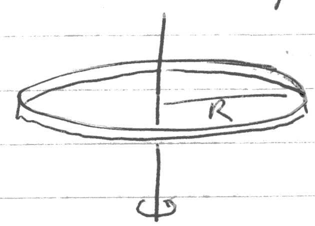
MOMENTS OF INERTIA

Note: If you extrude a shape along the axis, moment of inertia about that axis is the same as the non-extruded shape.

For a point mass: $I = mr^2$ (by definition)

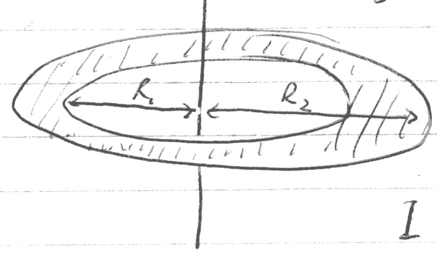
For a thin hoop about central axis
(or a thin cylinder about central axis (extruded thin hoop))



$$I = MR^2$$

(because every particle is R away from the center,
so $I = m_1R^2 + m_2R^2 + m_3R^2 + \dots = MR^2$)

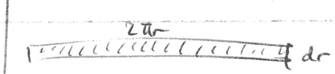
For an annular cylinder about central axis (extruded version)
or an annular ring:



Break it up into rings (hoops) from radius $= R_1$ to radius $= R_2$:

$$I = \int_{R_1}^{R_2} dI_{ring} = \int_{R_1}^{R_2} r^2\,dm$$

$dm = (2\pi r)(dr)\sigma$



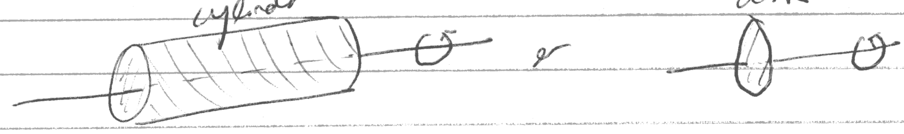
$$= \int_{R_1}^{R_2} r^2(2\pi r)\sigma\,dr = 2\pi\sigma \left.\frac{r^4}{4}\right|_{R_1}^{R_2}$$

$$= \frac{1}{2}\pi\sigma(R_2^4 - R_1^4) = \frac{1}{2}\cancel{\pi}\,\cancel{\frac{M}{\pi(R_2^2-R_1^2)}}\,(\cancel{R_2^2-R_1^2})(R_2^2+R_1^2)$$

$$\sigma = \frac{M}{A} = \frac{M}{\pi(R_2^2 - R_1^2)}$$

$$= \frac{1}{2}M(R_2^2 + R_1^2)$$

For a solid disk or ~~cyl~~ cylinder (~~ext~~ extruded version) about central axis:



Think rings (hoops) from $r=0$ to $r=R$:

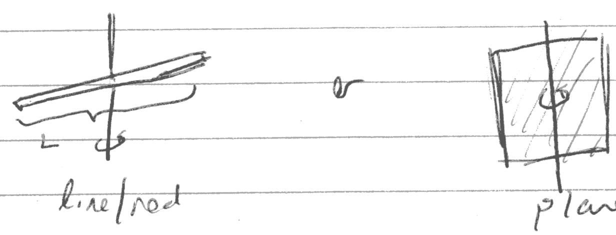
$$I = \int_0^R \underbrace{r^2\,dm}_{dI_{ring}} = 2\pi\sigma \int_0^R r^3\,dr = 2\pi\sigma \frac{r^4}{4}\Big|_0^R$$

$dm = (2\pi r)(dr)\sigma$

$$= \frac{\pi\sigma R^4}{2} = \frac{\cancel{\pi} R^{\cancel{4}2}}{2}\cdot\frac{M}{\cancel{\pi R^2}} = \frac{1}{2}MR^2$$

$\sigma = \frac{M}{A} = \frac{M}{\pi R^2}$

For a thin rod (basically a line) or a plane (extruded version) about a perpendicular line through its center



Integrate over length; each particle has $I = mr^2$

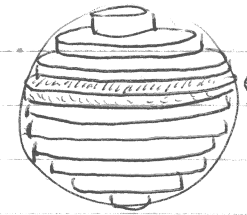
$$I = \int_{-\frac{L}{2}}^{\frac{L}{2}} mr^2 = \int_{-\frac{L}{2}}^{\frac{L}{2}} r^2\,dm = 2\sigma\int_0^{\frac{L}{2}} x^2\,dx = 2\sigma\,\frac{x^3}{3}\Big|_0^{\frac{L}{2}}$$

$r=x$ $\quad dm = \sigma\,dx$ $\quad \sigma = \frac{M}{L}$

$$= \frac{2M}{L}\,\frac{\left(\frac{L}{2}\right)^3}{3} = \frac{\cancel{2}ML^{\cancel{3}2}}{4\cancel{8}\cdot 3\cdot\cancel{L}} = \frac{1}{12}ML^2$$

MOMENTS OF INERTIA, CONT'D.

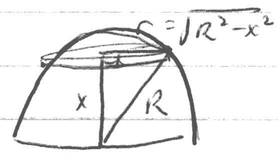
Solid sphere through its center
(using I of a solid disk through its center)



each disk has $I = \frac{1}{2}Mr^2$

add up all the disks' I's

$$I = \int_{-R}^{R} dI = \cancel{2}\int_0^R \cancel{\tfrac{1}{2}}\,dm\,r^2 = \int_0^R \pi\sigma(R^2-x^2)^2\,dx$$



$dm = (\pi r^2)(dx)\sigma$

$= \pi(R^2-x^2)\sigma\,dx$

$r^2 = (R^2-x^2)$

$$= \pi\sigma\int_0^R R^4 - 2x^2R^2 + x^4\,dx = \pi\sigma\left(R^4x - \frac{2R^2x^3}{3} + \frac{x^5}{5}\right)\Big|_0^R$$

$$\sigma = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3} = \frac{3M}{4\pi R^3}$$

$$= \frac{3\cancel{\pi}M}{4\cancel{\pi}R^3}\left(R^5 - \frac{2R^5}{3} + \frac{R^5}{5}\right) = \frac{\cancel{3}MR^{\cancel{5}\,2}}{4\cancel{R^3}}\left(\frac{15-10+3}{\cancel{15}\,5}\right)$$

$$= \frac{8MR^2}{4\cdot 5} = \frac{2}{5}MR^2$$

Solid sphere through its center (using triple cylindrical integrals).

$$I = \int_0^{2\pi} d\theta \int_0^R r \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} r^2\sigma\,dz\,dr$$

$$= 4\pi\sigma\int_0^R r^3 \int_0^{\sqrt{R^2-r^2}} dz\,dr = 4\pi\sigma\int_0^R r^3\sqrt{R^2-r^2}\,dr.$$

Let $r = R\sin\theta$, $dr = R\cos\theta\,d\theta$, $\theta = \sin^{-1}\left(\frac{r}{R}\right)$

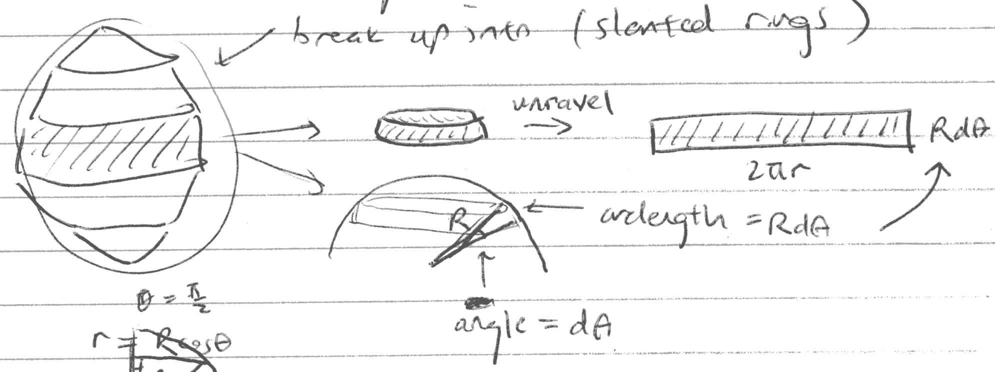
$$= 4\pi\sigma\int_{\sin^{-1}(0)}^{\sin^{-1}\left(\frac{R}{R}\right)} (R\sin\theta)^3\sqrt{R^2-R^2\sin^2\theta}\,(R\cos\theta)\,d\theta = 4\pi\sigma R^5\int_0^{\frac{\pi}{2}}\sin^3\theta\cos^2\theta\,d\theta$$

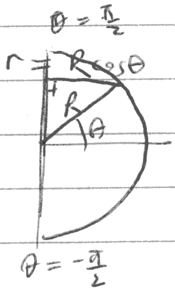
$$= 4\pi\sigma R^5\int_0^{\frac{\pi}{2}}\cos^2\theta\sin\theta - \cos^4\theta\sin\theta\,d\theta = \frac{\cancel{4\pi}R^{\cancel{5}\,2}(3M)}{\cancel{4\pi}\cancel{R^3}}\left(\frac{-\cos^3\theta}{3} + \frac{\cos^5\theta}{5}\right)\Big|_0^{\frac{\pi}{2}}$$

$\sigma = \frac{3M}{4\pi R^3}$ (from top of page)

$$= 3MR^2\left((0+0) - \left(-\frac{1}{3}+\frac{1}{5}\right)\right) = \frac{2}{\cancel{15}\,5}\left(\cancel{3}MR^2\right) = \frac{2}{5}MR^2$$

Hollow sphere (thin) through its center





$$dm = (2\pi r)(R\,d\theta)\sigma = 2\pi R\cos\theta R\sigma\,d\theta = 2\pi\sigma R^2\cos\theta\,d\theta$$

$$r^2 = (R\cos\theta)^2$$

$$I = \cancel{\quad} = \int dI_{hoop} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2\,dm = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} R^2\cos^2\theta\left(2\pi\sigma R^2\cos\theta\right)d\theta$$

$$= 2\cdot R^4\cdot 2\pi\sigma\int_0^{\frac{\pi}{2}}\cos^3\theta\,d\theta = 4\pi\sigma R^4\int_0^{\frac{\pi}{2}}\cos\theta - \sin^2\theta\cos\theta\,d\theta$$

$$\sigma = \frac{M}{A} = \frac{M}{4\pi R^2}$$

$$\frac{\cancel{4\pi}R^{\cancel{4}\,2}\cdot M}{\cancel{4\pi R^2}}\left(\sin\theta - \frac{\sin^3\theta}{3}\right)\Bigg|_0^{\frac{\pi}{2}} = MR^2\left(1-\frac{1}{3}\right) = \frac{2}{3}MR^2$$

Hollow sphere through its center using a surface integral

(jk, haven't learnt these yet :/)

Hoop about any diameter:

Use perpendicular axis theorem.

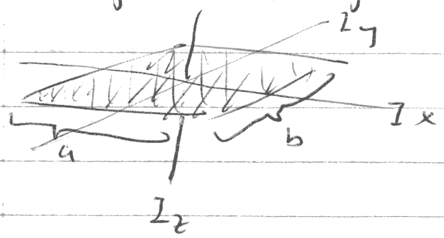
I_z is known: MR^2 (I through center axis of hoop)

$$I_z = I_x + I_y \Rightarrow 2I_x = I_z \Rightarrow I_x = \frac{I_z}{2} = \frac{MR^2}{2}$$

$I_x = I_y$ by symmetry

MOMENTS OF INERTIA, CONT'D.

Slab about perpendicular axis through center,
Using perpendicular axis theorem and I of a plane through axis through its center,



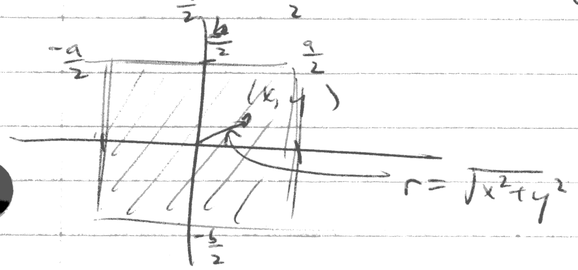
$I_x = \frac{1}{12} M a^2$ (I_{plane} with $L = a$)

$I_y = \frac{1}{12} M b^2$

$I_z = I_x + I_y = \frac{1}{12} M (a^2 + b^2)$

Slab about perpendicular axis through center (using double integral)

$$I = \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \left(\sqrt{x^2+y^2}\right)^2 \sigma \, dy \, dx = 4\sigma \int_0^{\frac{a}{2}} \int_0^{\frac{b}{2}} (x^2+y^2) \, dy \, dx$$



$$= 4\sigma \int_0^{\frac{a}{2}} \left(x^2 y + \frac{y^3}{3}\right)\Big|_0^{\frac{b}{2}} dx$$

$$= 4\sigma \int_0^{\frac{a}{2}} \frac{x^2 b}{2} + \frac{b^3}{24} \, dx$$

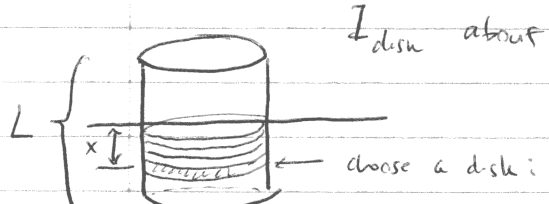
$$= 4\sigma \left(\frac{x^3 b}{6} + \frac{b^3 x}{24}\right)\Big|_0^{\frac{a}{2}} = \cancel{4}\left(\frac{a^3 b}{\cancel{48}_{12}} + \frac{b^3 a}{\cancel{48}_{12}}\right)$$

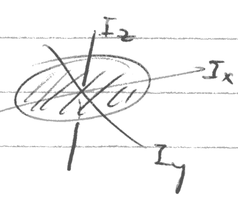
$$= \frac{1}{12} ab (a^2 + b^2) \sigma = \frac{1}{12} \cancel{ab} (a^2+b^2) \frac{M}{\cancel{ab}} = \frac{1}{12} M (a^2+b^2)$$

$$\sigma = \frac{M}{A} = \frac{M}{ab}$$

Solid cylinder or disk $(L=0)$ about central diameter

I_{disk} about its diameter: use perpendicular axis theorem,





$I_x = I_y$, $I_z = \frac{1}{2} M R^2$,

$\Rightarrow I_x = I_y = \frac{1}{4} M R^2$

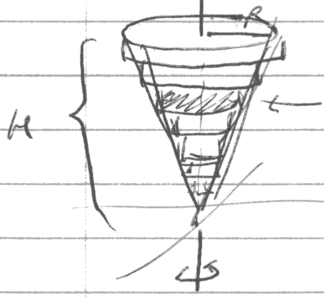
I_{disk} about its own axes is $\frac{1}{4} M R^2$. Using parallel axis theorem, I disk about center of cylinder is $\frac{1}{4} M R^2 + M x^2$, Sum these up: $dm = \pi R^2 \sigma dx$

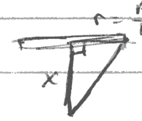
$$I = \int_{-\frac{L}{2}}^{\frac{L}{2}} I_{disk} = 2\int_0^{\frac{L}{2}} I_{disk} = 2\int_0^{\frac{L}{2}} \left(\frac{1}{4}R^2 + x^2\right) dm = 2\pi R^2 \sigma \int_0^{\frac{L}{2}} \left(\frac{1}{4}R^2 + x^2\right) dx$$

$$= 2\pi R^2 \sigma \left(\frac{1}{4} R^2 x + \frac{x^3}{3}\right)\Big|_0^{\frac{L}{2}} = \frac{2\cancel{\pi R^2} M}{\cancel{\pi R^2} L} = \frac{2M}{L}\left(\frac{R^2 L}{8} + \frac{L^3}{24}\right) = \frac{1}{4} M R^2 + \frac{1}{12} M L^2$$

$\sigma = \frac{M}{V} = \frac{M}{\pi R^2 L}$

Solid cone about central axis (using disks)





$r = \frac{R}{H}x$

$$I = \int_0^H dI_{disk} = \frac{1}{2}\int_0^H r^2 dm = \frac{\pi}{2}\sigma\int_0^H r^4\,dx$$

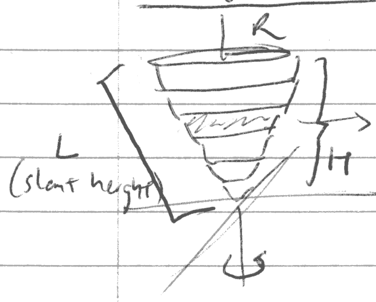
$$dm = (\pi r^2)(dx)\sigma$$

$$= \frac{\pi\sigma}{2}\int_0^H \left(\frac{R}{H}x\right)^4 dx = \frac{\pi\sigma R^4}{2H^4}\left(\frac{x^5}{5}\Big|_0^H\right)$$

$$\sigma = \frac{M}{V} = \frac{3M}{\pi R^2 H}$$

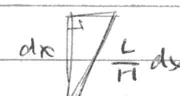
$$= \frac{\cancel{\pi} R^{\cancel{4}\,2}}{5\cdot 2\cancel{H^4}}\left(\frac{3M}{\cancel{\pi R^2 H}}\right)\cancel{H^5} = \frac{3}{10}MR^2$$

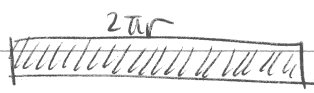
Hollow thin cone about central axis (using hoops/rings).



$r = \frac{R}{H}x$

↓ unravel





$\frac{L}{H}dx$

$$dm = (2\pi r)\left(\frac{L}{H}dx\right)\sigma$$

$$I = \int_0^H r^2 dm = \int_0^H r^2 2\pi r\left(\frac{L}{H}\right)\sigma\,dx = 2\pi\sigma\left(\frac{L}{H}\right)\int_0^H r^3\,dx$$

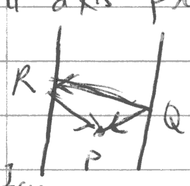
$$= 2\pi\sigma\left(\frac{L}{H}\right)\int_0^H\left(\frac{R}{H}\right)^3 x^3\,dx = 2\pi\sigma\frac{LR^3}{H^4}\left(\frac{x^4}{4}\right)\Big|_0^H = \frac{\cancel{2\pi}\cancel{L}R^{\cancel{3}\,2}}{\cancel{H^4}}\cdot\frac{M}{\cancel{\pi R L}}\cdot\frac{\cancel{H^4}}{\cancel{4}\,2}$$

$$\sigma = \frac{M}{A} = \frac{M}{\pi R L}$$

$$= \frac{1}{2}MR^2$$

Remember ∥ and ⊥ axis theorems.

∥-axis proof: $\vec{QP} = \vec{QR} + \vec{RP}$



$$I_2 = \int(\vec{PQ})^2 dm = \int(\vec{QR} + \vec{RP})^2 dm$$

$$= \int(\vec{QR})^2 dm + 2\int(\vec{QR}\cdot\vec{RP})dm + \int(\vec{RP})^2 dm$$

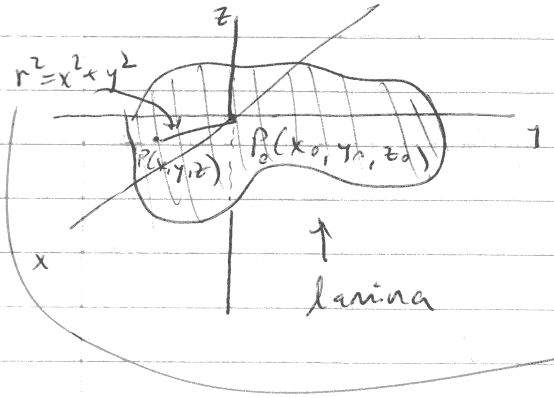
B/c distance b/t axes is fixed, let $\vec{QR}$ be constant h.

$$= h^2\int dm + \cancel{2h\int\vec{RP}\,dm}^{\,0} + \int(\vec{RP})^2 dm$$

$$= h^2 M + I_{cm} \qquad \text{QED.}$$

MOMENT OF INERTIAS, CONT'D.

Proof of perpendicular-axis theorem.

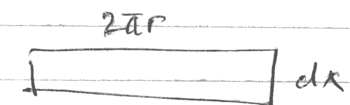


$$I_x = \int x^2\,dm$$
$$I_y = \int y^2\,dm$$
$$I_z = \int r^2\,dm = \int (x^2+y^2)\,dm$$
$$= \int x^2\,dm + \int y^2\,dm$$
$$= I_x + I_y$$

(only works for laminas, and only for three perpendicular axes, two of which lie on the plane, which pass through the same point)

$$I = \int r^2\,dm.$$

$$dm = 2\pi r$$



$$(2\pi r)(dx)\,\sigma$$

$$I = \int_0^L r^3\, 2\pi\sigma\,dx = 2\pi\sigma \int_0^L \left(\frac{Rx}{L}\right)^3 dx$$

$$= 2\pi\sigma\left(\frac{R}{L}\right)^3 \left(\frac{x^4}{4}\right)\Big|_0^L = \frac{\cancel{2}\pi\sigma}{\cancel{4}\,2}\,\frac{R^3}{L^3}\,L^4$$

$$= \frac{\pi\sigma R^3 L}{2}, \quad \sigma = \frac{M}{A} = \frac{M}{\pi R L}$$

$$= \frac{\pi \cancel{\sigma} R^{\cancel{3}\,2} \cancel{L}}{2}\,\frac{M}{\cancel{\pi R L}}$$

$$= \frac{1}{2}R^2 M$$

~~[illegible]~~ I of hollow cone integrating along slant height (and not regular height)

Derivation of Moment of Inertia of a Solid Cone using Spherical Coordinates

$$\int_0^{2\pi} d\theta \int_0^{\arctan\left(\frac{R}{H}\right)} \int_0^{H\sec\phi} \left((\rho\sin\phi)^2 \sigma \left(\rho^2 \sin\phi\right)\right) d\rho\, d\phi$$

$$= 2\pi\sigma \int_0^{\arctan\left(\frac{R}{H}\right)} \sin^3(\phi) \int_0^{H\sec\phi} \rho^4\, d\rho\, d\phi$$

$$= \sigma\frac{2\pi H^5}{5} \int_0^{\arctan\left(\frac{R}{H}\right)} \sin^3(\phi)\sec^5(\phi)\, d\phi$$

$$= \sigma\frac{2\pi H^5}{5} \int_0^{\arctan\left(\frac{R}{H}\right)} \tan^3(\phi)\sec^2(\phi)\, d\phi$$

$$= \sigma\frac{2\pi H^5}{5}\left(\frac{\tan^4\phi}{4}\right)\Bigg|_0^{\arctan\left(\frac{R}{H}\right)} = \frac{\sigma\pi H^5}{10}\left(\frac{R}{H}\right)^4$$

$$\sigma = \frac{M}{V} = \frac{3M}{\pi R^2 H}$$

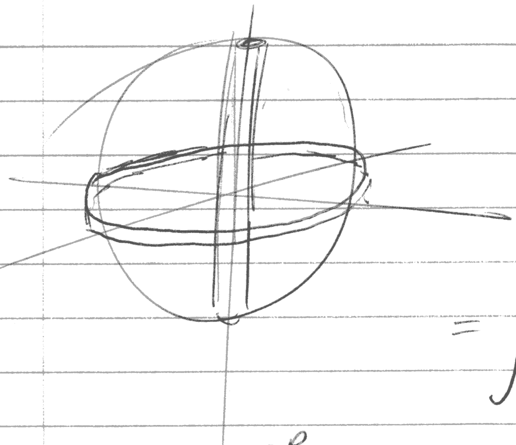
$$= \frac{3M}{\cancel{\pi R^2 H}}\,\frac{\cancel{\pi H^5}}{10}\,\frac{\cancel{R^4}^{\,2}}{\cancel{H^4}} = \frac{3}{10}MR^2$$

3/27/19

Add up thin cylinders to get hollow sphere

$dV = \underset{}{\overset{2\pi r}{\boxed{}}}\, dr \quad h$

$dm = \sigma 2\pi r h\, dr$



$$\int_0^R r^2 (2\pi r h \sigma)\, dr$$

$$h = 2\sqrt{R^2 - r^2}$$

$$= \int_0^R 4\pi r^3 \sqrt{R^2 - r^2}\, \sigma\, dr$$

$$= 4\pi\sigma \int_0^R r^3 \sqrt{R^2 - r^2}\, dr$$

let $r = R\sin A$,
$dr = R\cos\theta\, d\theta$
$\theta = \sin^{-1}\left(\frac{r}{R}\right)$

$$= 4\pi\sigma \int_0^{\frac{\pi}{2}} (R\sin\theta)^3 \sqrt{R^2 - (R\sin A)^2}\, R\cos\theta\, dA$$

$$= 4\pi\sigma R^5 \int_0^{\frac{\pi}{2}} \sin^3\theta \cos^2 A\, dA$$

(this is the same as the triple spherical integrals),

also ~~try adding up cylindrical shells~~ to get solid sphere

$$\int_0^R \frac{2}{3} r^2\, dm \qquad dm = 4\pi r^2\, dr\, \sigma$$

$$= \frac{8}{3}\pi\sigma \int_0^R r^4\, dm = \frac{8}{3}\pi\sigma \frac{R^5}{5} = \frac{8\pi\sigma R^5}{15}$$

$$\sigma = \frac{3M}{4\pi R^3} \qquad = \frac{8\pi R^5}{15} \cdot \frac{3M}{4\pi R^3} = \frac{2}{5}\pi R^3$$

~~$\lim_{\Delta h \to 0} \frac{2}{5}M(R+\Delta h) - \frac{2}{5}MR \to 0$~~

~~$\frac{2}{5}M(R+\Delta h) - \frac{2}{5}MR = \frac{2}{5}M\left(\frac{R+\Delta h}{R}\right)^3 (R+\Delta h) - \frac{2}{5}MR$~~

~~$M = \frac{4}{3}\pi(R+\Delta h)^3 \sigma$~~