}}^{\,P_1P_j} + \vec{\boldsymbol{F}}_1 \tag{19}$$

where we have used $\ddot{\vec{\boldsymbol{r}}}^{\,P_1P_1} \equiv 0$ and shifted the lower summation index to $j = 2$ since $j \neq i$ in all terms.

Multiplying (Eq. 19) by m_i/m_1 and subtracting the result from (Eq. 18), we eliminate the leading $\ddot{\vec{\boldsymbol{r}}}^{\,OP_1}$ term and obtain

$$m_i\ddot{\vec{\boldsymbol{r}}}^{\,P_1P_i} = \sum_{j=2}^{n} \frac{Gm_im_j}{|\vec{\boldsymbol{r}}^{\,P_iP_j}|^3}\,\vec{\boldsymbol{r}}^{\,P_iP_j} + \vec{\boldsymbol{F}}_i - \sum_{j=2}^{n} \frac{Gm_im_j}{|\vec{\boldsymbol{r}}^{\,P_1P_j}|^3}\,\vec{\boldsymbol{r}}^{\,P_1P_j} - \frac{m_i}{m_1}\vec{\boldsymbol{F}}_1 \qquad i \neq 1 \tag{20}$$

(Eq. 20) consists of $(n-1)$ vector equations that provide the motion of the particles relative to the reference point P_1. The motion of P_1, relative to origin O, is given by (Eq. 19).

In (Eq. 20), we can make all P_1 terms explicit by extracting the $j = 1$ term from the first summation on the right-hand side. This gives

$$\sum_{j=1}^{n} \frac{Gm_j}{|\vec{\boldsymbol{r}}^{\,P_iP_j}|^3}\,\vec{\boldsymbol{r}}^{\,P_iP_j} = \frac{Gm_1}{|\vec{\boldsymbol{r}}^{\,P_iP_1}|^3}\,\vec{\boldsymbol{r}}^{\,P_iP_1} + \sum_{j=2}^{n} \frac{Gm_j}{|\vec{\boldsymbol{r}}^{\,P_iP_j}|^3}\,\vec{\boldsymbol{r}}^{\,P_iP_j} \tag{21}$$

Substituting (Eq. 21) into (Eq. 20) and dividing through by m_i gives

$$\begin{aligned}
\ddot{\vec{\boldsymbol{r}}}^{\,P_1P_i} = \sum_{\substack{j=2 \\ j\neq i}}^{n} \frac{Gm_j}{|\vec{\boldsymbol{r}}^{\,P_iP_j}|^3}\,\vec{\boldsymbol{r}}^{\,P_iP_j} - \frac{Gm_1}{|\vec{\boldsymbol{r}}^{\,P_1P_i}|^3}\,\vec{\boldsymbol{r}}^{\,P_1P_i} + \cdots \\
\cdots \frac{\boldsymbol{F}_i}{m_i} - \sum_{j=2}^{n} \frac{Gm_j}{|\vec{\boldsymbol{r}}^{\,P_1P_j}|^3}\,\vec{\boldsymbol{r}}^{\,P_1P_j} - \frac{\boldsymbol{F}_1}{m_1} \qquad i = 2, 3, \ldots, n
\end{aligned}$$

1.4 Equations of Relative Motion for the n-Body Problem

Later, in the development of General Perturbations, we see that in the case of a planet in the Solar System and in the case of a close satellite about a non-spherical planet, a potential function U can be formed such that

$$U = U_0 + \mathcal{R}$$

where U_0 is the potential function due to the point-mass two-body problem and $\mathcal{R}$ is a potential function due to any other forces, which could include other attracting masses in the system or the oblateness of the planet about which the body revolves.

The term $\mathcal{R}$ is called the **Disturbing Function** and is usually an order of magnitude less than the two-body point mass potential term U_0. Under these conditions, General or Special Perturbation methods can be used.

Let us now consider the Equations of Relative Motion for the n-body problem. From (Eq. 21), with no external forces (so we set $\vec{\boldsymbol{F}}_1 = \vec{\boldsymbol{F}}_i = \vec{\boldsymbol{0}}$), we have

$$\ddot{\vec{\boldsymbol{r}}}^{P_1P_i} = \sum_{j=2}^{n} \frac{Gm_j}{|\vec{\boldsymbol{r}}^{P_iP_j}|^3}\vec{\boldsymbol{r}}^{P_iP_j} - \frac{Gm_1}{|\vec{\boldsymbol{r}}^{P_1P_i}|^3}\vec{\boldsymbol{r}}^{P_1P_i} - \sum_{j=2}^{n} \frac{Gm_j}{|\vec{\boldsymbol{r}}^{P_1P_j}|^3}\vec{\boldsymbol{r}}^{P_1P_j} \qquad i = 2, 3, \ldots, n \tag{22}$$

We note that the first summation in (Eq. 22) necessarily has $j \neq i$ since the particle cannot exert a force on itself (i.e., $\vec{\boldsymbol{f}}_{ii} = 0$), but the second summation may include $j = i$. On the other hand, we can extract the m_i term from the last summation and combine it with the $Gm_1\vec{\boldsymbol{r}}^{P_1P_i}$ term.

To see how the m_i term is removed, consider the case of $i = 3$. Then (Eq. 22) becomes

$$\begin{aligned}\ddot{\vec{\boldsymbol{r}}}^{P_1P_3} = &\sum_{j=2}^{n} \frac{Gm_j}{|\vec{\boldsymbol{r}}^{P_3P_j}|^3}\vec{\boldsymbol{r}}^{P_3P_j} - \frac{Gm_1}{|\vec{\boldsymbol{r}}^{P_1P_3}|^3}\vec{\boldsymbol{r}}^{P_1P_3} \ldots \\ &- \frac{Gm_2}{|\vec{\boldsymbol{r}}^{P_1P_2}|^3}\vec{\boldsymbol{r}}^{P_1P_2} - \frac{Gm_3}{|\vec{\boldsymbol{r}}^{P_1P_3}|^3}\vec{\boldsymbol{r}}^{P_1P_3} - \frac{Gm_4}{|\vec{\boldsymbol{r}}^{P_1P_4}|^3}\vec{\boldsymbol{r}}^{P_1P_4} - \cdots\end{aligned}$$

Continuing from Eq. (1.63), for the case $i = 3$, we obtain

$$\ddot{\vec{\boldsymbol{r}}}^{P_1P_3} = \sum_{j=2}^{n} \frac{Gm_j}{|\vec{\boldsymbol{r}}^{P_3P_j}|^3}\vec{\boldsymbol{r}}^{P_3P_j} - \frac{G(m_1+m_3)}{|\vec{\boldsymbol{r}}^{P_1P_3}|^3}\vec{\boldsymbol{r}}^{P_1P_3} - \sum_{j=2}^{n} \frac{Gm_j}{|\vec{\boldsymbol{r}}^{P_1P_j}|^3}\vec{\boldsymbol{r}}^{P_1P_j}$$

We emphasize the critical distinction that in Eq. (1.64), $j \neq i$ in either summation (in contrast to (Eq. 22), in which the second summation may have $j = i$). Thus, having extracted the m_i term from the second summation in (Eq. 22), we can write, for general i,

$$\ddot{\vec{\boldsymbol{r}}}^{P_1P_i} = \sum_{j=2}^{n} \frac{Gm_j}{|\vec{\boldsymbol{r}}^{P_iP_j}|^3}\vec{\boldsymbol{r}}^{P_iP_j} - \frac{G(m_1+m_i)}{|\vec{\boldsymbol{r}}^{P_1P_i}|^3}\vec{\boldsymbol{r}}^{P_1P_i} - \sum_{j=2}^{n} \frac{Gm_j}{|\vec{\boldsymbol{r}}^{P_1P_j}|^3}\vec{\boldsymbol{r}}^{P_1P_j} \qquad j \neq i,\ i = 2, 3, \ldots, n \tag{23}$$

Rearranging (Eq. 23), we obtain

$$\ddot{\vec{\boldsymbol{r}}}^{P_1P_i} + \frac{G(m_1+m_i)}{|\vec{\boldsymbol{r}}^{P_1P_i}|^3}\vec{\boldsymbol{r}}^{P_1P_i} = G\sum_{\substack{j=2\\ j\neq i}}^{n} m_j\left(\frac{\vec{\boldsymbol{r}}^{P_iP_j}}{|\vec{\boldsymbol{r}}^{P_iP_j}|^3} - \frac{\vec{\boldsymbol{r}}^{P_1P_j}}{|\vec{\boldsymbol{r}}^{P_1P_j}|^3}\right) \qquad j \neq i,\ i = 2, 3, \ldots, n \tag{24}$$

(Eq. 24) is identical in meaning (and similar in nomenclature) to the result found in Roy [19].

We make the following notes on (Eq. 24):

1. It provides the motion of the particles P_i having masses m_i with respect to the particle P_1, which has mass m_1.

2. If other particles P_j $(j \neq i)$ do not exist (or are vanishingly small), then the right-hand side is zero and Eq. (1.66) reduces to the two-body motion of P_i about P_1.

3. The right-hand side consists of perturbations from the P_j $(j \neq i)$ on the orbit of P_i about P_1. For example, in our Solar System, m_1 is the mass of the Sun and we have m_j/m_1 no larger than 10^{-3} — even for Jupiter — so the right-hand side effects are small.

4. For artificial Earth-orbiting satellites, the primary perturbing effects are due to the non-spherical Earth, atmospheric drag, lunar gravity, and solar gravity.

The Disturbing Function

Define a scalar function

$$\mathcal{R} = G\sum_{\substack{j=2\\ j\neq i}}^{n} m_j \left(\frac{1}{|\vec{\boldsymbol{r}}^{\,P_iP_j}|} - \frac{\vec{\boldsymbol{r}}^{\,P_1P_i}\cdot\vec{\boldsymbol{r}}^{\,P_1P_j}}{|\vec{\boldsymbol{r}}^{\,P_1P_j}|^3}\right) \qquad j\neq i,\ i=2,3,\dots,n.$$

where

$$\begin{aligned}
\vec{\boldsymbol{r}}^{\,P_1P_i} &= x_i\hat{\boldsymbol{e}}_1 + y_i\hat{\boldsymbol{e}}_2 + z_i\hat{\boldsymbol{e}}_3\\
\vec{\boldsymbol{r}}^{\,P_1P_j} &= x_j\hat{\boldsymbol{e}}_1 + y_j\hat{\boldsymbol{e}}_2 + z_j\hat{\boldsymbol{e}}_3\\
|\vec{\boldsymbol{r}}^{\,P_iP_j}| &= \Big[(x_j-x_i)^2+(y_j-y_i)^2+(z_j-z_i)^2\Big]^{1/2}\\
|\vec{\boldsymbol{r}}^{\,P_iP_j}|^3 &= \Big[(x_j-x_i)^2+(y_j-y_i)^2+(z_j-z_i)^2\Big]^{3/2}
\end{aligned}$$

We use $\mathcal{R}_i$ to denote the potential function due to the ith body. Next, we show that $\nabla\mathcal{R}_i$ equals the right-hand side of (Eq. 24), where the symbol ∇ ("nabla" or "del") denotes the gradient operator:

$$\nabla \equiv \text{grad} = \hat{\boldsymbol{e}}_1\frac{\partial}{\partial x_i} + \hat{\boldsymbol{e}}_2\frac{\partial}{\partial y_i} + \hat{\boldsymbol{e}}_3\frac{\partial}{\partial z_i}.$$

We note that for the ith body,

$$\begin{aligned}
\nabla\frac{1}{|\vec{\boldsymbol{r}}^{\,P_iP_j}|} &= \nabla\Big[(x_j-x_i)^2+(y_j-y_i)^2+(z_j-z_i)^2\Big]^{-1/2}\\
&= \hat{\boldsymbol{e}}_1\frac{\partial}{\partial x_i}\Big[(x_j-x_i)^2+(y_j-y_i)^2+(z_j-z_i)^2\Big]^{-1/2}\\
&\quad + \hat{\boldsymbol{e}}_2\frac{\partial}{\partial y_i}\Big[(x_j-x_i)^2+(y_j-y_i)^2+(z_j-z_i)^2\Big]^{-1/2}\\
&\quad + \hat{\boldsymbol{e}}_3\frac{\partial}{\partial z_i}\Big[(x_j-x_i)^2+(y_j-y_i)^2+(z_j-z_i)^2\Big]^{-1/2}.
\end{aligned}$$

Carrying out the differentiation, we obtain

$$\begin{aligned}
\nabla\frac{1}{|\vec{\boldsymbol{r}}^{\,P_iP_j}|} &= \hat{\boldsymbol{e}}_1\left(-\tfrac{1}{2}\right)\Big[(x_j-x_i)^2+(y_j-y_i)^2+(z_j-z_i)^2\Big]^{-3/2}2(x_j-x_i)(-1)\\
&\quad + \hat{\boldsymbol{e}}_2\left(-\tfrac{1}{2}\right)\Big[(x_j-x_i)^2+(y_j-y_i)^2+(z_j-z_i)^2\Big]^{-3/2}2(y_j-y_i)(-1)\\
&\quad + \hat{\boldsymbol{e}}_3\left(-\tfrac{1}{2}\right)\Big[(x_j-x_i)^2+(y_j-y_i)^2+(z_j-z_i)^2\Big]^{-3/2}2(z_j-z_i)(-1).
\end{aligned}$$

Simplifying,

$$\nabla\frac{1}{|\vec{\boldsymbol{r}}^{\,P_iP_j}|} = \frac{\vec{\boldsymbol{r}}^{\,P_jP_i}}{|\vec{\boldsymbol{r}}^{\,P_iP_j}|^3} = \frac{(x_j-x_i)\hat{\boldsymbol{e}}_1+(y_j-y_i)\hat{\boldsymbol{e}}_2+(z_j-z_i)\hat{\boldsymbol{e}}_3}{\Big[(x_j-x_i)^2+(y_j-y_i)^2+(z_j-z_i)^2\Big]^{3/2}}.$$

which can be written in vector form

$$\nabla\left(\frac{1}{|\vec{\boldsymbol{r}}^{\,P_iP_j}|}\right) = \frac{\vec{\boldsymbol{r}}^{\,P_iP_j}}{|\vec{\boldsymbol{r}}^{\,P_iP_j}|^3}.$$

3 Perturbations

In the case where a body only moves under the influence of a central gravitational force, the equation of motion gives us Keplerian motion. However, real celestial bodies are influenced by other effects such as additional gravitating bodies, oblateness etc. which create additional perturbing accelerations.

In [5], (Eq. 25) lists the general form of the equation of motion with perturbations which can be expressed in Earth-Centered Inertial (ECI) Cartesian coordinates as:

$$\frac{d^2\vec{\boldsymbol{r}}}{dt^2} = \vec{\boldsymbol{a}}_{gravity} + \vec{\boldsymbol{a}}_{3rd} + \vec{\boldsymbol{a}}_{SRP} + \vec{\boldsymbol{a}}_{D} + \vec{\boldsymbol{a}}_{sf} \tag{25}$$

where $\vec{\boldsymbol{a}}_{gravity}$ is the acceleration (per unit mass) resulting from gravity of the central body. The components of $\vec{\boldsymbol{a}}_{gravity}$ are $\{\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y,}, \frac{\partial\Phi}{\partial z}\}$. The perturbing accelerations are the result of the following:

- $\vec{\boldsymbol{a}}_{3rd}$ - gravity as a result of the third body
- $\vec{\boldsymbol{a}}_{SRP}$ - solar radiation pressure from the momentum of photons from the sun's radiation
- $\vec{\boldsymbol{a}}_{D}$ - atmospheric drag effects that oppose the motion of an object moving through the atmosphere
- $\vec{\boldsymbol{a}}_{sf}$ - sum of other small forces causing acceleration

The last value $\vec{\boldsymbol{a}}_{sf}$ is

$$\vec{\boldsymbol{a}}_{gf} = (\text{solid tides}) + (\text{ocean tides}) + \vec{\boldsymbol{a}}_{rel} + \vec{\boldsymbol{a}}_{ir} + \vec{\boldsymbol{a}}_{op} + \vec{\boldsymbol{a}}_{e} + \vec{\boldsymbol{a}}_{s}$$

where

- $\vec{\boldsymbol{a}}_{rel}$ results from the relativistic effects
- $\vec{\boldsymbol{a}}_{ir}$ results from Earth radiation (infrared)
- $\vec{\boldsymbol{a}}_{op}$ results from Earth albedo (optical)
- $\vec{\boldsymbol{a}}_{e}$ and $\vec{\boldsymbol{a}}_{s}$ result from Earth and solar Yarkovsky forces, respectively.

In the absence of perturbing forces, the solution to equation (Eq. **??**) describes the motion of a body by six constants of integration:

$$c_1, c_2, \ldots, c_6$$

These parameters are typically chosen to be a set of orbital elements, such as $\{a, e, i, \omega, \Omega, f\}$ as shown in figure 2 or an alternative representation such as equinoctial elements, depending on the formulation.

TODO: rewrite these

- a — **Semi-major axis:** Describes the size of the orbit; it is half the longest diameter of the elliptical orbit.
- e — **Eccentricity:** Measures the shape of the orbit, indicating how elongated it is; $e = 0$ is circular, $0 < e < 1$ is elliptical.
- i — **Inclination:** The angle between the orbital plane and the reference plane (usually the equatorial or ecliptic plane), measured at the ascending node.

- ω — **Argument of periapsis:** The angle from the ascending node to the closest approach point of the orbit (periapsis), measured within the orbital plane.
- Ω — **Right ascension of the ascending node (RAAN):** The angle from the reference direction (typically the vernal equinox) to the ascending node, measured in the reference plane.
- f — **True anomaly:** The angle between the periapsis and the current position of the satellite, measured in the direction of motion along the orbit.

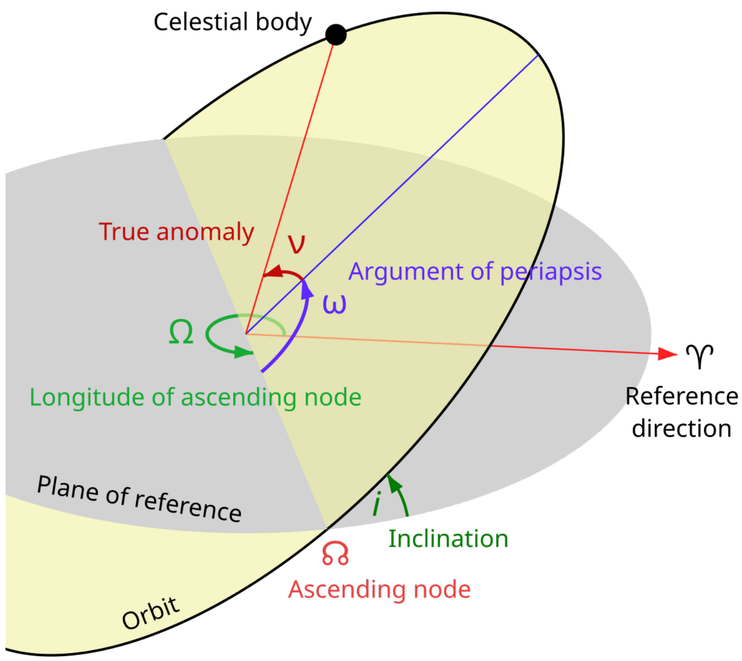

Figure 2: This figure is from the Wikipedia entry on orbital elements. The orbital plane (yellow) intersects a reference plane (gray). For Earth-orbiting satellites, the reference plane is usually the Earth's equatorial plane, and for satellites in solar orbits it is the ecliptic plane. The intersection is called the *line of nodes*, as it connects the reference body (the primary) with the ascending and descending nodes. The reference body and the vernal point (γ) establish a reference direction and, together with the reference plane, they establish a reference frame.

3.1 Variation of Parameters

This section on the variation of parameters method, which was developed by Lagrange in 1788[14]. An in-depth review of this method can be found in chapter 3.5 of Longuski, Hoots, Pollock IV[15], chapter 11 in Brouwer, Clemence[2], TODO: more citations.

The position and velocity of the body at any time t are then given by smooth functions of these constants and time:

$$\begin{aligned} x &= f_1(c_1, c_2, \ldots, c_6, t), & \dot{x} &= g_1(c_1, c_2, \ldots, c_6, t), \\ y &= f_2(c_1, c_2, \ldots, c_6, t), & \dot{y} &= g_2(c_1, c_2, \ldots, c_6, t), \\ z &= f_3(c_1, c_2, \ldots, c_6, t), & \dot{z} &= g_3(c_1, c_2, \ldots, c_6, t) \end{aligned}$$

which describe the Keplerian (unperturbed) motion. Since the elements c_k are constant in the unperturbed problem, the velocity components are simply the partial derivatives of the position functions with respect to time:

$$g_k = \frac{\partial f_k}{\partial t}, \quad \text{for } k = 1, 2, 3$$

In the presence of some perturbing force, the full equations of motion are:

$$\ddot{x} + \frac{\mu}{r^3}x = X, \quad \ddot{y} + \frac{\mu}{r^3}y = Y, \quad \ddot{z} + \frac{\mu}{r^3}z = Z \tag{26}$$

where (X, Y, Z) are the "perturbing" accelerations per unit mass.

Assuming the perturbing accelerations are conservative,

$$\vec{\boldsymbol{F}} = \vec{\boldsymbol{\nabla}}R$$

they can be written as the gradient of a scalar potential, which is called the disturbing function R:

$$X = \frac{\partial R}{\partial x}, \quad Y = \frac{\partial R}{\partial y}, \quad Z = \frac{\partial R}{\partial z}$$

In the case of perturbed motion, we're trying to satisfy equations 5 by the values of equation 3, but obviously the set $\{c_1, ..., c_6\}$ are not constant. So now, we should derive differential equations for this variable element set. Starting with

$$x = f_1(c_1, c_2, ..., c_6, t)$$

Notice the time dependence. To compute how x changes with time under time dependence, we use the chain rule. The total time derivative of x is:

$$\frac{dx}{dt} = \frac{\partial f_1}{\partial t} + \sum_{j=1}^{6} \frac{\partial f_1}{\partial c_j} \cdot \frac{dc_j}{dt}$$

The same applies for $\frac{dy}{dt}$ and $\frac{dz}{dt}$. At this point, we are faced with a system of only 3 equations but 6 unknowns: $\frac{dc_1}{dt}, \ldots, \frac{dc_6}{dt}$. Therefore, the system is **underdetermined**. There are infinitely many ways to choose the six functions $c_j(t)$ to satisfy the 3 equations of motion.

TODO: Add Gabe and Efroimsky gauge argument To make the system uniquely solvable, we must impose 3 additional gauge conditions [9].

$$\sum_j \frac{\partial f_1}{\partial c_j} \cdot \frac{dc_j}{dt} = 0$$
$$\sum_j \frac{\partial f_2}{\partial c_j} \cdot \frac{dc_j}{dt} = 0$$
$$\sum_j \frac{\partial f_3}{\partial c_j} \cdot \frac{dc_j}{dt} = 0$$

These choices eliminate the second terms in equation (Eq. 3.1), so that the first derivatives of the coordinates are simply the same as the first derivative of the unperturbed orbit

$$\frac{dx}{dt} = \frac{\partial f_1}{\partial t} = g_1$$
$$\frac{dy}{dt} = \frac{\partial f_2}{\partial t} = g_2$$
$$\frac{dz}{dt} = \frac{\partial f_3}{\partial t} = g_3$$

These expressions define the **osculating elements**. At each moment, the position and velocity vectors are consistent with Keplerian motion using the instantaneous orbital elements.

Continuing on, differentiate these expressions with respect to time again using the chain rule:

$$\ddot{x} = \frac{d}{dt}\left(\frac{\partial f_1}{\partial t}\right) = \frac{\partial^2 f_1}{\partial t^2} + \sum_{j=1}^{6} \frac{\partial g_1}{\partial c_j} \cdot \frac{dc_j}{dt}$$
$$\ddot{y} = \frac{\partial^2 f_2}{\partial t^2} + \sum_{j=1}^{6} \frac{\partial g_2}{\partial c_j} \cdot \frac{dc_j}{dt}$$
$$\ddot{z} = \frac{\partial^2 f_3}{\partial t^2} + \sum_{j=1}^{6} \frac{\partial g_3}{\partial c_j} \cdot \frac{dc_j}{dt}$$

Substituting the expressions for $\{\ddot{x}, \ddot{y}, \ddot{z}\}$ from above into the perturbed equation of motion in equation (Eq. 26)

$$\begin{aligned} \frac{\partial^2 f_1}{\partial t^2} - \frac{f_1}{r^3}\mu + \sum_{j=1}^{6} \frac{\partial g_1}{\partial c_j} \cdot \frac{dc_j}{dt} &= \frac{\partial R}{\partial x} \\ \frac{\partial^2 f_2}{\partial t^2} - \frac{f_2}{r^3}\mu + \sum_{j=1}^{6} \frac{\partial g_2}{\partial c_j} \cdot \frac{dc_j}{dt} &= \frac{\partial R}{\partial y} \\ \frac{\partial^2 f_3}{\partial t^2} - \frac{f_3}{r^3}\mu + \sum_{j=1}^{6} \frac{\partial g_3}{\partial c_j} \cdot \frac{dc_j}{dt} &= \frac{\partial R}{\partial z} \end{aligned} \tag{27}$$

Since f_1 is a solution of the unperturbed problem, it satisfies the equation:

$$\frac{\partial^2 f_1}{\partial t^2} + \frac{f_1}{r^3}\mu = 0$$

So the first two terms in equation (Eq. 27) are zero, and we are left with:

$$\sum_{j=1}^{6} \frac{\partial g_1}{\partial c_j} \frac{dc_j}{dt} = \frac{\partial R}{\partial x}$$
$$\sum_{j=1}^{6} \frac{\partial g_2}{\partial c_j} \frac{dc_j}{dt} = \frac{\partial R}{\partial y}$$
$$\sum_{j=1}^{6} \frac{\partial g_3}{\partial c_j} \frac{dc_j}{dt} = \frac{\partial R}{\partial z}$$

Switching notation from f_k, g_k to explicitly writing out the coordinates $\{x, y, z, \dot{x}, \dot{y}, \dot{z}\}$, we now collect the six first-order equations for the time derivatives $\frac{dc_j}{dt}$:

$$\frac{\partial x}{\partial c_1}\frac{dc_1}{dt}+\frac{\partial x}{\partial c_2}\frac{dc_2}{dt}+\frac{\partial x}{\partial c_3}\frac{dc_3}{dt}+\frac{\partial x}{\partial c_4}\frac{dc_4}{dt}+\frac{\partial x}{\partial c_5}\frac{dc_5}{dt}+\frac{\partial x}{\partial c_6}\frac{dc_6}{dt}=0 \tag{28}$$
$$\frac{\partial y}{\partial c_1}\frac{dc_1}{dt}+\frac{\partial y}{\partial c_2}\frac{dc_2}{dt}+\frac{\partial y}{\partial c_3}\frac{dc_3}{dt}+\frac{\partial y}{\partial c_4}\frac{dc_4}{dt}+\frac{\partial y}{\partial c_5}\frac{dc_5}{dt}+\frac{\partial y}{\partial c_6}\frac{dc_6}{dt}=0 \tag{29}$$
$$\frac{\partial z}{\partial c_1}\frac{dc_1}{dt}+\frac{\partial z}{\partial c_2}\frac{dc_2}{dt}+\frac{\partial z}{\partial c_3}\frac{dc_3}{dt}+\frac{\partial z}{\partial c_4}\frac{dc_4}{dt}+\frac{\partial z}{\partial c_5}\frac{dc_5}{dt}+\frac{\partial z}{\partial c_6}\frac{dc_6}{dt}=0 \tag{30}$$
$$\frac{\partial \dot{x}}{\partial c_1}\frac{dc_1}{dt}+\frac{\partial \dot{x}}{\partial c_2}\frac{dc_2}{dt}+\frac{\partial \dot{x}}{\partial c_3}\frac{dc_3}{dt}+\frac{\partial \dot{x}}{\partial c_4}\frac{dc_4}{dt}+\frac{\partial \dot{x}}{\partial c_5}\frac{dc_5}{dt}+\frac{\partial \dot{x}}{\partial c_6}\frac{dc_6}{dt}=\frac{\partial R}{\partial x} \tag{31}$$
$$\frac{\partial \dot{y}}{\partial c_1}\frac{dc_1}{dt}+\frac{\partial \dot{y}}{\partial c_2}\frac{dc_2}{dt}+\frac{\partial \dot{y}}{\partial c_3}\frac{dc_3}{dt}+\frac{\partial \dot{y}}{\partial c_4}\frac{dc_4}{dt}+\frac{\partial \dot{y}}{\partial c_5}\frac{dc_5}{dt}+\frac{\partial \dot{y}}{\partial c_6}\frac{dc_6}{dt}=\frac{\partial R}{\partial y} \tag{32}$$
$$\frac{\partial \dot{z}}{\partial c_1}\frac{dc_1}{dt}+\frac{\partial \dot{z}}{\partial c_2}\frac{dc_2}{dt}+\frac{\partial \dot{z}}{\partial c_3}\frac{dc_3}{dt}+\frac{\partial \dot{z}}{\partial c_4}\frac{dc_4}{dt}+\frac{\partial \dot{z}}{\partial c_5}\frac{dc_5}{dt}+\frac{\partial \dot{z}}{\partial c_6}\frac{dc_6}{dt}=\frac{\partial R}{\partial z} \tag{33}$$

These six equations, each first-order in $\frac{dc_j}{dt}$, form the complete system known as Equation 9 in Brouwer & Clemence[2]. They describe how the osculating elements evolve in time under a perturbing potential R. These six first order equations are exactly equivalent to the original second order three equations. What has been accomplished is that we've introduced a dependence on the orbital elements $\{c_1, ..., c_6\}$. T

We can obtain 6 new equations by multiplying Eqs 28 - 33 by $-\frac{\partial \dot{x}}{\partial c_j}, -\frac{\partial \dot{y}}{\partial c_j}, -\frac{\partial \dot{z}}{\partial c_j}, \frac{\partial x}{\partial c_j}, \frac{\partial y}{\partial c_j}, \frac{\partial z}{\partial c_j}$ successively.

Multiply Equation 28 by $-\frac{\partial \dot{x}}{\partial c_j}$

$$-\frac{\partial \dot{x}}{\partial c_j}\left(\frac{\partial x}{\partial c_1}\frac{dc_1}{dt}+\frac{\partial x}{\partial c_2}\frac{dc_2}{dt}+\frac{\partial x}{\partial c_3}\frac{dc_3}{dt}+\frac{\partial x}{\partial c_4}\frac{dc_4}{dt}+\frac{\partial x}{\partial c_5}\frac{dc_5}{dt}+\frac{\partial x}{\partial c_6}\frac{dc_6}{dt}\right)=0$$

Explicitly, for $c_j = c_1$:

$$-\frac{\partial \dot{x}}{\partial c_1}\left(\frac{\partial x}{\partial c_1}\frac{dc_1}{dt}+\frac{\partial x}{\partial c_2}\frac{dc_2}{dt}+\frac{\partial x}{\partial c_3}\frac{dc_3}{dt}+\frac{\partial x}{\partial c_4}\frac{dc_4}{dt}+\frac{\partial x}{\partial c_5}\frac{dc_5}{dt}+\frac{\partial x}{\partial c_6}\frac{dc_6}{dt}\right)=0$$

Multiplying through

$$-\left(\frac{\partial x}{\partial c_1}\frac{\partial \dot{x}}{\partial c_1}\right)\frac{dc_1}{dt}-\left(\frac{\partial x}{\partial c_2}\frac{\partial \dot{x}}{\partial c_1}\right)\frac{dc_2}{dt}-\left(\frac{\partial x}{\partial c_3}\frac{\partial \dot{x}}{\partial c_1}\right)\frac{dc_3}{dt}-\cdots$$
$$\cdots-\left(\frac{\partial x}{\partial c_4}\frac{\partial \dot{x}}{\partial c_1}\right)\frac{dc_4}{dt}-\left(\frac{\partial x}{\partial c_5}\frac{\partial \dot{x}}{\partial c_1}\right)\frac{dc_5}{dt}-\left(\frac{\partial x}{\partial c_6}\frac{\partial \dot{x}}{\partial c_1}\right)\frac{dc_6}{dt}=0$$

For $c_j = c_2$:

$$-\frac{\partial \dot{x}}{\partial c_2}\left(\frac{\partial x}{\partial c_1}\frac{dc_1}{dt}+\frac{\partial x}{\partial c_2}\frac{dc_2}{dt}+\frac{\partial x}{\partial c_3}\frac{dc_3}{dt}+\frac{\partial x}{\partial c_4}\frac{dc_4}{dt}+\frac{\partial x}{\partial c_5}\frac{dc_5}{dt}+\frac{\partial x}{\partial c_6}\frac{dc_6}{dt}\right)=0$$

Again multiplying through by the coefficient

$$-\left(\frac{\partial x}{\partial c_1}\frac{\partial \dot{x}}{\partial c_2}\right)\frac{dc_1}{dt}-\left(\frac{\partial x}{\partial c_2}\frac{\partial \dot{x}}{\partial c_2}\right)\frac{dc_2}{dt}-\left(\frac{\partial x}{\partial c_3}\frac{\partial \dot{x}}{\partial c_2}\right)\frac{dc_3}{dt}-\cdots$$
$$\cdots-\left(\frac{\partial x}{\partial c_4}\frac{\partial \dot{x}}{\partial c_2}\right)\frac{dc_4}{dt}-\left(\frac{\partial x}{\partial c_5}\frac{\partial \dot{x}}{\partial c_2}\right)\frac{dc_5}{dt}-\left(\frac{\partial x}{\partial c_6}\frac{\partial \dot{x}}{\partial c_2}\right)\frac{dc_6}{dt}=0$$

and so on for a total of 6 equations, which can succinctly be written as

$$-\left(\frac{\partial x}{\partial c_1}\frac{\partial \dot{x}}{\partial c_j}\right)\frac{dc_1}{dt}-\left(\frac{\partial x}{\partial c_2}\frac{\partial \dot{x}}{\partial c_j}\right)\frac{dc_2}{dt}-\left(\frac{\partial x}{\partial c_3}\frac{\partial \dot{x}}{\partial c_j}\right)\frac{dc_3}{dt}-\cdots$$
$$\cdots-\left(\frac{\partial x}{\partial c_4}\frac{\partial \dot{x}}{\partial c_j}\right)\frac{dc_4}{dt}-\left(\frac{\partial x}{\partial c_5}\frac{\partial \dot{x}}{\partial c_j}\right)\frac{dc_5}{dt}-\left(\frac{\partial x}{\partial c_6}\frac{\partial \dot{x}}{\partial c_j}\right)\frac{dc_6}{dt}=0$$

Multiply Equation 29 by $-\frac{\partial \dot{y}}{\partial c_j}$

$$-\left(\frac{\partial y}{\partial c_1}\frac{\partial \dot{y}}{\partial c_j}\right)\frac{dc_1}{dt}-\left(\frac{\partial y}{\partial c_2}\frac{\partial \dot{y}}{\partial c_j}\right)\frac{dc_2}{dt}-\left(\frac{\partial y}{\partial c_3}\frac{\partial \dot{y}}{\partial c_j}\right)\frac{dc_3}{dt}-\cdots$$
$$\cdots-\left(\frac{\partial y}{\partial c_4}\frac{\partial \dot{y}}{\partial c_j}\right)\frac{dc_4}{dt}-\left(\frac{\partial y}{\partial c_5}\frac{\partial \dot{y}}{\partial c_j}\right)\frac{dc_5}{dt}-\left(\frac{\partial y}{\partial c_6}\frac{\partial \dot{y}}{\partial c_j}\right)\frac{dc_6}{dt}=0$$

Multiply Equation 30 by $-\frac{\partial \dot{z}}{\partial c_j}$

$$-\left(\frac{\partial z}{\partial c_1}\frac{\partial \dot{z}}{\partial c_j}\right)\frac{dc_1}{dt}-\left(\frac{\partial z}{\partial c_2}\frac{\partial \dot{z}}{\partial c_j}\right)\frac{dc_2}{dt}-\left(\frac{\partial z}{\partial c_3}\frac{\partial \dot{z}}{\partial c_j}\right)\frac{dc_3}{dt}+\cdots$$
$$-\left(\frac{\partial z}{\partial c_4}\frac{\partial \dot{z}}{\partial c_j}\right)\frac{dc_4}{dt}-\left(\frac{\partial z}{\partial c_5}\frac{\partial \dot{z}}{\partial c_j}\right)\frac{dc_5}{dt}-\left(\frac{\partial z}{\partial c_6}\frac{\partial \dot{z}}{\partial c_j}\right)\frac{dc_6}{dt}=0$$

Multiply Equation 31 by $\frac{\partial x}{\partial c_j}$

Starting with $\frac{\partial x}{\partial c_1}$

$$\frac{\partial x}{\partial c_1}\left(\frac{\partial \dot{x}}{\partial c_1}\frac{dc_1}{dt}+\frac{\partial \dot{x}}{\partial c_2}\frac{dc_2}{dt}+\frac{\partial \dot{x}}{\partial c_3}\frac{dc_3}{dt}+\frac{\partial \dot{x}}{\partial c_4}\frac{dc_4}{dt}+\frac{\partial \dot{x}}{\partial c_5}\frac{dc_5}{dt}+\frac{\partial \dot{x}}{\partial c_6}\frac{dc_6}{dt}\right)=\frac{\partial R}{\partial x}\frac{\partial x}{\partial c_1}$$

Multiplying through

$$\left(\frac{\partial x}{\partial c_1}\frac{\partial \dot{x}}{\partial c_1}\right)\frac{dc_1}{dt}+\left(\frac{\partial x}{\partial c_1}\frac{\partial \dot{x}}{\partial c_2}\right)\frac{dc_2}{dt}+\left(\frac{\partial x}{\partial c_1}\frac{\partial \dot{x}}{\partial c_3}\right)\frac{dc_3}{dt}+\cdots$$
$$\cdots+\left(\frac{\partial x}{\partial c_1}\frac{\partial \dot{x}}{\partial c_4}\right)\frac{dc_4}{dt}+\left(\frac{\partial x}{\partial c_1}\frac{\partial \dot{x}}{\partial c_5}\right)\frac{dc_5}{dt}+\left(\frac{\partial x}{\partial c_1}\frac{\partial \dot{x}}{\partial c_6}\right)\frac{dc_6}{dt}=\frac{\partial R}{\partial x}\frac{\partial x}{\partial c_1}$$

So generally, with c_j

$$\left(\frac{\partial x}{\partial c_j}\frac{\partial \dot{x}}{\partial c_1}\right)\frac{dc_1}{dt}+\left(\frac{\partial x}{\partial c_j}\frac{\partial \dot{x}}{\partial c_2}\right)\frac{dc_2}{dt}+\left(\frac{\partial x}{\partial c_j}\frac{\partial \dot{x}}{\partial c_3}\right)\frac{dc_3}{dt}+\cdots$$
$$\cdots+\left(\frac{\partial x}{\partial c_j}\frac{\partial \dot{x}}{\partial c_4}\right)\frac{dc_4}{dt}+\left(\frac{\partial x}{\partial c_j}\frac{\partial \dot{x}}{\partial c_5}\right)\frac{dc_5}{dt}+\left(\frac{\partial x}{\partial c_j}\frac{\partial \dot{x}}{\partial c_6}\right)\frac{dc_6}{dt}=\frac{\partial R}{\partial x}\frac{\partial x}{\partial c_j}$$

Multiply Equation 32 by $\frac{\partial y}{\partial c_j}$

$$\frac{\partial y}{\partial c_1}\left(\frac{\partial \dot{y}}{\partial c_1}\frac{dc_1}{dt}+\frac{\partial \dot{y}}{\partial c_2}\frac{dc_2}{dt}+\frac{\partial \dot{y}}{\partial c_3}\frac{dc_3}{dt}+\frac{\partial \dot{y}}{\partial c_4}\frac{dc_4}{dt}+\frac{\partial \dot{y}}{\partial c_5}\frac{dc_5}{dt}+\frac{\partial \dot{y}}{\partial c_6}\frac{dc_6}{dt}\right)=\frac{\partial R}{\partial y}\frac{\partial y}{\partial c_1}$$

Multiply through

$$\begin{aligned}&\left(\frac{\partial y}{\partial c_j}\frac{\partial \dot{y}}{\partial c_1}\right)\frac{dc_1}{dt}+\left(\frac{\partial y}{\partial c_j}\frac{\partial \dot{y}}{\partial c_2}\right)\frac{dc_2}{dt}+\left(\frac{\partial y}{\partial c_j}\frac{\partial \dot{y}}{\partial c_3}\right)\frac{dc_3}{dt}+\cdots\\&\qquad\cdots+\left(\frac{\partial y}{\partial c_j}\frac{\partial \dot{y}}{\partial c_4}\right)\frac{dc_4}{dt}+\left(\frac{\partial y}{\partial c_j}\frac{\partial \dot{y}}{\partial c_5}\right)\frac{dc_5}{dt}+\left(\frac{\partial y}{\partial c_j}\frac{\partial \dot{y}}{\partial c_6}\right)\frac{dc_6}{dt}=\frac{\partial R}{\partial y}\frac{\partial y}{\partial c_j}\end{aligned}$$

Multiply Equation 33 by $\frac{\partial z}{\partial c_j}$

$$\frac{\partial z}{\partial c_1}\left(\frac{\partial \dot{z}}{\partial c_1}\frac{dc_1}{dt}+\frac{\partial \dot{z}}{\partial c_2}\frac{dc_2}{dt}+\frac{\partial \dot{z}}{\partial c_3}\frac{dc_3}{dt}+\frac{\partial \dot{z}}{\partial c_4}\frac{dc_4}{dt}+\frac{\partial \dot{z}}{\partial c_5}\frac{dc_5}{dt}+\frac{\partial \dot{z}}{\partial c_6}\frac{dc_6}{dt}\right)=\frac{\partial R}{\partial z}\frac{\partial z}{\partial c_1}$$

Multiply through

$$\begin{aligned}&\left(\frac{\partial z}{\partial c_j}\frac{\partial \dot{z}}{\partial c_1}\right)\frac{dc_1}{dt}+\left(\frac{\partial z}{\partial c_j}\frac{\partial \dot{z}}{\partial c_2}\right)\frac{dc_2}{dt}+\left(\frac{\partial z}{\partial c_j}\frac{\partial \dot{z}}{\partial c_3}\right)\frac{dc_3}{dt}+\cdots\\&\qquad\cdots+\left(\frac{\partial z}{\partial c_j}\frac{\partial \dot{z}}{\partial c_4}\right)\frac{dc_4}{dt}+\left(\frac{\partial z}{\partial c_j}\frac{\partial \dot{z}}{\partial c_5}\right)\frac{dc_5}{dt}+\left(\frac{\partial z}{\partial c_j}\frac{\partial \dot{z}}{\partial c_6}\right)\frac{dc_6}{dt}=\frac{\partial R}{\partial z}\frac{\partial z}{\partial c_j}\end{aligned}$$

Now add all of those equations together equations together

$$\begin{aligned}&\left(\frac{\partial x}{\partial c_j}\frac{\partial \dot{x}}{\partial c_1}+\frac{\partial y}{\partial c_j}\frac{\partial \dot{y}}{\partial c_1}+\frac{\partial z}{\partial c_j}\frac{\partial \dot{z}}{\partial c_1}-\frac{\partial x}{\partial c_1}\frac{\partial \dot{x}}{\partial c_j}-\frac{\partial y}{\partial c_1}\frac{\partial \dot{y}}{\partial c_j}-\frac{\partial z}{\partial c_1}\frac{\partial \dot{z}}{\partial c_j}\right)\frac{dc_1}{dt}+\\&\left(\frac{\partial x}{\partial c_j}\frac{\partial \dot{x}}{\partial c_2}+\frac{\partial y}{\partial c_j}\frac{\partial \dot{y}}{\partial c_2}+\frac{\partial z}{\partial c_j}\frac{\partial \dot{z}}{\partial c_2}-\frac{\partial x}{\partial c_2}\frac{\partial \dot{x}}{\partial c_j}-\frac{\partial y}{\partial c_2}\frac{\partial \dot{y}}{\partial c_j}-\frac{\partial z}{\partial c_2}\frac{\partial \dot{z}}{\partial c_j}\right)\frac{dc_2}{dt}+\\&\left(\frac{\partial x}{\partial c_j}\frac{\partial \dot{x}}{\partial c_3}+\frac{\partial y}{\partial c_j}\frac{\partial \dot{y}}{\partial c_3}+\frac{\partial z}{\partial c_j}\frac{\partial \dot{z}}{\partial c_3}-\frac{\partial x}{\partial c_3}\frac{\partial \dot{x}}{\partial c_j}-\frac{\partial y}{\partial c_3}\frac{\partial \dot{y}}{\partial c_j}-\frac{\partial z}{\partial c_3}\frac{\partial \dot{z}}{\partial c_j}\right)\frac{dc_3}{dt}+\\&\left(\frac{\partial x}{\partial c_j}\frac{\partial \dot{x}}{\partial c_4}+\frac{\partial y}{\partial c_j}\frac{\partial \dot{y}}{\partial c_4}+\frac{\partial z}{\partial c_j}\frac{\partial \dot{z}}{\partial c_4}-\frac{\partial x}{\partial c_4}\frac{\partial \dot{x}}{\partial c_j}-\frac{\partial y}{\partial c_4}\frac{\partial \dot{y}}{\partial c_j}-\frac{\partial z}{\partial c_4}\frac{\partial \dot{z}}{\partial c_j}\right)\frac{dc_4}{dt}+\\&\left(\frac{\partial x}{\partial c_j}\frac{\partial \dot{x}}{\partial c_5}+\frac{\partial y}{\partial c_j}\frac{\partial \dot{y}}{\partial c_5}+\frac{\partial z}{\partial c_j}\frac{\partial \dot{z}}{\partial c_5}-\frac{\partial x}{\partial c_5}\frac{\partial \dot{x}}{\partial c_j}-\frac{\partial y}{\partial c_5}\frac{\partial \dot{y}}{\partial c_j}-\frac{\partial z}{\partial c_5}\frac{\partial \dot{z}}{\partial c_j}\right)\frac{dc_5}{dt}+\\&\left(\frac{\partial x}{\partial c_j}\frac{\partial \dot{x}}{\partial c_6}+\frac{\partial y}{\partial c_j}\frac{\partial \dot{y}}{\partial c_6}+\frac{\partial z}{\partial c_j}\frac{\partial \dot{z}}{\partial c_6}-\frac{\partial x}{\partial c_6}\frac{\partial \dot{x}}{\partial c_j}-\frac{\partial y}{\partial c_6}\frac{\partial \dot{y}}{\partial c_j}-\frac{\partial z}{\partial c_6}\frac{\partial \dot{z}}{\partial c_j}\right)\frac{dc_6}{dt}\\&=\frac{\partial R}{\partial x}\frac{\partial x}{\partial c_j}+\frac{\partial R}{\partial y}\frac{\partial y}{\partial c_j}+\frac{\partial R}{\partial z}\frac{\partial z}{\partial c_j}\end{aligned}\tag{34}$$

The right-hand side of this is simply

$$\frac{\partial R}{\partial x}\frac{\partial x}{\partial c_j}+\frac{\partial R}{\partial y}\frac{\partial y}{\partial c_j}+\frac{\partial R}{\partial z}\frac{\partial z}{\partial c_j}=\frac{\partial R}{\partial c_j}.$$

On the left-hand side, the terms in parentheses are of the form

$$[c_j, c_k] = \frac{\partial x}{\partial c_j}\frac{\partial \dot{x}}{\partial c_k} - \frac{\partial x}{\partial c_k}\frac{\partial \dot{x}}{\partial c_j} + \frac{\partial y}{\partial c_j}\frac{\partial \dot{y}}{\partial c_k} - \frac{\partial y}{\partial c_k}\frac{\partial \dot{y}}{\partial c_j} + \frac{\partial z}{\partial c_j}\frac{\partial \dot{z}}{\partial c_k} - \frac{\partial z}{\partial c_k}\frac{\partial \dot{z}}{\partial c_j}.$$

This is the **Lagrange bracket**, frequently written as

$$[c_j, c_k] = \sum_i \left(\frac{\partial x_i}{\partial c_j}\frac{\partial \dot{x}_i}{\partial c_k} - \frac{\partial x_i}{\partial c_k}\frac{\partial \dot{x}_i}{\partial c_j} \right)$$

But in a more short-hand way, it can be found in literature [2] as

$$[c_j, c_k] = \sum_i \frac{\partial(x_i, \dot{x}_i)}{\partial(c_j, c_k)}.$$

For a given pair $\{c_j, c_k\}$ of orbital elements.

Note that equation (Eq. 34) is just one of the equations for a c_j. From the sheer size of 34, the benefit of using Lagrange brackets to compactly write and manipulate the equation is apparent. The 6 equations we have now for $j = \{1 \cdots 6\}$

$$[c_1, c_1]\frac{dc_1}{dt} + [c_1, c_2]\frac{dc_2}{dt} + [c_1, c_3]\frac{dc_3}{dt} + [c_1, c_4]\frac{dc_4}{dt} + [c_1, c_5]\frac{dc_5}{dt} + [c_1, c_6]\frac{dc_6}{dt} = \frac{\partial R}{\partial c_1}, \tag{35}$$

$$[c_2, c_1]\frac{dc_1}{dt} + [c_2, c_2]\frac{dc_2}{dt} + [c_2, c_3]\frac{dc_3}{dt} + [c_2, c_4]\frac{dc_4}{dt} + [c_2, c_5]\frac{dc_5}{dt} + [c_2, c_6]\frac{dc_6}{dt} = \frac{\partial R}{\partial c_2}, \tag{36}$$

$$[c_3, c_1]\frac{dc_1}{dt} + [c_3, c_2]\frac{dc_2}{dt} + [c_3, c_3]\frac{dc_3}{dt} + [c_3, c_4]\frac{dc_4}{dt} + [c_3, c_5]\frac{dc_5}{dt} + [c_3, c_6]\frac{dc_6}{dt} = \frac{\partial R}{\partial c_3}, \tag{37}$$

$$[c_4, c_1]\frac{dc_1}{dt} + [c_4, c_2]\frac{dc_2}{dt} + [c_4, c_3]\frac{dc_3}{dt} + [c_4, c_4]\frac{dc_4}{dt} + [c_4, c_5]\frac{dc_5}{dt} + [c_4, c_6]\frac{dc_6}{dt} = \frac{\partial R}{\partial c_4}, \tag{38}$$

$$[c_5, c_1]\frac{dc_1}{dt} + [c_5, c_2]\frac{dc_2}{dt} + [c_5, c_3]\frac{dc_3}{dt} + [c_5, c_4]\frac{dc_4}{dt} + [c_5, c_5]\frac{dc_5}{dt} + [c_5, c_6]\frac{dc_6}{dt} = \frac{\partial R}{\partial c_5}, \tag{39}$$

$$[c_6, c_1]\frac{dc_1}{dt} + [c_6, c_2]\frac{dc_2}{dt} + [c_6, c_3]\frac{dc_3}{dt} + [c_6, c_4]\frac{dc_4}{dt} + [c_6, c_5]\frac{dc_5}{dt} + [c_6, c_6]\frac{dc_6}{dt} = \frac{\partial R}{\partial c_6} \tag{40}$$

Which can be succinctly written as

$$\sum_{k=1}^{6} [c_j, c_k]\frac{dc_k}{dt} = \frac{\partial R}{\partial c_j}, \quad \text{for } j = 1, \ldots, 6$$

This is the compact form of **Equation (10)** in Brouwer & Clemence, giving the evolution of the osculating elements governed by the Lagrange brackets and the disturbing function.

There are 36 Lagrange brackets in these equations. However, we have the properties

$$[c_j, c_j] = 0,$$
$$[c_j, c_k] = -[c_k, c_j],$$

to reduce the number to 15 unique brackets.

If we were to form a matrix of Lagrange bracket coefficients in these equations, it would be antisymmetric, meaning the elements on the diagonal are zero and elements above the diagonal are equal but opposite in sign to those corresponding elements below the diagonal.

3.2 Properties of Lagrange Brackets

Proof: $[c_j, c_j] = 0$

Writing out the Lagrange Bracket

$$
\begin{aligned}
[c_j, c_j] = &\cancelto{0}{\left(\frac{\partial x}{\partial c_j}\frac{\partial \dot{x}}{\partial c_j} - \frac{\partial x}{\partial c_j}\frac{\partial \dot{x}}{\partial c_j}\right)} + \cancelto{0}{\left(\frac{\partial y}{\partial c_j}\frac{\partial \dot{y}}{\partial c_j} - \frac{\partial y}{\partial c_j}\frac{\partial \dot{y}}{\partial c_j}\right)} \\
&+ \cancelto{0}{\left(\frac{\partial z}{\partial c_j}\frac{\partial \dot{z}}{\partial c_j} - \frac{\partial z}{\partial c_j}\frac{\partial \dot{z}}{\partial c_j}\right)}.
\end{aligned}
$$

So clearly,

$$[c_j, c_j] = 0.$$

Proof: $[c_j, c_k] = -[c_k, c_j]$

Writing out the left-hand side bracket

$$[c_j, c_k] = \left(\frac{\partial x}{\partial c_j}\frac{\partial \dot{x}}{\partial c_k} - \frac{\partial x}{\partial c_k}\frac{\partial \dot{x}}{\partial c_j}\right) + \left(\frac{\partial y}{\partial c_j}\frac{\partial \dot{y}}{\partial c_k} - \frac{\partial y}{\partial c_k}\frac{\partial \dot{y}}{\partial c_j}\right) + \left(\frac{\partial z}{\partial c_j}\frac{\partial \dot{z}}{\partial c_k} - \frac{\partial z}{\partial c_k}\frac{\partial \dot{z}}{\partial c_j}\right) \tag{41}$$

Now, we can check it against the right-hand side bracket

$$
\begin{aligned}
-[c_k, c_j] &= -\left(\frac{\partial x}{\partial c_k}\frac{\partial \dot{x}}{\partial c_j} - \frac{\partial x}{\partial c_j}\frac{\partial \dot{x}}{\partial c_k}\right) + \left(\frac{\partial y}{\partial c_k}\frac{\partial \dot{y}}{\partial c_j} - \frac{\partial y}{\partial c_j}\frac{\partial \dot{y}}{\partial c_k}\right) + \left(\frac{\partial z}{\partial c_k}\frac{\partial \dot{z}}{\partial c_j} - \frac{\partial z}{\partial c_j}\frac{\partial \dot{z}}{\partial c_k}\right) \\
&= \left(\frac{\partial x}{\partial c_j}\frac{\partial \dot{x}}{\partial c_k} - \frac{\partial x}{\partial c_k}\frac{\partial \dot{x}}{\partial c_j}\right) + \left(\frac{\partial y}{\partial c_j}\frac{\partial \dot{y}}{\partial c_k} - \frac{\partial y}{\partial c_k}\frac{\partial \dot{y}}{\partial c_j}\right) + \left(\frac{\partial z}{\partial c_j}\frac{\partial \dot{z}}{\partial c_k} - \frac{\partial z}{\partial c_k}\frac{\partial \dot{z}}{\partial c_j}\right)
\end{aligned}
$$

which is equivalent to (Eq. 41), thus $[c_j, c_k] = -[c_k, c_j]$

Time Independence of Lagrange Brackets

We want to study the time evolution of Lagrange brackets to establish that the Lagrange brackets only depend on the elliptic elements.

Taking the time derivative

$$\frac{\partial}{\partial t}[p, q] = \frac{\partial}{\partial t}\sum_i \frac{\partial x^i}{\partial p}\frac{\partial \dot{x}^i}{\partial q} - \frac{\partial x^i}{\partial q}\frac{\partial \dot{x}^i}{\partial p}$$

Using the chain rule:

$$= \sum_i \left(\frac{\partial x^i}{\partial p}\frac{\partial^2 \dot{x}^i}{\partial t \partial q} + \frac{\partial^2 x^i}{\partial t \partial p}\frac{\partial \dot{x}^i}{\partial q}\right) - \left(\frac{\partial x^i}{\partial p}\frac{\partial^2 \dot{x}^i}{\partial t \partial q} + \frac{\partial^2 x^i}{\partial t \partial p}\frac{\partial \dot{x}^i}{\partial q}\right)$$

TODO: fix this derivation Let $(x_1, x_2, x_3) = (x, y, z)$ be the Cartesian coordinates and $v_i = \dot{x}_i$, $a_i = \ddot{x}_i$ their time derivatives. For any two orbital elements p, q define Lagrange's bracket

$$[p, q] = \sum_{i=1}^{3}\Big(\frac{\partial x_i}{\partial p}\frac{\partial v_i}{\partial q} - \frac{\partial x_i}{\partial q}\frac{\partial v_i}{\partial p}\Big). \tag{42}$$

4 Whittaker's Method

Still working on the left-hand side of the system of equations. We need a way to evaluate Lagrange brackets for a given set of orbital elements. Whittaker's method[23] for evaluating Lagrange brackets depends on how the brackets change with successive rotations.

4.1 First Rotation

Begin by rotating around the z axis through an angle Ω, which will bring the ascending node to the x-axis TODO: make diagram for this rotation

$$[p, q] = \frac{\partial(x, \dot{x})}{\partial(p, q)} + \frac{\partial(y, \dot{y})}{\partial(p, q)} + \frac{\partial(z, \dot{z})}{\partial(p, q)}.$$

Rotate around the z-axis through angle Ω to a "primed" axis:

$$\{x, y, z\} \longrightarrow \{x', y', z'\}.$$

This transformation between the coordinates is:

$$\begin{aligned} x &= x' \cos\Omega - y' \sin\Omega \\ y &= x' \sin\Omega + y' \cos\Omega \\ z &= z' \end{aligned}$$

In order to get the Lagrange bracket, we need the terms

$$\frac{\partial(x, \dot{x})}{\partial(p, q)}, \quad \frac{\partial(y, \dot{y})}{\partial(p, q)}, \quad \frac{\partial(z, \dot{z})}{\partial(p, q)}.$$

which consists of derivative with respect to p and q of our coordinates $\{x, y, z\}$.

Differentiation of x

Differentiation of the rotation in x with respect to p in equation (Eq. **??**) gives:

$$\frac{\partial x}{\partial p} = \frac{\partial x'}{\partial p} \cos\Omega - x' \sin\Omega \frac{\partial \Omega}{\partial p} - \frac{\partial y'}{\partial p} \sin\Omega + y' \cos\Omega \frac{\partial \Omega}{\partial p}.$$

Grouping terms by $\cos\Omega$ and $\sin\Omega$:

$$\frac{\partial x}{\partial p} = \left(\frac{\partial x'}{\partial p} - y' \frac{\partial \Omega}{\partial p} \right) \cos\Omega - \left(x' \frac{\partial \Omega}{\partial p} + \frac{\partial y'}{\partial p} \right) \sin\Omega.$$

This can be written as:

$$\frac{\partial x}{\partial p} = (A_1) \cos\Omega - (B_1) \sin\Omega,$$

where we have defined for compactness:

$$A_1 = \frac{\partial x'}{\partial p} - y' \frac{\partial \Omega}{\partial p}, \quad B_1 = x' \frac{\partial \Omega}{\partial p} + \frac{\partial y'}{\partial p}$$

Now, with respect to q, similarly:

$$\begin{aligned} \frac{\partial x}{\partial q} &= \frac{\partial x'}{\partial q} \cos\Omega - x' \sin\Omega \frac{\partial \Omega}{\partial q} - \frac{\partial y'}{\partial q} \sin\Omega - y' \cos\Omega \frac{\partial \Omega}{\partial q} \\ &= \left(\frac{\partial x'}{\partial q} - y' \frac{\partial \Omega}{\partial q} \right) \cos\Omega - \left(x' \frac{\partial \Omega}{\partial q} + \frac{\partial y'}{\partial q} \right) \sin\Omega \\ &= A_2 \cos\Omega - B_2 \sin\Omega, \end{aligned}$$

where

$$A_2 = \frac{\partial x'}{\partial q} - y' \frac{\partial \Omega}{\partial q},$$
$$B_2 = x' \frac{\partial \Omega}{\partial q} + \frac{\partial y'}{\partial q}.$$

Differentiation of y

From the coordinate transformation in equation **??**

$$y = x' \sin \Omega + y' \cos \Omega,$$

we differentiate with respect to p:

$$\frac{\partial y}{\partial p} = \frac{\partial x'}{\partial p} \sin \Omega + x' \cos \Omega \frac{\partial \Omega}{\partial p} + \frac{\partial y'}{\partial p} \cos \Omega - y' \sin \Omega \frac{\partial \Omega}{\partial p}.$$

Grouping terms by $\sin \Omega$ and $\cos \Omega$:

$$\frac{\partial y}{\partial p} = \left(\frac{\partial x'}{\partial p} - y' \frac{\partial \Omega}{\partial p} \right) \sin \Omega + \left(\frac{\partial y'}{\partial p} + x' \frac{\partial \Omega}{\partial p} \right) \cos \Omega.$$

We have the same coefficients as before in equation (Eq. 4.1):

$$\frac{\partial y}{\partial p} = (B_1) \cos \Omega + (A_1) \sin \Omega.$$

Similarly, for the derivative with respect to q
we have

$$\begin{aligned}
\frac{\partial y}{\partial q} &= \frac{\partial x'}{\partial q} \sin \Omega + x' \cos \Omega \frac{\partial \Omega}{\partial q} + \frac{\partial y'}{\partial q} \cos \Omega - y' \sin \Omega \frac{\partial \Omega}{\partial q} \\
&= \left(\frac{\partial x'}{\partial q} - y' \frac{\partial \Omega}{\partial q} \right) \sin \Omega + \left(\frac{\partial y'}{\partial q} + x' \frac{\partial \Omega}{\partial q} \right) \cos \Omega \\
&= (B_2) \cos \Omega + (A_2) \sin \Omega.
\end{aligned}$$

Differentiation of z

Since our rotation is around the z axis

$$z = z',$$

our derivatives are simply

$$\frac{\partial z}{\partial p} = \frac{\partial z'}{\partial p}, \quad \frac{\partial z}{\partial p} = \frac{\partial z'}{\partial p}$$

Velocity Derivative Terms

To fully evaluate a Lagrange bracket for a chosen pair of orbital elements $\{p, q\}$, we need the velocity derivative terms that appear in the terms of equation (Eq. 4.1)

Derivative of $\dot{x}$

Take a derivative of equation **??**

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial x}{\partial p}\right) &= \frac{\partial}{\partial t}\left(A_1 \cos\Omega - B_1 \sin\Omega\right) \\ &= \frac{\partial A_1}{\partial t}\cos\Omega - \frac{\partial B_1}{\partial t}\sin\Omega\end{aligned}$$

Note the lack of $\frac{\partial \Omega}{\partial t}$ because for osculating set of elements do not change with time as per our gauge choice. Taking the derivative of the coefficients A_1 and B_1 as they appear in equation (Eq. 4.1)

$$\frac{\partial}{\partial p}\left(\frac{\partial x}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial x'}{\partial p} - y'\frac{\partial \Omega}{\partial p}\right)\cos\Omega - \frac{\partial}{\partial t}\left(\frac{\partial y'}{\partial p} + x'\frac{\partial \Omega}{\partial p}\right)\sin\Omega$$

This becomes:

$$= \left(\frac{\partial}{\partial t}\left(\frac{\partial x'}{\partial p}\right) - \frac{\partial y^{'}}{\partial t}\frac{\partial \Omega}{\partial p}\right)\cos\Omega - \left(\frac{\partial}{\partial t}\left(\frac{\partial y'}{\partial p}\right) + \frac{\partial x^{'}}{\partial t}\frac{\partial \Omega}{\partial p}\right)\sin\Omega$$

Using $\frac{\partial x'}{\partial t} = \dot{x}'$ and $\frac{\partial y'}{\partial t} = \dot{y}'$:

$$\frac{\partial \dot{x}}{\partial p} = \left(\frac{\partial \dot{x}'}{\partial p} - \dot{y}'\frac{\partial \Omega}{\partial p}\right)\cos\Omega - \left(\frac{\partial \dot{y}'}{\partial p} + \dot{x}'\frac{\partial \Omega}{\partial p}\right)\sin\Omega$$

Therefore:

$$\frac{\partial \dot{x}}{\partial p} = C_1 \cos\Omega - D_1 \sin\Omega$$

where we define

$$\begin{aligned}C_1 &= \frac{\partial \dot{x}'}{\partial p} - \dot{y}'\frac{\partial \Omega}{\partial p}, \\ D_1 &= \frac{\partial \dot{y}'}{\partial p} + \dot{x}'\frac{\partial \Omega}{\partial p}.\end{aligned}$$

We also need the derivative of equation **??**.

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial x}{\partial q}\right) &= \frac{\partial}{\partial t}\left(A_2 \cos\Omega - B_2 \sin\Omega\right) \\ &= \frac{\partial}{\partial t}\left(\frac{\partial x'}{\partial q} - y'\frac{\partial \Omega}{\partial q}\right)\cos\Omega - \frac{\partial}{\partial t}\left(\frac{\partial y'}{\partial q} + x'\frac{\partial \Omega}{\partial q}\right)\sin\Omega \\ &= \left(\frac{\partial}{\partial q}\left(\frac{\partial x^{'}}{\partial t}\right) - \frac{\partial y^{'}}{\partial t}\frac{\partial \Omega}{\partial q}\right)\cos\Omega - \left(\frac{\partial}{\partial q}\left(\frac{\partial y^{'}}{\partial t}\right) + \frac{\partial x^{'}}{\partial t}\frac{\partial \Omega}{\partial q}\right)\sin\Omega\end{aligned}$$

Using $\frac{\partial x'}{\partial t} = \dot{x}'$ and $\frac{\partial y'}{\partial t} = \dot{y}'$:

$$\frac{\partial \dot{x}}{\partial q} = \left(\frac{\partial \dot{x}'}{\partial q} - \dot{y}'\frac{\partial \Omega}{\partial q}\right)\cos\Omega - \left(\frac{\partial \dot{y}'}{\partial q} + \dot{x}'\frac{\partial \Omega}{\partial q}\right)\sin\Omega$$

Therefore:

$$\frac{\partial \dot{x}}{\partial q} = C_2 \cos\Omega - D_2 \sin\Omega$$

where

$$C_2 = \frac{\partial \dot{x}'}{\partial q} - \dot{y}'\frac{\partial \Omega}{\partial q},$$
$$D_2 = \frac{\partial \dot{y}'}{\partial q} + \dot{x}'\frac{\partial \Omega}{\partial q}.$$

Derivative of $\dot{y}$

We start with

$$\frac{\partial}{\partial p}\left(\frac{\partial y}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial y'}{\partial p} + x'\frac{\partial \Omega}{\partial p}\right)\cos\Omega + \frac{\partial}{\partial t}\left(\frac{\partial x'}{\partial p} - y'\frac{\partial \Omega}{\partial p}\right)\sin\Omega.$$

Expanding, we get

$$= \left(\frac{\partial}{\partial p}\left(\frac{\partial y'}{\partial t}\right) + \frac{\partial x'}{\partial t}\frac{\partial \Omega}{\partial p}\right)\cos\Omega + \left(\frac{\partial}{\partial p}\left(\frac{\partial x'}{\partial t}\right) - \frac{\partial y'}{\partial t}\frac{\partial \Omega}{\partial p}\right)\sin\Omega.$$

Using $\frac{\partial x'}{\partial t} = \dot{x}'$ and $\frac{\partial y'}{\partial t} = \dot{y}'$, this becomes

$$\frac{\partial \dot{y}}{\partial p} = \left(\frac{\partial \dot{y}'}{\partial p} + \dot{x}'\frac{\partial \Omega}{\partial p}\right)\cos\Omega + \left(\frac{\partial \dot{x}'}{\partial p} - \dot{y}'\frac{\partial \Omega}{\partial p}\right)\sin\Omega.$$

As before, we define

$$C_1 = \frac{\partial x'}{\partial p} - y'\frac{\partial \Omega}{\partial p}, \quad D_1 = \frac{\partial \dot{y}'}{\partial p} + \dot{x}'\frac{\partial \Omega}{\partial p},$$

so that

$$\frac{\partial \dot{y}}{\partial p} = D_1 \cos\Omega + C_1 \sin\Omega.$$

Similarly, the derivative of equation **??**

$$\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial q}\right) = \frac{\partial}{\partial t}\left(B_2\cos\Omega + A_2\sin\Omega\right),$$

where A_2 and B_2 are the position coefficients defined earlier. Expanding:

$$= \frac{\partial}{\partial t}\left(\frac{\partial y'}{\partial q} + x'\frac{\partial \Omega}{\partial q}\right)\cos\Omega + \frac{\partial}{\partial t}\left(\frac{\partial x'}{\partial q} - y'\frac{\partial \Omega}{\partial q}\right)\sin\Omega.$$

Distributing the time derivative operation

$$= \left(\frac{\partial}{\partial q}\left(\frac{\partial y'}{\partial t}\right) + \frac{\partial x'}{\partial t}\frac{\partial \Omega}{\partial q}\right)\cos\Omega + \left(\frac{\partial}{\partial q}\left(\frac{\partial x'}{\partial t}\right) - \frac{\partial y'}{\partial t}\frac{\partial \Omega}{\partial q}\right)\sin\Omega.$$

Using $\frac{\partial x'}{\partial t} = \dot{x}'$ and $\frac{\partial y'}{\partial t} = \dot{y}'$:

$$\frac{\partial \dot{y}}{\partial q} = \left(\frac{\partial \dot{y}'}{\partial q} + \dot{x}'\frac{\partial \Omega}{\partial q}\right)\cos\Omega + \left(\frac{\partial \dot{x}'}{\partial q} - \dot{y}'\frac{\partial \Omega}{\partial q}\right)\sin\Omega.$$

We define, as before,

$$C_2 = \frac{\partial x'}{\partial q} - y'\frac{\partial \Omega}{\partial q}, \quad D_2 = \frac{\partial \dot{y}'}{\partial q} + \dot{x}'\frac{\partial \Omega}{\partial q},$$

so that

$$\frac{\partial \dot{y}}{\partial q} = D_2 \cos\Omega + C_2 \sin\Omega.$$

Derivative of $\dot{z}$

For the z-component, we simply have

$$\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial p}\right) = \frac{\partial}{\partial p}\left(\frac{\partial z}{\partial t}\right) = \frac{\partial \dot{z}}{\partial p}.$$

and

$$\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial q}\right) = \frac{\partial}{\partial q}\left(\frac{\partial z}{\partial t}\right) = \frac{\partial \dot{z}}{\partial q}.$$

using $\frac{\partial z}{\partial t} = \dot{z}$

Lagrange Bracket of First Rotation

We now have all terms in the Lagrange bracket:

$$[p, q] = \frac{\partial(x, \dot{x})}{\partial(p, q)} + \frac{\partial(y, \dot{y})}{\partial(p, q)} + \frac{\partial(z, \dot{z})}{\partial(p, q)}.$$

—

Looking at each term individually:
The first term:

$$\begin{aligned}\frac{\partial(x, \dot{x})}{\partial(p, q)} &= \frac{\partial x}{\partial p}\frac{\partial \dot{x}}{\partial q} - \frac{\partial x}{\partial q}\frac{\partial \dot{x}}{\partial p} \\ &= (A_1 \cos\Omega - B_1 \sin\Omega)(C_2 \cos\Omega - D_2 \sin\Omega) \\ &\quad - (A_2 \cos\Omega - B_2 \sin\Omega)(C_1 \cos\Omega - D_1 \sin\Omega)\end{aligned}$$

Multiplying out the terms

$$\begin{aligned}\frac{\partial(x, \dot{x})}{\partial(p, q)} &= (A_1C_2 - A_2C_1)\cos^2\Omega + (B_1D_2 - B_2D_1)\sin^2\Omega \\ &\quad + (-A_1D_2 - B_1C_2 + A_2D_1 + B_2C_1)\sin\Omega\cos\Omega\end{aligned}$$

The second term:

$$\begin{aligned}\frac{\partial(y, \dot{y})}{\partial(p, q)} &= \frac{\partial y}{\partial p}\frac{\partial \dot{y}}{\partial q} - \frac{\partial y}{\partial q}\frac{\partial \dot{y}}{\partial p} \\ &= (B_2 \cos\Omega + A_2 \sin\Omega)(D_2 \cos\Omega + C_2 \sin\Omega) \\ &\quad - (B_2 \cos\Omega + A_2 \sin\Omega)(C_1 \cos\Omega + A_1 \sin\Omega)\end{aligned}$$

Multiplying these out

$$\begin{aligned}\frac{\partial(y, \dot{y})}{\partial(p, q)} &= (B_1D_2 - B_2D_1)\cos^2\Omega + (A_1C_2 - A_2C_1)\sin^2\Omega \\ &\quad + (A_1D_2 + B_1C_2 - A_2D_1 - B_2C_1)\sin\Omega\cos\Omega.\end{aligned}$$

The third term: Since $z = z'$ and $\dot{z} = \dot{z}'$, we have

$$\frac{\partial(z, \dot{z})}{\partial(p, q)} = \frac{\partial(z', \dot{z'})}{\partial(p, q)}.$$

Substituting these back into the Lagrange bracket in equation (Eq. 4.1)

$$[p,q] = A_1C_2 - A_2C_1 + B_1D_2 - B_2D_1 + \frac{\partial(z,\dot z)}{\partial(p,q)}.$$

We can now simplify these terms by subsituting the coefficients from equations (Eq. 4.1), (Eq. **??**), (Eq. **??**), (Eq. **??**) The first two terms are

$$\begin{aligned} A_1C_2 - A_2C_1 = &\left(\frac{\partial x'}{\partial p} - y'\frac{\partial \Omega}{\partial p}\right)\left(\frac{\partial \dot x'}{\partial q} - \dot y'\frac{\partial \Omega}{\partial q}\right) \\ &- \left(\frac{\partial x'}{\partial q} - y'\frac{\partial \Omega}{\partial q}\right)\left(\frac{\partial \dot x'}{\partial p} - \dot y'\frac{\partial \Omega}{\partial p}\right) \end{aligned}$$

$$\begin{aligned} &= \frac{\partial(x',\dot x')}{\partial(p,q)} + \left(-y'\frac{\partial \dot x'}{\partial q} + \dot y'\frac{\partial x'}{\partial q} + y'\frac{\partial \dot x'}{\partial p} - \dot y'\frac{\partial x'}{\partial p}\right)\frac{\partial \Omega}{\partial p} \\ &+ \left(-\dot y'\frac{\partial x'}{\partial p} + y'\frac{\partial x'}{\partial p}\right)\frac{\partial \Omega}{\partial q}. \end{aligned}$$

The second two terms are

$$\begin{aligned} B_1D_2 - B_2D_1 &= \left(\frac{\partial y'}{\partial p} + x'\frac{\partial \Omega}{\partial p}\right)\left(\frac{\partial \dot y'}{\partial q} + \dot x'\frac{\partial \Omega}{\partial q}\right) - \left(\frac{\partial y'}{\partial q} + x'\frac{\partial \Omega}{\partial q}\right)\left(\frac{\partial \dot y'}{\partial p} + \dot x'\frac{\partial \Omega}{\partial p}\right) \\ &= \frac{\partial(y',\dot y')}{\partial(p,q)} + \left(x'\frac{\partial \dot y'}{\partial q} - \dot x'\frac{\partial y'}{\partial q}\right)\frac{\partial \Omega}{\partial p} + \left(\dot x'\frac{\partial y'}{\partial p} - x'\frac{\partial \dot y'}{\partial p}\right)\frac{\partial \Omega}{\partial q}. \end{aligned}$$

Putting the first four terms back into equation (Eq. **??**)

$$\begin{aligned} [p,q] = &\frac{\partial(x',\dot x')}{\partial(p,q)} + \frac{\partial(y',\dot y')}{\partial(p,q)} + \frac{\partial(z',\dot z')}{\partial(p,q)} \\ &+ \left(x'\frac{\partial \dot y'}{\partial q} + \dot y'\frac{\partial x'}{\partial q} - y'\frac{\partial \dot x'}{\partial q} - \dot x'\frac{\partial y'}{\partial q}\right)\frac{\partial \Omega}{\partial p} \\ &- \left(x'\frac{\partial \dot y'}{\partial p} + \dot y'\frac{\partial x'}{\partial p} - y'\frac{\partial \dot x'}{\partial p} - \dot x'\frac{\partial y'}{\partial p}\right)\frac{\partial \Omega}{\partial q}. \end{aligned}$$

The first 3 terms are simply the primed Lagrange bracket

$$[p,q]' = \frac{\partial(x',\dot x')}{\partial(p,q)} + \frac{\partial(y',\dot y')}{\partial(p,q)} + \frac{\partial(z',\dot z')}{\partial(p,q)}.$$

Then

$$\begin{aligned} [p,q] = [p,q]' &+ \left(x'\frac{\partial \dot y'}{\partial q} + \dot y'\frac{\partial x'}{\partial q} - y'\frac{\partial \dot x'}{\partial q} - \dot x'\frac{\partial y'}{\partial q}\right)\frac{\partial \Omega}{\partial p} \\ &- \left(x'\frac{\partial \dot y'}{\partial p} + \dot y'\frac{\partial x'}{\partial p} - y'\frac{\partial \dot x'}{\partial p} - \dot x'\frac{\partial y'}{\partial p}\right)\frac{\partial \Omega}{\partial q}. \end{aligned}$$

The last two terms after $[p,q]'$, can we succinctly write in their own Lagrange bracket

$$[p,q] = [p,q]' + \frac{\partial\left(\Omega,\, x'\dot y' - y'\dot x'\right)}{\partial(p,q)}.$$

4.2 Second Rotation

The next rotation will be about the new x' axis by the angle I as an inclination of the plane. The set of equations relating the new and old coordinates is

$$\begin{aligned} x' &= x'', \\ y' &= y'' \cos I - z'' \sin I, \\ z' &= y'' \sin I + z'' \cos I. \end{aligned}$$

From our last rotation, it now follows that

$$[p, q]' = [p, q]'' + \frac{\partial(I,\, y''\dot{z}'' - z''\dot{y}'')}{\partial(p, q)}.$$

But z'' and $\dot{z}''$ are zero in this case because the $x''y''$ plane coincides with the orbital plane. Hence

$$[p, q]' = [p, q]''.$$

4.3 Third Rotation

Finally, a rotation is made about the z'' axis by the angle $+\omega - \Omega$. Let the new coordinates be designated by X, Y. The XY plane coincides with the orbital plane, and the X axis is directed toward the perihelion. Application of (13) yields

$$\begin{aligned} [p, q]'' &= [p, q]''' + \frac{\partial(\tilde{\omega} - \Omega,\, X\dot{Y} - Y\dot{X})}{\partial(p, q)}, \\ [p, q]''' &= [p, q]''' + \frac{\partial(\tilde{\omega} - \Omega,\, G)}{\partial(p, q)}, \end{aligned}$$

if

$$G = \left[\mu a(1 - e^2)\right]^{1/2}, \quad H = G \cos I$$

Now we can see how the unprimed Lagrange Bracket changes throughout the successive rotations:

$$[p, q] = [p, q]''' + \frac{\partial(\tilde{\omega} - \Omega, G)}{\partial(p, q)} + \frac{\partial(\Omega, H)}{\partial(p, q)}.$$

Given a choice of a particular set (p, q), we can evaluate the last two terms. It still remains to evaluate the first term:

$$[p, q]''' = \frac{\partial X}{\partial p}\frac{\partial \dot{X}}{\partial q} - \frac{\partial X}{\partial q}\frac{\partial \dot{X}}{\partial p} + \frac{\partial Y}{\partial p}\frac{\partial \dot{Y}}{\partial q} - \frac{\partial Y}{\partial q}\frac{\partial \dot{Y}}{\partial p}.$$

The X and Y coordinates on the orbital plane can be evaluated using a Taylor series:

$$\begin{aligned} X &= X_0 + \dot{X}_0(t - T) + \frac{1}{2}\ddot{X}_0(t - T)^2 + \ldots, \\ Y &= Y_0 + \dot{Y}_0(t - T) + \frac{1}{2}\ddot{Y}_0(t - T)^2 + \ldots. \end{aligned}$$

Since Lagrange Brackets are time independent, we can evaluate them at mean anomaly $l = 0$, i.e., at perihelion:

$$l = nt + \epsilon - \tilde{\omega} = 0.$$

Since the Lagrange brackets are time-independent, we may evaluate at $l = 0$ (perihelion), with $l = nt + \epsilon - \tilde{\omega}$.

At perihelion:

$$X = a(1-e) - \frac{al^2}{2(1-e)^2} + \ldots, \qquad \dot{X} = -\frac{anl}{(1-e)^2} + \ldots,$$
$$Y = al\sqrt{\frac{1+e}{1-e}} + \ldots, \qquad \dot{Y} = an\sqrt{\frac{1+e}{1-e}} + \ldots.$$

The partial derivatives at perihelion are:

$$\frac{\partial X}{\partial a} = 1-e, \qquad \frac{\partial X}{\partial e} = -a, \qquad \frac{\partial X}{\partial(\epsilon-\tilde{\omega})} = 0,$$
$$\frac{\partial Y}{\partial a} = 0, \qquad \frac{\partial Y}{\partial e} = 0, \qquad \frac{\partial Y}{\partial(\epsilon-\tilde{\omega})} = a\sqrt{\frac{1+e}{1-e}},$$
$$\frac{\partial \dot{X}}{\partial a} = 0, \qquad \frac{\partial \dot{X}}{\partial e} = 0, \qquad \frac{\partial \dot{X}}{\partial(\epsilon-\tilde{\omega})} = -\frac{an}{(1-e)^2},$$
$$\frac{\partial \dot{Y}}{\partial a} = -\frac{1}{2}n\sqrt{\frac{1+e}{1-e}}, \qquad \frac{\partial \dot{Y}}{\partial e} = an(1+e)^{-1/2}(1-e)^{-3/2}, \qquad \frac{\partial \dot{Y}}{\partial(\epsilon-\tilde{\omega})} = 0.$$

It follows that:

$$\begin{aligned}[p,q]''' &= \frac{\partial(\epsilon-\tilde{\omega},a)}{\partial(p,q)} \cdot \frac{1}{2}\mu^{1/2}a^{-1/2} \\ &= \frac{\partial(\epsilon-\tilde{\omega},L)}{\partial(p,q)}\end{aligned}$$

Where $L = (\mu a)^{1/2}$
Combining with earlier results:

$$[p,q] = \frac{\partial(\epsilon-\tilde{\omega},L)}{\partial(p,q)} + \frac{\partial(\tilde{\omega}-\Omega,G)}{\partial(p,q)} + \frac{\partial(\Omega,H)}{\partial(p,q)}.$$

5. The derivatives of the Keplerian elements

From this expression the required set of Lagrange's brackets is easily obtained. Since

$$\begin{aligned}L &= (\mu a)^{1/2}, \\ G &= L(1-e^2)^{1/2}, \\ H &= G\cos I,\end{aligned}$$

it is found that, designating partial derivatives by subscripts and putting $na^2 = (\mu a)^{1/2}$ and $na = \mu^{1/2}a^{-1/2}$
We can take the derivatives

$$L_a = \frac{1}{2}na, \qquad G_a = \frac{1}{2}na(1-e^2)^{1/2}, \qquad H_a = \frac{1}{2}na(1-e^2)^{1/2}\cos I,$$
$$L_e = 0, \qquad G_e = -na^2e(1-e^2)^{-1/2}, \qquad H_e = -na^2e(1-e^2)^{-1/2}\cos I,$$
$$L_I = 0, \qquad G_I = 0, \qquad H_I = -na^2(1-e^2)^{1/2}\sin I.$$

There results:

$$
\begin{aligned}
[\epsilon, a] &= -[a, \epsilon] = +\frac{1}{2}na, \\
[\tilde{\omega}, a] &= -[a, \tilde{\omega}] = -\frac{1}{2}na\left[1 - (1 - e^2)^{1/2}\right], \\
[\Omega, a] &= -[a, \Omega] = -\frac{1}{2}na(1 - e^2)^{1/2}(1 - \cos I), \\
[\tilde{\omega}, e] &= -[e, \tilde{\omega}] = -na^2 e(1 - e^2)^{-1/2}, \\
[\Omega, e] &= -[e, \Omega] = +na^2 e(1 - e^2)^{-1/2}(1 - \cos I), \\
[\Omega, I] &= -[I, \Omega] = -na^2(1 - e^2)^{1/2}\sin I.
\end{aligned}
$$

All other brackets are zero.

Substitution of the Lagrange Brackers into equations 35-40 gives the sets of equations:

$$
\begin{aligned}
[\epsilon, a]\frac{da}{dt} &= \frac{\partial R}{\partial e}, \\
[\tilde{\omega}, a]\frac{da}{dt} + [\tilde{\omega}, e]\frac{de}{dt} &= \frac{\partial R}{\partial \tilde{\omega}}, \\
[\Omega, a]\frac{da}{dt} + [\Omega, e]\frac{de}{dt} + [\Omega, I]\frac{dI}{dt} &= \frac{\partial R}{\partial \Omega}, \\
[a, e]\frac{de}{dt} + [a, \tilde{\omega}]\frac{d\tilde{\omega}}{dt} + [a, \Omega]\frac{d\Omega}{dt} &= \frac{\partial R}{\partial a}, \\
[e, \tilde{\omega}]\frac{d\tilde{\omega}}{dt} + [e, \Omega]\frac{d\Omega}{dt} &= \frac{\partial R}{\partial e}, \\
[I, \Omega]\frac{d\Omega}{dt} &= \frac{\partial R}{\partial I}.
\end{aligned}
$$

The expressions for da/dt, de/dt, etc., that follow from these equations are:

$$
\begin{aligned}
\frac{da}{dt} &= \frac{2}{na}\frac{\partial R}{\partial e}, \\
\frac{de}{dt} &= -\frac{(1 - e^2)^{1/2}}{na^2 e}\left[1 - (1 - e^2)^{1/2}\right]\frac{\partial R}{\partial e} - \frac{(1 - e^2)^{1/2}}{na^2 e}\frac{\partial R}{\partial \tilde{\omega}}, \\
\frac{dI}{dt} &= -\frac{\tan\frac{1}{2}I}{na^2(1 - e^2)^{1/2}}\frac{\partial R}{\partial e} + \frac{1}{na^2(1 - e^2)^{1/2}\sin I}\frac{\partial R}{\partial \Omega}, \\
\frac{de}{dt} &= \frac{2}{na}\frac{\partial R}{\partial a} - \frac{(1 - e^2)^{1/2}}{na^2 e}\frac{\partial R}{\partial e} - \frac{(1 - e^2)^{1/2}}{na^2 e}\frac{\partial R}{\partial \tilde{\omega}} + \frac{\tan\frac{1}{2}I}{na^2(1 - e^2)^{1/2}}\frac{\partial R}{\partial \Omega}, \\
\frac{d\tilde{\omega}}{dt} &= \frac{(1 - e^2)^{1/2}}{na^2 e}\frac{\partial R}{\partial e} + \frac{\tan\frac{1}{2}I}{na^2(1 - e^2)^{1/2}}\frac{\partial R}{\partial \Omega}, \\
\frac{d\Omega}{dt} &= \frac{1}{na^2(1 - e^2)^{1/2}\sin I}\frac{\partial R}{\partial I}.
\end{aligned}
$$

5 The Disturbing Function

In the previous section, we've arrived at a set of differential equations describing the motion of a body in the presence of some undefined perturbing/disturbing force/potential. We have also looked at how to evaluate the left-hand side of the system of equations in terms of a set of orbital elements. In this section, we'll focus on the right hand side of the equations and derive some specific functions for the disturbing (perturbing) force in terms of the orbital elements so that we may have a system of equations capable of being solved.

5.1 Third Body Perturbations

From Fitzpatrick 9.2 The scenario:

A central mass (the Sun), M, at $\vec{R}_s$, a secondary smaller mass body (planet), m, located at $\vec{R}$. A third mass

$$\begin{aligned}
M\ddot{\vec{\boldsymbol{R}}}_s &= GMm\frac{\vec{\boldsymbol{r}}}{r^3} + GMm'\frac{\vec{\boldsymbol{r}}'}{r'^3} \\
m\ddot{\vec{\boldsymbol{R}}} &= Gmm'\frac{\vec{\boldsymbol{r}}' - \vec{\boldsymbol{r}}}{|\vec{\boldsymbol{r}}' - \vec{\boldsymbol{r}}|^3} - GmM\frac{\vec{\boldsymbol{r}}}{r^3} \\
m'\ddot{\vec{\boldsymbol{R}}}' &= Gm'm\frac{\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|^3} - Gm'M\frac{\vec{\boldsymbol{r}}'}{r'^3}
\end{aligned}$$

Let $\mu = G(M+m)$, $\mu' = G(M+m')$, $\tilde{\mu} = Gm$, and $\tilde{\mu}' = Gm'$.

$$\begin{aligned}
\ddot{\vec{\boldsymbol{r}}} + \mu\frac{\vec{\boldsymbol{r}}}{r^3} &= Gm'\left(\frac{\vec{\boldsymbol{r}}' - \vec{\boldsymbol{r}}}{|\vec{\boldsymbol{r}}' - \vec{\boldsymbol{r}}|^3} - \frac{\vec{\boldsymbol{r}}'}{r'^3}\right) \\
\ddot{\vec{\boldsymbol{r}}}' + \mu'\frac{\vec{\boldsymbol{r}}'}{r'^3} &= Gm\left(\frac{\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|^3} - \frac{\vec{\boldsymbol{r}}}{r^3}\right)
\end{aligned}$$

These right-hand sides can be written as gradients of disturbing functions:

$$\begin{aligned}
\ddot{\vec{\boldsymbol{r}}} + \mu\frac{\vec{\boldsymbol{r}}}{r^3} &= \nabla\mathcal{R} \\
\ddot{\vec{\boldsymbol{r}}}' + \mu'\frac{\vec{\boldsymbol{r}}'}{r'^3} &= \nabla'\mathcal{R}'
\end{aligned}$$

with disturbing functions:

$$\begin{aligned}
\mathcal{R}(\vec{\boldsymbol{r}}, \vec{\boldsymbol{r}}') &= \tilde{\mu}'\left(\frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|} - \frac{\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{r}}'}{r'^3}\right) \\
\mathcal{R}'(\vec{\boldsymbol{r}}, \vec{\boldsymbol{r}}') &= \tilde{\mu}\left(\frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|} - \frac{\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{r}}'}{r^3}\right)
\end{aligned}$$

1.5 Longuski/1.4.1 Vallado

We start with the form of the disturbing function from equation (9.8) in the book:

$$\mathcal{R}(\vec{\boldsymbol{r}}, \vec{\boldsymbol{r}}') = \tilde{\mu}'\left(\frac{1}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|} - \frac{\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{r}}'}{r'^3}\right)$$

Note: in equations 9.8 and 9.9,

$$\tilde{\mu}' = Gm', \quad \tilde{\mu} = Gm$$

Gradient of Scalar Potential

This can be conveniently expressed as the gradient of a scalar potential

Gradient of the Second term Term

Gradient of the First Term

We want to compute:

$$\nabla\left(\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\right)$$

Let:

$$s = |\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'| = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$$

Then:

$$\nabla\left(\frac{1}{s}\right) = \left(\frac{\partial}{\partial x}\hat{\mathbf{i}}+\frac{\partial}{\partial y}\hat{\mathbf{j}}+\frac{\partial}{\partial z}\hat{\mathbf{k}}\right)\left(s^{-1}\right)$$

Use the identity:

$$\nabla\left(s^{-1}\right) = -\frac{1}{s^2}\nabla s \quad \text{and} \quad \nabla s = \frac{\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}$$

So:

$$\nabla\left(\frac{1}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|}\right) = -\frac{\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'}{|\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|^3}$$

Component-wise Gradient Example

Let:

$$s = \sqrt{x^2+y^2+z^2} \quad \Rightarrow \quad \frac{1}{s} = (x^2+y^2+z^2)^{-1/2}$$

Then:

$$\nabla\left(\frac{1}{s}\right) = \nabla\left((x^2+y^2+z^2)^{-1/2}\right)$$

By the chain rule:

$$= -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2x\hat{\mathbf{i}} - \frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2y\hat{\mathbf{j}} - \frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2z\hat{\mathbf{k}}$$

$$= -(x^2+y^2+z^2)^{-3/2}(x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}) = -\frac{\vec{\boldsymbol{r}}}{r^3}$$

This is consistent with the known identity for the gradient of a Newtonian potential.
Let the disturbing function be defined as:

$$\mathcal{R} = Gm'\left[\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\phi}} - \frac{r\cos\phi}{r'^2}\right]$$

Let:

$$\zeta^2 = |\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}'|^2 = r^2+r'^2-2rr'\cos\phi$$

Factoring and Expansion

Factor out $\frac{1}{r'}$:

$$\mathcal{R} = \frac{Gm'}{r'}\left[\left(1+\left(\frac{r}{r'}\right)^2-2\left(\frac{r}{r'}\right)\cos\phi\right)^{-1/2} - \left(\frac{r}{r'}\right)\cos\phi\right]$$

Let:

$$\omega = \left(\frac{r}{r'}\right)^2 - 2\left(\frac{r}{r'}\right)\cos\phi$$

Then:

$$(1+\omega)^{-1/2} = 1-\frac{1}{2}\omega+\frac{3}{8}\omega^2-\frac{5}{16}\omega^3+\dots$$

Binomial Expansion Reference

For any $(1+x)^n$, we have:

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

In our case, $n = -\frac{1}{2}$, and $x = \omega$.

Explicit Terms for ω

$$\omega = \left(\frac{r}{r'}\right)^2 - 2\left(\frac{r}{r'}\right)\cos\phi$$

$$\omega^2 = \left(\left(\frac{r}{r'}\right)^2 - 2\left(\frac{r}{r'}\right)\cos\phi\right)^2 = \left(\frac{r}{r'}\right)^4 - 4\left(\frac{r}{r'}\right)^3\cos\phi + 4\left(\frac{r}{r'}\right)^2\cos^2\phi$$

$$\omega^3 = \left(\left(\frac{r}{r'}\right)^2 - 2\left(\frac{r}{r'}\right)\cos\phi\right)^3 = \left(\frac{r}{r'}\right)^6 - 6\left(\frac{r}{r'}\right)^5\cos\phi + 12\left(\frac{r}{r'}\right)^4\cos^2\phi - 8\left(\frac{r}{r'}\right)^3\cos^3\phi$$

$$\omega^4 = \left(\left(\frac{r}{r'}\right)^2 - 2\left(\frac{r}{r'}\right)\cos\phi\right)^4 = \left(\frac{r}{r'}\right)^8 - 8\left(\frac{r}{r'}\right)^7\cos\phi + 2\cdot\left(\frac{r}{r'}\right)^6\cos^2\phi - 32\left(\frac{r}{r'}\right)^5\cos^3\phi + 16\left(\frac{r}{r'}\right)^4\cos^4\phi$$

Putting It All Together

$$\begin{aligned}\mathcal{R} = \frac{Gm'}{r'}\Bigg[&1 + \left(\frac{r}{r'}\right)\cos\phi + \left(\frac{r}{r'}\right)^2\left(\frac{1}{2} + \frac{3}{2}\cos^2\phi\right) \\ &+ \left(\frac{r}{r'}\right)^3\left(-\frac{3}{2}\cos\phi + \frac{5}{2}\cos^3\phi\right) \\ &+ \left(\frac{r}{r'}\right)^4\left(\frac{3}{8} - \frac{15}{4}\cos^2\phi + \frac{35}{8}\cos^4\phi\right) + \cdots\Bigg]\end{aligned}$$

These coefficients match the form of Legendre polynomials:

$$P_0(\cos\phi),\ P_1(\cos\phi),\ P_2(\cos\phi),\ \dots$$

$$\begin{aligned}P_0(\cos\phi) &= 1 \\ P_1(\cos\phi) &= \cos\phi \\ P_2(\cos\phi) &= \frac{1}{2}\left(3\cos^2\phi - 1\right) \\ P_3(\cos\phi) &= \frac{1}{2}\left(5\cos^3\phi - 3\cos\phi\right) \\ P_4(\cos\phi) &= \frac{1}{8}\left(35\cos^4\phi - 30\cos^2\phi + 3\right)\end{aligned}$$

Trigonometric Power Relations

$$\cos^2\phi = \frac{1}{2}(1+\cos 2\phi) \tag{a}$$
$$\cos^3\phi = \frac{1}{4}(3\cos\phi+\cos 3\phi) \tag{b}$$
$$\cos^4\phi = \frac{1}{8}(3+4\cos 2\phi+\cos 4\phi) \tag{c}$$

Trig Expansion of P_2

$$P_2 = \frac{1}{2}\left(3\cos^2\phi - 1\right)$$

Substitute from (a):

$$= \frac{1}{2}\left(3\cdot\frac{1}{2}(1+\cos 2\phi) - 1\right) = \frac{1}{2}\left(\frac{3}{2}+\frac{3}{2}\cos 2\phi - 1\right) = \frac{1}{4}(1+3\cos 2\phi)$$

Trig Expansion of P_3

$$P_3 = \frac{1}{2}\left(5\cos^3\phi - 3\cos\phi\right)$$

Substitute from (b):

$$= \frac{1}{2}\left(5\cdot\frac{1}{4}(3\cos\phi+\cos 3\phi) - 3\cos\phi\right) = \frac{1}{2}\left(\frac{15}{4}\cos\phi+\frac{5}{4}\cos 3\phi - 3\cos\phi\right) = \frac{1}{8}(5\cos 3\phi+3\cos\phi)$$

Trig Expansion of P_4

$$P_4 = \frac{1}{8}\left(35\cos^4\phi - 30\cos^2\phi + 3\right)$$

Substitute from (a) and (c):

$$= \frac{1}{8}\left[35\cdot\frac{1}{8}(3+4\cos 2\phi+\cos 4\phi) - 30\cdot\frac{1}{2}(1+\cos 2\phi) + 3\right]$$

$$= \frac{1}{8}\left(\frac{105}{8}+\frac{140}{8}\cos 2\phi+\frac{35}{8}\cos 4\phi - 15 - 15\cos 2\phi + 3\right)$$

$$= \frac{1}{64}(105+140\cos 2\phi+35\cos 4\phi-120-120\cos 2\phi+24) = \frac{1}{64}(35\cos 4\phi+20\cos 2\phi+9)$$

Final Expressions

$$P_0(\cos\phi) = 1$$
$$P_1(\cos\phi) = \cos\phi$$
$$P_2(\cos\phi) = \frac{1}{4}(3\cos 2\phi+1)$$
$$P_3(\cos\phi) = \frac{1}{8}(5\cos 3\phi+3\cos\phi)$$
$$P_4(\cos\phi) = \frac{1}{64}(35\cos 4\phi+20\cos 2\phi+9)$$

Legendre Series Expansion of the Disturbing Function

The series:

$$\sum_{n=0}^{\infty} \left(\frac{r}{r'}\right)^n P_n(\cos\phi)$$

is convergent when:

$$\frac{r}{r'} < 1$$

Substituting into the disturbing function:

$$\mathcal{R} = \frac{Gm'}{r'}\left[1 + \left(\frac{r}{r'}\right)^2 P_2 + \left(\frac{r}{r'}\right)^3 P_3 + \left(\frac{r}{r'}\right)^4 P_4 + \cdots\right]$$

(Note: The P_1 terms cancel due to symmetry.)
Now we need to look at two terms in this disturbing function (in red)
To expand $\cos E$ in terms of the mean anomaly M, to order e^3, we use:

$$\begin{aligned}
\cos E &= \frac{1}{e}\left[1 - \frac{1}{2}e^2 - \sqrt{1-e^2}\cos M\right] \\
&= \cos M + \frac{e}{2}(\cos 2M - 1) + \frac{e^2}{8}(3\cos 3M - \cos M) \\
&= \cos M + \frac{e}{2}\cos 2M - \frac{e}{2} + \frac{3e^2}{8}\cos 3M - \frac{e^2}{8}\cos M
\end{aligned}$$

Rewriting $(r/r')^2$ using the expansion of $\cos E$

From Kepler's equation:

$$E - e\sin E = M$$

We also have:

$$\left(\frac{r}{r'}\right)^2 = \left(\frac{a(1 - e\cos E)}{r'}\right)^2$$

Substitute $\cos E$ from Eq. 6.28:

$$= \left(\frac{a}{r'}\right)^2 \left[1 + e^2 - 2e\cos M + 2e^2\cos^2 M - 2e^2\cos M + e^2\right]$$

Collecting terms:

$$\begin{aligned}
\left(\frac{r}{r'}\right)^2 &= \left(\frac{a}{r'}\right)^2 \left[1 + e^2 + 2e^2\cos^2 M - 4e\cos M - 2e^2\cos M\right] \\
&= \left(\frac{a}{r'}\right)^2 \left[1 + e^2 + 2e^2\cos^2 M - 2e(2+e)\cos M\right]
\end{aligned}$$

Further Simplification with Trig Identities

Using:

$$\cos^2 M = \frac{1}{2}(1 + \cos 2M)$$

$$\cos M\cos M = \frac{1}{2}[\cos(M+M) + \cos(M-M)] = \frac{1}{2}(\cos 2M + 1)$$

Then Eq. 6.37 becomes:

$$\left(\frac{r}{r'}\right)^2 = \left(\frac{a}{r'}\right)^2 \left[1 + e^2 + e^2\cos 2M - 2e(2+e)\cos M\right]$$

Structure of the Expansion

We observe from Eq. 6.40 that $(r/r')^2$ is the sum of terms of the form:

$$A_{pq}\cos(pM + qM)$$

where p and q are integers (positive, negative, or zero), and the coefficients $A_{pq} \equiv A_{pq}(a, e, i)$.
From Eq. 6.40:

$$\left(\frac{r}{r'}\right)^2 = \sum A_{pq}\cos(pM + qM')$$

where $p, q \in \mathbb{Z}$ and the coefficients depend on:

$$A_{pq} = A_{pq}(a, a', e, e')$$

6.1.2 The Factor $P_2(\cos\phi)$

From Eq. 6.42:

$$P_2(\cos\phi) = -\frac{1}{2} + \frac{3}{2}\cos^2\phi$$

Position Vector and Unit Vectors

The position vector in the perifocal frame:

$$\vec{\boldsymbol{r}} = \xi\mathbf{u}_P + \eta\mathbf{u}_Q = a\left[(\cos E - e)\mathbf{u}_P + \sqrt{1-e^2}\sin E\,\mathbf{u}_Q\right]$$

Perifocal Unit Vectors

$$\mathbf{u}_P = P_1\hat{\imath} + P_2\hat{\jmath} + P_3\hat{k}, \quad \mathbf{u}_Q = Q_1\hat{\imath} + Q_2\hat{\jmath} + Q_3\hat{k}, \quad \mathbf{u}_W = W_1\hat{\imath} + W_2\hat{\jmath} + W_3\hat{k}$$

With:

$$\begin{aligned}
P_1 &= \cos\Omega\cos\omega - \cos i\sin\Omega\sin\omega \\
P_2 &= \sin\Omega\cos\omega + \cos i\cos\Omega\sin\omega \\
P_3 &= \sin i\sin\omega \\
Q_1 &= -\sin\omega\cos\Omega - \cos\omega\cos i\sin\Omega \\
Q_2 &= -\sin\omega\sin\Omega + \cos\omega\cos i\cos\Omega \\
Q_3 &= \cos\omega\sin i \\
W_1 &= \sin i\sin\Omega, \quad W_2 = -\sin i\cos\Omega, \quad W_3 = \cos i
\end{aligned}$$

Angle Between Radius Vectors

From Eq. 6.44:

$$\cos\phi = \frac{\vec{\boldsymbol{r}}\cdot\vec{\boldsymbol{r}}'}{rr'} = \frac{\xi\xi'\mathbf{u}_P\cdot\mathbf{u}'_P + \xi\eta'\mathbf{u}_P\cdot\mathbf{u}'_Q + \eta\xi'\mathbf{u}_Q\cdot\mathbf{u}'_P + \eta\eta'\mathbf{u}_Q\cdot\mathbf{u}'_Q}{rr'}$$

$$\xi = a(\cos E - e), \quad \xi' = a'(\cos E' - e')$$

Use the expansion from Eq. 6.29:
A typical product is of the form:

$$\cos pM\cos qM' = \frac{1}{2}\left[\cos(pM + qM') + \cos(pM - qM')\right]$$

Dot Product Expansion

We also have from Eq. 6.48:

$$\mathbf{u}_P \cdot \mathbf{u}'_P = P_1 P'_1 + P_2 P'_2 + P_3 P'_3$$

And:

$$\cos i = 1 - 2\sin^2\left(\frac{i}{2}\right)$$

Summary of Terms

Thus, the disturbing function term:

$$\left(\frac{r}{r'}\right)^2 P_2(\cos\phi)$$

is ultimately composed of cosine sums of the form:

$$B_{pq}\cos(pM + qM')$$

with coefficients depending on orbital parameters:

$$B_{pq} = B_{pq}(a, a', e, e')$$

6.1.3 The $\left(\frac{r}{r'}\right)^2 P_2(\cos\phi)$ Term

We have been considering (as an illustration) what terms are involved in the disturbing function, $\mathcal{R}$, as expressed in Eq. (6.21), when we expand the Legendre term $\left(\frac{r}{r'}\right)^2 P_2(\cos\phi)$ as a function of the mean anomalies, M and M'. In particular, we have been examining the factor $P_2(\cos\phi)$. We must still consider the product of $\mathbf{u}_P \cdot \mathbf{u}'_P$ in Eq. (6.44), which can be written as

$$\mathbf{u}_P \cdot \mathbf{u}'_P = P_1 P'_1 + P_2 P'_2 + P_3 P'_3,$$

where, from Eq. (2.51),

$$\begin{aligned}
P_1 &= \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i,\\
P_2 &= \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i,\\
P_3 &= \sin\omega\sin i,
\end{aligned}$$

with analogous expressions for P'_1, P'_2, P'_3.

For convenience, let

$$\cos i = 1 - 2\sin^2\!\left(\tfrac{i}{2}\right)$$

In many applications, the inclination i is a small quantity so that

$$\begin{aligned}
\sin\!\left(\tfrac{i}{2}\right) &\approx \frac{i}{2} = \gamma,\\
\cos i &\approx 1 - 2\gamma^2,
\end{aligned}$$

where γ is used as the new inclination variable. Thus, for small i,

$$P_1 \approx \cos\Omega\cos\omega - \sin\Omega\sin\omega + 2\gamma^2\sin\Omega\sin\omega.$$

Using the identity

$$2\sin\omega\sin\Omega = \cos(\omega - \Omega) - \cos(\omega + \Omega),$$

we may write

$$P_1 \approx \cos(\Omega + \omega) + 2\gamma^2\sin\Omega\sin\omega.$$

From Eq. (6.55) we can show that

$$P_1 \approx (1-\gamma^2)\,\cos(\Omega+\omega) + \gamma^2\cos(\omega-\Omega),$$

and similarly

$$P_1' \approx (1-\gamma'^2)\,\cos(\Omega'+\omega') + \gamma'^2\cos(\omega'-\Omega').$$

We are interested only in the general form of the product $\mathbf{u}_P \cdot \mathbf{u}_P'$ as typified by the term $P_1 P_1'$. From Eqs. (6.57) and (6.58), we see that the term $P_1 P_1'$ consists of terms of the form

$$\cos(\Omega+\omega)\,\cos(\Omega'+\omega'),$$

which can be reduced to sums such as

$$\frac{1}{2}\Big[\cos\big(\Omega+\omega+\Omega'+\omega'\big) + \cos\big(\Omega+\omega-\Omega'-\omega'\big)\Big].$$

Thus, all products arising from $\mathbf{u}_P \cdot \mathbf{u}_P'$ and the other scalar products in Eq. (6.44) take the form

$$C_j\cos\big(j_1\Omega + j_2\Omega' + j_3\omega + j_4\omega'\big),$$

where the j_i $(i=1,2,3,4)$ are integers (positive, negative or zero) and the coefficients C_j are functions of the inclination variables γ and γ'. In general, $\cos^2\phi$ (which appears in the Legendre polynomial $P_2(\cos\phi) = -\frac{1}{2} + \frac{3}{2}\cos^2\phi$) and higher powers of $\cos\phi$ can be expressed as products of such functions and ultimately reduced by trigonometric identities to sums of cosines of multiple angles.

6.2 Form of the Perturbing Function

In the final analysis, the perturbation function $\mathcal{R}$ takes the form

$$\mathcal{R} = Gm'\sum_p C_p\big(a,a',e,e',\gamma,\gamma'\big)\,\cos\!\big(p_1M + p_2M' + p_3\Omega + p_4\Omega' + p_5\omega + p_6\omega'\big),$$

where the integers p_i $(i=1,2,3,4,5,6)$ may be positive, negative, or zero. Equation (6.59) can now be used in Lagrange's planetary equations (Eqs. 5.27 or 5.37) to obtain the perturbations in the orbital elements.

Let

$$M = nt+\sigma, \qquad M' = n't + \sigma',$$

so that

$$p_1M + p_2M' = (p_1n + p_2n')\,t + p_1\sigma + p_2\sigma'.$$

Denote the argument of the cosine in $\mathcal{R}$ by θ:

$$\theta = (p_1n+p_2n')\,t + p_1\sigma + p_2\sigma' + p_3\Omega + p_4\Omega' + p_5\omega + p_6\omega'.$$

We make the additional assumption that the orbital elements of the perturbing body m' can be considered constant. That is, the body of interest is assumed to have no significant effect on the motion of the perturbing body. Thus, Eq. (6.59) can be written as

$$\mathcal{R} = Gm'\sum_p C_p\cos\Big[(p_1n+p_2n')\,t + p_1\sigma + p_3\Omega + p_4\Omega' + p_5\omega + \theta_0\Big],$$

where θ_0 contains all the contributions due to combinations of p_2, p_4, p_6 with $\sigma', \Omega', \omega'$ and the summation refers to all p_i $(i=1,2,3,4,5,6)$. Taking partial derivatives of $\mathcal{R}$ with respect to the orbital elements yields

$$\begin{aligned}
\frac{\partial\mathcal{R}}{\partial\sigma} &= -\,Gm'\sum_p C_p\,p_1\sin\theta, && (p_1\neq 0),\\
\frac{\partial\mathcal{R}}{\partial\sigma} &= 0, && (p_1 = 0),\\
\frac{\partial\mathcal{R}}{\partial\Omega} &= -\,Gm'\sum_p C_p\,p_3\sin\theta, &&\\
\frac{\partial\mathcal{R}}{\partial\omega} &= -\,Gm'\sum_p C_p\,p_5\sin\theta &&
\end{aligned}$$

Partial Derivatives with respect to e, γ and a

We also have

$$\frac{\partial \mathcal{R}}{\partial e} = Gm' \sum_p \frac{\partial C_p}{\partial e} \cos\theta,$$

$$\frac{\partial \mathcal{R}}{\partial \gamma} = Gm' \sum_p \frac{\partial C_p}{\partial \gamma} \cos\theta.$$

Since $\gamma = \sin\left(\frac{i}{2}\right)$, the relation $\partial\gamma/\partial i = \frac{1}{2}\cos\left(\frac{i}{2}\right)$ allows us to write

$$\frac{\partial \mathcal{R}}{\partial i} = \frac{1}{2}\, Gm' \cos\left(\frac{i}{2}\right) \sum_p \frac{\partial C_p}{\partial \gamma} \cos\theta.$$

Differentiation with respect to a gives

$$\frac{\partial \mathcal{R}}{\partial a} = Gm' \sum_p \frac{\partial C_p}{\partial a} \cos\theta \;-\; Gm' \sum_p C_p \,(p_1 t)\, \frac{\partial n}{\partial a} \sin\theta,$$

where, in Eq. (6.72), we use the relation

$$\frac{\partial n}{\partial a} = -\,\frac{3\,n}{2a}.$$

As an example, consider the $\dot{\Omega}$ equation of Lagrange's planetary equations (Eqs. 5.27 or 5.37):

$$\dot{\Omega} = \frac{1}{a^2 n\nu\sqrt{1-e^2}\,\sin i}\,\frac{\partial \mathcal{R}}{\partial i}.$$

Substituting Eq. (6.71) into Eq. (6.73) yields

$$\dot{\Omega} = \frac{Gm' \cos\left(\frac{i}{2}\right)}{2\,a^2 n\nu\sqrt{1-e^2}\,\sin i} \sum_p \frac{\partial C_p}{\partial \gamma} \cos\theta,$$

where

$$\theta = (p_1 n + p_2 n')\,t + p_1\sigma + p_3\Omega + p_5\omega + \theta_0.$$

6.2 Short- and Long-Period Terms

If $p_1 n + p_2 n'$ is large, then perturbations to the orbital element have small amplitudes and short periods (high frequencies). Such perturbations are referred to as *short-period inequalities*. Conversely, if $p_1 n + p_2 n'$ is small, perturbations have large amplitudes and long periods (low frequencies); these are called *long-period inequalities*. From Lagrange's planetary equations (Eqs. 5.27 or 5.37) and Eqs. (6.65)–(6.72), all of the elements except the semimajor axis a exhibit secular as well as periodic changes (to first order in m').

Separation of $\dot{\Omega}$ into Secular and Periodic Parts

We can separate $\dot{\Omega}$ into two pieces as

$$\dot{\Omega} = A + \sum_p B_p \cos\left[(p_1 n + p_2 n')\,t + \theta_1\right],$$

where the p_1 and p_2 are not zero simultaneously. Integrating Eq. (6.76) yields

$$\Omega = \Omega_0 + At + \sum_p \frac{B_p}{(p_1 n + p_2 n')} \sin\left[(p_1 n + p_2 n')\,t + \theta_1\right],$$

where the subscript zero denotes fixed elements. The term At is a secular perturbation term. If $p_1 n + p_2 n' = 0$, we still have secular terms arising from Eq. (6.76). In this case we have a commensurability in the periods P and P' of the perturbed and perturbing bodies. If P and P' are the periods then

$$\frac{P'}{P} = \frac{n}{n'} = -\frac{p_2}{p_1},$$

where p_1 and p_2 are integers.

6.4 Stability of the Semimajor Axis

For the semimajor axis, Lagrange's equations yield

$$\begin{aligned} \dot{a} &= \frac{2}{an}\frac{\partial \mathcal{R}}{\partial \sigma} - \frac{2\,Gm'}{an}\sum_p C_p\, p_1 \sin\theta, && (p_1 \neq 0), \\ \dot{a} &= 0, && (p_1 = 0). \end{aligned}$$

Integrating Eq. (6.81) gives

$$\delta(a) = \frac{2\,Gm'}{a_0 n}\sum_p C_p \,\frac{p_1}{(p_1 n + p_2 n')}\, \cos\theta,$$

where

$$a = a_0 + \delta(a)$$

Thus, the semimajor axis oscillates about the mean value a_0 with period

$$P = \frac{2\pi}{p_1 n + p_2 n'}.$$

We have shown that the semimajor axis exhibits no secular change in the first-order theory; if it did, then the orbit would expand or contract indefinitely and the orbit would be unstable. In another situation, if the eccentricity were to increase secularly, a close approach with another planet could occur and disrupt the system. Lagrange showed that when all powers of e are included to first order in m' that the semimajor axis undergoes no secular change. Subsequent work by Poisson and Haretu extended this result to higher orders.

6 Other Disturbing Functions

7 Topics in Books

TODO: remove period and comma on equations TODO: reference fitzpatrick TODO: rewiret the ref of equation as (Eq. 14) [1] [2] [3] [4] [5] [6] [7] [8] [9] [10] [11] [12] [13] [15] [16] [17] [18] [19] [20] [21] [22] [?]

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