1. 隐式积分

$$
\begin{gathered}
Q_i^{n+1}=-\frac{K_{i+1/2}^{n+1}}{\Delta z_{i+1/2}}\left(\psi_i^{n+1}-\psi_{i+1}^{n+1}\right)-K_{i+1/2}^{n+1} \\
Q_{i-1}^{n+1}=-\frac{K_{i-1/2}^{n+1}}{\Delta z_{i-1/2}}\left(\psi_{i-1}^{n+1}-\psi_i^{n+1}\right)-K_{i-1/2}^{n+1} \\
\frac{\theta_i^{n+1}-\theta_i^n}{\Delta t}=-\frac{Q_{i-1}^{n+1}-Q_i^{n+1}}{\Delta z_i} \\
\theta_i^{n+1}=C_i^{n+1}\psi_i^{n+1}, \quad \theta_i^n=C_i^{n+1}\psi_i^n
\end{gathered} \tag{1}
$$

联立可得：

$$
\frac{C_i^{n+1}\psi_i^{n+1}-C_i^n\psi_i^n}{\Delta t}=-\frac{-\frac{K_{i-1/2}^{n+1}}{\Delta z_{i-1/2}}\left(\psi_{i-1}^{n+1}-\psi_i^{n+1}\right)-K_{i-1/2}^{n+1}+\frac{K_{i+1/2}^{n+1}}{\Delta z_{i+1/2}}\left(\psi_i^{n+1}-\psi_{i+1}^{n+1}\right)+K_{i+1/2}^{n+1}}{\Delta z_i} \tag{2}
$$

$$
(-C_i^{n+1}\psi_i^{n+1}+C_i^n\psi_i^n)\frac{\Delta z_i}{\Delta t}=-\frac{K_{i-1/2}^{n+1}}{\Delta z_{i-1/2}}\left(\psi_{i-1}^{n+1}-\psi_i^{n+1}\right)-K_{i-1/2}^{n+1}+\frac{K_{i+1/2}^{n+1}}{\Delta z_{i+1/2}}\left(\psi_i^{n+1}-\psi_{i+1}^{n+1}\right)+K_{i+1/2}^{n+1} \tag{3}
$$

联立公式，写成$a\psi_{i-1}^{n+1}+b\psi_i^{n+1}+c\psi_{i+1}^{n+1}=d$的形式

$$
\begin{gathered}
-\frac{K_{i-1/2}^{n+1}}{\Delta z_{i-1/2}}\psi_{i-1}^{n+1}+(\frac{C_i^{n+1}\Delta z_i}{\Delta t}+\frac{K_{i-1/2}^{n+1}}{\Delta z_{i-1/2}}+\frac{K_{i+1/2}^{n+1}}{\Delta z_{i+1/2}})\psi_i^{n+1}-\frac{K_{i+1/2}^{n+1}}{\Delta z_{i+1/2}}\psi_{i+1}^{n+1}= \\
\frac{C_i^n\Delta z_i}{\Delta t}\psi_i^n+K_{i-1/2}^{n+1}-K_{i+1/2}^{n+1}
\end{gathered} \tag{4}
$$

2. Crank-Nicolson

n+1时刻：

$$
\begin{gathered}
Q_i^{n+1}=-\frac{K_{i+1/2}^{n+1}}{\Delta z_{i+1/2}}\left(\psi_i^{n+1}-\psi_{i+1}^{n+1}\right)-K_{i+1/2}^{n+1} \\
Q_{i-1}^{n+1}=-\frac{K_{i-1/2}^{n+1}}{\Delta z_{i-1/2}}\left(\psi_{i-1}^{n+1}-\psi_i^{n+1}\right)-K_{i-1/2}^{n+1}
\end{gathered} \tag{5}
$$

n时刻：

$$
\begin{gathered}
Q_i^n=-\frac{K_{i+1/2}^n}{\Delta z_{i+1/2}}\left(\psi_i^n-\psi_{i+1}^n\right)-K_{i+1/2}^n \\
Q_{i-1}^n=-\frac{K_{i-1/2}^n}{\Delta z_{i-1/2}}\left(\psi_{i-1}^n-\psi_i^n\right)-K_{i-1/2}^n
\end{gathered} \tag{6}
$$

$$
\theta_i^{n+1}=C_i^{n+1}\psi_i^{n+1},\theta_i^n=C_i^{n+1}\psi_i^n \tag{7}
$$

联立公式，

$$
\begin{aligned}
Q_{i-1}^{n+1}-Q_i^{n+1}&=\left(-\frac{K_{i-1/2}^{n+1}}{\Delta z_{i-1/2}}(\psi_{i-1}^{n+1}-\psi_i^{n+1})-K_{i-1/2}^{n+1}\right)-\left(-\frac{K_{i+1/2}^{n+1}}{\Delta z_{i+1/2}}(\psi_i^{n+1}-\psi_{i+1}^{n+1})-K_{i+1/2}^{n+1}\right) \\
&=-\frac{K_{i-1/2}^{n+1}}{\Delta z_{i-1/2}}(\psi_{i-1}^{n+1}-\psi_i^{n+1})+\frac{K_{i+1/2}^{n+1}}{\Delta z_{i+1/2}}(\psi_i^{n+1}-\psi_{i+1}^{n+1})-(K_{i-1/2}^{n+1}-K_{i+1/2}^{n+1})
\end{aligned} \tag{8}
$$

$$
Q_{i-1}^n-Q_i^n=-\frac{K_{i-1/2}^n}{\Delta z_{i-1/2}}(\psi_{i-1}^n-\psi_i^n)+\frac{K_{i+1/2}^n}{\Delta z_{i+1/2}}(\psi_i^n-\psi_{i+1}^n)-(K_{i-1/2}^n-K_{i+1/2}^n) \tag{9}
$$

Crank-Nicolson方式，进行积分：

$$\frac{\theta_i^{n+1}-\theta_i^n}{\Delta t}=-\frac{Q_{i-1}^{n+1}-Q_i^{n+1}}{2\Delta z_i}-\frac{Q_{i-1}^n-Q_i^n}{2\Delta z_i} \tag{10}$$

$$\frac{-\theta_i^{n+1}+\theta_i^n}{\Delta t}=\frac{Q_{i-1}^{n+1}-Q_i^{n+1}}{2\Delta z_i}+\frac{Q_{i-1}^n-Q_i^n}{2\Delta z_i} \tag{11}$$

$$\begin{gathered}\frac{-C_i^{n+1}\psi_i^{n+1}+C_i^n\psi_i^n}{\Delta t}=\frac{1}{2\Delta z_i}\left[\left(-\frac{K_{i-1/2}^{n+1}}{\Delta z_{i-1/2}}(\psi_{i-1}^{n+1}-\psi_i^{n+1})+\frac{K_{i+1/2}^{n+1}}{\Delta z_{i+1/2}}(\psi_i^{n+1}-\psi_{i+1}^{n+1})-(K_{i-1/2}^{n+1}-K_{i+1/2}^{n+1})\right)\right]+\\ \frac{1}{2\Delta z_i}\left[\left(-\frac{K_{i-1/2}^n}{\Delta z_{i-1/2}}(\psi_{i-1}^n-\psi_i^n)+\frac{K_{i+1/2}^n}{\Delta z_{i+1/2}}(\psi_i^n-\psi_{i+1}^n)-(K_{i-1/2}^n-K_{i+1/2}^n)\right)\right]\end{gathered} \tag{12}$$

写成$a\psi_{i-1}^{n+1}+b\psi_i^{n+1}+c\psi_{i+1}^{n+1}=d$的形式，

$$\begin{gathered}(-\frac{K_{i-1/2}^{n+1}}{2\Delta z_{i-1/2}})\psi_{i-1}^{n+1}+(\frac{K_{i-1/2}^{n+1}}{2\Delta z_{i-1/2}}+\frac{K_{i+1/2}^{n+1}}{2\Delta z_{i+1/2}}+\frac{C_i^{n+1}\Delta z_i}{\Delta t})\psi_i^{n+1}+(-\frac{K_{i+1/2}^{n+1}}{2\Delta z_{i+1/2}})\psi_{i+1}^{n+1}=\\ \frac{C_i^n\Delta z_i}{\Delta t}\psi_i^n+\frac{K_{i-1/2}^n}{2\Delta z_{i-1/2}}(\psi_{i-1}^n-\psi_i^n)-\frac{K_{i+1/2}^n}{2\Delta z_{i+1/2}}(\psi_i^n-\psi_{i+1}^n)+(K_{i-1/2}^n-K_{i+1/2}^n)/2+(K_{i-1/2}^{n+1}-K_{i+1/2}^{n+1})/2\end{gathered} \tag{13}$$

3. Picard iteration

采用第m次的结果，更新K, C，然后重新求解ψ^{n+1}

$$\begin{gathered}-\frac{K_{i-1/2}^{n+1,m}}{\Delta z_{i-1/2}}\psi_{i-1}^{n+1,m+1}+\left(\frac{C_i^{n+1,m}\Delta z_i}{\Delta t}+\frac{K_{i-1/2}^{n+1,m}}{\Delta z_{i-1/2}}+\frac{K_{i+1/2}^{n+1,m}}{\Delta z_{i+1/2}}\right)\psi_i^{n+1,m+1}-\frac{K_{i+1/2}^{n+1,m}}{\Delta z_{i+1/2}}\psi_{i+1}^{n+1,m+1}=\\ C_i^{n+1,m}\frac{\Delta z_i}{\Delta t}\psi_i^n+K_{i-1/2}^{n+1,m}-K_{i+1/2}^{n+1,m}\end{gathered} \tag{14}$$

令$\delta^{m+1}=\psi^{n+1,m+1}-\psi^{n+1,m}$，

$$\begin{gathered}-\frac{K_{i-1/2}^{n+1,m}}{\Delta z_{i-1/2}}(\psi_{i-1}^{n+1,m}+\delta_{i-1}^{m+1})+\left(\frac{C_i^{n+1,m}\Delta z_i}{\Delta t}+\frac{K_{i-1/2}^{n+1,m}}{\Delta z_{i-1/2}}+\frac{K_{i+1/2}^{n+1,m}}{\Delta z_{i+1/2}}\right)(\psi_i^{n+1,m}+\delta_i^{m+1})-\frac{K_{i+1/2}^{n+1,m}}{\Delta z_{i+1/2}}(\psi_{i+1}^{n+1,m}+\delta_{i+1}^{m+1})=\\ C_i^{n+1,m}\frac{\Delta z_i}{\Delta t}\psi_i^n+K_{i-1/2}^{n+1,m}-K_{i+1/2}^{n+1,m}\end{gathered} \tag{15}$$

展开之后，可得：

$$\begin{gathered}-\frac{K_{i-1/2}^{n+1,m}}{\Delta z_{i-1/2}}\delta_{i-1}^{m+1}+\left(\frac{C_i^{n+1,m}\Delta z_i}{\Delta t}+\frac{K_{i-1/2}^{n+1,m}}{\Delta z_{i-1/2}}+\frac{K_{i+1/2}^{n+1,m}}{\Delta z_{i+1/2}}\right)\delta_i^{m+1}-\frac{K_{i+1/2}^{n+1,m}}{\Delta z_{i+1/2}}\delta_{i+1}^{m+1}=\\ \frac{K_{i-1/2}^{n+1,m}}{\Delta z_{i-1/2}}(\psi_{i-1}^{n+1,m}-\psi_i^{n+1,m})-\frac{K_{i+1/2}^{n+1,m}}{\Delta z_{i+1/2}}(\psi_i^{n+1,m}-\psi_{i+1}^{n+1,m})\\ C_i^{n+1,m}\frac{\Delta z_i}{\Delta t}(\psi_i^n-\psi_i^{n+1,m})+K_{i-1/2}^{n+1,m}-K_{i+1/2}^{n+1,m}\end{gathered} \tag{16}$$