CS5800 Algorithms
Fall 2021, Jameson, Hurst, Jean Paul, Komal – The Simplex Algorithm

The Simplex Algorithm

Simplex Algorithm

Simplex Algorithm is a technique in Linear Programming to solve Constraint Optimization Problem (COP). COP is a representation of a problem wherein we maximize or minimize a function (objective function) bounded by some constraints on the variables used in the objective function. The constraints in Simplex Algorithm are usually represented as inequalities and these inequalities form a bounded region in space where solution is feasible. We can plot these inequalities in a graph and find the solution of simple COP systems. However, as the complexity increases, plotting on graphs is not feasible and hence we need alternatives like the Simplex Algorithm that uses linear algebra to solve the problem.

Example Problem

Maximize $\mathbf{z = 4x_1 + 6x_2}$ where $x_1, x_2 \geq 0$
with constraints, $-x_1 + x_2 \leq 11$; $-x_1 - x_2 \geq -27$; $2x_1 + 5x_2 \leq 90$

The Algorithm

Step 1: To transform the given linear form into Standard Form

Simplex algorithm requires the constraints and the objective function to follow some standard conventions. They are as follows:

1. COP must be a maximization problem. Convert the objective function to the form $z - f(x) = 0$, where, $f(x)$ is a linear combination of the variables.
2. All the constraints must be linear and written in the form $f(x) \leq c$, where, $f(x)$ is a linear combination of the variables and c is a constant.
3. All variables must be non-negative.

If the system does not follow the above-mentioned rules, we can use linear algebra to transform the constraints and objective function to follow them.

From Example Problem: $-x_1 - x_2 \geq -27 \rightarrow x_1 + x_2 \leq 27$
$z = 4x_1 + 6x_2 \rightarrow z - 4x_1 - 6x_2 = 0$

Step 2: Determine the Slack Variable

We convert the constraint inequalities to equalities by introducing slack variables. The coefficient of slack variables will always be 1.

From Example Problem: $-x_1 + x_2 + s_1 = 11$

Step 3: Setting up the Tableau

We set up a tableau to perform operations on the problem as well as to check a solution for optimality. It consists of the coefficient corresponding to the linear constraint variables and the coefficients of the objective function.

	z	x_1	x_2	s_1	s_2	s_3	b
	1	-4	-6	0	0	0	0
s_1	0	-1	1	1	0	0	11
s_2	0	1	1	0	1	0	27
s_3	0	2	5	0	0	1	90

Step 4: Check Optimality

In Simplex Algorithm for maximization COP, the solution is said to be found when we do not have any negative values in the highlighted row of tableau which indicates the objective function.

Step 5: Identify the pivot variable
Provided the solution is not optimal, we find the pivot variable by looking at the highlighted row. We pick the smallest number and one of the values in this column will be the pivot variable. To find the indicator, divide the beta values of the constraints by their corresponding values from the column containing the possible pivot variable. The intersection of the row with the smallest non-negative indicator and the smallest negative value in the bottom row will become the pivot variable.

In our tableau, b refers to beta values, the column in red indicates the column containing the possible pivot variable, the row in blue indicates the smallest non-negative indicator row and the value is green is the pivot.

	z	x_1	x_2	s_1	s_2	s_3	b	Indicator
	1	-4	-6	0	0	0	0	
s_1	0	-1	1	1	0	0	11	11/1 = 11
s_2	0	1	1	0	1	0	27	27/1 = 27
s_3	0	2	5	0	0	1	90	90/5 = 18

Step 6: Create the new Tableau
Follow the below steps to create a new tableau:
1. Replace the indicator row variable with the pivot column variable.
2. Divide the indicator row by pivot
3. Filling up the remaining table

New tableau value = (Negative value in old tableau pivot column) x (value in new tableau pivot row) + (Old tableau value)

	z	x_1	x_2	s_1	s_2	s_3	b
	1	-10	0	6	0	0	66
x_2	0	-1	1	1	0	0	11
s_2	0	2	0	-1	1	0	16
s_3	0	7	0	-5	0	1	35

Repeat steps 4,5 and 6, till we find the optimal solution.

Step 7: Identify Optimal Values
Once the condition for optimality is satisfied, we can find the optimal values using basic variables. A basic variable is one with a single value of 1 in its column and the rest all should be zeros. The value in beta column against these basic variables is the optimal values for the variables.

	z	x_1	x_2	s_1	s_2	s_3	b
	1	0	0	0	8/3	2/3	132
x_2	0	0	1	0	-2/3	1/3	12
s_1	0	0	0	1	7/3	-2/3	14
x_1	0	1	0	0	5/3	-1/3	15

From table, $x_1 = 15$, $x_2 = 12$, $s_1 = 14$, $s_2 = 0$, $s_3 = 0$, $z = 132$

Applications
Applications for Simplex Algorithms range across multiple domains as this is primarily used for solving Constraint Optimization Problems. Some of the applications are as follows:
1. Maximization of profit given the limited sources of resources like time, equipment, etc.
2. Timetable scheduling to achieve optimal usage of faculty's time and school resources.
3. Job scheduling to achieve minimum wait time.