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PSET 5.

S59 # 1, 2, 4

1.) Suppose that $f(z)$ is entire and that the harmonic function $u(x,y) = \text{Re}[f(z)]$ has an upper bound u_0. Show that $u(x,y)$ must be constant through the plane.

If f is entire, then $g(z) = e^{f(z)}$ is entire (composition of entire functions is entire). If $v = \text{Im}[f(z)]$, then

$$g(z) = e^{u(x,y)} e^{iv(x,y)}$$

and $|g(z)| = e^{u(x,y)} \leq e^{u_0}$

(e^x is monotonically increasing and x has an upper bound)

By Liouville's thm., since $g(z)$ is analytic everywhere and $|g(z)|$ is bounded, then $g(z)$ is constant

$\Rightarrow |g(z)|$ is constant

$\Rightarrow e^{u(x,y)}$ is constant

$\Rightarrow u(x,y)$ is constant (since e^x is 1-to-1).

2.) Let a function f be continuous on a closed bounded region R, and let it be analytic and not constant through the interior of R. Assuming that $f(z) \neq 0$ anywhere in R, prove that $|f(z)|$ has a minimum value m in R which occurs on the boundary of R and never in the interior.

Let $g(z) = \frac{1}{f(z)}$ in the domain $D = R \setminus \partial R$ (interior of R). Since $f \neq 0$ in D and is analytic, g is analytic everywhere in D. Since f is not constant in D, g is not constant in D, and by Liouville's Thm, g has no maximum value in D.

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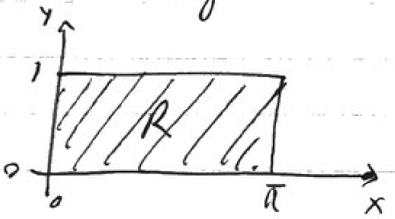
2, cont'd) This means that $\nexists z_0 \in D$ s.t. $|g(z)| \le |g(z_0)| \quad \forall z \in D$

$$\Rightarrow \nexists z_0 \in D \text{ s.t. } \left|\frac{1}{f(z)}\right| = \frac{1}{|f(z)|} \le \left|\frac{1}{f(z_0)}\right| = \frac{1}{|f(z_0)|} \quad \forall z \in D$$

$$\Rightarrow \nexists z_0 \in D \text{ s.t. } |f(z)| \ge |f(z_0)| \quad \forall z \in D$$

i.e., f has no minimum value in D. Since f is continuous on R, and R is closed and bounded, and doesn't achieve a minimum in its interior, it must achieve a minimum on the boundary of R.

4.) Let R be the region $0 \le x \le \pi$, $0 \le y \le 1$. Show that the modulus of the entire function $f(z) = \sin z$ has a maximum value in R at the boundary point $z = \frac{\pi}{2} + i$.



From S37, we know that:

$$|f(z)|^2 = |\sin z|^2 = \sin^2 x + \sinh^2 y \quad (\text{if } z = x+iy).$$

$|f(z)|$ should achieve a maximum when $|f(z)|^2$ achieves a maximum, and $|f(z)|^2$ should achieve a maximum value when $\sin^2 x$ and $\sinh^2 y$ achieve their maximums within R (since they are both positive values).

$\sin^2 x$ achieves a maximum when $|\sin x| = 1$, i.e., when $x = \frac{\pi}{2}(2n+1)$, $n \in \mathbb{Z}$. In R, it only achieves this once, at $n = \frac{1}{2}$ $(x = \frac{\pi}{2})$.

$\sinh^2 y$ achieves its maximum in R when $y = 1$, since $\sinh^2 y$ is monotonically increasing for $y > 0$.

Thus $|f(z)|$ reaches its maximum @ $x = \frac{\pi}{2}$, $y = 1$ $(z = \frac{\pi}{2} + i)$. This agrees w/ the result given by the Liouville thm.

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S65 # 2, 3, 11

2) Obtain the Taylor series:

$$e^z = e\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!} \qquad (|z-1|<\infty)$$

for $f(z) = e^z$ by:

a) Using $f^{(n)}(1)$, $n = 0, 1, 2, \ldots$

This is a taylor series centered @ $z = 1$.

$f^{(n)}(z) = e^z$, $n = 0, 1, 2, \ldots$

$f^{(n)}(1) = e$, $n = 0, 1, 2, \ldots$

using Taylor series formula:

$$f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \qquad \left(\begin{array}{l} z_0 = 1, \\ |z-z_0| < \infty \end{array}\right)$$

since f is entire.

$$= \sum_{n=0}^{\infty}\frac{e}{n!}(z-1)^n = e\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!}$$

b) writing $e^z = e^{z-1}e$.

From S64, we know that:

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!} \qquad (|z|<\infty)$$

make the substitution $z = z-1$

e^{z-1} analytic everywhere

$$e^{z-1} = \sum_{n=0}^{\infty}\frac{(z-1)^n}{n!} \qquad (|z-1|<\infty).$$

Thus $f(z) = ee^{z-1} = e\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!} \qquad (|z-1|<\infty).$

3.) Find the Maclaurin Series expansion of $f(z) = \frac{z}{z^4+4} = \frac{z}{4}\cdot\frac{1}{1+\left(\frac{z^4}{4}\right)}$.

From S64, we know that: $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad (|z|<1).$

Make the substitution: $z = -\frac{z^4}{4}$.

$$\frac{1}{1-\left(-\frac{z^4}{4}\right)} = \sum_{n=0}^{\infty}\left(-\frac{z^4}{4}\right)^n, \quad \left(\left|-\frac{z^4}{4}\right| < 1\right) \Rightarrow \left(\frac{|z|^4}{4} < 4 \Rightarrow |z| < \sqrt{2}\right)$$

$$\Rightarrow f(z) = \frac{z}{4}\cdot\frac{1}{1-\left(-\frac{z^4}{4}\right)} = \frac{z}{4}\sum_{n=0}^{\infty}\frac{(-1)^n z^{4n}}{4^n} = \sum_{n=0}^{\infty}\frac{(-1)^n z^{4n+1}}{4^{n+1}} \qquad (|z|<\sqrt{2}).$$

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11). Show that when $0<|z|<4$,

$$\frac{1}{4z-z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}$$

$$f(z) = \frac{1}{4z-z^2} = \frac{1}{4z} \cdot \underbrace{\frac{1}{1-\frac{1}{4}z}}$$

From S64, $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $(|z|<1)$

make substitution : $z = \frac{1}{4}z$

$$\frac{1}{1-\frac{1}{4}z} = \sum_{n=0}^{\infty} \left(\frac{1}{4}z\right)^n, \quad \left(\left|\frac{z}{4}\right|<1\right)$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{4^n}, \quad (|z|<4).$$

$$f(z) = \frac{1}{4z} \cdot \sum_{n=0}^{\infty} \frac{z^n}{4^n} \qquad (0<|z|<4)$$

$z \neq 0$ b/c of $\frac{1}{4z}$ term.

$$= \sum_{n=0}^{\infty} \frac{z^{n-1}}{4^{n+1}}$$

make substitution $n' = n-1$, $n = n'+1$

$$= \sum_{n'=-1}^{\infty} \frac{z^{n'}}{4^{n'+2}} = \frac{z^{-1}}{4^{-1+2}} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}$$

rename back to n

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§68 # 1, 4, 6, 7, 8, 10.

1.) Find the Laurent series that represents the function,

$f(z) = z^2 \sin\left(\frac{1}{z^2}\right)$. in the domain $0<|z|<\infty$.

From §64: $\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$ $(|z|<\infty)$.

Substituting: $z = \frac{1}{z^2}$:

$$\sin\frac{1}{z^2} = \sum_{n=0}^{\infty}(-1)^n \frac{\left(\frac{1}{z^2}\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \frac{z^{-4n-2}}{(2n+1)!} \quad \left(\begin{array}{l} z\neq 0 \text{ and} \\ \left|\frac{1}{z^2}\right|<\infty \end{array}\right)$$

thus $f(z) = z^2\sin\left(\frac{1}{z^2}\right) = \cancel{z^2}\sum_{n=0}^{\infty}\frac{(-1)^n z^{-4n}\cancel{z^{-2}}}{(2n+1)!}$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n z^{-4n}}{(2n+1)!} \quad (0<|z|<\infty)$$

4.) Give two Laurent series expansions in powers of z (i.e., centered @ 0) for the function $f(z) = \frac{1}{z^2(1-z)}$ and specify the regions in which those expansions are valid.

a) for ROC $|z|<1$, from §64:

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

$$\Rightarrow f(z) = \frac{1}{z^2}\cdot\frac{1}{1-z} = \frac{1}{z^2}\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} z^{n-2} \quad (0<|z|<1)$$

$\uparrow$ $z\neq 0$

b) $\frac{1}{z^2(1-z)} = \frac{-1}{z^3}\cdot\frac{1}{1-\frac{1}{z}}$ ← make substitution $z = \frac{1}{z}$ in expansion of $\frac{1}{1-z}$.

$$= -\frac{1}{z^3}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n, \quad (|z|\neq 0 \text{ and } \underbrace{\left|\tfrac{1}{z}\right|<1}_{\Downarrow\ \infty>|z|>1})$$

$\uparrow$ $z\neq 0$

$$= -\sum_{n=0}^{\infty} z^{-n-3}, \quad (1<|z|<\infty).$$

make substitution $n' = n+3$

$$= -\sum_{n'=3}^{\infty} z^{-n'}$$

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6.) Show that when $0 < |z-1| < 2$,

$$\frac{z}{(z-1)(z-3)} = -3\sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}.$$

(i.e., express in powers of $(z-1)$, i.e., Laurent series centered at 1).

$$\frac{z}{(z-1)(z-3)} = \frac{A}{z-1} + \frac{B}{z-3}.$$

Use heaviside cover-up: $A = -\frac{1}{2}$, $B = \frac{3}{2}$.

$$= -\frac{1}{2(z-1)} + \frac{3}{2(z-3)}$$

$$= -\frac{3}{4}\left(\frac{1}{\frac{3}{2} - \frac{z}{2}}\right) - \frac{1}{2(z-1)}$$

$$= -\frac{3}{4}\left(\underbrace{\frac{1}{1-\left(\frac{z-1}{2}\right)}}\right) - \frac{1}{2(z-1)}$$

use (Taylor) expansion for $\frac{1}{1-z}$, substituting $z = \frac{z-1}{2}$.

$$= -\frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n - \frac{1}{2(z-1)} \qquad \left(\left|\frac{z-1}{2}\right| < 1,\ |z-1| \neq 0\right)$$

$$= -\frac{3}{2^2}\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^n} - \frac{1}{2(z-1)} \qquad (0 < |z-1| < 2)$$

$$= -3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)} \qquad (0 < |z-1| < 2)$$

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§68 # 7, 8, 10.

7.) Let a denote a real number, where $-1 < a < 1$, and derive the Laurent series representation:

$$\frac{a}{z-a} = \sum_{n=1}^{\infty} \frac{a^n}{z^n} \qquad (|a| < |z| < \infty).$$

Case 1: $a = 0$

$$\frac{0}{z-a} = \sum_{n=1}^{\infty} \frac{0^n}{z^n} \quad \Rightarrow \quad 0 = 0 \ \checkmark \quad (0 < |z| < \infty)$$

doesn't work at $z=0$ b/c results in division by 0.

Case 2: $a \neq 0$:

$$\frac{a}{z-a} = \frac{a}{z} \cdot \underbrace{\frac{1}{1-\frac{a}{z}}}$$

↳ use Taylor series representation from §64 for $\frac{1}{1-z}$, substituting $z = \frac{a}{z}$

$$= \frac{a}{z} \cdot \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n \qquad \left(\left|\frac{a}{z}\right| < 1\right)$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^{n+1} \qquad (|a| < |z| < \infty)$$

Substitute $n' = n+1$

$$= \sum_{n'=1}^{\infty} \left(\frac{a}{z}\right)^{n'} \qquad (|a| < |z| < \infty)$$

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8.) Suppose that a series $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} c_{-n} z^{-n}$

converges to an analytic function $X(z)$ in some annulus, $R_1 < |z| < R_2$. That sum $X(z)$ is called the z-transform of $x[n]$ $(n \in \mathbb{Z})$. Show that if the annulus contains the unit circle $|z| = 1$, then the inverse z-transform of $X(z)$ can be written:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{i\theta}) e^{in\theta} \, d\theta, \quad n \in \mathbb{Z}.$$

In general, an analytic fn. in some annulus can be expressed as $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$, where $c_n = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}$

In this case, since $x[n]$ is the coefficient of a negative power of z, $x[n] = c_{-n}$.

By hypothesis, the domain is an annulus centered at the origin containing the unit circle. Thus let $C: z(\theta) = e^{i\theta}$ $(-\pi \le \theta \le \pi)$,

$$z'(\theta) = ie^{i\theta},$$
$$z_0 = 0.$$

Thus $x[n] = c_{-n} = \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{X(e^{i\theta})}{(e^{i\theta}-0)^{-n+1}} ie^{i\theta} \, d\theta$

$$= \frac{\cancel{i}}{2\pi \cancel{i}}\int_{-\pi}^{\pi} \frac{X(e^{i\theta})}{e^{-ni\theta}} \cdot \frac{\cancel{e^{i\theta}}}{\cancel{e^{i\theta}}} \, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{i\theta}) e^{ni\theta} \, d\theta.$$

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568 #10.

(10) a) Let $f(z)$ denote a function which is analytic in some annular domain about the origin that includes the unit circle $z = e^{i\phi}$ $(-\pi \le \phi \le \pi)$. Show that

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})\,d\phi + \frac{1}{2\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(e^{i\phi})\left[\left(\frac{z}{e^{i\phi}}\right)^n + \left(\frac{e^{i\phi}}{z}\right)^n\right]d\phi$$

where z is any point in the annular domain.

By hyp, the annular domain (centered @ the origin) contains the unit circle, so we can use this as our path of integration.

$$C: \; z(\phi) = e^{i\phi} \qquad -\pi \le \phi \le \pi$$
$$z'(\phi) = ie^{i\phi}$$
$$z_0 = 0.$$

Using the formulation for a Laurent series:

$$a_n = \frac{1}{2\pi \cancel{i}}\int_{-\pi}^{\pi} \frac{f(e^{i\phi})}{(e^{i\phi})^{n+\cancel{1}}}\cancel{ie^{i\phi}}\,d\phi = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(e^{i\phi})}{e^{in\phi}}\,d\phi,$$

$$b_n = \frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{f(e^{i\phi})}{(e^{i\phi})^{-n+\cancel{1}}}\cancel{ie^{i\phi}}\,d\phi = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})\cdot e^{in\phi}\,d\phi$$

$$f(z) = \sum_{n=0}^{\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{f(e^{i\phi})}{e^{in\phi}}\right)(z-0)^n d\phi + \sum_{n=1}^{\infty}\frac{f(e^{i\phi})\,e^{in\phi}}{(z-0)^n}\,d\phi$$

take out 0th term

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})\cancel{\left(\frac{z}{e^{in}}\right)^0}d\phi + \sum_{n=1}^{\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})\left(\frac{z}{e^{i\phi}}\right)^n d\phi$$
$$+ \sum_{n=1}^{\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})\left(\frac{e^{i\phi}}{z}\right)^n d\phi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})\,d\phi + \frac{1}{2\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(e^{i\phi})\left[\left(\frac{z}{e^{i\phi}}\right)^n + \left(\frac{e^{i\phi}}{z}\right)^n\right]d\phi.$$

10b). Write $u(\theta) = Re[f(e^{i\theta})]$ and show that

$$u(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\phi)\,d\phi + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} u(\phi)\cos(n(\theta-\phi))\,d\phi.$$

(thus ~~show~~ deriving one form of the Fourier series expansion of a real-valued fn. $u(\theta)$ on the interval $-\pi \leq \theta \leq \pi$.)

$u(\theta) = Re[f(e^{i\theta})]$

$$= Re\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})\,d\phi\right] + Re\left[\frac{1}{2\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(e^{i\phi})\left[\left(\frac{e^{i\theta}}{e^{i\phi}}\right)^n + \left(\frac{e^{i\phi}}{e^{i\theta}}\right)^n\right]d\phi\right]$$

From S42, we can bring Re inside of integral

(i.e, $Re\left[\int w(t)\,dt\right] = \int Re[w(t)]\,dt$,).

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} Re[f(e^{i\theta})]\,d\phi + \frac{1}{2\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} Re\left[f(e^{i\phi})\underbrace{\left[e^{in(\theta-\phi)} + e^{in(\phi-\theta)}\right]}_{=2\cos(n(\theta-\phi))}\right]d\phi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\phi)\,d\phi + \frac{2}{2\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} Re\left[\underbrace{f(e^{i\phi})}_{\text{complex}}\underbrace{\cos(n(\theta-\phi))}_{\text{real}}\right]d\phi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\phi)\,d\phi + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} u(\phi)\cos(n(\theta-\phi))\,d\phi.$$