ECE302 – HW6

Jonathan Lam

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Suppose in a binary communication system messages $X = 0$ and $X = 1$ occur with priori probabilities 0.25 and 0.75, respectively. Suppose we make observation $R = X + N$, where N is a continuous-valued r.v. with a uniform p.d.f.:

$$p_N(n) = \begin{cases} \frac{2}{3}, & -\frac{3}{4} < n < \frac{3}{4} \\ 0, & \text{else} \end{cases}$$

1. Define and plot the conditional p.d.f.s for H_0 and H_1.

   Let H_0 be the hypothesis that $X = 0$, and H_1 be the hypothesis that $X = 1$.

$$p_0(r) = p_R(r \mid H = H_0) = p_N(n)$$
$$p_1(r) = p_R(r \mid H = H_1) = \begin{cases} \frac{2}{3}, & \frac{1}{4} < r < \frac{7}{4} \\ 0, & \text{else} \end{cases}$$

2. Find the likelihood ratio test for the minimum probability of error.

   The MAP rule chooses H_0 if:

$$p_{R|H}(r \mid H_1)P_1 < p_{R|H}(r \mid H_0)P_0$$
$$P(r \mid H_1)(0.75) < P(r \mid H_0)(0.25)$$

   Since $P(r \mid H_1)$ and $P(r \mid H_0)$ are both equal to the constant value $\frac{2}{3}$ in their respective domains, then H_1 will always be chosen in its domain, and H_0 will only be chosen otherwise. I.e., the decision boundary is $\eta = r = \frac{1}{4}$:

$$\hat{H}(r) = \begin{cases} H_0, & -\frac{3}{4} < r < \frac{1}{4} \\ H_1, & \frac{1}{4} < r < \frac{7}{4} \end{cases}$$

3. Compute the corresponding probability of error.

$$
\begin{aligned}
P_{err} &= P(\hat{H} = 1 \cap H_0) + P(\hat{H} = 0 \cap H_1) \\
&= \left[\int_{\frac{1}{4}}^{\frac{7}{4}} P(r \mid H_0)\, dr\right] \left(\frac{1}{4}\right) + \left[\int_{-\frac{3}{4}}^{\frac{1}{4}} P(r \mid H_1)\, dr\right] \left(\frac{3}{4}\right) \\
&= \left[\int_{\frac{1}{4}}^{\frac{3}{4}} \frac{2}{3}\, dr\right] \left(\frac{1}{4}\right) + \left[\int_{\frac{1}{4}}^{\frac{1}{4}} \frac{2}{3}\, dr\right] \left(\frac{3}{4}\right) \\
&= \left[\frac{1}{2} \cdot \frac{2}{3}\right] \left(\frac{1}{4}\right) + 0 \left(\frac{3}{4}\right) \\
&= \frac{1}{12}
\end{aligned}
$$

4. Plot the receiver operating curve.

   The ROC is the locus of points $(P_F(\eta), P_D(\eta))$, where $P_F(\eta)$ is the conditionally probability of false alarm (Type I error), and $P_D(\eta)$ is the conditional probability of detection; both require the decision boundary.

$$
\begin{aligned}
P_F(\eta) = P(\hat{H} = 1 \mid H_0) = \int_{\eta}^{\infty} P(r \mid H_0)\, dr \\
P_D(\eta) = P(\hat{H} = 1 \mid H_1) = \int_{\eta}^{\infty} P(r \mid H_1)\, dr
\end{aligned}
$$

   The decision points of interest are $\eta = \{-\frac{3}{4}, \frac{1}{4}, \frac{3}{4}, \frac{7}{4}\}$. Since the distributions are uniform, we can expect that there will be a linear interpolation between these points on the ROC. Using the formulas above, we get the curve defined by:

$$
(1, 1), \left(\frac{1}{3}, 1\right), \left(0, \frac{2}{3}\right), (0, 0)
$$