MA347 – HW8

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Let $n \in \mathbb{N}$ and $\mathbb{K} = \mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$. For $A \in \mathrm{GL}(n, \mathbb{K})$, define $f_{A,b} : \mathbb{K}^n \to \mathbb{K}^n$ by $f_{A,b}(x) = Ax + b$. Then $\mathrm{Aff}(n, \mathbb{K}) = \{f_{A,b} : A \in \mathrm{GL}(n, \mathbb{K}), b \in \mathbb{K}^n\}$ is a group under composition.

1. Define $\varphi : \mathrm{GL}(n, \mathbb{K}) \to \mathrm{Aff}(n, \mathbb{K})$ by $\varphi(A) = f_{A,0}$, where 0 is the zero element of $\mathbb{K}^n$. Prove that φ is a group homomorphism. Find $\operatorname{Ker} \varphi$.

   Proof of Homomorphism. We already know that $\mathrm{GL}(n, \mathbb{K})$ and $\mathrm{Aff}(n, \mathbb{K})$ are groups, so all that is left to show is that the mapping φ satisfies the equality:

   $$\varphi(AB) = \varphi(A) \circ \varphi(B) \quad \forall A, B \in \mathrm{GL}(n, \mathbb{K})$$

   Let $A, B \in \mathrm{GL}(n, \mathbb{K})$. Then, $\forall x \in \mathbb{K}^n$:

   $$\begin{aligned}
   (\varphi(AB))(x) &= ABx + 0 && (\text{def. } \varphi, f_{AB,0}) \\
   &= A(Bx) + 0 && (\text{associativity of group } \mathrm{GL}(n, \mathbb{K})) \\
   &= f_{A,0}(Bx) && (\text{def. } f_{A,0}, \varphi) \\
   &= (\varphi(A))(Bx + 0) && (0 \text{ is identity of } \mathbb{K}^n) \\
   &= (\varphi(A))((\varphi(B))(x)) && (\text{def. } f_{B,0}, \varphi) \\
   &= (\varphi(A) \circ \varphi(B))(x) && (\text{def. } \circ)
   \end{aligned}$$

   Since the result of $\varphi(AB)$ and $\varphi(A) \circ \varphi(B)$ match for all values of $x \in \mathbb{K}^n$, then $\varphi(AB) = \varphi(A) \circ \varphi(B)$. $\square$

   The identity element of $\mathrm{Aff}(n, \mathbb{K}^n)$ is $e' = f_{I_n,0}$ (shown in HW5), so $\operatorname{Ker} \varphi = \varphi^{-1}(e') = \{I_n\}$ by inspection.

2. Define $\psi : \mathbb{K}^n \to \mathrm{Aff}(n, \mathbb{K})$ by $\psi(b) = f_{I_n,b}$, where I_n is the identity element of $\mathrm{GL}(n, \mathbb{K})$. Prove that ψ is a group homomorphism. Determine $\mathrm{Ker}\,\psi$.

   Proof of homomorphism. The method is the same as in the earlier problem, except that $\mathbb{K}^n$ is an additive group so we want the result (same homomorphism idea but denoted differently):

$$\psi(a + b) = \psi(a) \circ \psi(b)$$

   Let $a, b \in \mathbb{K}^n$. Then $\forall x \in \mathbb{K}^n$:

$$\begin{aligned}
(\psi(a + b))(x) &= I_n x + (a + b) && \text{(def. } \psi, f_{I_n,a+b}\text{)} \\
&= I_n x + (b + a) && \text{(commutativity of group } \mathbb{K}^n\text{)} \\
&= (I_n x + b) + a && \text{(associativity of group } \mathbb{K}^n\text{)} \\
&= ((\psi(b))(x)) + a && \text{(def. } f_{I_n,b}, \psi\text{)} \\
&= I_n((\psi(b))(x)) + a && (I_n \text{ is identity of } K^n) \\
&= (\psi(a))((\psi(b))(x)) && \text{(def. } f_{I_n,a}, \psi\text{)} \\
&= (\psi(a) \circ \psi(b))(x) && \text{(def. } \circ\text{)}
\end{aligned}$$

   Since the result of $\psi(a+b)$ and $\varphi(a) \circ \varphi(b)$ match for all values of $x \in \mathbb{K}^n$, then $\varphi(a + b) = \varphi(a) \circ \varphi(b)$. □

   Again, the identity element of the codomain is $e' = f_{I_n,0}$, so $\mathrm{Ker}\,\psi = \psi^{-1}(e') = \{0\}$ by inspection.