What is communication?

"Conveying information from a source to a receiver"

Ex.

Speech ⟷ sending pressure waves thru air

Writing ⟷ ink on paper

Radio ⟷ radio waves air/space

In all cases, we encode info

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Goal: Devise a mathematical formalism for modeling many schemes of information transfer

- How does noise impact communicate
- What is info (quant.)
- How fast can I send info?
- What is the most efficient/safe way to send info?

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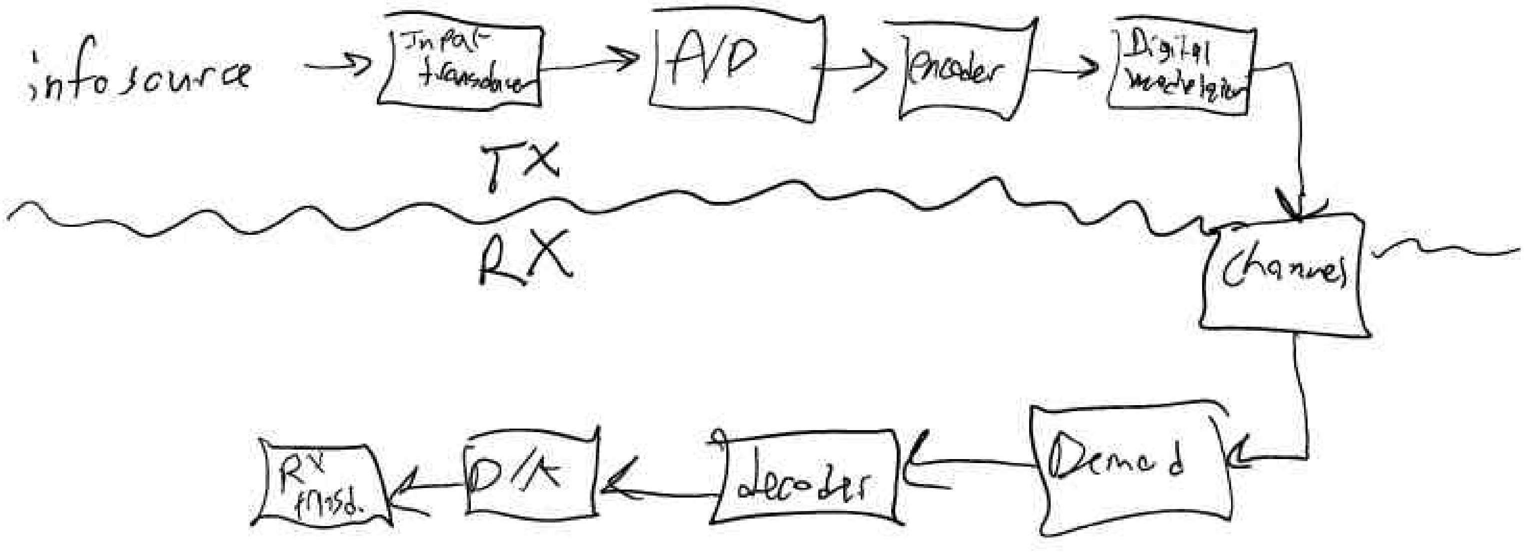
Avg. Comm system



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Def a cts-time signal is a member of the

$\mathcal{F}(\mathbb{R}, \mathbb{C})$ (often $\mathcal{F}(\mathbb{R}, \mathbb{R})$)

Def a disc. time signal . . . . .

$\mathcal{F}(\mathbb{Z}, \mathbb{C})$

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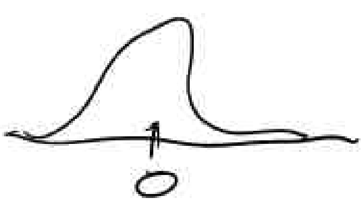
Def. A cts-time system is a member

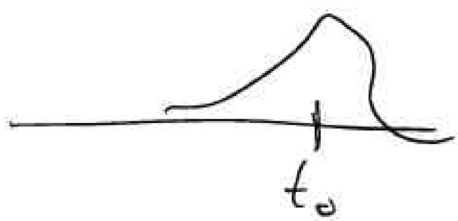
$$\mathcal{F}(\mathcal{F}(\mathbb{R},\mathbb{C}),\mathcal{F}(\mathbb{R},\mathbb{C}))$$

$$f(t) \to \boxed{\mathcal{H}} \to \mathcal{H}\{f\}(t)$$

Basic exs

$$f(t) \to \boxed{d_{t_0}} \to f(t-t_0)$$





$$x[n] \to \boxed{d_{n_0}} \to x[n-n_0]$$

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$x(t) \rightarrow x(-t)$, $x(t) \rightarrow x(at)$

$x(t) \rightarrow Re(x(t))$, $x(t) \rightarrow \angle x(t) = \arctan\left(\frac{Im(x(t))}{Re(x(t))}\right)$

Classification of signals

Periodic $x(t+T) = x(t)\ \forall t$, ex. $A\cos(\omega t + \varphi)$

$$x[n] = \exp(j\omega n)$$

$$x[n+T] = x[n],\ T \in \mathbb{Z}$$

$$\exp(j\omega(n+T)) = \exp(j\omega n)$$

need $\omega(n+T) = \omega n + 2\pi m$ for ~~any~~ some $m \in \mathbb{Z}$

$$\omega T = 2\pi m$$

$$2\pi f T = 2\pi m$$

$$f = \frac{m}{T},\ f \in \mathbb{Q}$$

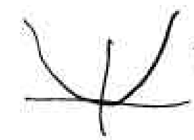
Causal signals $x(t) = 0, \; t < 0$

$x[n] = 0, \; n < 0$

anticausal $x(t) = 0, \; t > 0$

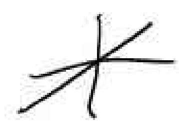
$x[n] = 0, \; n > 0$

Symmetries even: $x(t) = x(-t)$



x^2

odd: $x(t) = -x(-t)$



x

Hermitian (conjugate): $x(t) = x^*(-t)$

$x(t) = x_r(t) + i\, x_i(t)$

$x_r(t) = x_r(-t), \quad x_i(t) = -x_i(-t)$

$|x(t)| = |x(-t)|, \quad \angle x(t) = -\angle x(-t)$

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Def. The energy in a signal (if it is defined) is

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

or

$$\mathcal{E}_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

If a signal has finite energy, we call the signal "energy-type" or L^2 (cts)

ℓ^2 (discrete)

Def Power of a cts-time signal (if defined) is

$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

A signal is "power-type" if $0 < P_x < \infty$

X energy-type $\int^{\infty}$

x energy-type $\int_{-\infty}^{\infty} |x(t)|^2 dt \searrow \mathcal{E}_x < \infty$

$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt \rightarrow \frac{\mathcal{E}_x}{\infty} = 0$$

$$x(t) = A\cos \omega t \quad P_x = \frac{A^2}{T} \int_0^T \cos \omega t^2 dt = \frac{A^2}{2}$$

$$H(t) = \begin{cases} 0, t < 0 \\ 1, t \geq 0 \end{cases}, \quad u[n] = \begin{cases} 0, n < 0 \\ 1, n \geq 0 \end{cases}$$

$$\delta(t) \rightarrow \int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0), \quad \delta[n] = \begin{cases} 0, n \neq 0 \\ 1, n = 0 \end{cases}$$

$$= \begin{cases} 0, t \neq 0 \\ \infty, t = 0 \end{cases}$$

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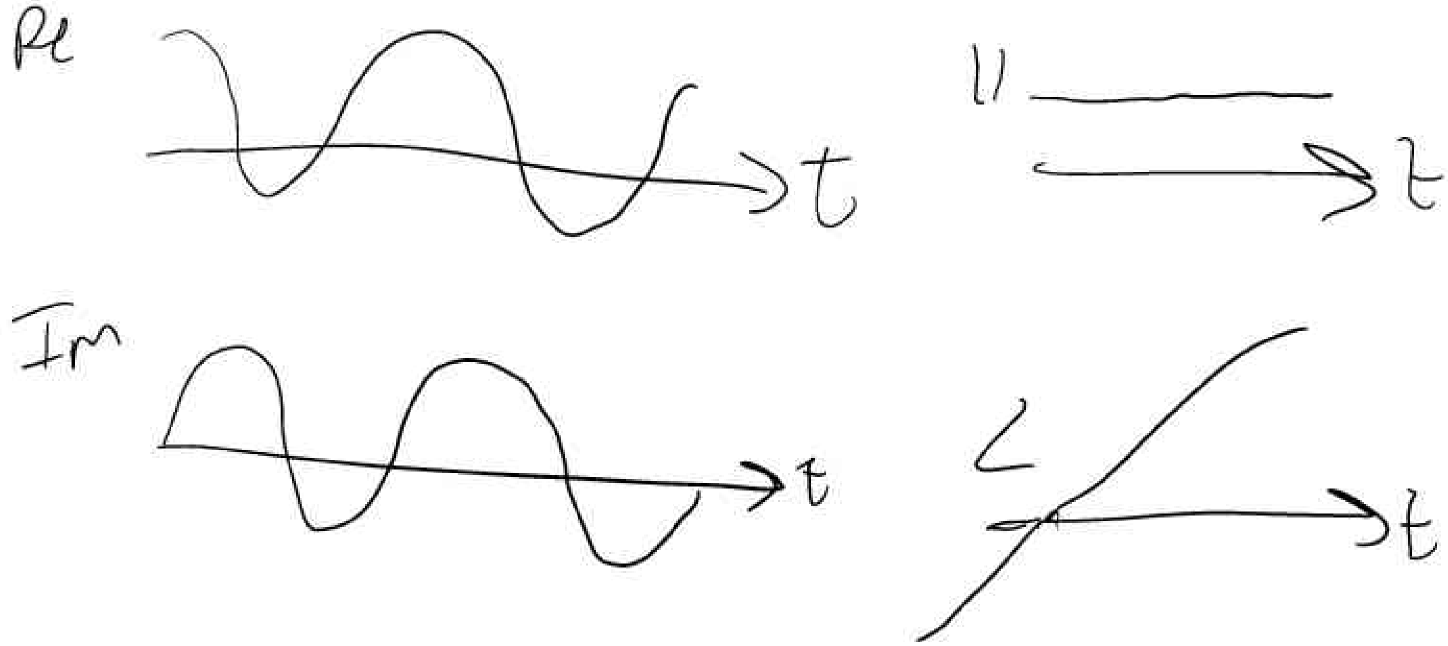
$$x(t) = \cos\omega t + j\sin\omega t$$



System classification

<u>Def</u> A system is linear iff $\forall x, y$ signals, $\forall a \in \mathbb{C}$

1) $\mathcal{H}\{ax\} = a\mathcal{H}\{x\}$ (homogeneity)

2) $\mathcal{H}\{x+y\} = \mathcal{H}\{x\} + \mathcal{H}\{y\}$ (superposition)

<u>Def</u> x is a signal cts time. Let x_{t_0} den. the signal s.t $x(t-t_0) = x_{t_0}(t)\ \forall t$.

System $\mathcal{H}$ is <u>time-invariant</u> iff $\forall$ cts. time x, we have

$$\mathcal{H}\{x_{t_0}\} = \left(\mathcal{H}\{x\}\right)_{t_0}$$

i.e. $\mathcal{H}\{x_{t_0}\}(t) = \mathcal{H}\{x\}(t-t_0)$

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1) $x(t) \rightarrow \boxed{d_{t_0}} \rightarrow x(t-t_0)$ LTI

2) $x(t) \rightarrow \boxed{S} \rightarrow x^2(t)$ TI

3) $x(t) \rightarrow \boxed{M} \rightarrow x(t)\cos\omega t$ LTI

Causal If output at time t_0 depends only on input values for $t \leq t_0$

$\mathcal{H}\{x\}[n] = x[n-1]$ Causal

$\mathcal{H}\{x\}(t) = x(t+\pi)$ noncausal

Convolution

$$(x*y)[n] = \sum_{k=-\infty}^{\infty} x[n-k]\,y[k]$$

$$(x*y)(t) = \int_{-\infty}^{\infty} x(t-\tau)\,y(\tau)\,d\tau$$

$x*y = y*x$, bilinear

$ax*y = x*ay = a(x*y)$

$$(x * \delta)[n] = \sum_k x[n-k]\,\delta[k] = x[n]$$

$$(x * \delta)(t) = \int x(\tau)\,\delta(t-\tau)\,d\tau = x(t)$$

Let $\mathcal{L}$ be a disc.-time LTI system

$$\mathcal{L}\{x\}[n] = \mathcal{L}\{x * \delta\}[n]$$

$$= \mathcal{L}\Big\{\sum_k x[k]\,\delta[n-k]\Big\}$$

$$= \sum_k x[k]\,\mathcal{L}\{\delta[n-k]\}$$

if $\mathcal{L}\{\delta[n]\} = h[n]$, then $= \sum_k x[k]\,h[n-k]$

$$= x * h$$

$$\boxed{\text{If } \mathcal{L} \text{ is LTI, the } \mathcal{L}\{x\} = x * h \text{ where } h = \mathcal{L}\{\delta\} \equiv \text{impulse response}}$$

true in disc. and cts

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$$x \rightarrow \boxed{\mathcal{L}} \rightarrow y$$

$$\equiv$$

$$x \rightarrow \boxed{h} \rightarrow y, \quad h = \mathcal{L}\{\delta\}$$

"duality of signals and systems"

system causal $\Longleftrightarrow$ impulse resp causal

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Ex. $\mathcal{H}$ system which integrates

$$\mathcal{H}\{x\}(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

$$\mathcal{H}\{\delta\}(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau = \begin{cases} 1, & t \geq 0 \\ 0, & \text{else} \end{cases}$$

$$= H(t)$$

"Complex Exponentials are the eigenfunctions of LTI systems"

$$Av = \lambda v$$

constant

if system just scales, eigenvector

$\mathcal{L}$ w/ imp. resp. h. (LTI)

$$\mathcal{L}\{e^{j(\omega t+\varphi)}\}(t) = \int_{-\infty}^{\infty} h(\tau)\, e^{j(\omega(t-\tau)+\varphi)}\, d\tau$$

$$= e^{j(\omega t+\varphi)} \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau$$

$-\infty$

input

Constant w.r.t. time

Fourier Series

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x periodic cts signal w/ period T

1) $\int_0^T |x(t)| dt$ ↘

2) x has only finitely many discontinuities

3) x has finitely many mins and maxs

$$X(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\omega n t}, \quad c_n = \frac{1}{T}\int_0^T x(t) e^{-j\omega n t} dt$$

↑ fourier coeffs.

$$\omega = \frac{2\pi}{T}$$

h is imp. resp, then $H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$

then $x * h$

$$= \sum c_n (e^{j\omega n t} * h)$$

$$= \sum c_n H(n\omega) e^{j\omega n t}$$

$$= y = \sum d_n e^{j\omega n t}, \quad d_n = c_n H(n\omega)$$

harmonics at the output are the same as at the input

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If x is real-valued, $c_n = c_{-n}^*$

<u>Parseval</u>

$$\frac{1}{T}\int_0^T |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2$$

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Many signals non-periodic

Same assumptions F.S. (over any finite interval)

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \mathcal{F}\{x\}(\omega)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega = \mathcal{F}^{-1}\{X\}(t)$$

$X : \mathbb{R} \to \mathbb{C}$, $|X(\omega)|$ mag. resp.

$\angle X(\omega)$ phase resp.

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Props

Linear

Duality $X(\omega) = \mathcal{F}\{x(t)\}$

$$\Downarrow$$

$$2\pi x(\omega) = \mathcal{F}\{X(-t)\}$$

Shift $\mathcal{F}\{x(t-t_0)\} = e^{-j\omega t_0} X(\omega)$

Convolution $\mathcal{F}\{x*y\} = X(\omega)\,Y(\omega)$

Modulation $\mathcal{F}\{x e^{j\omega_0 t}\} = X(\omega-\omega_0)$

Diff $\mathcal{F}\{\frac{d}{dt}x(t)\} = j\omega X(\omega)$

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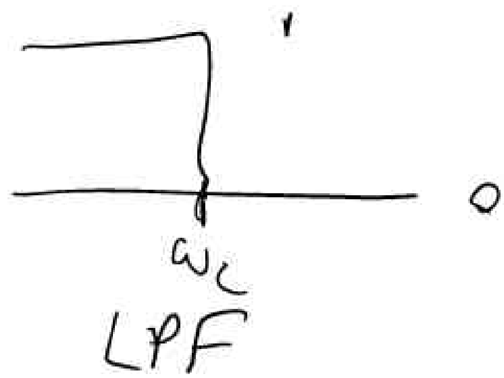
$$x \rightarrow \boxed{h} \rightarrow y$$

$$X(\omega) \rightarrow \boxed{H(\omega)} \rightarrow Y(\omega) = H(\omega)X(\omega)$$



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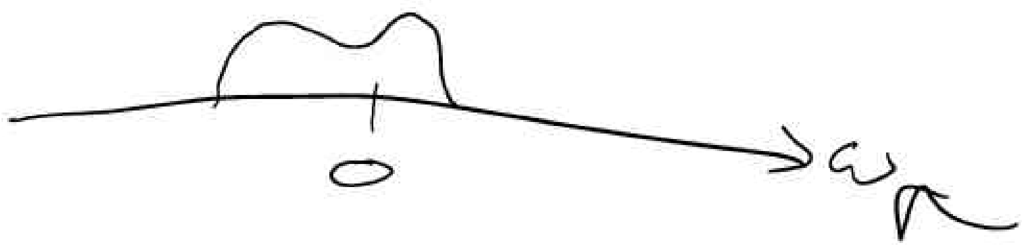
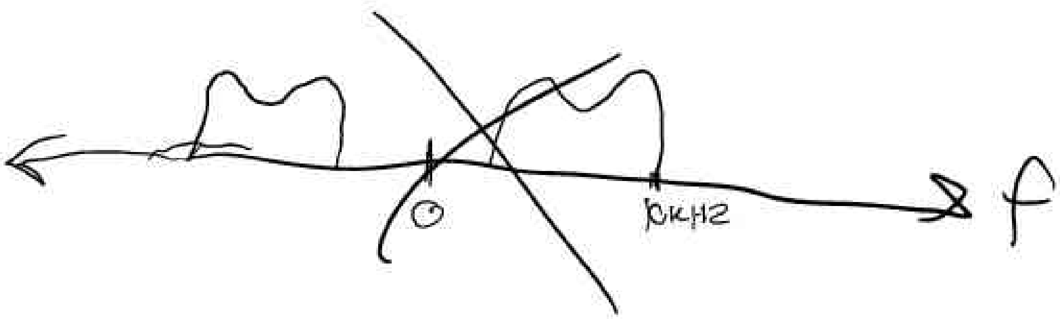
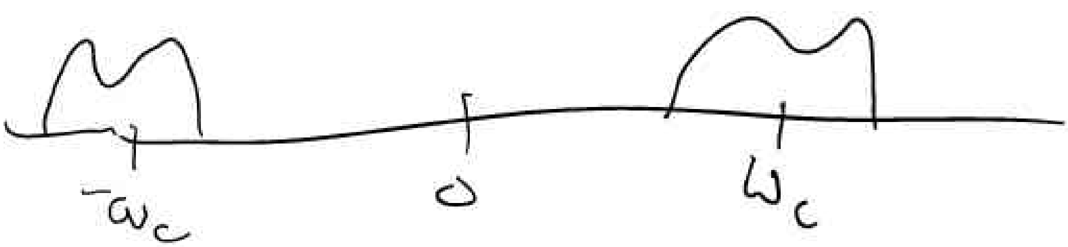
Baseband (**lowpass**) signal is a signal w/ spectrum located **around** $\omega = 0$





A **Bandpass** signal is one in which the signal's spectrum is **far** from DC



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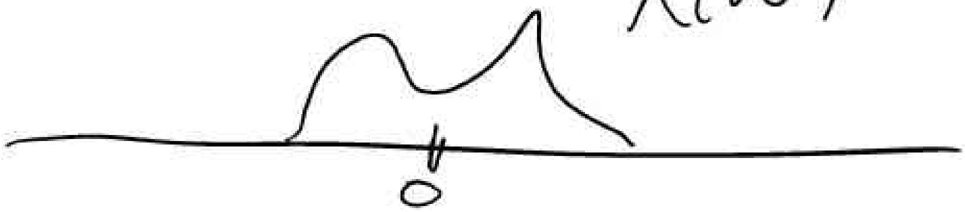
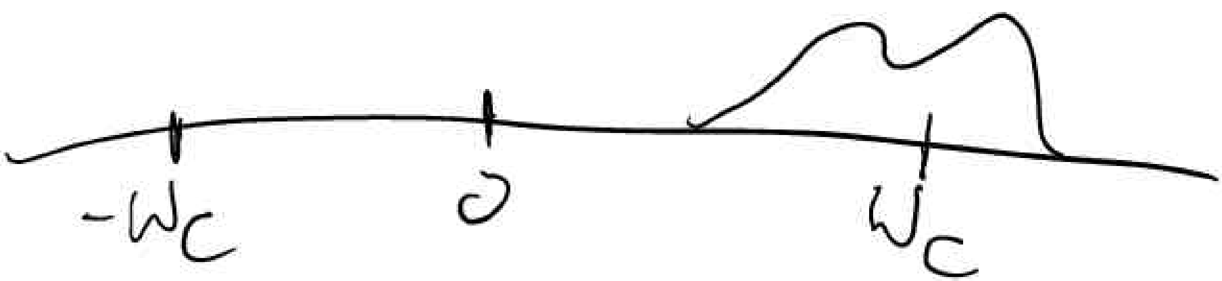
$X(\omega) \leftrightarrow x(t)$



$\downarrow \; x(t)e^{j\omega_c t}$



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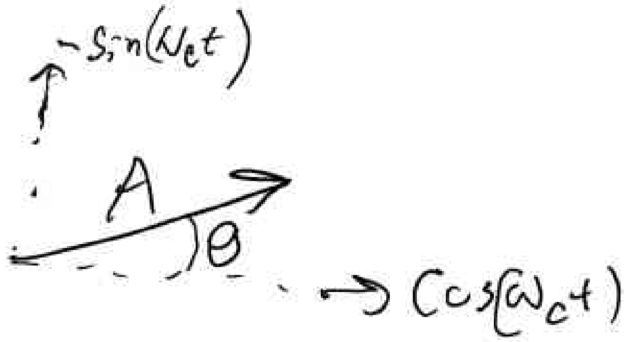
$x(t) = A\cos(\omega_c t + \theta)$

$Ae^{j\theta}$ "Phasor rep"

mult. by $e^{j\omega_c t}$ is implicit

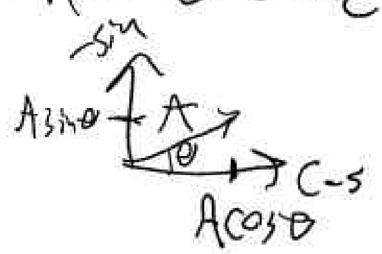


$$x(t) = A\cos(\omega_c t + \theta) = Re\left(Ae^{j(\omega_c t+\theta)}\right)$$

$$= Re\left(Ae^{j\omega_c t}e^{j\theta}\right)$$

$$= Re\left(A(\cos\omega_c t + j\sin\omega_c t)(\cos\theta + j\sin\theta)\right)$$

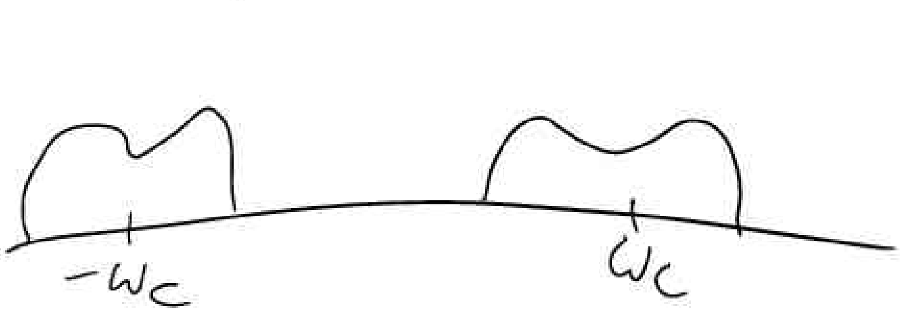
$$= A(\cos\theta\cos\omega_c t - \sin\theta\sin\omega_c t)$$



$$X(t) = X_c\cos\omega_c t - X_s\sin\omega_c t$$

X_c "inphase component"

X_s "quadrature component"



band pass signal

$$x(t) = \underset{a}{A(t)}\cos(\omega_c t + \theta(t))$$

$$x(t) = A(t)\cos(\omega_c t + \theta(t))$$

vary slower than $\omega_c t$

$$x(t) = \mathrm{Re}\left(A(t)e^{j(\omega t+\theta(t))}\right) = A(t)\cos\theta(t)\cos\omega_c t - A(t)\sin\theta(t)\sin\omega_c t$$

$$:= I(t)\cos\omega_c t - Q(t)\sin\omega_c t$$

The "Baseband equivalent"

$$x_{bb}(t) = A(t)e^{j\theta(t)} = I(t) + jQ(t)$$

$$\mathrm{Re}\left(x_{bb}(t)e^{j\omega_c t}\right) = x(t)$$

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The envelope: $A(t) = \sqrt{I^2(t) + Q^2(t)}$

Phase: $\theta(t) = \arctan(Q(t)/I(t))$

Hilbert Transform

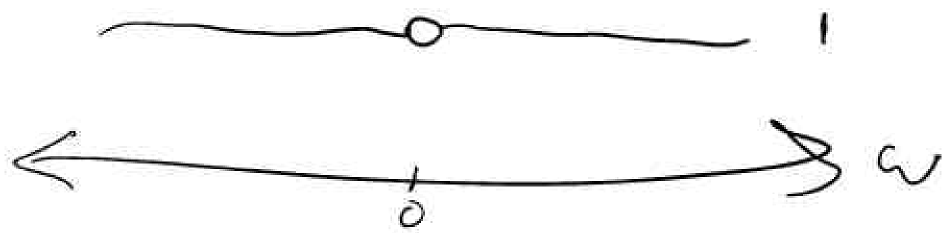
Def. The Hilbert transform of a cts-time signal w/ $X(0)=0$ is written $\hat{x}$, or $\hat{X}$ in freq. domain, and is def. by

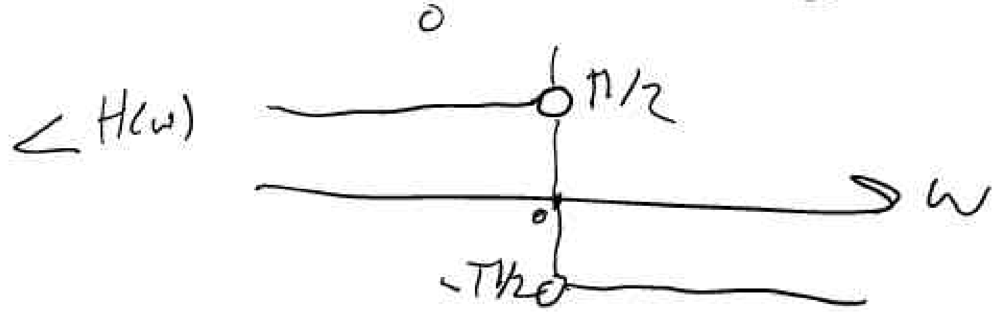
$$\hat{X}(\omega) = \begin{cases} -jX(\omega), & \omega \geq 0 \\ jX(\omega), & \omega < 0 \end{cases}$$

$$\hat{X}(\omega) = X(\omega)H(\omega), \quad H(\omega) = -j\,\mathrm{sgn}(\omega)$$

"sign"

$|H(\omega)|$





$H(\omega) = -j\,\mathrm{sgn}(\omega)$, is $h(t)$ real?

$$H(-\omega) = H(\omega)^*$$

So h is real

$$\mathcal{F}\{\mathrm{sgn}(t)\} = \frac{2}{j\omega}, \quad \mathcal{F}^{-1}\{\mathrm{sgn}(\omega)\} = \frac{-1}{\pi j t}$$

$$\mathcal{F}^{-1}\{-j\,\mathrm{sgn}(\omega)\} = \frac{1}{\pi t} = h(t)$$

TD: $\frac{1}{\pi t} * x$

FD: $-j\,\mathrm{sgn}(\omega)\,X(\omega)$

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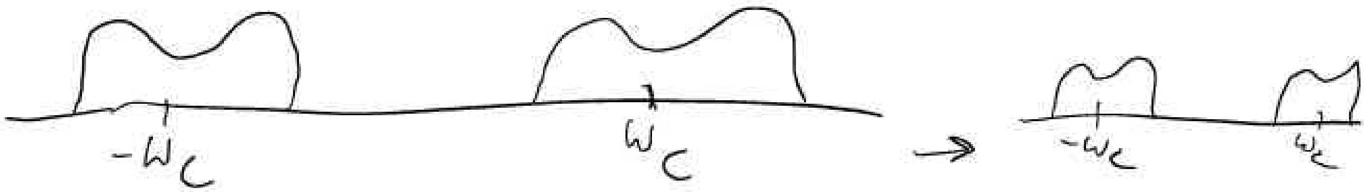
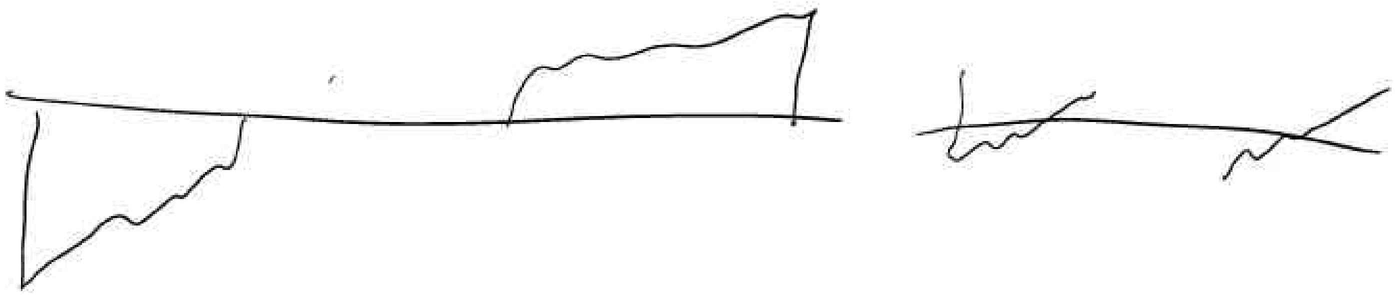
spectrum of a real signal





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Properties of HT

1) H.T. of an odd signal is even
H.T. of an even signal is odd

2) $\hat{\hat{x}}(t) = -x(t)$; $\hat{\hat{X}}(\omega) = (-j\,\mathrm{sgn}(\omega))^2 X(\omega)$
$= -X(\omega)$

3) $\mathcal{E}_{\hat{x}} = \mathcal{E}_x$

$$\int |-j\,\mathrm{sgn}(\omega) X(\omega)|^2 d\omega = \int |X(\omega)|^2 d\omega$$

4) $x \perp \hat{x}$.

$$\int_{-\infty}^{\infty} x(t) y(t) dt = 0$$

$$\int_{-\infty}^{\infty} x(t)\hat{x}(t) dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\hat{X}(\omega)^* d\omega \quad \text{(parseval)}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\left(-j\,\mathrm{sgn}(\omega) X(\omega)\right)^* d\omega$$

$$= \frac{1}{2\pi}\left(-j\int_{-\infty}^{0} |X(\omega)|^2 d\omega + j\int_{0}^{\infty} |X(\omega)|^2 \right)$$

$$= 0$$

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PARSEVAL

$$\int x(t)\, y(t)\, dt = \frac{1}{2\pi} \int X(\omega)\, Y^*(\omega)\, d\omega$$

$$\downarrow$$

$$\int |x(t)|^2\, dt = \frac{1}{2\pi} \int |X(\omega)|^2\, d\omega$$