APMA 4302 Methods - Homework 2

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Problem 1

Part (a)

We are given A is invertible, so A^{-1} exists.
Find an expression for $\mathbf{u} - \hat{\mathbf{u}}$:

$$\mathbf{u} - \hat{\mathbf{u}} = A^{-1}(\mathbf{b} - \hat{\mathbf{b}})$$

$$||\mathbf{u} - \hat{\mathbf{u}}|| = ||A^{-1}(\mathbf{b} - \hat{\mathbf{b}})|| \leq ||A^{-1}|| * ||(\mathbf{b} - \hat{\mathbf{b}})||$$

Find an expression for $\frac{1}{\mathbf{u}}$:

$$A\mathbf{u} = \mathbf{b}$$

$$||A\mathbf{u}|| = ||\mathbf{b}|| \leq ||A|| * ||\mathbf{u}||$$

$$\frac{1}{||\mathbf{u}||} \leq \frac{||A||}{||\mathbf{b}||}$$

Multiplying $\mathbf{u} - \hat{\mathbf{u}}$ by $\frac{1}{\mathbf{u}}$, since they are both smaller than their derived expressions above, will give a product smaller than the product of their derived expressions above. Thus:

$$\frac{||\mathbf{u} - \hat{\mathbf{u}}||}{||\mathbf{u}||} \leq \frac{||A||}{||\mathbf{b}||}||A^{-1}|| * ||(\mathbf{b} - \hat{\mathbf{b}})||$$

$$\frac{||\mathbf{u} - \hat{\mathbf{u}}||}{||\mathbf{u}||} \leq ||A|| * ||A^{-1}|| * \frac{||(\mathbf{b} - \hat{\mathbf{b}})||}{||\mathbf{b}||}$$

$$\frac{||\mathbf{u} - \hat{\mathbf{u}}||}{||\mathbf{u}||} \leq \kappa(A)\frac{||(\mathbf{b} - \hat{\mathbf{b}})||}{||\mathbf{b}||}$$

Part (b)

For the upper bound:
By definition:

$$\frac{||\mathbf{e}||}{||\mathbf{u}||} = \frac{||\mathbf{u} - \hat{\mathbf{u}}||}{||\mathbf{u}||}$$

As shown in Part (a) of this problem:

$$\frac{||\mathbf{u}-\hat{\mathbf{u}}||}{||\mathbf{u}||} \leq \kappa(A)\frac{||(\mathbf{b}-\hat{\mathbf{b}})||}{||\mathbf{b}||}$$

Using $\mathbf{r} \equiv \mathbf{b} - A\hat{\mathbf{u}}$ and $A\hat{\mathbf{u}} = \hat{\mathbf{b}}$:

$$\frac{||\mathbf{e}||}{||\mathbf{u}||} \leq \kappa(A)\frac{||\mathbf{r}||}{||\mathbf{b}||}$$

For the lower bound:
First form an expression for $||\mathbf{r}||$:

$$\mathbf{r} = \mathbf{b} - A\hat{\mathbf{u}} = A(\mathbf{u}-\hat{\mathbf{u}})$$

$$||\mathbf{r}|| \leq ||A|| * ||\mathbf{u}-\hat{\mathbf{u}}||$$

$$\frac{||\mathbf{r}||}{||A||} \leq ||\mathbf{u}-\hat{\mathbf{u}}||$$

Second, form an expression for $\frac{1}{||\mathbf{u}||}$:

$$\mathbf{u} = A^{-1}\mathbf{b}$$

$$||\mathbf{u}|| \leq ||A^{-1}|| * ||\mathbf{b}||$$

$$\frac{1}{||\mathbf{u}||} \geq \frac{1}{||A^{-1}|| * ||\mathbf{b}||}$$

Combine these two expressions together similarly to above, where the quantities from the first and second expressions that are on the smaller side of the inequality will be multiplied together to create a quantity that is certainly smaller than the other side of the inequality.

$$\frac{||\mathbf{r}||}{||A|| * ||A^{-1}|| * ||\mathbf{b}||} \leq \frac{||\mathbf{u}-\hat{\mathbf{u}}||}{||\mathbf{u}||}$$

$$\frac{1}{\kappa(A)}\frac{||\mathbf{r}||}{||\mathbf{b}||} \leq \frac{||\mathbf{e}||}{||\mathbf{u}||}$$

An interpretation of this result is that the range of the relative error on $\mathbf{u}$ given some small perturbation is large is the condition number is large and small if the condition number is small.

Problem 2

Part (a)

$$A\mathbf{v_j} = \lambda_j \mathbf{v_j}$$

We will handle the boundary conditions, which are simply zero at both ends of the 1D domain, by setting them within the $\mathbf{v_j}$ vector at their corresponding i row index.

Now using the discretized version of A:

$$A\mathbf{v_j}_i = \frac{1}{h^2}(-v_{i-1} + 2v_i - v_{i+1})$$

Let's plug in the given $\mathbf{v_j}$ and see if it gives us an eigenstructure.

$$A\mathbf{v_j}_i = \frac{1}{h^2}(-sin(j\pi h(i-1)) + 2sin(j\pi hi) - sin(j\pi h(i+1))).$$

Deploying a trigonometric identity:

$$A\mathbf{v_j}_i = \frac{2}{h^2}sin(j\pi hi)(1 - cos(j\pi h)) = \frac{2}{h^2}(1 - cos(j\pi h))\mathbf{v_j}_i = \lambda_j \mathbf{v_j}_i.$$

Sure enough, we encounter an eigenstructure. Thus, the provided $\mathbf{v_j}$ is indeed an eigenvector of A.

Part (b)

As shown in Part (a) of this problem, the eigenvalues of A corresponding to eigenvectors $\mathbf{v_j}$ are:

$$\lambda_i = \frac{2}{h^2}(1 - cos(j\pi h)).$$

Part (c)

$$\kappa(A) = ||A|| * ||A^{-1}||$$

Let's come up with an expression for $||A||$. By decomposing $A = Q\Lambda Q^T$ where Q is an orthogonal matrix and Λ is a diagonal matrix with the eigenvalues along the diagonal (this is possible because we know A is positive semi-definite):

$$||A\mathbf{x}||_2 = ||Q\Lambda Q^T Q\mathbf{y}||$$

where we have written any vector $\mathbf{x}$ as some other vector $\mathbf{y}$ rotated by orthogonal matrix Q. We will use the L2 norm here, assuming that is the definition of the condition number this problem is looking for. Since the norm of an orthogonal matrix is one (they only rotate vectors, but do not change their size):

$$||A\mathbf{x}||_2 = ||Q\Lambda\mathbf{y}||_2 = ||\Lambda\mathbf{y}||_2 = \sqrt{\sum_i \lambda_i^2 y_i^2}.$$

Similarly, we can write $||\mathbf{x}||$ as:

$$||\mathbf{x}||_2 = ||Q\mathbf{y}||_2 = \sqrt{\sum_i y_i^2}.$$

Now, we can use:

$$||A\mathbf{x}||_2 \leq ||A||_2 * ||\mathbf{x}||_2,$$

$$\sqrt{\sum_i \lambda_i^2 y_i^2} \leq ||A||_2 \sqrt{\sum_i y_i^2},$$

$$\frac{\sqrt{\sum_i \lambda_i^2 y_i^2}}{\sqrt{\sum_i y_i^2}} \leq ||A||_2.$$

Now we want to find the situation where this inequality achieves the equality condition. We can think about this as a sum on λ_i^2 with weights $w_i = \frac{y_i^2}{\sum_i y_i^2}$, such that $\sum_i w_i = 1$.

$$\sqrt{\sum_i w_i \lambda_i^2} \leq ||A||_2.$$

This sum will be maximized if the whole contribution comes from the largest λ_i, or the largest eigenvalue. Therefore:

$$||A||_2 = \lambda_{max}.$$

The same procedure can be completed for $A^{-1} = Q\Lambda^{-1}Q^T$ and will yield:

$$||A^{-1}||_2 = \frac{1}{\lambda_{min}}.$$

Thus, $\kappa(A) = \frac{\lambda_{max}}{\lambda_{min}}$. Using the eigenvalues derived in Part (b) of this problem:

$$\kappa(A) = \frac{1 - \cos(j_{max}\pi m^{-1})}{1 - \cos(j_{min}\pi m^{-1})}.$$

Deploying another trigonometric identity:

$$\kappa(A) = \frac{\sin^2(j_{max}\pi/(2m))}{\sin^2(j_{min}\pi/(2m))}.$$

j runs from 1 to m. λ_i is clearly maximized if the argument of sin^2 is $\pi/2$ to maximize sin^2. In this case, $j_{max} = m$. To minimize λ_i, the argument of sin^2 needs to get as close to zero as possible. Thus, $j_{min} = 1$. Seeing that this minimum case has the argument of sin^2 as a small number close to zero, we Taylor expand this sin^2 term about zero and find, if only keeping the first term, that it is approximately $(\pi/(2m))^2$. So:

$$\kappa(A) = \frac{\sin^2(\pi/2)}{(\pi/(2m))^2} \sim m^2.$$

This is true for large m, where the approximation of keeping the first term of this Taylor series expansion of the λ_{min} denominator term becomes valid (and then true for $m \to \infty$).

Problem 3

Part (a)

$$-u''(x) + \gamma u(x) = f(x), x \in [0, 1]$$
$$u(x) = sin(k\pi x) + c(x - 0.5)^3$$
$$(k\pi)^2 \sin(k\pi x) - 6c(x - 0.5) + \gamma \sin(k\pi x) + \gamma c(x - 0.5)^3 = f(x)$$

We will also need the boundary conditions for this problem. They can be found by plugging in $x = 0, 1$ into the manufactured solution for $u(x)$. At $x = 0$:

$$u(x = 0) = -c/8,$$

$$u(x = 1) = c/8.$$

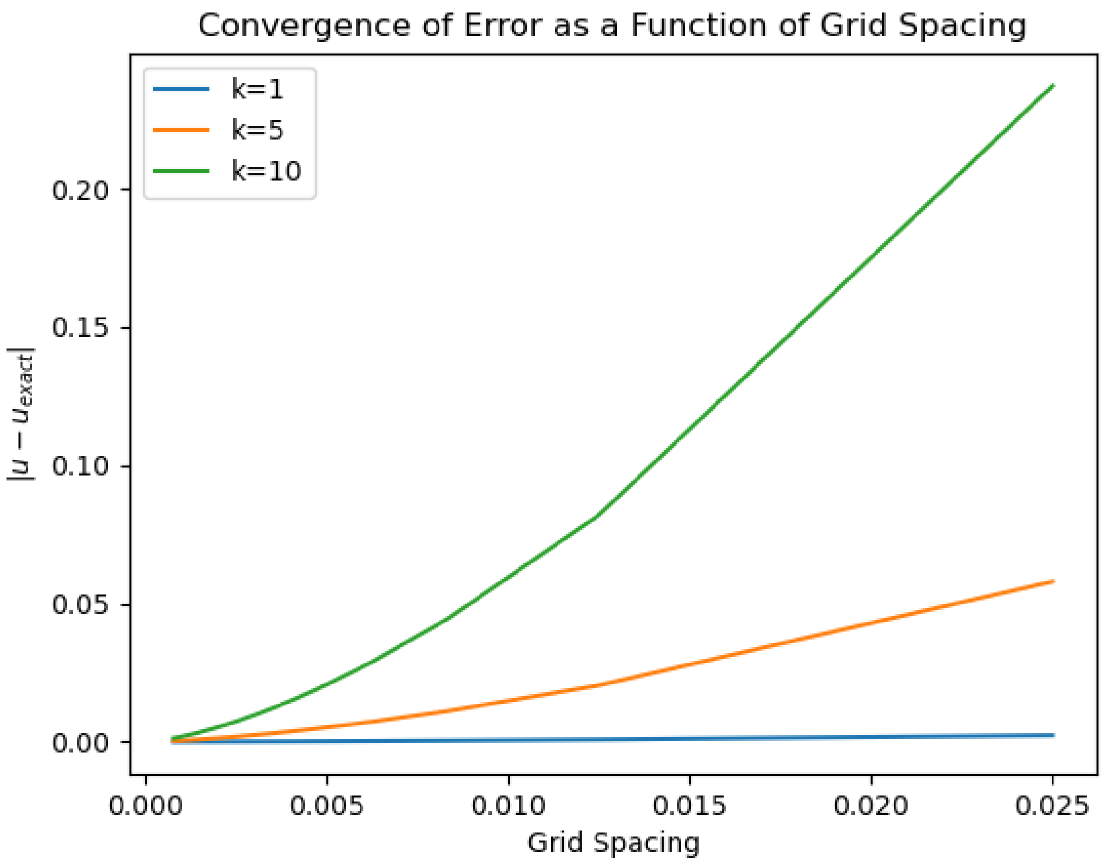

Figure 1: Convergence for Problem 3.

Part (c)

Fig. 1 shows roughly quadratic convergence (dropping h by a factor of two drops the error by about a factor of factor).

Problem 4

Parts (a) through (f.i) and (f.ii):

Note I know there seems to be something up since these plots all mostly look the same but I am tired and need to move on from this homework :).

For the explanation for Part c, I believe it becomes a direct solve but I didn't look into it that heavily. Note I changed c back to 3.0 after the problem that asked for it to be zero. Not sure what was desired with that.

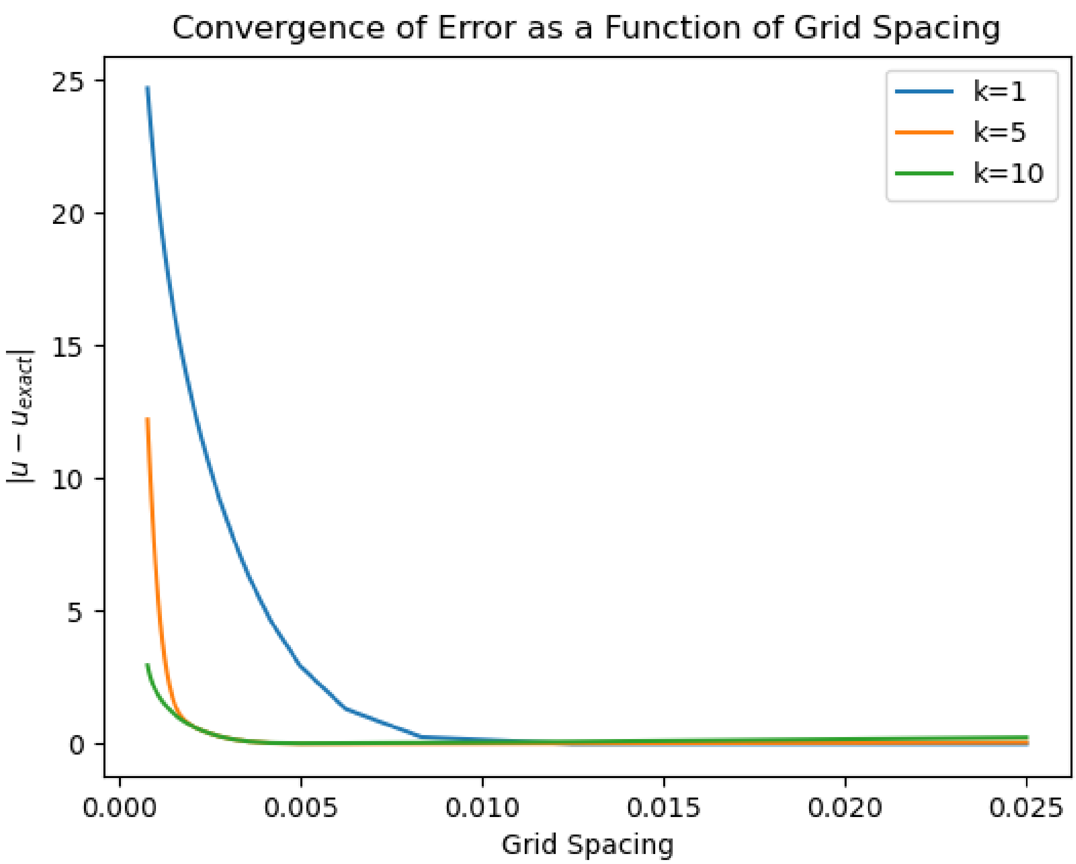

Figure 2: Convergence for Problem 4a.

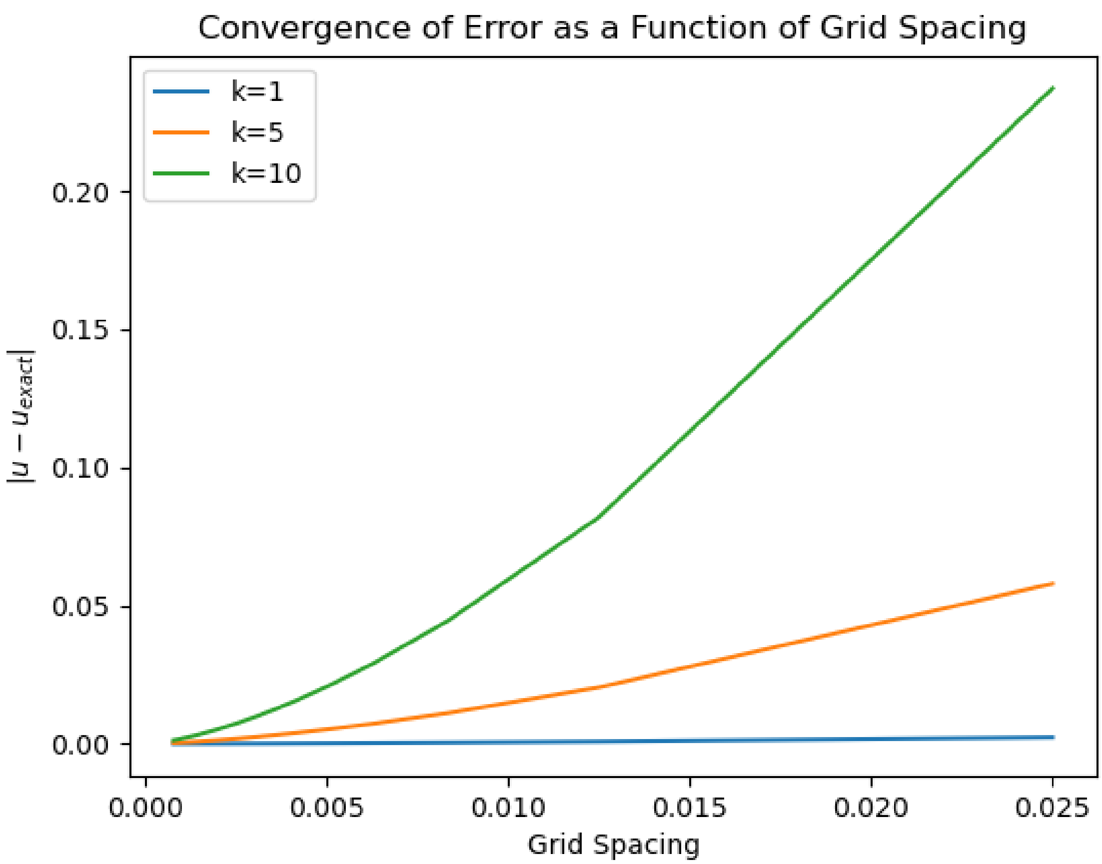

Figure 3: Convergence for Problem 4b.

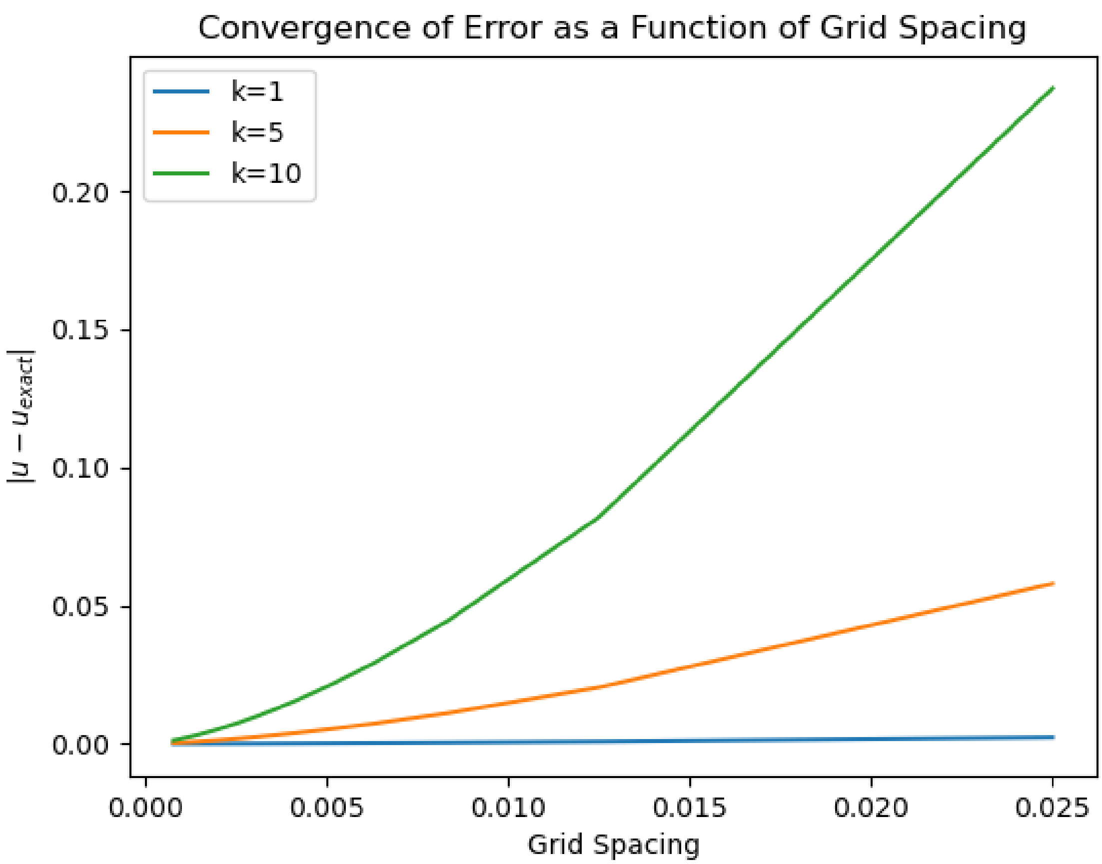

Figure 4: Convergence for Problem 4c.

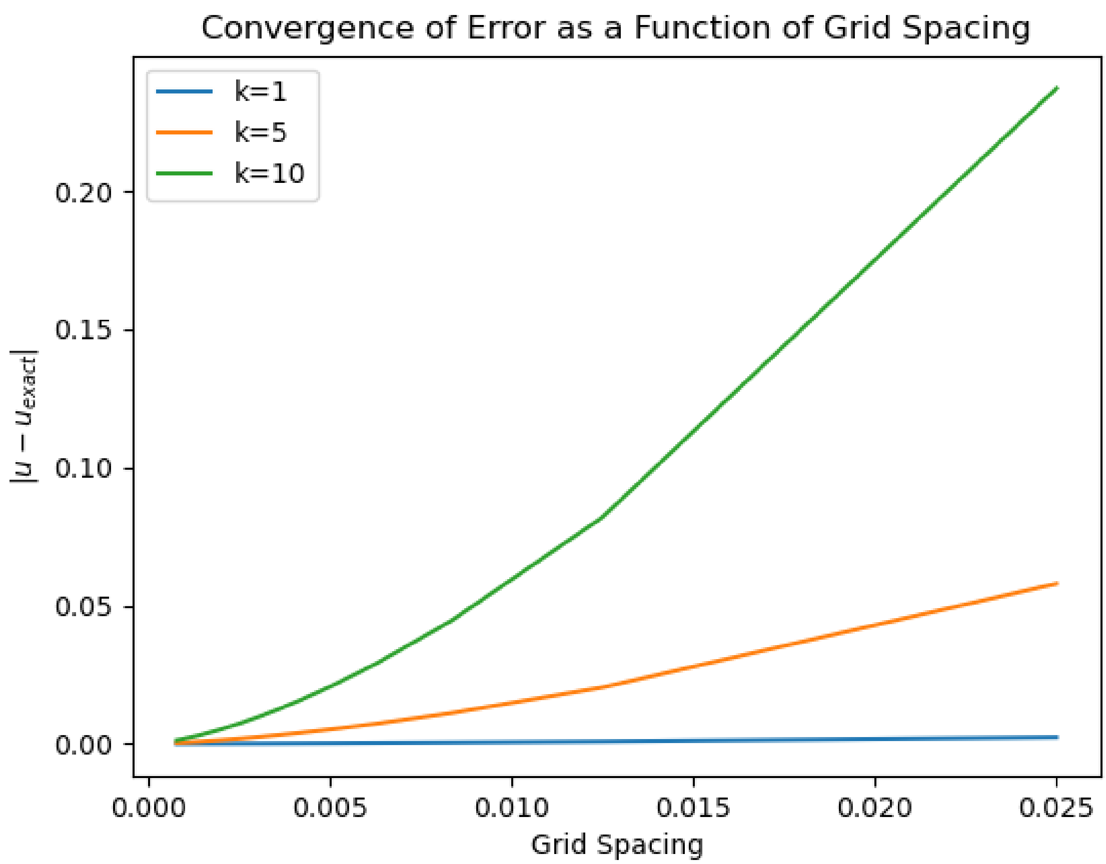

Figure 5: Convergence for Problem 4d.

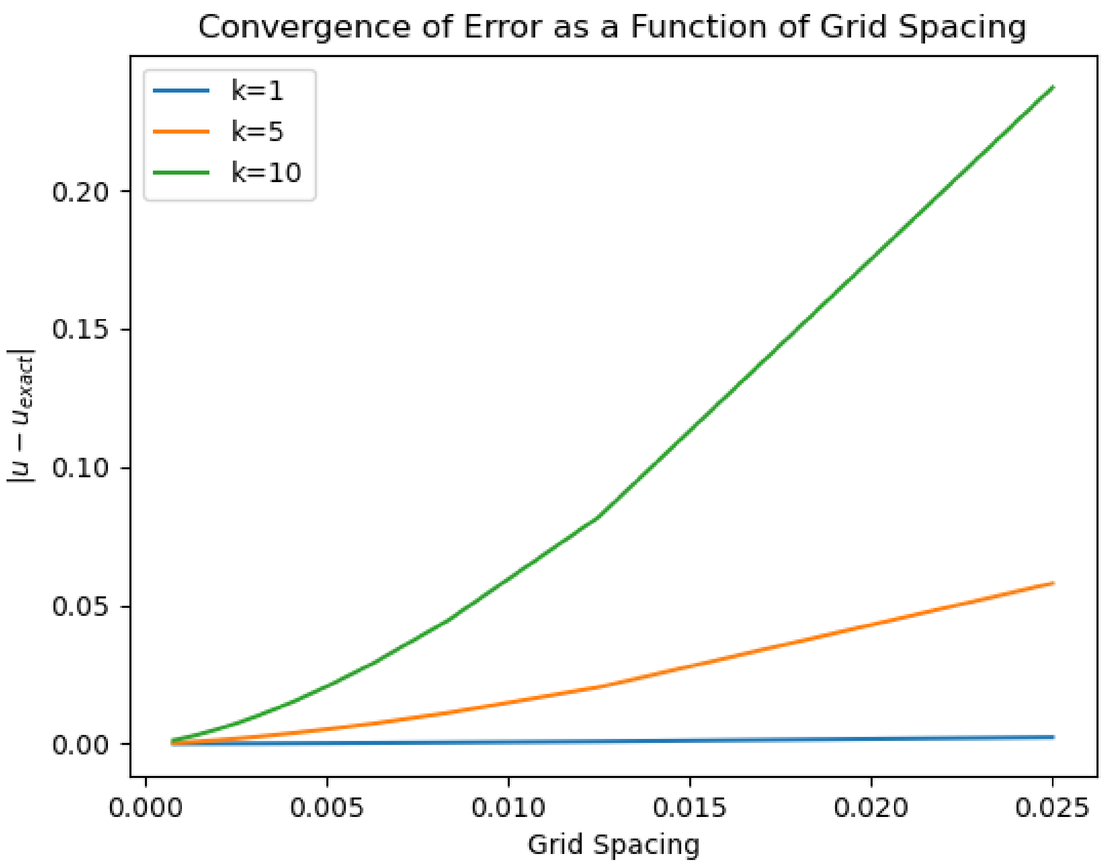

Figure 6: Convergence for Problem 4e.

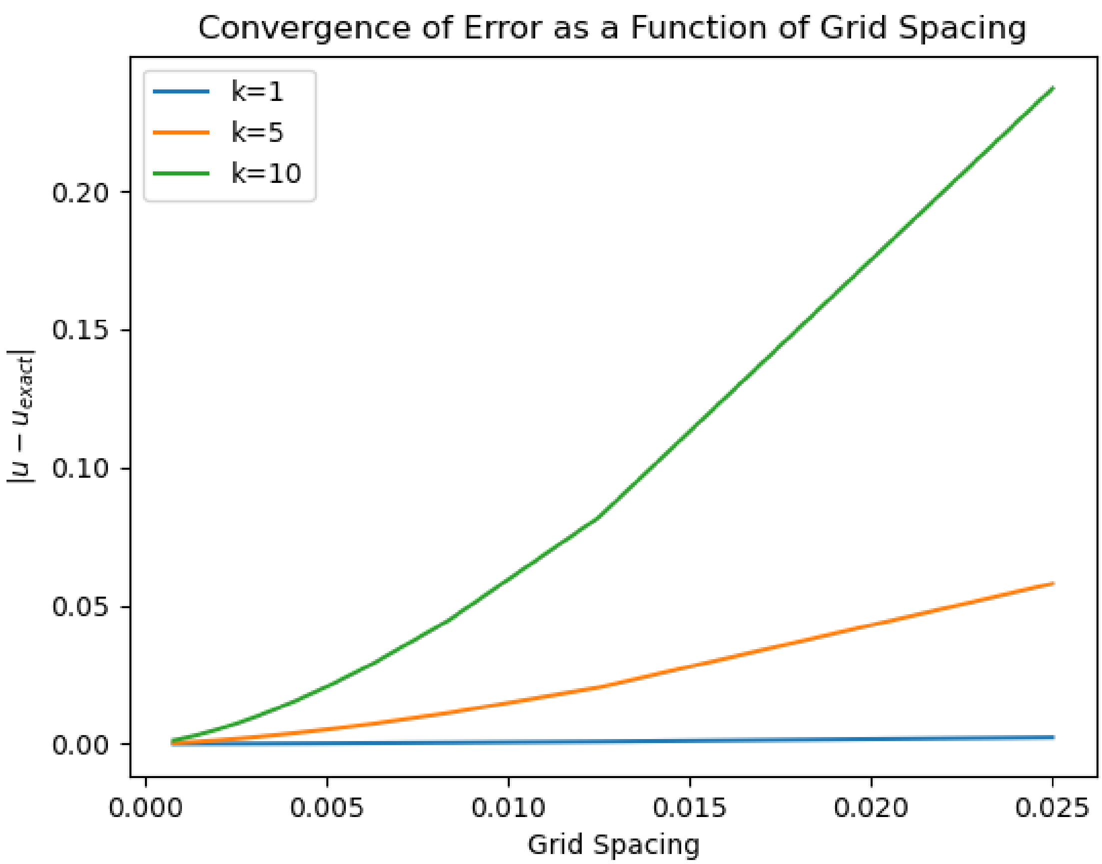

Figure 7: Convergence for Problem 4f with one processor.

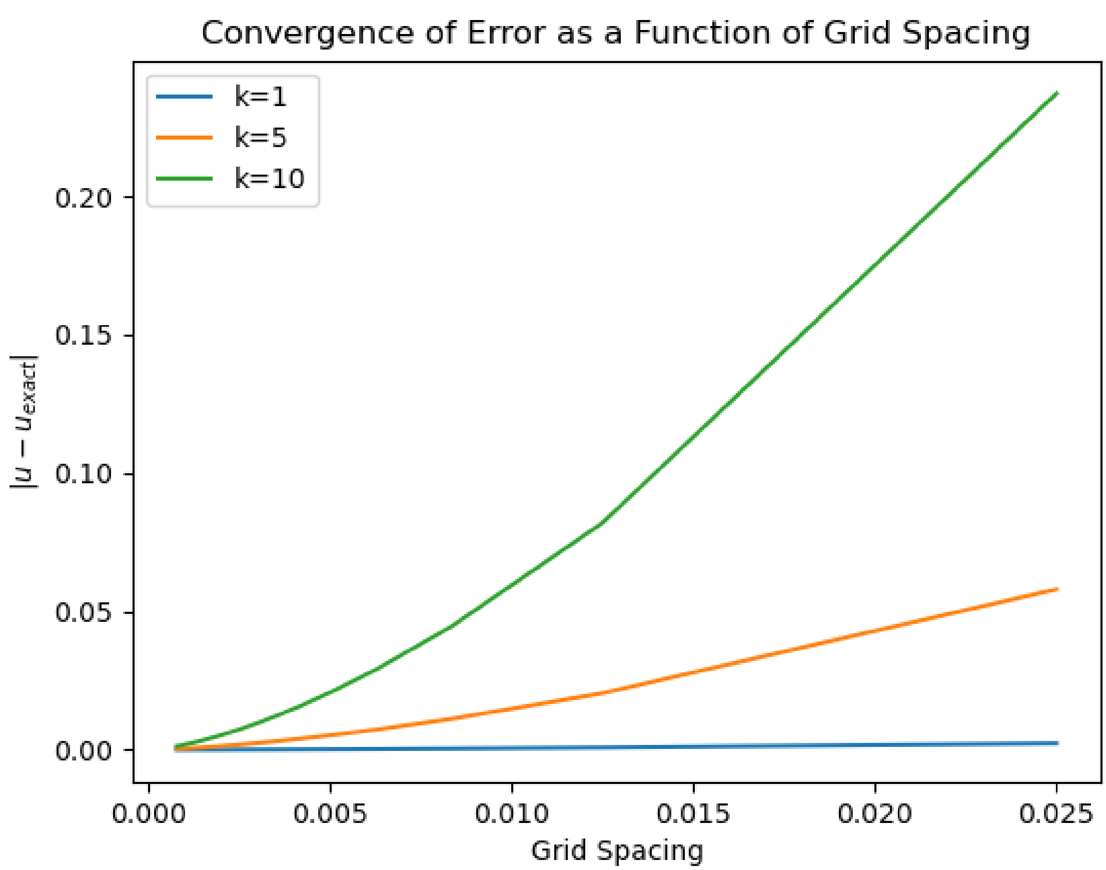

Figure 8: Convergence for Problem 4f with four processors.