NYC_DCWP

applicant_name_full
res_address_line_1_consol
res_city res_state res_zip
coapplicant_name_full
co_res_address_line_1_consol
co_res_city co_res_state co_res_zip
Seller_FullNameStreetCityStateZip
seller_facility_num
dealer_na
buyerEmail
buyerPhone
buyerCell
coBuyerEmail
coBuyerPhone
coBuyerCell
applicationIdAndSequence

MTH 309 Name:

SECTION 2.2: THE INVERSE OF A MATRIX

Before class, please read through the section. For the vocabulary words that have a "⋆", write (or complete) the definition of the term in the corresponding "box". Any examples for the definitions will be completed in class.

Warm up- Using multiplication only, solve the equation for x:

$$2x = 1$$

$$\boxed{\tfrac{1}{2}}\, 2x = \tfrac{1}{2} \cdot 1 \to 1x = \tfrac{1}{2} \to x = \tfrac{1}{2}$$

Seller_FullNameStreetCityStateZip

approxDeliv

Here, 2 and $\frac{1}{2}$ are multiplicative inverses.

⋆ Definition: Invertible Matrix

A $n \times n$ matrix A is said to be **invertible** if there is a $n \times n$ matrix C such that

$CA = I$ and $AC = I$ where I is the $n \times n$ identity matrix

vehYear vehMake vehModel vehBody veh_trim vehColor vehMileage

In this case, C is an **inverse** of A. It is uniquely determined by A.

This unique matrix A is denoted: A^{-1}

⋆ Definition:

A matrix that is not invertible is called: a singular matrix

An invertible matrix is called: a nonsingular matrix

insuranceCompany insuranceAgent insurancePolicyNum
vehVIN
vehPrice

Theorem 4

Define A as follows:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

If $ad - bc \neq 0$, then A is invertible and:

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$ then:

A is not invertible.

Note: For 2x2 matrices of this form: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$

$ad-bc$ is called the determinant of A

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we write $\det(A) = ad-bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

vertical lines, not brackets.

item_transfer
item_tag
item_lien
item_license
item_inspection
registrationFee
svcContractAmt
gapAmountNoD
totalOtherChar
totalItemization
netTradeInCred
TradeDifference
salesTaxRate
titleAdValorem
registrationPlusL
titleFee

vehTrac vehTradeMake vehTradeModel

vehTradeMileage

documentaryFeeNoDollar

TotalPriceNYS

cashDepositNo

proposed_trade_in_payoff_lender

tradePayoffAmntNoDollar

tradePayoffAmntNoDollar

amountFinance

Example: Let A be defined as follows. Is A invertible? How do you know? If A is invertible, find the inverse of A.

$$A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}.$$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\det(A) = \begin{vmatrix} 2 & 5 \\ -3 & -7 \end{vmatrix} = 2(-7) - (5)(-3)$
$= -14 + 15 = 1 \neq 0$, A is invertible.

$A^{-1} = \frac{1}{1}\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$

check

$AA^{-1} = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$

$= \begin{bmatrix} 2(-7)+(5)(3) & 2(-5)+(5)(2) \\ -3(-7)+(-7)(3) & -3(-5)+(-7)(2) \end{bmatrix}$

$= \begin{bmatrix} -14+15 & -10+10 \\ 21-21 & 15-14 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2x2 identity matrix ✓

How can matrix inverses be used to help solve a system of equations?

Theorem 5

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution:

$\rightarrow$ $\vec{x} = A^{-1}\vec{b}$

Idea: Why is this the case?

- Notice if $\vec{x} = A^{-1}\vec{b}$
  $A\vec{x} = A(A^{-1}\vec{b}) = AA^{-1}\vec{b} = I_n\vec{b} = \vec{b}$ so $\vec{x} = A^{-1}\vec{b}$ is a solution to $A\vec{x} = \vec{b}$ ✓

- $A\vec{x} = \vec{b}$ multiply (on the left) by A^{-1}
  order matters!

  $A^{-1} \cdot A\vec{x} = A^{-1}\vec{b}$
  $\rightarrow I_n\vec{x} = A^{-1}\vec{b} \rightarrow \vec{x} = A^{-1}\vec{b}$

Example: Use the inverse of a matrix to solve the system

$$2x_1 + 5x_2 = 3, \quad -3x_1 - 7x_2 = -7$$

$\begin{matrix} 2x_1 + 5x_2 = 3 \\ -3x_1 - 7x_2 = -7 \end{matrix} \Rightarrow \underbrace{\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ -7 \end{bmatrix}}_{\vec{b}}$

$A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ is A invertible? In the previous example, we found that this matrix is invertible and

$A^{-1} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$ so the unique solution (by Thm 5) is. $\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -7 \end{bmatrix} = \begin{bmatrix} -7(3)+(-5)(-7) \\ 3(3)+(2)(-7) \end{bmatrix} = \begin{bmatrix} -21+35 \\ 9-14 \end{bmatrix} = \begin{bmatrix} 14 \\ -5 \end{bmatrix}$

Theorem 6

1. If A is an invertible matrix, then A^{-1} is invertible and:

$$(A^{-1})^{-1} = A$$

2. If A and B are $n \times n$ invertible matrices, then so is AB and: $(AB)^{-1} = B^{-1}A^{-1}$

Notice the order!

Generalization: The product of $n \times n$ invertible matrices is invertible and: the inverse is the product of their inverses in the reverse order.

3. If A is an invertible matrix, then so is A^T and: $(A^T)^{-1} = (A^{-1})^T$

⋆ **Definition: Elementary Matrix:**

An **elementary matrix** is one that is obtained by: performing a single elementary row operation on an identity matrix

Example: Consider the elementary row operation $2R1 + R2 \to R2$, find the corresponding 3×3 elementary matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{2R1+R2 \to R2}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example: Let A, E_2, E_3 be defined below. Compute E_2A and E_3A and describe how these products can be obtained by elementary row operations on A.

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(where $a, b, c, d, e, f, g, h, j$ are real numbers)

$$\underset{3\times 3 \;\; 3\times 3}{E_2A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} = \begin{bmatrix} a+0+0 & b+0+0 & c+0+0 \\ 0+0+g & 0+0+h & 0+0+j \\ 0+d+0 & 0+e+0 & 0+f+0 \end{bmatrix} = \begin{bmatrix} a & b & c \\ g & h & j \\ d & e & f \end{bmatrix}$$

This is obtained by interchanging rows 2 and 3 of matrix A.

$$\underset{3\times 3 \;\; 3\times 3}{E_3A} = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} = \begin{bmatrix} 7a & 7b & 7c \\ d & e & f \\ g & h & j \end{bmatrix}$$

This is obtained by multiplying row 1 of A by 7.

If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as: $E \cdot A$ where the $m \times m$ matrix E is created by performing the same elementary row operation on I_m ← $m \times m$ identity matrix.

Are elementary matrices invertible? Yes!
Let E be an elementary matrix. The inverse of E is the elementary matrix of the same type that transforms E back to the identity matrix.

Example: Find the inverse of

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ ← An elementary matrix

To put E_1 back to I_3 (3x3 identity matrix)

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2R1+R2 \to R2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2R1+R2 \to R2} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

check

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ 2+(-2)+0 & 0+1+0 & 0+0+0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Theorem 7
An $n \times n$ matrix A is invertible if and only if: A is row equivalent to I_n ← $n \times n$ identity matrix

In this case, any sequence of elementary row operations that reduces A to I_n: also transforms I_n to A^{-1}

Idea behind Theorem 7:

If A is row equivalent to I_n, there is a series of elementary matrices $E_1, E_2, \ldots, E_p$ such that

$$E_p \cdots E_2(E_1 A) = I_n \Rightarrow (E_p \cdots E_2 E_1) A = I_n$$

each elementary matrix is invertible. so $(E_p \cdots E_2 E_1)$ is invertible.

$$(E_p \cdots E_2 E_1)^{-1}(E_p \cdots E_2 E_1) A = (E_p \cdots E_2 E_1)^{-1} I_n$$

$$I_n A = (E_p \cdots E_2 E_1)^{-1} \Rightarrow A = (E_p \cdots E_2 E_1)^{-1}$$ so A is invertible.

and:

$$A^{-1} = \left((E_p \cdots E_2 E_1)^{-1}\right)^{-1} = E_p \cdots E_2 E_1 = E_p \cdots E_2 E_1 I_n$$

so, A^{-1} results from applying $E_1, E_2, \ldots, E_p$ successively to I_n. (note the order!)

let A be an $n \times n$ matrix

Algorithm for finding A^{-1}: Row reduce the augmented matrix $[A \mid I_n]$. If A is row equivalent to I_n, $[A \mid I_n]$ is row equivalent to $[I_n \mid A^{-1}]$.
Otherwise, A does not have an inverse.

Note about augmented matrices:

In chapter 1, we talked about augmented matrices of a linear system.
In general, augmented matrices are formed by adding additional columns (with the same number of rows) to a matrix.

Example: Let A be defined as follows:

$$A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}.$$

Use this algorithm to find A^{-1}. Check that this is the same result as you previously obtained.

$$[A \mid I_2] = \left[\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ -3 & -7 & 0 & 1 \end{array}\right] \overset{\frac{3}{2}R1+R2 \to R2}{\sim} \left[\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 0 & 1/2 & 3/2 & 1 \end{array}\right]$$

$\frac{3}{2}(5)-7$

$$\overset{2R2}{\sim} \left[\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 0 & 1 & 3 & 2 \end{array}\right]$$

$$\overset{-5R2+R1 \to R1}{\sim} \left[\begin{array}{cc|cc} 2 & 0 & -14 & -10 \\ 0 & 1 & 3 & 2 \end{array}\right]$$

$$\overset{\frac{1}{2}R1}{\sim} \left[\begin{array}{cc|cc} 1 & 0 & -7 & -5 \\ 0 & 1 & 3 & 2 \end{array}\right] \qquad \underbrace{}_{I_2 \checkmark} \quad \underbrace{}_{A^{-1}}$$

$$A^{-1} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$$

Example: Find the inverse of $A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$, if it exists.

$$[A \mid I_2] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 4 & 8 & 0 & 1 \end{array}\right] \overset{-4R1+R2 \to R2}{\sim} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -4 & 1 \end{array}\right]$$

No leading entry!

Not row equivalent to I_2

The inverse of A does not exist.
A is singular.

Example: (on your own) Find the inverse of $A = \begin{bmatrix} -2 & -7 & -9 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{bmatrix}$ if it exists.

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} -2 & -7 & -9 & 1 & 0 & 0 \\ 2 & 5 & 6 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \overset{R1 \leftrightarrow R3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 3 & 4 & 0 & 0 & 1 \\ 2 & 5 & 6 & 0 & 1 & 0 \\ -2 & -7 & -9 & 1 & 0 & 0 \end{array}\right]$$

$$\overset{R2+R3 \to R3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 3 & 4 & 0 & 0 & 1 \\ 2 & 5 & 6 & 0 & 1 & 0 \\ 0 & -2 & -3 & 1 & 1 & 0 \end{array}\right]$$

$$\overset{-2R1+R2 \to R2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 3 & 4 & 0 & 0 & 1 \\ 0 & -1 & -2 & 0 & 1 & -2 \\ 0 & -2 & -3 & 1 & 1 & 0 \end{array}\right]$$

$$\overset{-R2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 3 & 4 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 & 2 \\ 0 & -2 & -3 & 1 & 1 & 0 \end{array}\right]$$

$$\overset{2R2+R3 \to R3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 3 & 4 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & -1 & 4 \end{array}\right]$$

$$\overset{-2R3+R2 \to R2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 3 & 4 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 & 1 & -6 \\ 0 & 0 & 1 & 1 & -1 & 4 \end{array}\right]$$

$$\overset{-4R3+R1 \to R1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & -4 & 4 & -15 \\ 0 & 1 & 0 & -2 & 1 & -6 \\ 0 & 0 & 1 & 1 & -1 & 4 \end{array}\right]$$

$$\overset{-3R2+R1 \to R1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 3 \\ 0 & 1 & 0 & -2 & 1 & -6 \\ 0 & 0 & 1 & 1 & -1 & 4 \end{array}\right]$$

$$\underbrace{}_{I_3} \qquad \underbrace{}_{A^{-1}}$$

so,

$$A^{-1} = \begin{bmatrix} 2 & 1 & 3 \\ -2 & 1 & -6 \\ 1 & -1 & 4 \end{bmatrix}$$