Halliday, Resnick, Walker - Chapter 15

Hooke's Law:

$$F = -kx \tag{1}$$

Newton's Second Law:

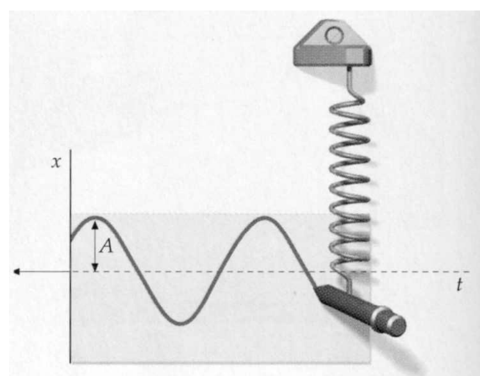
$$\Sigma \vec{F} = m\vec{a} \tag{2}$$

"A linear restoring force"

$$\begin{aligned}
F &= -kx && (3)\\
-kx &= ma && (4)\\
-kx &= m\left(\frac{d^2x(t)}{dt^2}\right) && (5)\\
-kx &= m\ddot{x} && (6)\\
-kx(t) &= m\ddot{x}(t) \leftarrow \text{"ordinary differential equation"} && (7)
\end{aligned}$$

x is an *unknown* function of t, but we know that the second derivative of it gives you back the function

"Test" solution: $x(t) = A\cos(\omega t + \delta)$
where ...
$\omega = \frac{2\pi}{T} = 2\pi f$



Try it as a solution...

$$x(t) = A\cos(\omega t + \delta) \tag{8}$$

$$\ddot{x}(t) = -A\omega^2\cos(\omega t + \delta) \tag{9}$$

$$-kx = m\ddot{x} \tag{10}$$
$$-kAcos(\omega t + \delta) = -Am\omega^2 \cos(\omega t + \delta) \tag{11}$$

Works if $k = m\omega^2$ since everything else cancels ...

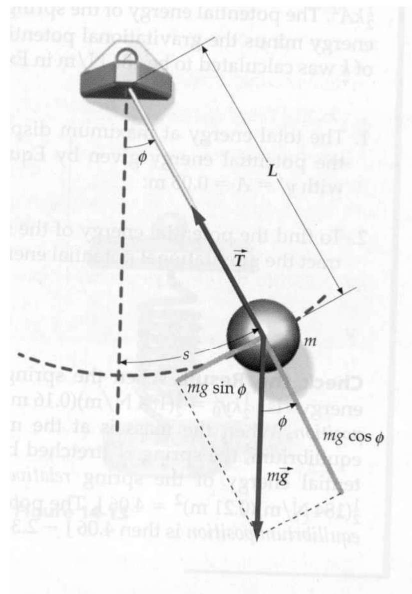
$$\omega = \sqrt{\frac{k}{m}} \equiv \omega_0 \tag{12}$$

where ω_0 is the "natural frequency".

<u>Note that the period is independent of the amplitude.</u>

A similar problem...

The Pendulum



$$\Sigma\tau = I\alpha(\text{Newton's 2nd Law}) \tag{13}$$

$$\alpha \equiv \ddot{\theta} \tag{14}$$

$$I = ml^2(\text{for a point mass}) \tag{15}$$

$$|\vec{\tau}| = |\vec{r} \times \vec{F}| = (mg\sin\theta)l \tag{16}$$

$\vec{\tau}$ is a "restoring" torque - it tends to pull the mass back to the center (equilibrium), smaller θ.

$$-mgl\sin\theta = (ml^2)\ddot{\theta} \tag{17}$$

Cancel...

$$\ddot{\theta} = \frac{-g}{l}\sin\theta \longleftarrow \text{Too hard to solve} \tag{18}$$

Limit our study to small θ where $\sin\theta \approx \theta$ (for θ less than about 10°, but must use radians!)

$$\ddot{\theta} = \frac{-g}{l}\theta \tag{19}$$

Which we've seen before with different variable names

So try...

$$\theta = \theta_0\cos(\omega t + \delta) \tag{20}$$

It works if...

$$-\theta_0\omega^2\cos(\omega t + \delta) = \frac{-g}{l}\theta_0\cos(\omega t + \delta), \qquad \omega = \sqrt{\frac{g}{l}} \tag{21}$$

That is. . .

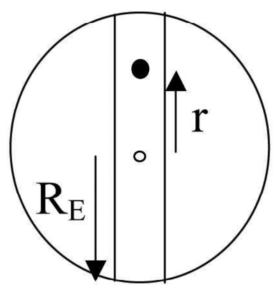
$$\theta = \theta_0 \cos(\sqrt{\frac{g}{l}}t + \delta) \tag{22}$$

where. . . θ_0 = starting amplitude, δ = starting phase

But note this is only approximately true, since we assumed $\sin\theta \approx \theta$

Another oscillation problem. . .

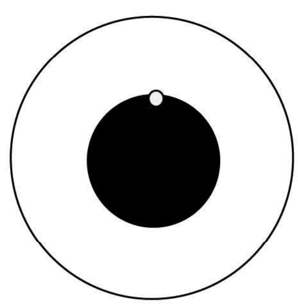
A hole drilled through the center of the earth.



Universal Law of Gravitation

$$F_{gravity} = \frac{GmM_E}{r^2} \tag{23}$$

But. . . If m is inside the Earth, only the part of the Earth closer to the center than him exerts a net force on him. (Claim for now, proof later)



Only shaded part of earth exerts net force.

So... not full "M_E" that pulls, only $\frac{4}{3}\pi r^3$ of the volume.

Mass of that piece = $\rho \cdot Vol = \frac{4}{3}\pi\rho r^3$

$$\Sigma\vec{F} = m\vec{a} \tag{24}$$

$$\frac{-G(\frac{4}{3}\pi\rho r^3)m}{r^2} = m\ddot{r} \tag{25}$$

Note the powers of r and note that the negative sign tends to pull towards the center (a "restoring" force).

$$-Cr = \ddot{r} \longleftarrow \text{Will result in SHM yet again} \tag{26}$$

$$C = G\frac{4}{3}\pi\rho \text{ (or can express in terms of } R_E,\, M_E) \tag{27}$$

Finish this up - solve for ω

Approximations We Will Use Often

For small $x\ldots$

$$\sqrt{1+x} \approx 1+\frac{x}{2} \tag{28}$$

$$\frac{1}{1+x} \approx 1-x \tag{29}$$

$$e^x \approx 1+x \tag{30}$$

(implying $\ln(1+x) \approx x$)

$$\sin(x) \approx x \tag{31}$$

(or $x-\frac{x^3}{6}$)

$$\cos(x) \approx 1-\frac{x^2}{2} \tag{32}$$

$$(1+x)^n \approx 1+nx \tag{33}$$

<u>Exercise:</u> Show that for $a << x$, the expression $\frac{1}{(x^2+a^2)^{\frac{5}{2}}} - \frac{1}{(x^2-a^2)^{\frac{5}{2}}}$ reduces to $-\frac{5a^2}{x^7}$

Add "Damping"

$$\Sigma \vec{F} = m\vec{a} \tag{34}$$
$$-kx - b\dot{x} = m\ddot{x} \tag{35}$$

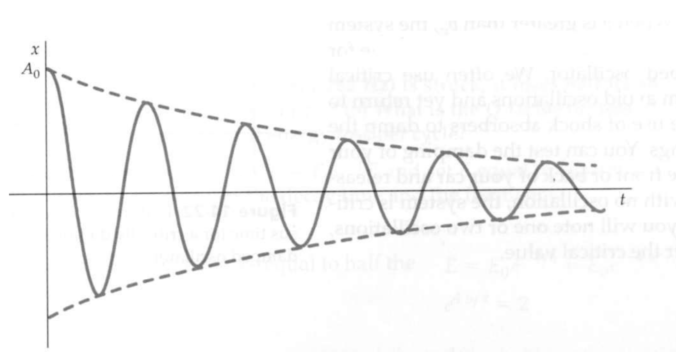
Try...

$$x(t) = Ae^{ct} \tag{36}$$
$$\dot{x} = Ace^{ct} \tag{37}$$
$$\ddot{x} = Ac^2e^{ct} \tag{38}$$

$$-kAe^{ct} - bAce^{ct} = mAc^2e^{ct} \tag{39}$$
$$-k - bc = mc^2 \tag{40}$$
$$mc^2 + bc + k = 0 \tag{41}$$

Quadratic in $c\ldots$

$$c = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} \tag{42}$$
$$x(t) = Ae^{\frac{-b}{2m}t}e^{\frac{\pm\sqrt{b^2-4mk}}{2m}t} \tag{43}$$



Note that $Ae^{\frac{-b}{2m}t}$ may be considered a decaying amplitude and that the term $b^2 - 4mk$ within $e^{\frac{\sqrt{b^2-4mk}}{2m}t}$ could be negative.

$$\begin{aligned}
&\text{Euler equation } \ldots && (44)\\
e^{i\theta} &= \cos\theta + i\sin\theta, i \equiv \sqrt{-1} && (45)\\
e^{i\omega t} &= \cos(\omega t) + i\sin\omega t \text{ (oscillation)} && (46)\\
&e^{at} \text{ (exponential growth/decay)} && (47)\\
&\text{so}\ldots\\
x(t) &= Ae^{\frac{-b}{2m}t}e^{i\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}t} && (48)\\
&\approx Ae^{\frac{-b}{2m}t}\cos(\omega t) && (49)
\end{aligned}$$

where...
$\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \approx \sqrt{\frac{k}{m}}$ is often a good approximation.

Add "Forcing" ("Driving")

$$-kx - b\dot{x} + F_0\cos(\omega t) = m\ddot{x} \tag{50}$$

We still get a solution like...

$$e^{-ct}\cos(\omega t) \text{ where } \omega \approx \omega_0 \tag{51}$$

This gradually goes away ("transient solution"), but we also get a solution like ...
$A\cos(\omega t - \delta)$ that persists ("steady state solution")
In this part of the solution, $\omega = \omega_d$ **NOT** ω_0.

VERY IMPORTANT!!
The driving frequency eventually wins out over the natural frequency.

$$dw \equiv \vec{F} \cdot d\vec{x} \tag{52}$$

$$P = \frac{dw}{dt} = \vec{F} \cdot \frac{d\vec{x}}{dt} \tag{53}$$

For *resonance* we want $P(t) > 0$ all the time.
$P(t)$ is positive if $\vec{F}(t)$, a cosine function is in phase with $\vec{v}(t)$, that is, also a cosine function.

$$v(t) = \cos(\omega t) \tag{54}$$

$$x(t) \propto \sin(\omega t) \tag{55}$$

$\Rightarrow$ At resonance $F(t)$ and $x(t)$ are 90° out of phase.

The steady-state solution (which does not depend on the initial conditions) can be written as

$$x(t) = A\cos(\omega t - \delta) \tag{56}$$

where the angular frequency ω is the same as that of the driving force and the amplitude A and phase constant δ are given by

$$A = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + b^2\omega^2}}, \tan\delta = \frac{b\omega}{m(\omega_0^2 - \omega^2)} \tag{57}$$

Comparing the o.d.e. and $x(t)$ we see that the *displacement* and the *driving force* oscillate with the same frequency but differ in phase by δ. The negative sign in the phase of $x(t)$is introduced so that the phase constant δ is positive. When the driving frequency ω is much less than the natural frequency ω_0, $\delta \approx 0$ – as can be seen from the $\tan(\delta)$ equation.

At resonance, $\delta = \pi/2$, and when ω is much greater than ω_0, $\delta = \pi$. At resonance, $\omega = \omega_0$, and the amplitude A is maximum.

The velocity of the object in the steady state is obtained by differentiating the $x(t)$ equation with respect to time:

$$v = \frac{dx}{dt} = -A\omega \sin(\omega t - \delta) \tag{58}$$

At resonance, the velocity is in phase with the driving force.

Damped Driven Oscillators (the details)

$$\Sigma F = -kx - bv + F_0 \cos(\omega t) = m\frac{dv}{dt} \tag{59}$$

$$\omega_0^2 = \frac{k}{m} \Rightarrow k = m\omega_0^2 \tag{60}$$

$$m\frac{d^2x}{dt} + b\frac{dx}{dt} + m\omega_0^2 x = F_0 \cos(\omega t) \tag{61}$$

Transient Solution (not derived here):

$$x = A_0 e^{-\frac{bt}{2m}} \cos(\omega' t + \delta) \tag{62}$$

Try the steady state solution,

$$x = A\cos(\omega t - \delta) \tag{63}$$

Plug this solution into the o.d.e.

$$m(-A\omega^2 \cos(\omega t - \delta)) + b(-A\omega \sin(\omega t - \delta)) + m\omega_0^2(A\cos(\omega t - \delta)) = F_0 \cos(\omega t) \tag{64}$$

$$(Am(\omega_0^2 - \omega^2)\cos(\omega t - \delta)) - A\omega b \sin(\omega t - \delta)) = F_0 \cos(\omega t) \tag{65}$$

$$(Am(\omega_0^2 - \omega^2)[\cos(\omega t)\cos\delta + \sin(\omega t)\sin\delta] - A\omega b[\sin(\omega t)\cos\delta - \cos(\omega t)\sin\delta] = F_0 \cos(\omega t) \tag{66}$$

$$(Am(\omega_0^2 - \omega^2)[\cos\delta + \tan(\omega t)\sin\delta] - A\omega b[\tan(\omega t)\cos\delta - \sin\delta] = F_0 \tag{67}$$

$$\Rightarrow Am(\omega_0^2 - \omega^2)\tan(\omega t)\sin\delta - A\omega b[\tan(\omega t)\cos\delta] = 0 \tag{68}$$

and

$$Am(\omega_0^2 - \omega^2)\cos\delta + A\omega b \sin\delta = F_0 \tag{69}$$

You get these equations because the previous line is $C\tan(\omega t) + D = F_0$. This can only be true FOR ALL t if both $C = 0$ and $D = F_0$.

$$m(\omega_0^2 - \omega^2)\sin\delta - b\omega\cos\delta = 0 \tag{70}$$

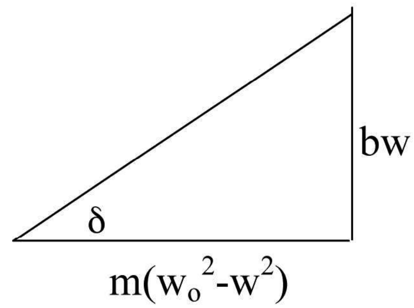
$$m(\omega_0^2 - \omega^2)\sin\delta = b\omega\cos\delta \tag{71}$$

$$\frac{\sin\delta}{\cos\delta} = \tan\delta = \frac{b\omega}{m(\omega_0^2 - \omega^2)} \tag{72}$$

$$Am(\omega_0^2 - \omega^2)\left[\frac{m(\omega_0^2 - \omega^2)}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + b^2\omega^2}}\right] + A\omega b\left[\frac{b\omega}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + b^2\omega^2}}\right] = F_0 \tag{73}$$

$$\frac{m^2(\omega_0^2 - \omega^2)^2 + b^2\omega^2}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + b^2\omega^2}} = \frac{F_0}{A} \tag{74}$$

$$\therefore A = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + b^2\omega^2}} \text{ as claimed.} \tag{75}$$

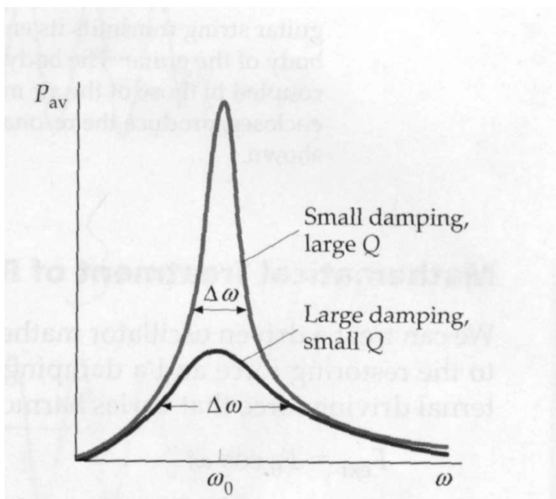


Resonance Curve

$$
\begin{aligned}
P &= Fv && (76)\\
&= (F_0 \cos\omega t)(A\frac{d}{dt}\cos(\omega t - \delta)) && (77)\\
&= (F_0 \cos\omega t)(-A\omega \sin(\omega t - \delta)) && (78)
\end{aligned}
$$

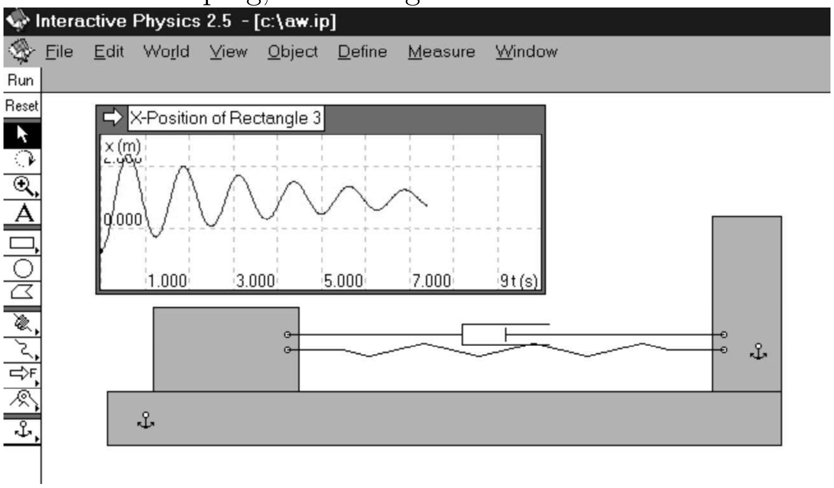
Take an average $\equiv \frac{1}{T}\int_0^T (thing)dt$ (not interested in moment by moment variations)

$$\bar{P} = \frac{1}{2}A\omega F_0 \sin\delta = \frac{1}{2}\frac{b\omega^2 F_0^2}{m^2(\omega^2-\omega_0^2)^2 + (b\omega)^2} \quad (79)$$

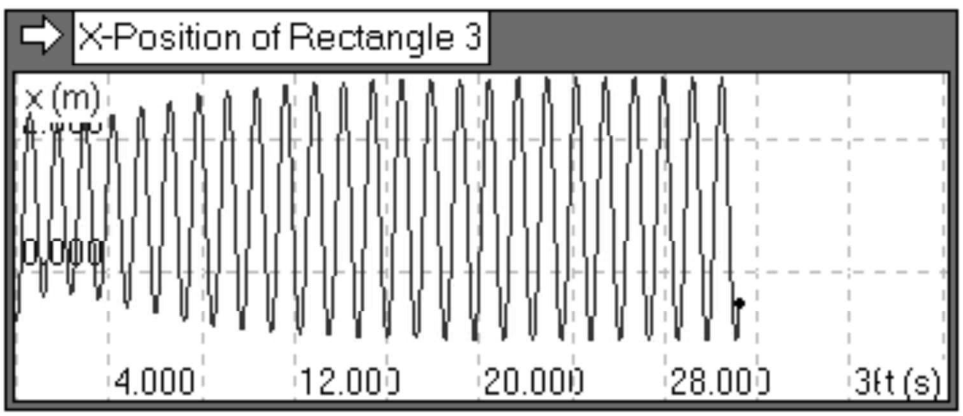


Interactive physics simulation

Mass = 2 kg, Spring constant = 50 N/m $\longrightarrow w = \sqrt{k/m} = 5$ radians/sec
Moderate damping, no driving...



Now add driving at 5 radians/sec...
(Do you expect resonance? Infinite amplitude?)

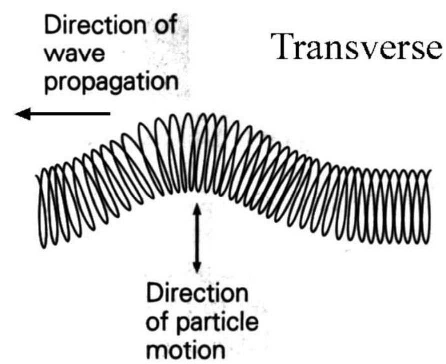


Note the region of transient behavior followed by steady state solution.

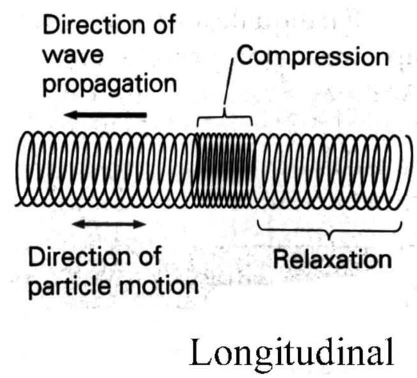
Waves

2 Kinds of Waves...

- Transverse
  Sideways 'pluck' on a taut string or slinky
  $\bar{d} \perp \bar{v}$
  (which d?, v?)



- Longitudinal
  Lateral pluck on a slinky
  $\bar{d} \| \bar{v}$

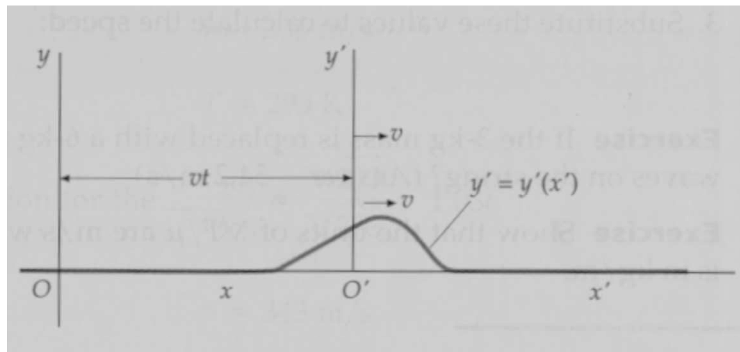


<u>Examples of each?</u>

What is a wave? (of the TRANSVERSE variety)

- "Vibration" of some quantity (e.g., pressure, temperature, electric field)
- Medium - matter (usually) that does the vibrating (e.g., air molecules, ... solid matter)
- Concept of propagating (travelling) vibration
- Self-sustaining (persists)
- Energy flow

Pulse on a string travelling at speed v

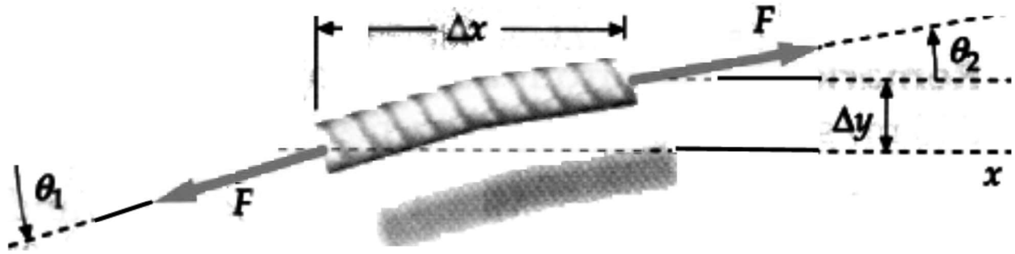


If the pulse is "small" ... It keeps its shape. (Claim!!)
So in primed frame, $y' = f(x')$. There is no time variation.
$y' = y$
$x' = x - vt$, @ $t = 0, x = x'$; as t grows, $x' < x$ for a given event
$y = f(x - vt)$

Why do waves propagate/regenerate?

- *Disturbance* at x produces ***restoring force*** at $x + dx$.
- *Restoring force* at $x + dx$ creates ***disturbance*** at $x + 2 \cdot dx$.
  - e.g. compressional (longitudinal) wave in wood. Think of one layer of molecules pushing on the next.

The Wave Equation



- Assume F to be constant over Δx (small pulse!)
- Apply $\Sigma\vec{F} = m\vec{a}$ to transverse (y) motion

$$F_{total,(net)} = F\sin(\theta+\Delta\theta) - F\sin\theta \tag{80}$$

$$\theta, \Delta\theta \ small \Rightarrow \sin\theta \approx \tan\theta \approx \frac{\text{opp}}{\text{adj}} = \frac{\Delta y}{\Delta x}$$

$$F_{total} = F\left(\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_{x}\right) \tag{81}$$

Partial derivatve?

$\frac{\partial y(x)}{\partial x}$ is just <u>some</u> function of $x\ldots$

$$F_{total} = F[g(x+\Delta x) - g(x)] \tag{82}$$

$$= F\left[\frac{g(x+\Delta x) - g(x)}{\Delta x}\right]\Delta x \tag{83}$$

$$= F(g'(x))\Delta x \tag{84}$$

$$= F\left(\frac{\partial^2 y}{\partial x^2}\right)\Delta x \tag{85}$$

APPLY $\Sigma\vec{F} = m\vec{a} = m\frac{\partial^2 y}{\partial t^2}$

What m?

$m = \mu\Delta x, \mu =$ Linear mass density (units?)

$$F\left(\frac{\partial^2 y}{\partial x^2}\right)\Delta x = \mu\Delta x\frac{\partial^2 y}{\partial t^2} \tag{86}$$

$$\Rightarrow \frac{\partial^2 y}{\partial t^2} = \left(\frac{F}{\mu}\right)\frac{\partial^2 y}{\partial x^2} \tag{87}$$

$\frac{F}{\mu}$? It has dimensions of velocity^2

$$\frac{\partial^2 y}{\partial t^2} = \left(\frac{F}{\mu}\right)\frac{\partial^2 y}{\partial x^2} \text{ (2nd order P.D.E.)} \tag{88}$$

$$\tag{89}$$

What would constitute a solution to this equation? Some $y(x,t)$ whose second space derivative is the same as its second time derivative.

$$y(x,t) = \sin(ax \pm bt) \tag{90}$$

$$\frac{\partial y}{\partial t} = b\cos(ax \pm bt) \tag{91}$$

$$\frac{\partial y}{\partial x} = a\cos(ax \pm bt) \tag{92}$$

$$\frac{\partial^2 y}{\partial t^2} = -b^2\sin(ax \pm bt) \tag{93}$$

$$\frac{\partial^2 y}{\partial x^2} = -a^2\sin(ax \pm bt) \tag{94}$$

Note that the space and time derivatives differ by a constant:

$$\frac{\partial^2 y}{\partial t^2} = \left(\frac{b^2}{a^2}\right)\frac{\partial^2 y}{\partial x^2} \tag{95}$$

Other solutions? Try $f(x \pm vt)$ for any function f.

$$y = f(x - vt) = f(p),\ \ (\text{for } p \equiv x - vt) \tag{96}$$

Use the chain rule a few times...
1st derivs:

$$\frac{\partial y}{\partial x} = \frac{df}{dp}\frac{\partial p}{\partial x} = \frac{df}{dp}(1) \tag{97}$$

$$\frac{\partial y}{\partial t} = \frac{df}{dp}\frac{\partial p}{\partial t} = \frac{df}{dp}(-v) \tag{98}$$

2nd dervis:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial y}{\partial x}\right] = \frac{d}{dp}\left[\frac{df}{dp}\right]\frac{\partial p}{\partial x} = \frac{d^2 f}{dp^2} \tag{99}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{\partial y}{\partial t}\right] = \frac{d}{dp}\left[\frac{df}{dp}(-v)\right]\frac{\partial p}{\partial t} = v^2\frac{d^2 f}{dp^2} \tag{100}$$

$$\tag{101}$$

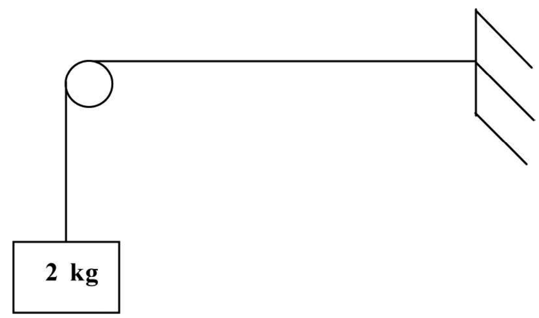
so...

$$\frac{\partial^2 y}{\partial t^2} = +v^2\frac{\partial^2 y}{\partial x^2} \tag{102}$$

The trial solution does work – so long as v is constant.

$\sin(x-vt), \cos(x-vt)$ satisfy both the wave equation & the "pulse shape rule" (i.e., "sliding" pulse has the form $f(x-vt)$) with $\frac{F}{\mu} = v^2$.

$v = \sqrt{\frac{F}{\mu}}$
Faster propagation speed if...?



$T = (2\text{kg})(9.8\frac{\text{m}}{m^2}) = 20N$

$\mu = 0.05\frac{\text{kg}}{\text{m}}$, so $v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{20}{0.05}} = 20\frac{\text{m}}{\text{s}}$
$y(x,t) = \sin(x-20t)$

$\Sigma\vec{F} = m\vec{a} \longrightarrow$ Applied to lateral (longitudinal) motion? $\Rightarrow$ Should give no motion for a transverse wave

$\Sigma\vec{F_L} = F\cos(\theta) - F\cos(\theta + \Delta\theta)$
small θ & small Δ θ?

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b) \tag{103}$$
$$\ldots a, b \text{ are small}$$
$$\cos(a) \sim 1 \approx 1 - \frac{a^2}{2!}(\text{a better approximation}) \tag{104}$$
$$\cos(b) \sim 1 \approx 1 - \frac{b^2}{2!} \tag{105}$$
$$\sin(a) \sim a - \frac{a^3}{3!} \tag{106}$$
$$\sin(b) \sim b - \frac{b^3}{3!} \tag{107}$$
$$\cos(\theta) - \cos(\theta - \Delta\theta) \approx 0 + O(\theta^2, \Delta\theta^2, \theta \cdot \Delta\theta) \tag{108}$$

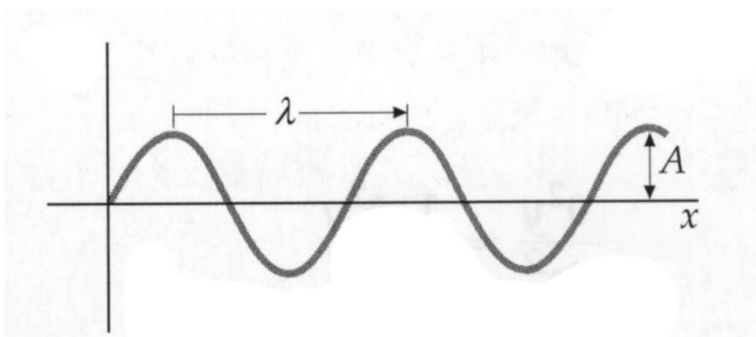
So no *net* lateral force.

General Sinusoid:
Periodic in space *and* time

$$A\sin(bx \pm ct) \tag{109}$$

$$A\cos(bx \pm ct) \tag{110}$$

$\sin(bx) \sim$ repeats every 2π in phase
Space variation (freeze time)



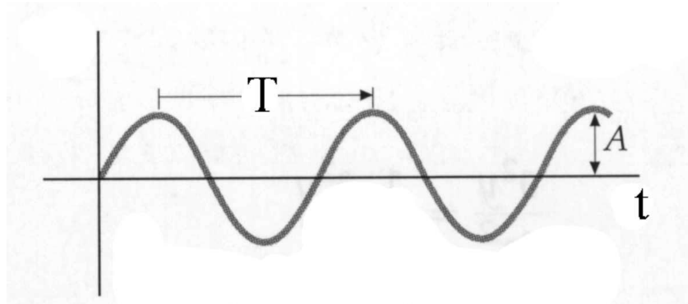
$$b = \frac{2\pi}{\lambda} \tag{111}$$

$$\sin\left(\frac{2\pi}{\lambda}x\right) = \sin\left(\frac{2\pi}{\lambda}(x+\lambda)\right) \tag{112}$$

$$= \sin\left(\frac{2\pi}{\lambda}x + 2\pi\right) \tag{113}$$

$$= \sin\left(\frac{2\pi}{\lambda}x\right) \tag{114}$$

Time Variation/Look at one point in space



$$\sin(ct) \tag{115}$$
$$c = \frac{2\pi}{T} \tag{116}$$

$$\sin\left(\frac{2\pi}{T}t\right) = \sin\left(\frac{2\pi}{T}(t+T)\right) \tag{117}$$
$$= \sin\left(\frac{2\pi}{T}t + 2\pi\right) \tag{118}$$
$$= \sin\left(\frac{2\pi}{T}t\right) \tag{119}$$
$$\Rightarrow y = A\sin\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t\right) \tag{120}$$
$$k \equiv \frac{2\pi}{\lambda}(\text{wave }\#), \omega \equiv \frac{2\pi}{T}(\text{angular frequency}) \tag{121}$$
$$\Rightarrow y = A\sin(kx - \omega t) \& \frac{\omega}{k} = v \tag{122}$$
$$\omega = \frac{2\pi}{T} = 2\pi f, k = \frac{2\pi}{\lambda} \tag{123}$$
$$\text{so if } \frac{\omega}{k} = v, \text{ then } \frac{(2\pi f)}{\left(\frac{2\pi}{\lambda}\right)} = f\lambda$$
$$v = f\lambda \tag{124}$$

$v_{sound,air} \approx 340\text{m/sec}, f_{voice} \approx 2000\text{hz}$
$\Rightarrow \lambda \approx 0.15\text{m}$

Superposition

Sum of valid solutions = a new valid solution

$$\frac{\partial^2}{\partial t^2} y_1 = \left(\frac{F}{\mu}\right) \frac{\partial^2}{\partial x^2} y_1 \tag{125}$$

$$\frac{\partial^2}{\partial t^2} y_2 = \left(\frac{F}{\mu}\right) \frac{\partial^2}{\partial x^2} y_2 \tag{126}$$

Add the two equations:

$$\frac{\partial^2}{\partial t^2}(y_1 + y_2) = \left(\frac{F}{\mu}\right) \frac{\partial^2}{\partial x^2}(y_1 + y_2) \tag{127}$$

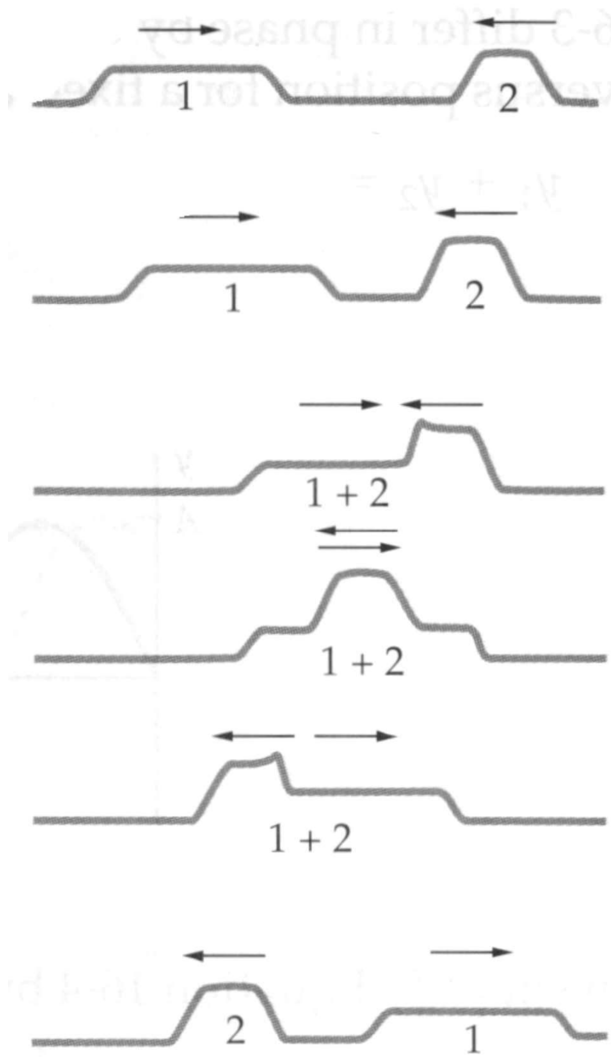
Rename $y_1 + y_2 = y_3$:

$$\frac{\partial^2}{\partial t^2} y_3 = \left(\frac{F}{\mu}\right) \frac{\partial^2}{\partial x^2} y_3 \tag{128}$$

so, the Σ of the solutions = a new solution.

<u>Why (as a matter of math) does the wave equation have this property?</u>

Superposition Principle



$$y_1 = f(x,t) \tag{129}$$

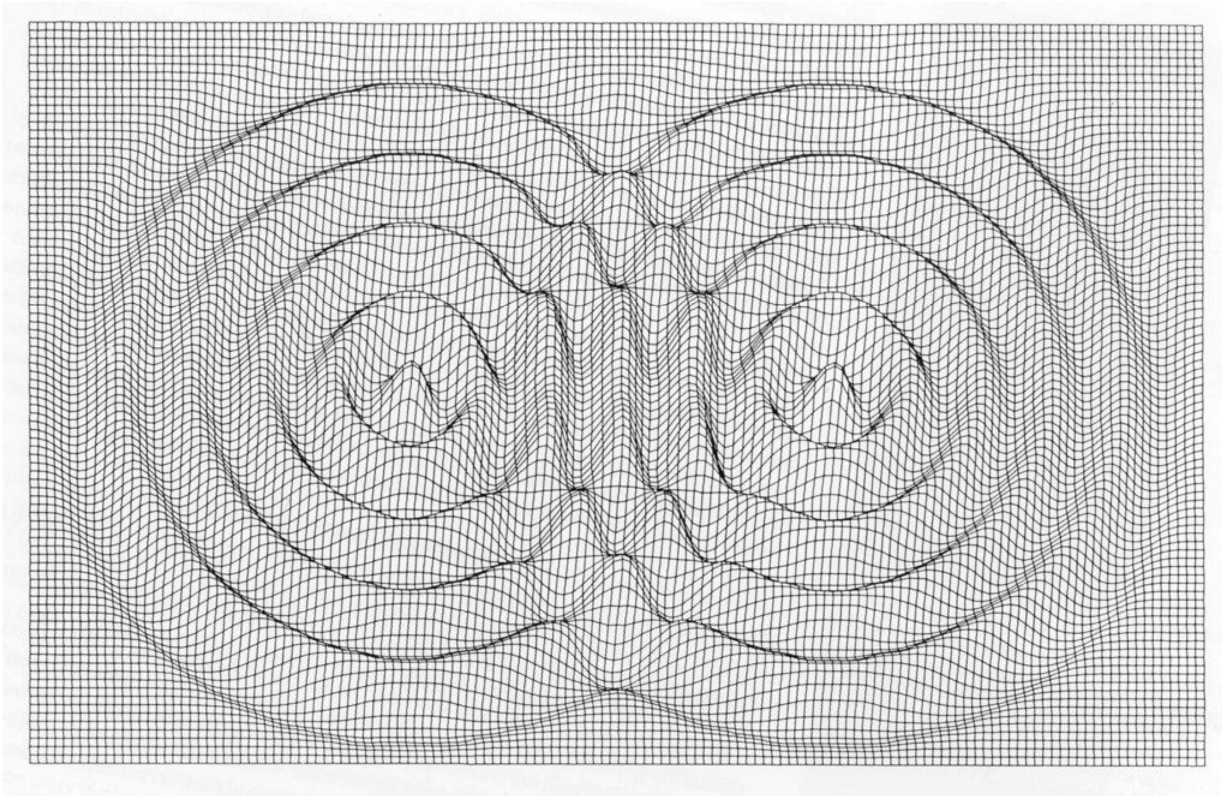
$$y_2 = g(x,t) \tag{130}$$

$$y(x,t) = y_1(x,t) + y_2(x,t) \tag{131}$$

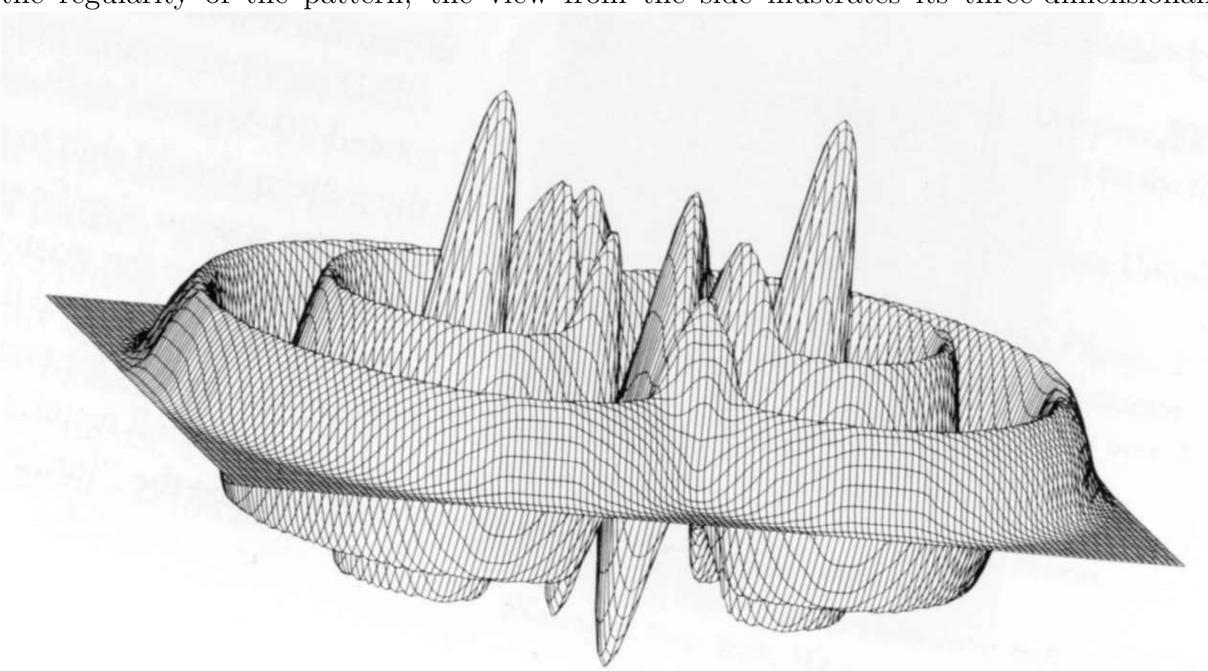
Here, the two waves largely "ignore" each other.

Not all waves have this propery.

True for our string?



When two pebbles are dropped in a pool of water, each generates a pattern of concentric ripples. Where two ripples overlap, the height of the water is the sum of the individual heights. The resulting disturbance is an "interference pattern". The view from above emphasizes the regularity of the pattern; the view from the side illustrates its three-dimensionality.



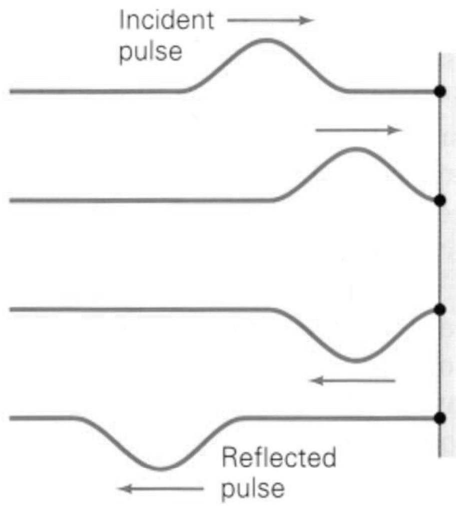
Note...

1. $\Sigma Solutions = Solution$
2. $v = v(T, u)$, but no dependence on λ, $y(x, t)$
3. Sum of several solutions (e.g., sine waves) can be (almost) any shape function.
   "Fourier Analysis"
   "Fourier Synthesis"
   So a pulse made up of sine waves won't disperse. *(explains why any shape solves wave equation)*

- Friction $\longrightarrow v = v(\lambda)$
- More complex physics $\longrightarrow$ "Dispersion"

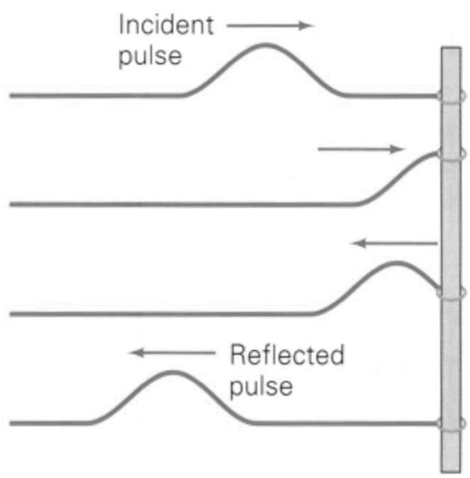
Reflections

Fixed end:
Upward tug of string on wall $\longrightarrow$ downward reaction force of wall on string.



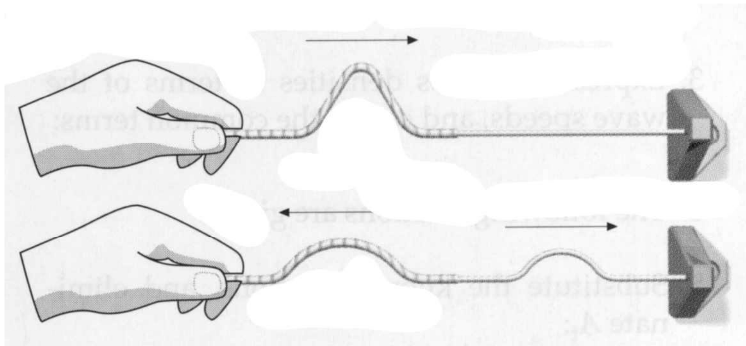
Phase shift of 180°

Free end:
Upward momentum of loop results in overshoot, tug upward on all string behind it.

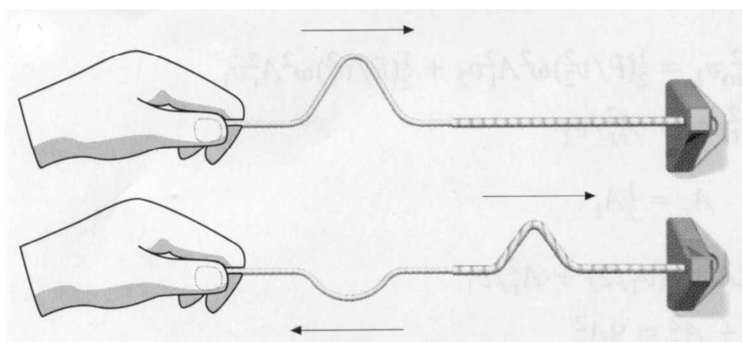


No phase shift

Propagation across a discontinuity

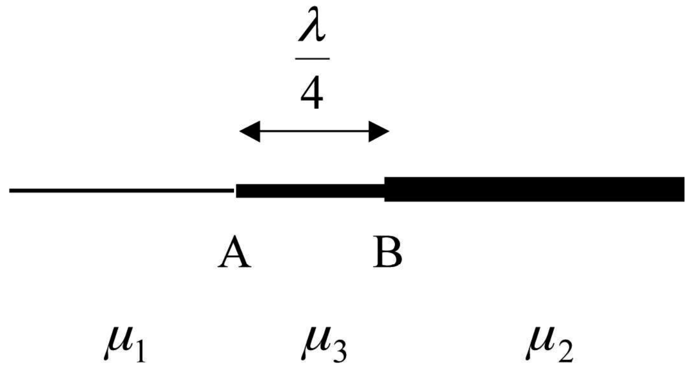


Transmission from a "heavy" rope to a "light" one results in no phase change in either the transmitted or reflected pulse.



Transmission from a light to heavy rope results in a 180-degree phase shift (an inversion) in the reflected pulse only.

Impedance Matching



$$\mu_3 = \sqrt{\mu_1 \mu_2} \tag{132}$$

Destructive interference between pulses reflected at A and pulses reflected at B produce no overall reflected pulse.
Therefore, maximum transmission to the right...

Note: this cancellation/destructive interference cannot work for single pulses, only for waves (trains of pulses).

Applications: visible light (lens coatings), transmission of information, energy with other E&M waves.

Standing Waves

$$y_1 = A\sin(kx - \omega t) \tag{133}$$

$$y_2 = A\sin(kx + \omega t) \tag{134}$$

Two waves with
propagation directions, but same speed.

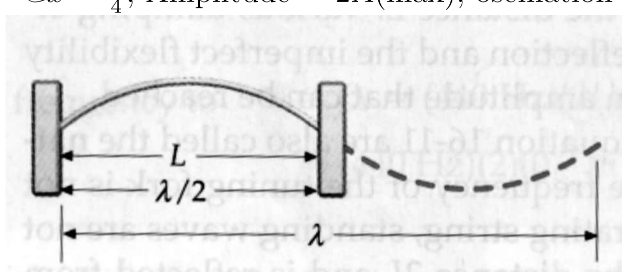
$$y_T \equiv y_1 + y_2 = A(\sin(kx - \omega t) + \sin(kx + \omega t)) \tag{135}$$

MAGIC!! (not really, just trig. identities)

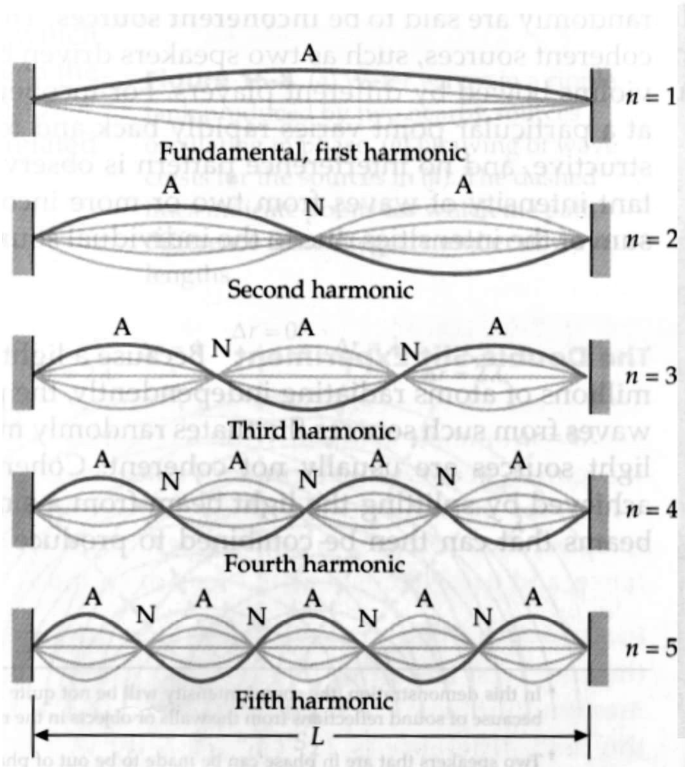
$$y_T = [2A\cos(\omega t)]sin(kx) \tag{136}$$

The bracketed term *can be thought of* as an amplitude that varies in time. The remaining term is then an oscillation, rather than a wave.

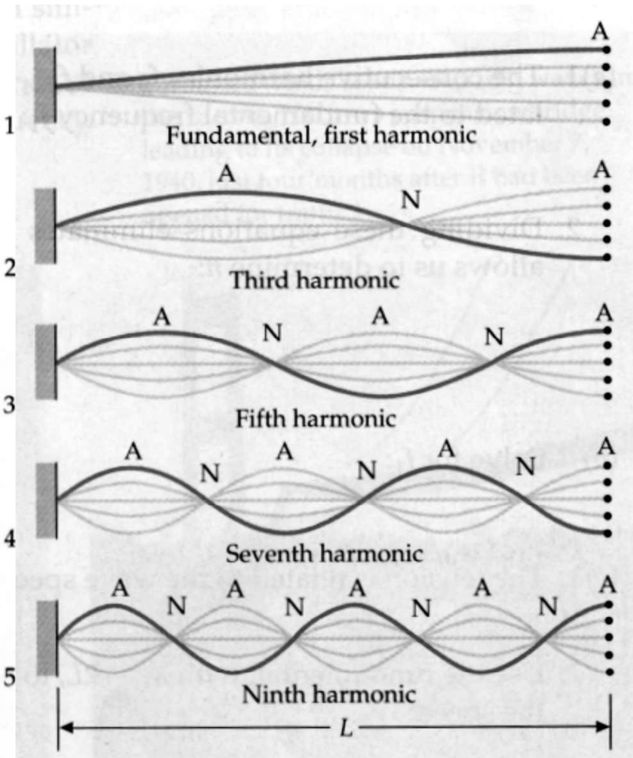
@$x = 0$, Amplitude $= 0$, no oscillation at all
@$x = \frac{\lambda}{2}$, Same
@$x = \frac{\lambda}{4}$, Amplitude $= 2A$(max), oscillation



"Nodes" @$0, \frac{\lambda}{2}, \lambda, etc. | x = \left(\frac{n}{2}\lambda\right), n = 0, 1, \ldots$
"Antinodes" @$\frac{\lambda}{4}, \frac{3\lambda}{4}, etc. | x = \left(\frac{2n+1}{4}\lambda\right), n = 0, 1, \ldots$

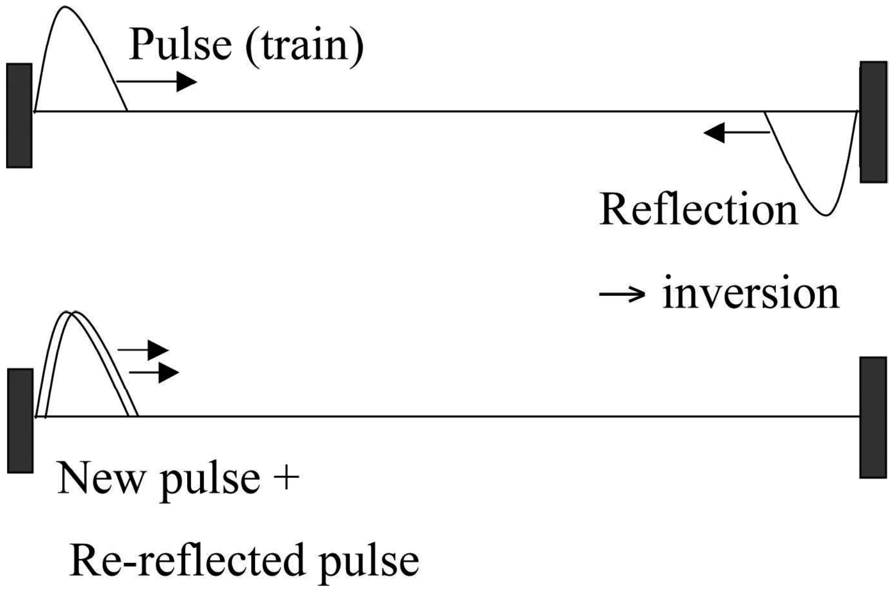


Two fixed ends (both nodes)



One fixed end, one free end (one node, one≈ antinode)

Standing Waves - Alternative Derivation



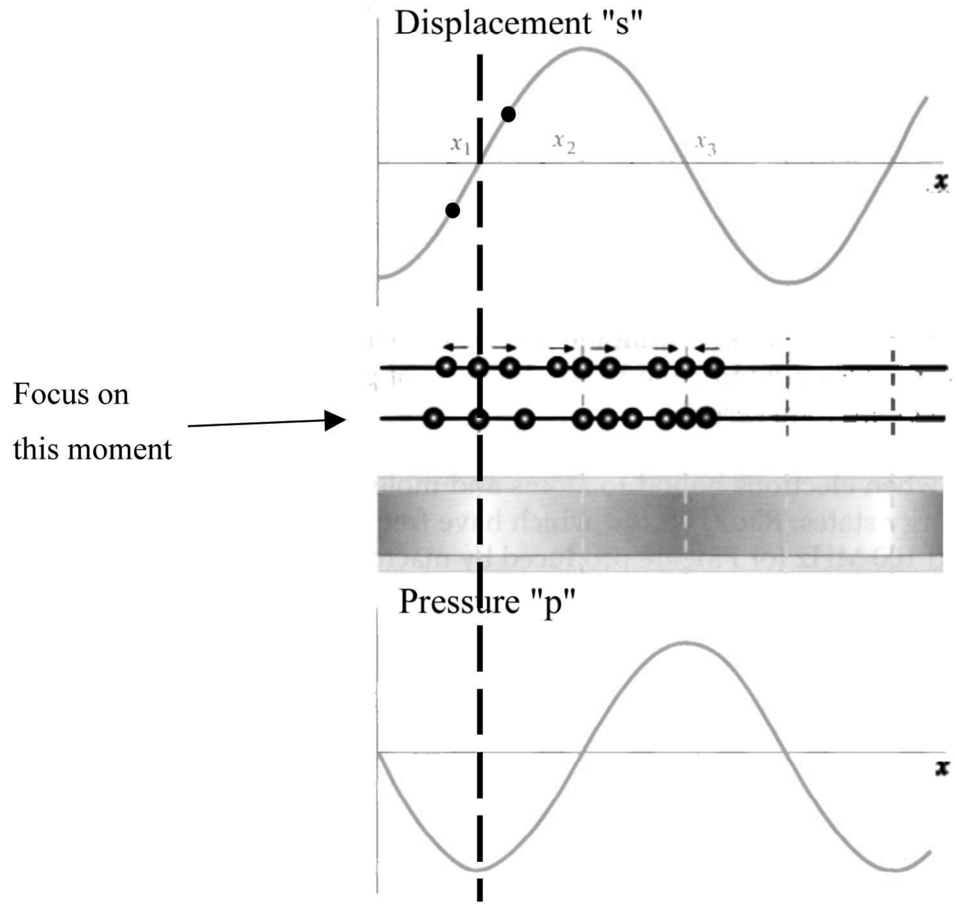
If the travel time (round trip) is equal to $T_{pulsetrain}$, there is *constructive interference* and *resonance*.

$$\text{Travel Time} = \frac{2L}{v} = T = \frac{1}{f} \tag{137}$$

$$nT = \frac{n}{f} \Rightarrow f = \frac{nv}{2L} = \frac{n}{2L}\sqrt{\frac{T}{\mu}} \tag{138}$$

(the same results as $L = \frac{n\lambda}{2}$)

Sound Waves, Longitudinal (compression)



$$s(x,t) = s_0 \sin(kx - \omega t)[\text{displacement}] \tag{139}$$

$$p(x,t) = p_0 \sin\left(kx - \omega t - \frac{\pi}{2}\right)[\text{pressure}] \tag{140}$$

90° phase difference between the two descriptions

Standing waves

(once again, w/ complex exponentials)

... we like the property $e^{a+b} = e^a e^b$ ...

$$Ae^{i(kx-\omega t)} + Ae^{i(kx+\omega t)} \tag{141}$$

$$y_T = Ae^{ikx}[e^{-i\omega t} + e^{i\omega t}] \tag{142}$$

$$y_T = [\cos(-\omega t) + i\sin(-\omega t)] + [\cos(\omega t) + i\sin(\omega t)] \tag{143}$$

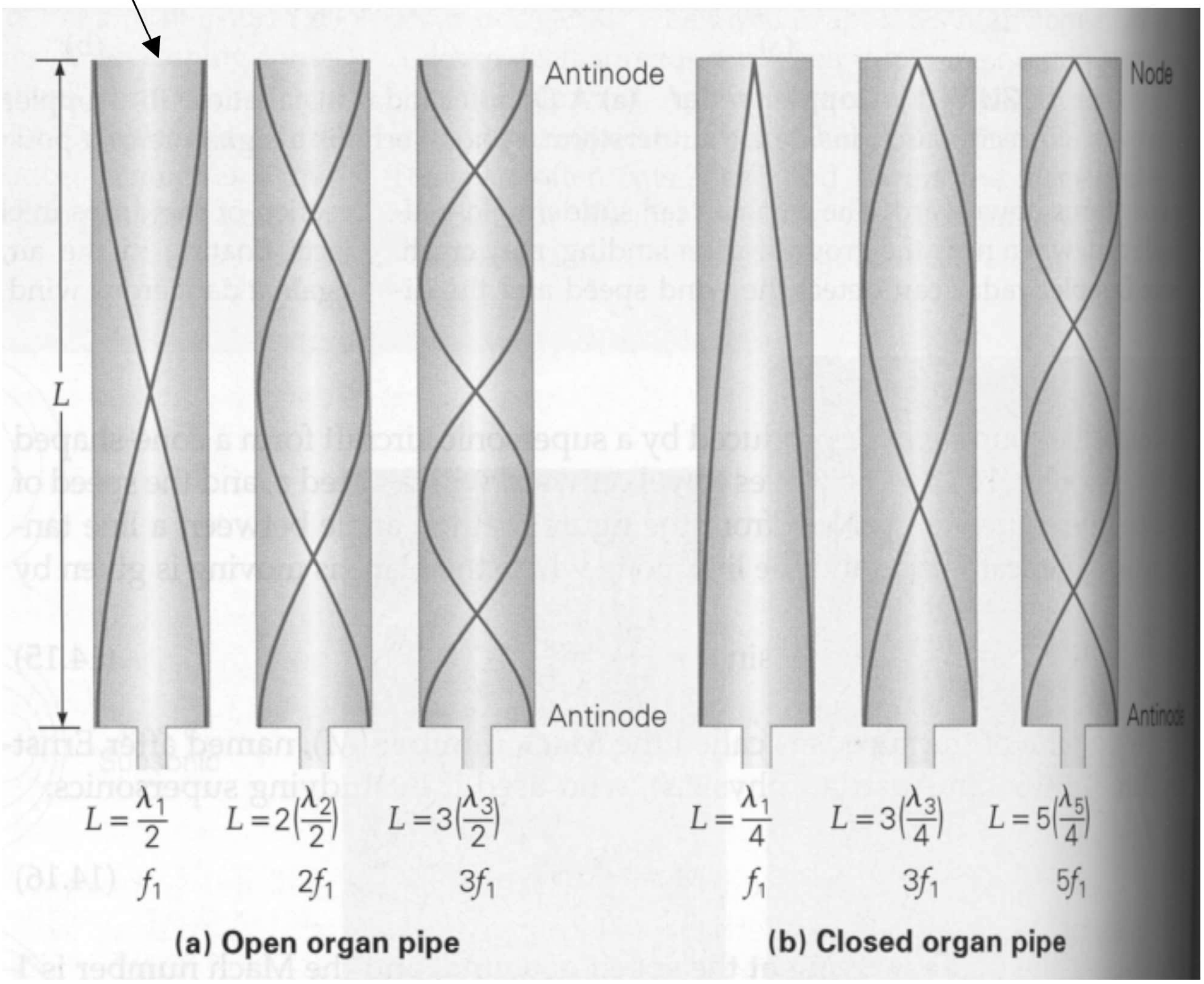
(using $\cos(-\omega t) = \cos(\omega t); \sin(-x) = -\sin(x)$)

$$\begin{aligned} y_T &= Ae^{ikx} 2\cos(\omega t) \qquad (144) \\ &= \text{(space function)(time function)} \end{aligned}$$

Standing waves in organ pipes

Antinode in displacement,

but node in pressure.



(a) Open organ pipe **(b) Closed organ pipe**

Beats

Superposition w/ different ω 's

$$p_1 = A\sin(\omega_1 t) \tag{145}$$
$$p_2 = A\sin(\omega_2 t) \tag{146}$$
$$p_T = 2A\cos\left(\frac{1}{2}[\omega_1 - \omega_2]t\right)\sin\left(\frac{1}{2}[\omega_1 + \omega_2]t\right) \tag{147}$$
$$\Delta\omega \equiv \omega_1 - \omega_2 \tag{148}$$
$$\bar{\omega} \equiv \frac{1}{2}[\omega_1 + \omega_2] \tag{149}$$
$$p_T = 2A\cos\left(\frac{1}{2}\Delta\omega t\right)\sin(\bar{\omega}t) \tag{150}$$

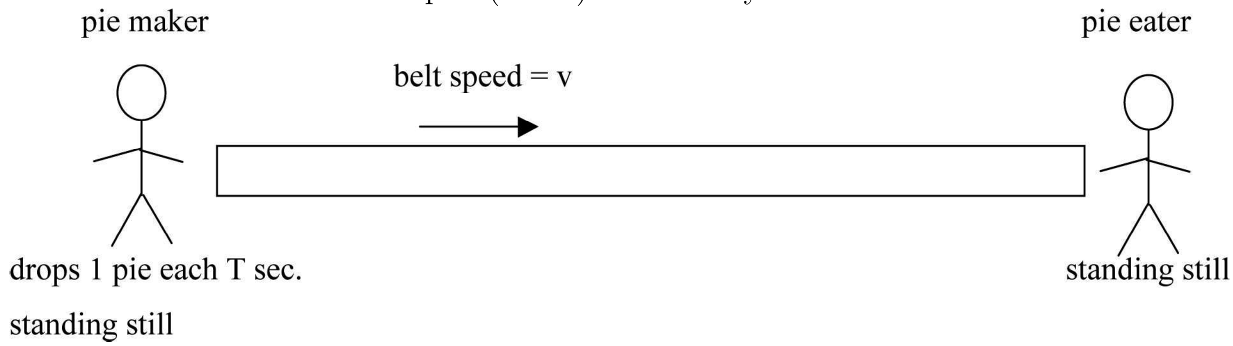
$$\omega_1 = 1000 \text{ rad/sec}, \omega_2 = 1001 \text{ rad/sec}$$
$$\bar{\omega} = 1000.5 \text{ rad/sec}, \Delta\omega = 1 \text{ rad/sec}$$
$$\text{Sounds like? } p_T = 2A\cos\left(\tfrac{1}{2}t\right)\sin(1000.5t)$$

But note that this "modulation" of the amplitude causes a "warble" at $\Delta\omega$, not $\Delta\omega/2$.

Exercise: Try this with complex exponentials. The math is easiest if you choose the two oscillations to be $e^{i\left(\omega - \frac{\Delta\omega}{2}\right)t}$, $e^{i\left(\omega + \frac{\Delta\omega}{2}\right)t}$.

Doppler Shift

with pies (crests) on a conveyer belt...

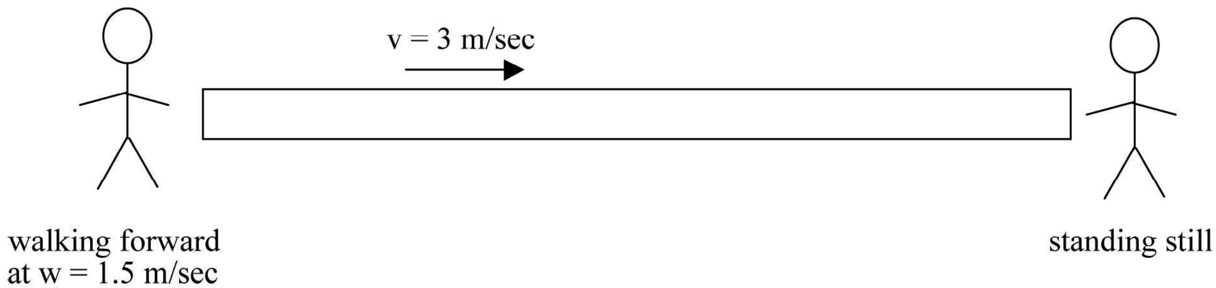


Pie dropped every T seconds goes $x = vT$ meters till the next pie is dropped.
vT = Actual spacing between pies on the belt = λ
What is the pie spacing as seen by the pie eater?
Same ("actual") spacing = λ
Example: Pie dropped each $\frac{1}{2}$ second.
($T = \frac{1}{2}$ sec., $f = 2$ hz)
Suppose $v = 3\ \frac{m}{sec}$
$x = vt$
$= [3\frac{\mathrm{m}}{\mathrm{sec}}][\frac{1}{2}\mathrm{sec}]$=1.5 m
This is the pie spacing as seen by both pie maker and pie eater.
For the pie eater,

$$v = f\lambda \tag{151}$$

$3 = f(1.5)$
$f = 2$ or $T = \frac{1}{2}$
No difference in f or λ.

Moving Source



Pie is (still) dropped every T ($\frac{1}{2}$) seconds.
The pie goes. . .
$x = vt = (3\frac{(m)}{(sec)})\left(\frac{1}{2}\text{sec}\right)$
=1.5 m on the belt til next one is dropped.
But. . . Where is the next one dropped?
During that $\frac{1}{2}$ second, the pie maker has moved
$x = \omega T = (1.5)\left(\frac{1}{2}\right) = \frac{3}{4}$ meters
2nd pie dropped 1.5-$\frac{3}{4} = \frac{3}{4}$ m behind 1st
$\lambda = \frac{3}{4}$ m$\sim$ actual spacing on the belt
Eater? Sees the belt approaching at 3 m/sec

- In the next second, 3 m of belt will pass the pie-eater.
- If the spacing of pies is $\frac{3}{4}$ m, then 3 meters of belt contains 4 pies – actually 4 pie SPACINGS.
- Hence the frequency is 4 hz.

That is. . .

$$v = f_{EATER}\lambda_{EATER}$$
$$3 = f_{EATER}\left(\frac{3}{4}\right), f_{EATER} = 4, f_{MAKER} = 2$$

Frequency shift . . .

Altogether. . .

$$v = f_M\lambda_M \text{but } "v" = v - \omega \tag{152}$$
$$\lambda_M = \frac{v-\omega}{f_M} \tag{153}$$
$$v = f_E\lambda_E, ("v \text{ is just } v"), f_E = \frac{v}{\lambda_E} = \frac{v}{v-\omega}f_M \tag{154}$$

(Note that λ's are the same - actual belt spacing.)

$$\begin{gathered}
f_E = \tfrac{1}{1-\frac{\omega}{v}} \cdot f_M \\
= \tfrac{1}{1-\frac{\frac{3}{2}}{3}} \cdot f_M \\
= \tfrac{1}{1-\frac{1}{2}} \cdot f_M \\
f_E = 2f_M \\
f_E = \tfrac{4}{\text{sec}}, f_M = \tfrac{2}{\text{sec}}
\end{gathered}$$

If the pie eater is walking, what changes?

Moving Receiver

$$f_{sent} = \frac{v}{\lambda} \tag{155}$$

Nothing "interesting" happens at the source.
Pie spacing (λ) is unaffected by receiver motion.

Pie eater applies the "usual" equation, $f = \frac{v}{\lambda}$, for the same λ. But what about v? Moving (approaching) pie eater sees a total of $v + w$ meters of belt pass each second.

$$f_{received} = \frac{v + w}{\lambda} = \frac{v\left(1 + \frac{w}{v}\right)}{\lambda} = f_{sent}\left(1 + \frac{w}{v}\right) \tag{156}$$

"Combo" of source and receiver motion:

$$f' = f\left(\frac{v \pm v_r}{v \mp v_s}\right) \tag{157}$$

Try this with # 's...

Moving Source

- source speed = 20 m/sec
- source frequency = 10 hz
- wave speed = 60 m/sec
- 10/sec and travel @ 60 m/s $\longrightarrow$ usual spacing of 1 per 6 m

But...
source moves @ 20 m/sec
t=0: the wave is emitted
t=0.1: next wave is emitted
The source has moved 2 m during this interval so the spacing is now reduced to 4 m

$$\lambda' = 4$$

Observer says... wave is travelling @ 60 m/sec,

$$\lambda = 4m \therefore f = \frac{v}{\lambda} = \frac{60}{4} = 15 \text{ hz}$$

DOPPLER ANALYSIS HINGES ON THE "TRUTH" OF THE RELATIONSHIP $v = f\lambda$ FOR THE SOURCE AND FOR THE RECEIVER.

(...two *different* truths ...)

$$v = f\lambda \tag{158}$$

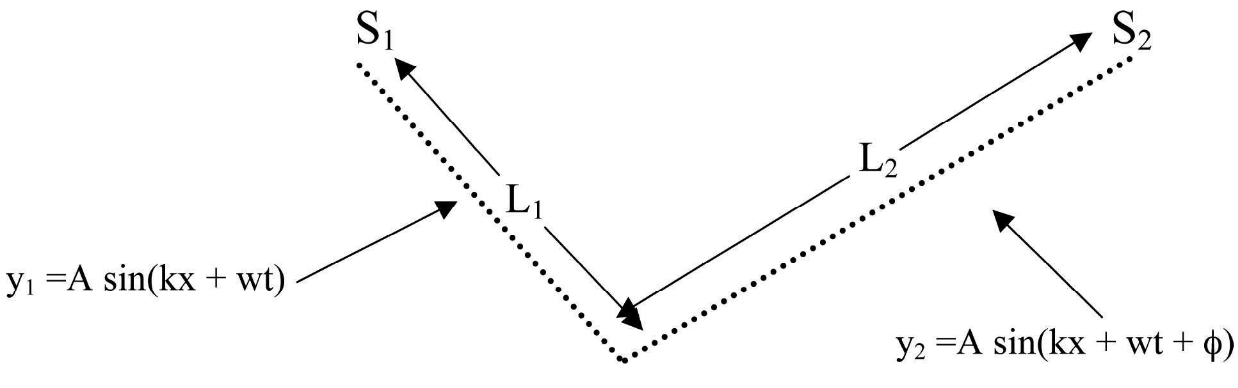
Why is this true?

$$= \frac{1}{T}\lambda \tag{159}$$

$$\longrightarrow \lambda = vT \tag{160}$$

This is just $x = vt$, where $t =$ one period of the motion, during which time a crest moves forward by one wavelength.

Interference



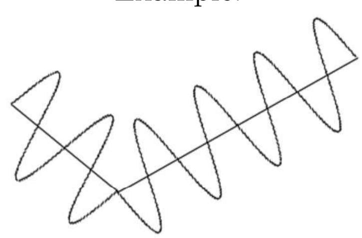
Define $x = 0$ at the observer (to get rid of x dependence)

$$y_T = A[\sin(\omega t) + \sin(\omega t + \phi)] \tag{161}$$

$$\phi = ? \tag{162}$$

$$\phi = \frac{L_2 - L_1}{\lambda} \cdot 2\pi \tag{163}$$

Example:



$\phi = \frac{(4\lambda)-(2\lambda)}{\lambda} \cdot 2\pi = 4\pi'' ='' 0$
$\Rightarrow$ constructive interference

Wave amplitude varies with distance

1-D wave (e.g., transverse wave on a string)
No effect (assuming no friction)
2-D wave (e.g., water wave on a pond)
Energy/pulse is a constant but spreads out over the (growing) circumference of a circle.

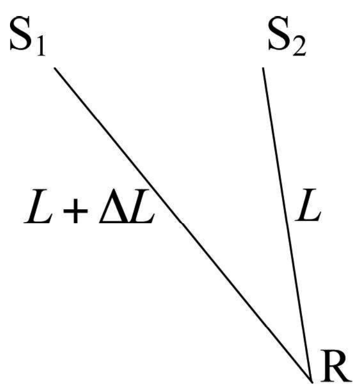
$$\frac{\text{Energy}}{\text{Length}} \propto \frac{1}{\text{circ.}} \sim \frac{1}{2\pi R} \tag{164}$$

$$\text{Energy} \propto \text{Amp}^2 \longrightarrow \text{Amp} \propto \frac{1}{\sqrt{R}} \tag{165}$$

3-D wave (e.g., sound wave in air)
Energy spreads out over the surface area of a sphere.

$$\frac{\text{Energy}}{\text{Area}} \propto \frac{1}{4\pi R^2} \Rightarrow \text{Amp} \propto \frac{1}{R} \tag{166}$$

The distance dependence of wave amplitude can "interfere" with interference!

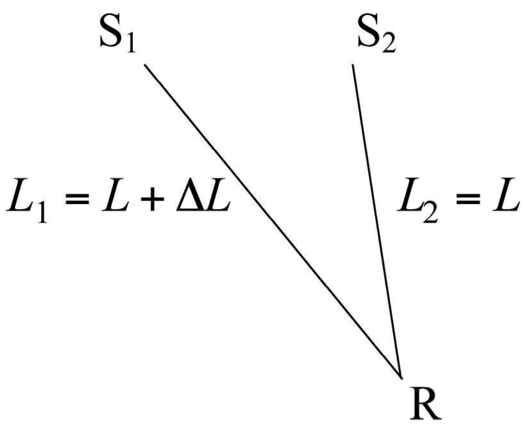


$$\text{Total} = A_1 \sin(\omega t) + A_2 \sin\left(\omega t + \frac{\Delta L}{\lambda} 2\pi\right) \tag{167}$$

Even if $\frac{\Delta L}{\lambda} 2\pi = \pi$, there is no complete destructive interference unless $A_1 = A_2$.

But, $A_1 \sim \frac{A}{L_1}$ and $A_2 \sim \frac{A}{L_2}$.

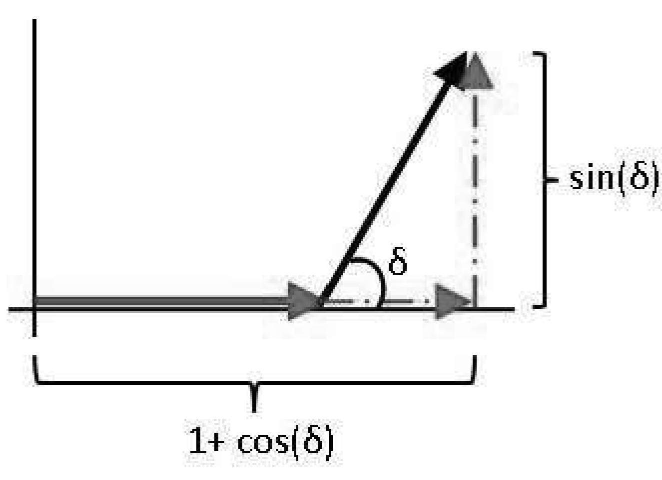
Interference - 2 sources, using complex exponentials



$$\frac{L_1 - L_2}{\lambda} \cdot 2\pi = \delta \tag{168}$$

$$\begin{aligned} y_T &= A\cos(\omega t) + A\cos(\omega t + \delta) && (169)\\ &= Ae^{i\omega t} + Ae^{i(\omega t+\delta)} && (170)\\ &= Ae^{i\omega t}(1 + e^{i\delta}) && (171) \end{aligned}$$

- The 1st term describes an oscillation arriving at R.
- The 2nd term is the amplitude of that oscillation.



$$\begin{aligned} (1 + e^{i\delta}) &= 1 + (\cos\delta + i\sin\delta) && (172)\\ &= (1 + \cos\delta) + i\sin\delta && (173) \end{aligned}$$

$$\begin{aligned} MAG &= \sqrt{(1+\cos\delta)^2 + (\sin\delta)^2} && (174)\\ &= \sqrt{1 + \cos^2\delta + 2\cos\delta + \sin^2\delta} && (175)\\ &= \sqrt{2 + 2\cos\delta} && (176) \end{aligned}$$

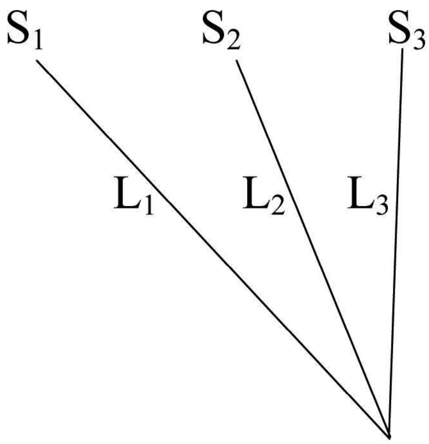
Exercise: Obtain the same result, $\sqrt{2 + 2\cos\delta}$, using trig. identities.

$$I \propto A^2 = 2 + 2\cos\delta \tag{177}$$

Some special cases...

$I = 4, \delta = 0 \longrightarrow$ (constructive interference)
$I = 0, \delta = \pi \longrightarrow$ (destructive interference)
$I = 2, \delta =$ random $\longrightarrow$ (see "Superposition with 'bad' (incoherent) waves" below)

Interference - 3 sources, using complex exponentials



$$L_1 = L_2 + \Delta L$$

$$L_3 = L_2 - \Delta L$$

$$\Rightarrow \delta = \frac{\Delta L}{\lambda} 2\pi \tag{178}$$

$$A\sin(\omega t - \delta) + A\sin(\omega t) + A\sin(\omega t + \delta) \tag{179}$$

or

$$Ae^{i(\omega t - \delta)} + Ae^{i\omega t} + Ae^{i(\omega t + \delta)} \tag{180}$$

$$y_T = Ae^{i\omega t}(e^{-i\delta} + 1 + e^{i\delta}) \tag{181}$$

[ignore $e^{i\omega t}$]

$$= [\cos(-\delta) + i\sin(-\delta) + 1 + \cos(\delta) + i\sin(\delta)] \tag{182}$$

$$y_T = Ae^{i\omega t}(2\cos\delta + 1) \tag{183}$$

$$I \propto \text{Amplitude}^2 \tag{184}$$

$$I = A^2(2\cos\delta + 1)^2 \tag{185}$$

Special cases...

$I_{max} = 9A^2$, for $\delta = \cos^{-1} 1 = 0$
$I_{min} = 0$, for $\delta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{4\pi}{3}$
$I_{random} = A^2\overline{(4\cos^2\delta + 4\cos\delta + 1)}^*$, for δ = random (see "Superposition with 'bad' (incoherent) waves" below) $\longrightarrow= A^2(2 + 0 + 1) = 3A^2$
*Note: Bar indicates average over all δ's, equivalent to a time average

Superposition with "bad" (incoherent) Waves

Start by assuming two waves w/ constant phase difference (at some constant $x = 0$).
$\cos(\omega t) + \cos(\omega t + \delta)$ = wave with amplitude $\sqrt{2 + 2\cos\delta}$, intensity $\propto (2 + 2\cos\delta)$.

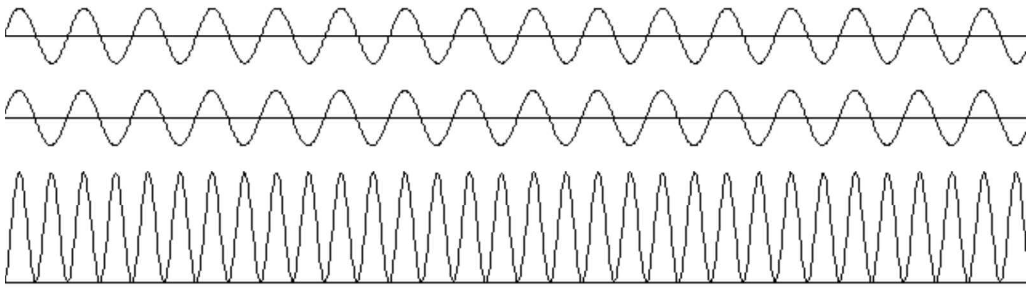
$$I \sim 2 + 2\cos\delta \tag{186}$$

$$\bar{I} = \frac{1}{T}\int_0^T I(t)dt \tag{187}$$

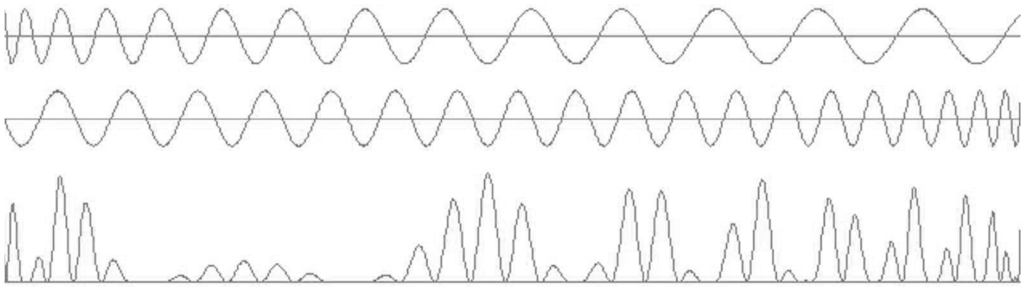
If δ = constant, $I \neq I(t), I = \bar{I} = 2 + 2\cos\delta$
But if δ varies randomly in time (cheap source), $\bar{I} = \bar{2} + \overline{2\cos\delta(t)} = 2$

N *incoherent* waves have a total intensity = sum of the individual intensities = $\sum_{i=1}^{n} A_i^2$. (Averaging kills the cross terms.)
N *coherent* waves have a total intensity = square of the total amplitude = $\left(\sum_{i=1}^{n} A_i\right)^2$. Consider the difference for the case of 100 equal amplitude sources!

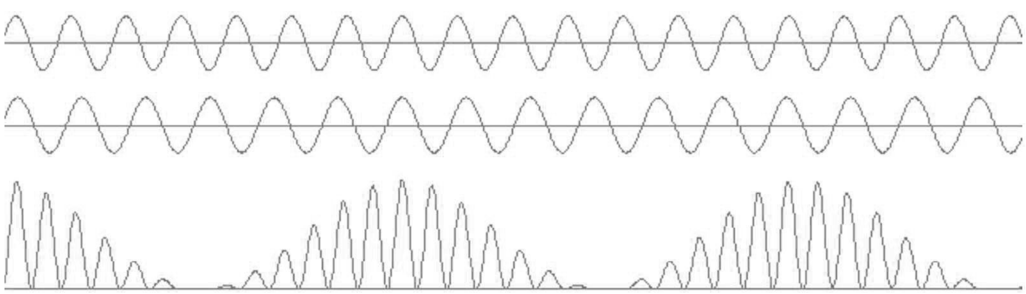
Example of coherent/incoherent combinations of waves



2 coherent waves (and their sum, squared), (*average intensity* $= \frac{1}{2} \cdot 4$)



2 incoherent waves (and sum, square), (*average intensity* $= \frac{1}{2} \cdot 2$



2 sine waves with different frequencies (& s.s.), (*average intensity* $= \frac{1}{2} \cdot 2$)

Things you should know about complex numbers...

[I'll use j, instead of i, just for a change of pace. EE's use j.]

$$\begin{aligned}
e^{j\omega nT} &= \cos(\omega nt) + j\sin(\omega nt), j = \sqrt{-1} && (188)\\
e^{j\omega t} + e^{-j\omega t} &= [\cos(\omega t) + j\sin(\omega t)] + [cos(-\omega t) + j\sin(-\omega t)] && (189)\\
&= \cos(\omega t) + j\sin(\omega t) + \cos(\omega t) - j\sin(\omega t) && (190)\\
&= 2\cos(\omega t) && (191)
\end{aligned}$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \tag{192}$$

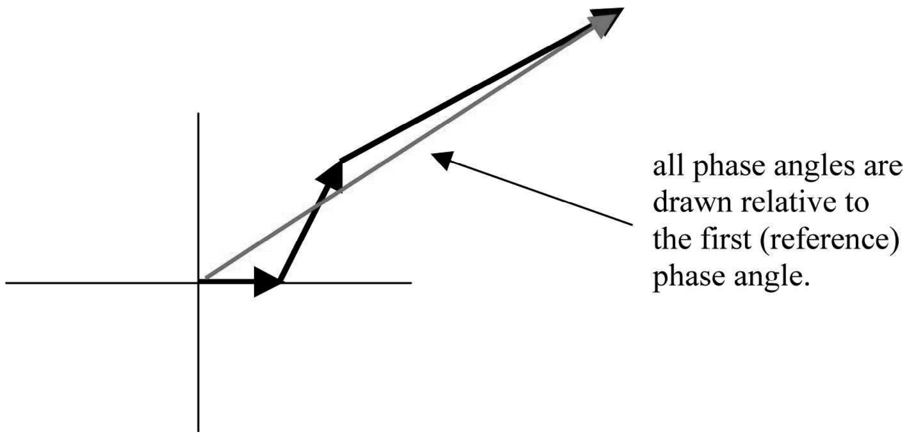
$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \tag{193}$$

What is a phasor?

A wave arriving at a receiver is characterized by 3 numbers: an amplitude, an intitial ($t = 0$) phase and a rate at which the phase changes (frequency).
A *phasor* is a rotating stick which is characterized by the same three numbers: the length of the stick, the angle the stick makes with the x-axis at $t = 0$, and a rate of rotation of the stick around the origin.

Adding waves?
Just add their phasors. Since all phasors rotate at the same rate, they don't change their relationship to each other, and we can ignore the rotation entirely. At that point each phasor only has two parameters - a length and angle. *It has become a vector.* Vectors can be added head to tail, as shown in the example below.
The length of the vector resultant is the amplitude of the total wave disturbance. The angle of the resultant with the x-axis is the initial phase of the total wave disturbance.

Sample phasor calculation



$$\cos(\omega t) + 2\cos\left(\omega t + \tfrac{\pi}{3}\right) + 5\cos\left(\omega t + \tfrac{\pi}{4}\right)$$

Find the length of the resultant vector.

x	**y**
1	0
2cos(60)	2sin(60)
5cos(45)	5sin(45)
Total x = 5.6	**Total y = 5.3**

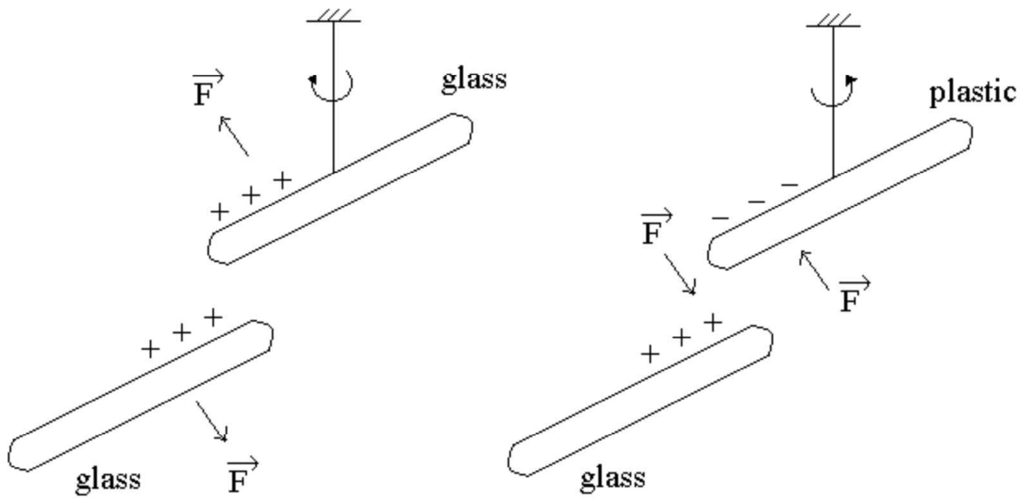
$$R = \sqrt{5.6^2 + 5.3^2} = 7.7$$

$$\tan\theta = \tfrac{5.3}{5.6}, \theta = 43^\circ$$

$$\text{Resultant} = 7.7\cos(\omega t + 43^\circ)$$

Electricity and Magnetism

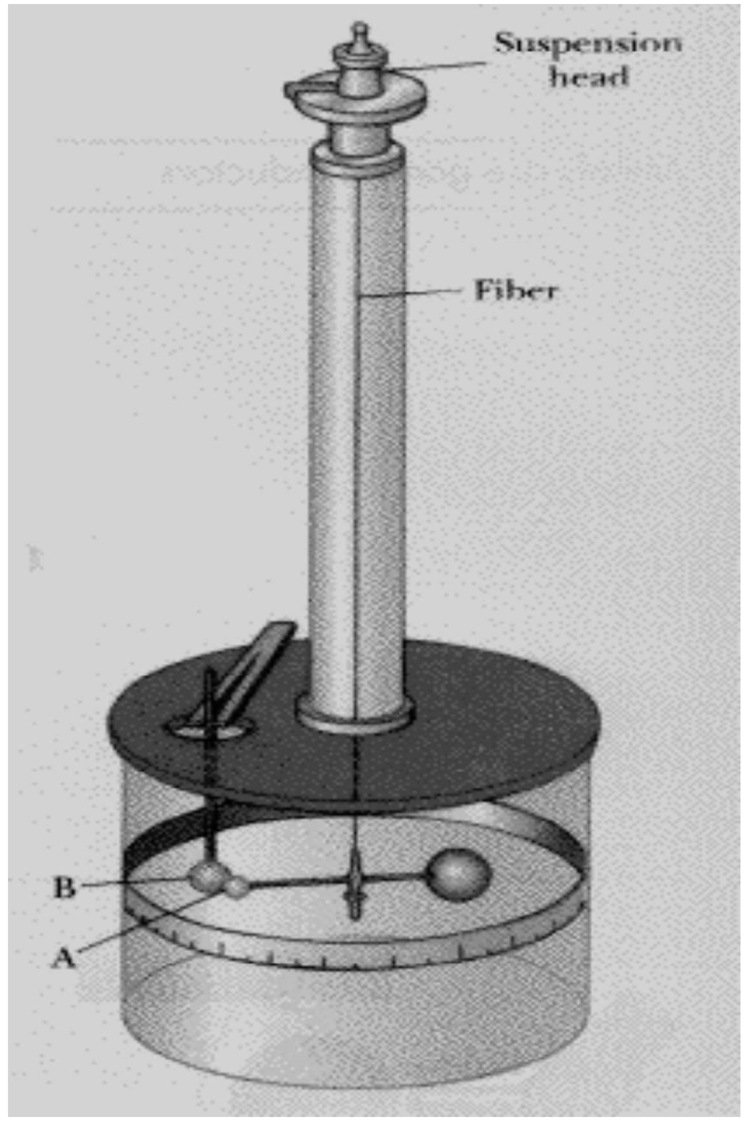
Electric Charge



Charges of like sign repel; charges of opposite sign attract

- 2 kinds/"signs" of charge
- a charge-charge interaction through a force

Coulomb's Torsion Balance



Coulomb's torsion balance, ∼ 1785

fiber= torsional spring

$$\tau = -k\theta \tag{194}$$

Measures force between charges on A, B

Electrostatics

- stationary or slowly moving charges

Coulomb's Law

$$|F_{electrical}| \propto \frac{q_1 q_2}{r^2} \tag{195}$$

$$|F| = \frac{k q_1 q_2}{r^2} \text{(for 2 point charges)} \tag{196}$$

r^2?

$r^{2+\in}, \in \leq 10^{-16}$

Compare to gravity, $|F| = G\frac{m_1 m_2}{r^2}$

In E& M the force can be attractive *or* repulsive.

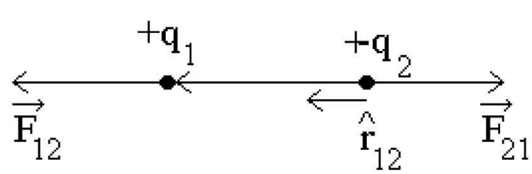
Choose $k = \frac{1}{4\pi\epsilon_0}$

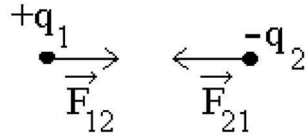
$$\begin{aligned} \text{Electric permitivity: } \epsilon_0 &= 8.854 \times 10^{-12} \frac{C^2}{Nm^2} \\ k &= 8.99 \times 10^9 \frac{N \cdot m^2}{C^2} \end{aligned}$$

(call it 9)

$$|F| = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \tag{197}$$

In vector form...





$$\vec{F}^*_{12} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}^2}\hat{r}_{12}, \text{where } \hat{r}_{12} = \text{the unit vector to 1 from 2} \tag{198}$$

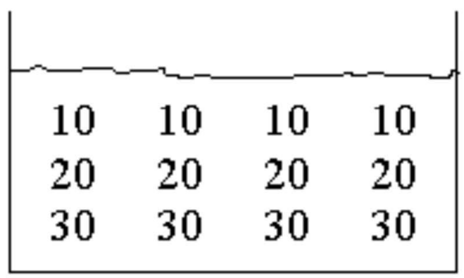
$$\vec{F}_{21} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{21}^2}\hat{r}_{21}, \text{where } \hat{r}_{21} = \text{the unit vector to 2 from 1} \tag{199}$$

*Note: $\vec{F_{12}}$ = force on # 1 from # 2
"equal and opposite" forces

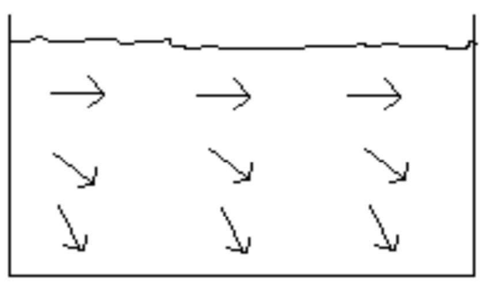
What is a "FIELD"?

Temperature(x, y, z) - a "scalar field"
Attaches a *number* to each point in 3-D space.

$\vec{p}(x, y, z)$ - a "vector field"
Attaches a *vector* to each point in space.



$\vec{v}(x, y, z)$: a fluid velocity vector
$\vec{v}(x, y, z, t)$: a series of snapshots if $v = v(t)$

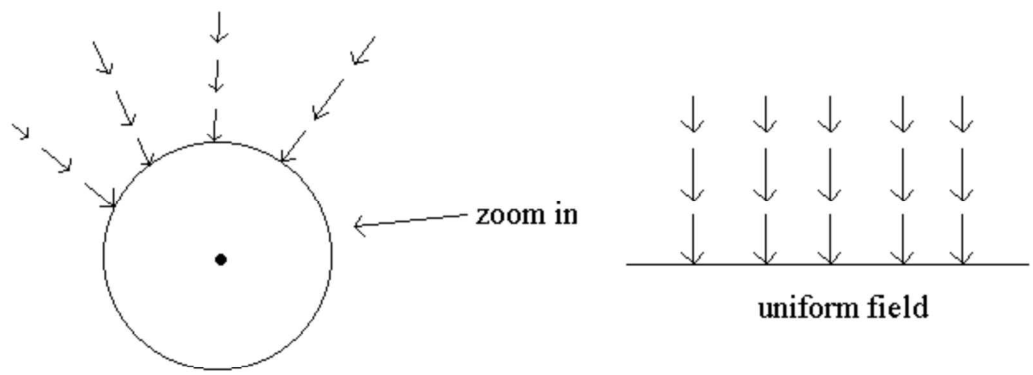


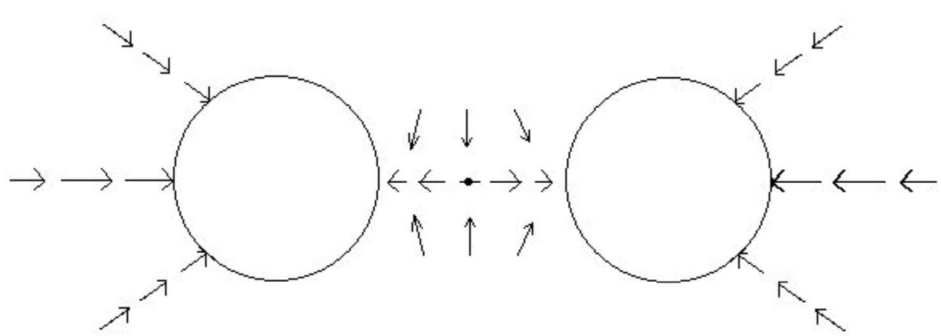
Gravitational Field

At each point in space . . .

$$\vec{g} = \frac{\vec{F}_{grav}}{m_0} \tag{200}$$

why define a test mass ?





$$\vec{F} = m\vec{g}_{\text{total}} \tag{201}$$

How can we determine the pattern?
What good is the pattern?

The Electric Field

$$\vec{g} \equiv \frac{\vec{F}}{m_0} \longrightarrow \text{Force per something} \quad (202)$$

$$\vec{E} \equiv \frac{\vec{F}}{q_0} \longrightarrow \text{("something" is a single relevant aspect of test particle)} \quad (203)$$

Remember, electric field is *basically* just a force
Don't let $\vec{E}$ become "abstract"!

$$\vec{E} \| \vec{F} \quad (204)$$

$$\vec{E} \propto \vec{F} \quad (205)$$

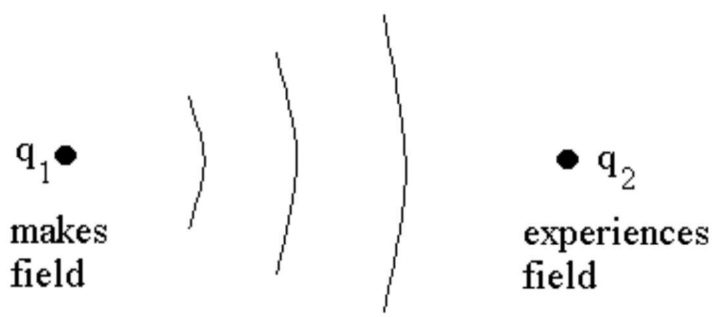
$$[E] = \frac{N}{C} \quad (206)$$

or . . .

$$\frac{\vec{F}}{q_1} = \frac{\frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}^2}\hat{r}_{12}}{q_1}, \quad (207)$$

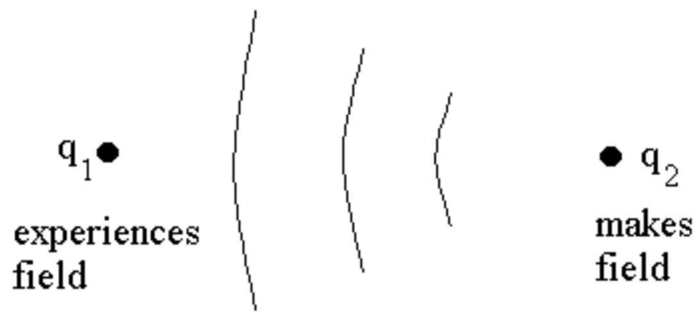
$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \quad (208)$$

How does $\vec{E}$ relate to the actual forces on charged particles?



$$\vec{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}\hat{r}, \quad (209)$$

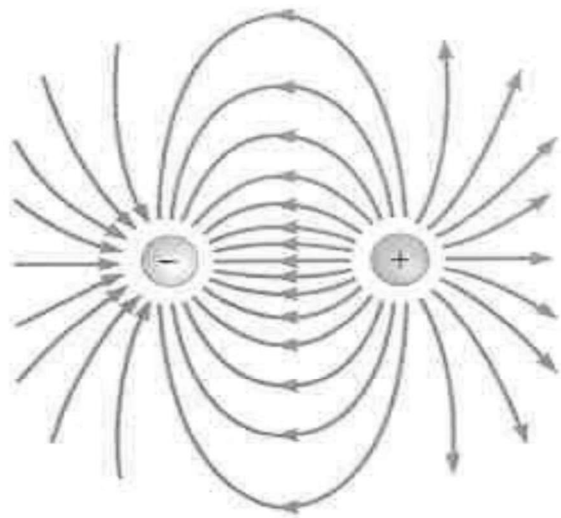
$$\vec{F} = q_2\vec{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}^2}\hat{r} \quad (210)$$

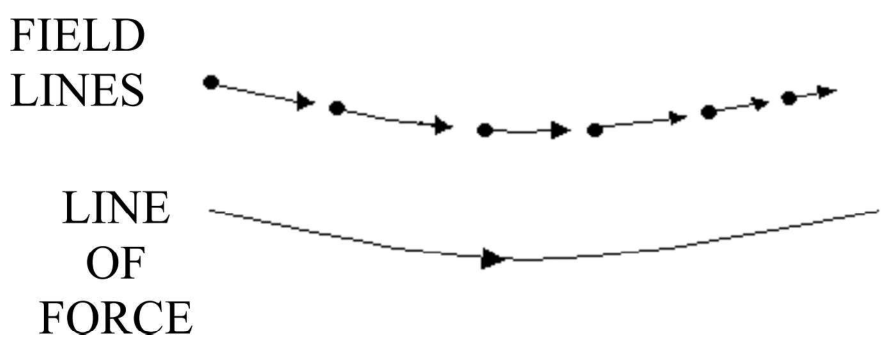


$$\vec{E}_2 = \frac{1}{4\pi\epsilon_0}\frac{q_2}{r^2}\hat{r}, \tag{211}$$

$$\vec{F} = q_1\vec{E}_2 = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r} \tag{212}$$

Dipole field - "Lines of force"





Something gained, something lost in moving from field lines to lines of force?

Actually determining motion in a non-uniform field

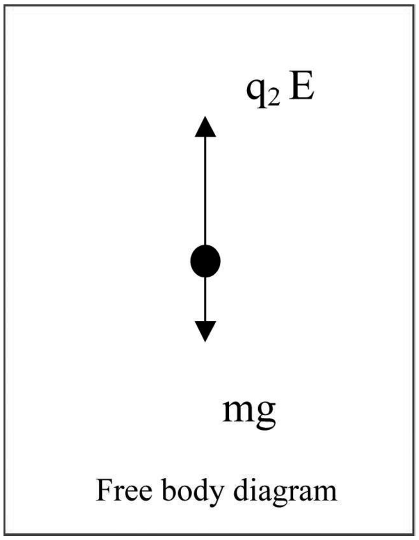
$$\vec{F} = q\vec{E}, \vec{E} = \vec{E}(x, y, z, t) \tag{213}$$

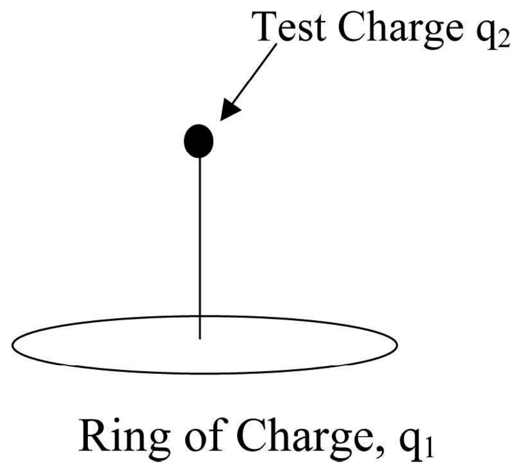
$$\longrightarrow \vec{a} = \frac{q\vec{E}}{m}\text{not constant} \tag{214}$$

- x_i, y_i, z_i and $v_{x_i}, v_{y_i}, v_{z_i}$ for charge q
- compute $\vec{E}$ at x_i, y_i, z_i
- compute $\vec{a}$ there
- $x_{new} = x_i + v_{x_i}\Delta t + \frac{1}{2}a_x\Delta t^2$
  $y_{new} = y_i + v_{y_i}\Delta t + \frac{1}{2}a_y\Delta t^2$
  small Δt (how small?)

Use of the kinematic equations in the last step is an approximation, since these are only true for constant acceleration.
Better to use Runge-Kutta to integrate Newton's 2^{nd} law directly.

Motion in a Non-Uniform $\vec{E}$ Field





$$|\vec{E}| = \frac{kq_1 z}{(z^2 + R^2)^{\frac{3}{2}}} \tag{215}$$

$$\Sigma \vec{F} = m\vec{a} \tag{216}$$

$$-mg + \frac{kq_1 q_2 z}{(z^2 + R^2)^{\frac{3}{2}}} = m\ddot{z} \tag{217}$$

A single 2^{nd} order differential equation

Define

$$\dot{z} = p \tag{218}$$

so

$$\dot{p} = \ddot{z} = -g + \frac{kq_1 q_2}{m} \frac{z}{(z^2 + R^2)^{\frac{3}{2}}} \tag{219}$$

(Note: The following are two 1^{st} order ode's, ready for numerical solution: $\dot{z} = p \& \dot{p} =$ the above ...)

Field From Point Charges

$$\frac{\vec{F}}{q} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} = \vec{E} \tag{220}$$

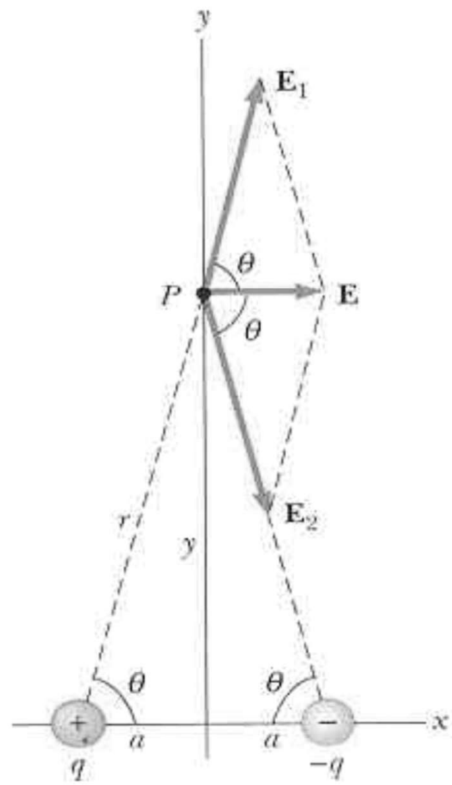
$$\vec{E}_{total} = \Sigma_{i=1}^{n}\vec{E}_i \tag{221}$$

Add $\vec{E}$ fields as we added $\vec{E}$ forces

CONVENTION…

The direction of $\vec{E}$ is the direction of the force on a positive test charge.

Electric Dipole



$$\vec{E} = \vec{E_1} + \vec{E_2} \tag{222}$$

$$|E_1| = |E_2| = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{q}{y^2+a^2} \tag{223}$$

Add vectorially.
Vertical components cancel by symmetry.

$$\text{Horiz. components} = \frac{1}{4\pi\epsilon_0}\frac{q}{y^2+a^2}\cos\theta \tag{224}$$

$$\cos\theta = \frac{a}{(y^2+a^2)^{\frac{1}{2}}} \tag{225}$$

$$\vec{E_T} = \frac{1}{4\pi\epsilon_0}\frac{2qa}{(y^2+a^2)^{\frac{3}{2}}}\hat{\imath} \text{ (valid only along this line.)} \tag{226}$$

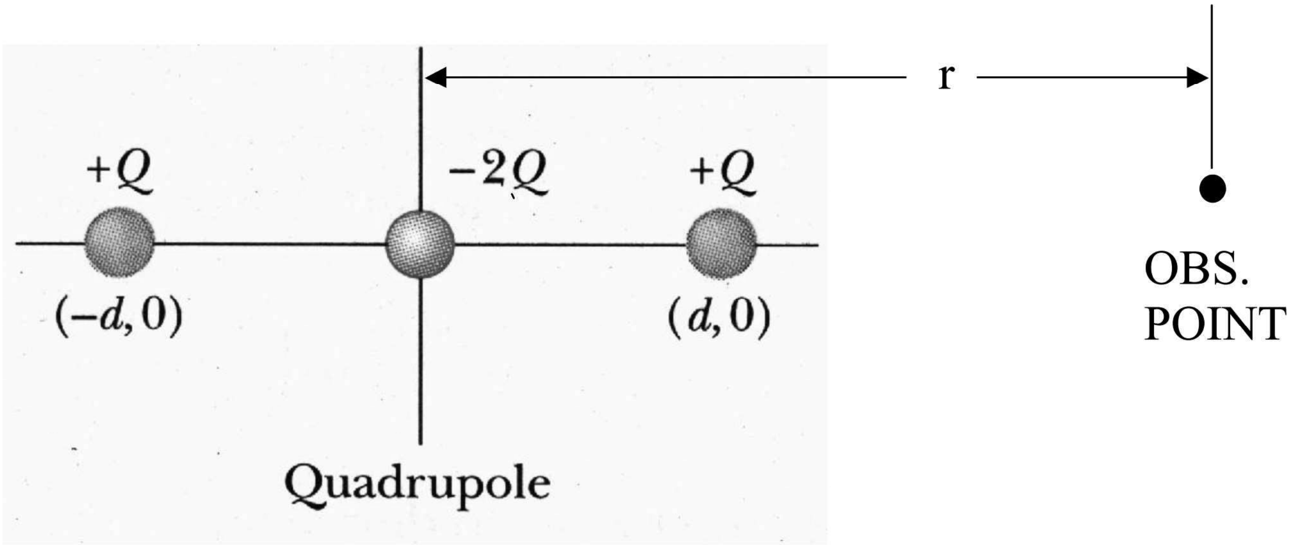
$$\text{If y >>a? Lowest order approximation} \tag{227}$$

$$|\vec{E}| = \frac{1}{4\pi\epsilon_0}\frac{q(2a)}{y^3} \tag{228}$$

$$q(2a) \equiv p \longleftrightarrow \text{Electric dipole moment} \tag{229}$$

Note the inverse distance^3 result.

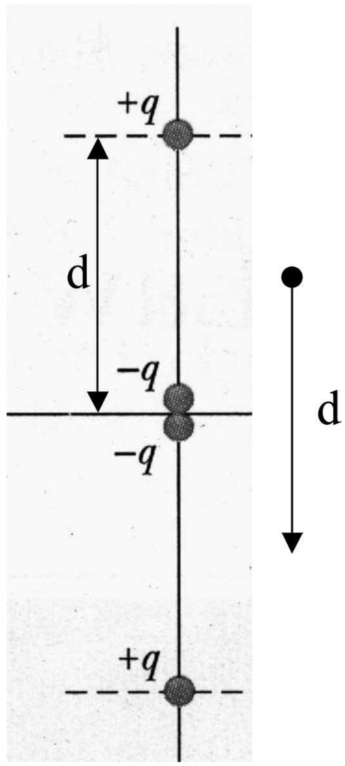
Electric Quadrupole



$$\begin{aligned}
\vec{E_{tot}} &= \vec{E_1} + \vec{E_2} + \vec{E_3} && (230)\\
&= k\left[\frac{q}{(r+d)^2} - \frac{2q}{r^2} + \frac{q}{(r-d)^2}\right] && (231)\\
\frac{1}{(r+d)^2} &= \frac{1}{(r^2+2rd+d^2)} \approx \frac{1}{r^2+2rd} && (232)\\
&= \frac{1}{r^2}\frac{1}{\left(1+\frac{2d}{r}\right)} \approx \frac{1}{r^2}\left(1-\frac{2d}{r}\right) && (233)\\
\therefore \vec{E_{tot}} &\approx \frac{kq}{r^2}\left[\left(1-\frac{2d}{r}\right) - 2 + \left(1+\frac{2d}{r}\right)\right] && (234)\\
&= 0 && (235)
\end{aligned}$$

Whoops! The field is small, but it isn't zero.
We didn't do the approximation carefully enough.

Another View of the Quadrupole



2 dipoles - 1 flipped upside-down & displaced by "d"

$$\vec{E}_{tot} = \vec{E}_{dip1} + \vec{E}_{dip2} \tag{236}$$

For each $\vec{E}_{dip}$ use the result for large r along the axis.

$$\rightarrow |\vec{E}_{tot}| \propto \frac{1}{r^4} \tag{237}$$

<u>Exercise:</u> Prove this result, find the proportionality constant.

More On Electric Charge

Charge is "quantized"

$$q = ne \tag{238}$$
$$(\pm)e = 1.602 \times 10^{-19} C \tag{239}$$

q=3n$\left(\frac{1}{3}e\right)$ Quarks
Charge on penny (of each sign)
$q = N(Ze), Z =$ atomic #
$\approx 10^{23}$ atoms (29 $\frac{\text{charges}}{\text{atom}} \times 1.6 \times 10^{-19}$)
$\approx 10^5 C$! little of this is "available."

Conservation of Charge

Not just a good idea, it's the law!

- Charge may be redistributed (may move from one object to another, or move within a single object)
- Sometimes charges cannot move freely (e.g., positive charges in nuclei, charges in 'insulators')
- Conservation even when charge is created or destroyed.

$$e^{+} + e^{-} \longrightarrow \gamma + \gamma \tag{240}$$
$$\mu \longrightarrow p + e^{-} + \gamma \tag{241}$$

Electric Fields from *Continuous* Charge Distributions

$$\vec{E}_{atP} = \int d\vec{E}_{atP} \quad (242)$$

$$E_x = \int dE_x \quad (243)$$

$$E_y = \int dE_y \quad (244)$$

etc.

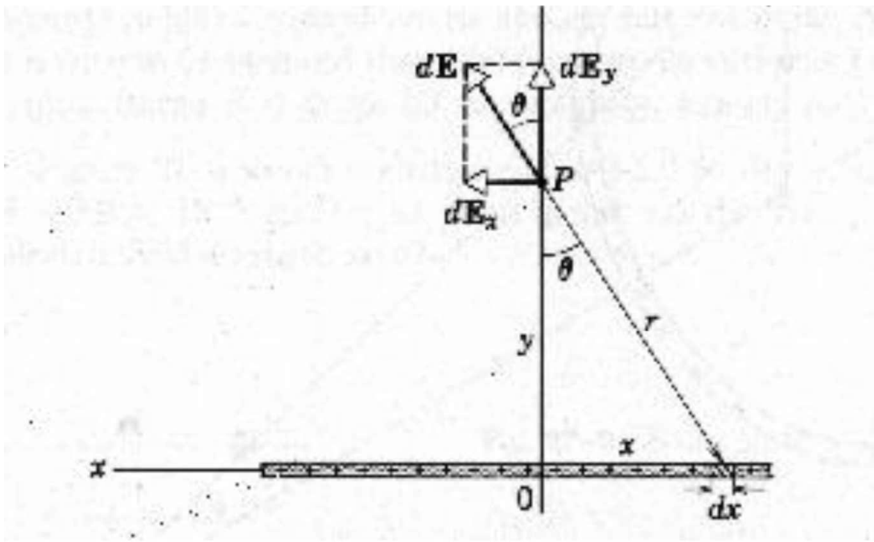
$$dE = \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2} \text{ (Really want to } \Sigma \text{ over each } \pm e) \quad (245)$$

- Compute dE for each (scalar) component
- r is the distance from the charge element dq to the point of observation, P.

OFTEN...

$$\begin{aligned} dq &= \lambda ds, \text{where } \lambda = \frac{\text{charge}}{\text{length}} && (246) \\ &= \sigma dA, \text{where } \sigma = \frac{\text{charge}}{\text{area}} && (247) \\ &= \rho dV, \text{where } \rho = \frac{\text{charge}}{\text{volume}} && (248) \end{aligned}$$

Infinite Line of Charge



$$dq = \lambda dx \tag{249}$$

$$dE = \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{\lambda dx}{y^2+x^2} \tag{250}$$

$$dE_y = dE\cos\theta \tag{251}$$

$$dE_x = dE\sin\theta \tag{252}$$

$$(E_x = 0, \text{ by symmetry})$$

$$x = y\tan\theta, dx = y\sec^2\theta d\theta \tag{253}$$

$$E = E_y = 2\int_{z=0}^{\infty} dE\cos\theta \tag{254}$$

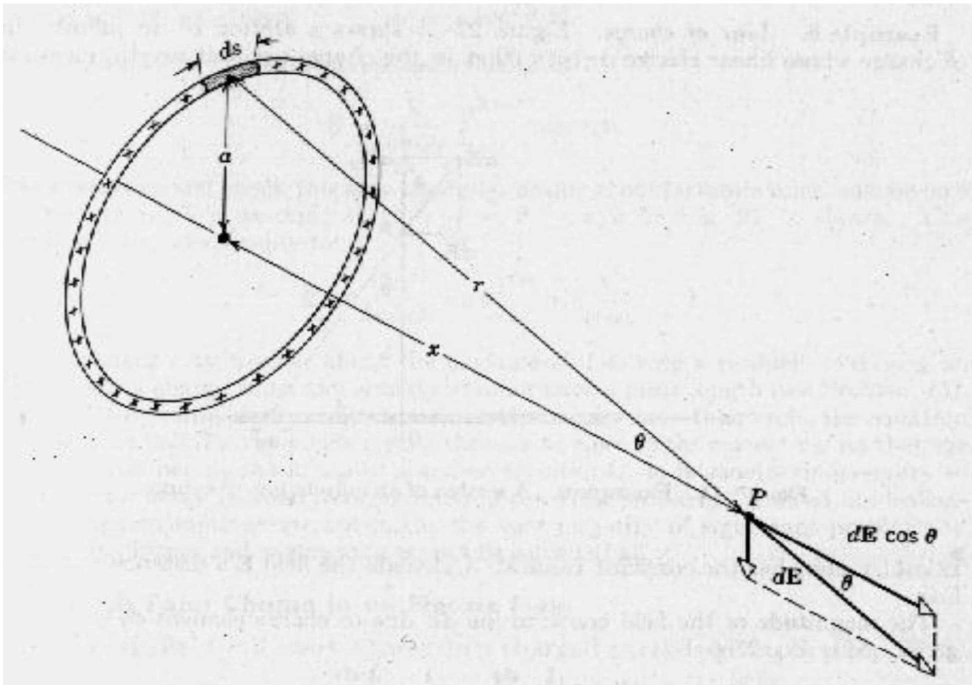
$$= \frac{\lambda}{2\pi\epsilon_0}\int_0^{\infty}\cos\theta\frac{dx}{y^2+x^2} \tag{255}$$

$$= \frac{\lambda}{2\pi\epsilon_0 y}\int_0^{\frac{\pi}{2}}\cos\theta d\theta \tag{256}$$

[careful! I omitted some algebra...]

$$= \frac{\lambda}{2\pi\epsilon_0 y}\hat{y} \tag{257}$$

Ring of Charge



$$dq = \lambda ds \tag{258}$$

$$dE = \frac{1}{4\pi\epsilon_0}\frac{\lambda ds}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{\lambda ds}{x^2+a^2} \tag{259}$$

$$E_y = E_z = 0, E_x \neq 0 \text{"symmetry"}$$

$$\cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2+a^2}} \tag{260}$$

$$\text{x-component} = dE\cos\theta$$

$$dE_x = \frac{x\lambda ds}{4\pi\epsilon_0(x^2+a^2)^{\frac{3}{2}}} \tag{261}$$

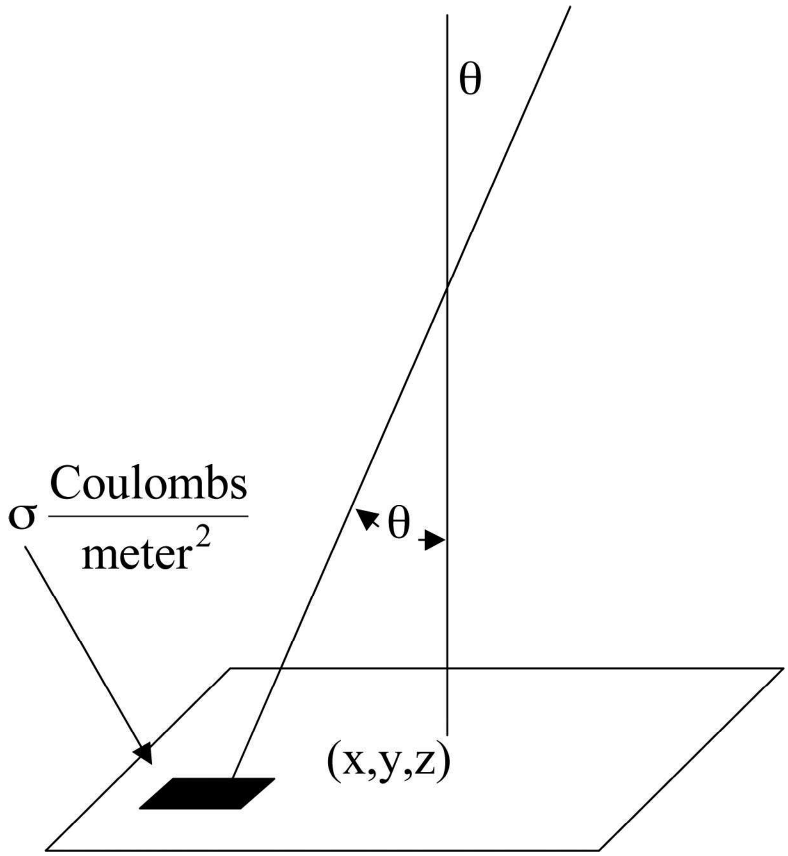
$$E_x = \int dE_x = () \int ds = ()2\pi a = \frac{qx}{4\pi\epsilon_0(x^2+a^2)^{\frac{3}{2}}} \tag{262}$$

$$\vec{E}_{ring} = \frac{qx}{4\pi\epsilon_0(x^2+a^2)^{\frac{3}{2}}}\hat{x} \tag{263}$$

$$\text{Check limiting values...}$$
$$\lim_{x \Rightarrow \infty} \vec{E} = \frac{q}{4\pi\epsilon_0} \lim_{x \Rightarrow \infty} \frac{x}{(x^2 + a^2)^{\frac{3}{2}}} \tag{264}$$
$$\frac{x}{x^3 \left(1 + \frac{a^2}{x^2}\right)^{\frac{3}{2}}} = \frac{1}{x^2} \tag{265}$$
$$\frac{1}{4\pi\epsilon_0} \frac{q}{x^2} \ \ldots \text{point charge result} \tag{266}$$

q may appear as...
$\lambda\cdot$ (Total length)
$\sigma\cdot$ (Total area)
$\rho\cdot$ (Total volume)

Infinite Insulating Sheet of Charge



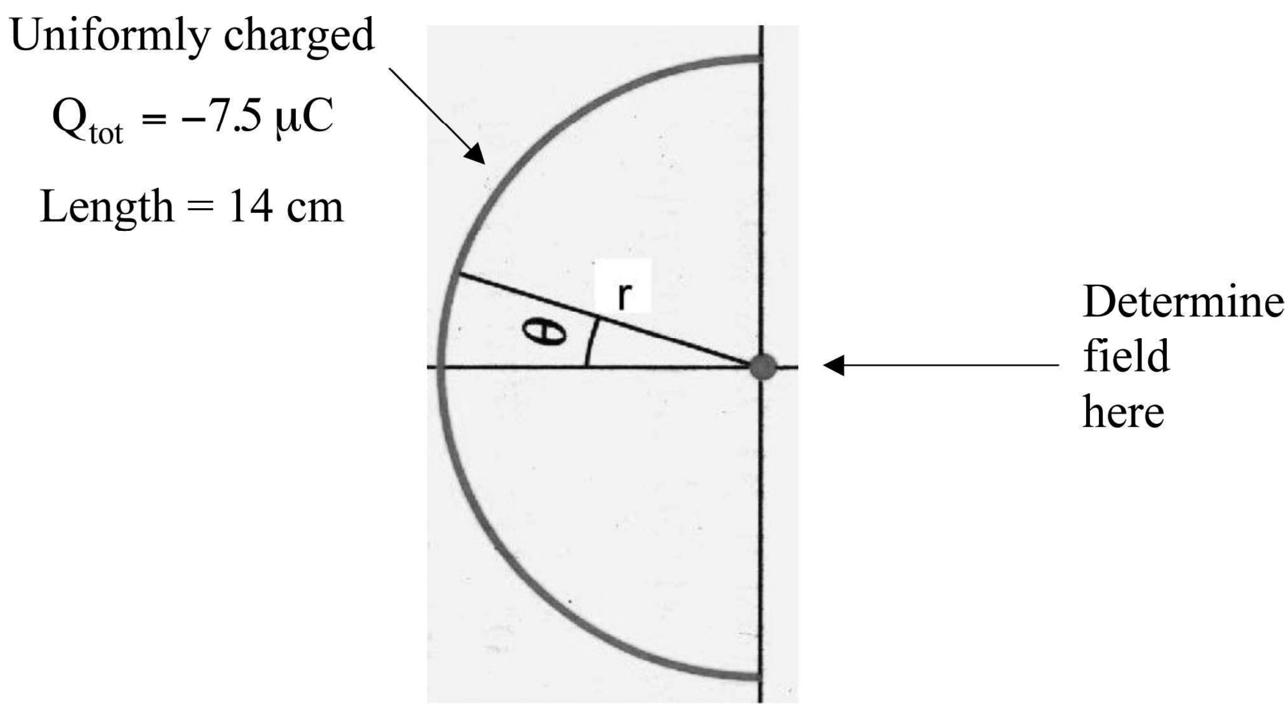
$$dq = \sigma dA = \sigma dxdy \tag{267}$$

$$E_x = E_y = 0 \tag{268}$$

$$dE_z = \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2}\cos\theta \tag{269}$$

$$dE_z = \frac{1}{4\pi\epsilon_0}\frac{\sigma dxdy}{(x^2+y^2+z^2)}\frac{z}{\sqrt{x^2+y^2+z^2}} \tag{270}$$

Finally integrate dx & dy from $-\infty$ to $+\infty$. (Interested in the setup, not the calculus.)



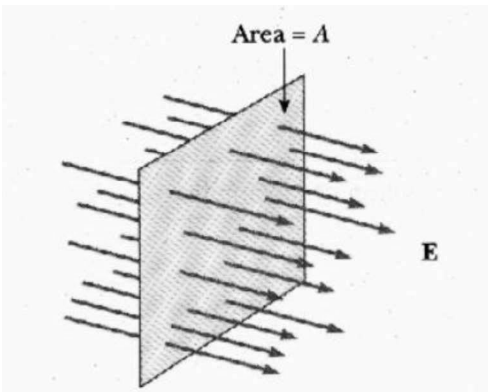
No net y-component

$$
\begin{aligned}
dE_x &= dE\cos\theta &(271)\\
E_x &= \frac{1}{4\pi\epsilon_0}\int\left(\frac{dq}{r^2}\cos\theta\right) &(272)\\
E_x &= \frac{1}{4\pi\epsilon_0}\int\left(\frac{\lambda ds}{r^2}\cos\theta\right), ds = rd\theta &(273)\\
&= \frac{1}{4\pi\epsilon_0}\lambda\frac{r}{r^2}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\cos\theta d\theta &(274)\\
&= \frac{\lambda}{4\pi\epsilon_0 r}\sin\theta|_{\frac{-\pi}{2}}^{\frac{\pi}{2}} &(275)\\
&= \frac{\lambda}{2\pi\epsilon_0 r} &(276)\\
&= \frac{\lambda(\pi r)}{4\pi\epsilon_0(\frac{r}{2})(\pi r)} &(277)\\
E_x &= \frac{1}{4\pi\epsilon_0}\frac{2\cdot Q_{tot}}{\pi r^2} &(278)\\
& &(279)
\end{aligned}
$$

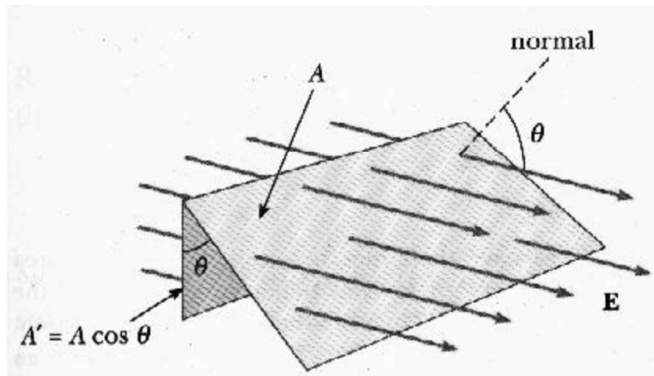
Gauss' Law

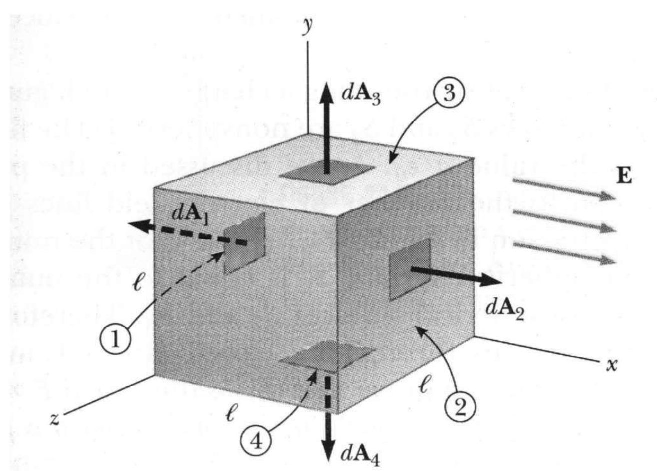
"Flux"

$$\Phi = \text{Rate of "fluid" passing through a loop} \tag{280}$$
$$\Phi = vA\cos\theta = \vec{v}\cdot\vec{A} \tag{281}$$



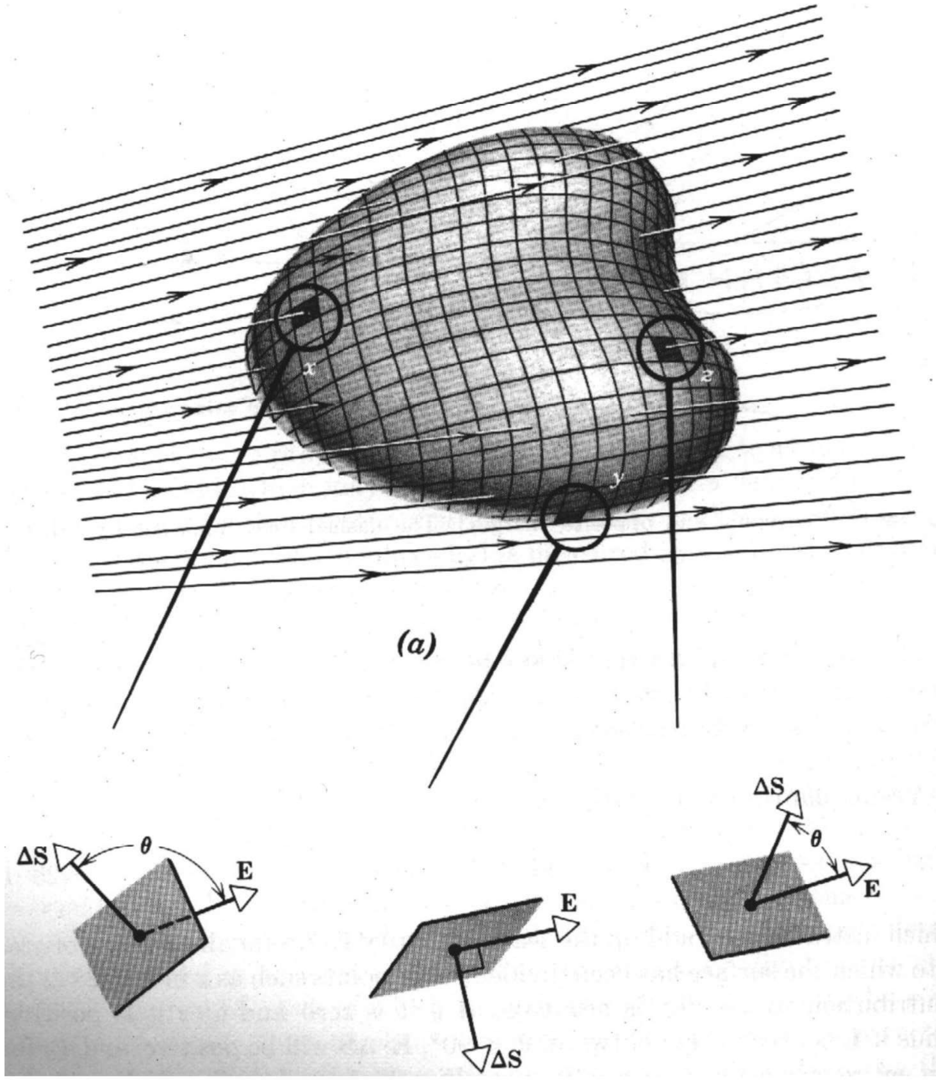
Field lines of a uniform electric field penetrating a plane of area A perpendicular to the field. The electric flux, Φ, through this area is equal to EA.





Net flux through closed surface = $\Sigma\vec{v}\cdot\vec{A}$ on the faces. $\Sigma\vec{v}\cdot\vec{A} = 0$ if no fluid is created/destroyed within the volume. In other words... no "sinks" or "sources."

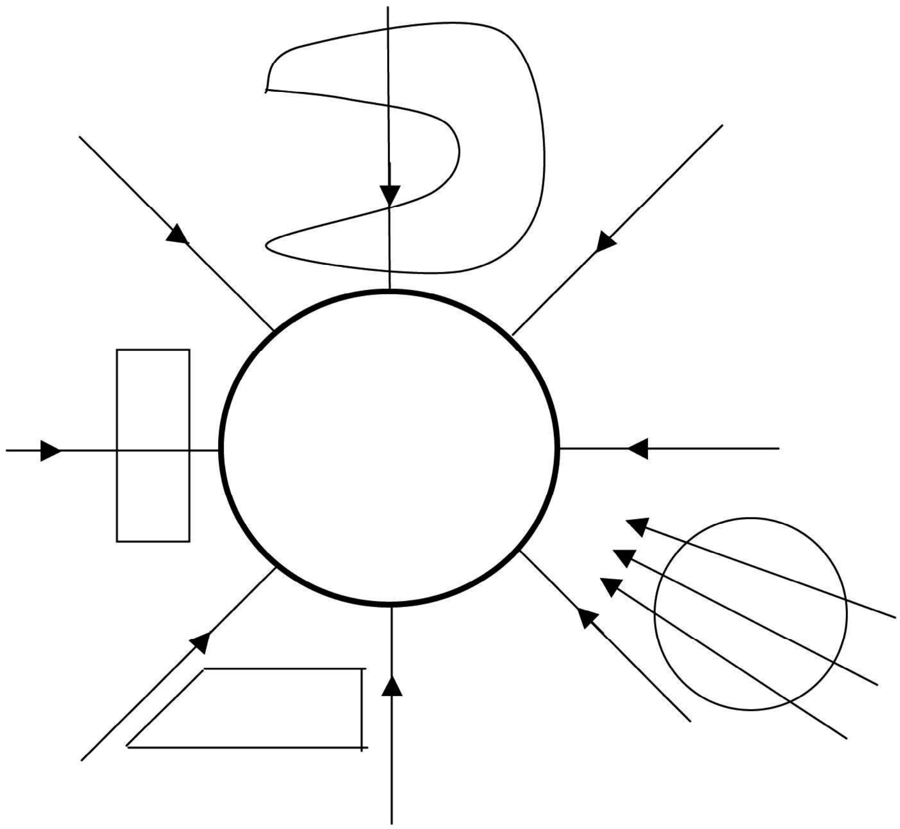
Arbitrary Closed Surface



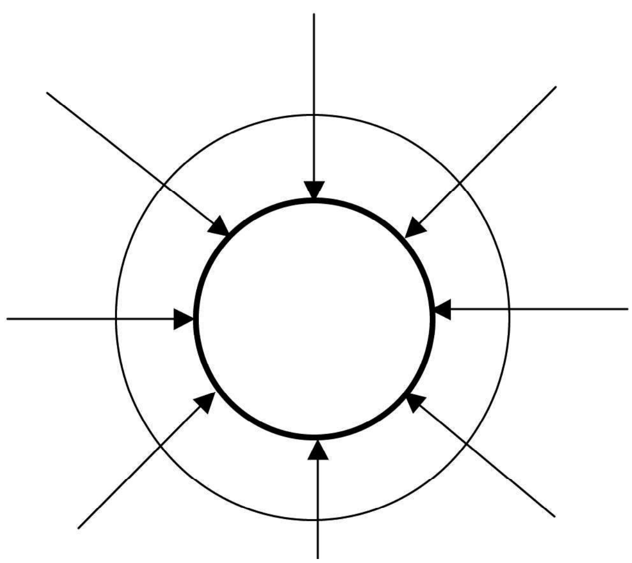
As book-keeping convention, all $\vec{S}$ vectors point <u>outward</u> from the surface. Cover the whole surface with "flat" patches of area $d\vec{S}$.

$$\Phi_{tot} = \int_{\text{closed surface}} \vec{v} \cdot d\vec{S} \tag{282}$$

Gravitational Flux



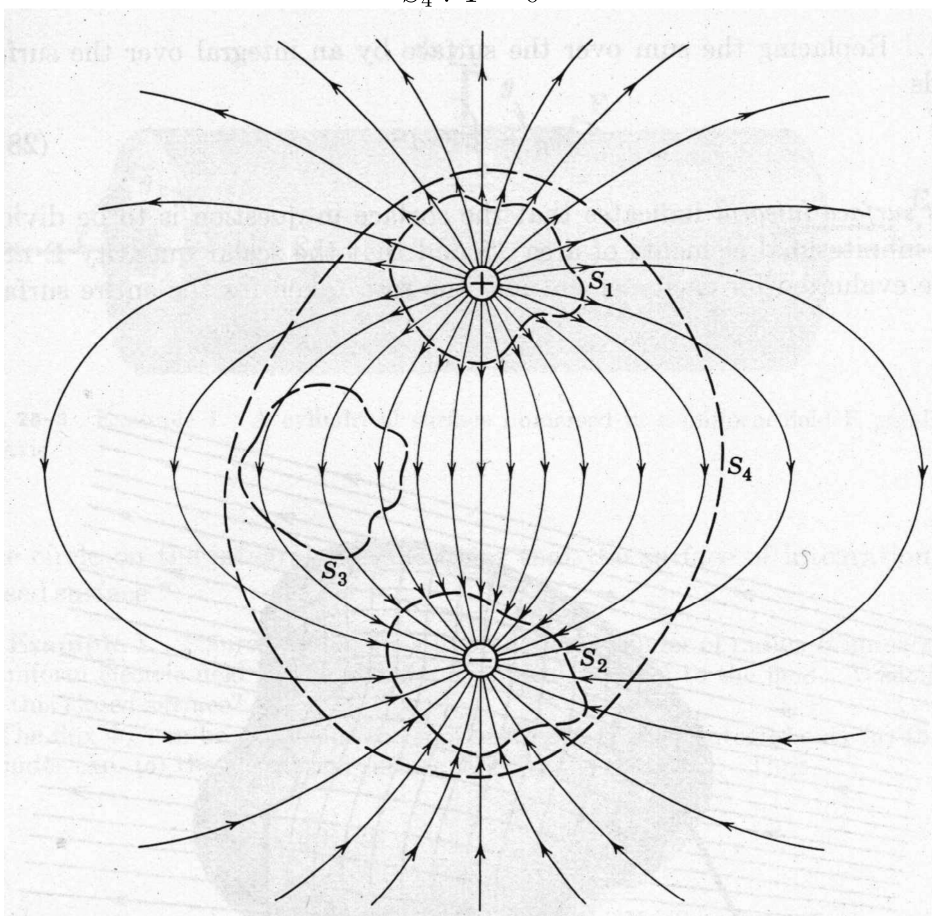
Net flux = 0 through each of the surrounding closed surfaces. Each field line that enters also leaves.



Net Flux $\neq 0$

Electric Flux

$S_1 : \Phi > 0$
$S_2 : \Phi < 0$
$S_3 : \Phi = 0$
$S_4 : \Phi = 0$

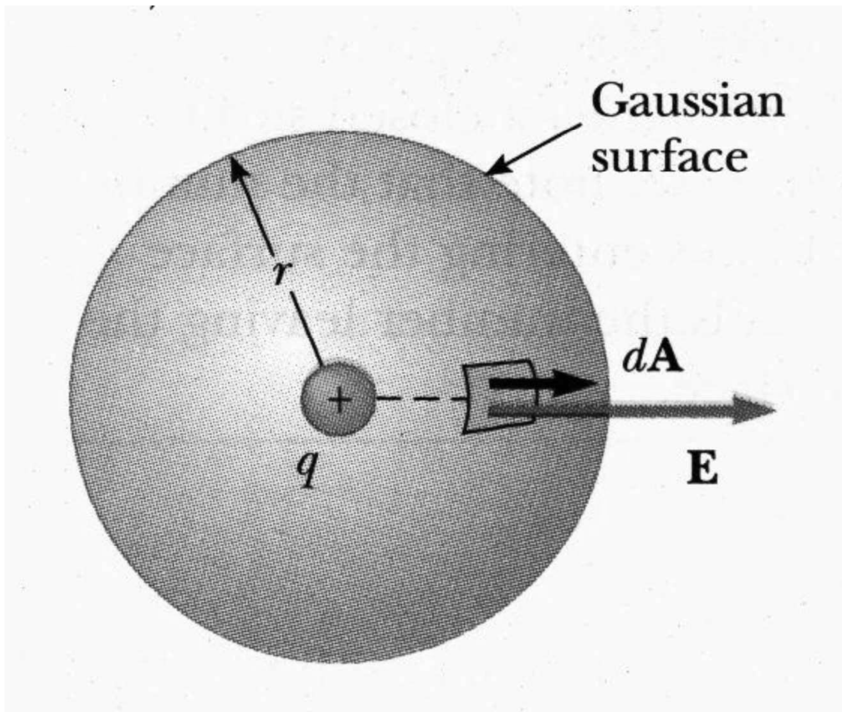


S_1 : encloses net positive charge
S_2 : encloses net negative charge
S_3, S_4 : enclose no net charge

Gauss' Law

$$\Phi_E = \frac{q}{\epsilon_0} \leftarrow q \text{ refers to the net enclosed charge} \tag{283}$$

$$\oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0} \tag{284}$$



$$\begin{aligned} \oint \vec{E} \cdot d\vec{A} &= E \int dA = EA && (285) \\ &= E(4\pi r^2) = \frac{q}{\epsilon_0} && (286) \\ |E| &= \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} && (287) \end{aligned}$$

So Gauss' Law is equivalent to Coulomb's Law.

Gauss' Law

- IS always true but
- IS NOT always useful (as a means of computing $\vec{E}$

If $|\vec{E}|$ is constant on the Gaussian surface & if $\cos\theta$ is constant on the Gaussian surface then...

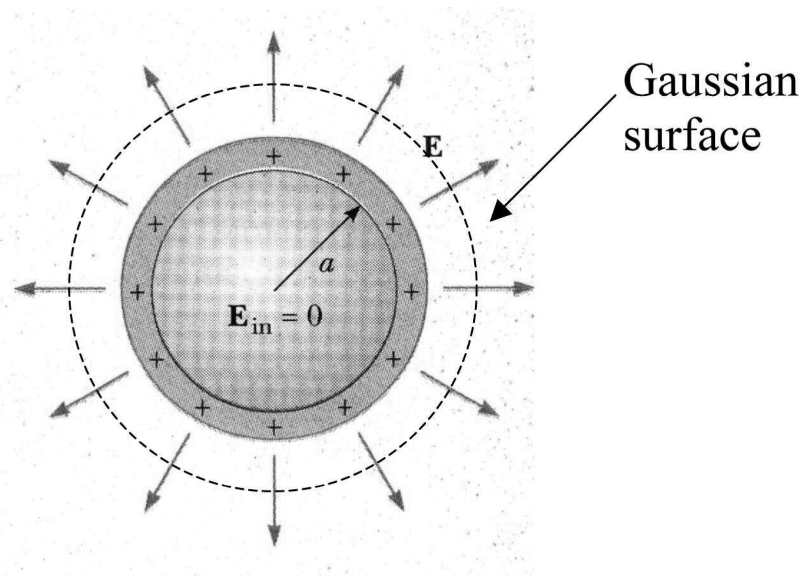
$$\oint \vec{E} \cdot d\vec{A} \equiv \oint |\vec{E}||d\vec{A}| \cos\theta \tag{288}$$

$$= |\vec{E}| \cos\theta \oint |d\vec{A}| \tag{289}$$

$$= |\vec{E}| \cos\theta A = \frac{q}{\epsilon_0} \tag{290}$$

can be solved for $|\vec{E}|$.
Else, $\vec{E}$ is buried in the integral.

Spherical Shell of Charge



Symmetry of charge distribution → radial field
Match the symmetry with a spherical Gaussian surface.
Surface outside shell...

$$\Phi = E(4\pi r^2) \tag{291}$$

$$\Phi = \frac{q}{\epsilon_0} \tag{292}$$

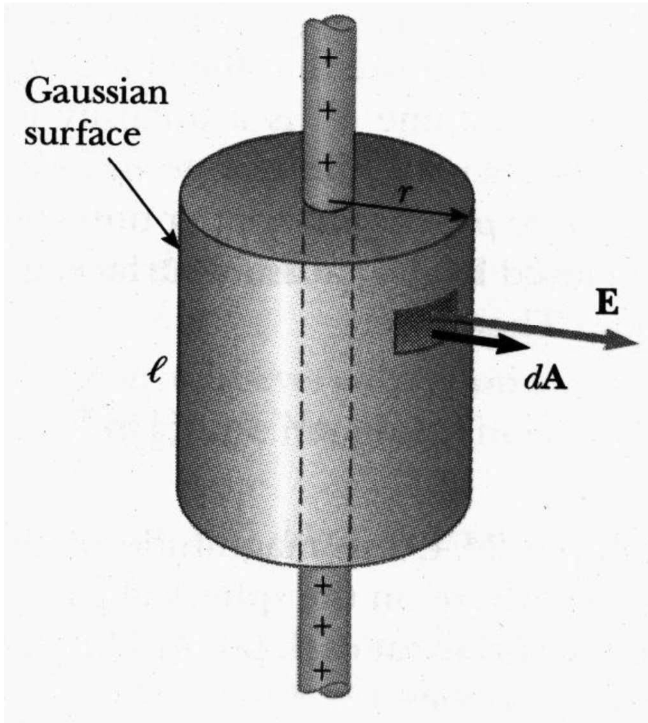
$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\text{(as if a point charge)} \tag{293}$$

Surface inside shell...

$$E(4\pi r^2) = \frac{0}{\epsilon_0} \tag{294}$$

$$E = 0 \text{ (Everywhere inside the shell)} \tag{295}$$

Infinite Line of Charge



λ C/m
Symmetry $\Rightarrow$ Radial Field
Choose circular cylinder as Gaussian surface
$\vec{E}\perp$ surface & $\vec{E}$ constant on the surface

$$\Phi = (2\pi r l)E \tag{296}$$

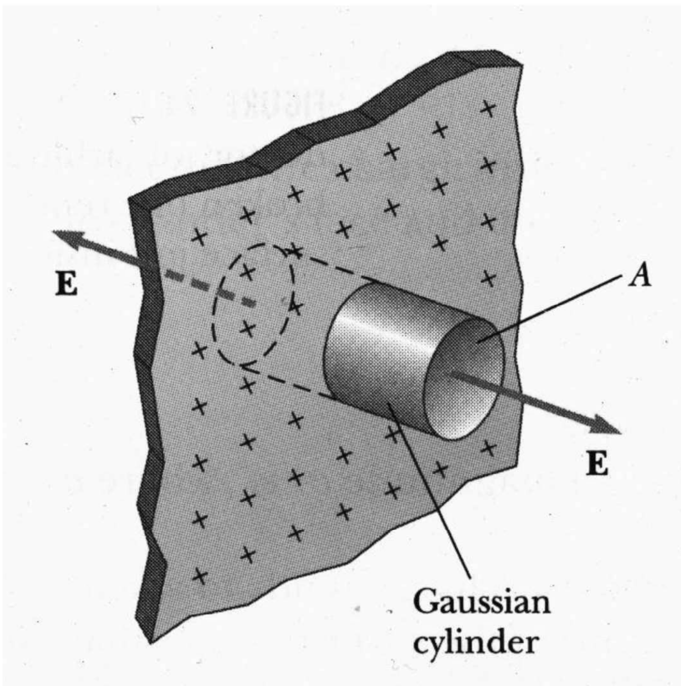
$$\Phi = \frac{q}{\epsilon_0} = \frac{\lambda l}{\epsilon_0} \tag{297}$$

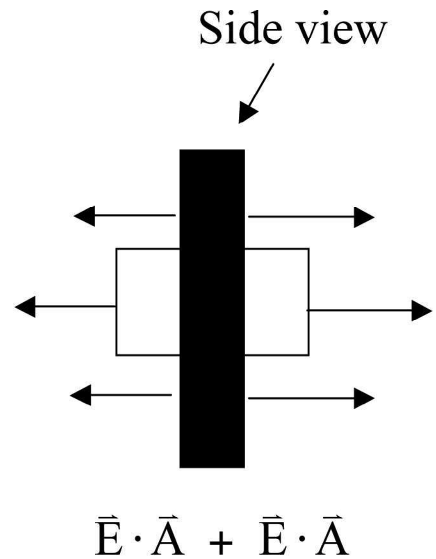
$$E = \frac{\lambda}{2\pi\epsilon_0 r} \tag{298}$$

Could you choose a different shape for the Gaussian surface?

*This is the first problem in which the Gaussian surface is made of distinct pieces (the "body" and the two "endcaps"). The total flux through the Gaussian surface is simply the sum of the fluxes through the individual pieces.

Infinite Insulating Sheet of Charge



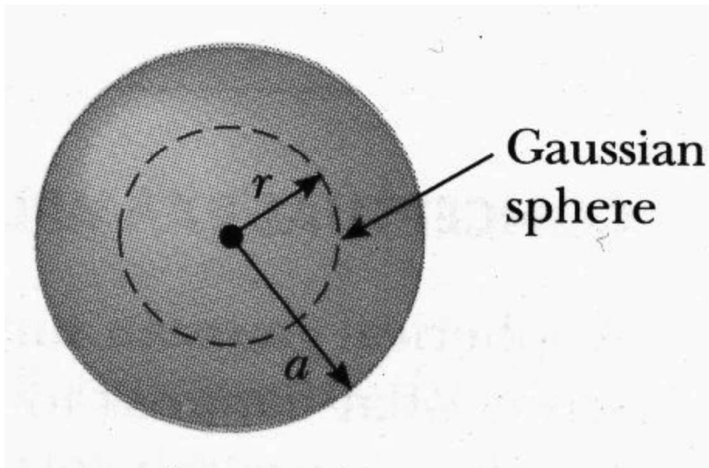


$$\oint \vec{E} \cdot d\vec{A} = \int_{\text{end cap on left}} + \int_{\text{end cap on right}} + \int_{\text{body of cylinder}} \tag{299}$$

$$2EA = \frac{\sigma A}{\epsilon_0} \tag{300}$$

$$E = \frac{\sigma}{2\epsilon_0} \tag{301}$$

Spherical Distribution of Charge



$\rho = \text{constant}$

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0} \tag{302}$$

$$E(4\pi r^2) = \frac{\rho \cdot \text{Vol}}{\epsilon_0} = \frac{\rho \frac{4}{3}\pi r^3}{\epsilon_0} \tag{303}$$

$$\therefore E = \frac{\rho}{3\epsilon_0} r \tag{304}$$

$$\text{or} \ldots Q = \frac{4}{3}\pi a^3 \cdot \rho \tag{305}$$

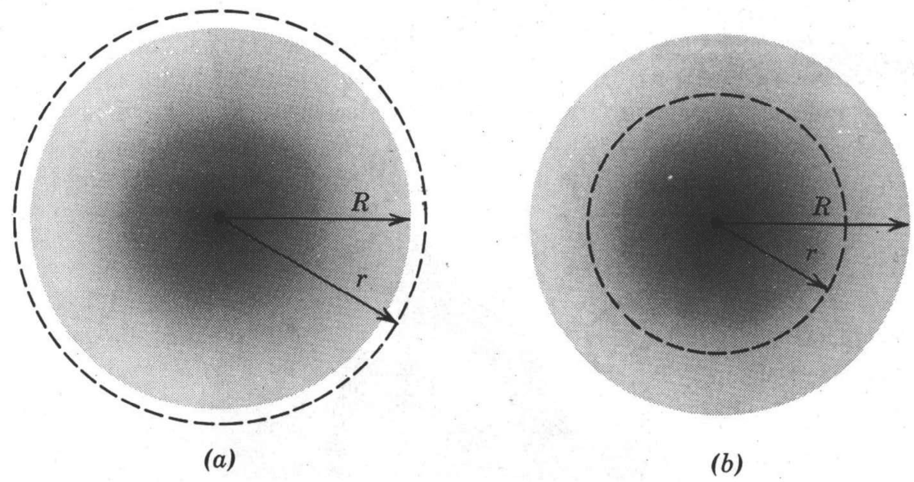
$$E = \frac{1}{4\pi\epsilon_0}\frac{Qr}{a^3} \tag{306}$$

If opp. test charge can move through this distribution,

$$F = -qE = -\frac{\rho q}{3\epsilon_0} r \text{ (S.H.M.)} \tag{307}$$

Note that the radius on both left and right hand side of Gauss' Law is "r" (and not "a"). On the left hand side this is so because we are working on the Gaussian surface, which is of radius r. On the right hand side, the charge continues up to r. Any charge further away (between r and a) is not "enclosed".

Suppose $\rho = \rho(r)$?



$$\int \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0} \tag{308}$$

$$E(4\pi r^2) = ? \tag{309}$$

$$= \frac{1}{\epsilon_0} \int \rho(r) dV \tag{310}$$

$$dV = 4\pi r^2 dr \tag{311}$$

$$\text{e.g., } \rho(r) = \frac{C}{r} \tag{312}$$

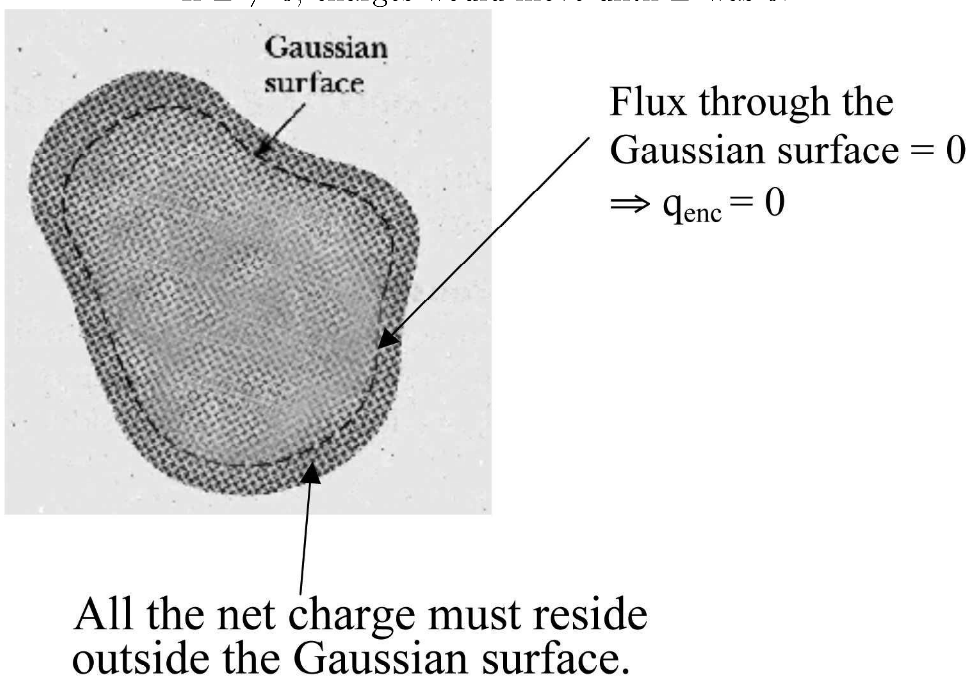
$$q_{enc} = \int_0^r \frac{C}{r} 4\pi r^2 dr \tag{313}$$

$$= 4\pi C \int_0^r r dr \text{ etc.} \tag{314}$$

On the left hand side of Gauss' Law, the radius of the Gaussian surface, "r" always appears (regardless of whether you are considering case a or case b).
On the right hand side of Gauss' Law, you integrate to find the total enclosed charge. Here, I am doing case b, so I only integrate up to the edge of the Gaussian surface, r, where the enclosed charge stops. For case a, I would integrate only up to R, since the enclosed charge stops at that radius, and doesn't continue up to r.

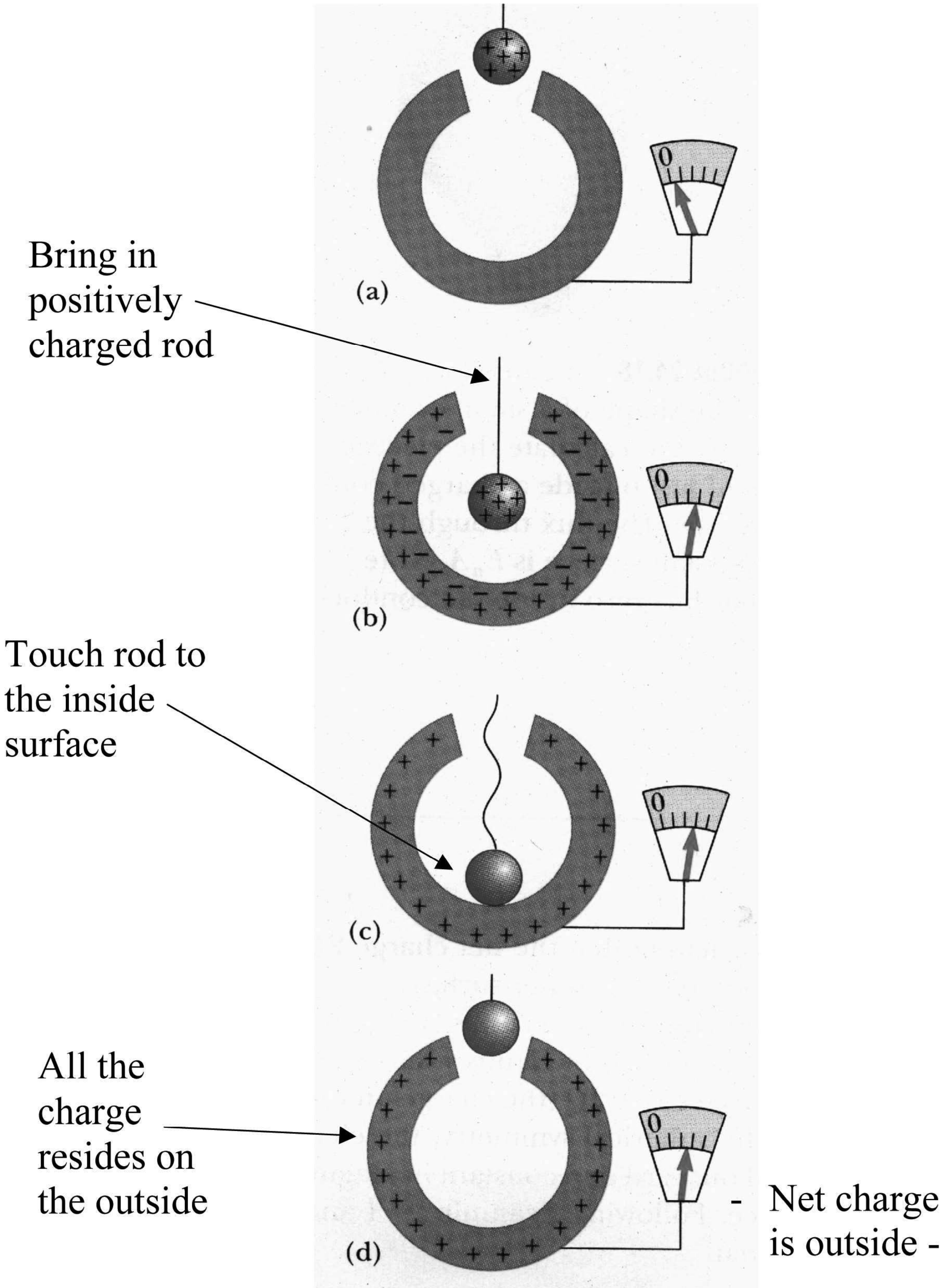
Even with Gauss' Law, you may still have to do an integral (usually an easy one).

CLAIM:
$\vec{E} = 0$ inside conductors
Proof:
If $\vec{E} \neq 0$, charges would move until $\vec{E}$ was 0!

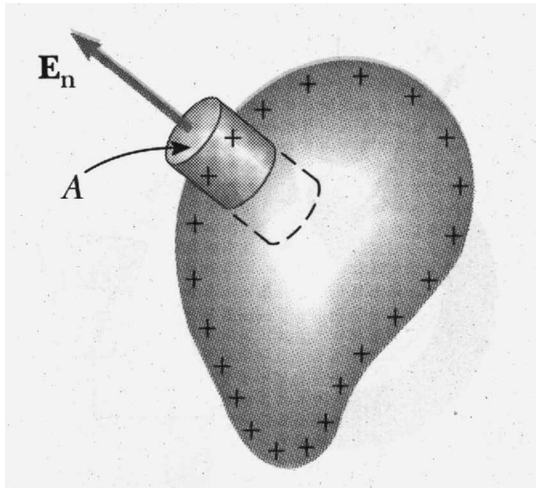


Move the Gaussian surface closer and closer to the outer surface.
Eventually the net charge is forced to reside on the outer surface.

Faraday's Ice Pail Experiment



$\vec{E}$ Just Outside A Conductor



Inside the conductor there is no $\vec{E}$ field, so no flux is coming out of the back endcap of the Gaussian surface.

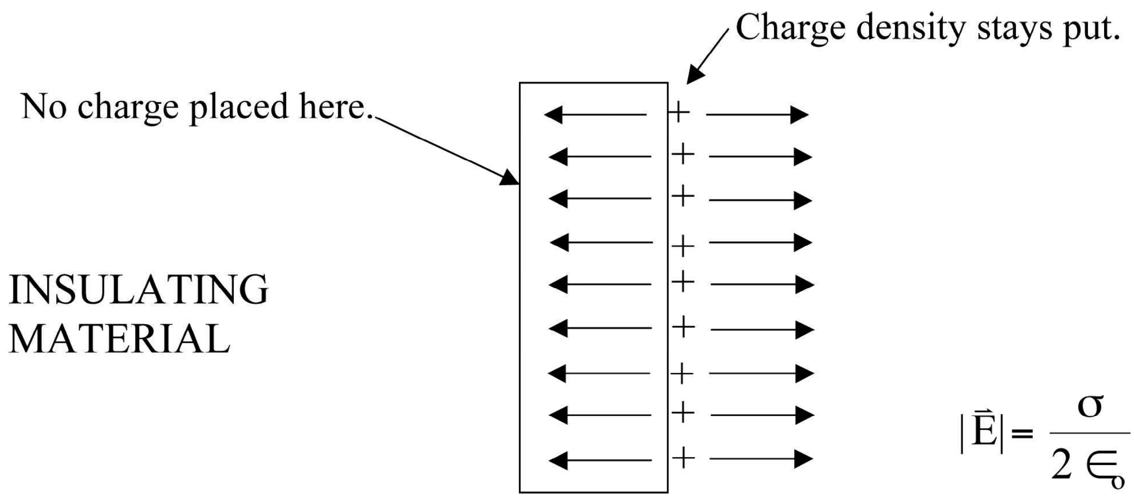
$$\oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0} \tag{315}$$

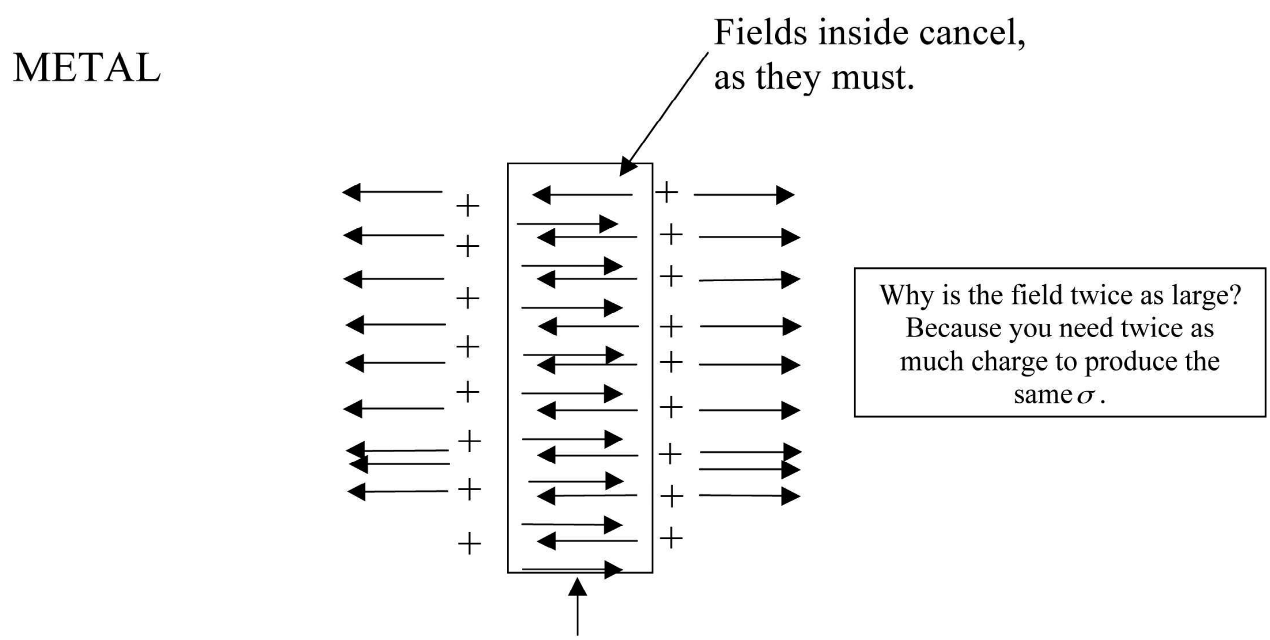
$$|\vec{E}|A = \frac{\sigma A}{\epsilon_0} \tag{316}$$

$$|\vec{E}| = \frac{\sigma}{\epsilon_0} \tag{317}$$

$$\sigma = \text{charge density in C/m}^2$$

Compare to infinite **insulating** sheet of charge.





Fields on either side add:

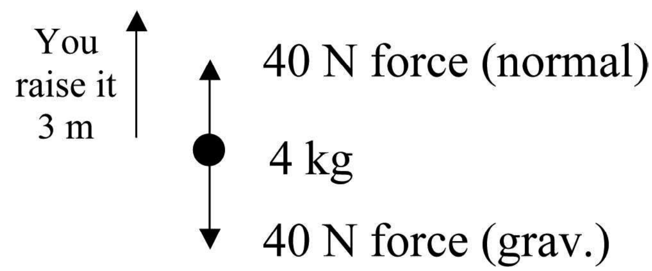
$$\frac{\sigma}{2\epsilon_o} + \frac{\sigma}{2\epsilon_o} = \frac{\sigma}{\epsilon_o}$$

Since all the charge in the metal moves to the outside, you get the same field as if there were two insulating layers.

Electric Potential Energy

Potential Energy:
When a force is conservative (path independent) you can define a "potential energy"
Example - gravitational force & energy

$$\Delta GPE = -W_{\text{by grav}} = -\int \vec{F} \cdot d\vec{r} \tag{318}$$



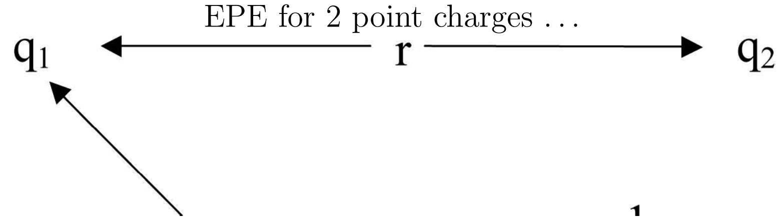
$$\begin{aligned} \Delta GPE &= mg(h_f - h_i) \\ &= 4(10)(3) \\ &= 120 \text{ J} \\ W_{bygrav} = \vec{F}_{grav} \cdot \vec{d}? & \\ &= 4(10)3 \cdot \cos 180^\circ \\ &= -120 \text{ J} \end{aligned}$$

Why negative? Gravity does negative work on the way up, but positive work on the way back down. Knowing the positive work *will* happen, you can pretend that the positive work is already 'owned' by the mass.

Same for the electric force ...

$$\begin{aligned} \Delta EPE &= EPE_f - EPE_i & (319) \\ &= -W_{byelec.} & (320) \\ &= -\int \vec{F} \cdot d\vec{s} = -\int q\vec{E} \cdot d\vec{s} & (321) \end{aligned}$$

EPE for 2 point charges ...



q_1 produces a field: $\frac{1}{4\pi\epsilon_0} \frac{q_1}{r^2} \hat{x}$

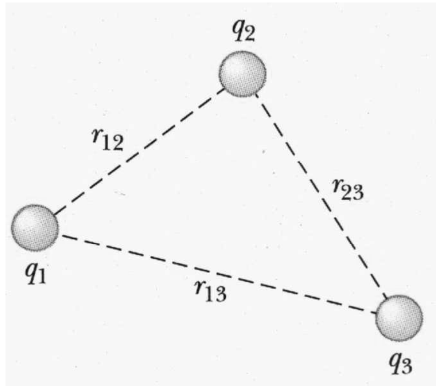
$$\Delta EPE = EPE_f - EPE_i \tag{322}$$
$$= -\int_{r_i}^{r_f} q_2 \left[\frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}\right] dr \tag{323}$$
$$= \frac{1}{4\pi\epsilon_0} q_1 q_2 \left(\frac{1}{r_f} - \frac{1}{r_i}\right) \tag{324}$$

if you <u>choose</u> r$_i = \infty$, EPE$_i = 0$,

$$EPE = \frac{q_1 q_2}{4\pi\epsilon_0 r} \tag{325}$$
$$GPE = \frac{G m_1 m_2}{r} \tag{326}$$

q_1, q_2 pos. $\rightarrow$ EPE pos. $\rightarrow$ W$_{\text{by E}}$ = negative. Why?

EPE of System of Charges



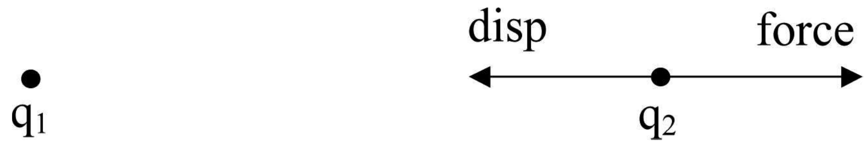
$$EPE = \frac{1}{4\pi\epsilon_0}\left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right] \tag{327}$$

Review the energy expenditures ...
q_1 first $\rightarrow$ "Free"
q_2 next $\rightarrow \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}}$
q_3 last $\rightarrow \frac{1}{4\pi\epsilon_0}\frac{q_1 q_3}{r_{13}} + \frac{1}{4\pi\epsilon_0}\frac{q_2 q_3}{r_{23}}$
Independent of order.
Where is the energy stored? (If you believe in potential energy at all...)

Checking the Signs

Creating a system of particles from particles initially at ∞.

1. VOID
2. Drag in q_1 from ∞ at a constant velocity. No work is required for this step.
3. Drag in q_2 from ∞. Since q_1, q_2 of like sign, charge repel. Work is required for this step.



Work done by $\vec{F}_{elec}$ is negative. Total work = negative

Electric Potential (not EPE)

$$V_p = \frac{\Delta EPE_p}{q_0} \tag{328}$$

Existing collection of charges...
Bring in a positive test charge, q_0 to point p
Measure change in EPE.

$$\frac{\Delta EPE}{q_0} = V_p, \text{electric potential at point p} \tag{329}$$

V_p =zero? $\Rightarrow$ No net attraction or repusion on a new test charge during trip from ∞. V_p =positive? $\Rightarrow$ Positive test charge will be repelled from existing charge distribution, positive work is done to bring it in, V_p further increases.

Units of Potential

$$[\text{Electric Potential}] = \text{"Volt"} = \frac{J}{C} \tag{330}$$

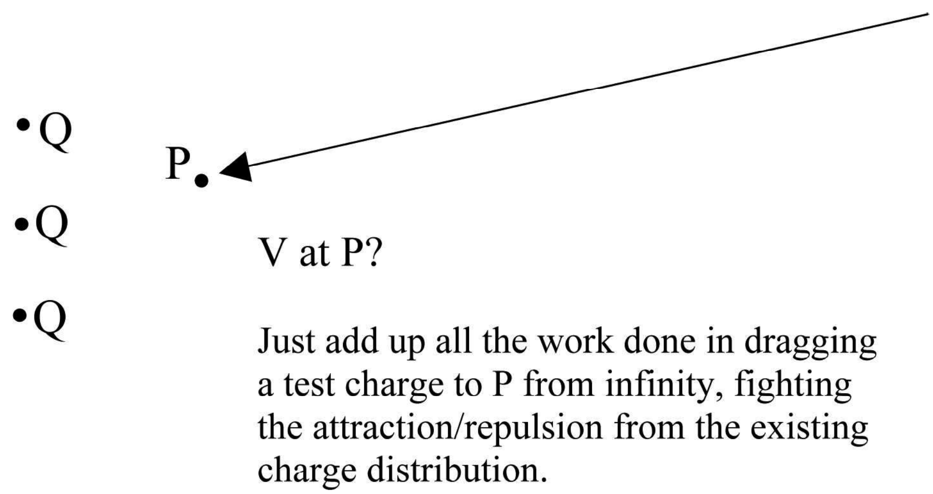
"Voltage"

$$\Delta V = V_b - V_a \tag{331}$$
$$V = \frac{EPE}{q} \tag{332}$$
$$\tag{333}$$

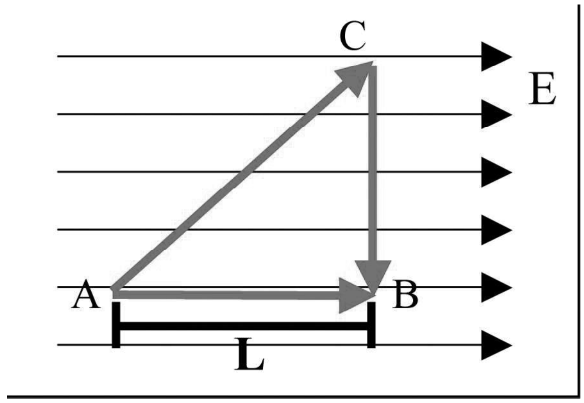
$$\begin{aligned} EPE &= \text{can have units of Joules, or...} \\ &= qV = (\#\text{electric charges})(\text{Volts}) \\ &= \text{electron-Volts} \end{aligned}$$

What is V?

- A positional energy per unit Q.
- The total work in dragging a charge from ∞ to near an existing charge distribution, per unit Q.
- You get to pick the "zero point."



P.E. only if Path Independent



$$V_c - V_a = \int_a^c \vec{E} \cdot d\vec{s} \tag{334}$$

$$= -\int_a^c Eds\cos(\pi - \theta) \tag{335}$$

$$E\cos\theta \int_a^c ds = E\cos\theta \frac{L}{\cos\theta} = EL \tag{336}$$

$$V_b - V_c = -\int_c^b \vec{E} \cdot d\vec{s} = 0 \tag{337}$$

$$V_b - V_a = (V_b - V_c) + (V_c - V_a) = 0 + EL = EL \tag{338}$$

Potential for a Collection of Charges

$$V_a - V_b = -\int \vec{E} \cdot d\vec{s} = -\int_{r_a}^{r_b} Edr \text{ (Radial field)} \tag{339}$$

$$= \frac{-q}{4\pi\epsilon_0} \int_{r_a}^{r_b} \frac{1}{r^2} dr \tag{340}$$

$$= \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r_b} - \frac{1}{r_a} \right), r_a = \infty, V_a = 0 \tag{341}$$

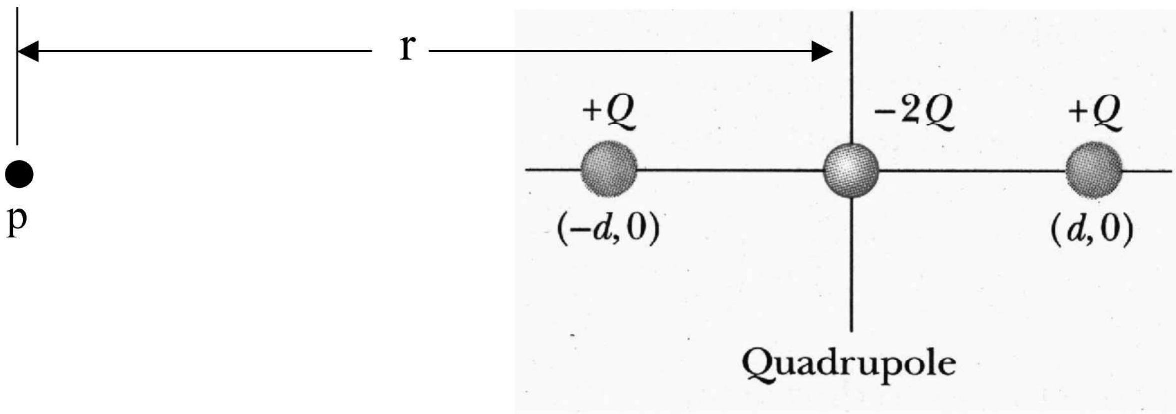
$$\tag{342}$$

$$V = \frac{1}{4\pi\epsilon_0} \frac{q}{r} \tag{343}$$

$$V_{\text{total}} = V_1 + V_2 + \ldots + V_n \tag{344}$$

$$= \frac{1}{4\pi\epsilon_0} \Sigma \frac{q_i}{r_i} (\text{ scalar sum}) \tag{345}$$

Potential for Electric Quadrupole



$$\begin{aligned}
V &= \Sigma V_i && (346)\\
&= \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r-d}+\frac{Q}{r+d}-\frac{2Q}{r}\right) && (347)\\
&= \frac{1}{4\pi\epsilon_0}\frac{2Qd^2}{r(r^2-d^2)} && (348)\\
&= \frac{1}{4\pi\epsilon_0}\frac{2Qd^2}{r^3\left(1-\frac{d^2}{r^2}\right)} && (349)
\end{aligned}$$

$d << r \Rightarrow$

$$V = \frac{1}{4\pi\epsilon_0}\frac{T}{r^3} \tag{350}$$

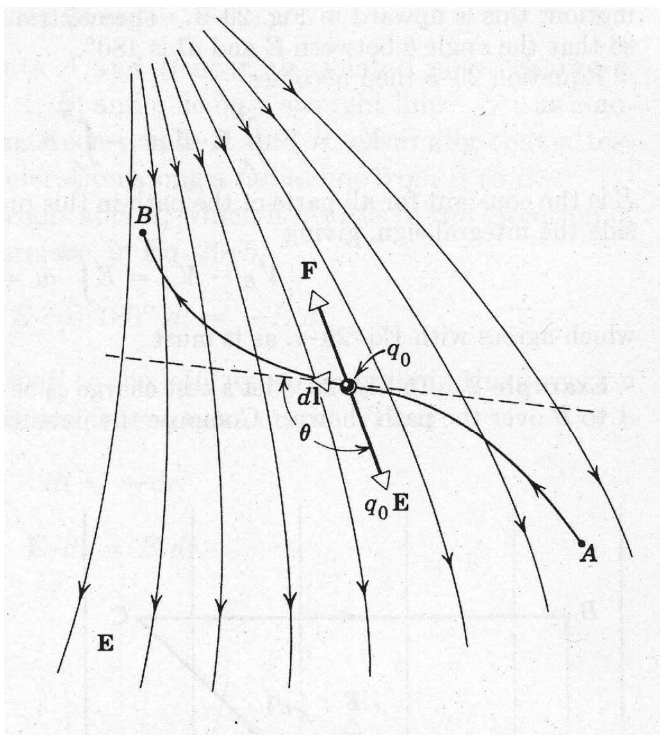
for $T \equiv 2qd^2$ "Electric quadruople moment"

V from E

$$W = F\Delta x \to W = \int_a^b \vec{F} \cdot d\vec{s} \tag{351}$$
$$= q_0 \int_a^b \vec{E} \cdot d\vec{s} \tag{352}$$

$$V_b - V_a = \frac{U_b - U_a}{q_0} = \frac{-W_{ab}}{q_0} \tag{353}$$
$$= -\int_a^b \vec{E} \cdot d\vec{s} \tag{354}$$

Choose point a @ ∞, $V_a = 0$



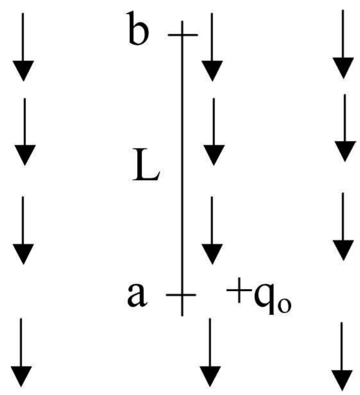
$$V_p = -\int_\infty^P \vec{E} \cdot d\vec{s} \tag{355}$$

Later, $\vec{E}$ from V ...

Why Bother with V?

$$\vec{E} \to V \tag{356}$$
$$V \to \vec{E} \tag{357}$$

V is scalar, easier to compute than $\vec{E}$ directly



$$W_{by\vec{E}} = F\Delta y = (-q_0 E)L \tag{358}$$
$$\Delta EPE = -W \tag{359}$$
$$V_b - V_a = \frac{EPE_b - EPE_a}{q_0} = \frac{-W_{ab}}{q_0} = EL \tag{360}$$

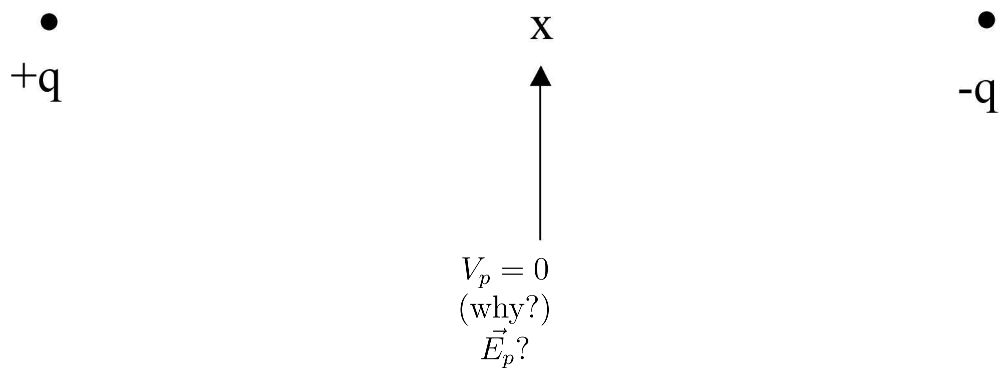
$V_b = V_a + EL \Rightarrow$ Potential at b is higher

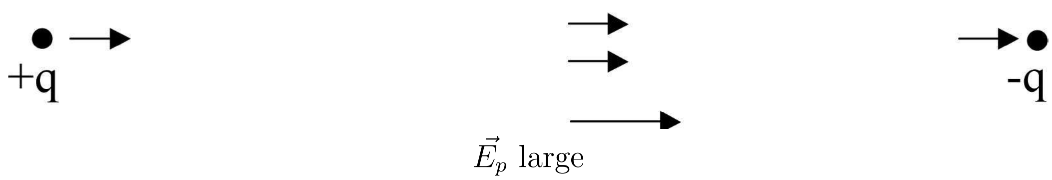
Why?

Let go of charge at b.

DOES $V_p = 0$ IMPLY $\vec{E}_p = 0$?

<u>NO</u>





E field = 0 means . . .

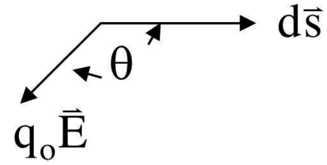
No net force at that spot.

Potential = 0 means . . .

No total work in dragging a charge from infinity to that spot.

These aren't similar!

$\vec{E}$ from V



$$dW = \vec{F}\cdot d\vec{s} = (q_0\vec{E})\cdot d\vec{s} \tag{361}$$
$$= q_0\vec{E}ds\cos\theta \tag{362}$$

<u>**Also...**</u>

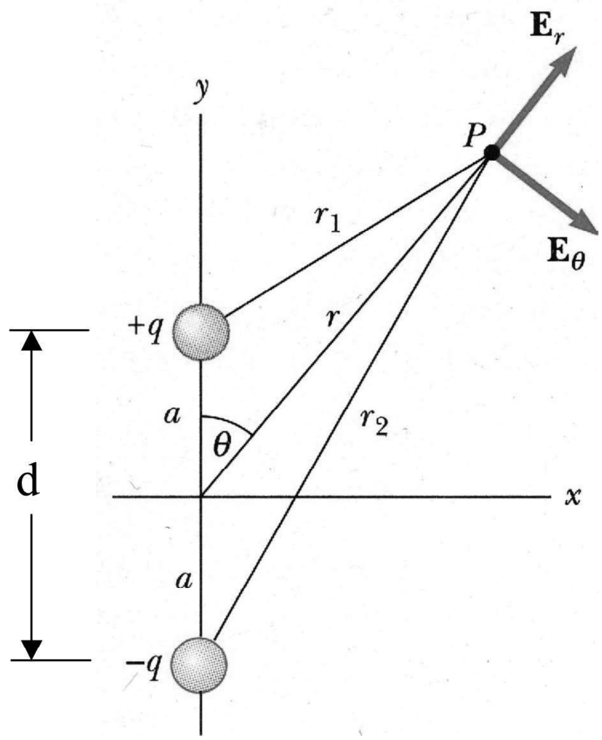
$$\Delta V = \frac{\Delta EPE}{q_0} = \frac{-W}{q_0} \tag{363}$$
$$-dW = q_0 dV \tag{364}$$
$$-q_0 dV = q_0 E ds\cos\theta \tag{365}$$
$$-\frac{dV}{ds} = E\cos\theta = E_{\vec{s}} \tag{366}$$

(Potential gradient) $\rightarrow \left(\frac{dV}{ds}\right)_{max} = E$

$$E_x = -\frac{\partial V}{\partial x} \tag{367}$$
$$E_y = -\frac{\partial V}{\partial y} \tag{368}$$
$$E_z = -\frac{\partial V}{\partial z} \tag{369}$$

The rate of change of V in y-direction = E in that direction

$\vec{E}$ from V, for the dipole



$$V_p = V_1 + V_2 = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_1} + \frac{-q}{r_2}\right) \tag{370}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{r_2 - r_1}{r_1 r_2} \tag{371}$$

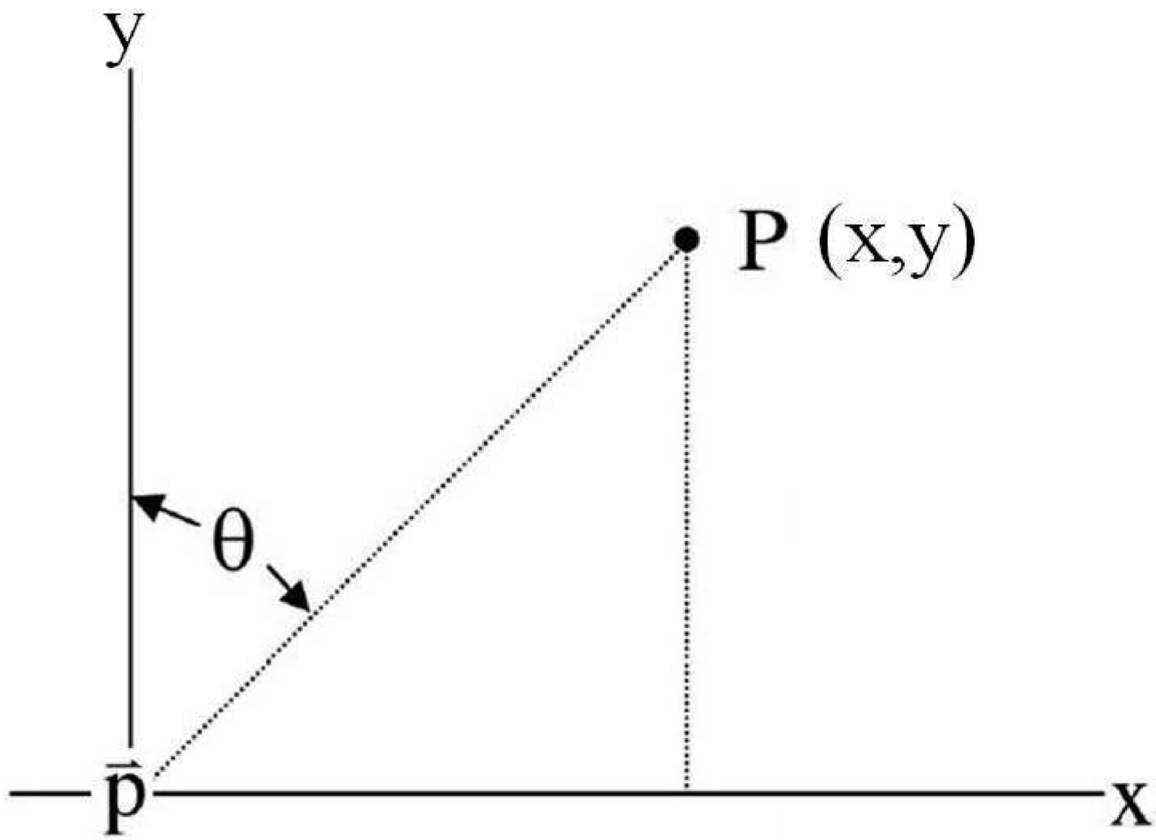
If $r >> d$, $r_2 - r_1 \sim d\cos\theta \& r_1 r_2 \sim r^2$

$$V \simeq \frac{q}{4\pi\epsilon_0}\frac{d\cos\theta}{r^2} \tag{372}$$

$$V \simeq \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2}\text{for } p \equiv qd. \tag{373}$$

Special cases: $\theta = 0°, 90°, 180°$

$\vec{E}$ from V, for the dipole



$$V = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2} \tag{374}$$

$$r = \sqrt{x^2+y^2} \tag{375}$$

$$\cos\theta = \frac{y}{\sqrt{x^2+y^2}} \tag{376}$$

$$V = \frac{p}{4\pi\epsilon_0}\frac{y}{(x^2+y^2)^{\frac{3}{2}}} \tag{377}$$

$$E_y = -\frac{\partial V}{\partial y} \tag{378}$$

$$= \frac{-p}{4\pi\epsilon_0}\frac{x^2-2y^2}{(x^2+y^2)^{\frac{5}{2}}} \tag{379}$$

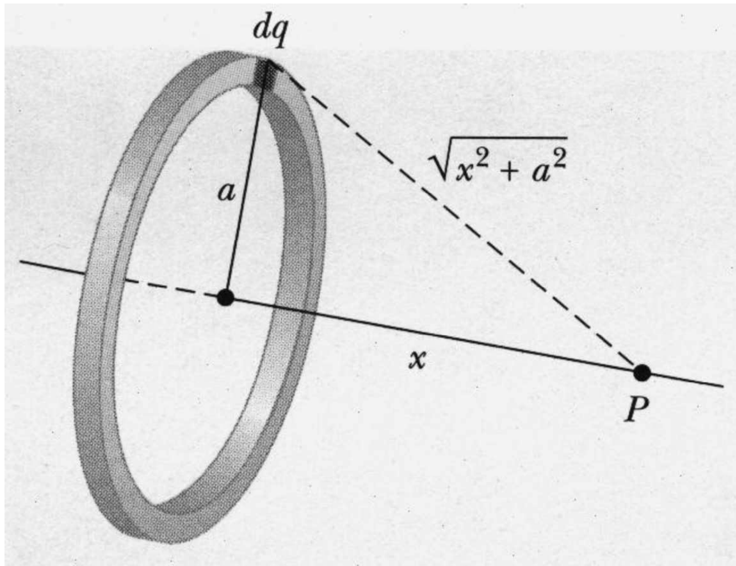
When $x = 0$,

$$\Rightarrow E_y = \frac{1}{4\pi\epsilon_0}\frac{2p}{y^3} \tag{380}$$

When $y = 0$,

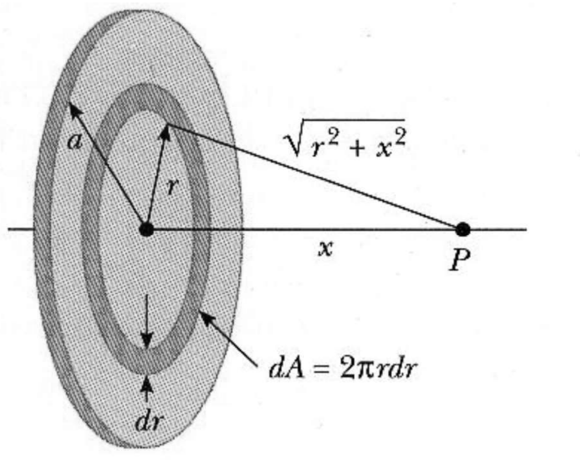
$$\Rightarrow E_y = -\frac{1}{4\pi\epsilon_0}\frac{p}{x^3} \tag{381}$$

V for Continuous Distributions of Charge



$$
\begin{aligned}
V &= \frac{1}{4\pi\epsilon_0}\int\frac{dq}{r} &(382)\\
&= \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int dq &(383)\\
&= \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{a^2+x^2}} &(384)
\end{aligned}
$$

V for an Insulating Disk of Charge



$$dq = \sigma(2\pi r)dr \tag{385}$$

$$\begin{aligned} dV &= \frac{1}{4\pi\epsilon_0}\frac{dq}{L} &(386)\\ &= \frac{1}{4\pi\epsilon_0}\frac{\sigma 2\pi r dr}{\sqrt{r^2+x^2}} &(387)\\ & &(388)\end{aligned}$$

$$\begin{aligned} V &= \int dV &(389)\\ &= \frac{\sigma}{2\epsilon_0}\int_0^a \frac{r}{\sqrt{r^2+x^2}}dr &(390)\\ &= \frac{\sigma}{2\epsilon_0}\left(\sqrt{a^2+x^2}-x\right) &(391)\end{aligned}$$

Remember $\sqrt{1+\epsilon} \approx 1 + \frac{\epsilon}{2}$
$x >> a \Rightarrow$

$$V \simeq \frac{1}{4\pi\epsilon_0}\frac{\sigma\pi a^2}{x} \tag{392}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{x} \tag{393}$$

$\vec{E}$ from V

$$V = \frac{\sigma}{2\epsilon_0}\left(\sqrt{a^2+x^2}-x\right) \tag{394}$$

$\vec{E} = -\vec{\nabla} V \ldots$

$$E_x = -\frac{\partial V}{\partial x} \tag{395}$$

$$E_y = -\frac{\partial V}{\partial y} \tag{396}$$

$$E_z = -\frac{\partial V}{\partial z} \tag{397}$$

$$\tag{398}$$

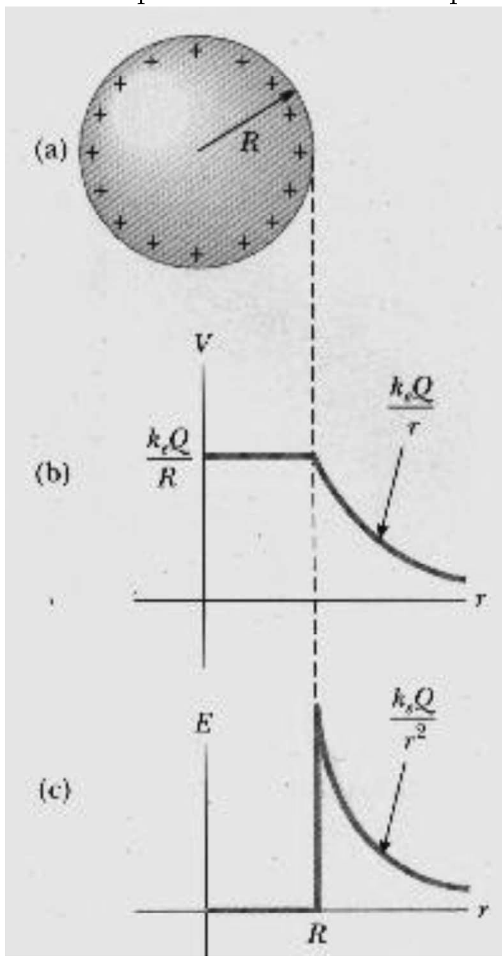
$E_y = E_z = 0$, by symmetry, or $\ldots \frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$

$$E_x = -\frac{\sigma}{2\epsilon_0} \frac{d}{dx} \left(\sqrt{x^2 + a^2} - x \right) \tag{399}$$

$$= \frac{\sigma}{2\epsilon_0} \left(1 - \frac{x}{\sqrt{x^2 + a^2}} \right) \tag{400}$$

Isolated Conductors

Same for spherical shell or solid sphere



The excess charge distributes itself so all points on the conductor are at the same potential.

Proof:

$\sim$ No currents

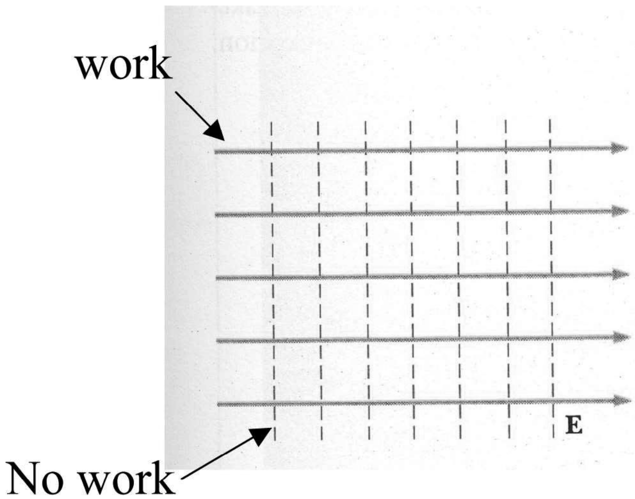
or

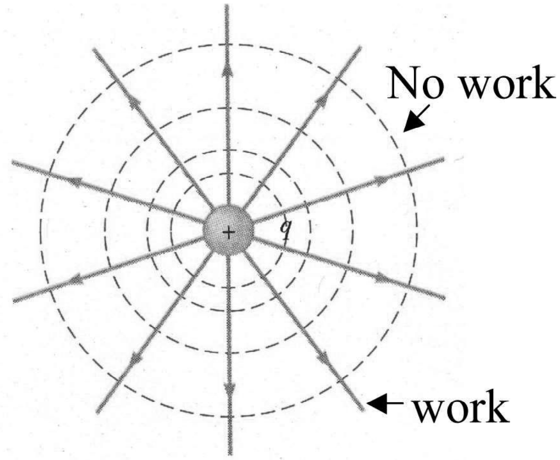
$\vec{E} = 0$ inside

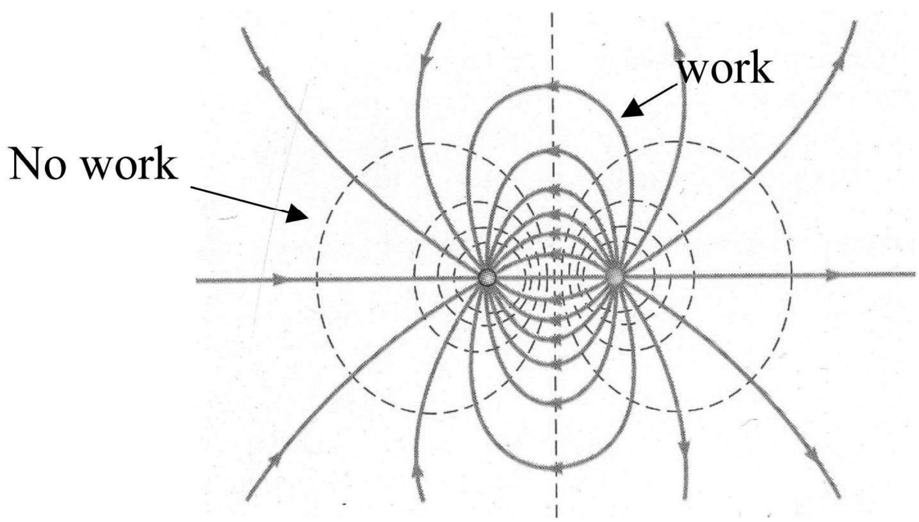
$\Rightarrow V_p - V_a = 0$

$\Rightarrow V$=constant

Equipotential Surfaces

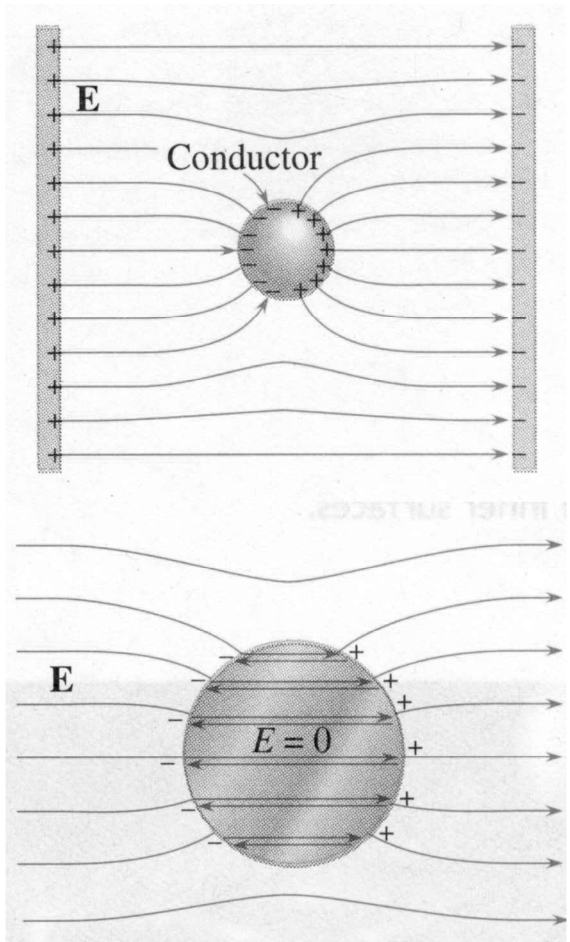






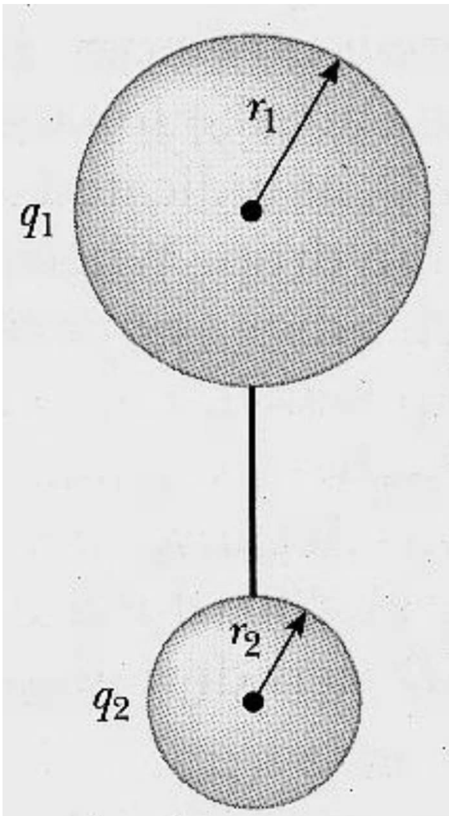
Similar to the use of lines of force

Conductor in External Field



Charges migrate in the conductor so the E field is zero everywhere in the interior.
Exercise: Sketch a number of equipotentials.

Charge Distribution of Non-Spherical Conductor



$$V = \frac{1}{4\pi\epsilon_0}\frac{q_1}{R_1} = \frac{1}{4\pi\epsilon_0}\frac{q_2}{R_2} \tag{401}$$

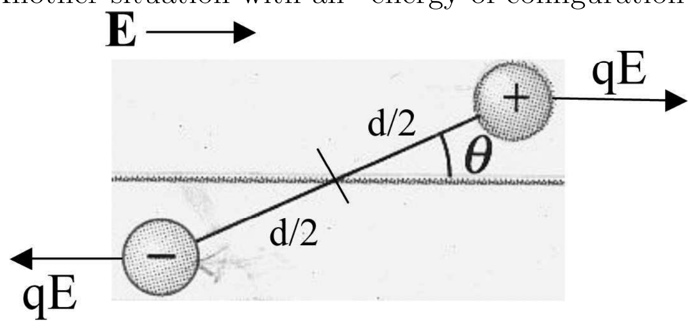
$$\frac{q_1}{R_1} = \frac{q_2}{R_2} \tag{402}$$

$$\frac{\sigma_1}{\sigma_2} = \frac{\frac{q_1}{4\pi R_1^2}}{\frac{q_2}{4\pi R_2^2}} \tag{403}$$

$$\Rightarrow \frac{\sigma_1}{\sigma_2} = \frac{R_2}{R_1} \tag{404}$$

Work Done On Dipole

Another situation with an "energy of configuration":



$$
\begin{aligned}
W &= \int dw && (405)\\
&= \int_{\theta_0}^{\theta} \tau d\theta && (406)\\
&= \int_{\theta_0}^{\theta} pE\sin\theta d\theta && (407)\\
&= pE(\cos\theta_0 - \cos\theta) && (408)
\end{aligned}
$$

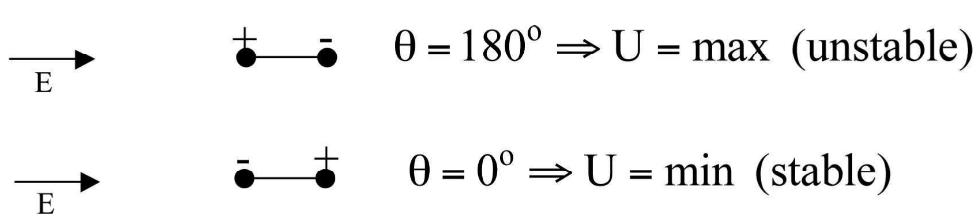
Note:

$$
\begin{aligned}
|\vec{\tau}| &= \text{force*moment arm}\\
|\vec{\tau}| &= 2(qE)(\frac{d}{2}\sin\theta)\\
&= qdE\sin\theta\\
&= pE\sin\theta
\end{aligned}
$$

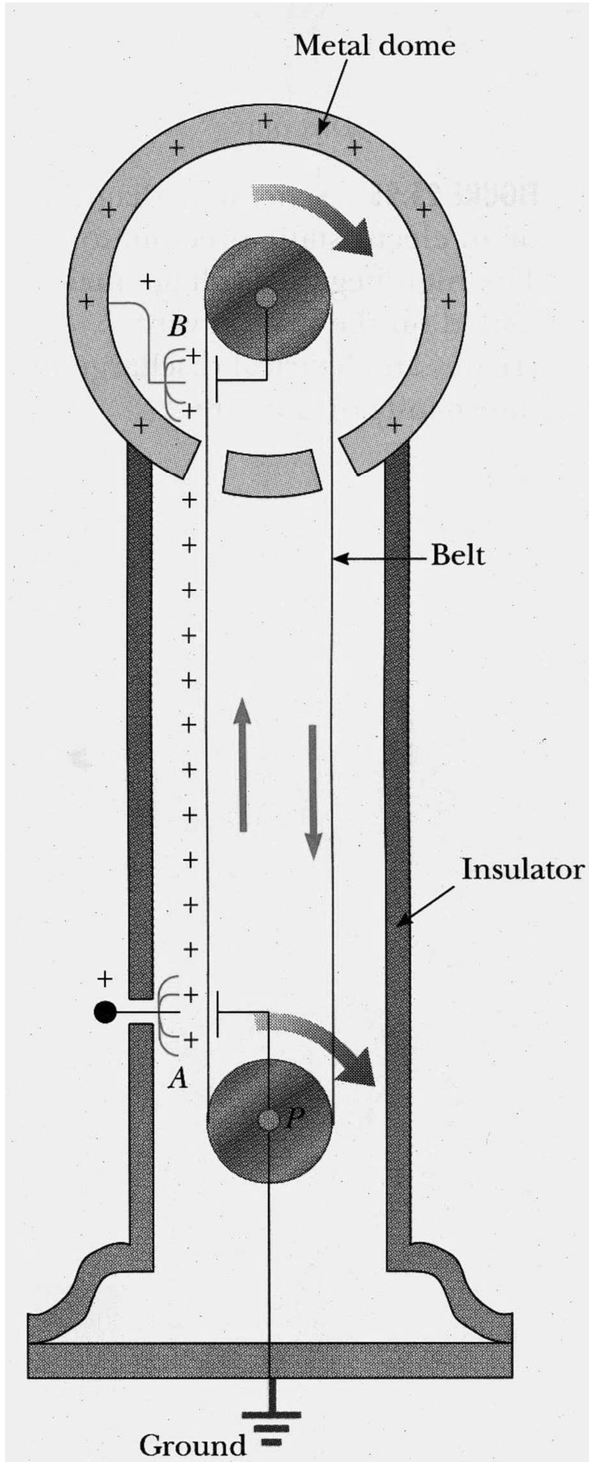
Work Done Changes P.E.

$$W = U - U_0 \tag{409}$$

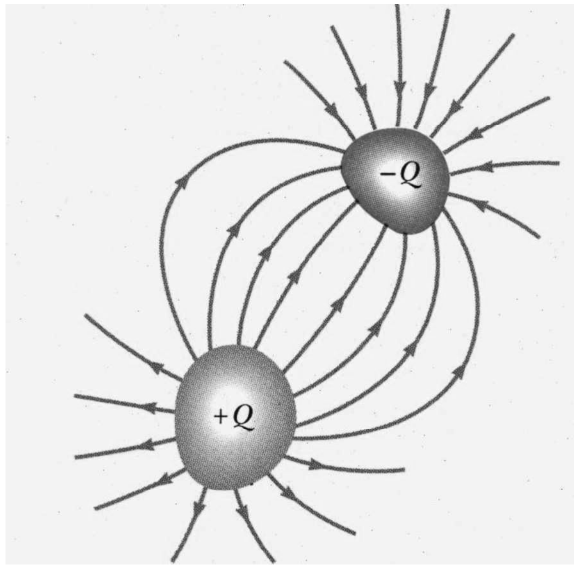
$$\Rightarrow U = -pE\cos\theta \tag{410}$$



Electrostatic Generator (Accelerator)



Capacitors



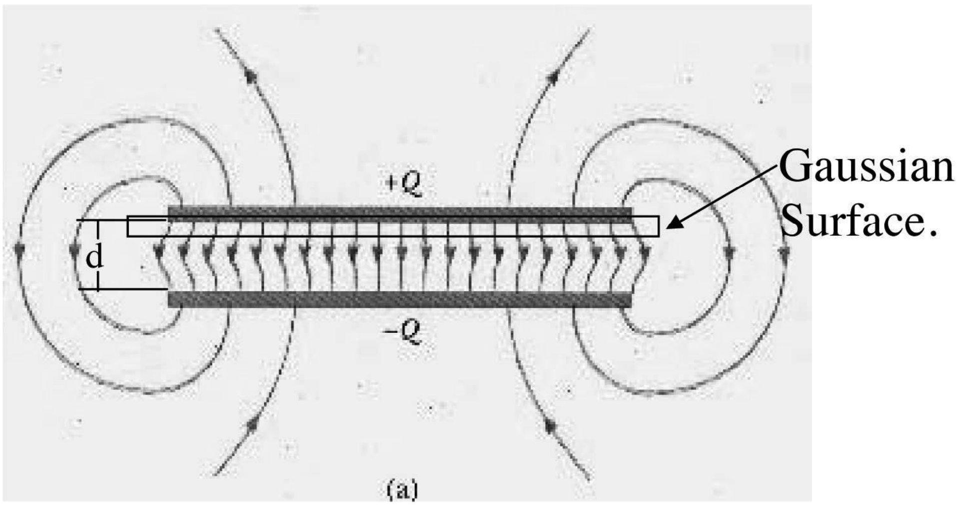
Any 2 isolated conductors with equal & opposite charge

$$q = CV \tag{411}$$

$$[\mathrm{C}] = \frac{\text{Coulomb}}{\text{Volt}} \equiv \text{Farad} \tag{412}$$

Why? A positive charge brought near an isolated object raises its potential.
A negative charge lowers it.
⇒ Larger common q, larger V (Exploit for ability to store energy.) *or even one conductor (later)

Parallel Plate Capacitor



Calculating C...

1. Assume charge q
2. Calculate $\vec{E}$ (Gauss' Law)
   $\epsilon_0 \Phi = q, \epsilon_0(EA) = q, A =$ Plate area
   $E = \frac{q}{\epsilon_0 A}$
3. $\vec{E} \rightarrow V$ Calculation
   $V = \int_{+}^{-} E ds = \frac{q}{\epsilon_0 A} \int ds = \frac{qd}{\epsilon_0 A}$
4. Form $C = q/V$
   $\frac{q}{V} = \frac{\epsilon_0 A}{d}$ depends on geometric factors only

Spherical Capacitor

2 concentric spheres (shells) - radii a, b
Gaussian surface = Sphere of intermediate radius r

$$\epsilon_0 \Phi = q \tag{413}$$

$$\epsilon_0 E A = q, \tag{414}$$

$$\text{where } A = 4\pi r^2$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$$

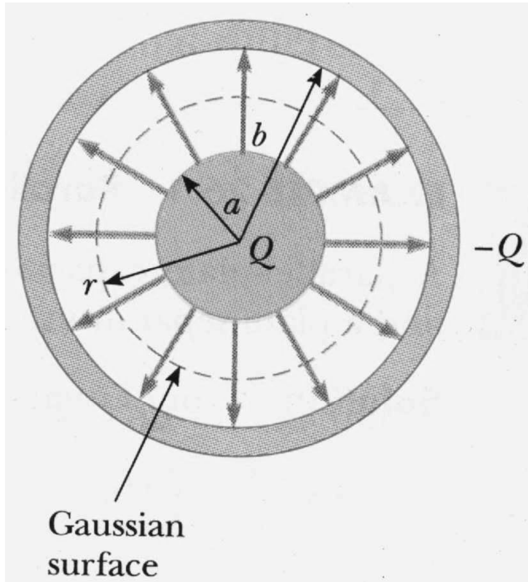
Special Case...
Radius $b \to \infty$, $a = R$
So only 1 sphere...
$C \to 4\pi\epsilon_0 R$

$$\begin{aligned} V &= \int E ds && (415)\\ &= \frac{q}{4\pi\epsilon_0}\int_a^b \frac{dr}{r^2} && (416)\\ &= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) && (417)\\ &= \frac{q}{4\pi\epsilon_0}\left(\frac{b-a}{ab}\right) && (418)\end{aligned}$$

$$C = 4\pi\epsilon_0 \frac{ab}{b-a} \tag{419}$$

Earth as a capacitor
$4\pi\epsilon_0 R \sim 7 \times 10^{-4} F \sim 710\ \mu F$

Cylindrical Capacitor



2.

$$\Phi = \frac{q}{\epsilon_0} \tag{420}$$

$$EA = \frac{q}{\epsilon_0} \tag{421}$$

$$A = 2\pi r L \tag{422}$$

(ignore fringing)

$$E = \frac{q}{2\pi\epsilon_0 L r} \tag{423}$$

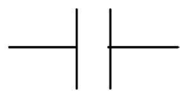
3.

$$\begin{aligned} \Delta V &= \int E ds && (424) \\ &= \frac{q}{2\pi\epsilon_0 L}\int_a^b \frac{dr}{r} && (425) \\ &= \frac{q}{2\pi\epsilon_0 L}\ln\left(\frac{b}{a}\right) && (426) \end{aligned}$$

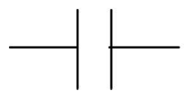
4.

$$\begin{aligned} C &= \frac{q}{\Delta V} && (427) \\ &= \frac{2\pi\epsilon_0 L}{\ln\left(\frac{b}{a}\right)} && (428) \end{aligned}$$

Energy Storage...



@ some $t \Rightarrow V' = \frac{q'}{C}$



$$du = V'dq' \\ = \frac{q'}{C}dq'$$

Start from $q = 0$, transfer until $q = q$

$$U = \int dU \tag{429}$$

$$= \int_0^q \frac{q'}{C}dq' \tag{430}$$

$$= \frac{q^2}{2C}, \qquad C = \frac{q}{V} \tag{431}$$

$$\Rightarrow U = \frac{1}{2}CV^2 \tag{432}$$

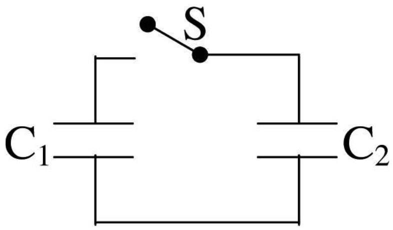
In a parallel plate capacitor, $\vec{E}$ is uniform, and the energy storage in space is uniform...

$$u = \frac{U}{\text{Vol}} = \frac{U}{Ad} = \frac{\frac{1}{2}CV^2}{Ad} \ \& \ C = \frac{\epsilon_0 A}{d} \tag{433}$$

$$u = \frac{1}{2}\epsilon_0 \left(\frac{V}{d}\right)^2 \tag{434}$$

$$u = \frac{1}{2}\epsilon_0 E^2 \rightarrow \text{Very general result.} \tag{435}$$

Wherever there is $\vec{E}, u$, there is energy density.



C_1, charged first, then C_2 inserted

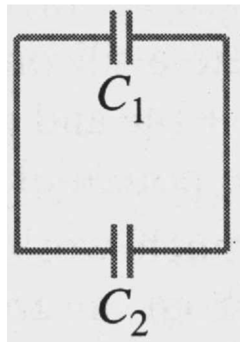
$$U_i = \frac{1}{2}C_1V_0^2 \;\&\; U_f = \frac{1}{2}C_1V^2 + \frac{1}{2}C_2V^2 \tag{436}$$

$$q = q_1 + q_2 \Rightarrow CV_0 = C_1V_1 + C_2V_2, V_1 = V_2 \tag{437}$$

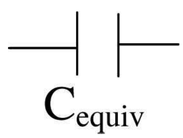
#'s $\Rightarrow U_f < U_i$ where?

Combinations of Capacitors

"Parallel" combinations

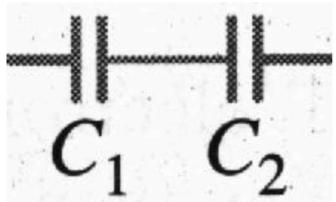


$$q_1 = C_1 V \tag{438}$$
$$q_2 = C_2 V \tag{439}$$
$$q = q_1 + q_2 \tag{440}$$
$$q = C_1 V + C_2 V \tag{441}$$
$$= (C_1 + C_2)V \tag{442}$$
$$= C_{\text{equiv.}} V \tag{443}$$
$$C_{\text{equiv.}} = C_1 + C_2 \tag{444}$$



Generally, $C_T = \Sigma C_i$

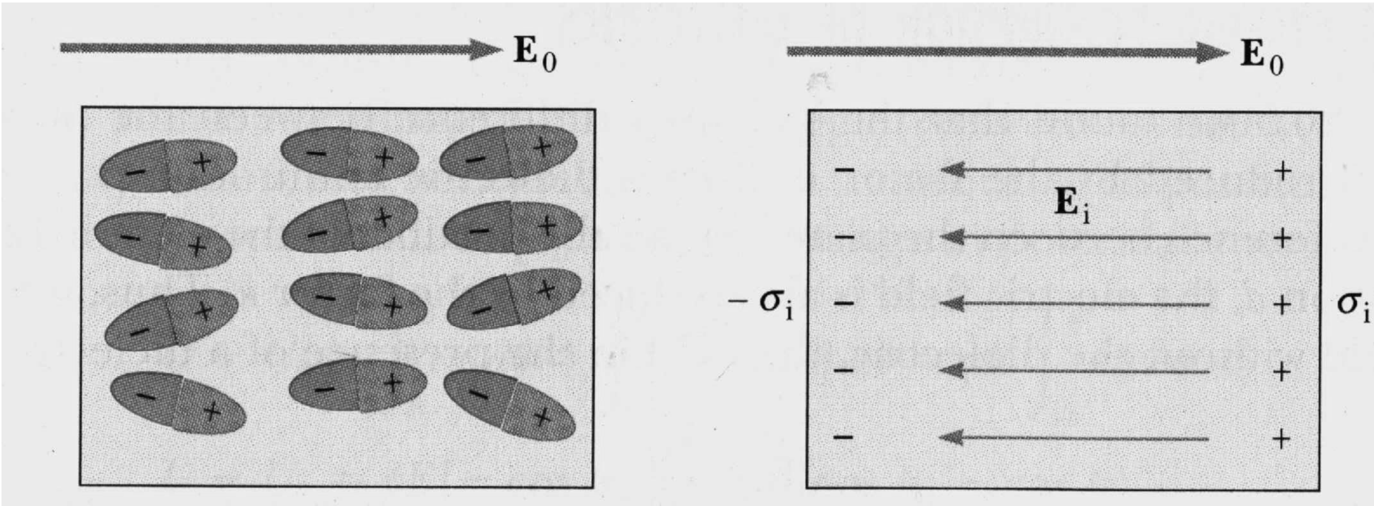
"Series" Combination



$$q = C_1 V_1 \tag{445}$$
$$q = C_2 V_2 \tag{446}$$
$$V = V_1 + V_2 \tag{447}$$
$$V = \frac{q}{C_1} + \frac{q}{C_2} \tag{448}$$
$$= q\left(\frac{1}{C_1} + \frac{1}{C_2}\right) \tag{449}$$
$$= q\frac{1}{C_{\text{equiv.}}} \tag{450}$$
$$\frac{1}{C_{\text{equiv}}} = \frac{1}{C_1} + \frac{1}{C_2} \tag{451}$$

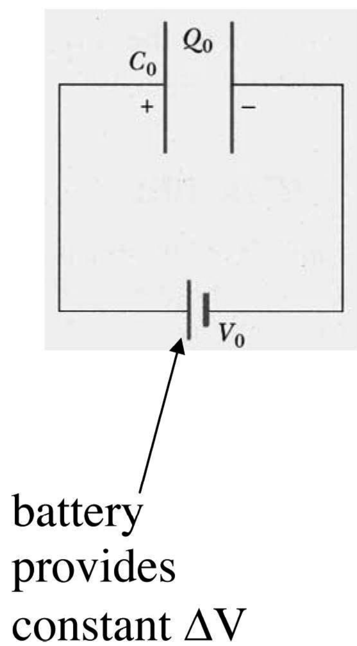
Generally, $\frac{1}{C_T} = \Sigma \frac{1}{C_i}$

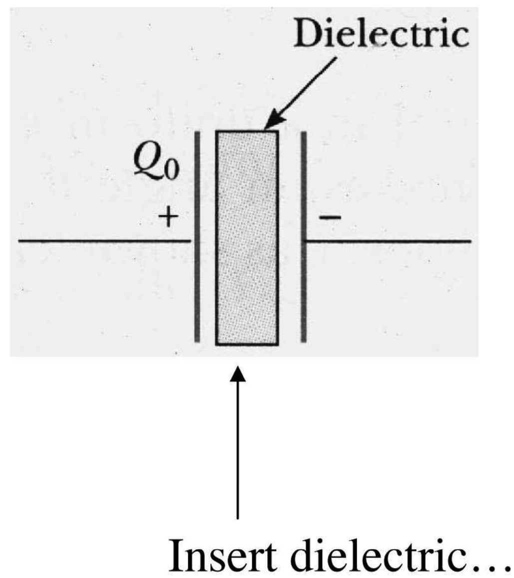
Dielectrics



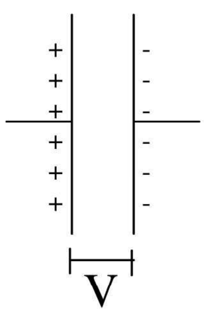
Molecules without permanent dipole moments acquire them by induction in the external field.
Slight separation between positive and negative charges...
⇒ surface charges
⇒ additional field
⇒ reduction of field strength inside the material

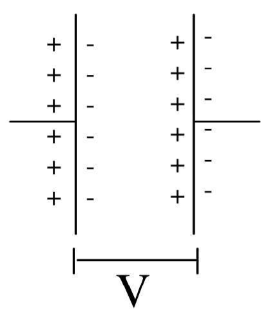
Dielectric In A Capacitor





V across the capactior remains constant, and the charge on the plates increases. Why?





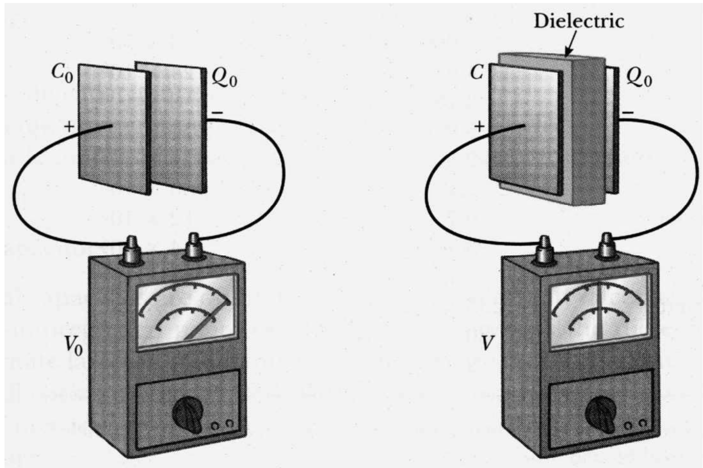
Whoops!
V can't change, but if $\vec{E}$ is smaller, it will have to …
More charge needed

q bigger for the same V
$\Rightarrow$ increase in C
$C_0 \to k_e C$, k_e: dielectric constant… material property
Similarly…

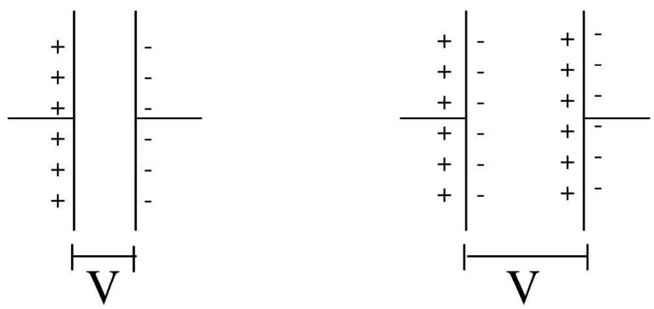
$$E \to \frac{E_0}{k} = \frac{1}{4\pi\epsilon_0 k_e}\frac{q}{r^2} \tag{452}$$

Reduction in E… Includes original field & opposite dielectric field

Repeat the experiment with the battery removed...



insert dielectric
q constant
V decreases by k_e
$C = \frac{q}{V}$ if q is constant, C & V vary inversely



q must remain constant, so V must decrease

C_0 is the capacitance in the absence of the dielectric. That is, the capacitance *increases* by the factor k when the dielectric completely fills the region between the plates. For a parallel-plate capacitor, where $C_0 = \epsilon_0 \frac{A}{d}$, we can express the capacitance when the capacitor is filled with a dielectric as

$$C = k\frac{\epsilon_0 A}{d} \tag{453}$$

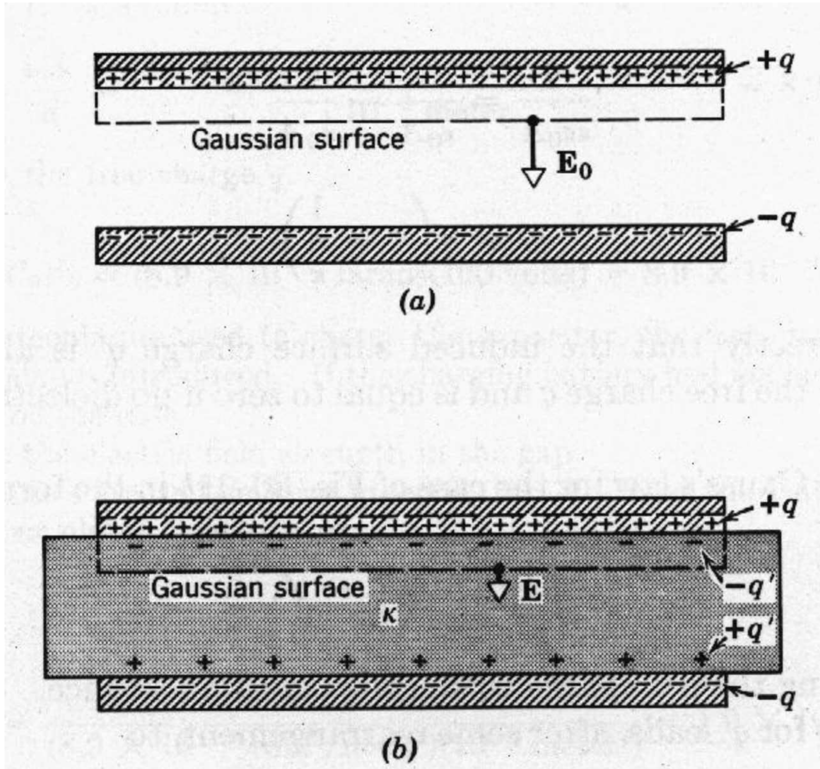
The dielectric strength equals the maximum electric field that can exist in a dielectric without electrical breakdown.

If another experiment is performed in which the dielectric is introduced while the potential difference remains constant by means of a battery, the charge increases to a value $Q = kQ_0$. The additional charge is supplied by the battery and the capacitance still increases by the factor k.

Material	Dielectric Constant (k	Dielectric Strength (V/m)
Vacuum	1.00000	-
Air (dry)	1.00059	3×10^6
Bakelite	4.9	24×10^6
Fused quartz	3.78	8×10^6
Pirex glass	5.6	14×10^6
Polystyrene	2.56	24×10^6
Teflon	2.1	60×10^6
Neoprene rubber	6.7	12×10^6
Nylon	3.4	14×10^6
Paper	3.7	16×10^6
Strontium	235	8×10^6
Water	80	-
Silicone oil	2.5	15×10^6

Dielectric Constants and Dielectric Strengths of Various Materials at Room Temperature.

Gauss' Law & Dielectrics



$$\epsilon_0 \Phi = q - q' \tag{454}$$

$$\epsilon_0 E A = q - q' \tag{455}$$

$$E = \frac{q}{\epsilon_0 A} - \frac{q'}{\epsilon_0 A} \tag{456}$$

$$\text{Also...} \tag{457}$$

$$E = \frac{E_0}{k_e} = \frac{q}{\epsilon_0 A k_e} \tag{458}$$

$$q' = q\left(1 - \frac{1}{k_e}\right) \tag{459}$$

$$\epsilon_0 \int \vec{E} \cdot d\vec{A} = q - q' \tag{460}$$

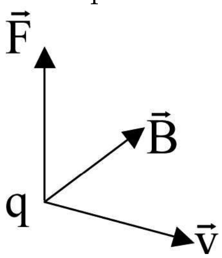
$$\epsilon_0 \int k_e \vec{E} \cdot d\vec{A} = q, \tag{461}$$

$$q \text{ is free charge only} \tag{462}$$

Magnetic Fields

Electric charge $\rightarrow \vec{E} \rightarrow$ force on other charges.
Yes.
Magnetic charge $\rightarrow \vec{B} \rightarrow$ force on other magnetic charges.
NO!
There are no "magnetic monopoles."
Moving electric charge $\rightarrow \vec{B} \rightarrow$ force on other moving charges

Experiment:



Given a $\vec{B}$ ...
$\vec{F}_{\text{charge}} = q\vec{v} \times \vec{B}$
$\vec{v}$ = velocity of the charge moving through the external field.
$\vec{B}$ = external field

$$\vec{F} \text{ not } \parallel \text{ to } \vec{B}$$
$$|\vec{F}| \propto \vec{v}$$
$$\vec{F} \perp \vec{v} \Rightarrow \vec{F} \cdot \overline{displacement} = 0$$
$$\Rightarrow \text{No work done by } \vec{B} \text{ field.}$$
$$\vec{B} \text{ can only alter the direction of } \vec{v}.$$

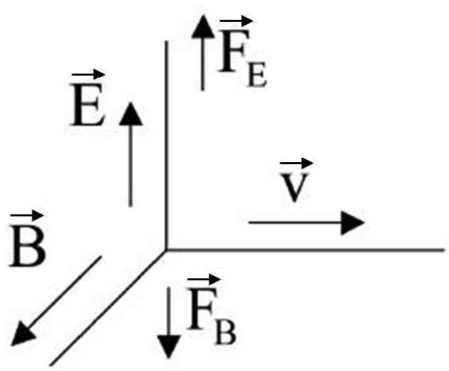
$$[\vec{B}] = \left[\frac{\vec{F}}{q\vec{v}}\right] \tag{463}$$
$$= \frac{N}{C\frac{\text{m}}{\text{sec}}} = \frac{N}{A \cdot m} \equiv T, \text{"Tesla"} \tag{464}$$

1T = 10^4 Gauss

Location	Magnetic Field (T)
Surface of neutron star (calc.)	10^4
Near a superconducting magnet	5
Near a large electromagnet	1
Near a small bar magnet	10^{-2}
At the surface of the Earth	10^{-4}
In interstellar space	10^{-10}
In a magnetically shielded room	10^{-14}

Approximate values

Typical Values of Some Magnetic Fields.

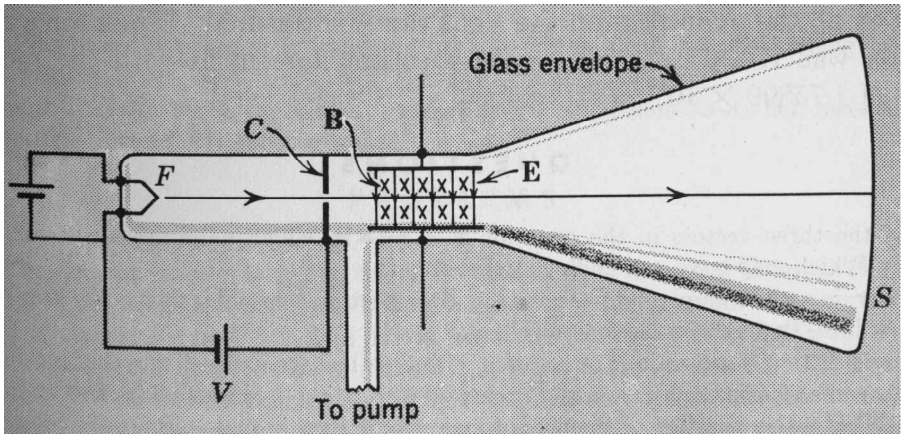


"Lorentz Force"

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B} \tag{465}$$

$\Sigma\vec{F} = \vec{0}$?

$$|q\vec{E}| = |q\vec{v} \times \vec{B}| \rightarrow v = \frac{E}{B}\text{velocity selector} \tag{466}$$



If $\vec{E}$ only,

$$\Sigma F = qE = ma \rightarrow a = \frac{qE}{m} \tag{467}$$

$$x = L = vt \rightarrow t = \frac{L}{v} \tag{468}$$

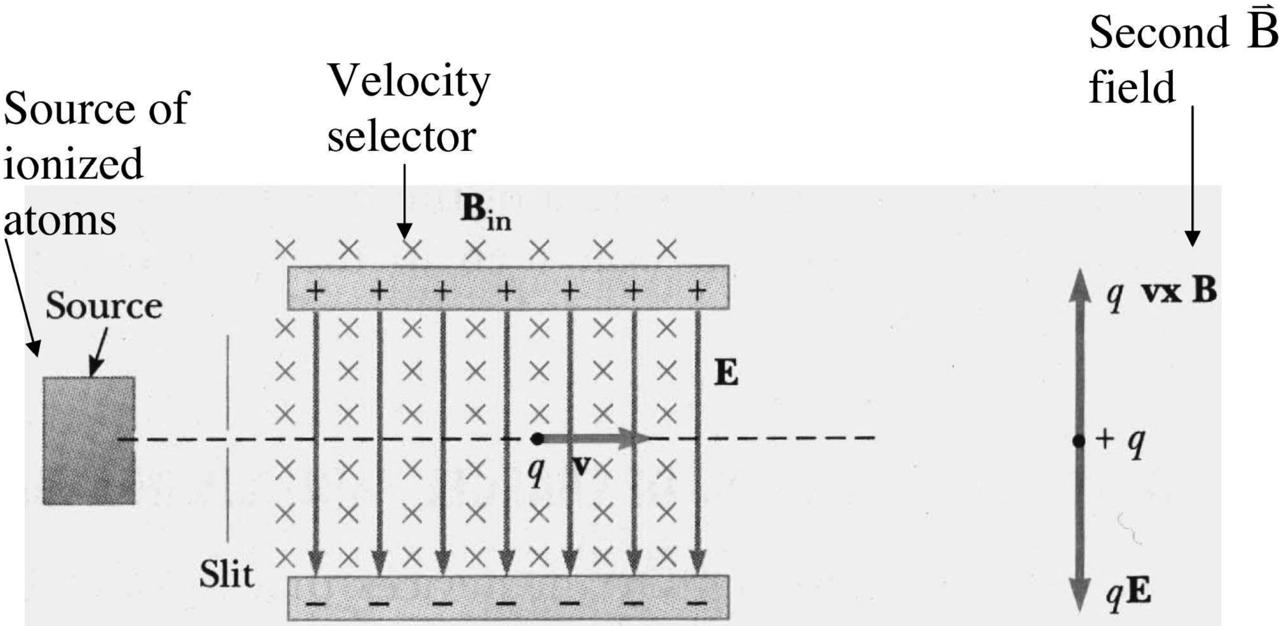
$$y = \frac{1}{2}at^2 = \frac{1}{2}\left(\frac{qE}{m}\right)\left(\frac{L}{v}\right)^2 \tag{469}$$

$$\rightarrow \text{Deflection } y = \frac{qEL^2}{2mv^2} \tag{470}$$

Add $\vec{B}$, adjust until there is no deflection.

$$v = \frac{E}{B} \Rightarrow y = \frac{qEL^2}{2m\left(\frac{E}{B}\right)^2} \Rightarrow \underline{\frac{q}{m} = \frac{2yE}{B^2L^2}} \tag{471}$$

Mass Spectrometer



$$F = qvB = \frac{mv^2}{r} \Rightarrow r = \frac{mv}{qB} \tag{472}$$

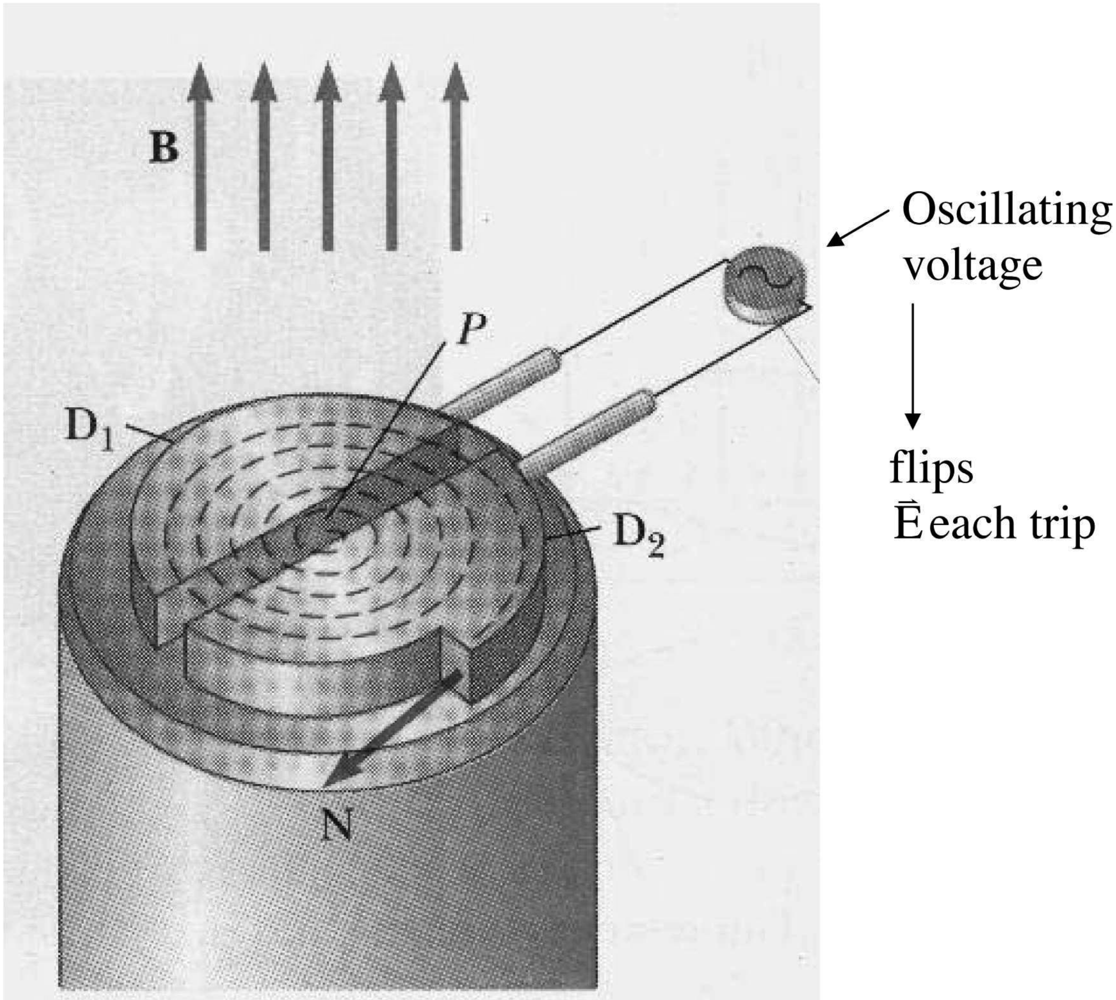
$\Rightarrow$ Radius of circular motion depends on the particle's mass (473)

$$\omega = \frac{v}{r} = \frac{qB}{m} \tag{474}$$

$$f = \frac{\omega}{2\pi} = \frac{qB}{2\pi m} \tag{475}$$

Independent of $|\vec{v}|$

Cyclotron



Trip time is independent of radius

$$v = \frac{qB}{2\pi m} \text{ Resonance.} \tag{476}$$

+++

Algorithm for the motion of a charge in a B field

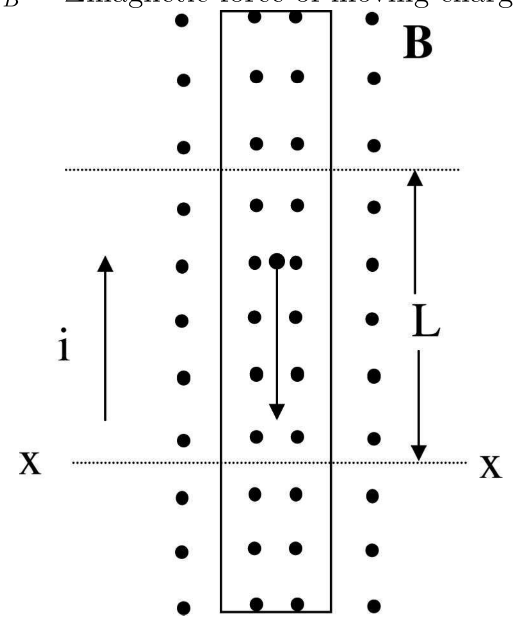
Given:
x_i, y_i, z_i
$v_{x_i}, v_{y_i}, v_{z_i}$
B_x, B_y, B_z
$\vec{F} = q\vec{v} \times B, F_x = q(v_y B_z - v_z B_y)$

- At current x, y, z compute B_x, B_y, B_z
- Compute current F_x, F_y, F_z
- $a = \frac{qE}{m}$, etc.
- $v_{x_f} = v_{x_i} + a_x \Delta t$

*Better to do this with Euler's method, or Runge-Kutta.
$x_f = x_i + v_{i_z}\Delta t + \frac{1}{2}a_x \Delta t^2$

Magnetic Force On A Current

$$\vec{F}_B = \Sigma \text{magnetic force of moving charges}$$



$$F = (-Ne)\vec{v}_d \times \vec{B} \tag{477}$$

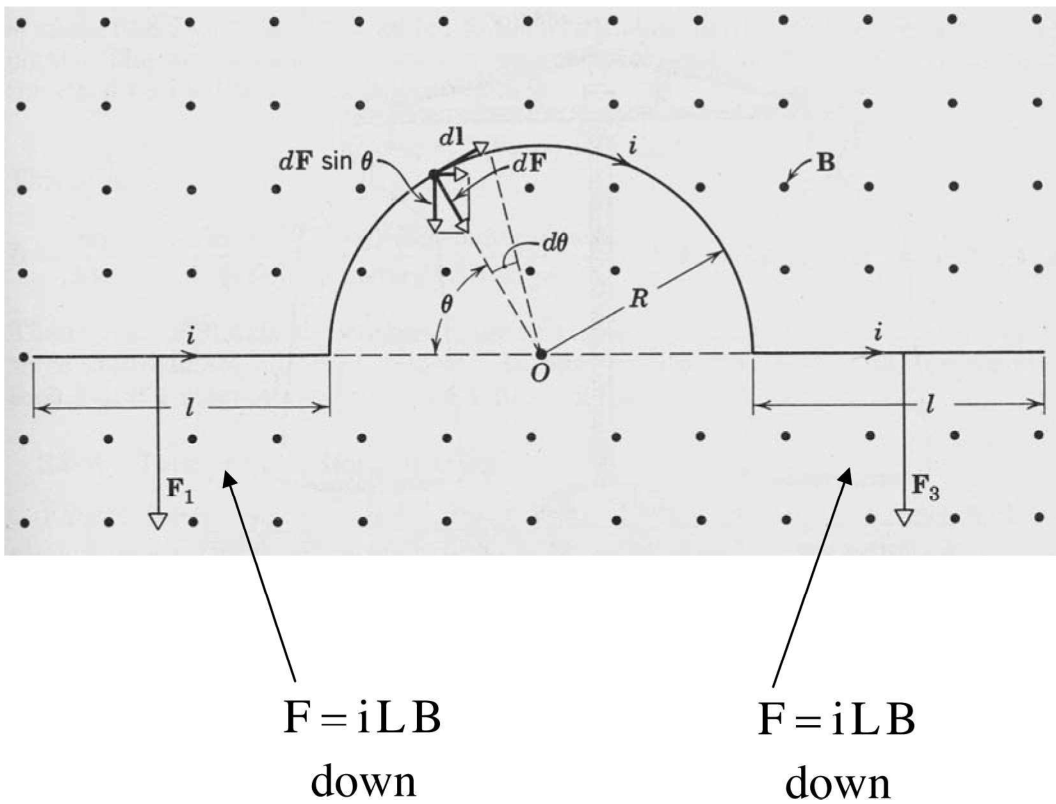
$$N = nAL \tag{478}$$

$$i = nAev_d, \vec{v}_d \text{ is opposite to } \vec{L} \text{ and absorbs negative sign} \tag{479}$$

$$\vec{F} = i\vec{L} \times \vec{B} \tag{480}$$

$$\vec{L} \perp \vec{B} \Rightarrow |\vec{F}| = iLB \tag{481}$$

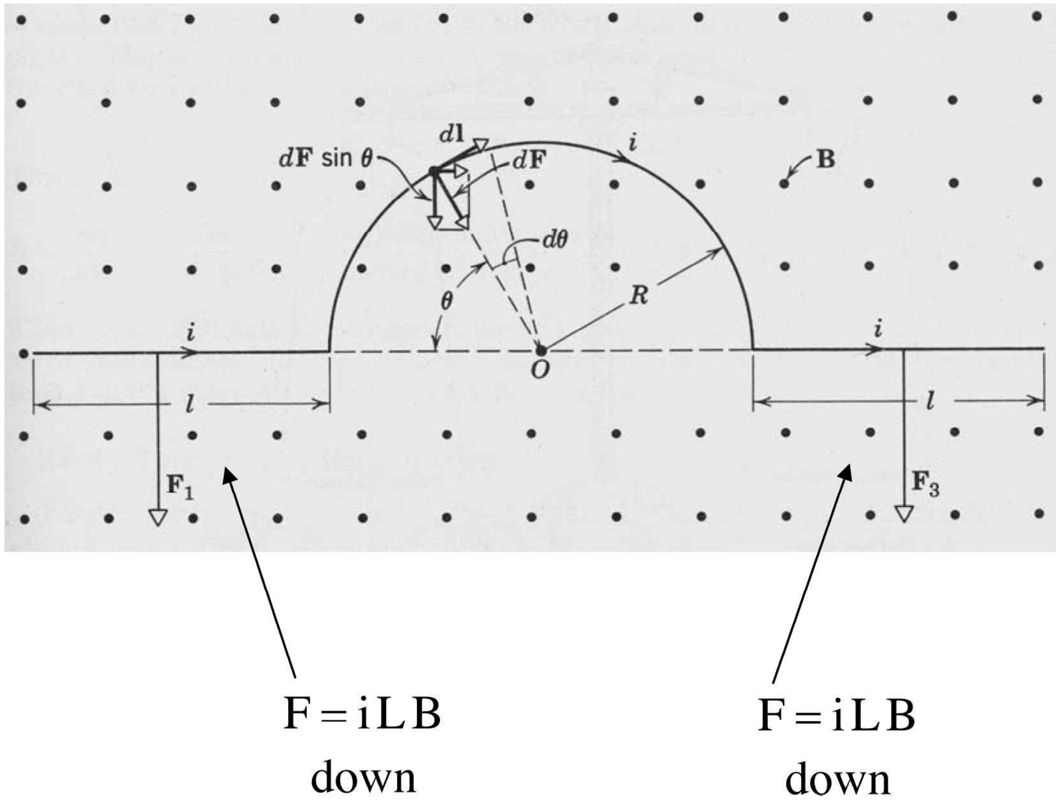
$$d\vec{F} = id\vec{s} \times \vec{B} \tag{482}$$



$$dF = iBds, \text{radially inward} \tag{483}$$

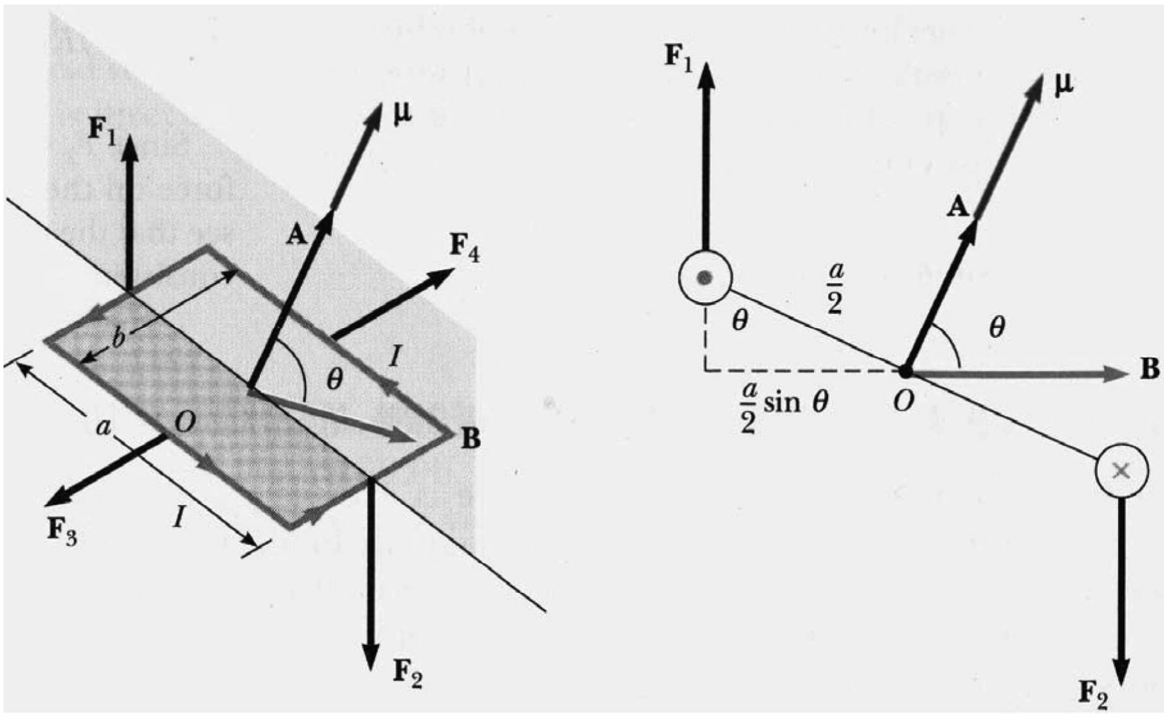
$$\int dF \sin\theta = iBR \int_0^\pi \sin\theta d\theta = 2iBR \tag{484}$$

$$F_{\text{total}} = iB(2L + 2R) \tag{485}$$



$$F_{\text{total}} = iB(2L + 2R) \tag{486}$$

Net Torque on a Current Loop

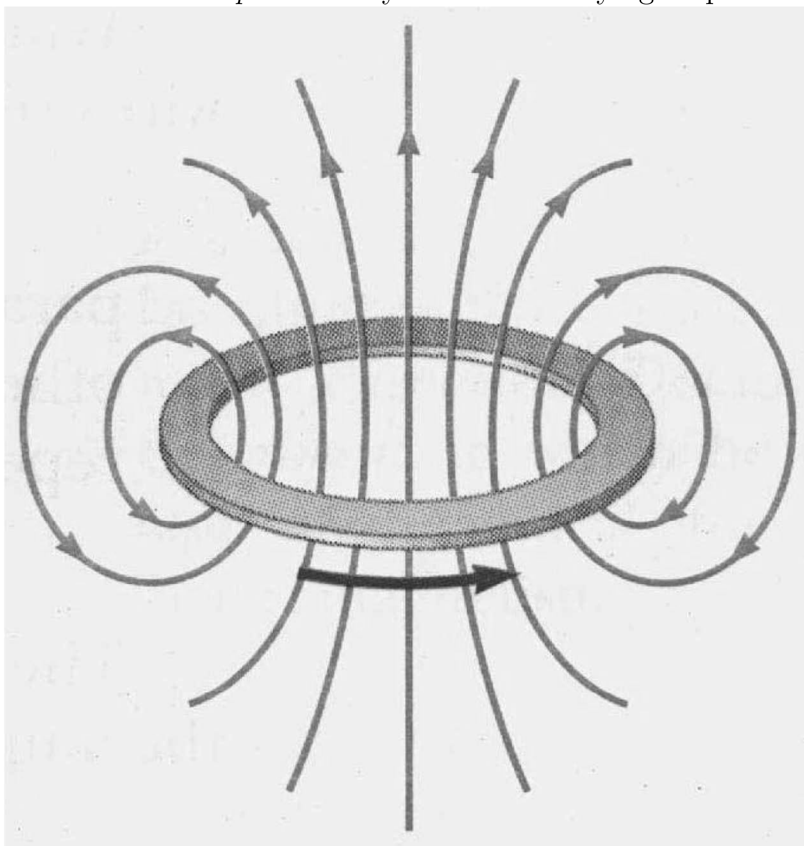


$$F_1 = ibB \tag{487}$$
$$F_2 = ibB \tag{488}$$
$$\text{moment arm} = \frac{a}{2}\sin\theta \tag{489}$$
$$\tau = 2(ibB)\left(\frac{a}{2}\sin\theta\right) = iabB\sin\theta \tag{490}$$
$$\text{Area of loop} = ab = A \tag{491}$$

$$F_3 + F_4 \sim \text{same line of action}$$
$$\Rightarrow \text{no torque from these two force}$$
$$\tau = NiAB\sin\theta \tag{492}$$

Result is independent of the loop shape

What $\vec{B}$ is *produced* by a current-carrying loop?

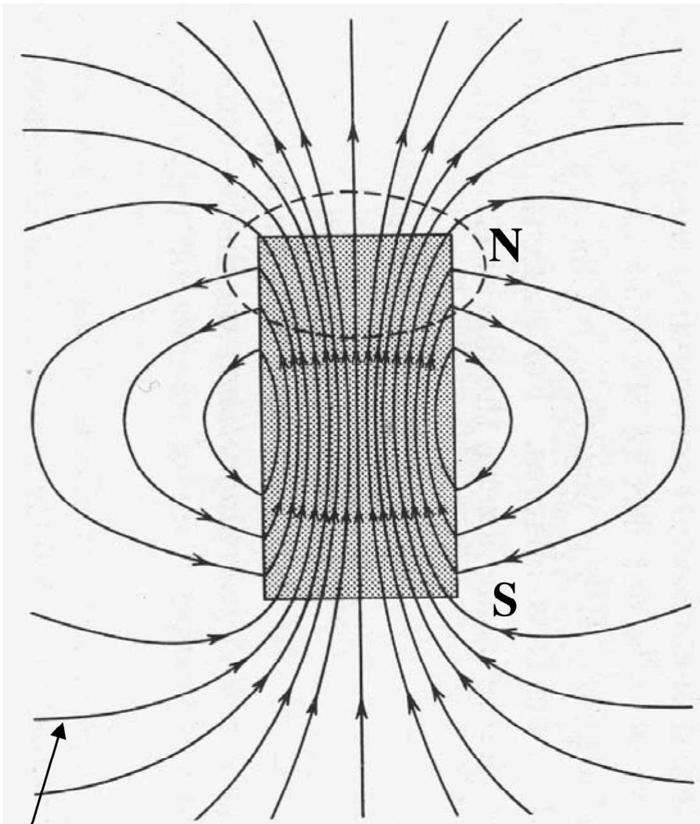


Note that far from the loop, the field lines are identical in form to those of an electric dipole.

- No magnetic monopoles (point magnets)

- Magnetic dipole is fundamental

Bar
Magnet



- like poles repel
- opposite poles attract.

Magnetic field lines are <u>not</u> lines of force.

Nor are they lines of motion.

Note: Earth's "N" pole is a magnetic "S" pole.

Magnetic Dipole

You already know that *electric dipoles* tend to orient with an external $\vec{E}$ field.
Now you know that *current carrying coils* ("magnetic dipoles") tend to orient with an external $\vec{B}$ field.

Both electric and magnetic dipoles like to line up with external fields.

$$|\vec{\tau}| = |\vec{p} \times \vec{E}| = pE\sin\theta = qdE\sin\theta \tag{493}$$

compare to

$$\tau = NiAB\sin\theta \tag{494}$$

Define...

$$\mu = NiA \text{ "magnetic dipole moment"} \tag{495}$$

$\vec{\mu}$ right-hand rule with current

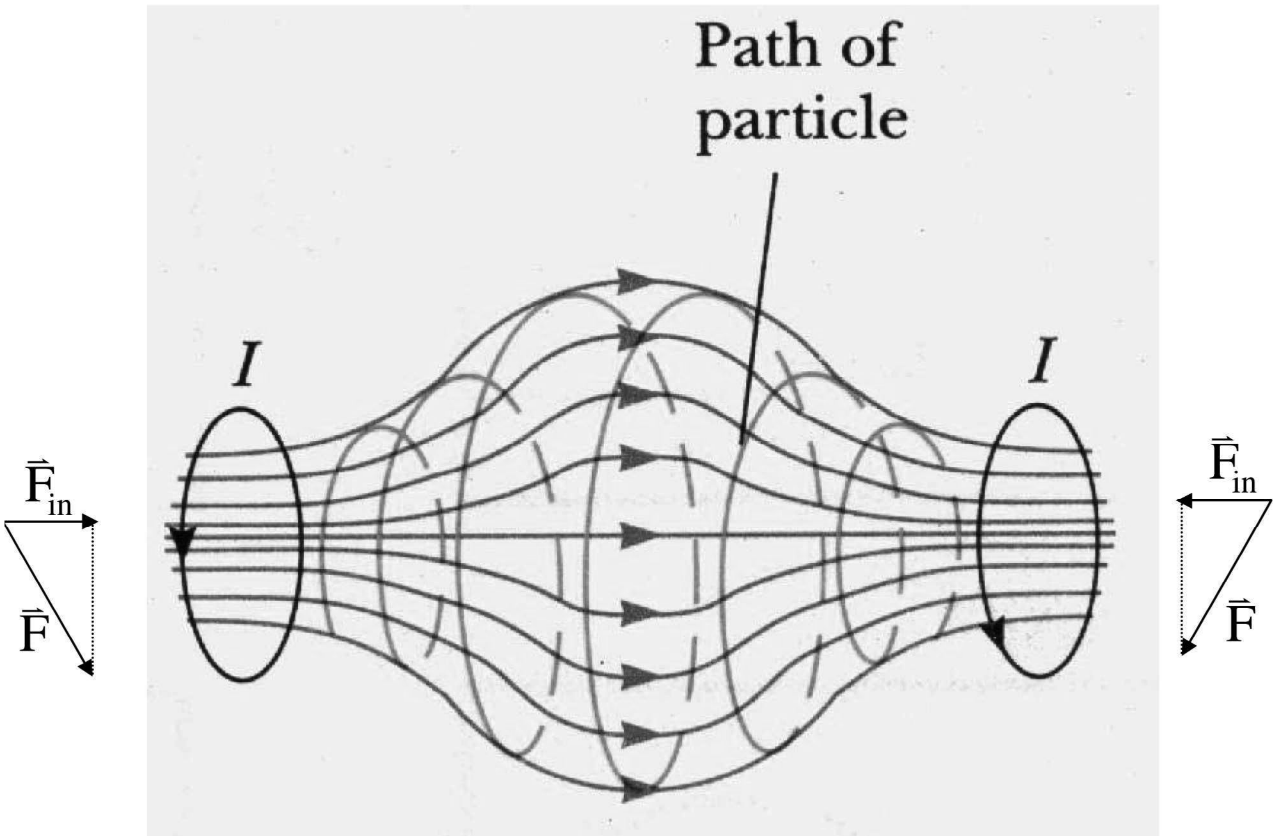
$$\vec{\tau} = \vec{\mu} \times \vec{B} \tag{496}$$

&

$$U = -\vec{\mu} \cdot \vec{B} \tag{497}$$

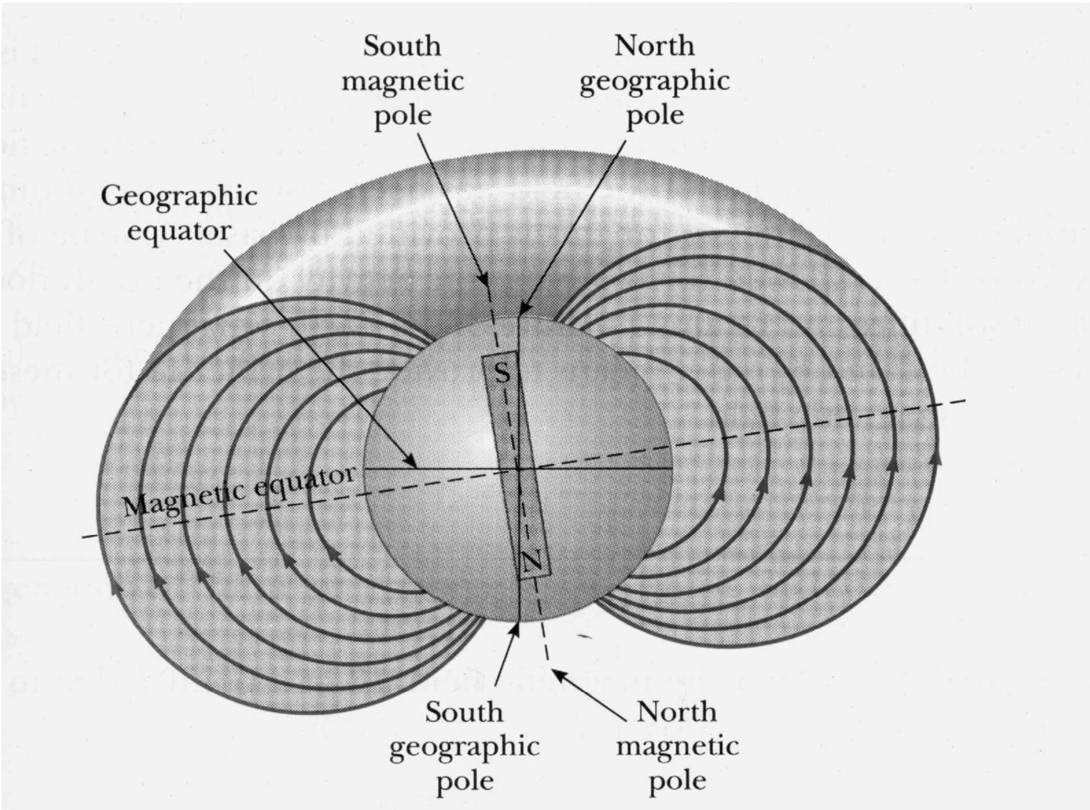
Warning!
In general, $\vec{B}$ is not conservative, so there is no magnetic potential energy. $\vec{B}$ *can't* be conservative since it gives rise to a velocity-dependent force (similar to friction).

Magnetic Mirror



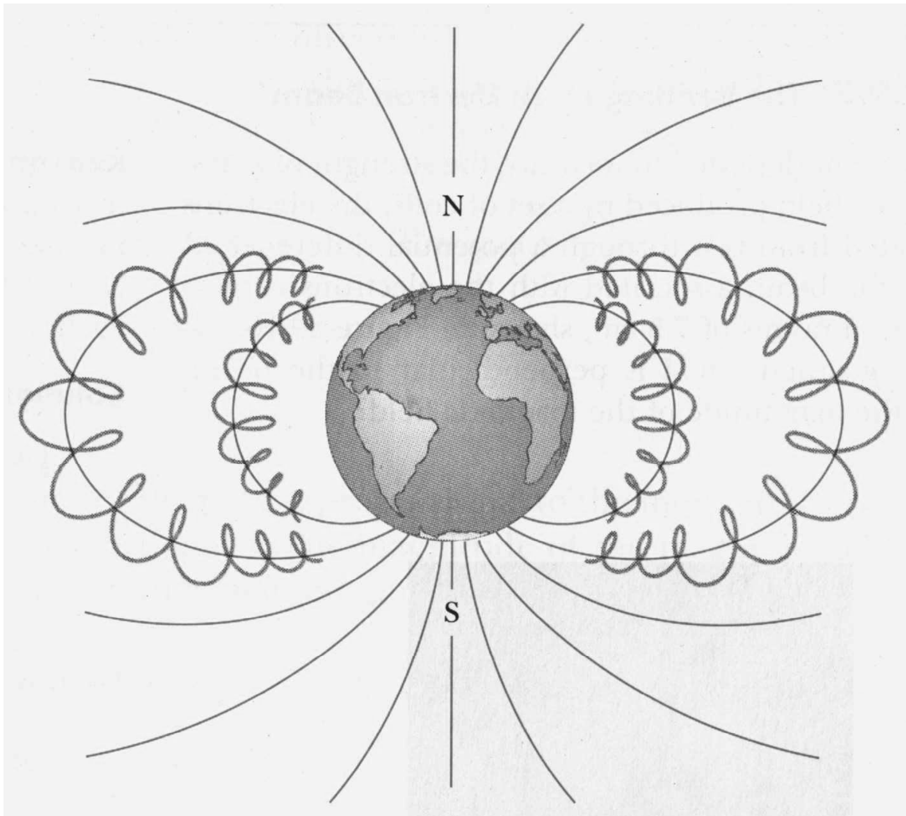
A charged particle bounces back and forth between the pinched ends of the field.
But how to create a non-uniform $\vec{B}$?

Earth's Magnetic Field



The earth's field is not due to magnetic rocks
($T_{\text{earth}} > T_{\text{iron's demagnetization}}$)
"Dynamo Effect"?
(rotating conducting fluid)

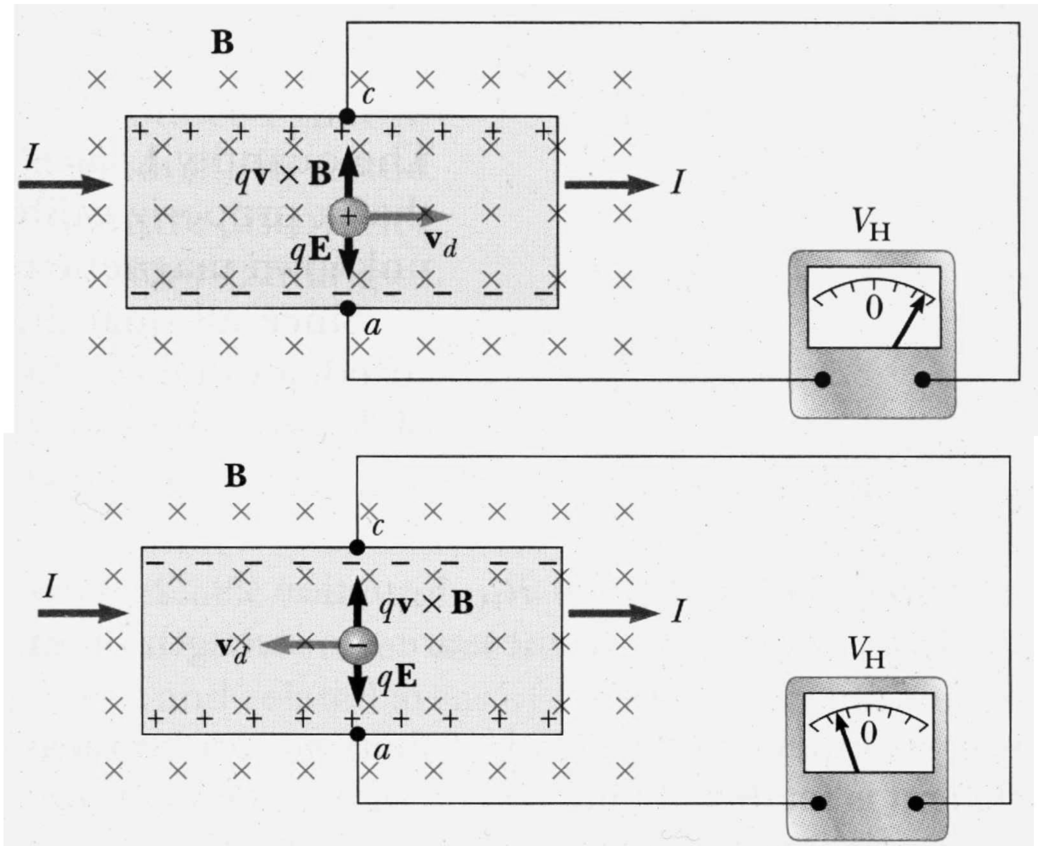
Trapped e's & p's



Van Allen Radiation Belts
"Bent" Magnetic Mirror

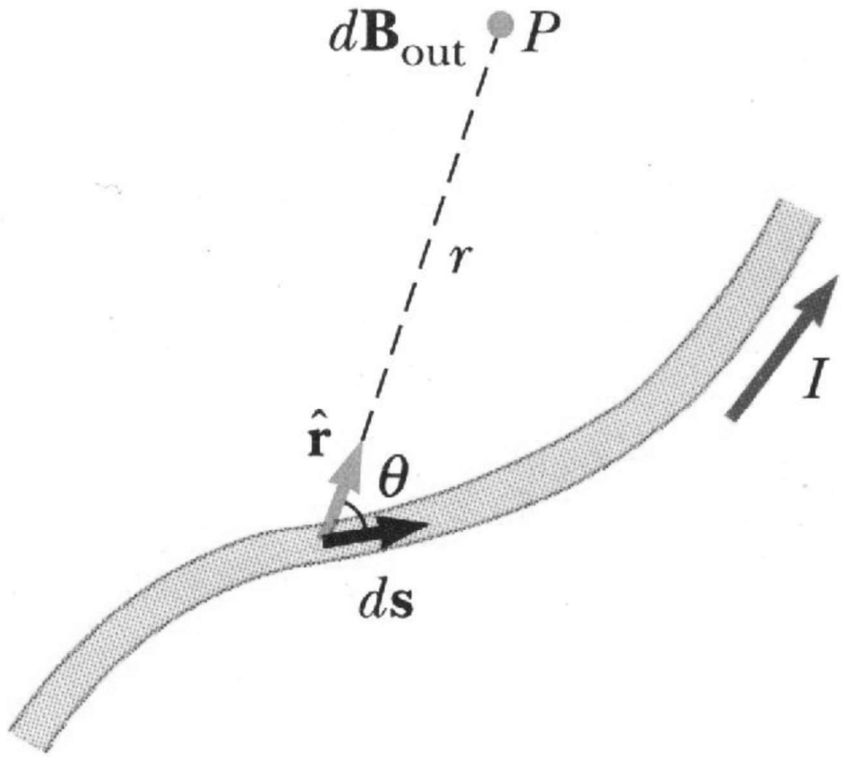
Hall Effect

Which charges (positive or negative) actually move in a conductor?



Sign of $V_H \Rightarrow$ Type of charge carrier

Biot-Savart Law

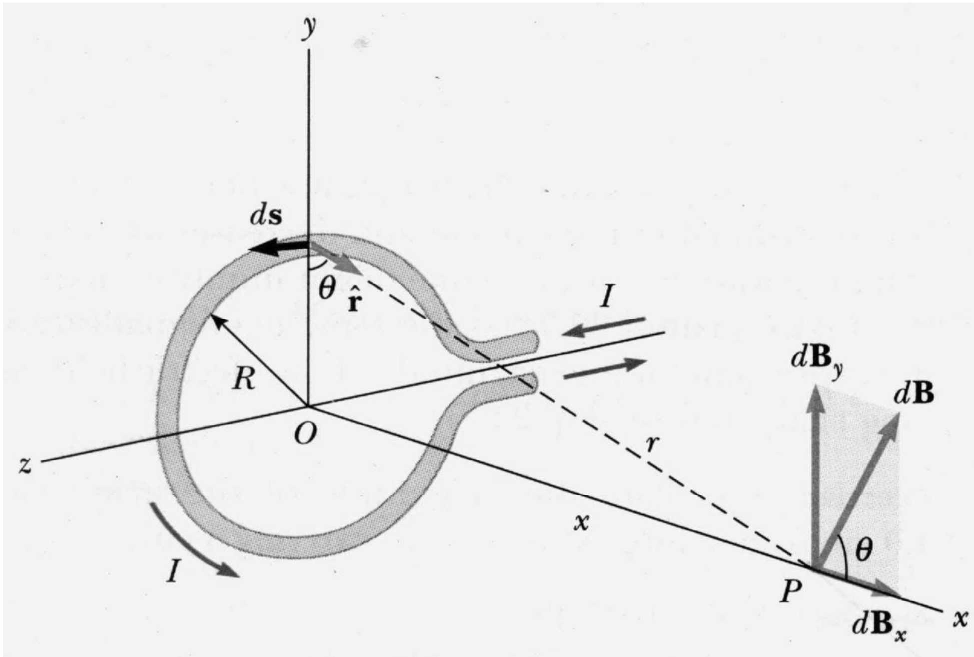


$$d\vec{B}_{atP} = \frac{\mu_0}{4\pi}\frac{Id\vec{s}\times\hat{r}}{r^2} \tag{498}$$

$$|d\vec{B}| = \frac{\mu_0}{4\pi}\frac{Ids\sin\theta}{r^2} \tag{499}$$

$$\vec{B} = \int d\vec{B} \tag{500}$$

Circular Current Loop



$$B = \int dB_x = \int dB \cos\theta \quad (501)$$

(y component totals to 0)

$$= \frac{\mu_0 I}{4\pi} \int \frac{ds \sin 90^\circ}{r^2} \cos\theta \quad (502)$$

$$r = \sqrt{R^2 + x^2}, \cos\theta = \frac{R}{r} = \frac{R}{\sqrt{R^2 + x^2}} \quad (503)$$

$$B = \frac{\mu_0 I}{4\pi} \frac{R}{(R^2 + x^2)^{\frac{3}{2}}} \int ds, \int ds = 2\pi R \quad (504)$$

$$B = \frac{\mu_0 I R^2}{2(R^2 + x^2)^{\frac{3}{2}}}, \text{ direction by the right-hand rule} \quad (505)$$

$$\text{At the center of the loop } B = \frac{\mu_0 I}{2R} \quad (506)$$

$$B = \frac{\mu_0 I R^2}{2(R^2 + x^2)^{\frac{3}{2}}} \quad (507)$$

For $x >> R$,

$$B = \frac{\mu_0 I R^2}{2x^3} \quad (508)$$

$$N \text{ Loops} \Rightarrow I \to NI \tag{509}$$

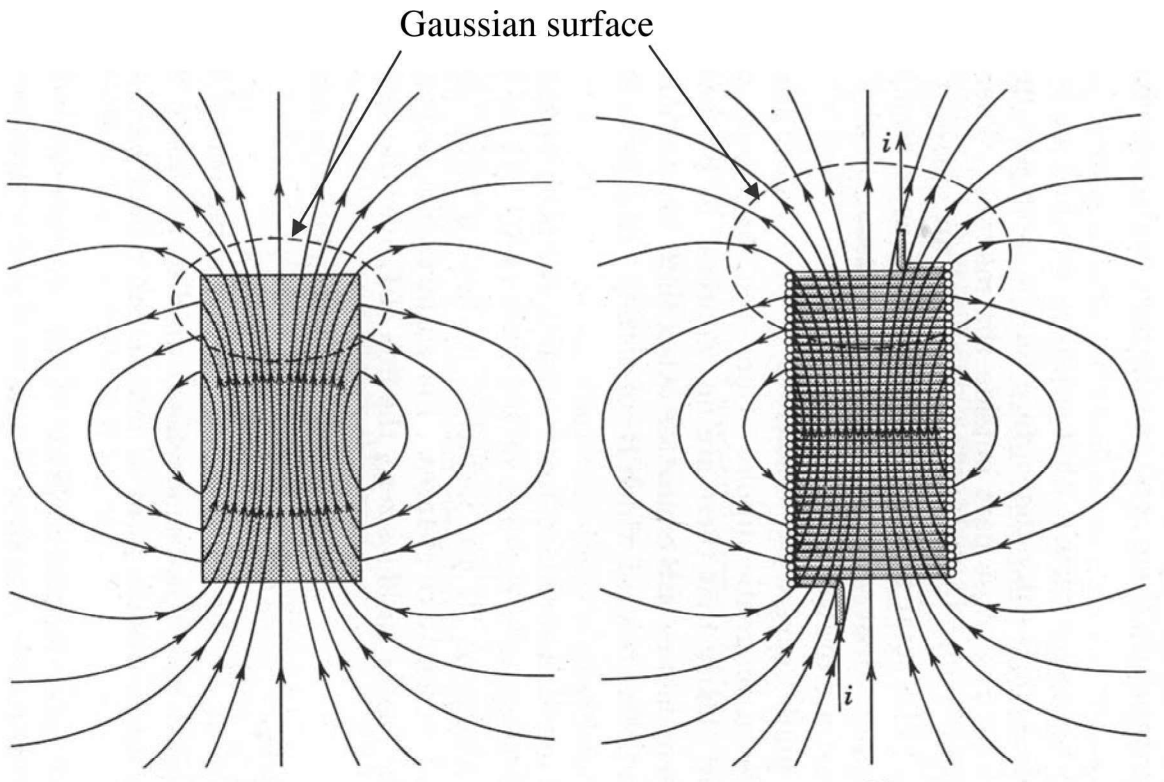
$$A = \pi R^2 \tag{510}$$

$$B = \frac{\mu_0}{2\pi}\frac{NIA}{x^3} = \frac{\mu_0}{2\pi}\frac{\mu}{x^3} \quad \mu \equiv \text{magnetic dipole moment} \tag{511}$$

Similar to $E = \frac{1}{2\pi\epsilon_0}\frac{p}{x^3}$ for electric dipole
A current loop acts like a magnetic dipole

- experiences a torque in an external $\vec{B}$ field
- produces a similar field

Gauss's Law For Magnetism



$$\Phi_B = \oint \vec{B} \cdot d\vec{A} = 0 \tag{512}$$

compare to...

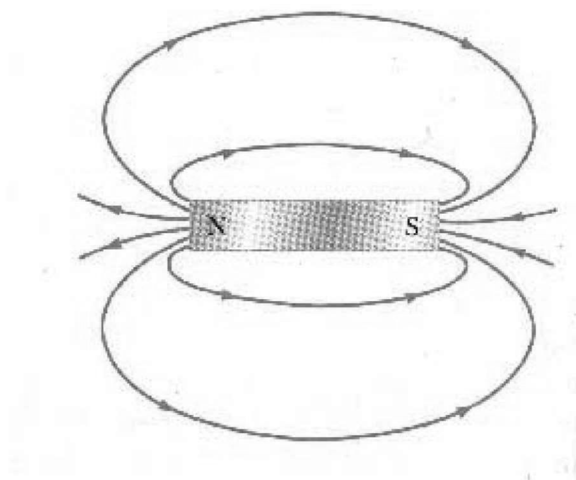
$$\Phi_E = \frac{q}{\epsilon_0} \tag{513}$$

No isolated magnetic charges (monopoles).

No Isolated Magnetic "Charges"

Magnetic

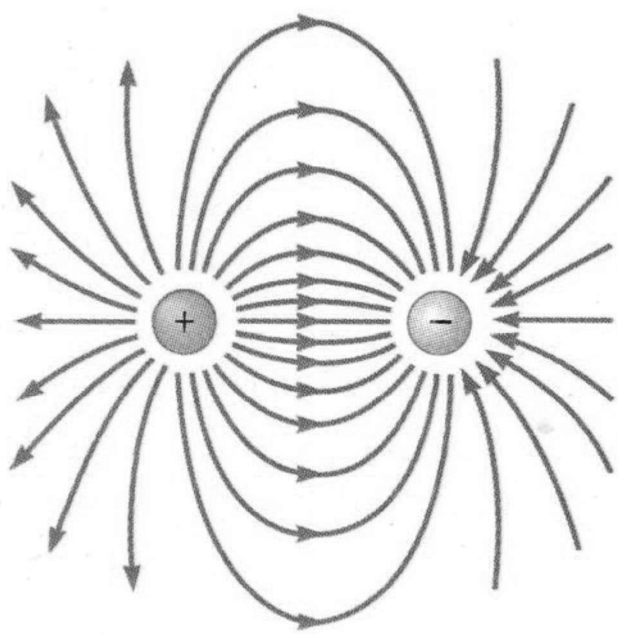
Dipole



Compare to

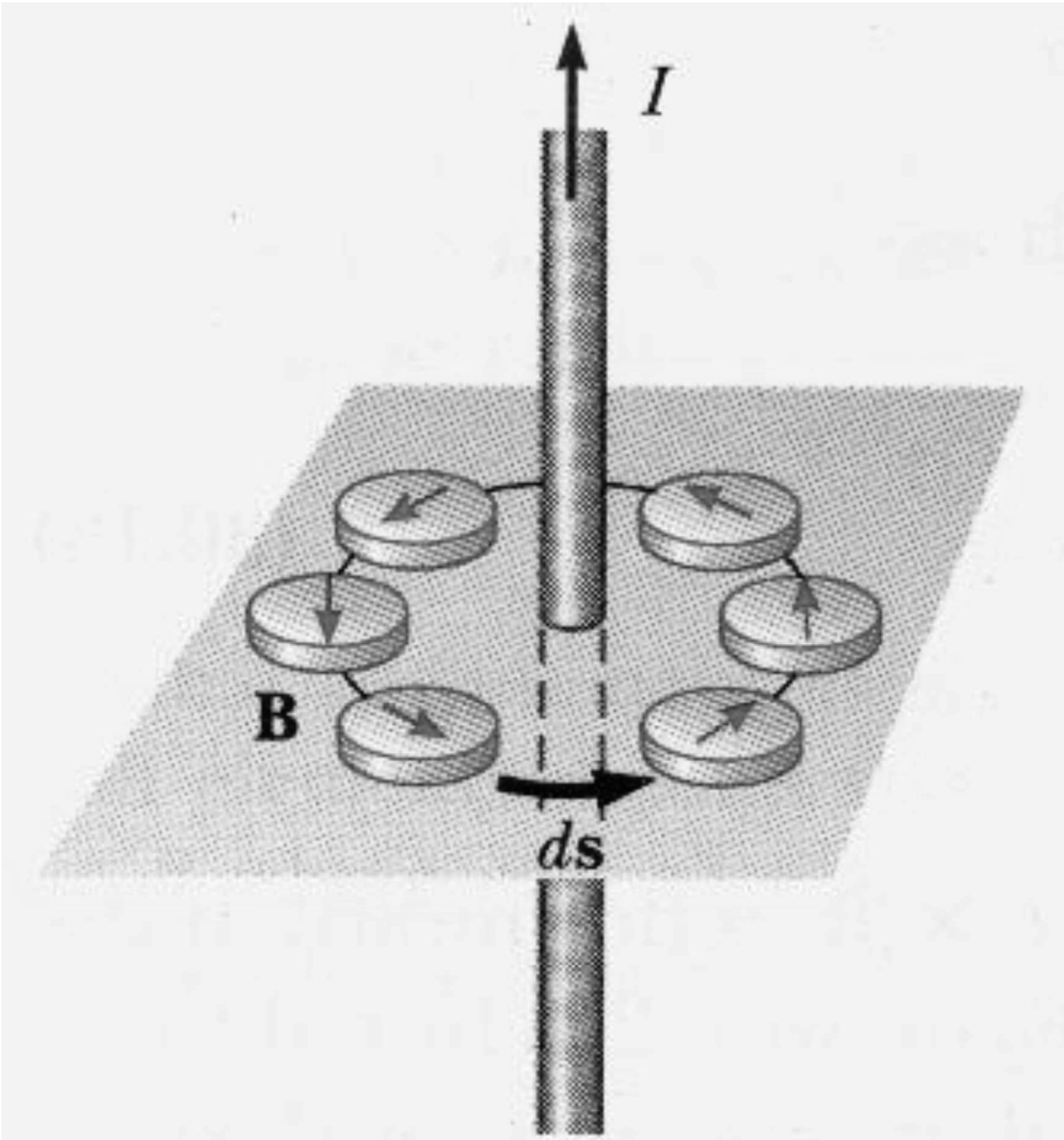
Electric

Dipole…



Magnetic dipoles also exist at the scale of individual atoms.

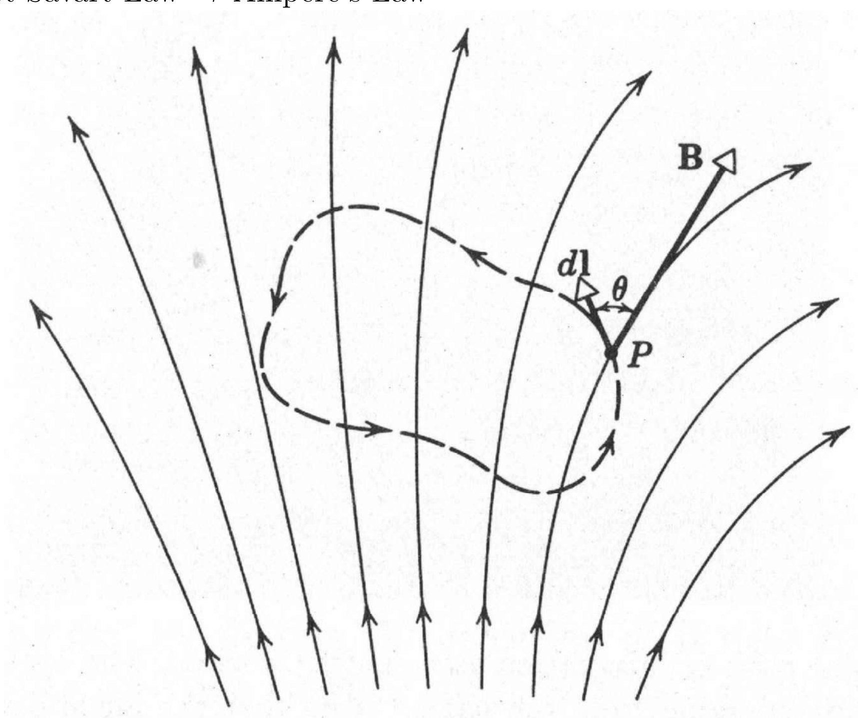
Ampere's Law



Current produces a $\vec{B}$ field tangent to circles around the wire
- Oersted 1820 -

Ampere's Law

Coulomb's Law → Gauss' Law
Biot-Savart Law → Ampere's Law

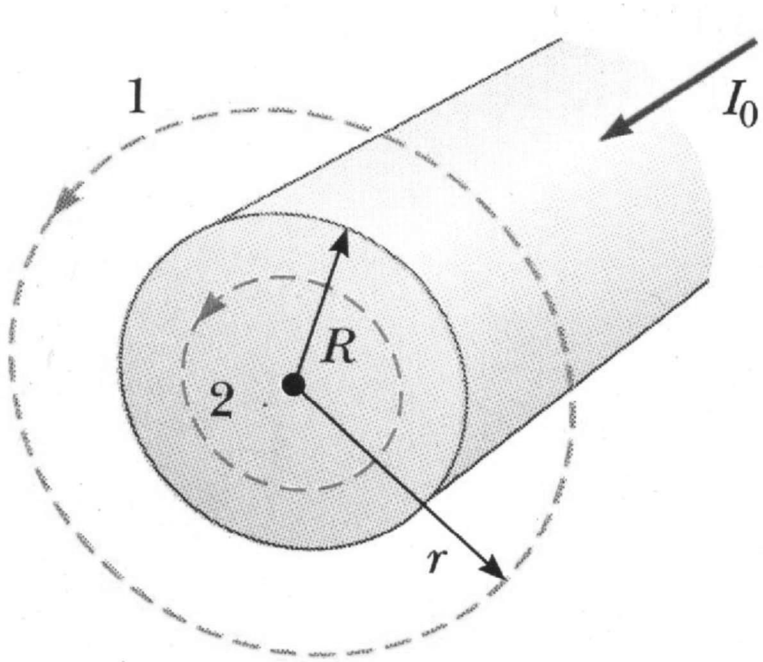


$$\oint \vec{B} \cdot d\vec{l} = \mu_0 i \tag{514}$$

where $\oint$ is the line integral around any closed curve, and i is the net current encircled by that curve
(Compare to Gauss' Law, which uses a surface integral, and finds the net number of field lines piercing the surface.)

Ampere's Law For A Long Wire

Choose Amperian path of a circle

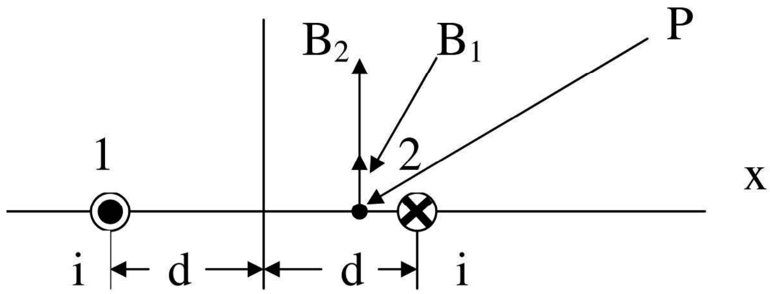


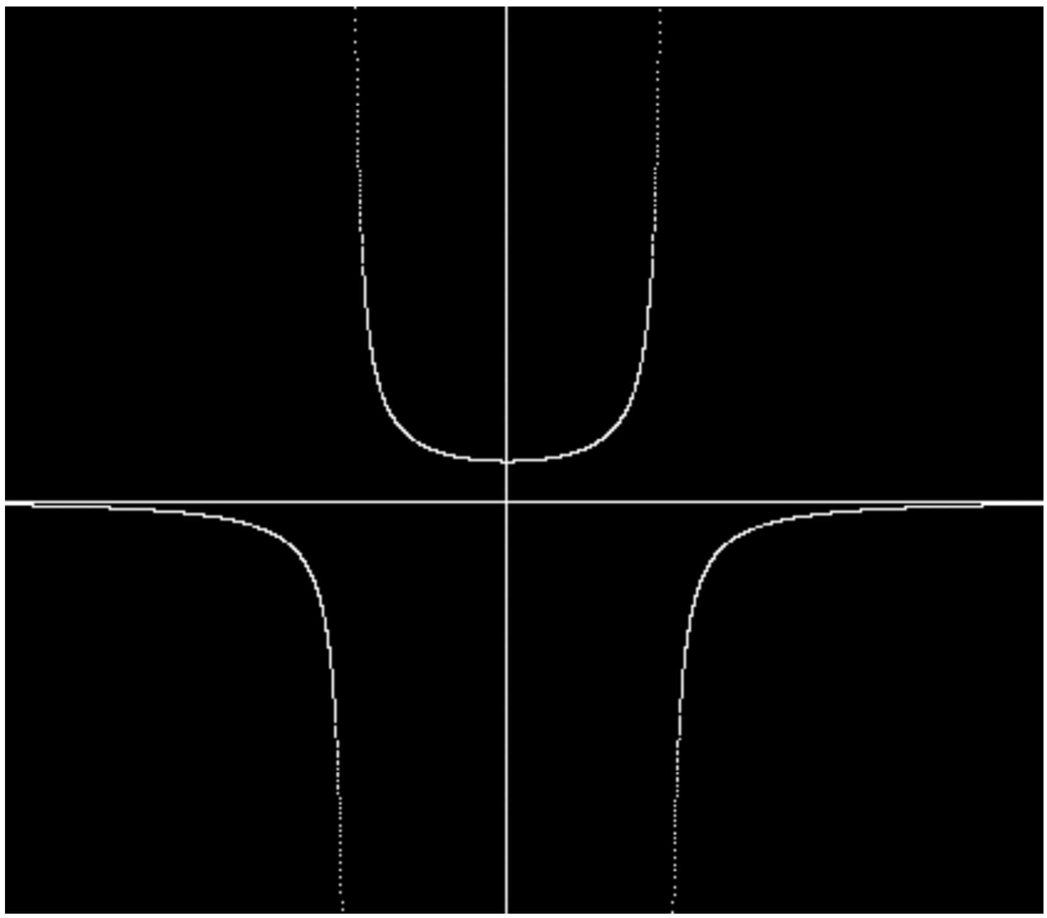
(for now, focus on B <u>outside</u> the wire)

$$\int \vec{B}\cdot d\vec{s} = \int B ds \tag{515}$$
$$= B\int ds \tag{516}$$
$$= B(2\pi r) \tag{517}$$
$$B(2\pi r) = \mu_0 i \tag{518}$$
$$B = \frac{\mu_0 i}{2\pi r} \tag{519}$$

Easier than using Biot-Savart!

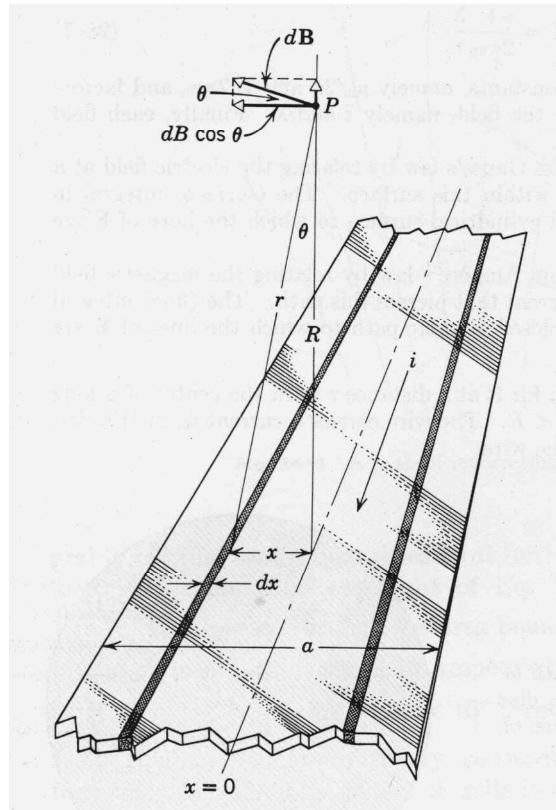
Magnetic Field Between Two Parallel Wires





$$\begin{aligned} B &= B_1 + B_2 && (520) \\ \text{(looking at mags. only)} & && \\ &= \frac{\mu_0 i}{2\pi(d+x)} + \frac{\mu_0 i}{2\pi(d-x)} && (521) \\ \text{(assuming the currents don't disturb each other)} & && \end{aligned}$$

B field from Current-Carrying Metal Strip



Long wire result (from Ampere's Law or integrating Biot-Savart):

$$dB = \frac{\mu_0}{2\pi}\frac{idx/a}{R\sec\theta} \tag{522}$$

$$r = \frac{R}{\cos\theta} \tag{523}$$

$$= R\sec\theta \tag{524}$$

$$\Sigma dB_{\text{rest}} = 0 \tag{525}$$

$$B = \int dB\cos\theta \tag{526}$$

$$= \frac{\mu_0 i}{2\pi aR}\int\frac{dx}{\sec^2\theta} \tag{527}$$

$$x = R\tan\theta \tag{528}$$

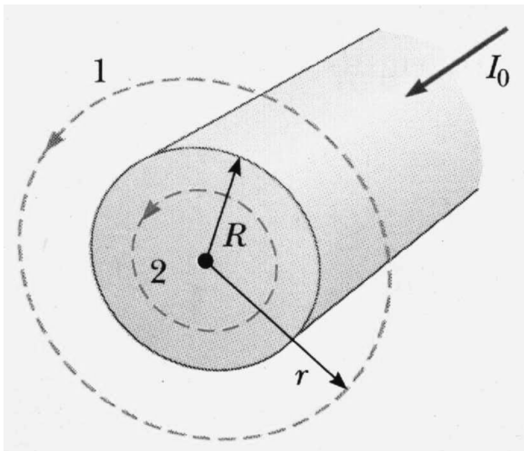
$$dx = R\sec^2\theta d\theta \tag{529}$$

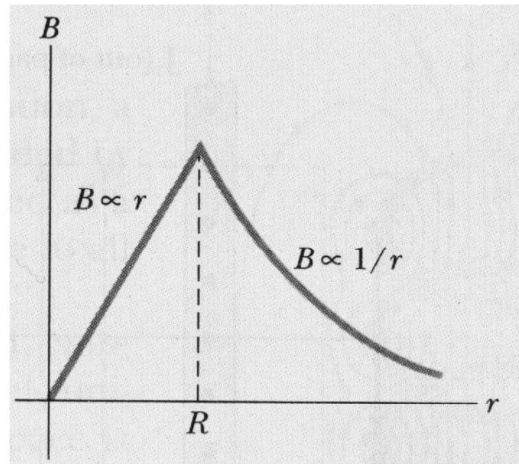
$$B = \frac{\mu_0 i}{2\pi aR}\int\frac{R\sec^2\theta d\theta}{\sec^2\theta} = \frac{\mu_0 i}{2\pi a}\int_{-\tan^{-1}(a/2R)}^{\tan^{-1}(a/2R)} d\theta \tag{530}$$

$$B = \frac{\mu_0 i}{\pi a}\tan^{-1}\left(\frac{a}{2R}\right) \tag{531}$$

$$\text{Large } R \rightarrow \frac{\mu_0 i}{2\pi R}$$

Long Wire With Uniform Current Distribution





Inside:

$$B(2\pi r) = \mu_0 i \frac{\pi r^2}{\pi R^2} \tag{532}$$

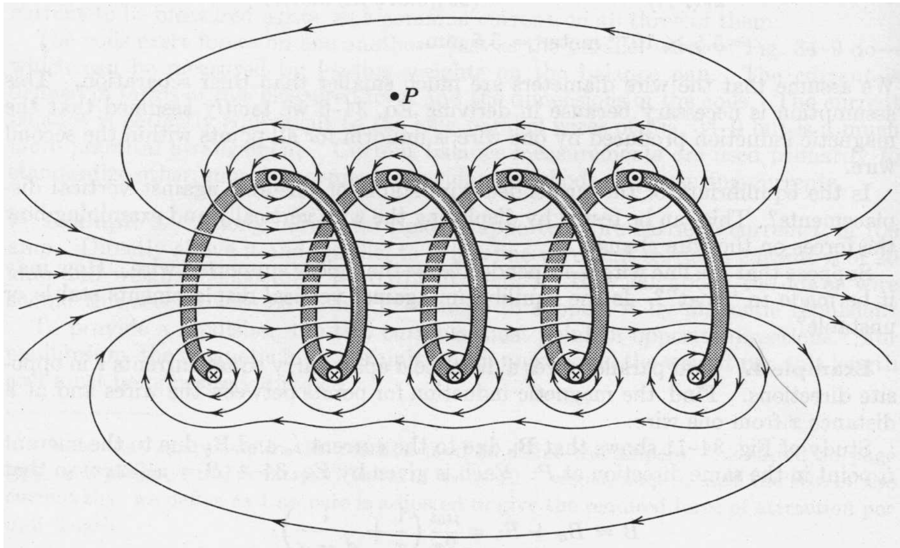
$$B = \frac{\mu_0 i}{2\pi R^2} r \tag{533}$$

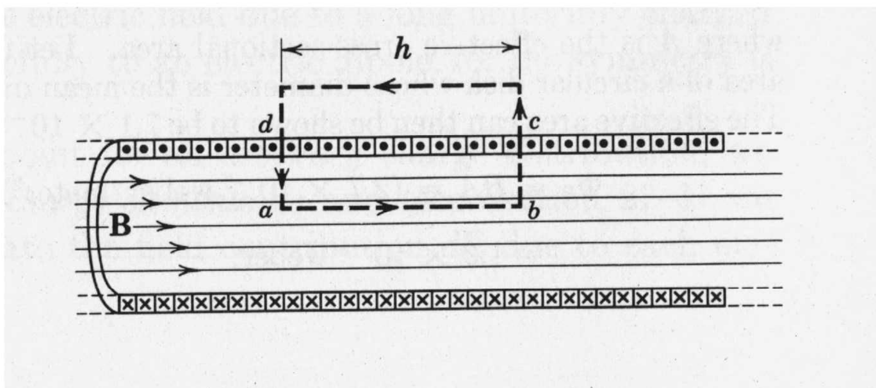
Outside

$$B(2\pi r) = \mu_0 i \tag{534}$$

$$B = \frac{\mu_0 i}{2\pi r} \tag{535}$$

Long Solenoid





$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i \tag{536}$$

$$\int_a^b + \int_b^c + \int_c^d + \int_d^a = \int_a^b \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{total}} \tag{537}$$

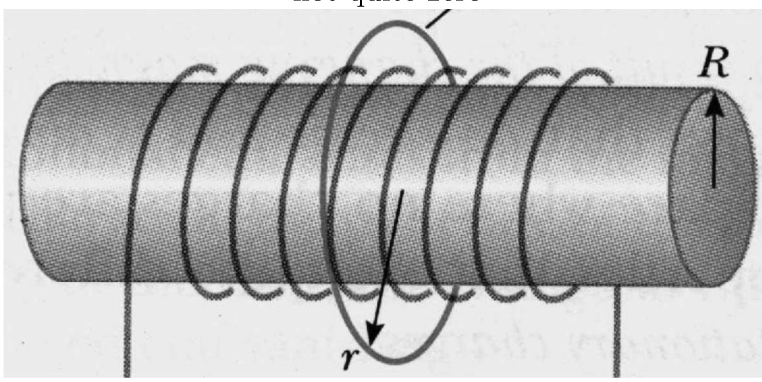
$$n \equiv \frac{\#\text{ turns}}{\text{length}} \tag{538}$$

$$Bh = \mu_0 (inh) \tag{539}$$

$$B = \mu_0 n i \tag{540}$$

Field Outside Solenoid

- not quite zero -



$$B(2\pi r) = \mu_0 i \tag{541}$$

$$B_{\text{ext}} = \frac{\mu_0 i}{2\pi r} \tag{542}$$

$$\frac{B_{\text{ext}}}{B_{\text{int}}} = \frac{\frac{\mu_0 i}{2\pi r}}{\mu_0 i n} = \frac{1}{2\pi r n} \tag{543}$$

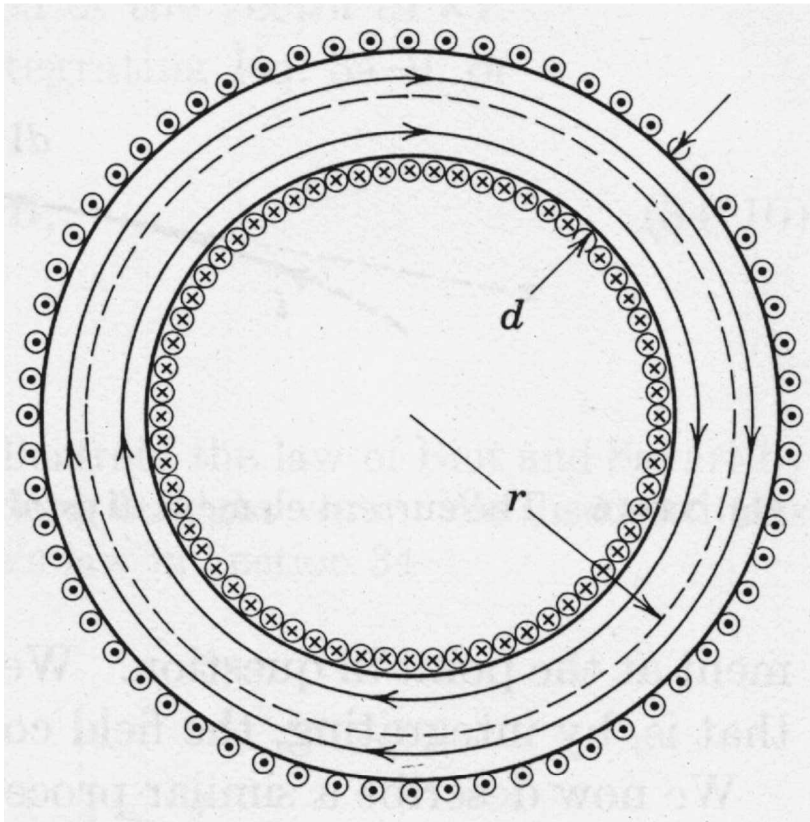
If adjacent turns touch...

$$n = \frac{1}{2}R, (2R\text{=wire diameter}) \tag{544}$$

$$\frac{B_{\text{ext}}}{B_{\text{int}}} = \frac{2R}{2\pi r} \tag{545}$$

$$2R \approx 1\text{mm} \leq 10^{-3} \tag{546}$$

Toroid (Solenoid As A Doughnut)



$$B(2\pi r) = \mu_0 i N \tag{547}$$

$$N = \text{Total turns} \tag{548}$$

$$B = \frac{\mu_0 i N}{2\pi r} \tag{549}$$

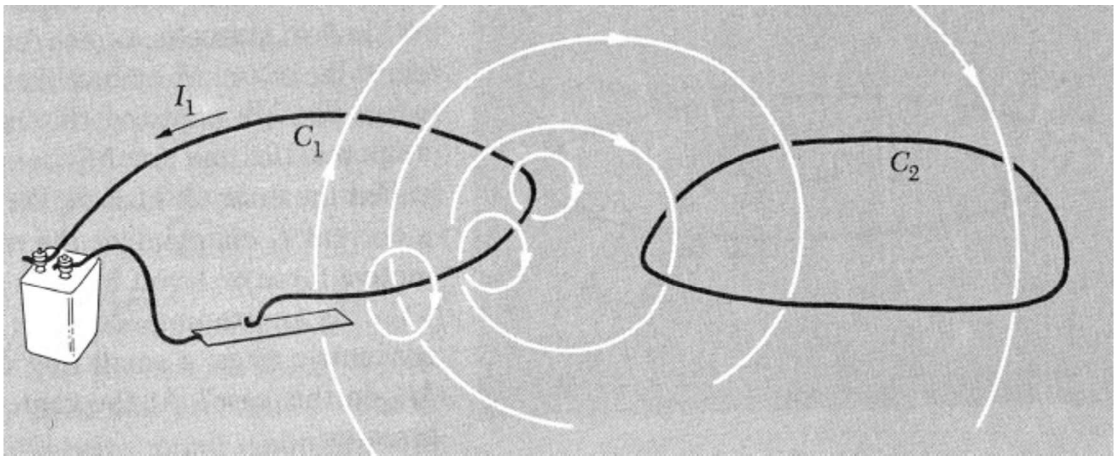
Unlike solenoid, $B = B(r)$
Outside, $B = 0$

Faraday's Law

Changing the number of B field lines passing through a loop induces an emf around the loop.

$$\Phi_B = \int \vec{B} \cdot d\vec{A} \tag{550}$$

$$[\Phi_B] = \text{Tesla} \cdot \text{meter}^2 = \text{Weber} \tag{551}$$

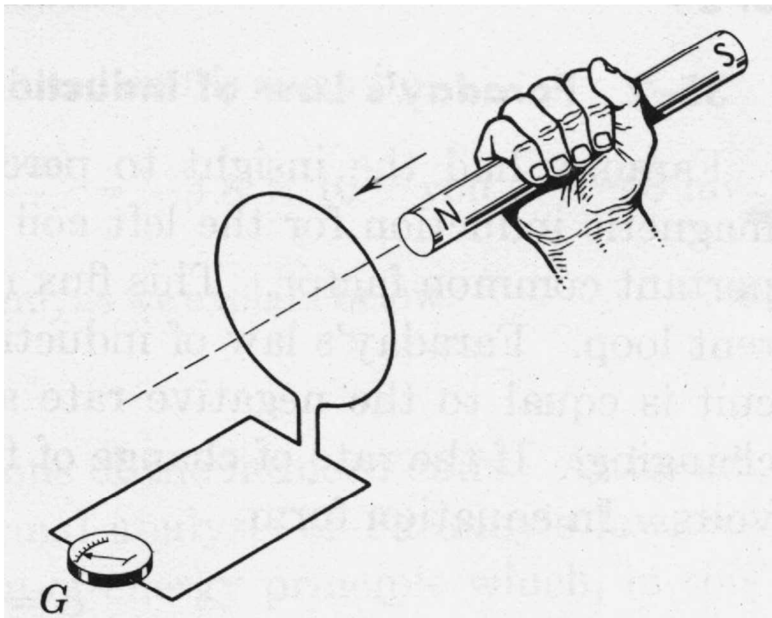


Faraday's Law: Induced emf = negative rate of change of magnetic flux through the circuit

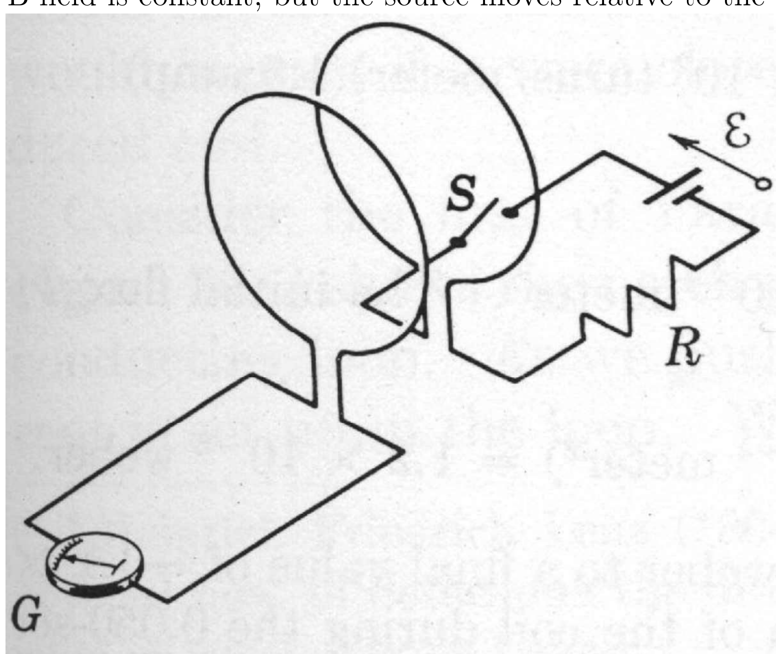
$$\epsilon = -\frac{d}{dt}\Phi_B \tag{552}$$

$$\epsilon = -N\frac{d\Phi_B}{dt}\text{coil} \tag{553}$$

Two Ways to Change Φ_B Through A Circuit



The B field is constant, but the source moves relative to the loop.



The B field itself changes in time, the geometry doesn't change.

$$\frac{-d\Phi_B}{dt} = \epsilon \Rightarrow \text{induced}\vec{E}\text{field} \Rightarrow \text{current} \tag{554}$$

Induced E's are <u>not</u> created by q's.
⇒ E's don't start on positive q's or end on negative q's.
⇒ Closed field lines.

$$V - V_a = -\int_a^b \vec{E} \cdot d\vec{s} \tag{555}$$

$$\text{if } a = b, \oint \vec{E} \cdot d\vec{s} = 0 \tag{556}$$

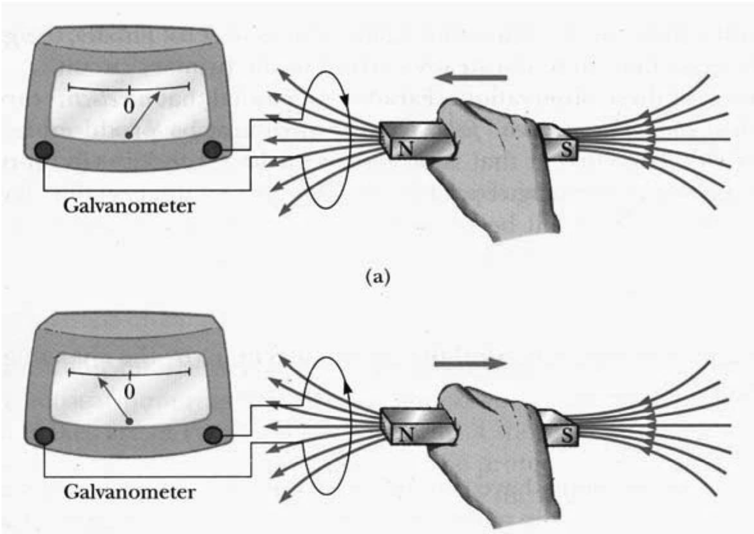
$$\Rightarrow \text{conservative force}$$

$$\text{Where } \vec{E} \text{ is induced by changing } \vec{B}, \oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi}{dt} \neq 0 \tag{557}$$

$$\tag{558}$$

$\Rightarrow$ This $\vec{E}$ is not conservative, and not associated with a potential energy.

Lenz' Law

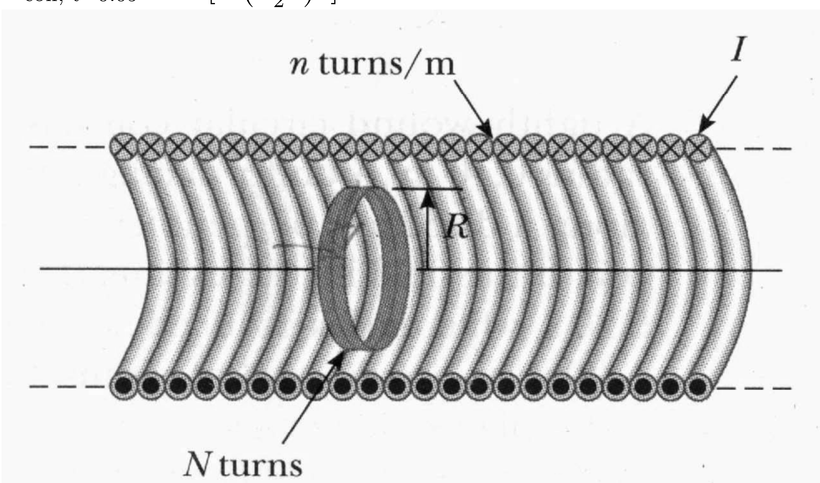


Induced current in a closed loop appears in a direction that opposes the change that produced it.
Opposition is not to the *field* from the bar magnet, but to *the change in the field*

$$\epsilon = -N\frac{d\Phi_B}{dt} \quad (559)$$

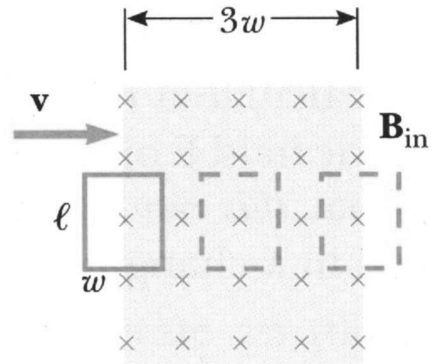
Faraday's Law Example

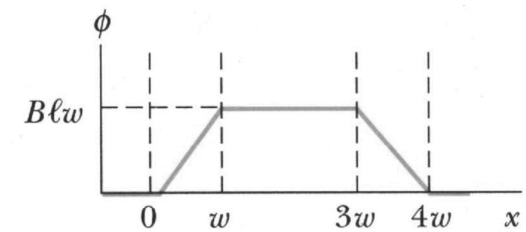
$\Phi_{\text{coil}} = B_{\text{sol}} A_{\text{coil}}$, where $B_{\text{sol}} = \mu_0 i n$
Given: $i_{\text{sol}}(t = 0sec) = 1.5$ Amp, $i_{\text{sol}}(t = 0.05sec) = -1.5$ Amp
$d_{\text{sol}} = 3.0$ cm (note: this won't be used!)
$n_{\text{sol}} = 200$ turns/cm
$B_{\text{sol}} = (4\pi\times10^{-7})(1.5\text{A})(200/\text{cm}) \left(100\frac{cm}{m}\right) = 3.8 \times10^{-2}$ T
$\Phi_{\text{coil, t=0.05}} = B[\pi \left(\frac{d_{\text{coil}}}{2}\right)^2] = 1.2\times10^{-5}$ Wb



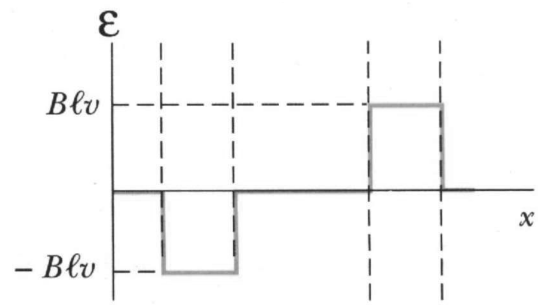
$$N_{\text{coil}} = 100 \text{ turns}$$
$$d_{\text{coil}} = 2 \text{ cm}$$
$$|\epsilon_{\text{coil}}| = N_{\text{coil}}\frac{\Delta\Phi}{\Delta t} = (100)\frac{2\times1.2\times10^{-5}\text{ Wb}}{0.05\text{ sec}} = 48 \text{ mV}$$

Loop Pulled Through Constant $\vec{B}$

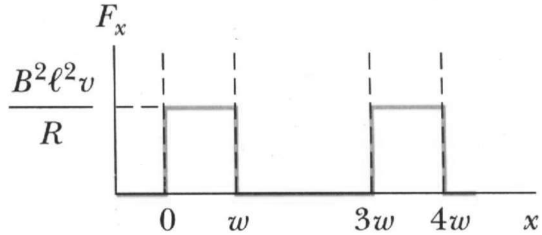




(b)

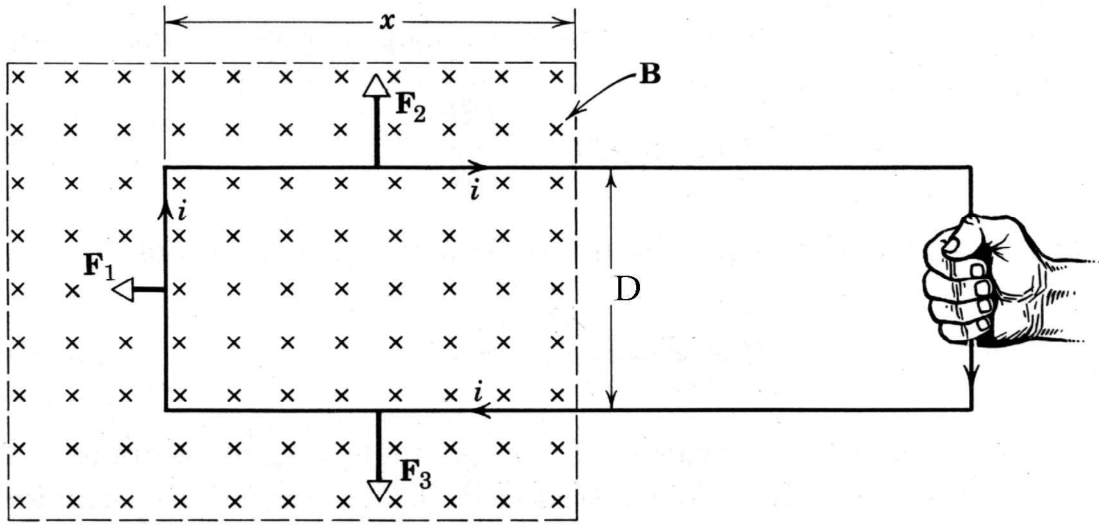


(c)



Power $= \frac{d}{dt}$work $= \frac{d}{dt}(QV) = IV = I^2R = \frac{V^2}{R}$

Calculation of Motional Emf



$$\Phi = BDx \tag{560}$$

$$\epsilon = -\frac{d\Phi}{dt} = -BD\frac{dx}{dt} = +BD\left(-\frac{dx}{dt}\right) = +BDv \tag{561}$$

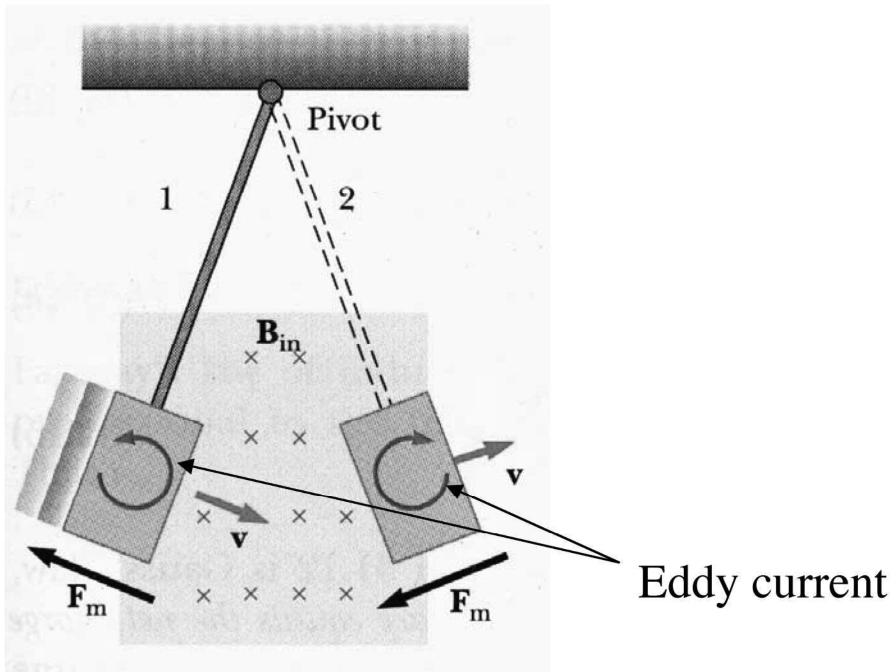
$$i = \frac{\epsilon}{R} = \frac{BDv}{R} \Rightarrow \vec{F}_1, \vec{F}_2, \vec{F}_3 \tag{562}$$

$$\vec{F}_1 = i\vec{L} \times \vec{B} = iDB = \frac{B^2D^2v}{R} \tag{563}$$

$$P = F_1 v = \frac{B^2D^2v^2}{R}, \text{ Work done at this rate by external agent} \tag{564}$$

$$P = i^2R = \left(\frac{BDv}{R}\right)^2 R = \frac{B^2D^2v^2}{R}, \text{ is the resulting Joule heating} \tag{565}$$

Eddy Currents



Lenz' Law:
⇒ opposition to motion
⇒ magnetic breaking
Eddy Currents:
⇒ heating

Reduce heating by lamination or cutting slices
⇒ longer current paths
⇒ larger resistance
⇒ smaller $\frac{V^2}{R}$

Eddy Current Brakes

Maxwell's Equations (so far)

$$\text{Gauss' law for electricity}$$
$$\oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0} \tag{566}$$
$$\text{Gauss' law for magnetism}$$
$$\oint \vec{B} \cdot d\vec{A} = 0 \tag{567}$$
$$\text{Faraday's law of induction}$$
$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \tag{568}$$
$$\text{Ampere's law}$$
$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i \tag{569}$$

We will add an additional term to Ampere's Law $\propto \frac{d\Phi_E}{dt}$
$\Rightarrow B(t) \rightarrow E, E(t) \rightarrow B$

In "differential" form...
Use Stokes' Theorem:

$$\oint \vec{F} \cdot d\vec{s} = \int (\vec{\nabla} \times \vec{F}) \cdot d\vec{A} \tag{570}$$

(true for any vector field F)
Apply to B field...

$$\oint \vec{B} \cdot d\vec{s} = \int (\vec{\nabla} \times \vec{B}) \cdot d\vec{A} = \mu_0 i \equiv \mu_0 \int \vec{j} \cdot d\vec{A} \tag{571}$$

(this defines the current density vector, $\vec{j}$)
so... $\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$
Similarly, we had...

$$\oint \vec{E} \cdot d\vec{s} = 0 \text{ (for } \vec{E} \text{ produced by charges)} \tag{572}$$

now...

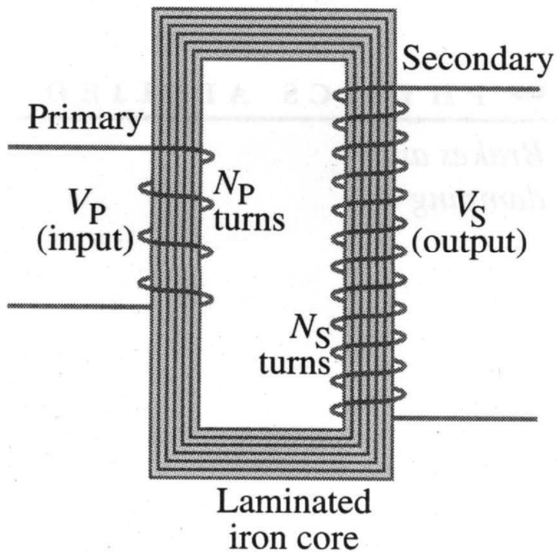
$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \text{ (for } \vec{E} \text{ produced by induction)} \tag{573}$$

Using Stokes'

$$\oint \vec{E} \cdot d\vec{s} = \int (\vec{\nabla} \times \vec{E}) \cdot d\vec{A} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A} \tag{574}$$

So $\vec{\nabla} \times \vec{E} = -\frac{d\vec{B}}{dt}$

Transformer



The ferromagnetic core keeps the field lines inside

Faraday's Law... ϵmf per turn is the same since the fluxes are the same.

$$\frac{d\Phi}{dt}|_{\text{primary}} = \frac{d\Phi}{dt}|_{\text{secondary}} \tag{575}$$

$$\epsilon_p = \epsilon_s \tag{576}$$

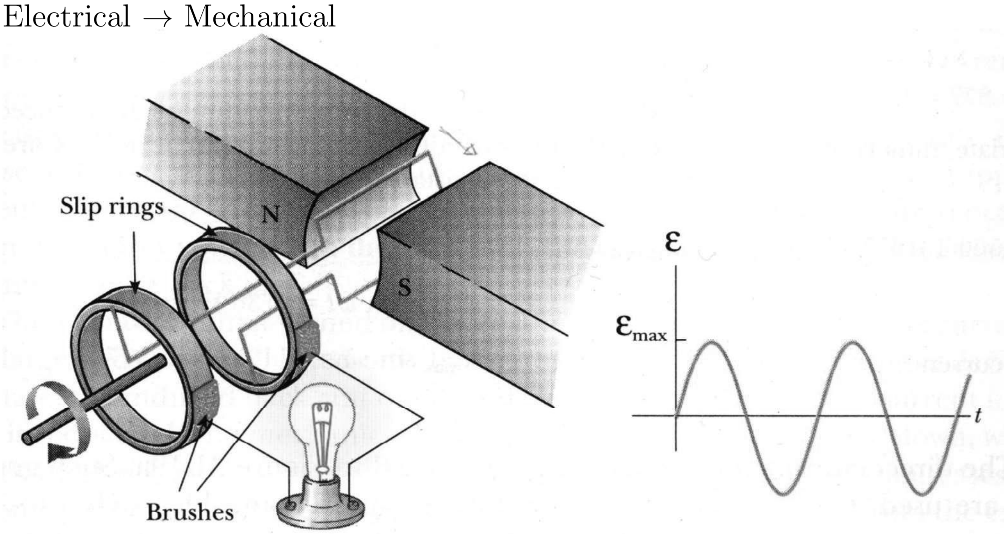
$$\frac{V_p}{N_p} = \frac{V_s}{N_s} \tag{577}$$

$$\Rightarrow V_s = V_p\left(\frac{N_s}{N_p}\right) \tag{578}$$

Step up, step down

AC Generator/Motor

Mechanical $\rightarrow$ Electrical
Electrical $\rightarrow$ Mechanical

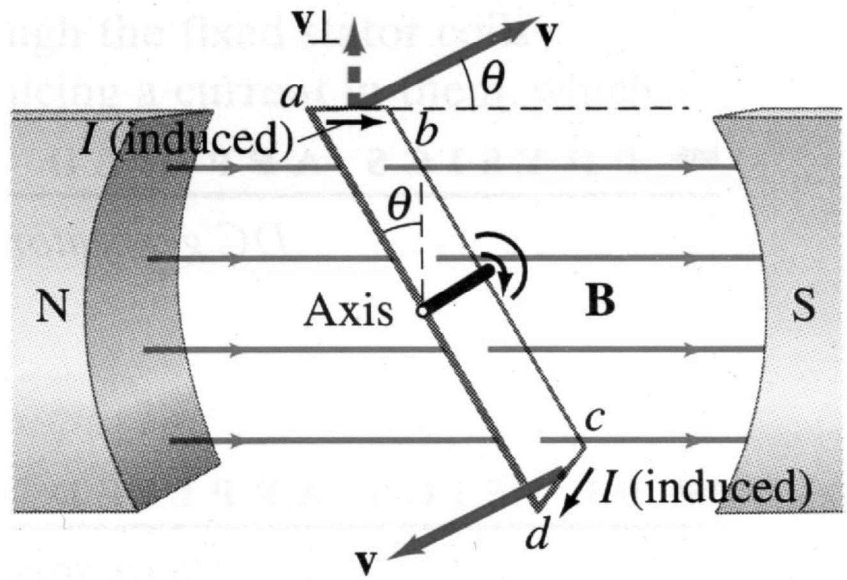


Motor:
Apply voltage to loop $\Rightarrow$ Torque on current loop $\Rightarrow$ Rotation. Torque "switches" at the critical moment.

Generator:
Rotate loop $\Rightarrow$ magnetic flux change through loop $\Rightarrow \epsilon = -\frac{d}{dt}\Phi_B$

AC Generator

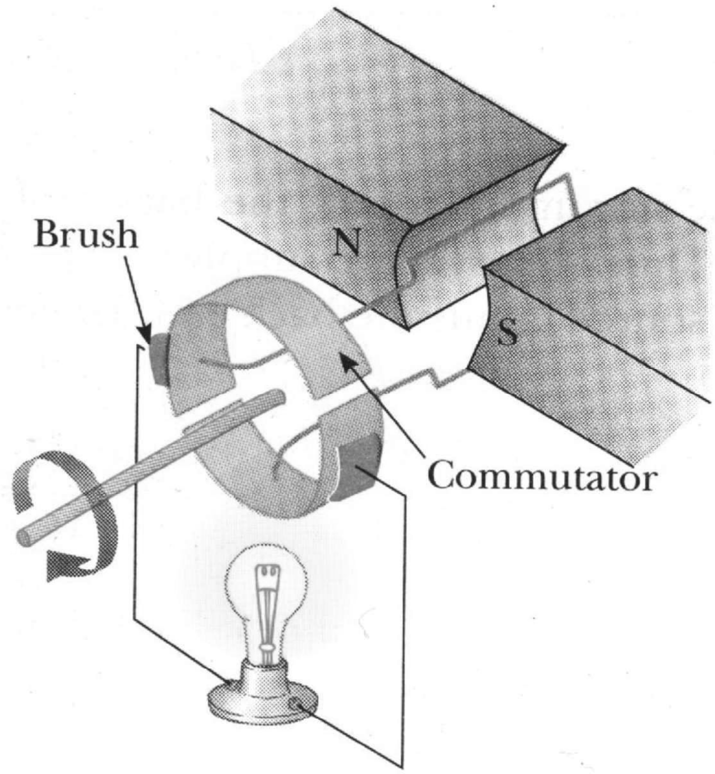
Mechanical $\rightarrow$ Electrical
To produce a driving signal $\epsilon = \epsilon_m \sin(\omega t)\ldots$

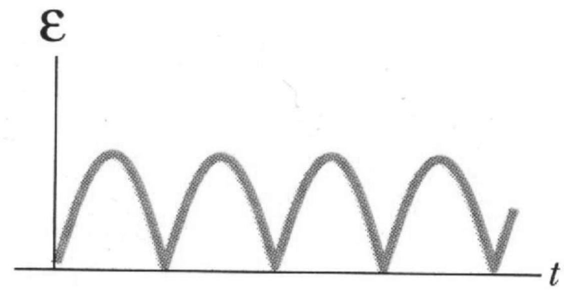


Motional ϵmf/Faraday's Law

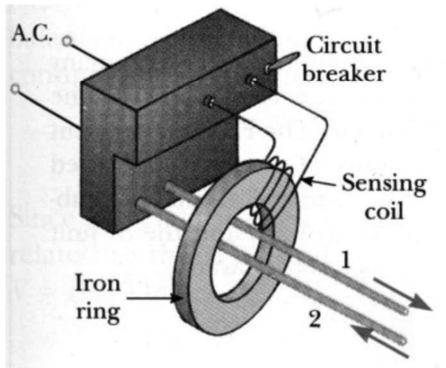
$$\begin{aligned} \epsilon &= -\frac{d}{dt}\Phi_B && (579)\\ &= -\frac{d}{dt}(BA\cos\theta) && (580)\\ &= BA\sin\theta && (581)\\ &= BA\sin(\omega t) && (582) \end{aligned}$$

DC Generator/Motor





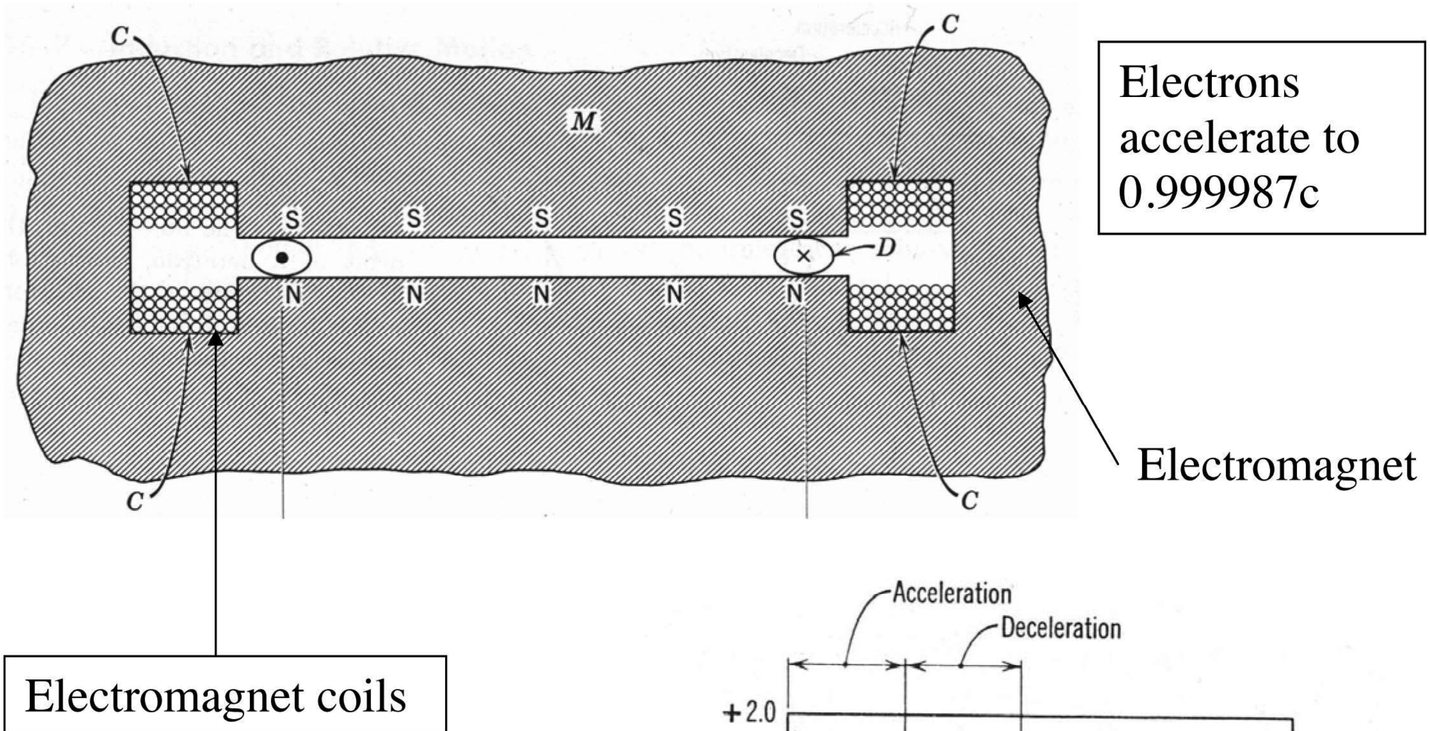
Ground Fault Interrupter

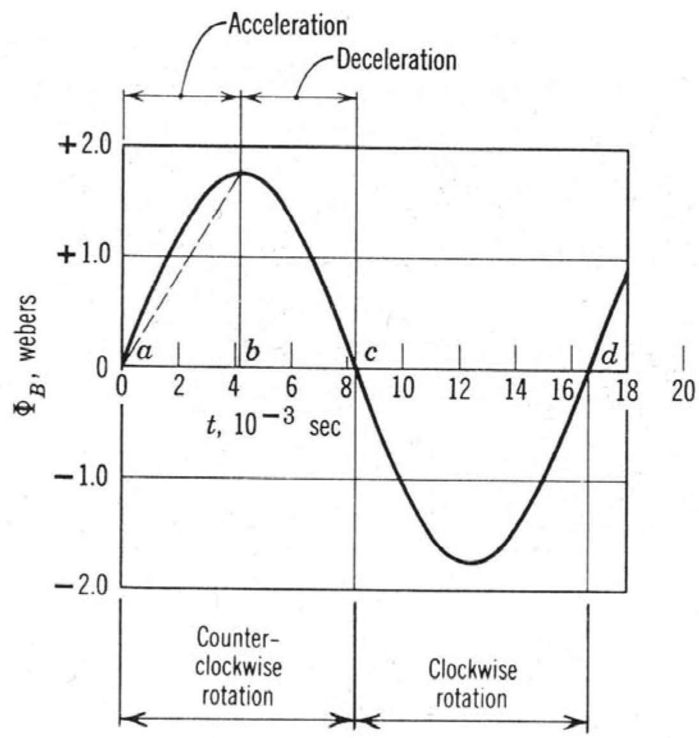


Ordinarily:
Net $i = 0$
$\Rightarrow$ No flux $\Rightarrow$ no $\frac{d}{dt}$ flux

If "fault":
$\Rightarrow$ Net $i \sim A \sin \omega t$
$\Rightarrow$ Flux $\propto \frac{d}{dt}$ flux

Betatron



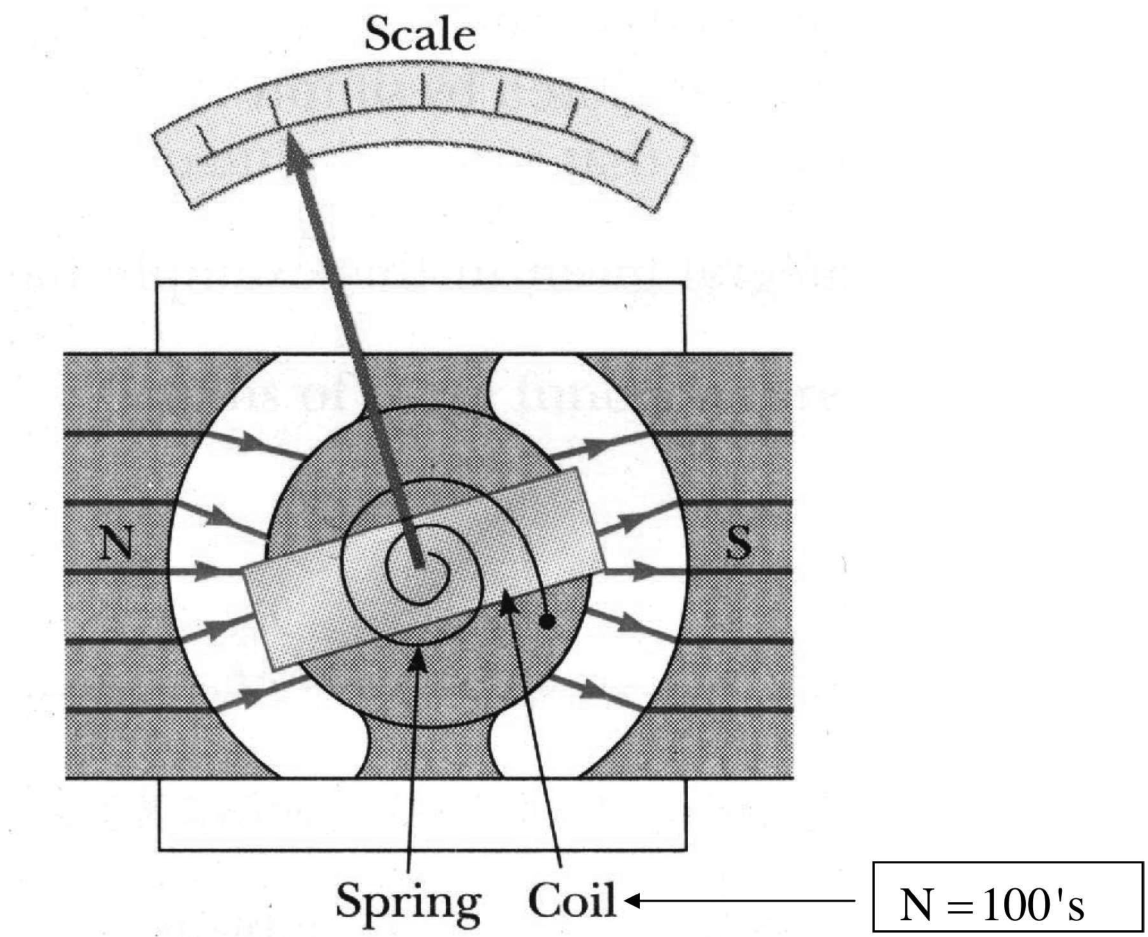


Electromagnet...

1. Guides in circular paths
2. $\frac{dB}{dt} \Rightarrow E$ which accelerates electrons along the path
3. Maintains constant r
4. Applies restoring force

Galvanometer

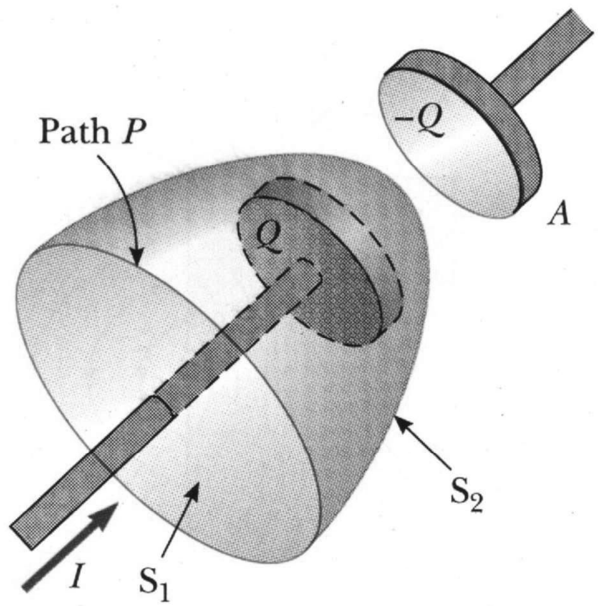
- Ammeter
- Voltmeter



$$\tau = NiAB\sin\theta \tag{583}$$

Spring for counter-torque $\Rightarrow$ Reading $\propto i$
How to measure V?

Displacement Current



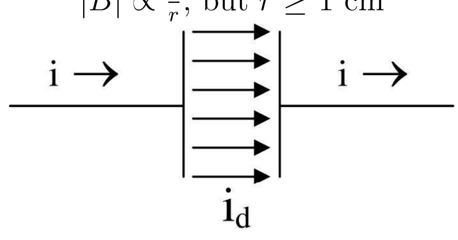
Ampere's law considers a closed curve (path) P and the surface area S bounded by that curve. It concerns itself with B along the curve and the current piercing the surface. *But there are an infinite number of bounding surfaces and they don't all have the same piercing current!* (S_1 pierced by "I" in the wire, S_2 has no piercing current.)

The fix: there is no real current between the capacitor plates, but the I in the connecting wires does produce a growing $\vec{E}$ field between the plates, so define ...

$$i_d \equiv \epsilon_0 \frac{d\Phi_E}{dt} \tag{584}$$

$$\Rightarrow \oint \vec{B} \cdot d\vec{l} = \mu_0(i + i_d) \text{ Revised Ampere's Law} \tag{585}$$

Why a late discovery?
$i_{\mathrm{real}} \sim$ produces relatively large $\vec{B}$ near the thin wire.
$|\vec{B}| \propto \frac{1}{r}, r \geq 0.1$ mm
$i_{\mathrm{disp}} \sim$ prodcues small $\vec{B}$, since current spread out over entire cross-section of capacitor plates.
$|\vec{B}| \propto \frac{1}{r}$, but $r \geq 1$ cm



B Induced Between The Plates of A Capacitor

Assume that $\vec{E}$ is growing between the plates ...

Use Ampere's Law

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} \quad (586)$$

$$\frac{d\Phi}{dt} > 0 \quad (587)$$

$$\oint \vec{B} \cdot d\vec{s} > 0 \quad (588)$$

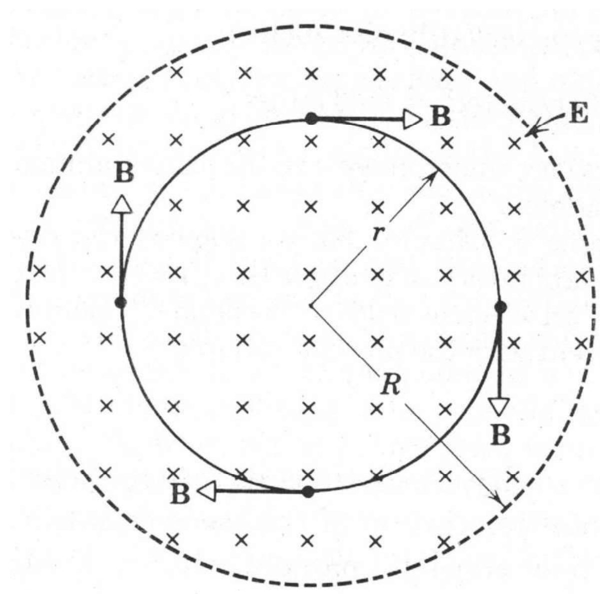
$\vec{B}$ direction

$$B(2\pi r) = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} = \mu_0 \epsilon_0 \frac{d}{dt}(E\pi R^2) \quad (589)$$

$$= \mu_0 \epsilon_0 \pi R^2 \frac{dE}{dt}, r \geq R \quad (590)$$

$$B(2\pi r) = \mu_0 \epsilon_0 \pi r^2 \frac{dE}{dt}, r \leq R \quad (591)$$

Match at $r = R$



Maxwell's Equations in *Integral* Form

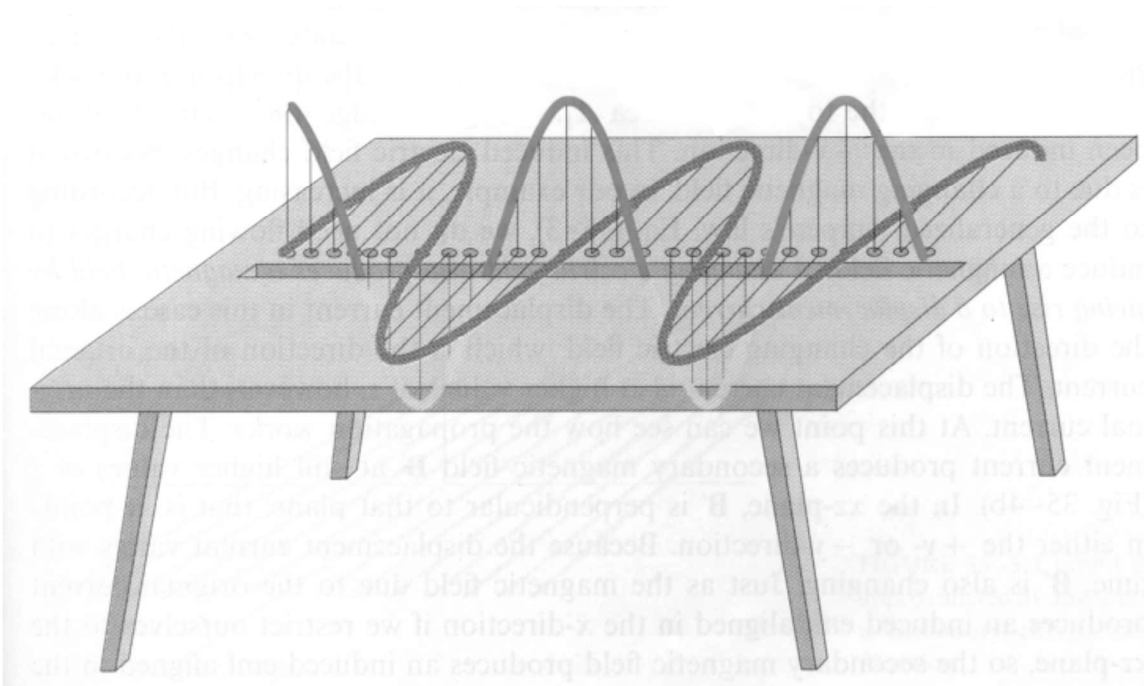
Gauss' law for electricity	$\oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$	Surface integral
Gauss' law for magnetism	$\oint \vec{B} \cdot d\vec{A} = 0$	Surface integral
Faraday's law of induction	$\oint \vec{E} \cdot d\vec{s} = \frac{-d\Phi_B}{dt}$	Line integral
Ampere's law (as extended by Maxwell)	$\oint \vec{B} \cdot d\vec{s} = \mu_o i + \mu_o \epsilon_o \frac{d\Phi_E}{dt}$	Line integral

Maxwell's Equations in *Differential* Form

Gauss' law for electrcity	$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$	Divergence
Gauss' law for magnetism	$\vec{\nabla} \cdot \vec{B} = 0$	Divergence
Faraday's law of induction	$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	Curl
Ampere's law (as extended by Maxwell)	$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$	Curl

$\vec{E}$ and $\vec{B}$ are "coupled" through Maxwell's Eqns.

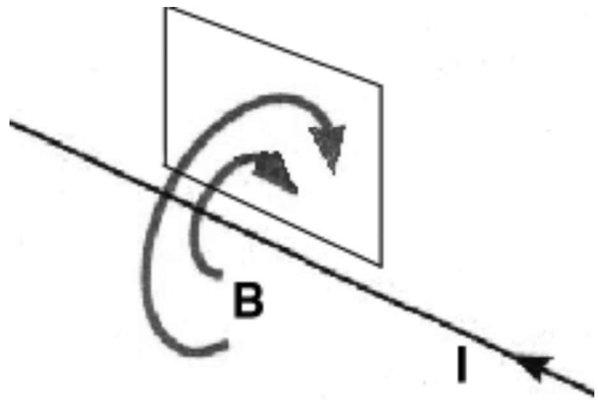
(Don't take this figure literally! It is only intended to suggest the tight coupling between E and B fields.)



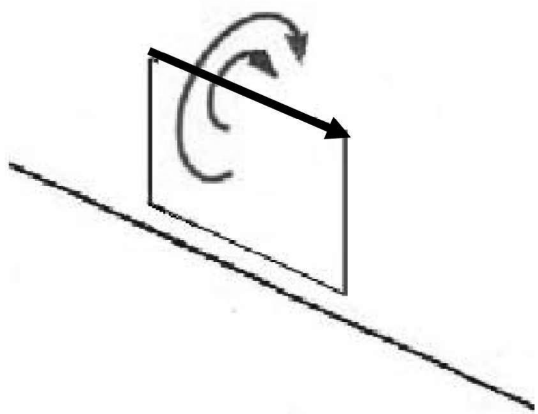
$$\oint \vec{B} \cdot d\vec{s} = \mu_0 \epsilon_0 \frac{d}{dt} \iint \vec{E} \cdot d\vec{A} \tag{592}$$

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \iint \vec{B} \cdot d\vec{A} \tag{593}$$

Electromagnetic Waves - Hand-waving Proof

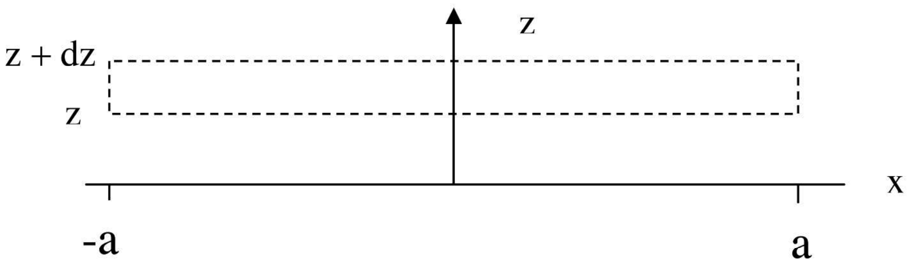


If $i(t)$ grows $\Rightarrow \Phi_B$ grows $\Rightarrow$ induced $\vec{E}$
Then ...



$\vec{E}$ field at top of rectangle.
$\vec{E}$ changing in time $\Rightarrow$ displacement $I \Rightarrow \vec{B}$ produced
New $\vec{B}$ is at larger z than the original.
$\vec{E}, \vec{B}$ are $\perp$ to each other & to the propagation direction

Electromagnetic Waves - Real Proof



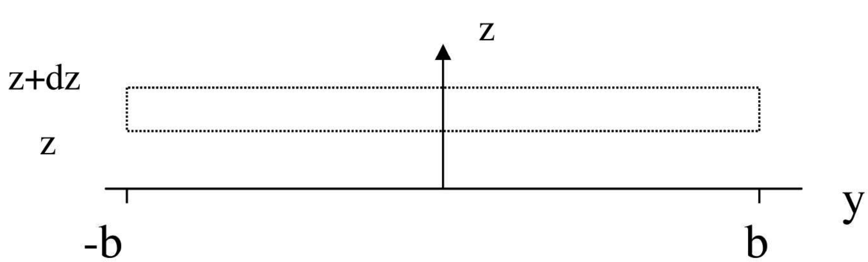
Assume fields vary with z only.
Rectange is infinitesimally thin.
B is perpendicular to xz plane.

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d}{dt}\Phi_B, |d\vec{s}| = dx \tag{594}$$

$$E_x(z+dz)[2a] - E_x(z)[2a] = -\frac{d}{dt}B_y[2adz] \tag{595}$$

$$\frac{\partial E_x}{\partial z}dz[2a] = -\frac{\partial B_y}{\partial t}dz[2a] \tag{596}$$

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} \tag{597}$$



Notice we are in the yz plane now.
Through nearly identical derivation:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 \epsilon_0 \frac{d}{dt} \Phi_E \Rightarrow \frac{\partial B_y}{\partial z} = -\mu_0 \epsilon_0 \frac{\partial E_x}{\partial t} \tag{598}$$

$$\frac{\partial B_y}{\partial t} = -\frac{\partial E_x}{\partial z} \text{ (from last page)} \tag{599}$$

$$\text{Take} \frac{\partial}{\partial z} \Rightarrow \frac{\partial^2 B_y}{\partial z \partial t} = -\frac{\partial^2 E_x}{\partial z^2} \tag{600}$$

$$\frac{\partial B_y}{\partial z} = -\mu_0 \epsilon_0 \frac{\partial E_x}{\partial t} \text{ (from this page)} \tag{601}$$

$$\text{Take } \frac{\partial}{\partial t} \Rightarrow \frac{\partial^2 B_y}{\partial z \partial t} = -\mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2} \tag{602}$$

Eliminate $\partial^2 B$ between these eqns.

$$\Rightarrow \frac{\partial^2 E_x}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2} \tag{603}$$

similarly:

$$\frac{\partial^2 B_y}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 B_y}{\partial t^2} \tag{604}$$

Each is a wave equation:

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial z^2} \tag{605}$$

$$\Rightarrow E_x = E_0 \cos(kz - \omega t + \phi) \tag{606}$$

$$\text{where } v^2 = \frac{1}{\mu_0 \epsilon_0} \tag{607}$$

$$v = 3 \times 10^8 \text{m/sec} \equiv'' c'' \tag{608}$$

$$E_x = E_0 \cos(kz - \omega t) \tag{609}$$

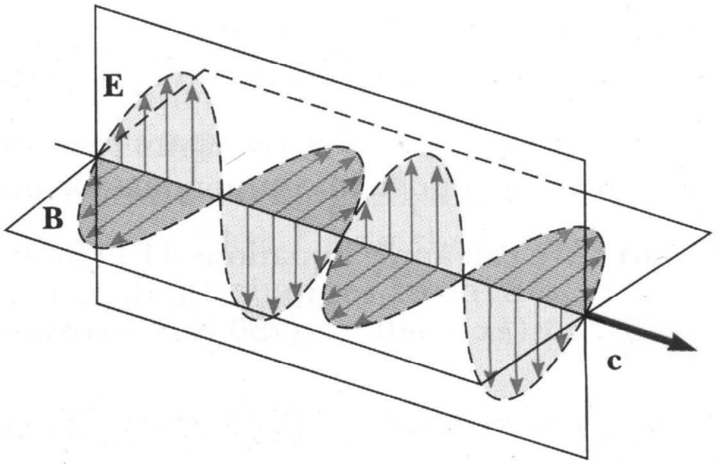
$$B_y = B_0 \cos(kz - \omega t) \tag{610}$$

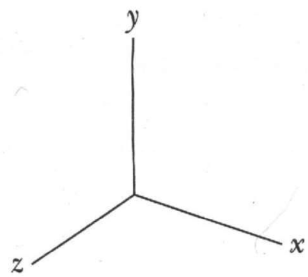
$$\text{Use the equation:} \frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} \tag{611}$$

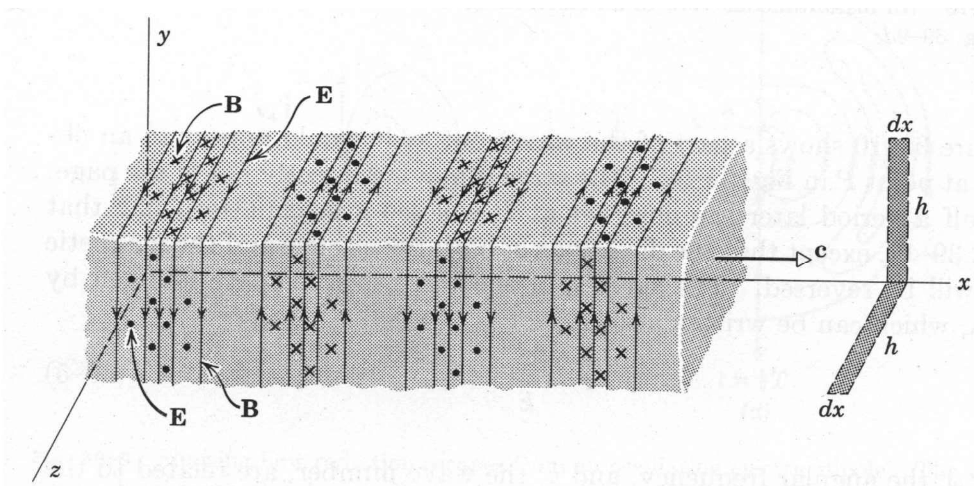
$$-kE_0 \sin(kx - \omega t) = -\omega B_0 \sin(kx - \omega t) \tag{612}$$

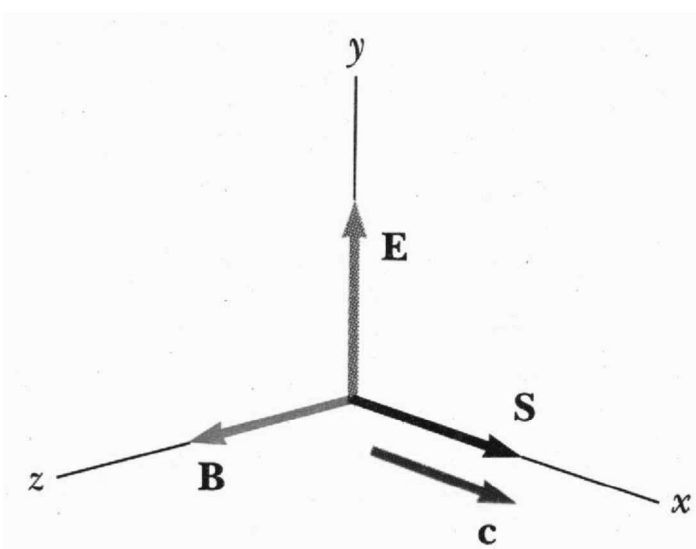
$$kE_0 = \omega B_0 \tag{613}$$

$$\frac{\omega}{k} = \frac{2\pi f}{\left(\frac{2\pi}{\lambda}\right)} = f\lambda = c = \frac{E_0}{B_0} = \frac{|\vec{E}|}{|\vec{B}|} \tag{614}$$

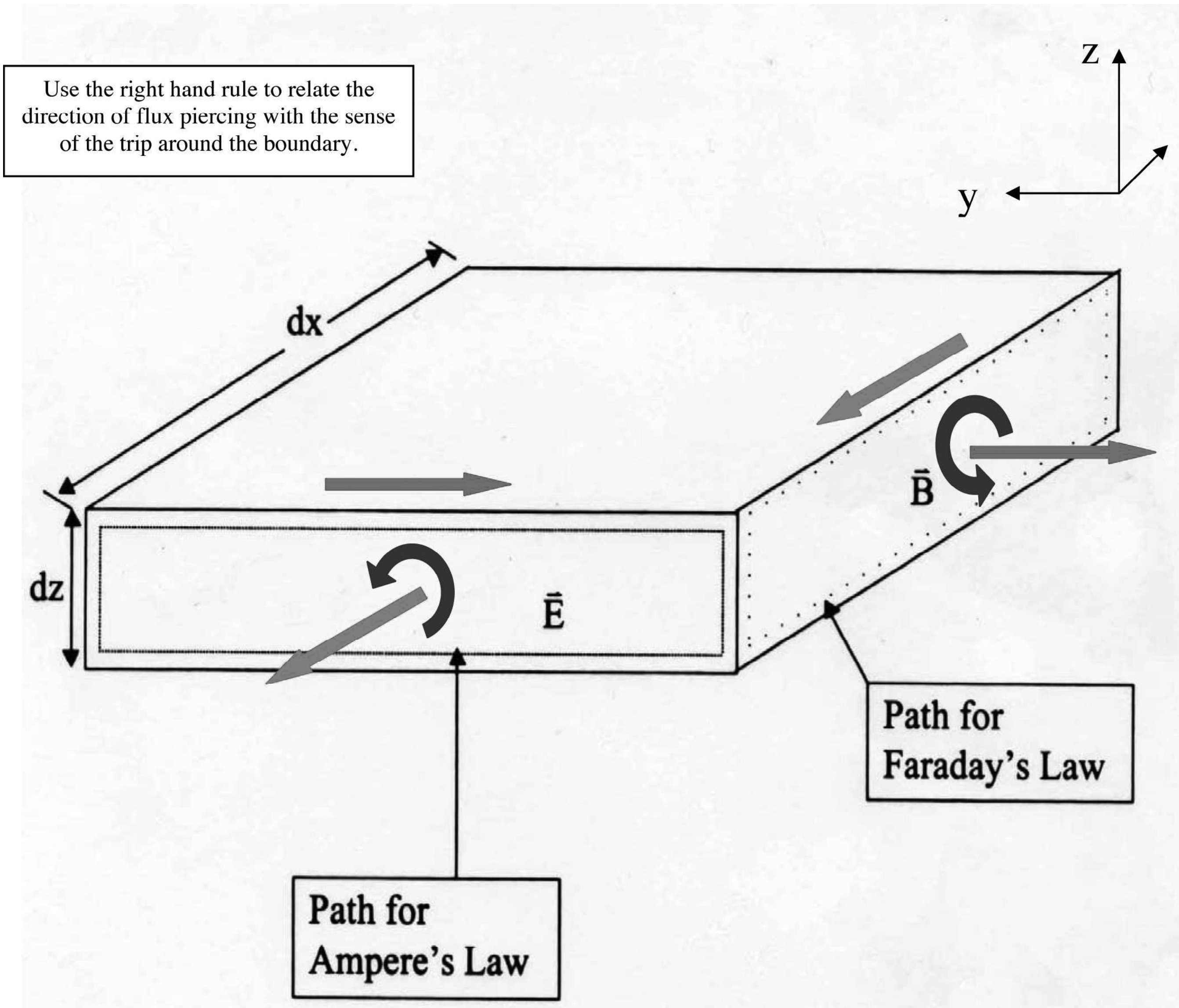

E
B
c


y
z
x


y
B
E
dx
h
c
x
h
dx
E
B
z


y
E
S
z
B
x
c

Wave Propagation & Direction



$$\oint \vec{B} \cdot d\vec{l} = \epsilon_0 \mu_0 \frac{d\Phi_E}{dt} \qquad (615)$$

$$-B_y(z+dz) + B_y(z) = \epsilon_0 \mu_0 \frac{d\Phi_E}{dt} \qquad (616)$$

$$-\frac{\partial B_y}{\partial z} \propto \frac{d\Phi_E}{dt} \qquad (617)$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt} \qquad (618)$$

$$E_y(z+dz) - E_y(z) = -\frac{d\Phi_B}{dt} \qquad (619)$$

$$-\frac{\partial E_y}{\partial z} \propto \frac{d\Phi_B}{dt} \qquad (620)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (621)$$

$$\text{if } \vec{E} = E_0 \hat{i} \cos(kz - \omega t) \qquad (622)$$

$\vec{\nabla} \times \vec{E} =$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix}$$

$$= \hat{j}\frac{\partial E_x}{\partial z} - \hat{k}\cancelto{0}{\frac{\partial E_x}{\partial y}}, E_x \neq f(y) \tag{623}$$

$$-\frac{\partial \vec{B}}{\partial t} = -\hat{j}kE_0 \sin(kz - \omega t) \tag{624}$$

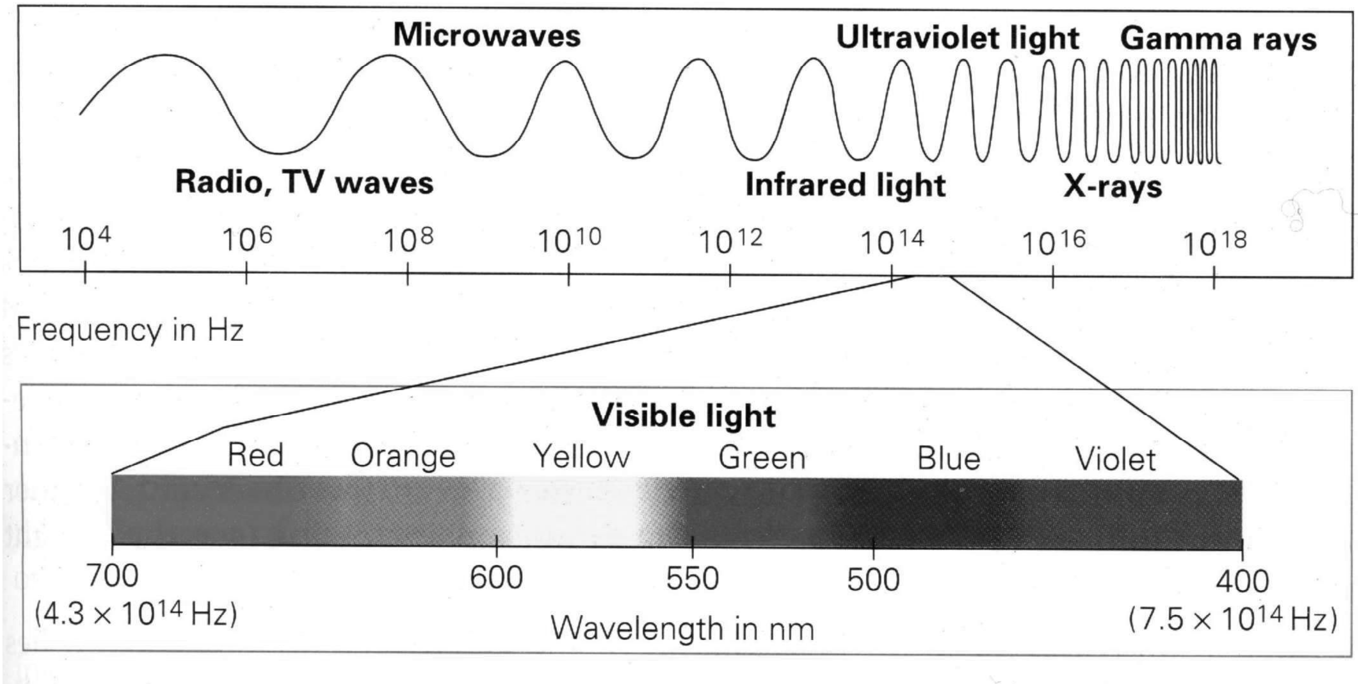
$$\vec{B} = +\hat{j}\int (kE_0 \sin(kz - \omega t))dt \tag{625}$$

$$= +\hat{j}\frac{k}{\omega}E_0 \cos(kz - \omega t) \tag{626}$$

$$= +\hat{j}\left(\frac{1}{c}E_0\right)\cos(kz - \omega t) \tag{627}$$

$\vec{E}, \vec{B} \perp$ to each other & to z

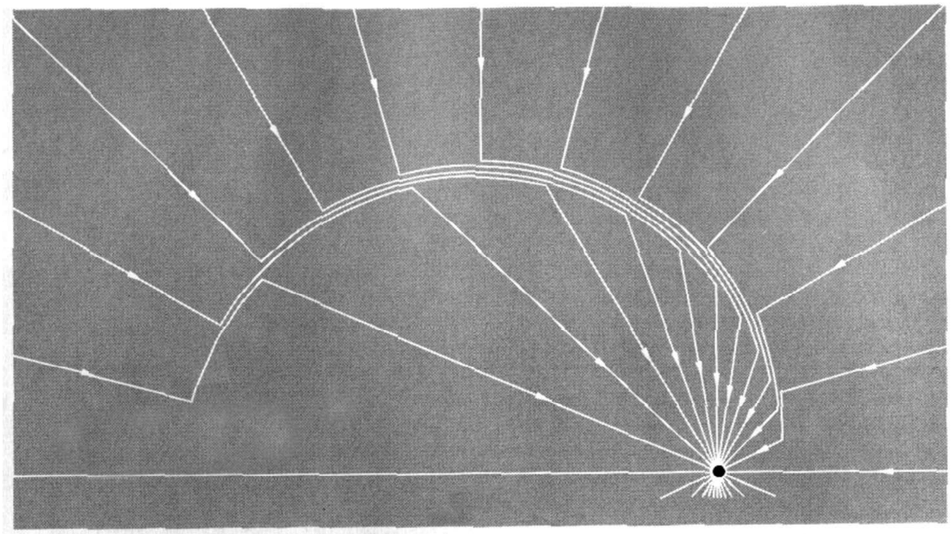
The Electronmagnetic Spectrum



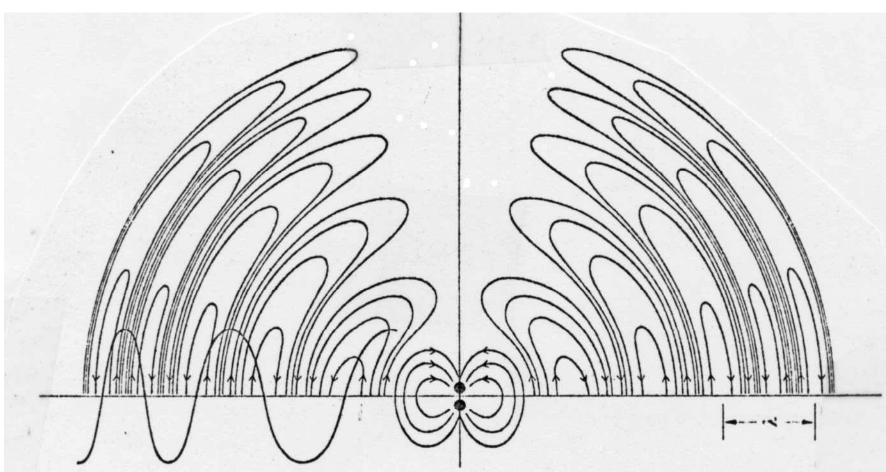
$$v = f\lambda = c = \text{constant}$$
$$\Rightarrow f, \lambda \text{ are inversely related}$$

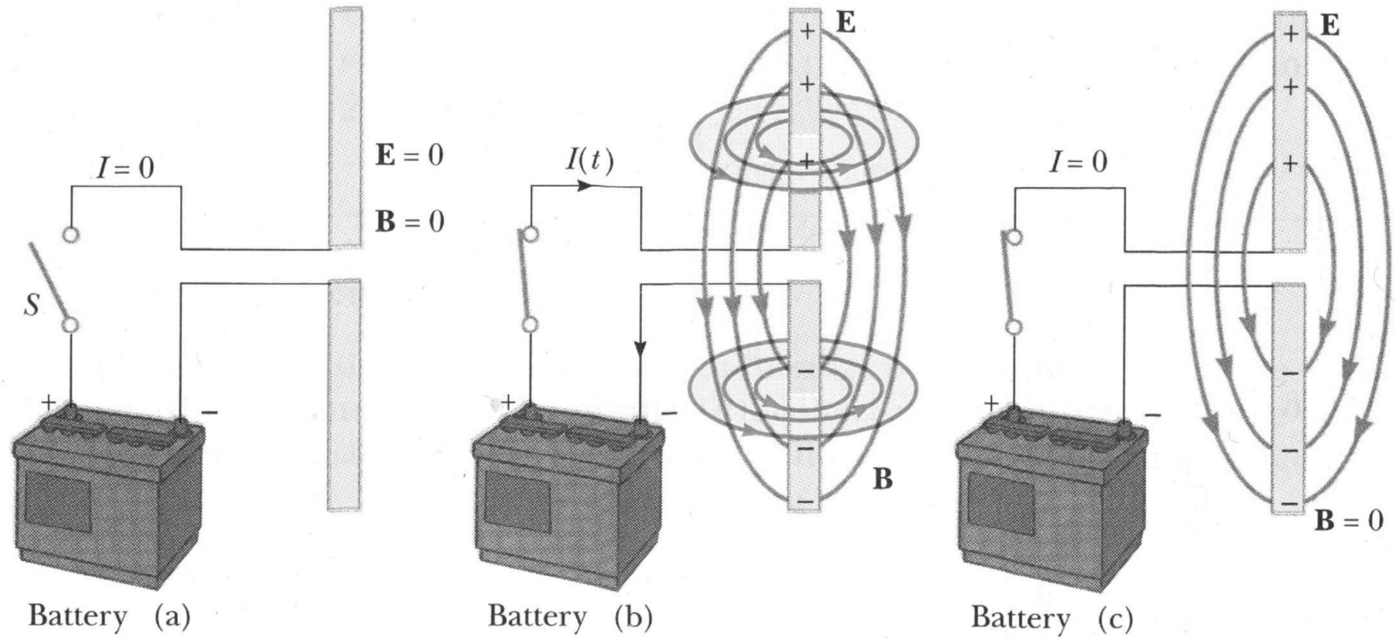
How To Start A Wave?

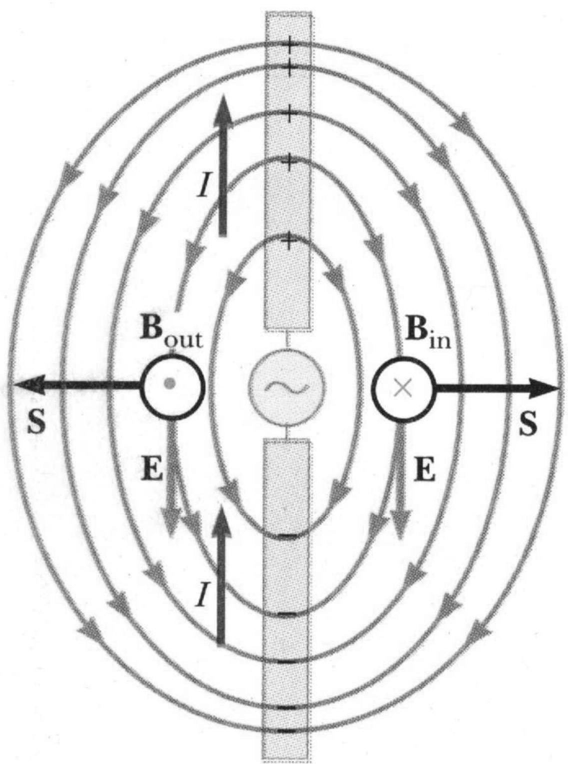
Accelerate a stationary charge, then let it rest:



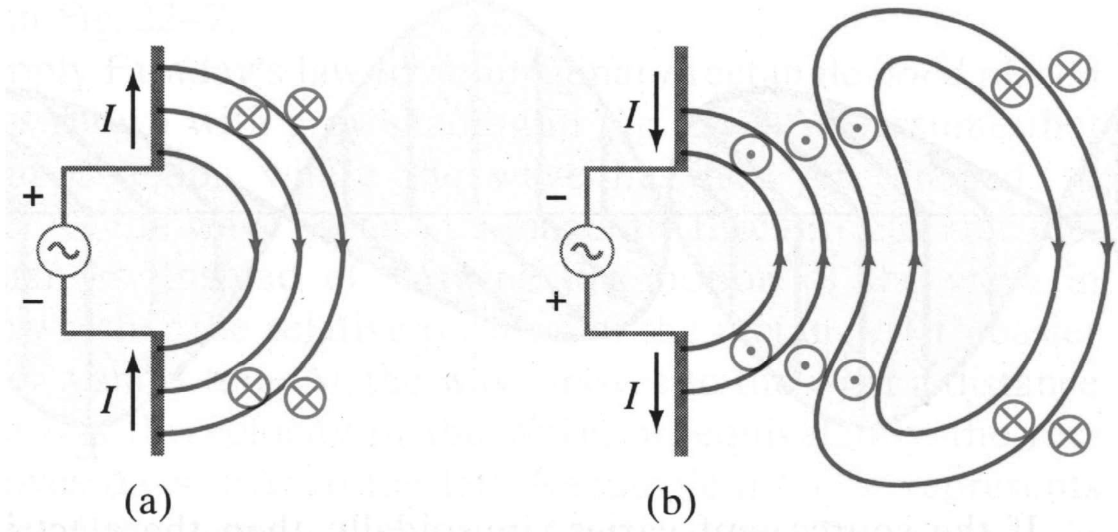
Information as to where the charge is currently located (trace back to the center from the radial field lines) moves out at the finite speed "c". Acceleration produces a "kink" in the field lines.
Now imagine a charged particle that accelerates smoothly, back and forth, as $\sin(\omega t)$. Kinks become waves. . .




I = 0
E = 0
B = 0
S
+
−
Battery (a)
I(t)
E
B
+
−
Battery (b)
I = 0
E
B = 0
+
−
Battery (c)


I
B_out
B_in
S
S
E
E
I

$\vec{E}$ Field From A Dipole Transmitting Antenna



The universe updates itself at a finite speed.

Retarded Potential:

$$V(x', y', z', t) = \frac{1}{4\pi\epsilon_0} \int \frac{dq(x, y, z, t - \frac{r}{c})}{r} \tag{628}$$

Electromagnetic Oscillation in a Cavity

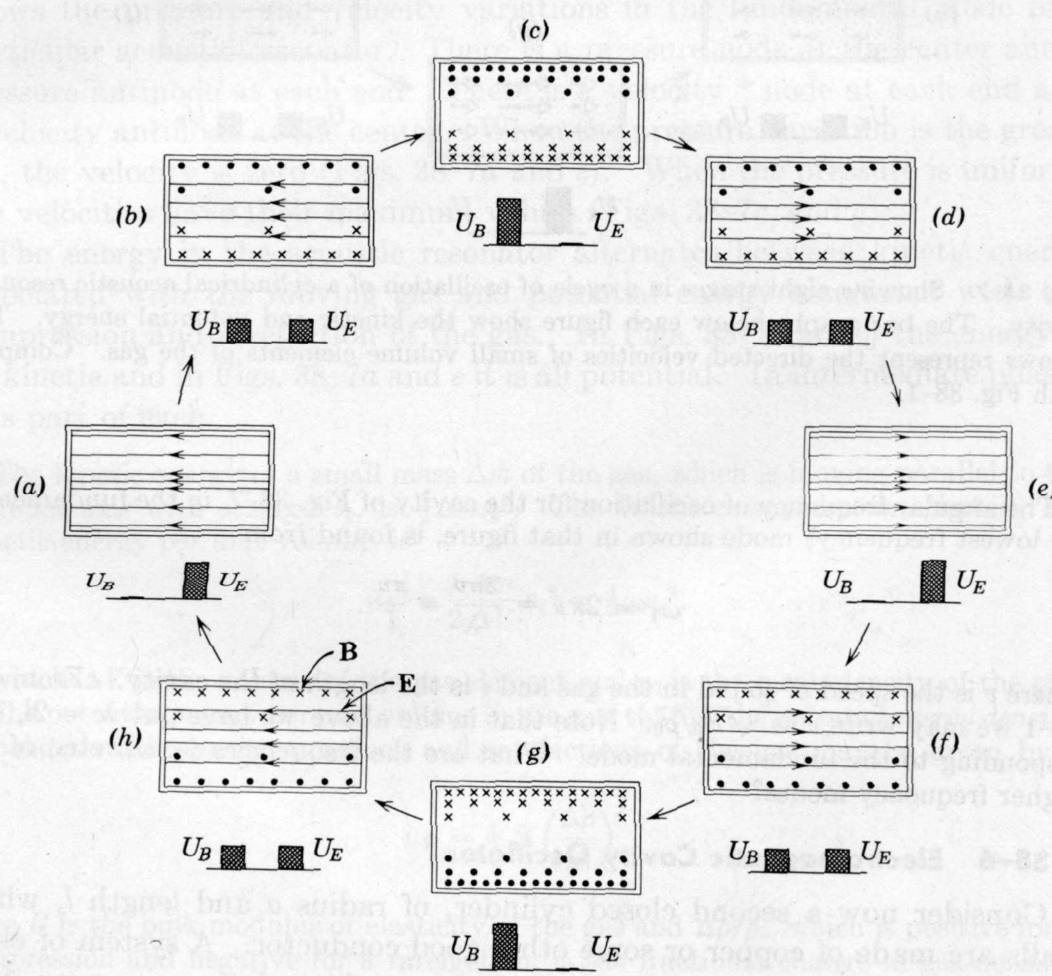


Fig. 38–8 Showing eight stages in a cycle of oscillation of a cylindrical electromagnetic resonant cavity. The bar graphs below each figure show the stored electric and magnetic energy. The dots and crosses represent circular lines of **B**; the horizontal lines represent **E**.

Electromagnetic Oscillation in a Circuit

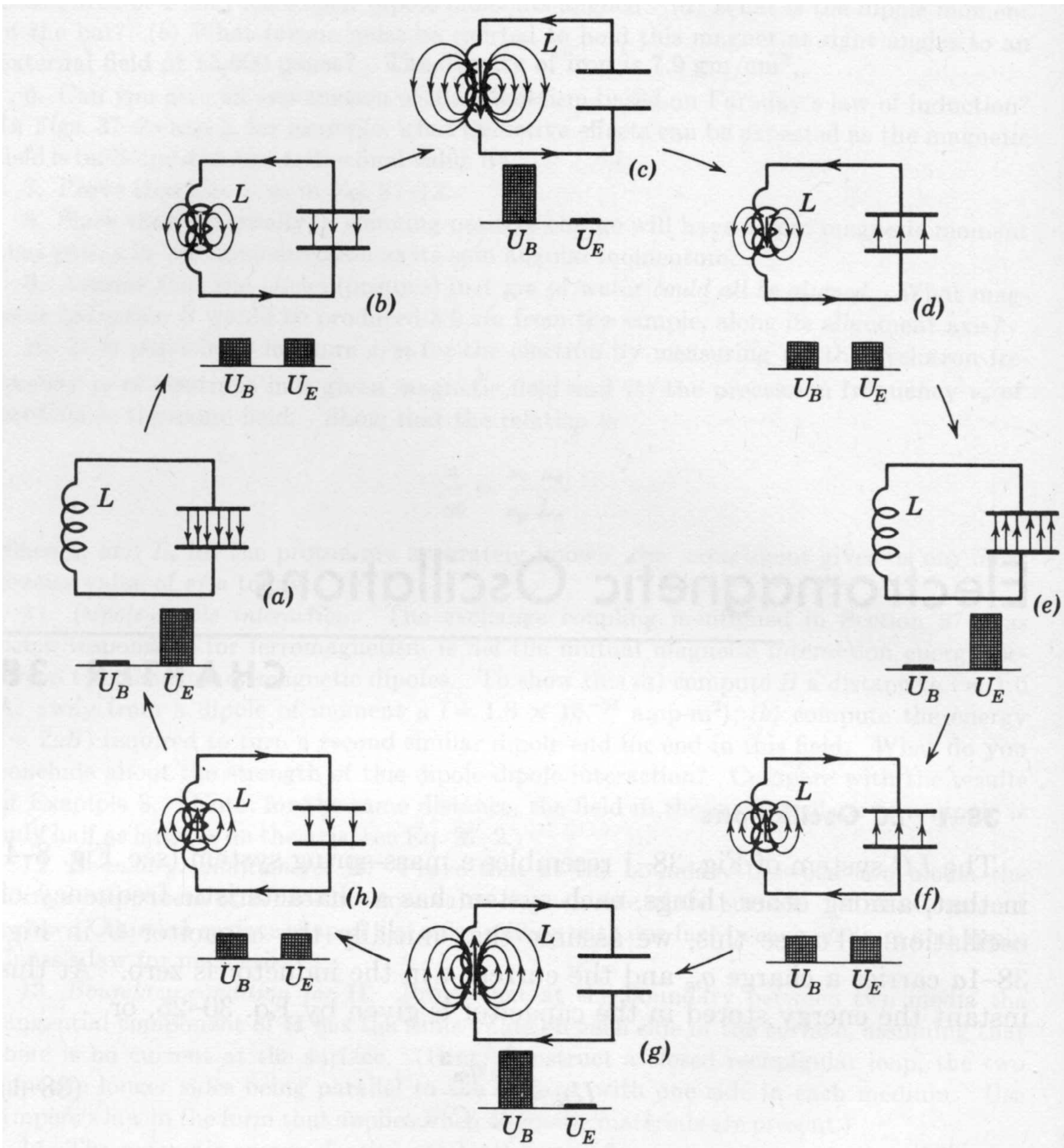


Fig. 38–1 Showing eight stages in a cycle of oscillation of an LC circuit. The bar graphs below each figure show the stored magnetic and electric potential energy. The vertical arrows on the inductor axis show the current.

Energy Transport

Poynting Vector - rate of energy flow/unit area

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B} \tag{629}$$

Direction?

$$\hat{i} \times \hat{j} = \hat{k} \tag{630}$$

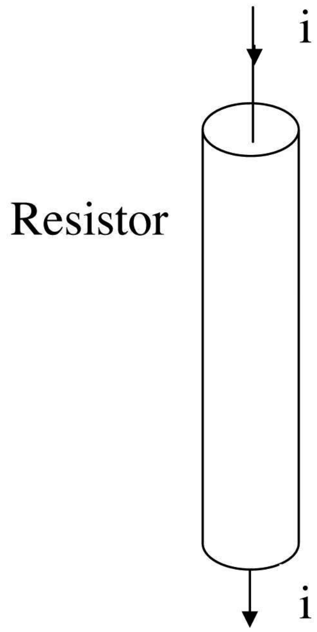
Magnitude

$$\begin{aligned} dU &= dU_B + dU_E & (631)\\ &= (U_B + U_E)Adz & (632)\\ &= \left(\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2\right)Adz & (633)\\ &= \left(\frac{1}{2}\epsilon_0 E(cB) + \frac{1}{2\mu_0}B\left(\frac{E}{c}\right)\right)Adz & (634)\\ &= EB\frac{Adz}{\mu_0 c} & (635) \end{aligned}$$

divide by A, $\frac{dz}{c}$ (transit time)

$$\Rightarrow |\vec{S}| = \frac{EB}{2\mu_0} \tag{636}$$

$$S = \frac{EB}{2\mu_0} \tag{637}$$



$\vec{E}$ along resistor $\vec{B}$ Azimuthal

$$\Rightarrow \vec{E} \times \vec{B} \text{ radially inward} \tag{638}$$

$$S = \frac{EB}{\mu_0}, E = \frac{V}{L}, V = IR$$

$$\Rightarrow S = \frac{iRB}{\mu_0 L} \tag{639}$$

$$R = \frac{\rho L}{A} = \frac{\rho L}{\pi a^2}, B = \frac{\mu_0 i}{2\pi a}$$

$$\Rightarrow S = \frac{i^2 \rho}{2\pi^2 a^2} \tag{640}$$

$$\int \vec{S} \cdot d\vec{A} = -SA = -i^2 R \tag{641}$$

$$\int \vec{B} \cdot d\vec{s} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} \quad (642)$$

$$B(2\pi R) = \mu_0 \frac{d}{dt}(E\pi R^2) \quad (643)$$

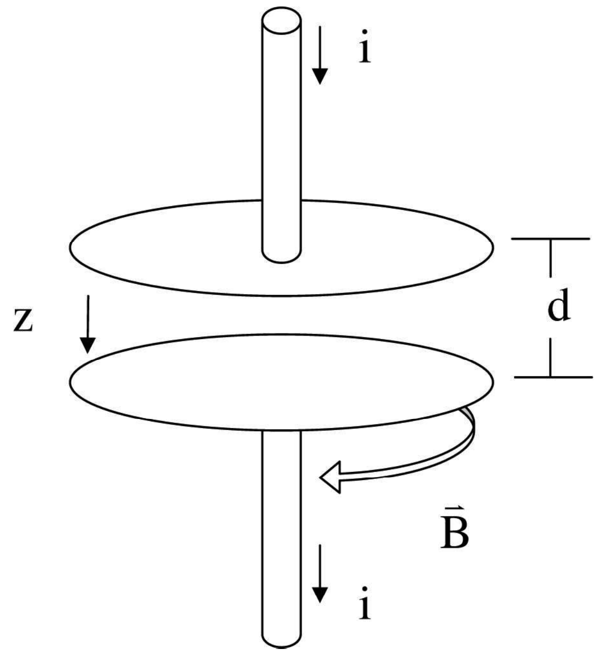
$$B = \frac{1}{2}\mu_0 \epsilon_0 R \frac{dE}{dt} \quad (644)$$

$$E = \frac{\sigma}{\epsilon_0} \quad (645)$$

$$\frac{|\vec{E} \times \vec{B}|}{\mu_0} = \frac{\frac{1}{2}\mu_0 \epsilon_0 R \frac{dE}{dt}\left(\frac{\sigma}{\epsilon_0}\right)}{\mu_0} \quad (646)$$

$$|\vec{S}| = \frac{1}{2} R\sigma \frac{dE}{dt} \quad (647)$$

$$\vec{S} \cdot d\vec{A} = SA = S(2\pi R d) = \pi R^2 d\sigma \frac{dE}{dt} \quad (648)$$



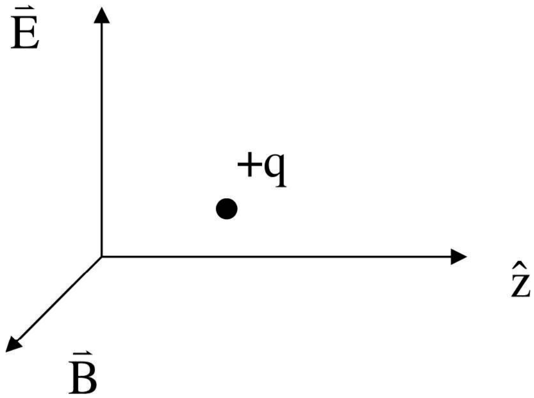
Compare to...

$$\int \frac{d}{dt}(\text{stored energy}) = \frac{d}{dt}(\text{volume} \times \frac{1}{2}\epsilon_0 E^2) \quad (649)$$

$$= Ad\frac{d}{dt}\left(\frac{1}{2}\epsilon_0 E^2\right) \quad (650)$$

$$= Ad\epsilon_0 E \frac{dE}{dt} = \pi R^2 d\sigma \frac{dE}{dt} \quad (651)$$

Momentum in EM Waves



If $\vec{E}$ up at $^{+}$q at $t = 0$
$\Rightarrow$ force up $\Rightarrow$ velocity up
$\vec{B}$, in phase, points out
$\Rightarrow q\vec{v} \times \vec{B}$ force points in $+\hat{z}$ direction

when fields reverse, same direction of force $\sim +\hat{z}$

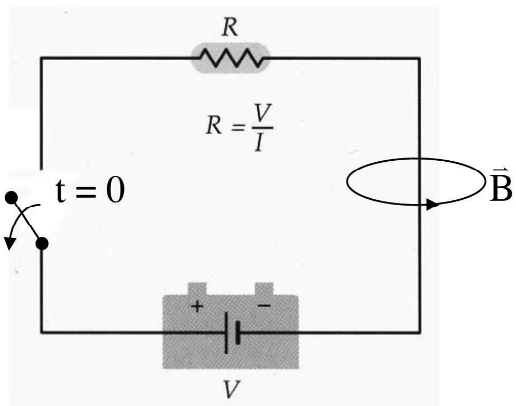
Net force always in $+\hat{z}$
$\Rightarrow$ momentum transfer
if U = incident energy:
$p = \frac{U}{c}$ (total absorbed), $p = \frac{2U}{c}$ (total reflected)

Faraday's Law

$\Rightarrow$ "unexpected" circuit behavior

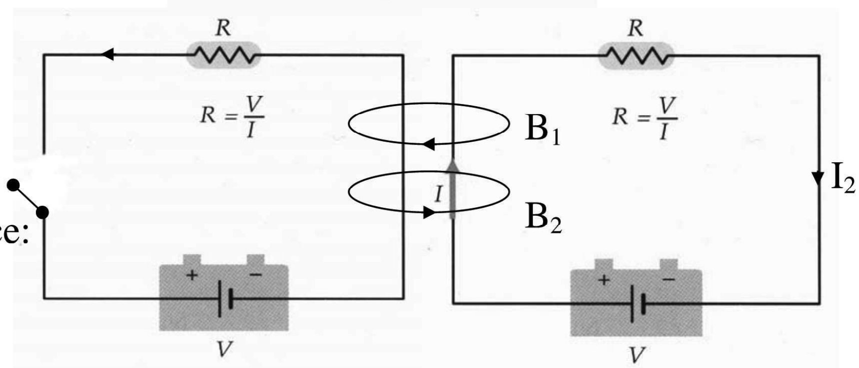
Inductance

Self Inductance:



Changing I in a circuit induces an EMF in the same circuit.

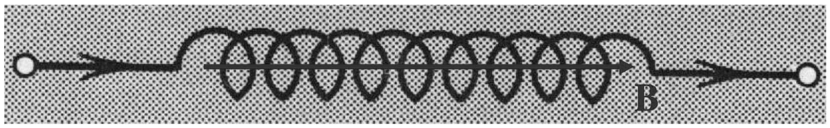
Mutual Inductance:



Changing I in either circuit $\Rightarrow$ induced EMF in the other.

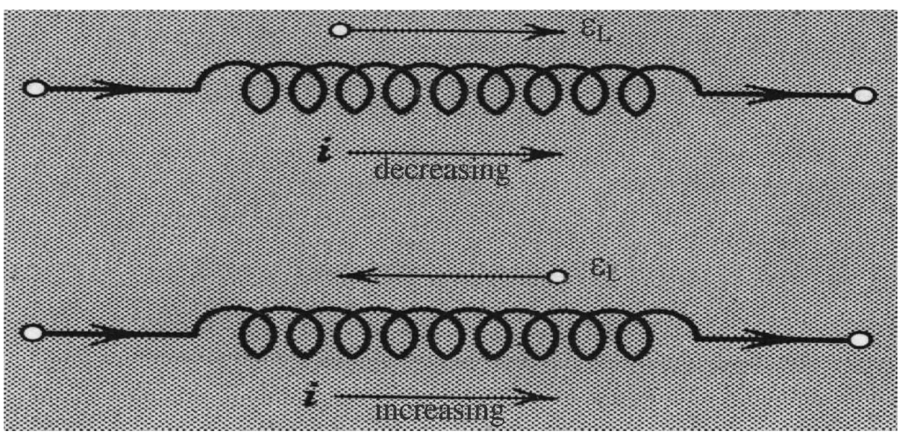
Inductor

Self Inductance:



"obvious"

$$\Phi_B \propto i \quad (652)$$
$$= Li \quad (653)$$
$$\text{Faraday's Law} \Rightarrow \epsilon = -\frac{d\Phi_B}{dt} = -L\frac{di}{dt} \quad (654)$$
$$\epsilon_L \equiv -L\frac{di}{dt} \text{ (no other voltage difference across the coil)} \quad (655)$$
$$[L] = \frac{\text{V} \cdot \text{sec}}{\text{Amp}} = \text{" Henry"}$$



Calculating Inductance

Start with definition:

$$\epsilon = L\frac{di}{dt} = \frac{d}{dt}(Li) \tag{656}$$

Then add Faraday's Law

$$\epsilon = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}(N\Phi_B) \tag{657}$$

$$\Rightarrow L = \frac{N\Phi_B}{i} \tag{658}$$

Inductance of Solenoid

$$B = \mu_0 ni, n = \text{density of turns} \tag{659}$$

$$\Phi_B = \mu_0 niA \tag{660}$$

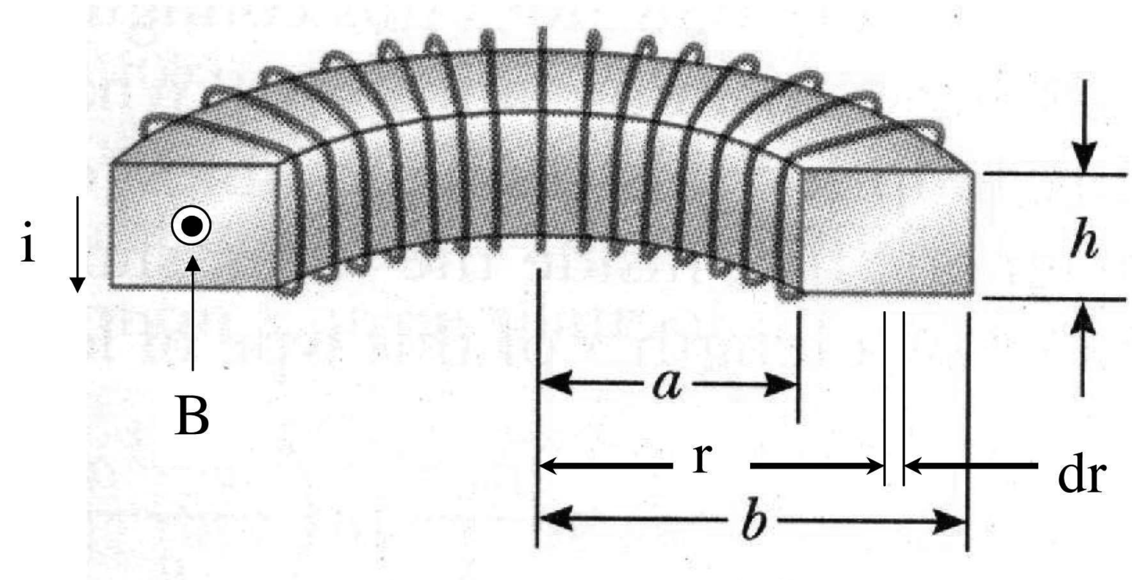
$$N\Phi_B = (nl)\mu_0 niA \tag{661}$$

$$= \mu_0 n^2 lAi \tag{662}$$

$$L = \frac{N\Phi_B}{i} = \mu_0 n^2 lA \tag{663}$$

$$\text{or } \frac{L}{l} = \mu_0 n^2 A \text{ (only geometrical factors)} \tag{664}$$

Inductance of A Toroid



$$B = \frac{\mu_0 i N}{2\pi r} \tag{665}$$

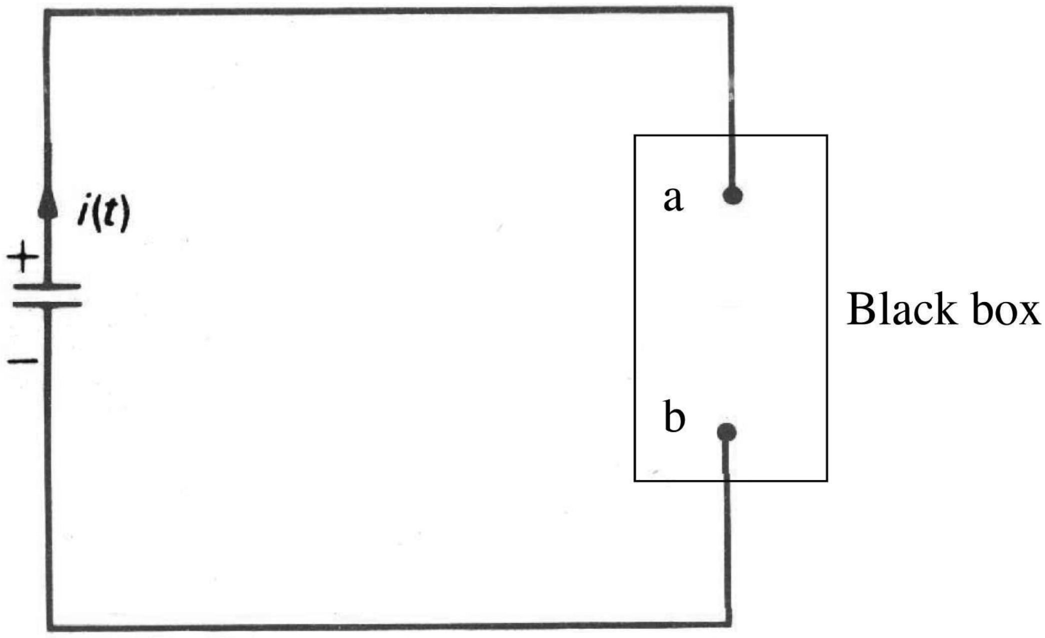
$$\Phi_B = \int \vec{B} \cdot d\vec{A} = \int_a^b Bhdr = \frac{\mu_0 i N h}{2\pi} \int_a^b \frac{dr}{r} \tag{666}$$

$$= \frac{\mu_0 i N h}{2\pi} \ln \frac{b}{a} \tag{667}$$

$$L = \frac{N\Phi}{i} = \frac{\mu_0 N^2 h}{2\pi} \ln \frac{b}{a} \tag{668}$$

(Note: Geometrical factors only)

Energy Transfers in a Circuit



$$dU_{\text{box}} = dqV_{ab} \quad (669)$$

$$= idtV_{ab} \quad (670)$$

$$P = \frac{dU}{dt} = iV_{ab} \quad (671)$$

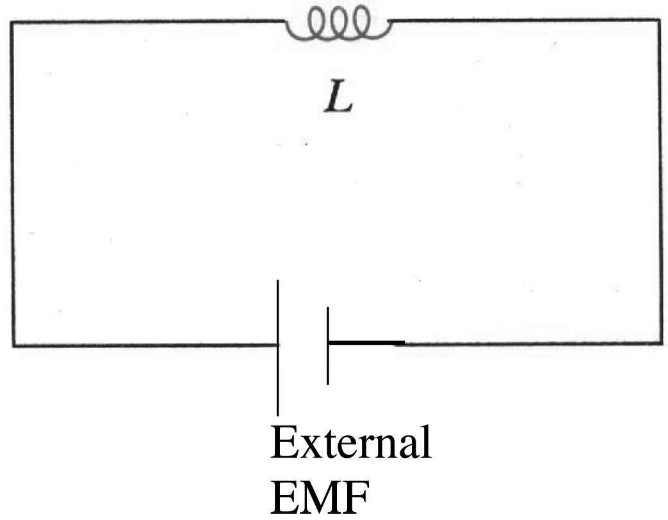
$$[P] = [i][V] = \text{Amp} \cdot \text{Volt} = \frac{C}{sec} \cdot \frac{J}{C} = \text{Watt} \quad (672)$$

Box contains resistor?

$$R = \frac{V}{i} \quad (673)$$

$$\Rightarrow P = i^2R = \frac{V^2}{R} \text{ Joule's Law} \quad (674)$$

Energy In Inductors



$$\epsilon = -L\frac{dI}{dt} \tag{675}$$

$$\text{Power} = IV$$

$$\text{Recall: } W = qv, P = \dot{W} = \dot{q}V$$

$$\dot{W} = LI\frac{dI}{dt} \tag{676}$$

Work done by external EMF positive, opposes induced EMF.

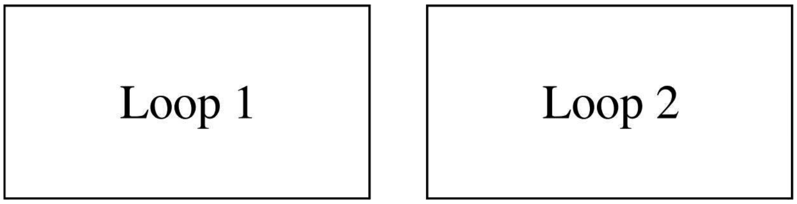
$$dW = LIdI \tag{677}$$

$$\int dW = L\int IdI \tag{678}$$

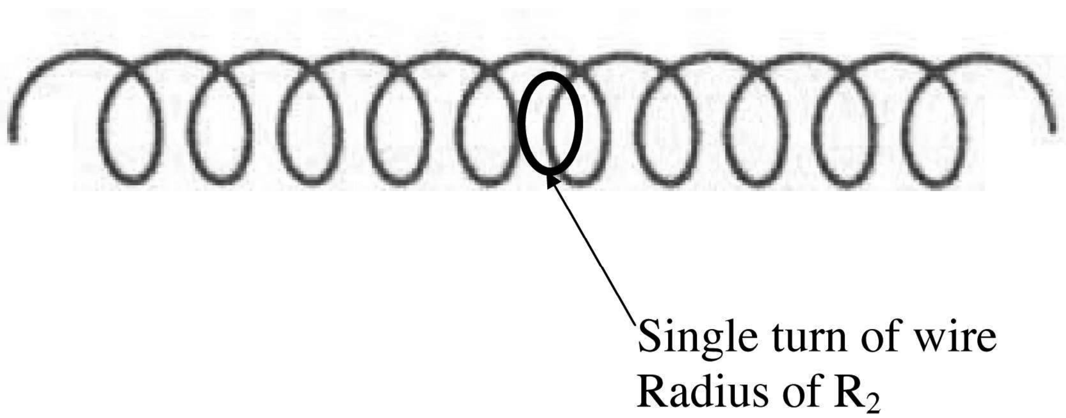
$$\partial U = \frac{1}{2}LI^2\Big|_{I_i}^{I_f} = \frac{1}{2}L(I_f^2 - I_i^2) \text{ (compare } U_{cap} = \frac{Q^2}{2C}) \tag{679}$$

Paradox? Constant $I \rightarrow$ stored energy

Mutual Inductance



$$\Phi_1 = L_1 i_1 + M_{12} i_2 \tag{680}$$
$$\Phi_2 = L_2 i_2 + M_{21} i_1 \tag{681}$$



Solenoid

$$B_{\text{solenoid}} = \mu_0 n_1 i_1 \tag{682}$$
$$\begin{aligned} \Phi_B &= BA_{\text{loop}} = \mu_0 n_1 i_1 \pi R_2^2 && (683) \\ &\equiv Mi_1 && (684) \\ &\rightarrow M\mu_0 n_1 \pi R_2^2 && (685) \end{aligned}$$

Notice how hard it is to compute M_{21} instead of M_{12}!

Energy In $\vec{B}$ Fields Generally

Solenoid:

$$L = \mu_0 A n^2 1 \tag{686}$$

$$U = \frac{1}{2} L I^2 = \frac{1}{2} \mu_0 A I^2 n^2 1 \tag{687}$$

$$\text{Also know} B_{\text{sol}} = \mu_0 n I \tag{688}$$

$$U = \frac{B^2}{2\mu_0} A 1 \tag{689}$$

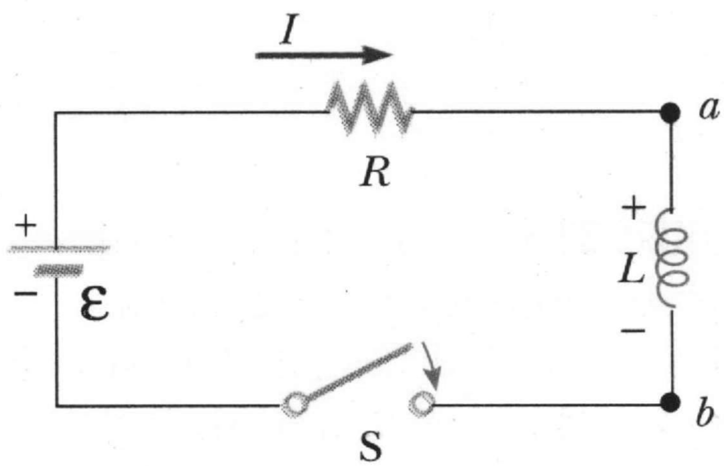
$$u_b = \frac{B^2}{2\mu_0} \text{ (compare } u_E = \frac{\epsilon_0 E^2}{2}) \tag{690}$$

Trick to find L

1. Find $\vec{B}$
2. Integrate u_B over "all space" to get U
3. U also $= \frac{1}{2} L I^2$

LR circuit

(compare RC circuit)



Growing $i \rightarrow$ EMF opposing change

$$\Rightarrow i < \frac{\epsilon}{R} \text{ so long as change occurs} \tag{691}$$

$$-iR - L\frac{di}{dt} + \epsilon = 0 \tag{692}$$

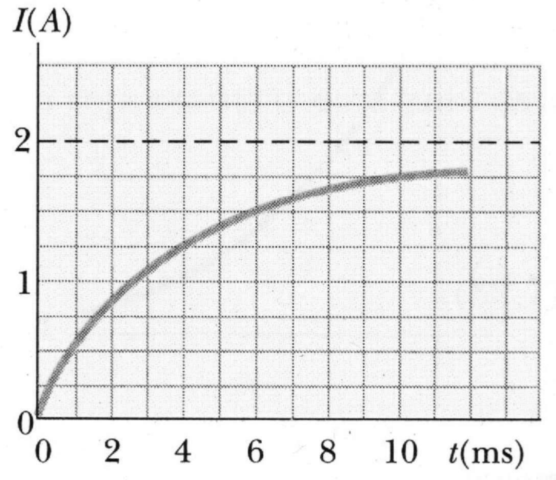
$$\frac{di}{dt} + \frac{R}{L}i - \frac{\epsilon}{L} = 0\text{"ansatz"}i = Ae^{bt} + C \tag{693}$$

$$i = Ae^{-\frac{R}{L}t} + \frac{\epsilon}{R} \tag{694}$$

$$i(0) = 0 \Rightarrow A = -\frac{\epsilon}{R} \tag{695}$$

$$i(t) = \frac{\epsilon}{R}(1 - e^{-Rt/L}) = \frac{\epsilon}{R}(1 - e^{-\frac{t}{\tau_L}}), \tau_L = \frac{L}{R} \tag{696}$$

From i(t), can compute voltages across the two elements, which must total to the battery voltage.



$$V_R = iR \tag{697}$$
$$= \frac{\epsilon}{R}(1 - e^{-\frac{t}{\tau_L}}) \cdot R \tag{698}$$
$$= \epsilon(1 - e^{\frac{t}{\tau_L}}) \tag{699}$$

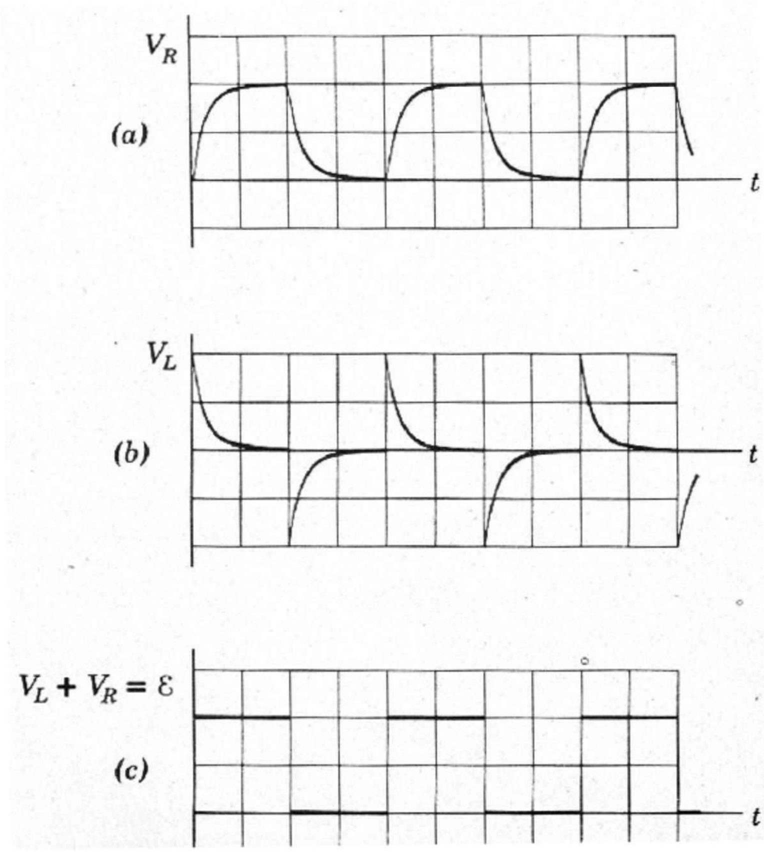
$$V_L = L\frac{di}{dt} \tag{700}$$
$$= L \cdot \left(\frac{\epsilon}{L}e^{-\frac{t}{\tau_L}}\right) \tag{701}$$
$$= \epsilon(e^{-\frac{t}{\tau_L}}) \tag{702}$$

Flip switch to b $\Rightarrow$

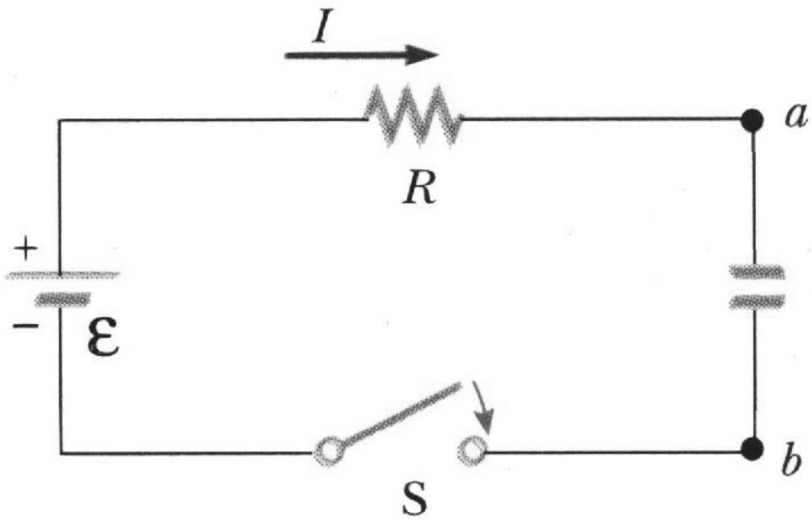
$$L\frac{di}{dt} + iR = 0 \tag{703}$$

$$i(t) = i_0 e^{-\frac{t}{\tau_L}} \tag{704}$$

Feed square wave to circuit:



RC circuit



$$\epsilon = iR + \frac{q}{C} \text{ (charging)} \tag{705}$$

$$i = \frac{dq}{dt}, \epsilon = \frac{dq}{dt}R + q\frac{1}{C} \tag{706}$$

$$\int \frac{dq}{q - \epsilon C} = \int -\frac{dt}{RC} \tag{707}$$

$$\int \frac{dz}{z} = -\frac{1}{RC}\int dt \tag{708}$$

$$\ln z = -\frac{t}{RC}, z = e^{-\frac{t}{RC}} = q - \epsilon C \tag{709}$$

$$q = \epsilon C + e^{-\frac{t}{RC}}, q(0) \neq 0! \; error!! \tag{710}$$

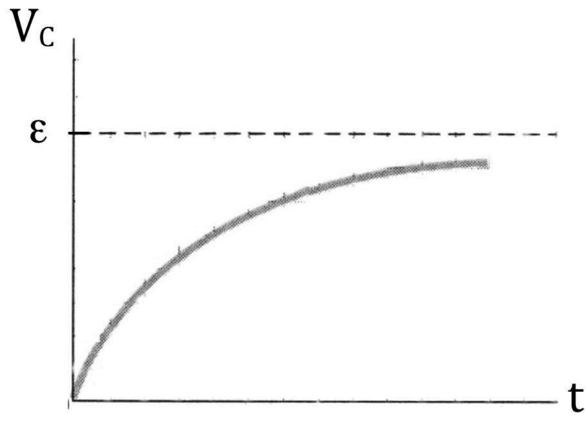
$$\int \frac{dz}{z} = -\frac{1}{RC}\int dt \tag{711}$$

$$\ln z = -\frac{t}{RC} + \text{constant} \tag{712}$$

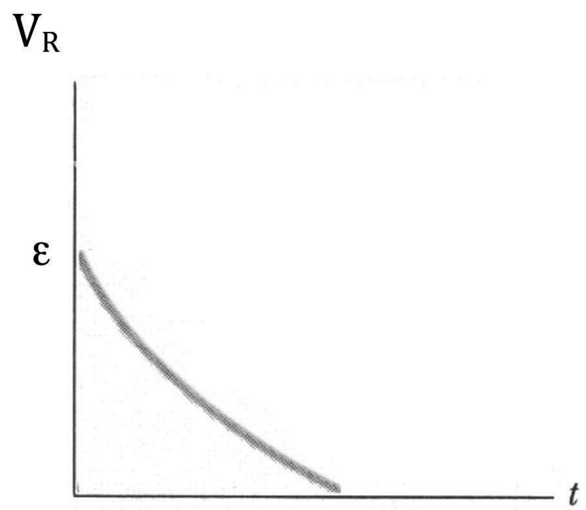
$$e^{\ln z} = e^{(-\frac{t}{RC}+C)} \tag{713}$$

$$z = q - \epsilon C = e^{-\frac{t}{RC}}e^{C} = ke^{-\frac{t}{RC}} \tag{714}$$

$$q = \epsilon C + ke^{-\frac{t}{RC}} \tag{715}$$



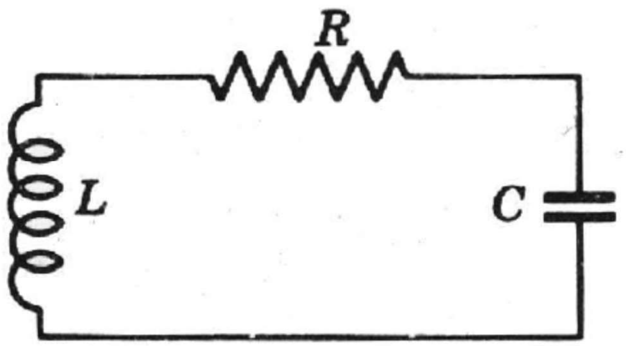
$$q = 0 \text{ at } t = 0? \tag{716}$$
$$q(0) = \epsilon C + ke^0 \tag{717}$$
$$\Rightarrow k = -\epsilon C \tag{718}$$
$$\Rightarrow q(t) = \epsilon C(1 - e^{-\frac{t}{RC}}) \tag{719}$$
$$RC \equiv \tau_c \tag{720}$$
$$\text{Then } i(t) = \frac{dq}{dt} = \frac{\epsilon}{R} e^{-\frac{t}{RC}} \tag{721}$$



For each element ...

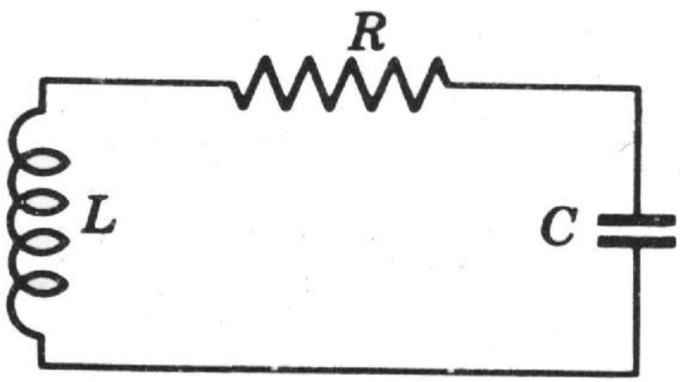
$$V_{\text{cap}} = \frac{q}{C} = \epsilon(1 - e^{-\frac{t}{\tau_L}}) \tag{722}$$
$$V_{\text{Res}} = \epsilon e^{-\frac{t}{\tau_L}} \tag{723}$$
$$V_{\text{cap}} + V_{\text{res}} = \epsilon \tag{724}$$

RLC Circuit



Solution = "transient" + "steady state"
i goes as driving frequency:

$$i = i_m \sin(\omega t - \phi) \tag{725}$$



$$U = U_B + U_E \tag{726}$$

$$= \frac{1}{2}LI^2 + \frac{q^2}{2C} \text{ (no storage in the resistor)} \tag{727}$$

$$\frac{dU}{dt} = -I^2R \tag{728}$$

$$\Rightarrow LI\frac{dI}{dt} + \frac{q}{C}\frac{dq}{dt} = -I^2R \tag{729}$$

$$\Rightarrow Lq'' + Rq' + \frac{1}{C}q = 0 \tag{730}$$

Alternatively. . .

$$\Sigma\Delta V = 0 \Rightarrow -L\frac{dI}{dt} - IR - \frac{Q}{C} = 0 \tag{731}$$

Looks like. . .

$$mx'' = -kx - bv \tag{732}$$

$$mx'' + bx' + kx = 0 \tag{733}$$

spring mass with friction

General Solution

$$q(t) = q_0 e^{-\frac{Rt}{2L}} \cos(\omega' t + \phi) \tag{734}$$

$$\omega\text{'} = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} \tag{735}$$

"Damped Oscillations"

LC Circuit?

$$LI\frac{dI}{dt} + \frac{q}{C}I = 0 \tag{736}$$

$$q'' + \frac{1}{LC}q = 0, (\text{mx"+kx=0}) \tag{737}$$

$$q(t) = q_0 \cos(\omega t + \phi) \tag{738}$$

$$\text{plug in}\omega = \frac{1}{\sqrt{LC}}$$

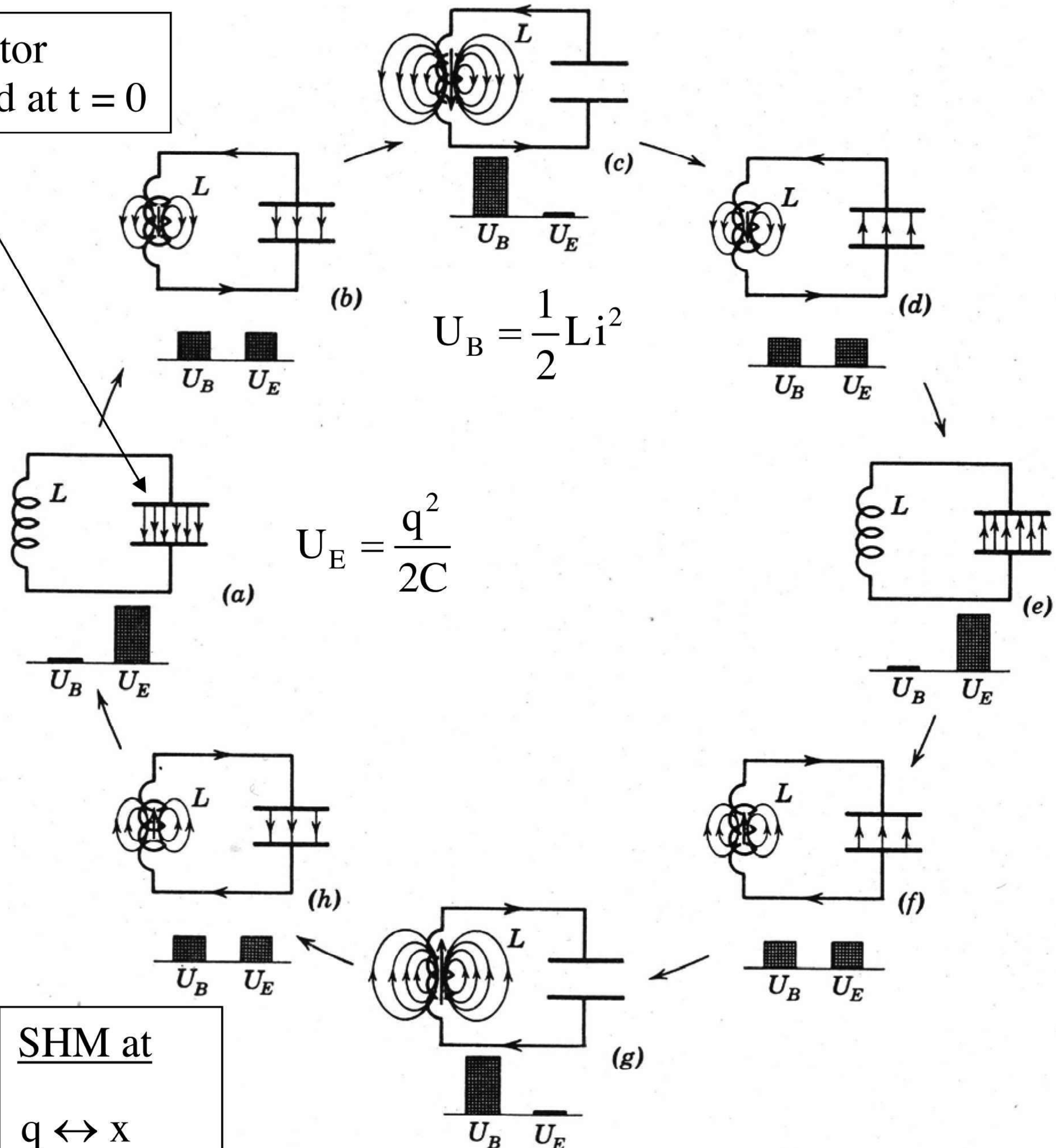
$$U_E = \frac{q^2}{LC} = \frac{q_0^2}{LC}\cos^2(\omega t + \phi) \tag{739}$$

$$U_B = \frac{1}{2}LI^2 = \frac{1}{2}L\omega^2 q_0^2 \sin^2(\omega t + \phi) \tag{740}$$

Electromagnetic Oscillations

(LC Circuits)

Capacitor charged at $t = 0$



$$U_B = \frac{1}{2} L i^2$$

$$U_E = \frac{q^2}{2C}$$

$$\omega = \sqrt{\frac{1}{LC}}$$

<u>SHM at</u>

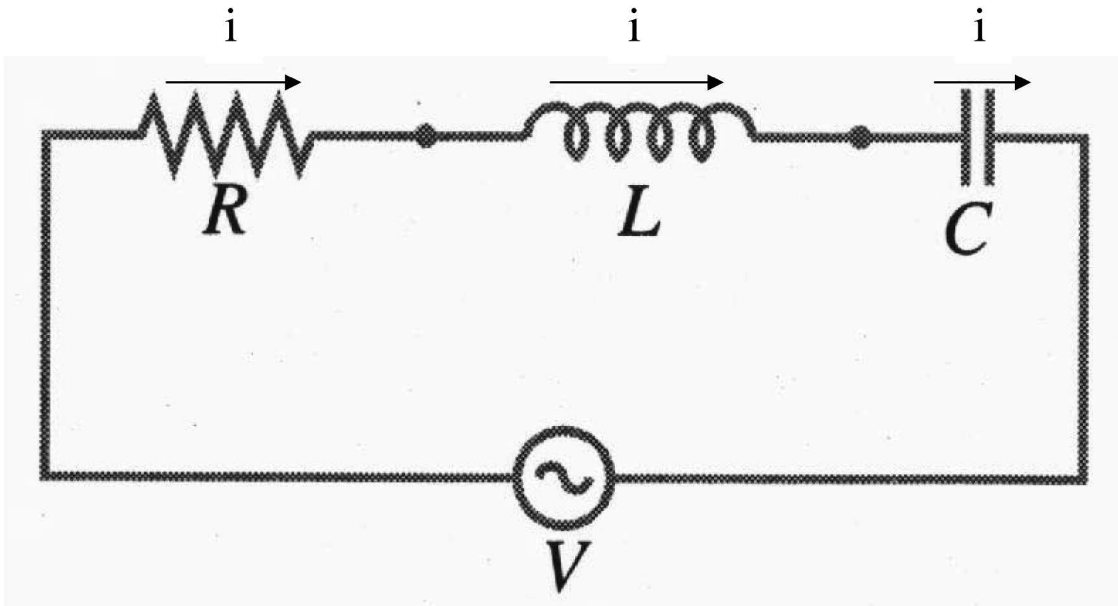
$q \leftrightarrow x$

$\frac{1}{C} \leftrightarrow k$

$i \leftrightarrow v$

$L \leftrightarrow m$

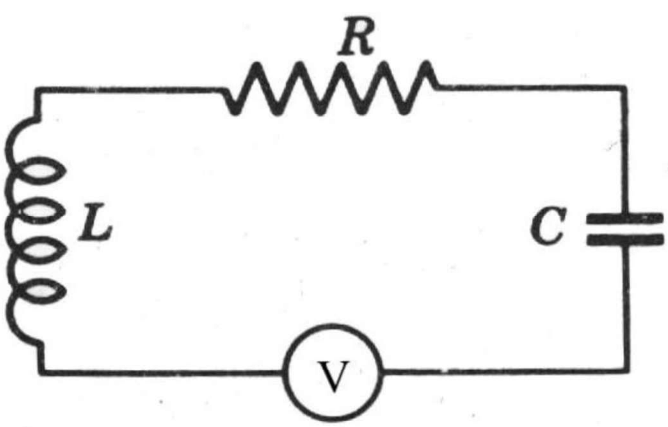
RLC Circuit



Solution = "transient" + "steady state"
"steady state" goes as driving frequency

$$i = i_m \sin(\omega t - \phi) \tag{741}$$

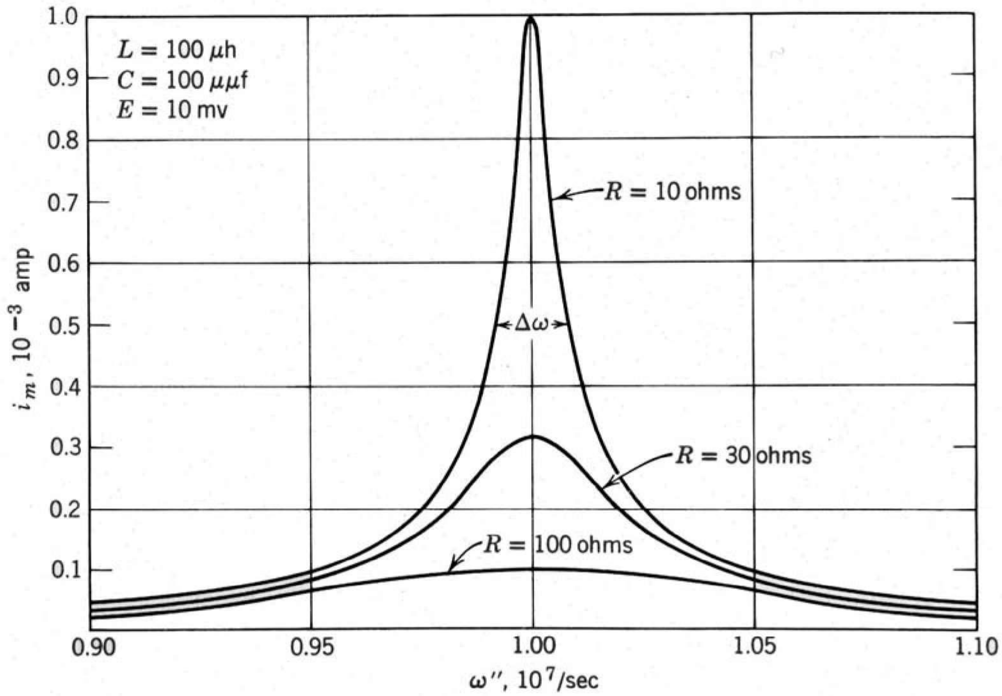
Add driving force $F = F_m \cos \omega'' t$ "MISSING PICT"
Forced Oscillations/Resonance



Answer:

$$i = i_n \sin(\omega'' t - \phi) \tag{742}$$

Sinusoidal driving: $\epsilon = \epsilon_m \cos \omega'' t$



$$\omega = \frac{1}{\sqrt{LC}} = \omega_m$$

$\Rightarrow$ resonance condition

Resonance in matching broadcast frequency to that of internal oscillator.

	RLC Circuit	Harmonic Motion
Variable	Q	x
Multiplier of variable	I/C	k
Multiplier of d/dt (variable)	R	b (friction coefficient)
Multiplier of $\mathrm{d}^2/\mathrm{dt}^2$ (variable)	L	m

$$\omega\text{'} = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} \tag{743}$$

For harmonic motion?

$$\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} = 0? \tag{744}$$

$$\Rightarrow b = 2\sqrt{km} \tag{745}$$

$$m = 2, k = 50$$

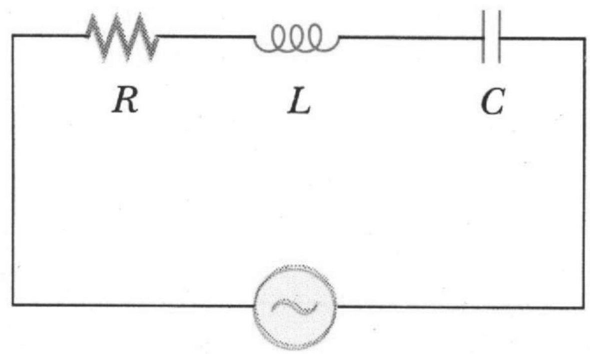
$$\Rightarrow b = 20$$

"critically damped"

$$b = 1$$

$$b = 25$$

AC Circuits



$$V = V_m \sin(\omega t) \quad (746)$$

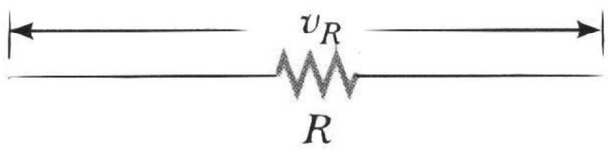
$$i = i_m \sin(\omega t - \phi) \quad (747)$$

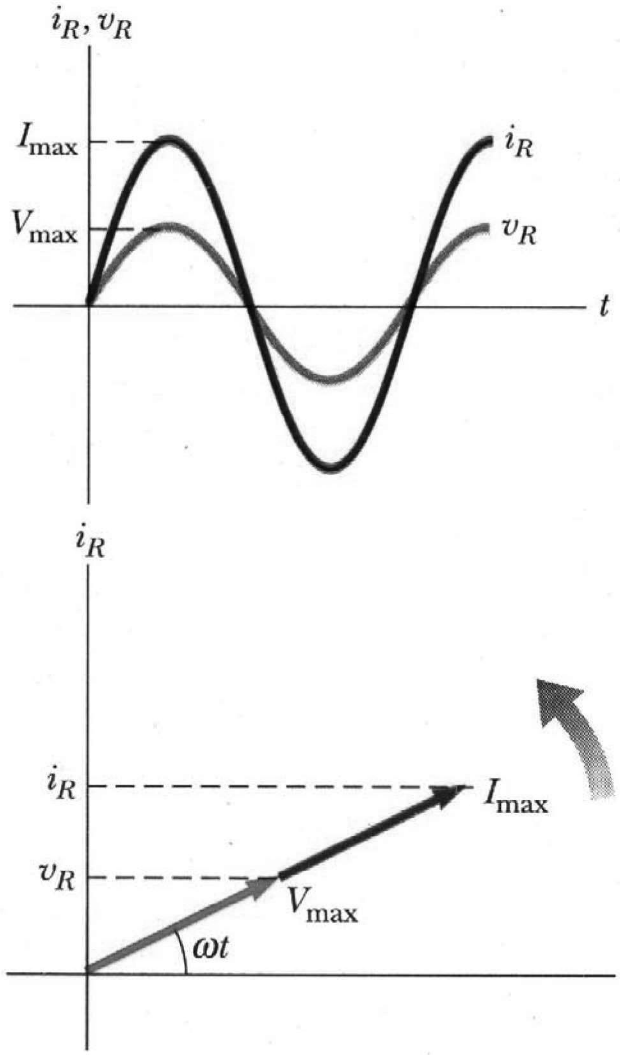
i and V are not necessarily in phase

Elements

Resistors:

$$V_R = iR = i_m R \sin(\omega t - \phi) \tag{748}$$



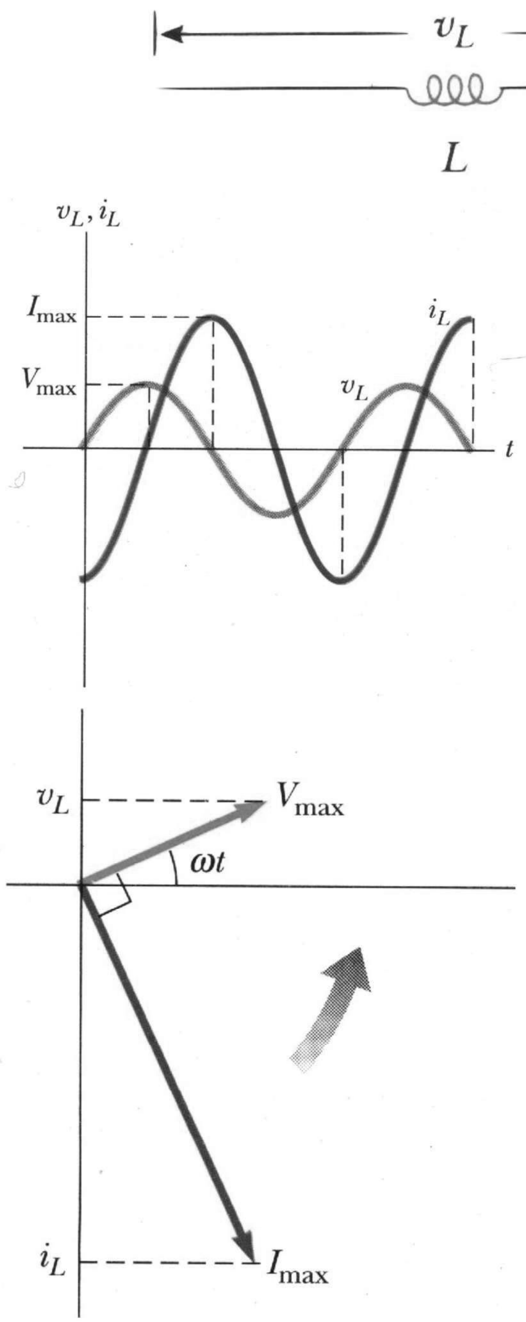


V_R and i are in phase

Phasor diagram: relative phase = 0

Inductor:

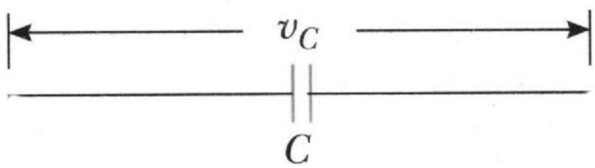
$$V_L = L\frac{di}{dt} = Li_m\omega\cos(\omega t - \phi) \tag{749}$$

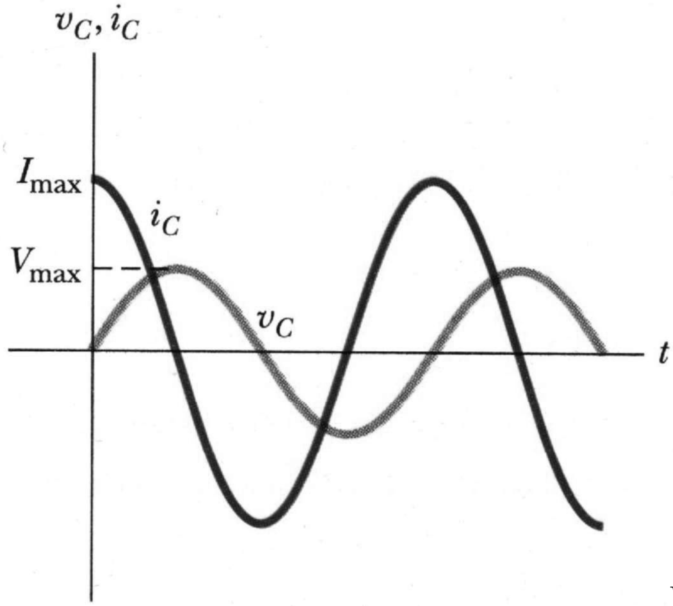


Phasor diagram: 90° phase difference between i and V_L
$X_L = \omega L$ Inductive Reactance
$(V_L)_{\max} = i_m X_L$ (like Ohm's Law)

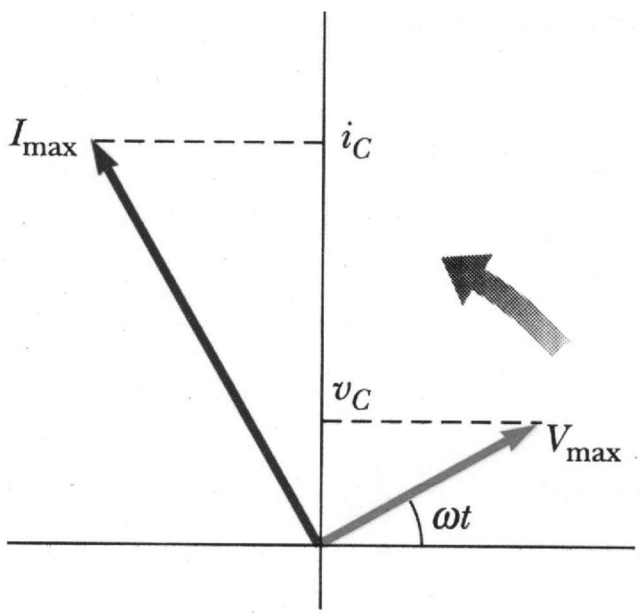
Capacitor:

$$V_C = \frac{q}{C} = \frac{\int i dt}{C} = \frac{i_m}{\omega C}\cos(\omega t - \phi) \tag{750}$$





V_c and i are 90° out of phase (i leads V_C)



$$X_C = \frac{1}{\omega C} \tag{751}$$

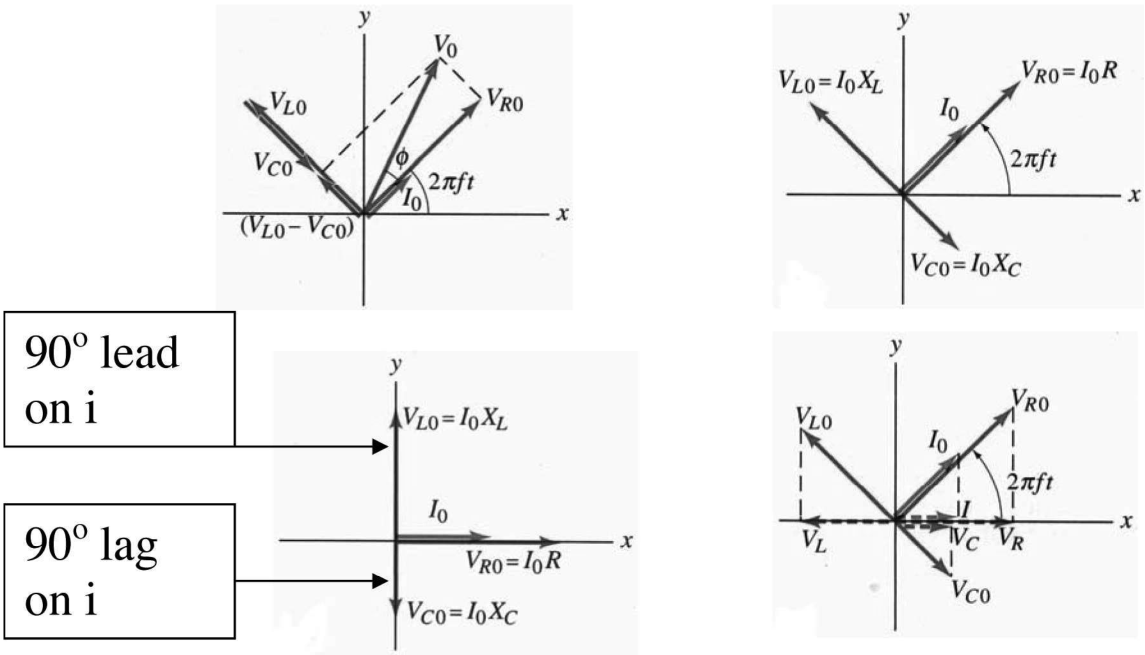
Capacitance Reactance

$$V_C = i_m X_C \sin(\omega t - \phi - \frac{\pi}{2}) \tag{752}$$

$$(V_C)_{\max} = i_m X_C \tag{753}$$

Double ω
Double X_C
Cut current amplitude in half.

Graphical RLC Solution



$$\Sigma \Delta V's = 0 \tag{754}$$

$$\Sigma_4 \text{Phasors} = \vec{0} \tag{755}$$

$$\Sigma \text{R,L,C phasors} = -\text{phasors on source} \tag{756}$$

$$\epsilon = \sqrt{(V_R)^2_{\max} + [(V_L)_{\max} - (V_C)_{\max}]^2} \tag{757}$$

$$= \sqrt{(i_m R)^2 + (i_m X_L - i_m X_C)^2} \tag{758}$$

$$= i_m \sqrt{R^2 + (X_L - X_C)^2} \tag{759}$$

$$\tan\phi = \frac{(V_L)_{\max} - (V_C)_{\max}}{(V_R)_{\max}} \tag{760}$$

$$= \frac{X_L - X_C}{R} \tag{761}$$

Power Delivered To Load

(Resistor, Motor, etc.)

$$
\begin{aligned}
P_{\text{by EMF}} &= \epsilon I && (762)\\
&= V_0 \sin(\omega t) \cdot I_m \sin(\omega t - \phi) && (763)\\
&= V_0 I_m [\sin^2(\omega t)\cos\phi - \sin(\omega t)\cos(\omega t)\sin\phi] && (764)
\end{aligned}
$$

$$\bar{P} = \frac{1}{2} V_0 I_m \cos\phi \quad (765)$$

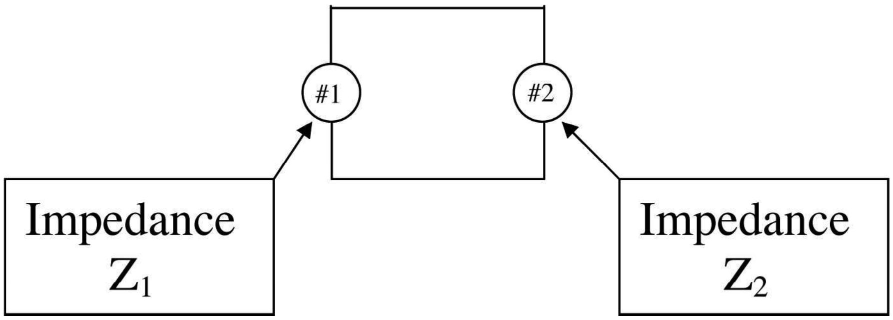
Note: $\bar{P} \rightarrow$ Time average, $\cos\phi \rightarrow$ Power factor
From $\tan\phi$ equation

$$\Rightarrow \cos\phi = \frac{R}{\sqrt{R^2 + (X_L - X_C)^2}} \quad (766)$$

Maximum power delivered when the circuit is purely resistive, or $X_L = X_C \Rightarrow$ Add caps!!
Resonance:

$$
\begin{aligned}
X_L &= X_C && (767)\\
\omega L &= \frac{1}{\omega C} && (768)\\
\omega &= \frac{1}{\sqrt{LC}} && (769)
\end{aligned}
$$

Impedance Matching



$$I = \frac{\epsilon}{z_{\text{total}}} \tag{770}$$

$$z_{\text{total}} = \sqrt{[(X_{C1} + X_{C2}) - (X_{L1} + X_{L2})]^2 + (R_1 + R_2)^2} \tag{771}$$

$$\langle P_2 \rangle = I^2 R_2 \tag{772}$$

$$= \frac{\epsilon^2 R^2}{[X_{C1} + X_{C2} - (X_{L1} + X_{L2})]^2 + (R_1 + R_2)^2} \tag{773}$$

Maximized for:

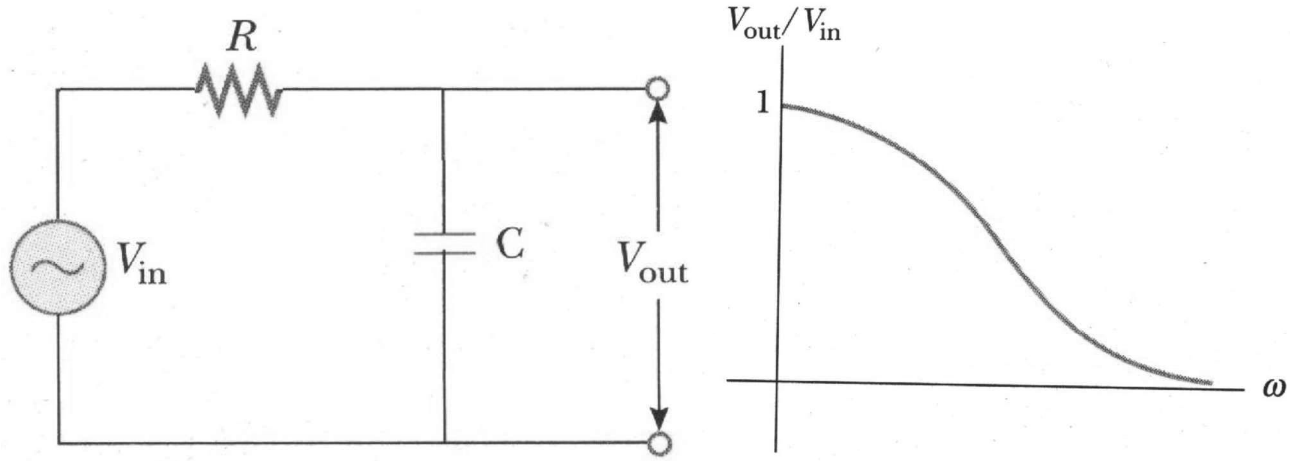
$$R_2 = R_1 \tag{774}$$

and

$$X_{C2} - X_{L2} = -(X_{C1} - X_{L1}) \tag{775}$$

For voltmeter don't have good match

Low Pass Filter

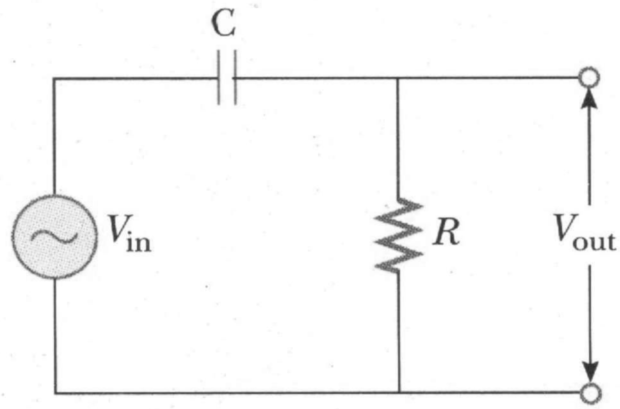


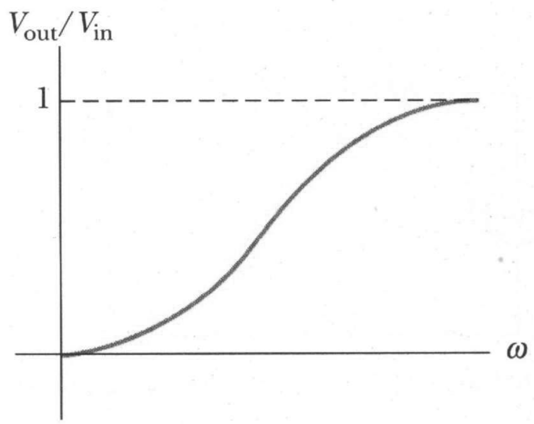
$$V_{\text{out}} = I_m X_C = \frac{I_m}{\omega C} \tag{776}$$

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{\frac{1}{\omega C}}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \tag{777}$$

HighPass Filter

1V @ 20 hz
1V @ 2 khz
1V @ 20 khz
0.001V @ 20 hz
0.6V @ 2 khz
0.95V @ 20 khz
Capacitor blocks "DC"





$$V_{in} = I_m z \tag{778}$$

$$= I_m \sqrt{R^2 + \left(\frac{1}{\omega c}\right)^2} \tag{779}$$

$$V_{out} = I_m R \tag{780}$$

$$\frac{V_{out}}{V_{in}} = \frac{R}{\sqrt{R^2 + \left(\frac{1}{\omega c}\right)^2}} \tag{781}$$