Evaluation with environments

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1 Motivation

Evaluation with substitution is not efficient because it forces the re-evaluation of the substituted expression every time it is encountered. A more efficient involves an environment model, where variable values are evaluated and stored in an environment when bound and looked-up when encountered.

(Is evaluation with substitution considered normal order evaluation? This seems similar to normal/applicative order evaluation described in SICP 1.1.5.)

2 Overview

The irreducible judgment (for internal expressions) in Hazel is not $d\ \mathsf{val}$, but rather $E \vdash d\ \mathsf{final}$. Thus, final expressions evaluate to themselves. Variables evaluate to the final value that they are bound to (assuming they are bound; otherwise they are free and thus final). Lambdas evaluate to a closure type. The evaluation of a let-expression or function application extends the current environment with the newly-bound variable. For function applications, the current environment is first extended with the closure environment before binding the new variable. When extending an environment ($E :: E'$ or $E, x \leftarrow d$), bindings on the right overwrite bindings on the left. The following metatheorem states that environments only include final terms.

Theorem 1 *If the variable binding $x \leftarrow d$ exists in E, then d final.*

This can be proved by induction on an empty environment by observing that all terms added to an environment must be final.

Big-step semantics

The judgment rules for evaluating variables, lambdas (which evaluate to closures), function application, `let`-expressions (very similar to function application), and a sample binary operator are shown.

$\boxed{E \vdash d \Downarrow d'}$ Internal expression d evaluates to d' given environment E

$$\frac{E \vdash d \ \mathsf{final}}{E \vdash d \Downarrow d}\ \text{EvalB-Final} \qquad \frac{}{E, x \leftarrow d \vdash x \Downarrow d}\ \text{EvalB-Var}$$

$$\frac{}{E \vdash (\lambda x : \tau . d) \Downarrow [E](\lambda x : \tau . d)}\ \text{EvalB-Lam}$$

$$\frac{E \vdash d_2 \Downarrow d_2' \qquad E :: E', x \leftarrow d_2' \vdash d_1 \Downarrow d}{E \vdash [E'](\lambda x : \tau . d_1)(d_2) \Downarrow d}\ \text{EvalB-Ap}$$

$$\frac{E \vdash d_2 \Downarrow d_2' \qquad E, x \leftarrow d_2' \vdash d_1 \Downarrow d}{E \vdash \texttt{let}\ x = d_2\ \texttt{in}\ d_1 \Downarrow d}\ \text{EvalB-Let}$$

$$\frac{E \vdash d_1 \Downarrow d_1' \qquad E \vdash d_2 \Downarrow d_2'}{E \vdash d_1 + d_2 \Downarrow d_1' + d_2'}\ \text{EvalB-Op}$$

Small-step semantics

The small-step evaluation judgments equivalent to the above big-step judgments are shown below.

$\boxed{E \vdash d \to d'}$ Internal expression d takes an instruction transition to d' given environment E

$$\frac{}{E, x \leftarrow d \vdash x \to d}\ \text{EvalS-Var} \qquad \frac{}{E \vdash (\lambda x : \tau.d) \to ([E]\lambda x : \tau.d)}\ \text{EvalS-Lam}$$

$$\frac{E \vdash d_1 \to d_1'}{E \vdash d_1(d_2) \to d_1'(d_2)}\ \text{EvalS-Ap}_1 \qquad \frac{E \vdash d_2 \to d_2'}{E \vdash d_1(d_2) \to d_1(d_2')}\ \text{EvalS-Ap}_2$$

$$\frac{E \vdash d_2\ \mathsf{final} \qquad E :: E' :: x \leftarrow d_2 \vdash d_1 \to d_1'}{E \vdash ([E']\lambda x : \tau.d_1)(d_2) \to ([E']\lambda x : \tau.d_1')(d_2)}\ \text{EvalS-Ap}_3$$

$$\frac{E \vdash d_2\ \mathsf{final} \qquad E :: E' :: x \leftarrow d_2 \vdash d_1\ \mathsf{final}}{E \vdash ([E']\lambda x : \tau.d_1)(d_2) \to d_1}\ \text{EvalS-Ap}_4$$

$$\frac{E \vdash d_2 \to d_2'}{E \vdash \texttt{let}\ x = d_2\ \texttt{in}\ d_1 \to \texttt{let}\ x = d_2'\ \texttt{in}\ d_1}\ \text{EvalS-Let}_1$$

$$\frac{E \vdash d_2\ \mathsf{final} \qquad E, x \leftarrow d_2 \vdash d_1 \to d_1'}{E \vdash \texttt{let}\ x = d_2\ \texttt{in}\ d_1 \to \texttt{let}\ x = d_2\ \texttt{in}\ d_1'}\ \text{EvalS-Let}_2$$

$$\frac{E \vdash d_2\ \mathsf{final} \qquad E, x \leftarrow d_2 \vdash d_1\ \mathsf{final}}{E \vdash \texttt{let}\ x = d_2\ \texttt{in}\ d_1 \to d_1}\ \text{EvalS-Let}_3$$

$$\frac{E \vdash d_1 \to d_1'}{E \vdash d_1 + d_2 \to d_1' + d_2}\ \text{EvalS-Op}_1 \qquad \frac{E \vdash d_2 \to d_2'}{E \vdash d_1 + d_2 \to d_1 + d_2'}\ \text{EvalS-Op}_2$$

$$\frac{}{E \vdash \underline{n_1} + \underline{n_2} \to \underline{n_1 + n_2}}\ \text{EvalS-Op}_3$$