PSET 5.

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Complex Analysis
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§53 #2, 4-7.

2. Let C_1 denote the POSCC of the square whose sides lie along the lines $x = \pm 2$, $y = \pm 1$, let C_2 be the POSCC $|z| = 4$. Point out why $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$ when:

a) $f(z) = \frac{1}{3z^2+1}$

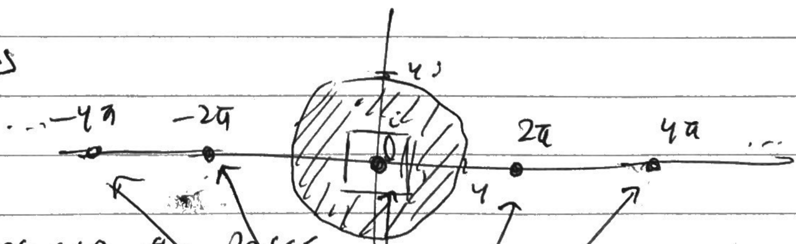
f is only non-analytic when $3z^2+1 = 0$, i.e., $z = \left(-\frac{1}{3}\right)^{\frac{1}{2}}$ at which points $|z| < 1$. Thus f is analytic on the square C_1 and further from the origin, and thus in the region on and between C_1 and C_2. By the principle of deformation of paths ~~(PODOP)~~ (PDP), $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$.

b) $f(z) = \frac{z+2}{\sin\left(\frac{z}{2}\right)}$

f is analytic everywhere except where $\sin\left(\frac{z}{2}\right) = 0$, since it is the ~~composition~~ quotient of entire functions. This occurs when $z = 2\pi k$, $n \in \mathbb{Z}$, i.e.,

$$\sin\left(\frac{z}{2}\right) = \frac{e^{\frac{iz}{2}} - e^{-\frac{iz}{2}}}{2i} = 0 \Rightarrow e^{\frac{iz}{2}} = e^{-\frac{iz}{2}} \Rightarrow e^{iz} = 1 \Rightarrow z = 2\pi k, \; k \in \mathbb{Z}$$

None of the singular points lies in the region on and between C_1 and C_2, i.e.,



thus f is analytic everywhere in the region on and between the POSCCs C_1 and C_2, so the PDP applies to show that $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$.

Singular points (none in region on or between C_1, C_2).

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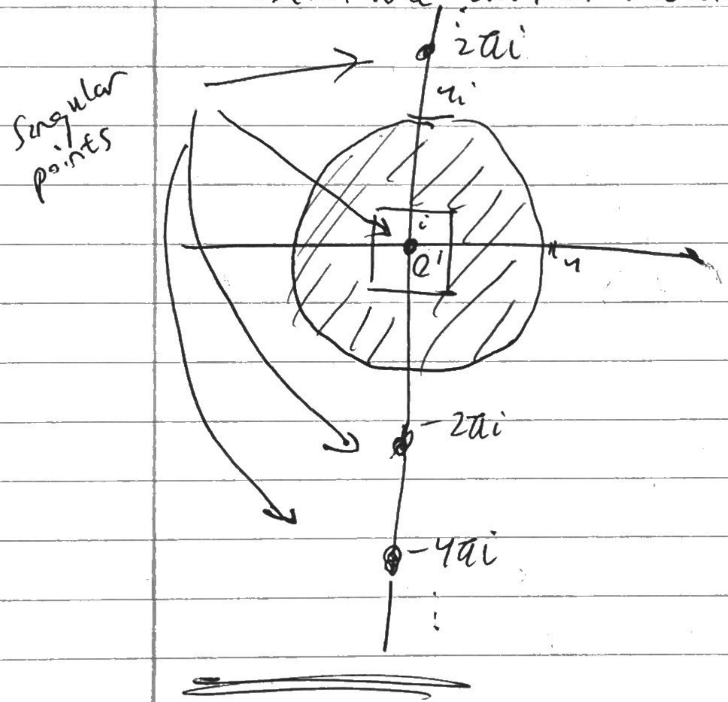
2c) $f(z) = \frac{z}{1-e^z}$.

Since this is the quotient of two analytic functions, it is analytic everywhere except where $1-e^z = 0$, i.e, when $e^z = 1$, i.e, where $z = 2\pi i k$, $k \in \mathbb{Z}$. As with part (b), none of these singular points lies in the region on and between the POSCCs C_1 and C_2, thus PDP applies to obtain the desired result: $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$.



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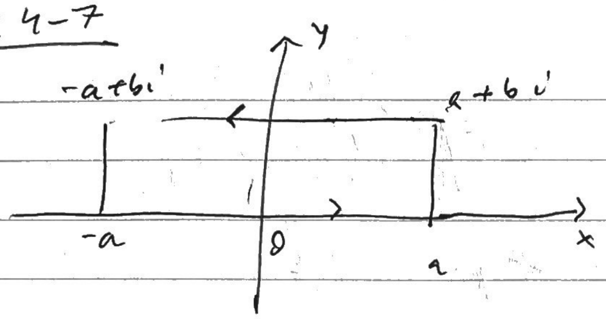
553 #4-7

4.)



Show that the sum of the integrals of e^{-z^2} along the lower and upper horizontal legs can be written

$$2\int_0^a e^{-x^2}dx - 2e^{b^2}\int_0^a e^{-x^2}\cos 2bx\,dx.$$

lower leg: $z(x) = x, \quad z'(x) = dx$
$-a \le x \le a$

$$\int f(z)dz = \int_{-a}^a e^{-x^2}\cdot 1\,dx$$

e^{-x^2} is even, so this can be written:

$$= 2\int_a^a e^{-x^2}dx$$

upper leg: $z(x) = -x + bi \quad z'(x) = -dx.$
$-a \le x \le a$

$$\int f(z)dz = \int_{-a}^a e^{-(-x+bi)^2}\cdot(-1)\,dx = -\int_{-a}^a e^{-(x^2-b^2-2xbi)}dx$$

$$= -\int_{-a}^a e^{+b^2}e^{-x^2}e^{i(2xb)}dx = -e^{b^2}\int_{-a}^a e^{-x^2}(\cos(2xb)+i\sin(2xb))\,dx$$

$$= -e^{b^2}\int_{-a}^a \underbrace{\underbrace{e^{-x^2}}_{\text{even}}\underbrace{\cos 2bx}_{\text{even}}}_{\text{even}}\,dx - ie^{b^2}\int_{-a}^a \underbrace{\underbrace{e^{-x^2}}_{\text{even}}\underbrace{\sin 2bx}_{\text{odd}}}_{\text{odd}}\,dx$$

$$= -2e^{b^2}\int_0^a e^{-x^2}\cos 2bx\,dx \cancel{- ie^{b^2}(0)}$$

Thus the sum of the integrals on the horizontal legs (in the correct orientations) is:

$$2\int_0^a e^{-x^2}dx - 2e^{b^2}\int_0^a e^{-x^2}\cos 2bx\,dx.$$

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Show that the sum of the integrals along the vertical legs can be written

$$ie^{-a^2}\int_0^b e^{y^2}e^{-i2ay}dy - ie^{-a^2}\int_0^b e^{y^2}e^{i2ay}dy.$$

Left leg: $z(y) = -a + yi, \quad 0 \le y \le b$ ← this is in the reverse direction, so negate the value later.

$z'(y) = i\,dy.$

$$-\int f(z)\,dz = -\int_0^b e^{-(-a+yi)^2}\cdot i\,dy$$

$$= -i\int_0^b e^{-(a^2-y^2-2ayi)}dy = -ie^{-a^2}\int_0^b e^{y^2}e^{i2ay}dy.$$

Right leg: $z(y) = a + yi \quad 0 \le y \le b.$

$z'(y) = i\,dy$

$$\int f(z)\,dz = \int_0^b e^{-(a+yi)^2}\,i\,dy = i\int_0^b e^{-(a^2-y^2+2ayi)}dy$$

$$= ie^{-a^2}\int_0^b e^{y^2}e^{-i2ay}dy.$$

Sum of integrals on left and right legs:

$$ie^{-a^2}\int_0^b e^{y^2}e^{-i2ay}dy - ie^{-a^2}\int_0^b e^{y^2}e^{i2ay}dy$$

$f(z) = e^{-z^2}$ is the composition of two entire functions, so it is analytic everywhere. Since the rectangle (C) is a SCC and f is AOIC, then $\int_C f(z)\,dz = 0$.

Thus:

$$\underbrace{\int_C f(z)\,dz}_{=0} = 2\int_0^a e^{-x^2}dx - 2e^{b^2}\int_0^a e^{-x^2}\cos 2bx\,dx + ie^{-a^2}\int_0^b e^{y^2}e^{-i2ay}dy - ie^{-a^2}\int_0^b e^{y^2}e^{i2ay}dy$$

$$\Rightarrow \frac{2e^{b^2}\int_0^a e^{-x^2}\cos 2bx\,dx}{2e^{b^2}} = \frac{2\int_0^a e^{-x^2}dx + ie^{-a^2}\int_0^b e^{y^2}\left(e^{-2ayi} - e^{2ayi}\right)dy}{2e^{b^2}}$$

$$\int_0^a e^{-x^2}\cos 2bx\,dx = e^{-b^2}\int_0^a e^{-x^2}dx + e^{-a^2}e^{-b^2}\int_0^b e^{y^2}\left(\frac{e^{(2ay)i} - e^{(-2ay)i}}{2i}\right)dy$$

$$= e^{-b^2}\int_0^a e^{-x^2}dx + e^{-(a^2+b^2)}\int_0^b e^{y^2}\sin 2ay\,dy.$$

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553 #4b, -7.

4b.) Accept that $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$ (**), $\left|\int_0^b e^{y^2}\sin 2ay\,dy\right| \le \int_0^b e^{y^2}dy$. (*)

Take the limit of the expression from 4a as $a \to \infty$ to get the desired integration formula.

$$\int_0^\infty e^{-x^2}\cos 2bx\,dx = \lim_{a\to\infty}\int_0^a e^{-x^2}\cos 2bx\,dx = \lim_{a\to\infty}\left[e^{-b^2}\int_0^a e^{-x^2}dx + e^{-(a^2+b^2)}\int_0^b e^{y^2}\sin 2ay\,dy\right]$$

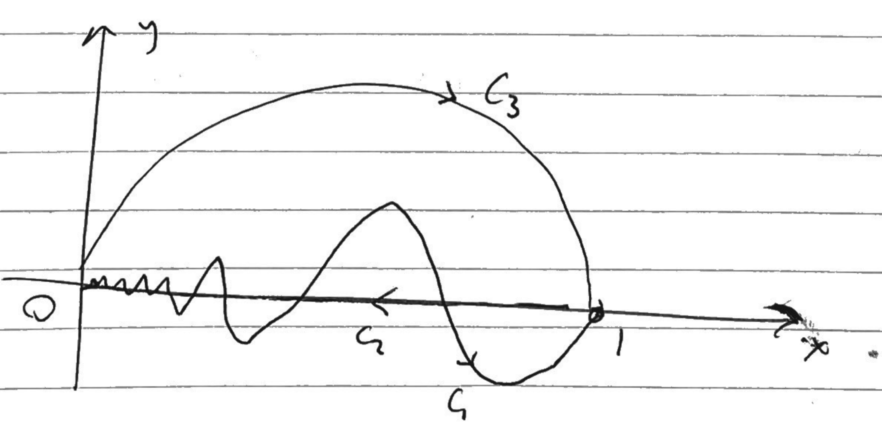
$$= \underbrace{e^{-b^2}\int_0^\infty e^{-x^2}dx}_{=\frac{\sqrt{\pi}}{2}\ (**)} + \underbrace{\underbrace{\lim_{a\to\infty}\left(e^{-a^2}\right)}_{=0}\ e^{-b^2}\underbrace{\int_0^b e^{y^2}\sin 2ay\,dy}_{\text{bounded by } (*)}}_{\text{limit of 0 \& (some bounded finite value)} = 0}$$

$$= \frac{\sqrt{\pi}}{2}e^{-b^2} + 0 = \frac{\sqrt{\pi}}{2}e^{-b^2}.$$

5.) It was established in a previous exercise that the arc C_1 along the graph of the function

$$y(x) = \begin{cases} x^3\sin\left(\frac{\pi}{x}\right), & 0 < x \le 1 \\ 0 & x = 0 \end{cases}$$

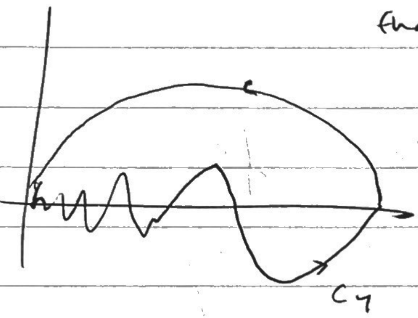
is a smooth arc. Let C_2 denote the line segment along the real axis from $z = 1$ to the origin, and let C_3 denote any smooth arc from $z = 1$ back to the origin not intersecting C_1 or C_2.



Show that if f is entire, then

$$\int_{C_1} f(z)\,dz = \int_{C_3} f(z)\,dz, \qquad \int_{C_2} f(z)\,dz = -\int_{C_3} f(z)\,dz.$$

Let $C_4 = C_1 - C_3$. Then C_4 is a SCC (it is already established that C_1 is smooth, so this is a contour).



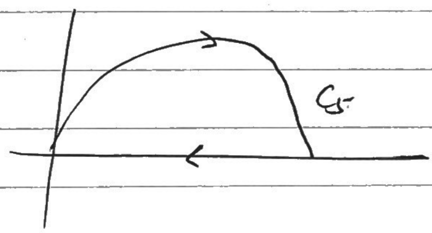
Since f is entire, then it is AOIC,

and $\int_{C_4} f(z)\,dz = \int_{C_1} f(z)\,dz - \int_{C_3} f(z)\,dz = 0$

C-G

$\Rightarrow \int_{C_1} f(z)\,dz = \int_{C_3} f(z)\,dz.$ (*)

Similarly, let $C_5 = C_2 + C_3$. Then C_5 is a (NO) SCC, and by the same reasoning:



C-G

$$\underset{\text{SCC}}{\int_{C_5}} \underset{\text{AOIC}}{f(z)\,dz} = \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz = 0$$

$\Rightarrow \int_{C_2} f(z)\,dz = -\int_{C_3} f(z)\,dz.$ (**)

Thus, we can conclude that

$$\int_{C_1 + C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

$$= \underset{(*)}{\int_{C_3} f(z)\,dz} - \underset{(**)}{\int_{C_3} f(z)\,dz} = 0$$

even though the curve $C = C_1 + C_2$ self-intersects infinitely many times.

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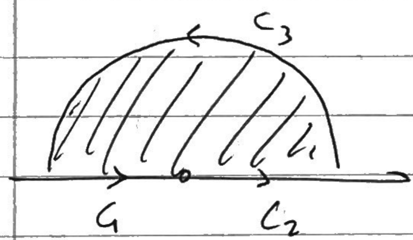
S53 # 6, 7

6.) Let C denote the POSCC of the half-disk $0 \le r \le 1$, $0 \le \theta \le \pi$, and let $f(z)$ be a cont. fn. defined on that half disk by writing $f(0)=0$ and using the branch

$$f(z) = \sqrt{r}\, e^{i\frac{\theta}{2}} \qquad \left(r>0,\ -\frac{\pi}{2} < \theta < \frac{3\pi}{2}\right)$$

of the multiple-valued fn. $z^{\frac{1}{2}}$. Show that $\int_C f(z)\,dz = 0$ by evaluating separately the integrals of $f(z)$ over the semicircle and the two radii that make up C. Why does C-G not apply here?

C-G doesn't apply since f is not analytic at $z=0$, which lies on the curve.



in the negative direction

$$z(r) = re^{i\pi}, \quad 0 \le r \le 1, \quad z' = -1\,dr$$

$$C_1: -\int_0^1 \sqrt{r}\, e^{i\left(\frac{\pi}{2}\right)} (-1)\,dr$$

$$= \int_0^1 \sqrt{r} \cdot i\,dr = \frac{2i}{3} r^{\frac{3}{2}}\Big|_0^1 = \frac{2i}{3}.$$

C_2: $z(r) = re^{i0}$, $0 \le r \le 1$, $z' = 1$

$$\int_0^1 \sqrt{r}\, e^{i\left(\frac{0}{2}\right)} \cdot 1\,dr = \int_0^1 \sqrt{r}\,dr = \frac{2}{3} r^{\frac{3}{2}}\Big|_0^1 = \frac{2}{3}.$$

C_3: $z(\theta) = e^{i\theta}$, $z' = ie^{i\theta}\,d\theta$, $0 \le \theta \le \pi$

$$\int_0^\pi \sqrt{1}\, e^{i\left(\frac{\theta}{2}\right)} \cdot ie^{i\theta}\,d\theta = \int_0^\pi ie^{i\frac{3\theta}{2}}\,d\theta = \frac{2}{3} e^{i\frac{3\theta}{2}}\Big|_0^\pi$$

use antiderivative

$$= \frac{2}{3}\left(e^{-\frac{3\pi i}{2}} - e^{0}\right) = \frac{2}{3}(-i-1)$$

$$\int_{C_1+C_2+C_3} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz$$

$$= \frac{2i}{\cancel{3}} + \frac{2}{\cancel{3}} - \frac{2i}{\cancel{3}} - \frac{2}{\cancel{3}} = 0.$$

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7.) Show that if C is a POSCC, then the area of the region enclosed by C can be written $\frac{1}{2i}\int_C \bar{z}\,dz$.

In Sec. 50, we derived expression (4), which can be applied to any SCC on any complex-valued function:

$$\int_C f(z)\,dz = \iint_R (-v_x - u_y)\,dA + i\iint_R (u_x - v_y)\,dA.$$

Since this only used the basic concepts of parametric integration and Green's theorem, and doesn't require analyticity of f, it may be used with $f(z) = \bar{z}$. Begin with the proposed formula:

$$\frac{1}{2i}\int_C \bar{z}\,dz = \frac{1}{2i}\left(\iint_R (-v_x - u_y)\,dA + i\iint_R (u_x - v_y)\,dA\right)$$

$$\left(\begin{array}{llll} f(z) = x - iy & u = x, & u_x = 1, & u_y = 0, \\ & v = -y, & v_y = -1, & v_x = 0. \end{array}\right)$$

$$= \frac{1}{2i}\left(\cancel{\iint_R (-0-0)\,dA} + i\iint_R (1-(-1))\,dA\right)$$

$$= \frac{1}{2i}\, i\iint_R 2\,dA = \frac{2i}{2i}\iint_R dA = A. \checkmark$$

This confirms the hypothesis.

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S57 # 1, 2a, 3.

1.) Let C denote the POSCC of the square whose sides lie on the lines $x = \pm 2$, $y = \pm 2$. Evaluate the integrals:

a) $$\underbrace{\int_C}_{\text{POSCC}} \frac{\overbrace{e^{-z}}^{\text{AOIC}}\,dz}{z - \underbrace{\left(\frac{\pi i}{2}\right)}_{\text{interior point}}} \underset{\text{CIF}}{=} 2\pi i\left(e^{-\frac{\pi i}{2}}\right) = 2\pi i(-i) = -2\pi(-1) = 2\pi$$

b) $$\int_C \frac{\cos z}{z(z^2+8)}\,dz = \underbrace{\int_C}_{\text{POSCC}} \frac{\overbrace{\left(\frac{\cos z}{z^2+8}\right)}^{\text{AOIC (poles lie outside enclosed region)}}}{\underbrace{(z-0)}_{\text{interior point}}}\,dz \underset{\text{CIF}}{=} 2\pi i\left(\frac{\cos 0}{0^2+8}\right) = \frac{\pi i}{4}$$

c) $$\int_C \frac{z\,dz}{2z+1} = \underbrace{\int_C}_{\text{POSCC}} \frac{\overbrace{\frac{z}{2}}^{\text{AOIC}}}{\underbrace{z+\frac{1}{2}}_{-\frac{1}{2}\text{ is an interior point}}} \underset{\text{CIF}}{=} 8\pi i\left(\frac{-\frac{1}{2}}{\cancel{8}}\right) = -\frac{\pi i}{2}$$

d) $$\int_C \frac{\cosh z}{z^4}\,dz = \underbrace{\int_C}_{\text{POSCC}} \frac{\overbrace{\cosh z}^{\text{AOIC}}}{\underbrace{(z-0)^4}_{\text{interior point}}}\,dz \underset{\text{(extended) CIF}}{=} \frac{2\pi i}{3!}\left(\frac{d^3}{dz^3}\cosh z\right)\Bigg|_{z=0}$$

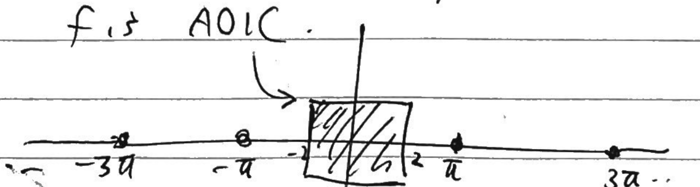
$$\left(\frac{d^3}{dz^3}\cosh z = \sinh z\right)$$

$$= \frac{2\pi i}{6}\sinh(0) = \frac{2\pi i}{6}(0) = 0.$$

e) $$\underbrace{\int_C}_{\text{POSCC}} \frac{\overbrace{\tan\left(\frac{z}{2}\right)}^{\text{AOIC}}}{\underbrace{(z-x_0)^2}_{\text{interior point}}}\,dz \quad (-2 < x_0 < 2)$$

~~tan is analytic when cos ≠~~

$\tan = \frac{\sin}{\cos}$ is analytic when $\cos \neq 0$, i.e., when $\frac{z}{2} \neq (2n+1)\frac{\pi}{2}$ i.e., when $z \neq (2n+1)\pi$. None of these singularities lie on C or in the enclosed region, thus f is AOIC.



$$\underset{\text{CIF}}{=} \frac{2\pi i}{1!}\frac{d}{dz}\left(\tan\left(\frac{z}{2}\right)\right)\Bigg|_{x_0}$$

$$= \cancel{2}\pi i\left(\sec^2\left(\frac{z}{2}\right)\cdot\frac{1}{\cancel{2}}\right)\Bigg|_{x_0}$$

$$= \pi i \sec^2\left(\frac{x_0}{2}\right).$$

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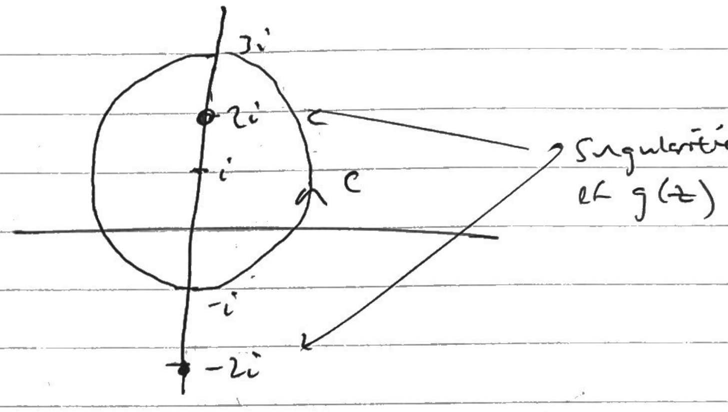
2a) Find the value of the integral of $g(z)$ around the circle $|z-i| = 2$ in the positive sense when

a) $g(z) = \dfrac{1}{z^2+4}$



Roots of $z = (-4)^{\frac{1}{2}}$
$= \pm 2i$

let $\cancel{g}\, h(z) = \dfrac{1}{z+2i}$

Then h is AOIC

$$\int_C \frac{1}{z^2+4}\,dz = \int_C \frac{\left(\frac{1}{z+2i}\right)}{z-2i}\,dz = 2\pi i\left(\frac{1}{2i+2i}\right) = \frac{8\pi i}{24i} = \frac{\pi}{2}.$$

posCC — interior point — CIF — AOIC

3.) Let C be the circle $|z| = 3$. in the positive orientation.
Show that if $g(z) = \displaystyle\int_C \frac{2s^2-s-2}{s-z}\,ds \quad (|z| \neq 3)$

then ~~[illegible]~~ $g(2) = 8\pi i$. What is $g(z)$ when $|z| > 3$.

$$g(2) = \int_C \frac{(2s^2-s-2)}{s-2}\,ds = 2\pi i(2(2^2)-2-2) = 2\pi i(4) = 8\pi i.$$

AOIC — CIF — posCC — interior to $|z| < 3$

When $|z| > 3$, then the integrand becomes AOIC (ratio of polynomials, denominator $\neq 0$ since z is exterior to the circle). Thus, by C-G, the integral is zero.

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S57 # 4, 5, 7, 10.

4) Let C be any POSCC, and write $g(z) = \int_C \frac{s^3 + 2s}{(s-z)^3} ds$.

Show that $g(z) = 6\pi i z$ when z is inside C and $g(z) = 0$ when z is outside.

when z is inside C: $g(z) = \int_C \frac{s^3+2s}{(s-z)^3} \overset{\text{CIF}}{=} \frac{2\pi i}{2!}\left(\frac{d^2}{ds^2} s^3 + 2s\right)\Big|_z$

AOIC (polynomial)

POSCC

interior to C by hyp.

$= \pi i (6s)\big|_z = 6\pi i z$.

when z is outside C, then the integrand becomes AOIC (ratio of polynomials, denominator $\neq 0$), thus by C-G the integral evaluates to 0.

5.) Show that if f is AOIC, where C is a SCC and z_0 is not on C, then: $\int_C \frac{f'(z)\,dz}{z-z_0} = \int_C \frac{f(z)\,dz}{(z-z_0)^2}$

Let $g = f'$. If f AOIC, then g AOIC (since g is analytic where f is). Assume z_0 is within C and C is a POSCC.

Case 1.

$$\int_C \frac{g(z)}{z-z_0} dz \overset{\text{CIF}}{=} 2\pi i\, g(z_0) = \frac{2\pi i}{1!} f'(z) \overset{\text{CIF}}{=} \int_C \frac{f(z)\,dz}{(z-z_0)^2}$$

AOIC

POSCC

interior point

AOIC

POSCC

interior point

Now, assume z_0 is within C and C is a NOSCC,

Case 2.

Then $\int_C \frac{g(z)}{z-z_0} dz = -2\pi i\, g(z_0) = \frac{-2\pi i}{1!} f'(z_0) = \int_C \frac{f(z)}{(z-z_0)^2} dz$

for much of the same reasoning as case 1, except that the orientation is reversed

Case 3.

Assume z_0 is not an interior point of the region enclosed by C. Then the integrand becomes AOIC and by C-G the integral evaluates to 0 (for both integrals).

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7.) Let C be the unit circle $z = e^{i\theta}$ $(-\pi \le \theta \le \pi)$.

First, show that for any real constant a;

$$\int_C \frac{e^{az}}{z}\,dz = 2\pi i.$$

$$\int_C \frac{e^{az}}{z-\cancel{\theta}}\,dz \overset{\text{CIF}}{=} 2\pi i\left(e^{a(0)}\right) = 2\pi i\left(e^{0}\right) = 2\pi i. \quad (*)$$

$\uparrow$ Pos cc $\quad$ $\uparrow$ interior point $\quad$ ← ADic

Then write this integral in terms of θ to derive the integration formula $\int_0^\pi e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = \pi$.

$z = e^{i\theta}$, $-\pi \le \theta \le \pi$

$z' = ie^{i\theta}\,d\theta$

$$\int_{-\pi}^{\pi} \frac{e^{a(\exp i\theta)}}{\cancel{e^{i\theta}}}\, i\cancel{e^{i\theta}}\,d\theta = i\int_{-\pi}^{\pi} e^{a(\cos\theta + i\sin\theta)}\,d\theta$$

$$= i\int_{-\pi}^{\pi} e^{a\cos\theta} e^{ia\sin\theta}\,d\theta$$

$$= i\int_{-\pi}^{\pi} e^{a\cos\theta}\left(\cos(a\sin\theta) + i\sin(a\sin\theta)\right)d\theta$$

$$= i\int_{-\pi}^{\pi} \underbrace{\underbrace{e^{a\cos\theta}}_{\text{even}}\underbrace{\cos(a\sin\theta)}_{\text{even}}}_{\text{even}}\,d\theta + i^2\int_{-\pi}^{\pi} \underbrace{\underbrace{e^{a\cos\theta}}_{\text{even}}\underbrace{\sin(a\sin\theta)}_{\text{odd}}}_{\text{odd}}\,d\theta$$

$$= \frac{\cancel{2i}\int_0^\pi e^{a\cos\theta}\cos(a\sin\theta)\,d\theta}{\cancel{2i}} + \cancel{i^2(0)} \overset{(\text{by } *)}{=} \frac{2\pi i}{2i}$$

$$\Rightarrow \int_0^\pi e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = \pi.$$

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S57 #10.

10.) Let f be an entire function s.t. $|f(z)| \leq A|z|$ $\forall z \in \mathbb{C}$, where A is a fixed positive number. Show that $f(z) = a_1 z$, where a_1 is a complex constant.

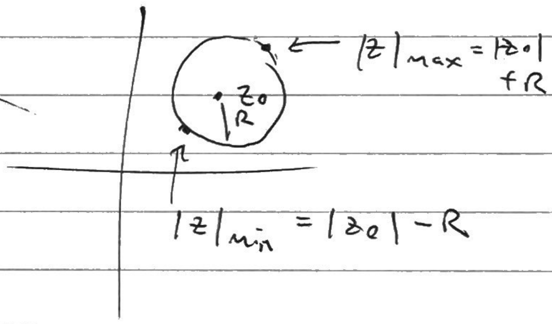
Fix some point $z_0 \in \mathbb{C}$ and some $R \in \mathbb{R}^+$. Suppose C is a POSCC circle centered at z_0 w/ radius R.

Let M_R denote the maximum value of $|f(z)|$ on the circle C_R.

On C_R, $|z_0| - R \leq |z| \leq |z_0| + R$

$$\Rightarrow A(|z_0| - R) \leq |f(z)| = A|z| \leq A(|z_0| + R)$$

Thus $M_R \leq A(|z_0| + R)$.



Since f is entire, then f is AOIC, and we may apply Cauchy's inequality:

$$|f^{(n)}(z_0)| \leq \frac{n!\, M_R}{R^n} = \frac{n!\, A(|z_0| + R)}{R^n}$$

Note that this result must hold true for any choice of n, R, and z_0. Thus we can use this result to find $|f^{(n)}(z_0)|$ using a circle of any radius centered at z_0. Let $n = 2$, and let R approach ∞:

$$\lim_{R \to \infty} |f''(z_0)| \left(= |f''(z_0)|\right) = \lim_{R \to \infty} \frac{(2!)\, A(|z_0| + R)}{R^2}$$

we can rewrite this infinite limit in another way:

$$|f''(z_0)| = \lim_{R \to 0} \left(\frac{2A\left(|z_0| + \frac{1}{R}\right)}{\left(\frac{1}{R}\right)^2} \right) \cdot \frac{R^2}{R^2} = \lim_{R \to 0} \frac{2A(|z_0|R^2 + R)}{1}$$

$$= 2A(\cancel{|z_0|}(0)^2 + \cancel{0}) = 0 \Rightarrow |f''(z_0)| = 0 \Rightarrow f''(z_0) = 0$$

$\forall z_0 \in \mathbb{C}$.

Thus f must be a zeroth- or first-order polynomial of z, or else its second derivative would be nonzero. Thus $f(z) = a_1 z$, where $a_1 \in \mathbb{C}$.