Nonstandard finite element methods

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Abstract

We present the principles of conforming and nonconforming finite element methods (FEMs) and virtual element methods (VEMs). The virtual element method is the main part of this course.

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Contents

1. Primal variational problem

1.1. Abstract variational problem

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial\Omega$ where $d = 2, 3$. V is a Hilbert space on Ω with the norm $\|\cdot\|_V$. Let $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ be a bilinear form. Given a functional $f \in V'$, the abstract variational problem is to find $u \in V$ such that

$$a(u, v) = \langle f, v \rangle, \quad \forall v \in V, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ is the duality pair between V' and V.

Lemma 1.1 (Lax-Milgram)**.** *If the bilinear form $a(\cdot, \cdot)$ on $V \times V$ satisfies*

(1) Continuity: $a(u, v) \leq C\|u\|_V \|v\|_V, \ \forall u, v \in V$;
(2) Coercivity (or ellipticity): $a(v, v) \geq \alpha \|v\|_V^2, \ \forall v \in V$,

then for any $f \in V'$, the problem (1.1) *has a unique solution $u \in V$ satisfying*

$$\|u\|_V \leq \frac{1}{\alpha} \|f\|_{V'},$$

i.e. is well-posed.

The simplest case is the bilinear form $a(\cdot, \cdot)$ is symmetric and positive definite on V. Then it defines a new inner product on V. The Lax-Milgram lemma is simply the Riesz representation theorem.

Next, we display some examples.

1.2. Poisson problem

The first one is the Poisson problem with homogeneous boundary condition:

$$\begin{cases} -\Delta u = f & \text{in } H^{-1}(\Omega), \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$.

The corresponding variational form is to find $u \in V$ s.t.

$$a(u, v) = \langle f, v \rangle, \quad \forall v \in V, \tag{1.2}$$

where $a(u, v) = (\nabla u, \nabla v), \ \langle f, v \rangle = (f, v), \ V = H_0^1(\Omega), \ V' = H^{-1}(\Omega), \ \|\cdot\|_V = |\cdot|_1$ (or $\|\cdot\|_1$).

Due to the Poincaré-Friedrichs inequality

$$\|v\| \leq C|v|_1, \quad \forall v \in H_0^1(\Omega), \tag{1.3}$$

we obtain the coercivity of $a(\cdot, \cdot)$. Thus the Lax-Milgram lemma guarantees the well-posedness of the variational problem (1.2).

1.3. Biharmonic equation

The second one is the Biharmonic equation with homogeneous boundary condition:

$$\begin{cases} \Delta^2 u = f & \text{in } H^{-2}(\Omega), \\ u = 0, \ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where n is the outer unit normal to Ω and $H^{-2}(\Omega)$ is the dual space of $H_0^2(\Omega)$.

The corresponding variational form is to find $u \in V$ s.t.

$$a(u, v) = \langle f, v \rangle, \quad \forall v \in V, \tag{1.4}$$

where $\langle f, v \rangle = (f, v), \ V = H_0^2(\Omega), \ V' = H^{-2}(\Omega), \ \|\cdot\|_V = |\cdot|_2$ (or $\|\cdot\|_2$),

$$a(u, v) = (\Delta u, \Delta v),$$

or

$$a(u, v) = (\nabla^2 u, \nabla^2 v).$$

Using the Poincaré-Friedrichs inequality (1.3) twice, we have

$$\|v\|_2 \leq |v|_2 = \|\Delta v\|, \quad \forall v \in H_0^2(\Omega).$$

Then we obtain the coercivity of $a(\cdot, \cdot)$. Thus the Lax-Milgram lemma guarantees the well-posedness of the variational problem (1.4).

1.4. Linear elasticity problem

The third one is the linear elasticity problem with homogeneous boundary condition:

$$\begin{cases} -\operatorname{div}\big(2\mu\varepsilon(u)+\lambda \operatorname{div} u I\big) = f & \text{in } [H^{-1}(\Omega)]^d, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where I is the unit matrix with size $d \times d$, u and f are vector-valued functions, and

$$\operatorname{div} u = \nabla \cdot u, \quad (\nabla u)_{ij} = \frac{\partial u_i}{\partial x_j}, \quad \varepsilon(u) = \big(\nabla u + (\nabla u)^\top\big)/2.$$

Here, μ and λ are the Lamé constants satisfying $\mu_1 \le \mu \le \mu_2$ and $\lambda > 0$.

The corresponding variational form is to find $u \in V$ s.t.

$$a(u,v) = \langle f, v\rangle, \quad \forall v \in V, \tag{1.5}$$

where

$$a(u,v) = 2\mu\big(\varepsilon(u), \varepsilon(v)\big) + \lambda(\operatorname{div} u, \operatorname{div} v),$$

or

$$a(u,v) = \mu(\nabla u, \nabla v) + (\mu+\lambda)(\operatorname{div} u, \operatorname{div} v),$$

and $\langle f, v\rangle = (f,v)$, $V = [H_0^1(\Omega)]^d$, $V' = [H^{-1}(\Omega)]^d$, $\|\cdot\|_V = |\cdot|_1$ (or $\|\cdot\|_1$).

Due to the Korn inequality [4]

$$|v| \le C\|\varepsilon(v)\|, \quad \forall v \in [H_0^1(\Omega)]^d,$$

and the Poincaré-Friedrichs inequality (1.3), we obtain the coercivity of $a(\cdot,\cdot)$. Thus the Lax-Milgram lemma guarantees the well-posedness of the variational problem (1.5).

2. Abstract discrete problem

2.1. Abstract discrete problem

Let $\mathcal{T}_h$ is a mesh over Ω. V_h is a finite dimensional subspace with the norm $\|\cdot\|_{V_h}$. Here $V_h \subset V$ or $\not\subseteq V$ but locally on $K \in \mathcal{T}_h$ it holds $V_h|_K \subset V|_K$. Given $f_h \in V_h'$, we consider the discrete problem: to find $u_h \in V_h$ s.t.

$$a_h(u_h, v_h) = \langle f_h, v_h\rangle_h, \quad \forall v_h \in V_h, \tag{2.1}$$

where $a_h(\cdot,\cdot)$ is the discrete bilinear form on $V_h \times V_h$ and $\langle\cdot,\cdot\rangle_h$ is the duality pair between V_h' and V_h. If $V_h \subset V$, the discrete method is conforming, otherwise it is nonconforming.

Obviously we have the following corollary of Lax-Milgram lemma.

Lemma 2.1. *If the bilinear form $a_h(\cdot,\cdot)$ on $V_h \times V_h$ satisfies*

(1) Continuity: $a_h(u_h, v_h) \le C\|u_h\|_{V_h}\|v_h\|_{V_h}$, $\forall u_h, v_h \in V_h$;
(2) Coercivity (or ellipticity): $a(v_h, v_h) \ge \alpha\|v_h\|_{V_h}^2$, $\forall v_h \in V_h$,

then for any $f_h \in V_h'$, the problem (2.1) has a unique solution $u_h \in V_h$ satisfying

$$\|u_h\|_{V_h} \le \frac{1}{\alpha}\|f_h\|_{V_h'}.$$

2.2. Conforming FEMs

For the conforming FEM, we have $V_h \subset V$. Then we can set $a_h(\cdot,\cdot) = a(\cdot,\cdot)$ and $\langle f_h, v_h\rangle_h = \langle f, v_h\rangle$ and $\|\cdot\|_{V_h} = \|\cdot\|_V$. We given the conforming FEM: to find $u_h \in V_h$ s.t.

$$a(u_h, v_h) = \langle f, v_h\rangle, \quad \forall v_h \in V_h. \tag{2.2}$$

Next we assume that the conditions of Lemmas 1.1 and 2.1 are satisfied, i.e. the problems (1.1) and (2.1) are well-posed. Let u is the solution to problem (1.1) and u_h the discrete solution to problem (2.1). We have an abstract error bound as follows.

Lemma 2.2 (Cea's lemma)**.** *It holds*

$$\|u - u_h\|_V \le C \inf_{v_h \in V_h} \|u - v_h\|_V. \tag{2.3}$$

Some conforming elements are listed as follows:

1. H^1-conforming Lagrange elements
2. H^2-conforming elements: Argyris element, Bell element, etc.
3. $H(\text{div})$-conforming Raviart-Thomas elements, BDM element
4. $H(\text{curl})$-conforming Nédélec elements of I type and II type
5. ...

see the references [3, 8].

3. Nonconforming FEM for Poisson problem

3.1. Crouzeix-Raviart element

For the Crouzeix-Raviart element, the triple (K, V_K, Φ_K) is defined by

1. K is a simplex;
2. The shape function space $V_K = \mathbb{P}_1(K)$
3. The set Φ_K of degrees of freedom is given by

$$v(x_e) \quad \text{or} \quad \frac{1}{|e|}\int_e v \mathrm{d}s, \quad e \subset \partial K,$$

where x_e is the barycenter of edge or face e.

We define the global discrete space V_h by

$$V_h = \{v_h \in L^2(\Omega);\ v_h|_K \in V_K,\ \forall K \in \mathcal{T}_h,\ \int_e [v_h]\mathrm{d}s = 0,\ \forall e \in \mathcal{E}_h\}, \tag{3.1}$$

where $\mathcal{E}_h$ is the set of all edges or faces in $\mathcal{T}_h$ and $[v_h]$ is the jump of v_h defined by

$$[v_h]|_e = v_h|_{K^+} - v_h|_{K^-}, \quad e \subset K^+ \cap K^-, \quad K^+, K^- \in \mathcal{T}_h.$$

Assume the mesh $\mathcal{T}_h$ is shape-regular and consists of simplexes. By the standard scaling arguments, we have the interpolation error estimates: $\forall v \in H^m(\Omega)$ with $m = 1, 2$,

$$\|v - I_h v\|_K + h_K |v - I_h v|_{1,K} \le C h_K^m |v|_{m,K}, \quad \forall K \in \mathcal{T}_h, \tag{3.2}$$

where h_K is the diameter of element K.

3.2. Nonconforming discretization

In this subsection, we use the Crouzeix-Raviart element to discrete the Poisson problem (1.2). To this end, we let

$$a_h(u_h, v_h) = \sum_{K \in \mathcal{T}_h} (\nabla u_h, \nabla v_h)_K, \quad \langle f_h, v_h \rangle_h = (f, v_h), \quad u_h, v_h \in V_h.$$

Then the nonconforming FEM for Poisson problem is to find $u_h \in V_h$ s.t.

$$a_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h. \tag{3.3}$$

To discuss the well-posedness of the discrete problem (3.3), we introduce a discrete semi-norm

$$|v_h|_{1,h} = \Big(\sum_{K \in \mathcal{T}_h} |v_h|_{1,K}^2 \Big)^{1/2}.$$

It is easily verified that $|\cdot|_{1,h}$ is a norm on V_h. Then the continuity and coercivity of $a_h(\cdot,\cdot)$ is immediate,i.e.

$$a_h(u_h, v_h) \le |u_h|_{1,h}|v_h|_{1,h}, \quad a_h(v_h, v_h) \ge |v_h|_{1,h}^2, \quad \forall u_h, v_h \in V_h.$$

According to Lax-Milgram lemma, the discrete problem (3.3) is well-posed.

Lemma 3.1 (The second Strang lemma)**.** *Let u is the solution to the Poisson problem* (1.2) *and u_h the solution to the discrete problem* (3.3)*. We have*

$$|u-u_h|_{1,h} \le 2\inf_{v_h\in V_h}|u-v_h|_{1,h} + \sup_{w_h\in V_h}\frac{a_h(u,w_h)-(f,w_h)}{|w_h|_{1,h}}. \tag{3.4}$$

Proof. First for arbitrary $v_h \in V_h$, we have the triangle inequality

$$|u-u_h|_{1,h} \le |u-v_h|_{1,h} + |v_h-u_h|_{1,h}. \tag{3.5}$$

For convenience, set $\delta_h = v_h - u_h \in V_h$. Due to the coercivity of $a_h(\cdot,\cdot)$, we have

$$\begin{aligned}|\delta_h|_{1,h}^2 &\le a_h(\delta_h,\delta_h)\\ &= a_h(v_h,\delta_h) - a_h(u_h,\delta_h)\\ &= a_h(v_h,\delta_h) - (f,\delta_h) \quad \text{(use (2.1))}\\ &= a_h(v_h-u,\delta_h) + \big(a_h(u,\delta_h)-(f,\delta_h)\big).\end{aligned}$$

Thus we have

$$|\delta|_{1,h} \le |u-v_h|_{1,h} + \sup_{w_h\in V_h}\frac{a_h(u,w_h)-(f,w_h)}{|w_h|_{1,h}},$$

which, together with the triangle inequality (3.5), yields

$$|u-u_h|_{1,h} \le 2|u-v_h|_{1,h} + \sup_{w_h\in V_h}\frac{a_h(u,w_h)-(f,w_h)}{|w_h|_{1,h}}.$$

Since v_h is arbitrary, we take the infimum to obtain (3.4). ☐

The first term in error bound (3.4) is the approximation error from discrete space V_h, which can be bounded by the interpolation error, i.e.

$$\inf_{v_h\in V_h}|u-v_h|_{1,h} \le |u-I_hu|_{1,h} \le Ch|u|_2. \tag{3.6}$$

where $h = \max_{K\in\mathcal{T}_h} h_K$. The second term in error bound (3.4) is the consistency error which arises from the nonconformity of discrete space V_h, which is estimated in following lemma.

Lemma 3.2.

$$\sup_{w_h\in V_h}\frac{a_h(u,w_h)-(f,w_h)}{|w_h|_{1,h}} \le Ch|u|_2. \tag{3.7}$$

Proof. By integration by parts, we obtain

$$a_h(u,w_h)-(f,w_h) = \sum_{K\in\mathcal{T}_h}\int_{\partial K}\frac{\partial u}{\partial \boldsymbol{n}_K}w_h\mathrm{d}s = \sum_{e\in\mathcal{E}_h}\int_e\frac{\partial u}{\partial \boldsymbol{n}_e}[w_h]\mathrm{d}s, \quad \forall w_h\in V_h, \tag{3.8}$$

where $\boldsymbol{n}_K$ is the outer unit normal to K and $\boldsymbol{n}_e$ is the unit normal associated to edge or face e.

Let $P_K\nabla u$ is the average value of ∇u on element K and $P_e w_h$ the average value of w_h on edge or face e. Suppose $e\subset\partial K$, the weak continuity of w_h implies

$$\int_e\frac{\partial u}{\partial n_e}[w_h]\mathrm{d}s = \int_e(\nabla u - P_K\nabla u)\cdot\boldsymbol{n}_e[w_h-P_ew_h]\mathrm{d}s \le \|\nabla u - P_K\nabla u\|_e\|[w_h-P_ew_h]\|_e. \tag{3.9}$$

Combining the trace inequality and Poincaré inequality leads to

$$\|\nabla u - P_K\nabla u\|_e \le C(h_K^{-1/2}\|\nabla u - P_K\nabla u\|_K + h_K^{1/2}|u|_{2,K}) \le Ch_K^{1/2}|u|_{2,K}. \tag{3.10}$$

For interior edge or face e shared by two elements K_1 and K_2, we use the trace inequality and Poincaré-Friedrichs inequality to obtain

$$\|[w_h-P_ew_h]\|_e \le C(h_K^{-1/2}\|w_h-P_ew_h\|_{K_1\cup K_2} + h_K^{1/2}\|\nabla_h w_h\|_{K_1\cup K_2}) \le Ch_K^{1/2}\|\nabla_h w_h\|_{K_1\cup K_2}, \tag{3.11}$$

where ∇_h is the discrete version of ∇. For boundary edge or face, the adjustment is obvious.

Substituting (3.9)-(3.11) into (3.8), we obtain

$$a_h(u,w_h)-(f,w_h) \le Ch|u|_2|w_h|_{1,h}, \quad \forall w_h\in V_h.$$

Then we obtain (3.7). ☐

Constituting (3.6) and (3.7) into (3.4), we obtain the error estimate for the nonconforming FEM (3.3), i.e.

$$|u-u_h|_{1,h} \le Ch|u|_2.$$

4. Conforming VEM for Poisson problem

4.1. Polytopal meshes

Let $\mathcal{T}_h$ be a decomposition of Ω into non-overlapping polytope (polygon/polyhedron) and h stand for the maximum of diameters of elements in $\mathcal{T}_h$. The following assumption on the mesh $\mathcal{T}_h$ is standard:

H0. There exists a positive constant ρ such that, for every $K \in \mathcal{T}_h$,

- for every edge/face e, the diameters h_K and h_e of K and e satisfy $h_e \ge \rho h_K$, (4.1)
- K is star-shaped with respect to all the points of a disc/ball with radius $\ge \rho h_K$, (4.2)
- for $d=3$, each face e is star-shaped with respect to all the points of a disc with radius $\ge \rho h_e$. (4.3)

Let $\mathcal{E}_h$ denote the set of edges/faces of $\mathcal{T}_h$. For any $K \in \mathcal{T}_h$, $\boldsymbol{n}_K$ ($\boldsymbol{t}_K$) always denotes its exterior unit normal (anticlockwise tangential) vector; we shall use the notation $\boldsymbol{n}_e$ ($\boldsymbol{t}_e$) for a unit normal (tangential) vector of an edge/face $e \in \mathcal{E}_h$, whose orientation is chosen arbitrarily but fixed for interior edges/faces and coinciding with the exterior normal (tangential) of Ω for boundary edges/faces. When $d=2$, if the normal vector of e is $\boldsymbol{n}=(n_1,n_2)^\top$, we define the corresponding tangential vector $\boldsymbol{t}=(-n_2,n_1)^\top$.

For an internal edge/face e shared by K, $L \in \mathcal{T}_h$ such that $\boldsymbol{n}_e$ points from K to L, we define the jump of function v through the edge/face e by

$$[v]|_e = (v|_K)|_e - (v|_L)|_e.$$

For the boundary edge/face e, set $[v]|_e = v|_e$.

Virtual triangulation

In fact, for each $K \in \mathcal{T}_h$, there exists a "virtual triangulation" $\mathcal{T}_K$ of K by connecting the center of the disc/ball and vertices of K such that $\mathcal{T}_K$ is shape-regular, quasi-uniform and consists of simplexes. The corresponding mesh size of $\mathcal{T}_K$ is comparable to h_K. Then we get a virtual mesh $\widetilde{\mathcal{T}}_h$ consisting of such virtual triangulations of elements. Moreover, the virtual mesh $\widetilde{\mathcal{T}}_h$ is shape-regular.

For $v \in H_0^1(\Omega)\cap H^2(\Omega)$, we introduce the Lagrange interpolation $I_h^c v \in H_0^1(\Omega)$ on the virtual mesh$\widetilde{\mathcal{T}}_h$. Then we have the interpolation error estimates.

Lemma 4.1. *For any $v \in H_0^1(\Omega)\cap H^s(\Omega)$ with $2 \le s \le k+1$, we have*

$$\|v - I_h^c v\|_K + h_K |v - I_h^c v|_{1,K} \le C h_K^s |v|_{s,K}, \quad \forall K \in \mathcal{T}_h. \tag{4.4}$$

4.2. Some tools

In this subsection, we introduce some preliminary results from the reference [6].

Bramble-Hilbert estimates

Mesh condition (4.2) implies the following Bramble-Hilbert estimates: For $K \in \mathcal{T}_h$ and $m \ge 0$,

$$\inf_{q\in\mathbb{P}_k(K)} |v-q|_{m,K} \le C h_K^{s-m} |v|_{s,K}, \quad \forall v \in H^s(K), \tag{4.5}$$

where $0 \le m \le s \le k+1$.

According to (4.5), for any $v \in H^s(\Omega)$ there a piecewise polynomial $v_\pi \in \mathbb{P}_k^{dc}(\mathcal{T}_h)$ s.t.

$$|v - v_\pi|_{m,K} \le C h_K^{s-m} |v|_{s,K}, \quad \forall K \in \mathcal{T}_h, \tag{4.6}$$

where $0 \le m \le s \le k+1$.

L^2-projection

For any $v \in H^s(K)$ with $0 \le s \le k+1$, by $P_k^K v$ we denote the L^2 projection of v onto the space $\mathbb{P}_k(K)$. We have the projection error estimates

$$|v - P_k^K v|_{m,K} \le C h_K^s |v|_{s,K}. \tag{4.7}$$

where $0 \le m \le s \le k+1$.

Poincaré-Friedrichs inequalities

Mesh condtion (4.2) implies the following Poincaré-Friedrichs inequalities: for $K \in \mathcal{T}_h$,

$$\|v\|_K \lesssim h_K^{-d/2} \int_K v \mathrm{d}x + h_K |v|_{1,K}, \quad \forall v \in H^1(K), \tag{4.8}$$

$$\|v\|_K \lesssim h_K^{1-d/2} \int_{\partial K} v \mathrm{d}x + h_K |v|_{1,K}, \quad \forall v \in H^1(K), \tag{4.9}$$

Trace inequality

For $K \in \mathcal{T}_h$, we have the standard trace inequality

$$\|v\|_{\partial K} \lesssim h_K^{-1/2} \|v\|_K + h_K^{1/2} |v|_{1,K}, \quad \forall v \in H^1(K). \tag{4.10}$$

Some inequalities for polynomials

Lemma 4.2 (Inverse inequality)**.** *For $K \in \mathcal{T}_h$, we have*

$$|p|_{1,K} \leq C h_K^{-1} \|p\|_K, \quad \forall p \in \mathbb{P}_k(K), \tag{4.11}$$

$$|p|_{\infty,K} \leq C h_K^{-d/2} \|p\|_K, \quad \forall p \in \mathbb{P}_k(K). \tag{4.12}$$

Lemma 4.3. *Given any $p \in \mathbb{P}_{k-2}(K)$ $(k \geq 2)$, there exists $q \in \mathbb{P}_k(K)$ s.t.*

$$\Delta q = p \quad \text{in } K, \quad |q|_{1,K} \leq C h_K \|p\|_K. \tag{4.13}$$

The trace of H^1 space

For each function $v \in C^\infty(\overline{K})$, we define its trace $\gamma_0 v$ as the restriction of v on ∂K where γ_0 is called the trace operator, which can be continuously extended to $H^1(K)$. The boundary space $H^{1/2}(\partial\Omega)$ is given by

$$H^{1/2}(\partial K) = \gamma_0(H^1(K)),$$

with the norm

$$\|\mu\|_{1/2,\partial K} = \inf_{\gamma_0 v = \mu, v \in H^1(K)} \|v\|_{1,K}.$$

According to the definition, it is easy to see

$$\|v\|_{1/2,\partial K} \leq \|v\|_{1,K}, \quad \forall v \in H^1(K).$$

The trace of H(div) space

We define the space

$$H(\mathrm{div}, K) = \{\phi \in [L^2(K)]^d : \ \mathrm{div}\phi \in L^2(K)\}.$$

For the trace of space $H(\mathrm{div}, K)$, we have the inequality

$$\|\phi \cdot \boldsymbol{n}_K\|_{-\frac{1}{2},K} \leq (\|\phi\|_K^2 + \|\mathrm{div}\phi\|_K^2)^{1/2}. \tag{4.14}$$

where $\|\phi \cdot \boldsymbol{n}_K\|_{-\frac{1}{2},K}$ is the norm on space $H^{-1/2}(\partial K)$ which is the dual space of $H^{1/2}(\partial K)$. For the details on space $H(\mathrm{div}, K)$, see [9].

4.3. Virtual element space

Next we restrict to the two dimension case $d = 2$. Let K be a polygon with n edges. For convenience, we define a boundary space on ∂K by

$$\mathbb{B}_k(\partial K) = \{v \in C^0(\partial K) : \ v|_e \in P_k(e), \ \forall e \in \mathcal{E}(K)\}, \quad k \geq 1,$$

where $\mathcal{E}(K)$ is the set of edges of polygon K. On a polygon K with n edges, the local shape function space is defined by

$$V_k(K) = \{v \in H^1(K) : \ \Delta v \in \mathbb{P}_{k-2}(K), \ v|_{\partial K} \in \mathbb{B}_k(\partial K)\}, \quad k \geq 1, \tag{4.15}$$

where $\mathbb{P}_{-1}(K) = \{0\}$. Obvious it holds $\mathbb{P}_k(K) \subset V_k(K)$.

We recall the well-posedness of the nonhomogeneous Dirichlet problem. Specially, for any given $f \in P_{k-2}(K)$ and $g \in \mathbb{B}_k(\partial K)$, there exits a unique solution $v \in H^1(\Omega)$, s.t.

$$-\Delta v = f \quad \text{in } K, \quad v = g \quad \text{on } \partial K. \tag{4.16}$$

By the existence and uniqueness of the solution to problem (4.16), we define a bijective mapping from (f, g) to v. Then we have

$$\dim V_k(K) = \dim \mathbb{P}_{k-2}(K) + \dim \mathbb{B}_k(\partial K) = \frac{k(k-1)}{2} + nk. \tag{4.17}$$

For space $V_k(K)$, we define the degrees of freedom (DOF) as

$$\bullet \text{ values of } v(a), \qquad \text{vertex } a \text{ of } K, \tag{4.18}$$

$$\bullet \text{ moments } \frac{1}{|e|}\int_e vq\mathrm{d}s, \quad q \in \mathbb{P}_{k-2}(e), \quad \text{edge } e \text{ of } K, \tag{4.19}$$

$$\bullet \text{ moments } \frac{1}{|K|}\int_K vq\mathrm{d}K, \quad q \in \mathbb{P}_{k-2}(K). \tag{4.20}$$

We have the unisolvence of DOF as follows.

Lemma 4.4 (Unisolvence)**.** *The DOF* (4.18)-(4.20) *are unisolvent for the space* $V_k(K)$*.*

Proof. It is easy to verify that the dimension of $V_k(K)$ is equal to the number of DOF (4.18)-(4.20). It remains to show that for any function $v \in V_k(K)$, if all the DOF (4.18)-(4.20) vanishe, then $v = 0$.

Since $v|_{\partial K} \in \mathbb{B}_k(\partial K)$ is determined by the DOF (4.18)-(4.19), we have $v \in H_0^1(K)$. By using the integration by parts, we obtain

$$|v|_{1,K}^2 = (\nabla v, \nabla v)_K = -(v, \Delta v)_K = 0,$$

where we have used the fact that $\Delta v \in \mathbb{P}_{k-2}(K)$. Thus we obtain $\nabla v = 0$, which, with $v \in H_0^1(K)$, leads to $v = 0$. $\square$

Remark 4.1. The edge DOF (4.19) can be changed to the values of v at $(k-1)$ internal points on each edge, which, together the values of v at vertices, still determine the boundary values of v on ∂K. For example, we can take the values of v at $(k-1)$ internal Gauss-Lobatto points on each edge as DOF.

We define the global virtual element space V_h by

$$V_h = \{v \in H_0^1(\Omega) : \ v|_K \in V_k(K), \ \forall K \in \mathcal{T}_h\}.$$

The global DOF is an immediate extension of the local DOF (4.18)-(4.20).

Inverse inequality

We present an inverse inequality on the VE space V_h.

Lemma 4.5. *For any* $v_h \in V_h$*, it holds*

$$\|\Delta v_h\|_K \le C h_K^{-1} |v_h|_{1,K}, \quad \forall K \in \mathcal{T}_h. \tag{4.21}$$

Proof. We use the bubble function on a given polygon $K \in \mathcal{T}_h$ to prove the inverse inequality. Let λ_e be the linear function associated with the edge e of K defined by setting $\lambda_e = -\alpha(\boldsymbol{x} - \boldsymbol{x}_e) \cdot \boldsymbol{n}_K/|e|$, such that $\lambda_e = 0$ on e, where the constant $\alpha > 0$ is chosen to make sure $\|\lambda_e\|_{\infty,K} = 1$ and $\boldsymbol{x}_e$ is the midpoint of e. b_K is the bubble function on K obtained by multiplying all the edge functions λ_e twice, so that $b_K > 0$ in K and b_K vanishes on ∂K.

Since $\Delta v_h \in P_{k-2}(K)$ in K, by the norm equivalence on finite dimensional space we have

$$C\|\Delta v_h\|_K^2 \le (b_K \Delta v_h, \Delta v_h)_K = -(\nabla(b_K \Delta v_h), \nabla v_h)_K \le |b_K \Delta v_h|_{1,K} |v_h|_{1,K},$$

Then by using the inverse inequality (4.11) on polynomial space, we obtain

$$\|\Delta v_h\|_K^2 \le C h_K^{-1} \|b_K \Delta v_h\|_K |v_h|_{1,K} \le C h_K^{-1} \|\Delta v_h\|_K |v_h|_{1,K},$$

which yields (4.21). $\square$

A minimum energy principle

The following minimum energy principle is useful for bounding the H^1-norm of a VE function.

Lemma 4.6 ([6]). *For any given VE function $v_h \in V_k(K)$ where $K \in \mathcal{T}_h$, then the inequality*

$$|v_h|_{1,K} \le |w|_{1,K} \tag{4.22}$$

holds for all $w \in H^1(K)$ satisfying $P_{k-2}^K(w - v_h) = 0$ in K and $w - v_h = 0$ on ∂K.

A local Lagrange interpolation

For $v_h \in V_k(K)$, let $w_h \in H^1(K)$ be the k-order Lagrange interpolation function of v_h on the virtual triangulation $\widetilde{\mathcal{T}}_K$ of K such that $w_h = v_h$ on ∂K and $w_h = 0$ at all interior nodes of K. From the mesh assumptions and scaling, it follows

$$h_K^{-1}\|w_h\|_K + |w_h|_{1,K} \le C h_K^{-1/2}\|v_h\|_{\partial K}. \tag{4.23}$$

Upper bound on H^1-norm

We present a upper bound on the H^1-norm of VE functions.

Lemma 4.7. *For any given $K \in \mathcal{T}_h$, we have*

$$|v_h|_{1,K} \le C\big(h_K^{-1}\|P_{k-2}^K v_h\|_K + h_K^{-1/2}\|v_h\|_{\partial K}\big), \quad \forall v_h \in V_k(K). \tag{4.24}$$

Proof. The proof is similar to the one of [5, Lemma 2.19], although the definition of $V_k(K)$ is a little different from that in [5]. For clearness, we give the main steps for the proof.

For any given function $v_h \in V_k(K)$, let $w_h \in H^1(K)$ be the k-order Lagrange interpolation function of v_h on $\widetilde{\mathcal{T}}_K$ satisfying (4.23). Then we have $w_h = v_h$ on ∂K and

$$h_K^{-1}\|w_h\|_K + |w_h|_{1,K} \le C h_K^{-1/2}\|v_h\|_{\partial K}. \tag{4.25}$$

Let $b_K \in H_0^1(K)$ be the bubble function given in the proof of Lemma 4.5. By the equivalence of norms on polynomial space, we have

$$C_1\|p\|_K^2 \le (p, p b_K)_K \le C_2\|p\|_K^2, \quad \forall p \in P_{k-1}(K). \tag{4.26}$$

Let $\eta = w_h + p b_K$ where the polynomial $p \in P_{k-2}(K)$ is determined by

$$(\eta - v_h, q)_K = 0, \quad \forall q \in P_{k-2}(K),$$

i.e.,

$$(p b_K, q)_K = (v_h - w_h, q)_K = (P_{k-2}^K v_h - w_h, q)_K, \quad \forall q \in P_{k-1}(K). \tag{4.27}$$

Observing $v_h = \eta$ on ∂K and $P_{k-2}^K(v_h - \eta) = 0$ in K, we use Lemma 4.6 to obtain

$$|v_h|_{1,K} \le |\eta|_{1,K} \le |w_h|_{1,K} + |p b_K|_{1,K}.$$

Further, we use the inverse inequality (4.11) and (4.25)-(4.27) to obtain

$$\begin{aligned} |v_h|_{1,K} &\le C(|w_h|_{1,K} + h_K^{-1}\|p\|_K) \\ &\le C(|w_h|_{1,K} + h_K^{-1}\|P_{k-2}^K v_h - w_h\|_K) \\ &\le C(h_K^{-1}\|P_{k-2}^K v_h\|_K + h_K^{-1}\|w_h\|_K + |w_h|_{1,K}) \\ &\le C(h_K^{-1}\|P_{k-2}^K v_h\|_K + h_K^{-1/2}\|v_h\|_{\partial K}). \end{aligned}$$

The proof is complete. $\square$

Interpolation operator

For any function $v \in H_0^1(\Omega) \cap H^2(\Omega)$, we define the interpolation $I_h v$ by requiring that the values of DOF (4.18)-(4.20) of $I_h v$ are equal to the corresponding ones of v. Before estimating the interpolation error, we show an approximation result from the VE space V_h.

Lemma 4.8. *For any $v \in H_0^1(\Omega) \cap H^s(\Omega)$ with $2 \le s \le k+1$, there exists a VE function $w_h \in V_h$ s.t.*

$$\|v - w_h\|_K + h_K |v - w_h|_{1,K} \le C h_K^s |v|_{s,K}, \quad \forall K \in \mathcal{T}_h, \tag{4.28}$$

and

$$w_h = I_h^c v \quad \text{on } \partial K, \quad \forall K \in \mathcal{T}_h,$$

where $I_h^c v$ is the Clément interpolation of v on the virtual triangulation $\widetilde{\mathcal{T}}_h$.

Proof. Let $w_K \in H^1(K)$ be the solution of

$$\Delta w_K = \Delta v_\pi \quad \text{in } K, \quad w_K = I_h^c v \quad \text{on } \partial K,$$

where v_π is the piecewise polynomial approximation of v with estimates (4.6). Seting $w_h|_K = w_K$, we $w_h \in V_h$. Next, we should show that the function w_h satisfies the requirements. Since $I_h^c v - w_h \in H_0^1(K)$, by integration by parts, we obtain

$$\begin{aligned} |I_h^c v - w_h|_{1,K}^2 &= \big(\nabla(I_h^c v - w_h), \nabla(I_h^c v - w_h)\big)_K = -\big(\Delta(I_h^c v - w_h), I_h^c v - w_h\big)_K \\ &= -\big(\Delta(I_h^c v - v_\pi), I_h^c v - w_h\big)_K = \big(\nabla(I_h^c v - v_\pi), \nabla(I_h^c v - w_h)\big)_K \\ &\le |I_h^c v - v_\pi|_{1,K} |I_h^c v - w_h|_{1,K}. \end{aligned}$$

Thus we obtain

$$|I_h^c v - w_h|_{1,K} \le |I_h^c v - v_\pi|_{1,K} \le |v - I_h^c v|_{1,K} + |v - v_\pi|_{1,K}.$$

which, with the triangle inequality, leads to

$$|v - w_h|_{1,K} \le 2|v - I_h^c v|_{1,K} + |v - v_\pi|_{1,K}.$$

Then we use the estimates (4.4) and (4.6) to obtain

$$|v - w_h|_{1,K} \le C h_K^{s-1} |v|_{s,K}. \tag{4.29}$$

For $\|v - w_h\|_K$, we use the triangle inequality and Poincaré-Friedrichs inequality (4.9) to obtain

$$\begin{aligned} \|v - w_h\|_K &\le \|v - I_h^c v\|_K + \|I_h^c v - w_h\|_K \\ &\le \|v - I_h^c v\|_K + C h_K |I_h^c v - w_h|_{1,K} \\ &\le \|v - I_h^c v\|_K + C h_K (|v - I_h^c v|_{1,K} + |v - w_h|_{1,K}), \end{aligned} \tag{4.30}$$

where we have also used that fact that $I_h^c v - w_h \in H_0^1(K)$ in K. Combining the estimates (4.4) and (4.29), we get

$$\|v - w_h\|_K \le C h_K^s |v|_{s,K}.$$

The proof is complete. □

Then we present the following interpolation error estimates.

Lemma 4.9. *For any $v \in H_0^1(\Omega) \cap H^s(\Omega)$ with $2 \le s \le k+1$, it holds*

$$\|v - I_h v\|_K + h_K |v - I_h v|_{1,K} \le C h_K^s |v|_{s,K}, \quad \forall K \in \mathcal{T}_h. \tag{4.31}$$

Proof. Let $w_h \in V_h$ be the VE function with the estimates (4.28) and satisfy $w_h = I_h^c v$ on $e \in \mathcal{E}_h$, see Lemma (4.8). since $I_h v \in P_k(e)$ on each edge $e \in \mathcal{E}_h$, the definitions of interpolation operators I_h and I_h^c implies $I_h v = I_h^c v$ on on each edge $e \in \mathcal{E}_h$. Thus for any given $K \in \mathcal{T}_h$, we have $w_h - I_h v \in H_0^1(K)$ in K.

By integration by parts, we obtain

$$|w_h - I_h v|_{1,K}^2 = -\big(w_h - I_h v, \Delta(w_h - I_h v)\big)_K = -\big(w_h - v, \Delta(w_h - I_h v)\big)_K \le \|v - w_h\|_K \|\Delta(w_h - I_h v)\|_K,$$

where we have used the fact that $\Delta(w_h - I_h v) \in \mathbb{P}_{k-2}(K)$ in K and $(I_h v, q)_K = (v, q)_K$ for $q \in \mathbb{P}_{k-2}(K)$. Then the inverse inequality (4.21) implies

$$|w_h - I_h v|_{1,K} \leq C h_K^{-1} \|v - w_h\|_K. \tag{4.32}$$

Thus we have

$$|v - I_h v|_{1,K} \leq |v - w_h|_{1,K} + |w_h - I_h v|_{1,K} \leq |v - w_h|_{1,h} + C h_K^{-1} \|v - w_h\|_K,$$

which, together with the estimates (4.28), leads to

$$|v - I_h v|_{1,K} \leq C h_K^{s-1} |v|_{s,K}.$$

For $\|v - I_h v\|_K$, we recall the fact that $w_h - I_h v \in H_0^1(K)$ and use the Poincaré-Friedrichs inequality (4.9) to obtain

$$\|v - I_h v\|_K \leq \|v - w_h\|_K + \|w_h - I_h v\|_K \leq \|v - w_h\|_K + C h_K |w_h - I_h v|_{1,K},$$

which, together with the estimates (4.28) and (4.32), leads to

$$\|v - I_h v\|_K \leq C h_K^s |v|_{s,K}.$$

The proof is complete. $\square$

4.4. The discrete bilinear form

Because the VE function $v_h \in V_h$ has no explicit expression, we only can use the DOF information to compute the stiff matrix, load vector, etc. We observe that the local DOF allow us to compute exactly $a^K(v, q)$ for $v \in V_h$ and $q \in \mathbb{P}_k(K)$, where $a^K(v, q) = (\nabla v, \nabla q)_K$. Indeed, we have

$$a^K(v, q) = -(v, \Delta q)_K + \int_K v \frac{\partial q}{\partial \boldsymbol{n}_K} \mathrm{d}s. \tag{4.33}$$

Since $\Delta q \in \mathbb{P}_{k-2}(K)$ and $v|_e \in P_k(e)$ for all $e \in \mathcal{E}(K)$, the two terms in the above equation can be computed exactly by the DOF of v, i.e. $a^K(v, q)$ is computable.

H^1-projection operator

We define a projection operator $\Pi_K^\nabla : V_k(K) \to \mathbb{P}_k(K)$ by finding $\Pi_K^\nabla v$ satisfying

$$\left\{ \begin{array}{rcl} a^K(\Pi_K^\nabla v, q) & = & a^K(v, q), \quad \forall q \in \mathbb{P}_k(K), \\ \int_{\partial K} \Pi_K^\nabla v \mathrm{d}s & = & \int_{\partial K} v \mathrm{d}s. \end{array} \right. \tag{4.34}$$

where $v \in V_k(K)$. Obviously we have $\Pi_K^\nabla q = q$ for $q \in \mathbb{P}_k(K)$. Since $a_K(v, q)$ is computable, the projection $\Pi_K^\nabla v$ is also computable by the DOF of v. According to the definition, we have the boundedness of H^1 projection

$$|\Pi_K^\nabla v_h|_{1,K} \leq |v_h|_{1,K}, \quad \forall v_h \in V_k(K). \tag{4.35}$$

The discrete bilinear form

We define the local discrete bilinear form $a_K(\cdot, \cdot)$ by

$$a_h^K(u_h, v_h) = a^K(\Pi_K^\nabla u_h, \Pi_K^\nabla v_h) + S^K(u_h - \Pi_K^\nabla u_h, v_h - \Pi_K^\nabla v_h), \quad u_h, v_h \in V_k(K),$$

where the bilinear form $S^K(\cdot, \cdot)$ is given by

$$S^K(u_h, v_h) = h_K^{-2} (P_{k-2}^K u_h, P_{k-2}^K v_h)_K + h_K^{-1} \int_{\partial K} u_h v_h \mathrm{d}s, \quad u_h, v_h \in \mathrm{Ker}(\Pi_K^\nabla) \cap V_k(K).$$

Obviously $S^K(\cdot, \cdot)$ is computable by the DOF (4.18)-(4.20) for $V_k(K)$. Further, we have an equivalence between $S^K(\cdot, \cdot)$ and H^1-seminorm on the kernel space of Π_K^∇.

Lemma 4.10. *For any $v_h \in Ker(\Pi_K^\nabla) \cap V_k(K)$, we have*

$$C_1 |v_h|_{1,K}^2 \leq S_K(v_h, v_h) \leq C_2 |v_h|_{1,K}^2. \tag{4.36}$$

Proof. According to Lemma 4.24, the first inequality in (4.36) is immediate. On the other hand, the property of P_{k-2}^K and trace inequality imply

$$S^K(v_h, v_h) \le h_K^{-2}\|v_h\|_K^2 + h_K^{-1}\|v_h\|_{\partial K}^2 \le C(h_K^{-2}\|v_h\|_K^2 + |v_h|_{1,K}^2).$$

Since $v_h \in \mathrm{Ker}(\Pi_K^\nabla)$, we have $\int_{\partial K} v_h \mathrm{d}s = 0$, which, with the Poincar'e-Friedrichs inequality (4.9), leads to

$$\|v_h\|_K \le Ch_K|v_h|_{1,K}.$$

Thus we have

$$S^K(v_h, v_h) \le C|v_h|_{1,K}^2.$$

□

The global discrete bilinear form is given by

$$a_h(u, v) = \sum_{K\in\mathcal{T}_h} a_h^K(u, v), \quad u, v \in V_h.$$

Lemma 4.11 (Consistency and stability)**.** *For any $K \in \mathcal{T}_h$, the bilinear form $a_h(\cdot,\cdot)$ satisfies*

- *k-consistency: For all $q \in \mathbb{P}_k(K)$ and $v_h \in V_k(K)$,*

$$a_h^K(v_h, q) = a^K(v_h, q). \tag{4.37}$$

- *Stability: There exist two positive constants α_* and α^* independent of h, such that*

$$\alpha_* a^K(v_h, v_h) \le a_h^K(v_h, v_h) \le \alpha^* a^K(v_h, v_h), \quad \forall v_h \in V_k(K). \tag{4.38}$$

Proof. Since $\Pi_K^\nabla q = q$ for $q \in \mathbb{P}_k(K)$, the consistency (4.37) is immediate. The property (4.36) of $S^K(\cdot,\cdot)$ and the H^1-orthogonality of Π_K^∇ imply the the stability (4.38). □

Remark 4.2 (Another choice of $S^K(\cdot,\cdot)$)**.** *We give another choice of $S^K(\cdot,\cdot)$ which also the classical one in the literature. Let ϕ_i be the basis function associated to the i-th DOF χ_i, s.t*

$$\chi_i(\phi_j) = \delta_{ij}, \quad i, j = 1, 2, \dots \dim V_k(K).$$

Then we define the bilinear form $S^K(\cdot,\cdot)$ by

$$S^K(\phi_i - \Pi_K^\nabla\phi_i, \phi_j - \Pi_K^\nabla\phi_j) = \sum_{r=1}^{\dim V_k(K)} \chi_r(\phi_i - \Pi_K^\nabla\phi_i)\chi_r(\phi_j - \Pi_K^\nabla\phi_j).$$

*The mesh assumption **H0** guarantees $a^K(\phi_i, \phi_i) \sim 1$. As a consequence, we have*

$$C_1 a^K(v_h, v_h) \le S^K(v_h, v_h) \le C_2 a^K(v_h, v_h), \quad \forall v_h \in Ker(\Pi_K^\nabla) \cap V_k(K). \tag{4.39}$$

4.5. The right-hand side

For $k \ge 2$, we set $f_h = P_{k-2}^K f$ on each $K \in \mathcal{T}_h$. We define the right-hand side by

$$\langle f_h, v_h\rangle_h = (f_h, v_h), \quad \forall v_h \in V_h.$$

Since

$$(f_h, v_h)_K = (f, P_{k-2}^K v_h)_K, \quad \forall K \in \mathcal{T}_h,$$

the right-hand side is computable by the DOF of v_h. Further, the L^2-orthogonality and the error estimates (4.7) of P_{k-2}^K imply

$$(f - f_h, v_h)_K = (f - P_{k-2}^K f, v_h - P_0^K v_h)_K \le \|f - P_{k-2}^K f\|_K\|v_h - P_0^K v_h\|_K \le Ch_K^k|f|_{k-1,K}|v_h|_{1,K}.$$

Thus, for $k \ge 2$ we obtain

$$(f, v_h) - \langle f_h, v_h\rangle_h \le Ch^k|f|_{k-1}|v_h|_1, \quad \forall f \in H^{k-1}(\Omega),\ v_h \in V_h. \tag{4.40}$$

For $k = 1$, let $f_h = P_0^K f$ on $K \in \mathcal{T}_h$ and define the right-hand side

$$\langle f_h, v_h\rangle_h = \sum_{K\in\mathcal{T}_h} (P_0^K f, \overline{v}_h)_K,$$

where $\overline{v}_h|_K = \int_{\partial K} v_h \mathrm{d}s$. By using the Poincaré-Friedrichs inequality (4.9), we obtain

$$(f, v_h)_K - (f_h, \overline{v}_h)_K = (f, v_h - \overline{v}_h)_K \le \|f\|_K\|v_h - \overline{v}_h\|_K \le Ch_K\|f\|_K|v_h|_{1,K}.$$

Thus for $k = 1$ we have the error bound

$$(f, v_h) - \langle f_h, v_h\rangle_h \le Ch\|f\||v_h|_1, \quad \forall f \in L^2(\Omega),\ v_h \in V_h. \tag{4.41}$$

4.6. The discrete problem

With the above preparations, we present the VEM for Poisson problem (1.2): to find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = \langle f_h, v_h \rangle_h, \quad \forall v_h \in V_h. \tag{4.42}$$

The stability (4.38) of $a_h(\cdot,\cdot)$ implies its continuity and coercivity. The discrete problem (4.42) is well-posed.

4.7. Error analysis for H^1-norm

Let $\mathbb{P}_k^{dc}(\mathcal{T}_h)$ stand for the discontinuous piecewise polynomial space. For convenience, we define a discrete seminorm by

$$|v|_{1,h} = \Big(\sum_{K \in \mathcal{T}_h} |v|_{1,K} \Big).$$

We present an abstract error bound for the VEM (4.42).

Theorem 4.12. *Assume that u is the solution to Poisson problem (1.2), u_h the solution to the discrete problem (4.42), and u_π any approximation of u from space $\mathbb{P}_k^{dc}(\mathcal{T}_h)$, we have*

$$|u - u_h|_1 \le C(|u - I_h u|_1 + |u - u_\pi|_{1,h} + \|f - f_h\|_{V_h'}), \tag{4.43}$$

where

$$\|f - f_h\|_{V_h'} := \sup_{v_h \in V_h} \frac{(f, v_h) - \langle f_h, v_h \rangle_h}{|v_h|_1}.$$

Proof. Setting $\delta_h = I_h u - u_h$, we have

$$\begin{aligned}
\alpha_* |\delta_h|_1^2 &\le \alpha_* a(\delta_h, \delta_h) \le a_h(\delta_h, \delta_h) \quad \text{(use (4.38))} \\
&= a_h(I_h u, \delta_h) - a_h(u_h, \delta_h) \\
&= \sum_{K \in \mathcal{T}_h} a_h^K(I_h u, \delta_h) - \langle f_h, \delta_h \rangle_h \quad \text{(use (4.42))} \\
&= \sum_{K \in \mathcal{T}_h} \big(a_h^K(I_h u - u_\pi, \delta_h) + a_h^K(u_\pi, \delta_h) \big) - \langle f_h, \delta_h \rangle_h \\
&= \sum_{K \in \mathcal{T}_h} \big(a_h^K(I_h u - u_\pi, \delta_h) + a^K(u_\pi, \delta_h) \big) - \langle f_h, \delta_h \rangle_h \quad \text{(use (4.37))} \\
&= \sum_{K \in \mathcal{T}_h} \big(a_h^K(I_h u - u_\pi, \delta_h) + a^K(u_\pi - u, \delta_h) \big) + a(u, \delta_h) - \langle f_h, \delta_h \rangle_h \quad \text{(use (1.2))} \\
&= \sum_{K \in \mathcal{T}_h} \big(a_h^K(I_h u - u_\pi, \delta_h) + a^K(u_\pi - u, \delta_h) \big) + (f, \delta_h) - \langle f_h, \delta_h \rangle_h.
\end{aligned}$$

We use the stability (4.38) of $a_h(\cdot,\cdot)$ and the continuity of $a^K(\cdot,\cdot)$ in above inequality to get

$$|\delta_h|_1^2 \le C(|I_h u - u_\pi|_{1,h} + |u - u_\pi|_{1,h} + \|f - f_h\|_{V_h'})|\delta_h|_1,$$

Then the inequality (4.43) follows by the triangle inequality. □

Substituting interpolation estimates (4.31), estimates (4.6) of local polynomial approximation, approximation error estimates (4.40)-(4.41) of f_h into the error bound (4.43), we obtain the optimal convergence of the VEM.

Corollary 4.13. *Assume $u \in H_0^1(\Omega) \cap H^{k+1}(\Omega)$ and $f \in H^{k-1}(\Omega)$, we have*

$$|u - u_h|_1 \le C h^k (|u|_{k+1} + |f|_{k-1}). \tag{4.44}$$

4.8. Error analysis for L^2-norm

In this section, by using the dual technique we analyze the L^2 error for the VEM.

A modified VE space [1]

In order to obtain the optimal L^2 convergence for the VEM, we need to modify the original VE space such that the L^2-projection is computable. To this end, we introduce an augmented VE space by

$$\widetilde{V}_k(K) = \{v_h \in H^1(\Omega): \ \Delta v_h \in \mathbb{P}_k(K), \ v_h|_{\partial K} \in \mathbb{B}_k(\partial K)\}, \quad K \in \mathcal{T}_h$$

with an adjustment of DOF (4.18)-(4.20). Then by using the H^1-projection operator Π_K^{∇}, we define the desired VE space by

$$W_k(K) = \{v_h \in \widetilde{V}_k(K): \ (v_h, q) = (\Pi_K^{\nabla} v_h, q)_K, \ \forall q \in \mathbb{P}_k(K)/\mathbb{P}_{k-2}(K)\}. \tag{4.45}$$

with the original DOF (4.18)-(4.20), where $\mathbb{P}_k(K)/\mathbb{P}_{k-2}(K)$ denotes the subspace consisting of polynomials L^2-orthogonal to $\mathbb{P}_{k-2}(K)$ in $\mathbb{P}_k(K)$. Observing that Π_K^{∇} is still computable by the DOF (4.18)-(4.20) and preserves the polynomials of up to order k, we have $\mathbb{P}_k(K) \subset W_k(K)$.

For $K \in \mathcal{T}_h$ with n edges, obviously we have

$$\dim W_k(K) > \dim \widetilde{V}_k(K) - (2k+1) = nk + k(k-1)/2.$$

Since $W_k(K) \subset \widetilde{V}_k(K)$, the unisolvence of DOF for $\widetilde{V}_k(K)$ and the restriction condition in definition of $W_k(K)$ imply the unisolvence of DOF for $W_k(K)$. Thus the dimension of $W_k(K)$ is

$$\dim W_k(K) = nk + k(k-1)/2,$$

is equal to the number of DOF (4.18)-(4.20). We note that the L^2-projection $P_k^K : W_k(K) \to \mathbb{P}_k(K)$ is computable by the DOF (4.18)-(4.20). The global VE space is denoted by W_h. Let I_h^W be the interpolation operator onto the space W_h. We present the interpolation error estimates in the following lemma.

Lemma 4.14. *For any $v \in H_0^1(\Omega) \cap H^s(\Omega)$ with $2 \le s \le k+1$, it holds*

$$\|v - I_h^W v\|_K + h_K |v - I_h^W v|_{1,K} \le C h_K^s |v|_{s,K}, \quad \forall K \in \mathcal{T}_h. \tag{4.46}$$

Proof. Let $\widetilde{I}_h$ be the interpolation operator onto $\widetilde{V}_h$. Similar to the proof of Lemma 4.9, we can obtain the estimates

$$\|v - \widetilde{I}_h v\|_K + h_K |v - \widetilde{I}_h v|_{1,K} \le C h_K^s |v|_{s,K}, \quad \forall K \in \mathcal{T}_h. \tag{4.47}$$

For any given $K \in \mathcal{T}_h$, the properties of $\widetilde{I}_h$ and I_h^W imply

$$\widetilde{I}_h v - I_h^W v \in H_0^1(K), \quad \Pi_K^{\nabla}(\widetilde{I}_h v) = \Pi_K^{\nabla}(I_h^W v),$$

and

$$(\widetilde{I}_h v - I_h^W v, q)_K = 0, \quad q \in \mathbb{P}_{k-2}(K).$$

For convenience, we set $w = \Delta(\widetilde{I}_h v - I_h^W v) \in P_k(K)$. Then we have

$$\begin{aligned}
|\widetilde{I}_h v - I_h^W v|_{1,K}^2 &= -(\widetilde{I}_h v - I_h^W v, w)_K \\
&= -(\widetilde{I}_h v - I_h^W v, w - P_{k-2}^K w)_K \\
&= -(\widetilde{I}_h v - \Pi_K^{\nabla}(I_h^W v), w - P_{k-2}^K w)_K \\
&= -(\widetilde{I}_h v - \Pi_K^{\nabla}(\widetilde{I}_h v), w - P_{k-2}^K w)_K \\
&\le \|\widetilde{I}_h v - \Pi_K^{\nabla}(\widetilde{I}_h v)\|_K \|w - P_{k-2}^K w\|_K \\
&\le C h_K |\widetilde{I}_h v - \Pi_K^{\nabla}(\widetilde{I}_h v)|_{1,K} \|w - P_{k-2}^K w\|_K.
\end{aligned}$$

Recalling the boundedness of Π_K^{∇} and P_{k-2}^K and $\Pi_K^{\nabla} q = q$ for $q \in \mathbb{P}_k(K)$, we obtain

$$\begin{aligned}
|\widetilde{I}_h v - I_h^W v|_{1,K}^2 &\le C h_K |\widetilde{I}_h v - \Pi_K^{\nabla}(\widetilde{I}_h v)|_{1,K} \|\Delta(\widetilde{I}_h v - I_h^W v)\|_K \\
&= C h_K |\widetilde{I}_h v - v_\pi - \Pi_K^{\nabla}(\widetilde{I}_h v - v_\pi)|_{1,K} \|\Delta(\widetilde{I}_h v - I_h^W v)\|_K \\
&\le C h_K |\widetilde{I}_h v - v_\pi|_{1,K} \|\Delta(\widetilde{I}_h v - I_h^W v)\|_K,
\end{aligned} \tag{4.48}$$

where $v_\pi \in \mathbb{P}_k(K)$ is the local approximation of v with estimates (4.6). Similar arguments as in the proof of Lemma 4.5, we obtain the inverse inequality

$$\|\Delta(\widetilde{I}_h v - I_h^W v)\|_K \leq Ch_K^{-1}|\widetilde{I}_h v - I_h^W v|_{1,K}. \tag{4.49}$$

Combining (4.48) and (4.49) yields

$$|\widetilde{I}_h v - I_h^W v|_{1,K} \leq C|\widetilde{I}_h v - v_\pi|_{1,K} \leq C(|v - \widetilde{I}_h v|_{1,K} + |v - v_\pi|_{1,K}),$$

which, with the triangle inequality, leads to

$$|v - I_h^W v|_{1,K} \leq C(|v - \widetilde{I}_h v|_{1,K} + |v - v_\pi|_{1,K}).$$

Using the interpolation error estimates (4.47) and estimates (4.6), we get

$$|v - I_h^W v|_{1,K} \leq Ch_K^s|v|_{s,K}. \tag{4.50}$$

For $\|v - I_h^W v\|_K$, we use the Poincaré-Friedrichs inequality (4.9) to obtain

$$\begin{aligned}\|v - I_h^W v\|_K &\leq \|v - \widetilde{I}_h v\|_K + \|\widetilde{I}_h v - I_h^W v\|_K \\ &\leq \|v - \widetilde{I}_h v\|_K + Ch_K|\widetilde{I}_h v - I_h^W v|_{1,K} \\ &\leq \|v - \widetilde{I}_h v\|_K + Ch_K(|v - \widetilde{I}_h v|_{1,K} + |v - I_h^W v|_{1,K}).\end{aligned}$$

Using the above interpolation estimates (4.47) and (4.50), we get

$$\|v - I_h^W v\|_K \leq Ch_K^s|v|_{s,K}.$$

The proof is complete. □

Modified VEM

With the modified VE space W_h, we define the VEM for Poisson problem (1.2): find $u_h \in W_h$ s.t.

$$a_h(u_h, v_h) = (f_h, v_h), \quad \forall v_h \in W_h. \tag{4.51}$$

where $f_h|_K = P_k^K f$ for $K \in \mathcal{T}_h$.

Similar to the previous discussion, we still have the consistency (4.37) and stability (4.38) of $a_h(\cdot,\cdot)$ on W_h. Thus $a_h(\cdot,\cdot)$ is continuous and coercive on W_h, which guarantees the well-posedness of the discrete problem (4.51). Similar to Theorem 4.12, we have the error bound (4.43). Then by the interpolation error estimates (4.47), local approximation estimates (4.6) and projection error estimates (4.7), we obtain the optimal convergence in H^1 norm, as in (4.13).

An auxiliary problem

We introduce an auxiliary problem: find $\psi \in H_0^1(\Omega)$ s. t.

$$-\Delta\psi = u - u_h \quad \text{in } \Omega, \tag{4.52}$$

where u is the exact solution to Poisson problem (1.2) and u_h the numerical solution of VEM (4.51). Obviously this problem is well-posedness, Further, if the domain Ω is convex, we have the regularity result:

$$\|\psi\|_2 \leq C\|u - u_h\|. \tag{4.53}$$

L^2-error bound

Similar to Theorem 4.12, we also have an error bound for L^2-error.

Theorem 4.15. *Assume that u is the solution to Poisson problem* (1.2)*, u_h the solution to the discrete problem* (4.51)*, and u_π any approximation of u from space $\mathbb{P}_k^{dc}(\mathcal{T}_h)$, we have*

$$\|u - u_h\| \leq Ch(|u - u_h|_1 + |u - u_\pi|_{1,h} + \|f - f_h\|), \tag{4.54}$$

Proof. Recalling the Poisson problem (1.2) and discrete problem (4.51), the auxiliary problem (4.52) implies

$$\begin{aligned}\|u-u_h\|^2 &= (u-u_h,-\Delta\psi) = a(u-u_h,\psi)\\ &= a(u-u_h,\psi-I_h\psi)+a(u-u_h,I_h\psi)\\ &= a(u-u_h,\psi-I_h\psi)+(f,I_h\psi)-a(u_h,I_h\psi)\\ &= a(u-u_h,\psi-I_h\psi)+(f-f_h,I_h\psi)+a_h(u_h,I_h\psi)-a(u_h,I_h\psi).\end{aligned}\tag{4.55}$$

For the first term in (4.55), we use the interpolation error estimates (4.47) to

$$a(u-u_h,\psi-I_h\psi)\le|u-u_h|_1|\psi-I_h\psi|_1\le Ch|u-u_h|_1|\psi|_2.\tag{4.56}$$

For the second one in (4.55), we use the property of L^2-projection and interpolation error estimates (4.47) to obtain

$$\begin{aligned}(f-f_h,I_h\psi) &= (f-f_h,I_h\psi-\psi)+\sum_{K\in\mathcal{T}_h}(f-f_h,\psi-P_0^K\psi)_K\\ &\le\|f-f_h\|\|\psi-I_h\psi\|+\sum_{K\in\mathcal{T}_h}\|f-f_h\|_K\|\psi-P_0^K\psi\|_K\\ &\le Ch\|f-f_h\|\|\psi\|_2.\end{aligned}\tag{4.57}$$

For the last one in (4.55), we use the consistency (4.37) and stability (4.38) of $a_h(\cdot,\cdot)$ to obtain

$$\begin{aligned}a_h(u_h,I_h\psi)-a(u_h,I_h\psi) &= \sum_{K\in\mathcal{T}_h}\left(a_h^K(u_h,I_h\psi-P_0^K\psi)-a^K(u_h,I_h\psi-P_0^K\psi)\right)\\ &= \sum_{K\in\mathcal{T}_h}\left(a_h^K(u_h-u_\pi,I_h\psi-P_0^K\psi)-a^K(u_h-u_\pi,I_h\psi-P_0^K\psi)\right)\\ &\le C\sum_{K\in\mathcal{T}_h}|u_h-u_\pi|_{1,K}|\psi-P_0^K\psi|_{1,K}.\end{aligned}$$

Then we use the L^2-projection error estimates to obtain

$$a_h(u_h,I_h\psi)-a(u_h,I_h\psi)\le Ch|u_h-u_\pi|_{1,h}|\psi|_2.\tag{4.58}$$

Substituting (4.56)-(4.58) into (4.55), we get

$$\|u-u_h\|^2\le Ch(|u-u_h|_1+|u-u_\pi|_{1,h}+\|f-f_h\|)\|\psi\|_2,$$

which, together with the regularity result (4.53), yields (4.54). The proof is complete. □

By using Theorem (4.15), we obtain the optimal convergence in L^2 norm.

Corollary 4.16. *Assume $u\in H_0^1(\Omega)\cap H^{k+1}(\Omega)$ and $f\in H^k(\Omega)$, we have*

$$|u-u_h|_1\le Ch^{k+1}(|u|_{k+1}+|f|_k).\tag{4.59}$$

4.9. Virtual element in 3D

Let $d=3$ and K be a polyhedron in $\mathcal{T}_h$, For each face F of K, we introduce a face VE function $W_k(F)$ by (4.45). Then we define the boundary VE space on ∂K by

$$W_k(\partial K)=\{v\in H^1(\partial K):\ v|_F\in W_k(F),\ \forall F\in\mathcal{F}(K)\},$$

where $\mathcal{F}(K)$ denotes the set of faces of K. Then we define the local VE space $V_k(K)$ by

$$V_k(K)=\{v\in H^1(K):\ \Delta v\in P_{k-2}(K),\ v|_{\partial K}\in W_k(\partial K)\}.\tag{4.60}$$

Obvious it holds $\mathbb{P}_k(K)\subset V_k(K)$.

For space $V_k(K)$, we define the degrees of freedom (DOF) as

$$\bullet \text{ values of } v(a), \qquad \text{vertex } a \text{ of } K, \tag{4.61}$$

$$\bullet \text{ moments } \frac{1}{|e|}\int_e vq\mathrm{d}s, \quad q \in \mathbb{P}_{k-2}(e), \quad \text{edge } e \text{ of } K, \tag{4.62}$$

$$\bullet \text{ moments } \frac{1}{|f|}\int_f vq\mathrm{d}s, \quad q \in \mathbb{P}_{k-2}(f), \quad \text{face } f \text{ of } K, \tag{4.63}$$

$$\bullet \text{ moments } \frac{1}{|K|}\int_K vq\mathrm{d}K, \quad q \in \mathbb{P}_{k-2}(K). \tag{4.64}$$

We have the unisolvence of DOF as follows.

Lemma 4.17 (Unisolvence)**.** *The DOF* (4.61)-(4.64) *are unisolvent for the space* $V_k(K)$*.*

The process of error analysis for the 3D VE is similar to the 2D case, but is more complicated. For the details, we refer to the reference [1].

5. Nonconforming VEM for Poisson problem

5.1. Nonconforming VE space

To fix the idea, next we only discuss the 3D case. For the 2D case, the extension is immediate. On a polyhedron K with n faces, the local shape function space for the nonconforming VE [2] is defined by

$$V_k^{nc}(K) = \{v \in H^1(K);\ \Delta v \in \mathbb{P}_{k-2}(K),\ \frac{\partial v}{\partial \boldsymbol{n}_K}\Big|_F \in \mathbb{P}_{k-1}(F)\}. \tag{5.1}$$

Obviously it holds $\mathbb{P}_k(K) \subset V_k(K)$.

We recall the well-posedness of the nonhomogeneous Neumann problem [9]. Specially, for any given $f \in \mathbb{P}_{k-2}(K)$ and $g \in \mathbb{P}_{k-1}^{dc}(\partial K)$, there exists a unique solution $v \in H^1(\Omega)/\mathbb{R}$, s.t.

$$-\Delta v = f \quad \text{in } K, \qquad \frac{\partial v}{\partial \boldsymbol{n}_K} = g \quad \text{on } \partial K,$$

if and only if the data (f, g) satisfy the consistency condition

$$\int_K f\mathrm{d}x + \int_{\partial K} g\mathrm{d}s = 0.$$

By some basic discussions, we obtain

$$\dim V_k^{nc}(K) = \dim \mathbb{P}_{k-2}(K) + \dim \mathbb{P}_{k-1}^{nc}(\partial K) = \frac{k(k-1)}{2} + nk.$$

For space $V_k^{nc}(K)$, we define the degrees of freedom (DOF) as

$$\bullet \text{ moments } \frac{1}{|F|}\int_F vq\mathrm{d}s, \quad q \in \mathbb{P}_{k-2}(F), \quad \text{face } F \text{ of } K, \tag{5.2}$$

$$\bullet \text{ moments } \frac{1}{|K|}\int_K vq\mathrm{d}K, \quad q \in \mathbb{P}_{k-2}(K). \tag{5.3}$$

We have the unisolvence of DOF as follows.

Lemma 5.1 (Unisolvence)**.** *The DOF* (5.2)-(5.3) *are unisolvent for the space* $V_k^{nc}(K)$*.*

Proof. It is easy to verify that the dimension of $V_k^{nc}(K)$ is equal to the number of DOF (5.2)-(5.3). It remains to show that for any function $v \in V_k^{nc}(K)$, if all the DOF (5.2)-(5.3) vanish, then $v = 0$.

By using the integration by parts, the vanishing DOF imply

$$|v|_{1,K}^2 = (\nabla v, \nabla v)_K = -(v, \Delta v)_K + \int_{\partial K} v \frac{\partial v}{\partial \boldsymbol{n}_K}\mathrm{d}s = 0,$$

since $\Delta v \in \mathbb{P}_{k-2}(K)$ in K and $\frac{\partial v}{\partial \boldsymbol{n}_K}\big|_F \in \mathbb{P}_{k-1}(F)$ for face F of K. Thus we obtain $\nabla v = 0$, which, with the fact that $\int_{\partial K} v\mathrm{d}s = 0$, leads to $v = 0$. □

We define the global virtual element space V_h^{nc} by

$$V_h^{nc} = \{v \in H^1(\Omega):\ v|_K \in V_k^{nc}(K),\ \forall K \in \mathcal{T}_h,\ \int_F [v]q\mathrm{d}s = 0,\ \forall q \in \mathbb{P}_{k-1}(F),\ \forall F \in \mathcal{E}_h\}.$$

The global DOF is an immediate extension of the local DOF (5.2)-(5.3). We note that $V_h^{nc} \not\subseteq H_0^1(\Omega)$.

Interpolation

For any function $v \in H_0^1(\Omega)$, we define the interpolation $I_h v$ by requiring that the values of DOF (5.2)-(5.3) of $I_h v$ are equal to the corresponding ones of v. We present the following interpolation error estimates.

Lemma 5.2. *For any $v \in H_0^1(\Omega) \cap H^s(\Omega)$ with $1 \le s \le k+1$, it holds*

$$\|v - I_h v\|_K + h_K |v - I_h v|_{1,K} \le C h_K^s |v|_{s,K}, \quad \forall K \in \mathcal{T}_h. \tag{5.4}$$

Proof. Let $v_\pi \in \mathbb{P}_k^{dc}(\mathcal{T}_h)$ be the piecewise polynomial approximation of v with estimates (4.6). For any given $K \in \mathcal{T}_h$, by integration by parts we have

$$|v_\pi - I_h v|_{1,K}^2 = -(v_\pi - I_h v, \Delta(v_\pi - I_h v))_K + \int_{\partial K} (v_\pi - I_h v) \frac{\partial (v_\pi - I_h v)}{\partial \boldsymbol{n}_K} \mathrm{d}s.$$

Recalling the fact that $\Delta(v_\pi - I_h v) \in \mathbb{P}_{k-2}(K)$ in K and $\frac{\partial (v_\pi - I_h v)}{\partial \boldsymbol{n}_K}\big|_F \in \mathbb{P}_{k-1}(F)$ for faces F of K, the property of I_h implies that

$$|v_\pi - I_h v|_{1,K}^2 = -(v_\pi - v, \Delta(v_\pi - I_h v))_K + \int_{\partial K} (v_\pi - v) \frac{\partial (v_\pi - I_h v)}{\partial \boldsymbol{n}_K} \mathrm{d}s = (\nabla(v_\pi - v), \nabla(v_\pi - I_h v))_K.$$

Thus we obtain

$$|v_\pi - I_h v|_{1,K} \le |v - v_\pi|_{1,K},$$

which, together with the triangle inequality and estimates (4.6), leads to

$$|v - I_h v|_{1,K} \le C h_K^s |v|_{s,K}.$$

For $\|v - I_h v\|_K$, we use the triangle inequality and Poincaré-Friedrichs inequality (4.9) to obtain

$$\|v - I_h v\|_K \le C h_K |v - I_h v|_{1,K} \le C h_K^s |v|_{s,K}.$$

The proof is complete. □

Inverse inequality

By the same argument as in proof of Lemma (4.5), we obtain an inverse inequality on the nonconforming VE space V_h^{nc}.

Lemma 5.3. *For any $v_h \in V_h^{nc}$, it holds*

$$\|\Delta v_h\|_K \le C h_K^{-1} |v_h|_{1,K}, \quad \forall K \in \mathcal{T}_h. \tag{5.5}$$

5.2. The discrete bilinear form

Similar to the discussion on (4.33) for the conforming VE, for any $v \in V_h^{nc}$ and $q \in \mathbb{P}_k(K)$ the bilinear form $a^K(v, q)$ is computable by the DOF (5.2)-(5.3). Then we can define a H^1-projection operator $\Pi_K^\nabla : V_k^{nc}(K) \to \mathbb{P}_k(K)$ by (4.34). Thus the H^1-projection operator Π_K^∇ is also computable.

With the help of Π_K^∇, we define the local discrete bilinear form $a_h^K(\cdot,\cdot)$ by

$$a_h^K(u_h, v_h) = a^K(\Pi_K^\nabla u_h, \Pi_K^\nabla v_h) + S^K(u_h - \Pi_K^\nabla u_h, v_h - \Pi_K^\nabla v_h), \quad u_h, v_h \in V_k^{nc}(K),$$

where the bilinear form $S^K(\cdot,\cdot)$ is given by

$$S^K(u_h, v_h) = h_K^{-2}(P_{k-2}^K u_h, P_{k-2}^K v_h)_K + h_K^{-1} \sum_{F \in \mathcal{E}(K)} \int_F P_{k-1}^F u_h P_{k-1}^F v_h \mathrm{d}s, \quad u_h, v_h \in \mathrm{Ker}(\Pi_K^\nabla) \cap V_k^{nc}(K).$$

Obviously $S^K(\cdot,\cdot)$ is computable by the DOF (5.2)-(5.3) for $V_k^{nc}(K)$.

For any face F of K, let $\mathbb{R}_F^2$ be the 2D affine space passing through F, $\mathcal{F}_F^1(K) := \{F' \in \mathcal{F}^1(K) : F' \subset \mathbb{R}_F^2\}$ and

$$\lambda_F := \boldsymbol{n}_F^\top \frac{\boldsymbol{x} - \boldsymbol{x}_F}{h_K},$$

where $\boldsymbol{x}_F$ is the barycenter of F. Apparently $\lambda_F|_F = 0$. Define the face bubble function

$$b_F := \Big(\prod_{F' \in \mathcal{F}^1(K) \setminus \mathcal{F}_F^1(K)} \lambda_{F'} \Big) \Big(\prod_{F' \in \mathcal{F}_F^1(K)} \prod_{e \in \mathcal{F}^1(F')} \boldsymbol{n}_{F',e}^\top \frac{\boldsymbol{x} - \boldsymbol{x}_e}{h_K} \Big)$$

for each face $F \in \mathcal{F}^1(K)$.

For $S^K(\cdot,\cdot)$, we have the norm equivalence on the kernel space of Π_K^∇ as follows.

Lemma 5.4. *For any $v_h \in Ker(\Pi_K^{\nabla}) \cap V_k^{nc}(K)$, it holds that*

$$C_1|v_h|_{1,K}^2 \le S^K(v_h, v_h) \le C_1|v_h|_{1,K}^2. \tag{5.6}$$

Proof. For convenience, let $\phi = \nabla v_h$ on a given $K \in \mathcal{T}_h$ where $v_h \in \mathrm{Ker}(\Pi_K^{\nabla}) \cap V_k^{nc}(K)$. Then we have

$$\mathrm{div}\phi \in \mathbb{P}_{k-2}(K) \quad \text{in } K, \quad \phi \cdot \boldsymbol{n}_K \in \mathbb{P}_{k-1}(F) \quad \text{on each face } F \in \mathcal{E}(K).$$

For a polynomial q on a face F, we define its extension by setting

$$E_K(q)(\boldsymbol{x}) = q(\boldsymbol{x}_p), \quad \boldsymbol{x} \in \mathbb{R}_F^3,$$

where $\boldsymbol{x}_p$ is the projection of $\boldsymbol{x}$ onto the plane $\mathbb{R}^2$ through the face F and $\mathbb{R}_F^3 = \{\boldsymbol{x} \in \mathbb{R}^3 : \ \boldsymbol{x}_p \in F\}$. Obviously the extension $E_K(q)$ is a polynomial in $\mathbb{R}_F^3$. For simplicity, we assume that there are not two or more faces of K which share a common plane.

Let b_F be the face bubble function associated to F such that b_F is a polynomial and

$$b_F = 0 \quad \text{on } \partial K/F, \quad \|b_F\|_{\infty,K} \sim 1.$$

For the details on face bubble functions, see [7]. Then we introduce a continuous piecewise polynomial

$$w_F(\boldsymbol{x}) = \begin{cases} b_F^2 E_K(\phi \cdot \boldsymbol{n}_K), & \boldsymbol{x} \in \mathbb{R}_F^3, \\ 0, & \boldsymbol{x} \notin \mathbb{R}_F^3, \end{cases}$$

such that

$$\|w_F\|_K \le C h_K^{1/2} \|\phi \cdot \boldsymbol{n}_K\|_F. \tag{5.7}$$

We use the the inverse inequality and (5.7) to obtain

$$\begin{aligned}
\int_F \phi \cdot \boldsymbol{n}_K w_F \mathrm{d}s &= \int_{\partial K} \phi \cdot \boldsymbol{n}_K w_F \mathrm{d}s \\
&= (\phi, \nabla w_F)_K + (\mathrm{div}\phi, w_F)_K \\
&\le \|\phi\|_K |w_F|_{1,K} + \|\mathrm{div}\phi\|_K \|w_F\|_K \\
&\le C(h_K^{-1}\|\phi\|_K + \|\mathrm{div}\phi\|_K)\|w_F\|_K \\
&\le C(h_K^{-1/2}\|\phi\|_K + h_K^{1/2}\|\mathrm{div}\phi\|_K)\|\phi \cdot \boldsymbol{n}_K\|_F,
\end{aligned}$$

which, together with the inequality

$$\|\phi \cdot \boldsymbol{n}_K\|_F^2 \le C \int_F \phi \cdot \boldsymbol{n}_K w_F \mathrm{d}s,$$

leads to

$$\|\phi \cdot \boldsymbol{n}_K\|_F \le C h_K^{-1/2}(\|\phi\|_K + h_K \|\mathrm{div}\phi\|_K).$$

Thus we have

$$\begin{aligned}
|v_h|_{1,K}^2 &= (\nabla v_h, \phi)_K \\
&= -(v_h, \mathrm{div}\phi)_K + \sum_{F \in \mathcal{E}(K)} \int_F v_h \phi \cdot \boldsymbol{n}_K \mathrm{d}s \\
&= -(P_{k-2}^0 v_h, \mathrm{div}\phi)_K + \sum_{F \in \mathcal{E}(K)} \int_F P_{k-1}^F v_h \phi \cdot \boldsymbol{n}_K \mathrm{d}s \\
&\le \|P_{k-2}^0 v_h\|_K \|\mathrm{div}\phi\|_K + \sum_{F \in \mathcal{E}(K)} \|P_{k-1}^F v_h\|_F \|\phi \cdot \boldsymbol{n}_K\|_F \\
&\le C(S^K(v,v))^{1/2}(\|\phi\|_K^2 + h_K^2\|\mathrm{div}\phi\|_K^2)^{1/2},
\end{aligned} \tag{5.8}$$

which, together with the inverse inequality (5.5), yields

$$C_1|v_h|_{1,K}^2 \le S^K(v_h, v_h). \tag{5.9}$$

On the other hand, the Poincaré-Friedrichs (4.9), trace inequality (4.10) and the boundedness of L^2-projection imply

$$\|P_{k-2}^K v_h\|_K \le \|v_h\|_K \le Ch_K|v_h|_{1,K},$$

$$\|P_{k-1}^F v_h\|_F \le \|v_h\|_F \le C(h_K^{-1/2}\|v\|_K + h_K^{1/2}|v_h|_{1,K}) \le Ch_K^{1/2}|v_h|_{1,K},$$

since $\Pi_K^\nabla v_h = 0$. Thus we have

$$S^K(v_h, v_h) \le C_2|v_h|_{1,K}^2.$$

The proof is complete. □

Remark 5.1. In fact, there exists another simple but classical choice of $S^K(\cdot,\cdot)$ as presented in Remark4.2.

The global discrete bilinear form is given by

$$a_h(u,v) = \sum_{K\in\mathcal{T}_h} a_h^K(u,v), \quad u,v \in V_h^{nc}.$$

Similar to Lemma 4.11, we obtain the consistency and stability of $a_h(\cdot,\cdot)$ on space V_h^{nc}.

Lemma 5.5 (Consistency and stability)**.** *For any $K \in \mathcal{T}_h$, the bilinear form $a_h(\cdot,\cdot)$ satisfies*

- *k-consistency: For all $q \in \mathbb{P}_k(K)$ and $v_h \in V_k^{nc}(K)$,*

$$a_h^K(v_h, q) = a^K(v_h, q). \tag{5.10}$$

- *Stability: There exist two positive constants α_* and α^* independent of h, such that*

$$\alpha_* a^K(v_h, v_h) \le a_h^K(v_h, v_h) \le \alpha^* a^K(v_h, v_h), \quad \forall v_h \in V_k^{nc}(K). \tag{5.11}$$

5.3. The right-hand side

The right-hand side is the same as in Section 4.5, i.e. on each K,

$$\langle f_h, v_h\rangle_h = \begin{cases} (P_0^K f, \overline{v}_h)_K, & k=1, \\ (P_{k-2} f, v_h)_K, & k\ge 2. \end{cases}$$

Moreover, for $f \in H^{k-1}(\Omega)$ we have

$$(f, v_h) - \langle f_h, v_h\rangle_h \le Ch^k|f|_{k-1}|v_h|_{1,h}, \quad \forall v_h \in V_h^{nc}. \tag{5.12}$$

5.4. The discrete problem

With the above preparations, we present the nonconforming VEM for Poisson problem (1.2): to find $u_h \in V_h^{nc}$ such that

$$a_h(u_h, v_h) = \langle f_h, v_h\rangle_h, \quad \forall v_h \in V_h^{nc}. \tag{5.13}$$

By using the weak continuity of nonconforming VE, we can show that $|\cdot|_{1,h}$ is indeed a norm on V_h^{nc}. The stability (5.11) of $a_h(\cdot,\cdot)$ implies its continuity and coercivity. The discrete problem (5.13) is well-posed.

5.5. Error analysis for H^1-norm

We present an abstract error bound for the nonconforming VEM (5.13). For convenience, we extend the definition of $a(\cdot,\cdot)$ to the sum space $H_0^1(\Omega) + V_h^{nc}$ by

$$a(u,v) = \sum_{K\in\mathcal{T}_h} a^K(u,v), \quad u,v \in H_0^1(\Omega) + V_h^{nc}.$$

Theorem 5.6. *Assume that u is the solution to Poisson problem* (1.2)*, u_h the solution to the discrete problem* (5.13)*, and u_π any approximation of u from space $\mathbb{P}_k^{dc}(\mathcal{T}_h)$, we have*

$$|u - u_h|_{1,h} \le C(|u - I_h u|_{1,h} + |u - u_\pi|_{1,h} + \|f - f_h\|_{V_h'} + E_h), \tag{5.14}$$

where

$$\|f - f_h\|_{V_h'} := \sup_{v_h\in V_h} \frac{(f, v_h) - \langle f_h, v_h\rangle_h}{|v_h|_{1,h}}, \quad E_h = \sup_{v_h \in V_h^{nc}} \frac{a(u, v_h) - (f, v_h)}{|v_h|_{1,h}}.$$

Proof. Setting $\delta_h = I_h u - u_h$, we have

$$\begin{aligned}
\alpha_* |\delta_h|_{1,h}^2 &\le \alpha_* a(\delta_h, \delta_h) \le a_h(\delta_h, \delta_h) \quad (\text{use } (5.11)) \\
&= a_h(I_h u, \delta_h) - a_h(u_h, \delta_h) \\
&= \sum_{K \in \mathcal{T}_h} a_h^K(I_h u, \delta_h) - \langle f_h, \delta_h \rangle_h \quad (\text{use } (5.13)) \\
&= \sum_{K \in \mathcal{T}_h} \big(a_h^K(I_h u - u_\pi, \delta_h) + a_h^K(u_\pi, \delta_h) \big) - \langle f_h, \delta_h \rangle_h \\
&= \sum_{K \in \mathcal{T}_h} \big(a_h^K(I_h u - u_\pi, \delta_h) + a^K(u_\pi, \delta_h) \big) - \langle f_h, \delta_h \rangle_h \quad (\text{use } (5.10)) \\
&= \sum_{K \in \mathcal{T}_h} \big(a_h^K(I_h u - u_\pi, \delta_h) + a^K(u_\pi - u, \delta_h) \big) + a(u, \delta_h) - \langle f_h, \delta_h \rangle_h \\
&= \sum_{K \in \mathcal{T}_h} \big(a_h^K(I_h u - u_\pi, \delta_h) + a^K(u_\pi - u, \delta_h) \big) + (f, \delta_h) - \langle f_h, \delta_h \rangle_h + a(u, \delta_h) - (f, \delta_h).
\end{aligned}$$

We use the stability (5.11) of $a_h(\cdot, \cdot)$ and the continuity of $a^K(\cdot, \cdot)$ in above inequality to get

$$|\delta_h|_{1,h}^2 \le C(|I_h u - u_\pi|_{1,h} + |u - u_\pi|_{1,h} + \|f - f_h\|_{V_h'} + E_h)|\delta_h|_{1,h},$$

Then the inequality (5.14) follows easily by the triangle inequality. □

Remark 5.2. *By contrast, for the nonconforming VE there exists a consistency error E_h in the error bound* (5.6)*, which arises from the conformity.*

For the consistency error, we use the same arguments as in the proof of Lemma (3.2) to estimate it.

Lemma 5.7. *Assume $u \in H_0^1(\Omega) \cap H^{k+1}(\Omega)$ is the exact solution to the Poisson problem* (1.2)*. It holds*

$$\sup_{v_h \in V_h^{nc}} \frac{a(u, v_h) - (f, v_h)}{|v_h|_{1,h}} \le Ch^k |u|_{k+1}. \tag{5.15}$$

Proof. By integration by parts, we obtain

$$a(u, v_h) - (f, v_h) = \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial u}{\partial \boldsymbol{n}_K} v_h \mathrm{d}s = \sum_{e \in \mathcal{E}_h} \int_F \frac{\partial u}{\partial \boldsymbol{n}_F} [v_h] \mathrm{d}s, \quad \forall v_h \in V_h^{nc}, \tag{5.16}$$

where $\boldsymbol{n}_K$ is the outer unite normal to K and $\boldsymbol{n}_F$ is the unit normal associated to face F.

Suppose $F \subset \partial K$, the weak continuity of v_h implies

$$\int_F \frac{\partial u}{\partial \boldsymbol{n}_F} [v_h] \mathrm{d}s = \int_F (\nabla u - P_{k-1}^K \nabla u) \cdot \boldsymbol{n}_F [v_h - P_0^F v_h] \mathrm{d}s \le \|\nabla u - P_{k-1}^K \nabla u\|_F \|[v_h - P_0^F v_h]\|_F. \tag{5.17}$$

Combining the trace inequality (4.10) and L^2-projection error estimates (4.7) leads to

$$\|\nabla u - P_{k-1}^K \nabla u\|_F \le C(h_K^{-1/2} \|\nabla u - P_{k-1}^K \nabla u\|_K + h_K^{1/2} |\nabla u - P_{k-1}^K \nabla u|_{1,K}) \le Ch_K^{k-1/2} |u|_{k+1,K}. \tag{5.18}$$

For interior face F shared by two elements K_1 and K_2, we use the trace inequality (4.10)and Poincaré-Friedrichs inequality (4.8) to obtain

$$\|[v_h - P_0^F v_h]\|_F \le C(h_K^{-1/2} \|v_h - P_0^F v_h\|_{K_1 \cup K_2} + h_K^{1/2} \|\nabla_h v_h\|_{K_1 \cup K_2}) \le Ch_K^{1/2} \|\nabla_h v_h\|_{K_1 \cup K_2}. \tag{5.19}$$

For boundary face, the adjustment is obvious.

Substituting (5.17)-(5.19) into (5.16), we obtain

$$a(u, v_h) - (f, v_h) \le Ch^k |u|_{k+1} |v_h|_{1,h}, \quad \forall v_h \in V_h^{nc}.$$

Then we obtain (5.15). □

Substituting interpolation estimates (5.4), estimates (4.6) of local polynomial approximation, approximation error estimates (5.12) of f_h and the estimate on consistency error (5.15) into the error bound (5.14), we obtain the optimal convergence of the nonconforming VEM.

Corollary 5.8. *Assume $u \in H_0^1(\Omega) \cap H^{k+1}(\Omega)$ and $f \in H^{k-1}(\Omega)$, we have*

$$|u - u_h|_{1,h} \le Ch^k (|u|_{k+1} + |f|_{k-1}). \tag{5.20}$$

5.6. *Error analysis for L^2-norm*

Similar to the discussions in Section 4.8, by using the dual technique we also obtain the optimal convergence for the nonconforming VEM in the L^2-norm.

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