Alg-21
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$a \in G \quad \underline{a^m = a \cdots a}, \; m > 0$

$= e \quad$ for $m = 0$

$= (a^{-1})^{-m} \quad m < 0$

$a^{m+n} = a^m \cdot a^n$

$(a^{-1})^m = (a^m)^{-1}$

$(a^m)^n = a^{mn}$

#1 HW set 16

$f: G \to G'$ isomorphism, $k \in \mathbb{N}$

$g \in G$ $A := \{a \in G : a^k = g\}$

$B := \{b \in G' : b^k = f(g)\}$

$\Rightarrow |A| = |B|$

i.e. find $\varphi: A \to B$ a bijection

Take $\varphi: A \to B$ defined by

$\varphi(a) = f(a)$ for $a \in A$.

Take $\varphi : A \to B$
$\varphi(a) = f(a)$ for $a \in A$.
Show that φ is a bijection
φ is injective & surjective

$$f(a^m) = f(\underbrace{\overbrace{a \times a \times \cdots \times a}^{\text{NO}}}_{\text{times}}) = \underbrace{\overbrace{f(a) \times \cdots \times f(a)}^{\text{NO}}}_{m}$$

NO

Do not write each
$\underbrace{a^m = a \cdots a}_{\text{times}}$, use induc

a

$f(a_1) = f(a_2)$, $a_1, a_2 \in A$
$a_1 = a_2$ as f is injective
on G
so on A

Surjection. Let $b \in B$
to find $a \in A$ s.t.
$b = f(a)$. $b \in B \Rightarrow$
$b^k = f(g)$. $\qquad \underline{a^k = g} \Rightarrow a \in A$

$f(\ldots) \quad \ldots \quad f(a^k) \quad f(a)^k$ by f...

$f(a) = b \qquad f(a^k) = f(a)^k$ by f is homo

$= f(g) = b^k.$

$(xax^{-1})^n = xa^nx^{-1} \quad \forall a \in \mathbb{Z}.$

First for $n \in \mathbb{N}$, induction

$\underbrace{xax^{-1}xax^{-1} \cdots xax^{-1}}$

$n=1 \quad (xax^{-1})^1 = xax^{-1}$

Assume $(xax^{-1})^k = xa^kx^{-1}$
for $n=k$.

Now $(xax^{-1})^{k+1} = (xax^{-1})^k(xax^{-1})$

$= (xa^kx^{-1})(xax^{-1})$

$= (xa^k)(x^{-1}x)ax^{-1})$

$= xa^k e\, ax^{-1}$

$= xa^k a x^{-1}$

$= xa^{k+1}x^{-1}$

G is finite $\quad S$ is finite

G is group ...
S is a G-set
$t \in Gs \Rightarrow$ Prove $\boxed{G_s = G_t \quad |G_s| = |G_t|}$

&

& $\sum_{t \in Gs} \frac{1}{|Gt|} = 1$ ⟵ to prove.

P $\boxed{\text{If } \sum_{t \in Gs} \frac{1}{|Gt|} = 1}$ then $|Gt| = |Gs|$

$t \in Gs \Rightarrow t = xs$ for some $x \in G$

$Gt = Gxs = Gs$

$AB = \{ab : a \in A, b \in B\}$

$Gx = G. \Leftarrow x \in G.$

Elementary divisors and invariant factors of a finite abelian group

finite abelian group of ...

group

① A is finite abelian group of order $n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ where

$p_1 \le p_2 \le \cdots \le p_k$ are primes ; $n_i \ge 1$
$i = 1 \cdots k$.

Thm $A = \bigoplus_{i=1}^{k} A_i$ where A_i is cyclic group of order $p_i^{n_i}$

$p_1^{n_1}, \cdots, p_k^{n_k}$ are called elementary divisors of A.

② Every finite abelian group A is direct sum of cyclic groups B_j of order m_j s.t. $m_j \mid m_{j+1}$

where $A = \bigoplus_{j=1}^{t} B_j$ $m_1 \mid m_2 \mid m_3 \mid \cdots \mid m_t$

$$|A| = m_1 m_2 \cdots m_t$$

$m_1 \cdots m_t$ are called invariant facts.

Thm A, B are finite abelian groups

Thm

groups

(1) A & B are isomorphic $\Longleftrightarrow$ they have the same invariant facts

(2) A & B are iso $\Longleftrightarrow$ they have the same elementary divisors

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E H, K are groups
then $H \times K \simeq H \oplus K$.

$(h, k) \longrightarrow h+k$. is iso

H & K are subgroup of G

then HK is a subgroup $\Longleftrightarrow$ $HK = KH$

$\Longrightarrow$ Assume HK is a subgroup
To show $HK = KH$

$x \in HK$, to show $x \in KH$

$x = hk$, $h \in H$, $k \in K$, $x^{-1} \in HK$

as HK is a subgroup.

as HK is a subgroup.

$x^{-1} = k^{-1}h^{-1} \in HK$.

$= h_1 k_1$

$x = k_1^{-1} h_1^{-1} \in KH$; similarly

show that $y \in KH$ & HK is a group, then $y \in HK$

$(\Longleftarrow)$ If $HK = KH$, then show that HK is a subgroup

$h_1 k_1, h_2 k_2 \in HK$

$(h_1 k_1)(h_2 k_2) = h_1 (k_1 h_2) k_2$

$k_1 h_2 \in KH = HK$. $\exists$

$h_3 \in H, k_3 \in K$ s.t. $k_1 h_2 = h_3 k_3$

$(h_1 k_1)(h_2 k_2) = h_1 (h_3 k_3) k_2$

$= (h_1 h_3)(k_3 k_2) \in HK$

Back to elementary divisors and invariant factors.

invariant factors.

Ex $A = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{25}$

$|A| = 2 \cdot 2 \cdot 4 \cdot 8 \cdot 3 \cdot 3 \cdot 3 \cdot 5 \cdot 25$

$= 2 \cdot 2 \cdot 2^2 \cdot 2^3 \cdot 3 \cdot 3 \cdot 3 \cdot 5 \cdot 5^2$

So the elementary divisors of A are $2, 2, 2^2, 2^3, 3, 3, 3, 5, 5^2$

$$\begin{array}{cccc} 2 & 2 & 2^2 & 2^3 \\ & 3 & 3 & 3 \\ & & 5 & 5^2 \end{array}$$

$2 \quad 6 \quad 60 \quad 600$

$2 \mid 6, \ 6 \mid 60, \ 60 \mid 600$

So the invariant factors are 2, 6, 60 and 600

2 6 60 ...

$$
\begin{array}{ccccccc}
A \simeq \mathbb{Z}_2 & \oplus & \mathbb{Z}_6 & \oplus & \mathbb{Z}_{60} & \oplus & \mathbb{Z}_{600} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
B_1 & & B_2 & & B_3 & & B_4
\end{array}
$$

$$|B_1| \mid |B_2| \mid |B_3| \mid |B_4|$$

Ex Ab. groups	Elem. Divs	Invariant factors
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$	2, 2, 3, 3	6, 6 $\mathbb{Z}_6 \oplus \mathbb{Z}_6$
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9$	2, 2, 9	2, 18 $\mathbb{Z}_2 \oplus \mathbb{Z}_{18}$
$\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$	$\cancel{4}\ 2^2$, 3, 3	3, 12 $\mathbb{Z}_3 \oplus \mathbb{Z}_{12}$
$\mathbb{Z}_4 \oplus \mathbb{Z}_9$	2^2, 3^2	36 $\mathbb{Z}_{36}$