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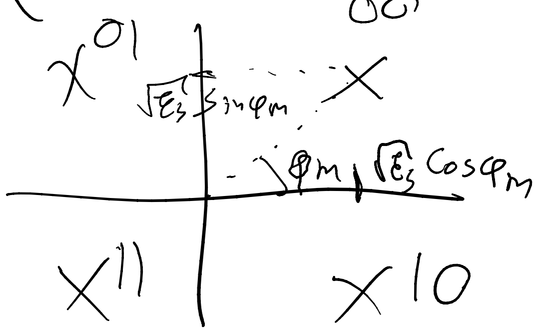
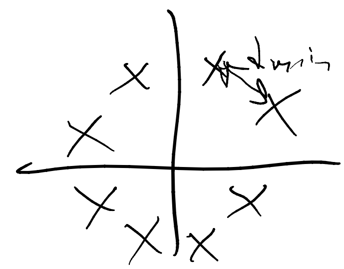
PSK $u_m(t) = p(t)\cos(\omega_c t + \varphi_m)$, $0 \le t \le T_s$

$$\varphi_m = \frac{2\pi m}{M}$$

4-PSK



8-PSK



equal energy, equally spaced

- Gray coding → most common error is 1 bit

$$d_{min} = 2\sqrt{E_s}\sin(\pi/M)$$

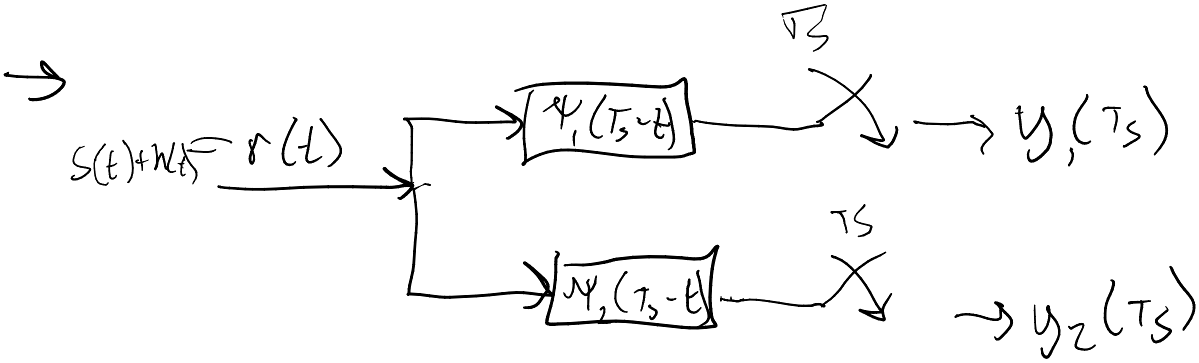
Equal-energy symbols

if equiprobable → we can use Union Bound
(and AWGN)

$\{\psi_1, \psi_2\}$ onb, $\psi_1(t) = \frac{1}{\sqrt{\mathcal{E}_p 2}} \cos \omega_c t$

$\psi_2(t) = \frac{-1}{\sqrt{2\mathcal{E}_p}} \sin \omega_c t$

← I, Q demod



$$y = \begin{pmatrix} \sqrt{E_s} \cos \varphi_m + n_I \\ \sqrt{E_s} \sin \varphi_m + n_Q \end{pmatrix}$$

$$n_I = \int_0^{T_s} n(t) \psi_1(t)\, dt, \quad n_Q = \int_0^{T} n(t) \psi_2(t)\, dt$$

$$\mathbb{E}[n_I] = 0, \quad \mathbb{E}[n_Q] = 0, \quad \mathbb{E}[n_I n_Q] = 0$$

$$y = \begin{pmatrix} \cos \varphi_m \\ \sin \varphi_m \end{pmatrix} + \begin{pmatrix} n_I \\ n_Q \end{pmatrix} = \vec{s} + \vec{n}$$

← 0-mean
covariance?

$\mathbb{E}[|n \ldots|^2] = \mathbb{E}[n^2] = \int^{T_s} \int^{T_s}$

$$\mathbb{E}[(n_I-0)^2] = \mathbb{E}[n_I^2] = \int_0^{T_s}\int_0^{T_s} \mathbb{E}[n(t)n(\tau)]\,\psi_1(t)\psi_1(\tau)\,dt\,d\tau$$

$$= \int_0^{T_s}\int_0^{T_s} \delta(t-\tau)\frac{N_0}{2}\,\psi_1(t)\psi_1(\tau)\,dt\,d\tau$$

$$= \frac{N_0}{2}\int_0^{T_s} \psi_1(\tau)^2\,d\tau$$

$$= N_0/2$$

Same derivation shows $\sigma_Q^2 = N_0/2$

$$\Sigma = \begin{pmatrix} N_0/2 & 0 \\ 0 & N_0/2 \end{pmatrix} = \frac{N_0}{2} I$$

indep. isotropic 2-D Gaussian noise

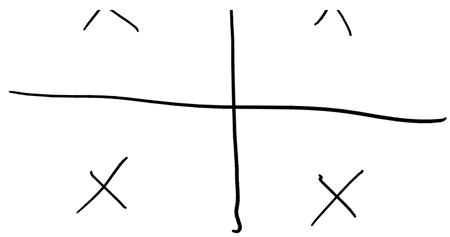
Because every signal has equal energy

ML = LS

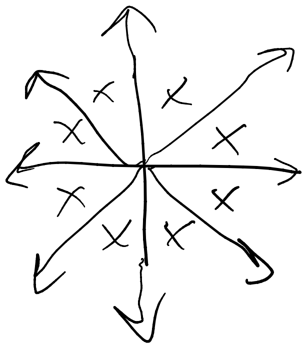
we are fine to <u>just compute</u> the phase of incoming signal

Why? What are the LS boundaries

quadrants



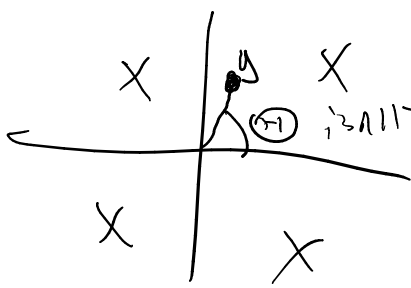
4 quadrants



"LS bdries for PSK are lines of constant phase" (all lines are of constant phase, just important that they're lines)

My LS = ML decision is based only on

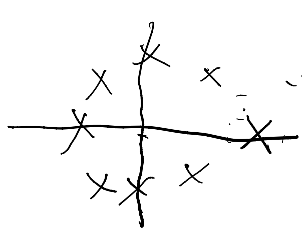
$$\Theta = \tan^{-1}\left(\frac{y_2}{y_1}\right), \quad \text{where } y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\varphi_m + n_I \\ \sin\varphi_m + n_Q \end{pmatrix}$$



Θ is all I care about

P_{error} for Coherent PSK

(using LS)

wlog, say I have  $\varphi_m = 0$ is on the constellation and say I transmit $(\sqrt{\mathcal{E}_s}, 0) \rightarrow$ the symbol w/ $\varphi = 0$

$$y = (\sqrt{\mathcal{E}_s} + n_I, n_Q)$$

$$y \sim \left(N(\sqrt{\mathcal{E}_s}, N_0/2), N(0, N_0/2) \right)$$

indep., so

$$f_y(y_1, y_2) = f_y(y_1) f_y(y_2) = \frac{1}{\pi N_0} e^{-\left((y_1 - \sqrt{\mathcal{E}_s})^2 + y_2^2\right)/N_0}$$

we want $\Theta = \tan^{-1}(y_2/y_1)$, let $R = \sqrt{y_1^2 + y_2^2}$

Jacobian $= R$, $y_1 = R\cos\Theta$, $y_2 = R\sin\Theta$

$$|J| = \begin{vmatrix} \cos\Theta & -R\sin\Theta \\ \sin\Theta & R\cos\Theta \end{vmatrix} = R$$

$$f_{R,\Theta}(r, \theta) = \frac{r}{\pi N_0} \exp\left(-(r^2 + \mathcal{E}_s - 2\sqrt{\mathcal{E}_s}\, r\cos\theta)/N_0\right)$$

we don't care about r, what is $f_\Theta(\theta)$?

$\int^{\infty}$

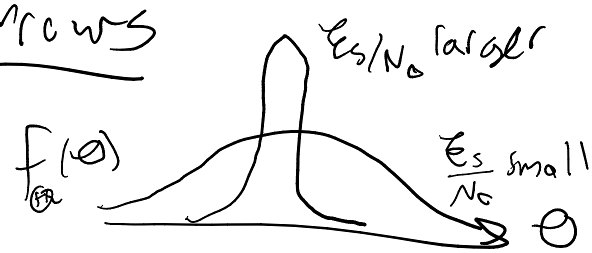
$$f_{\Theta}(\theta) = \int_0^\infty f_{R,\Theta}(r,\theta)\, dr$$

trust

$$= \frac{1}{2\pi} e^{-\frac{\mathcal{E}_s}{N_0}\sin^2\theta} \underbrace{\int_0^\infty r \exp\left(-\left(r - \sqrt{\frac{2\mathcal{E}_s}{N_0}}\cos\theta\right)^2 / 2\right) dr}_{\text{depends on } \theta \text{, unfortunately}}$$

can solve numerically

but - we can see that as $\frac{\mathcal{E}_s}{N_0} = SNR_s \nearrow$, distribution narrows



Your book - gives an approx. for P_{error} for PSK for arbitrary M

$M=2$: $Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$ just binary antipodal

#symbols

$M=4$: 2 independent binary antipodal

$$P_{error}^{M=4} = 1 - \left(1 - P_{error}^{M=2}\right)^2 =$$

$$P_{error} = 1 - (1 - P_{error}) =$$

$$2\,Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\left(1 - \frac{1}{2} Q\left(\sqrt{\frac{2\mathcal{E}_s}{N_0}}\right)\right)$$

generally, very complicated

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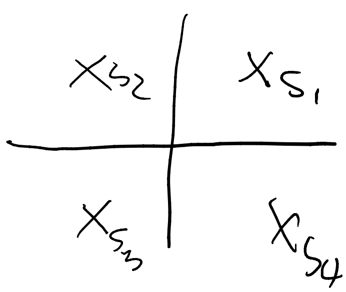
Not done w/ PSK yet

— It's a problem that we need coherence
huge waste of power

Solution: DIFFERENTIAL Phase encoding

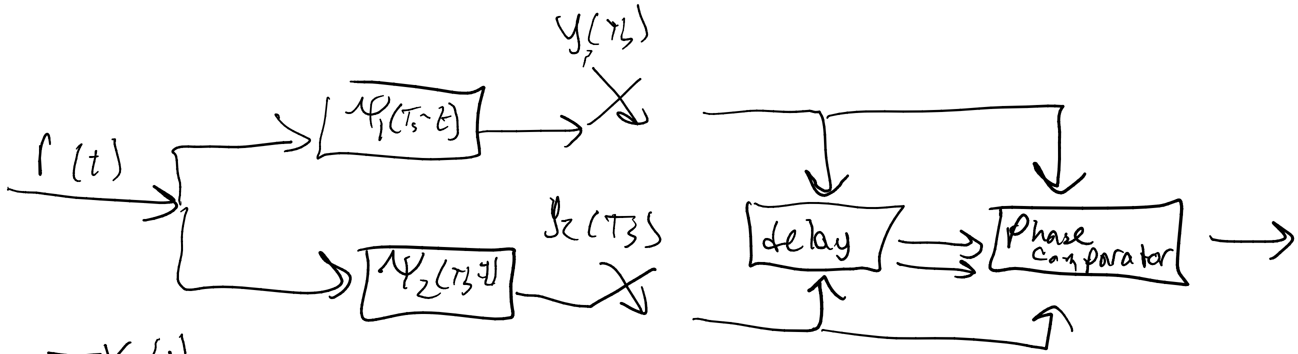
instead of encoding info in φ_m for a given symbol
encode the info in $(\varphi_m - \varphi_{m-1})$ the phase shift



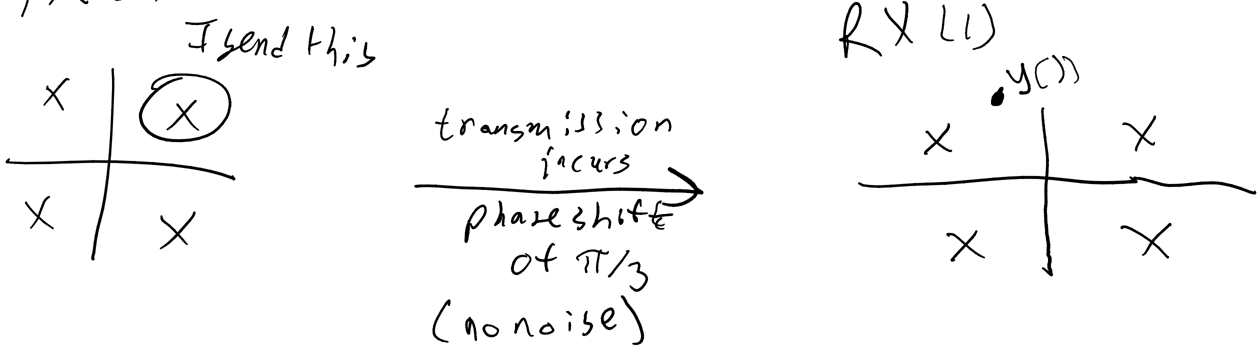
$S_1 \to S_2 \quad \Delta\varphi = \pi/2$
$S_2 \to S_3 \quad \Delta\varphi = \pi/2$
same bit-string being sent

$S_1 \to S_3 \quad \Delta\varphi = \pi$
$S_3 \to S_1 \quad \Delta\varphi = \pi$
same as well

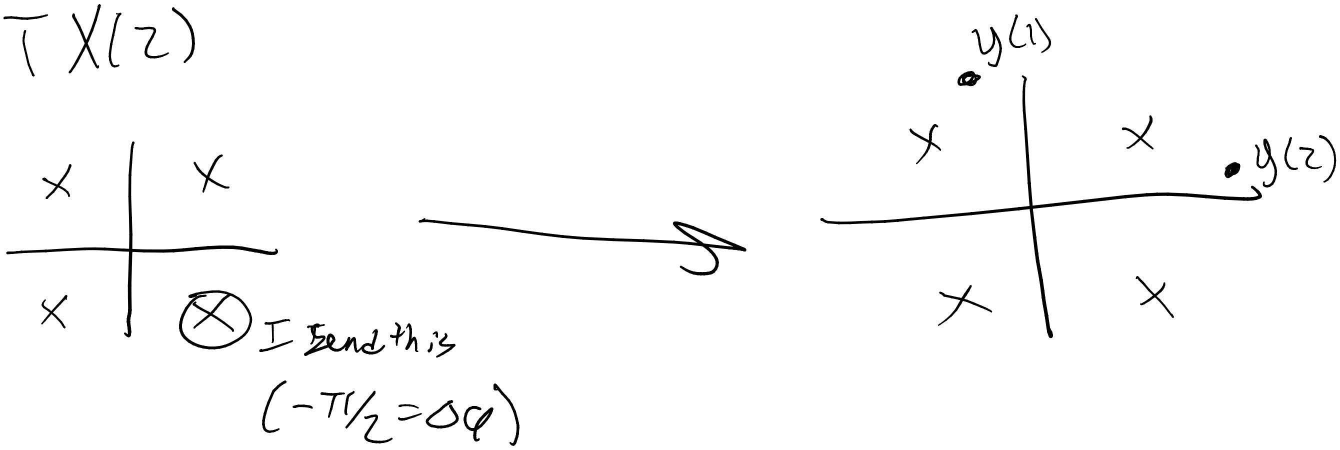
For M-DPSK (M stars in a PSK constellation, differential encoding) there are M possible phaseshifts → M $\Delta\varphi$ symbols



TX(1)



TX(2)



$$\angle y(2) - \angle y(1)$$

$$= -\pi/2 = \angle s(2) - \angle s(1)$$

even noncoherently, this scheme works.

- It took me 2 transmissions to send 1 data string
- It will take me N+1 transmissions to send N data strings

P_{error} — see book (if you dare)

~ little bit worse than PSK in terms of error

but you save on power

$$\begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix} \to \begin{pmatrix} H_1 \\ \vdots \\ H_n \end{pmatrix} \xrightarrow{diff(v)} \begin{pmatrix} H_2 - H_1 \\ \vdots \\ H_n - H_{n-1} \end{pmatrix}$$

LS on this

H closer to which?

1 smaller than $\vec{H}$

QAM

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Quadrature Amplitude Modulation

The culmination of I/Q constellations

$$u_m(t) = A_m p(t) \cos w_c t - B_m p(t) \sin w_c t$$

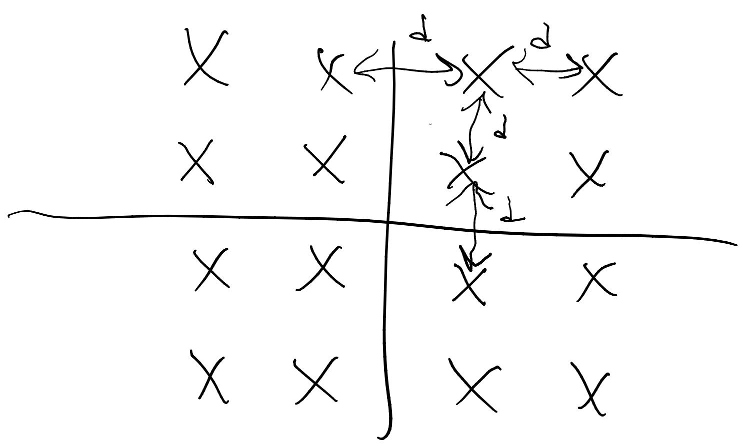
$$\psi_1(t) = \sqrt{\frac{1}{\mathcal{E}_s}}\, p(t) \cos w_c t \qquad \psi_2(t) = -\sqrt{\frac{1}{\mathcal{E}_s}}\, p(t) \sin w_c t$$

$\{(A_m, B_m),\ m = 1, \dots, M\}$ correspond to bit sequences

M pairs, M symbols

QAM allows any shape, but almost always means square, equidistant "nearest neighbors"

16-QAM



here, Θ and R both matter, not equal energy

can have very complicated shape

$\#$ distinct values of $A_m = M_A$

$\#$ distinct values of $B_m = M_B$

rectangular QAM: $M = M_A M_B$

encode $\log_2 M$ bits

$\log_2 M_A$ in phase, $\log_2 M_B$ in quadrature

technically: binary antipodal, orth, PSK are all QAM special cases

WIFI usually uses 64-QAM (square)

For square M-QAM:

avg energy per bit

$$P_{error} \leq 4Q\left(\sqrt{\frac{3\,\mathcal{E}_{bar}\,\log_2 M}{(M-1)N_0}}\right)$$

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Couple of things we've ignored:

next 1 lectures

1) what is pulse $p(t)$? How do we pick a good one?

2) What is going on in the channel? (intersymbol interference)

last 6 lectures

3) How do we encode info we're sending

Channels

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we've more or less ignored channels
aside from +AWGN

$$TX \xrightarrow[s(t)]{+n(t)} r(t) = s(t) + n(t) \; RX$$

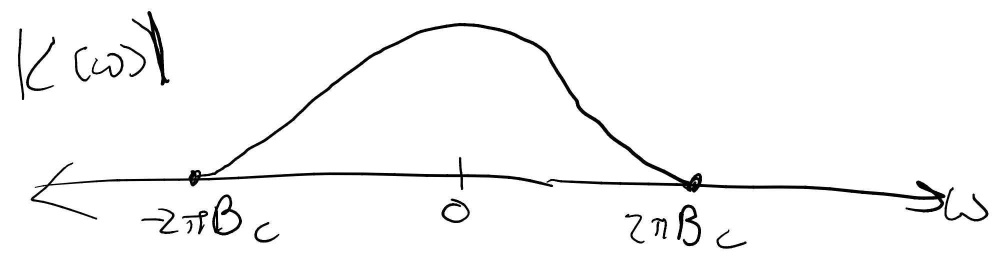
really ... not true at all

<u>a more realistic model</u> (channel is a LTI system)

$$TX \xrightarrow{g(t)} \boxed{\text{Channel } C(\omega)} \xrightarrow{h(t)} \bigoplus \rightarrow r(t)$$

$$\uparrow n(t)$$

maybe bandlimited channel



+ Phaseshift (be a problem in PSK, QAM)

Issue of bandlimited channels:

- If I send a signal thru a band limited channel the output signal is bandlimited

$g(t) \longrightarrow \boxed{C(\omega)} \longrightarrow h(t) \leftarrow$ bandlimited $\Rightarrow$ infinite in time

Compact in time $\longrightarrow$ band unlimited in frequency

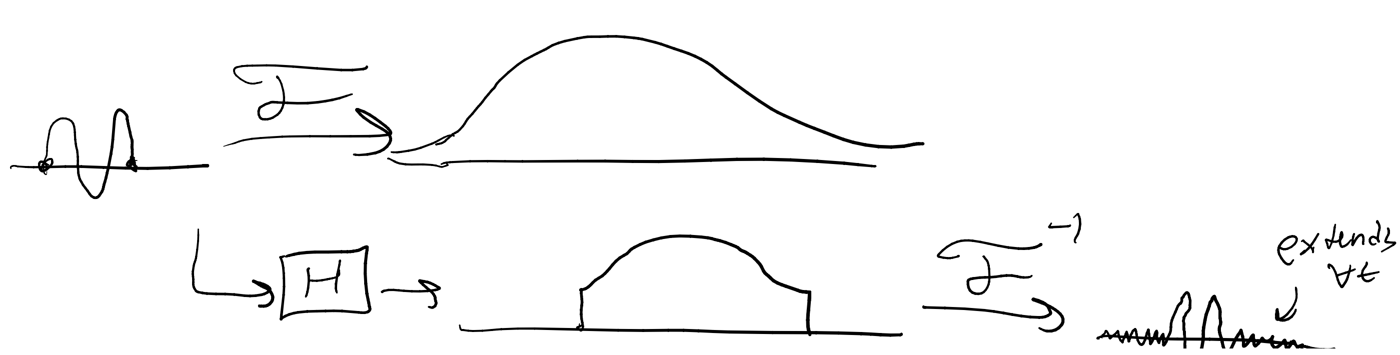
(compact) bandlimited in freq $\longrightarrow$ infinite in time

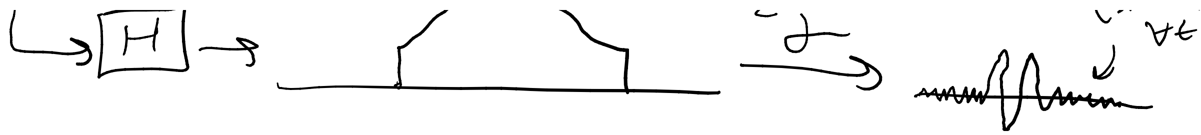
I send $A_n p(t)\cos\omega_c t - B_n p(t)\sin\omega_c t,\ 0 \le t \le T_s$

This is a finite-time (band unlimited) symbol

W/ no channel $\rightarrow$ symbols exist at separate time-intervals

W/ bandlimited channel $\rightarrow$ every symbol is infinite in time





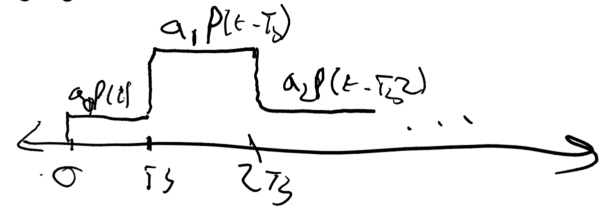
Develop a theory to handle this issue we've ignored

Ex. I send M-ary PAM signals $\{s_i = a_i\, p(t)\}_{i=1}^{M}$

$p(t)$ is defined on $(0, T_s)$

a sequence of symbols $v(t)$ can be written as

$$v(t) = \sum_{n=-\infty}^{\infty} a_n\, p(t - nT)$$



now apply channel $c(t)$, and AWGN

Call $p(t) * c(t) = h(t)$

then

$$r(t) = v(t) * c(t) + n(t)$$

$$= \sum_{n=-\infty}^{\infty} a_n\, h(t - nT) + n(t)$$

not constrained to $(nT, (n+1)T)$

it's like PAM with $s_i = a_i h(t)$ instead of $s_i = a_i p(t)$

issue is $h(t)$ is not compact in time!

issue is $h(t)$ is not compact in time!

If we know the channel, then we know $c(t)$

so we know $p(t) * c(t) = h(t)$

so we can define a matched filter using h rather than p (this makes more sense)

$$r(t) \longrightarrow \boxed{h(-t)} \longrightarrow y(t) = \left(r(t) * h(-t)\right)$$

$$= \left(\sum a_n h(t) * h(-t) + n(t) * h(-t)\right)$$

define $X(t) = h(t) * h(-t)$

$w(t) = n(t) * h(-t)$

$$\text{so } y(t) = \sum_{n=-\infty}^{\infty} a_n X(t - nT) + w(t)$$

$$y(mT) = \sum_{n=-\infty}^{\infty} a_n X(mT - nT) + w(mT)$$

now - if X were $p * p$ (like in no-channel case)

then $p*p(mT - nT) = 0$ if $m \neq n$

then $p * p(mT-nT) = 0$ if $m \neq n$

in compact time case — totally avoid intersymbol interference, but here

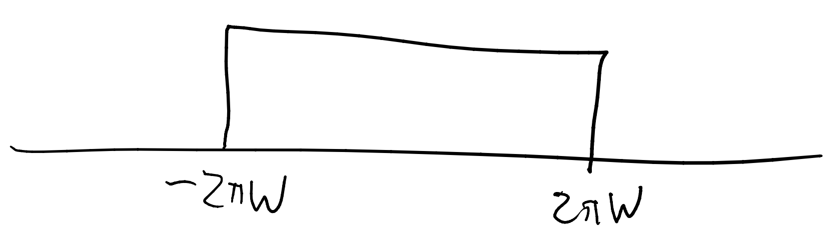
$$y(mT) = y_m = \underbrace{a_m X_0}_{\text{Signal we want}} + \underbrace{\sum_{n \neq m} a_n X_{m-n}}_{\substack{\text{ISI} \\ \text{(intersymbol interference)}}} + \underbrace{w_m}_{\text{AWGN}}$$

How are we going to avoid ISI

answer: We need to understand the Channel

Simplification: Let

$$|C(\omega)| = 1, \quad |\omega| \leq 2\pi W$$



just a LPF

~linear phase, const magnitude

in reality, most these things look more like

$|C(\omega)|$ well-approximated in most cases by a rect filter

$$y_m = a_m x_0 + \sum_{n \neq m} a_n x_{m-n} + w_m$$

What we want is $x_{m-n} = 0 \;\; \forall n \neq m$

Zero ISI

of course, also want $x_0 \neq 0$

wlog, let's just say $x_0 = 1$

Need to design $\underline{p(t)}$ so that this is true

as $x(t) = [p(t) * c(t)] * ((p(-t) * c(-t))$ no control over $c(t)$

$$x_{m-n} = x(mT - nT)$$
$$x_0 = x(0)$$

$\rightarrow$ desire $x(nT) = \begin{cases} 1, & n=0 \\ 0, & \text{else} \end{cases}$

Nyquist Criterion for Zero ISI

Theorem $x(nT) = \begin{cases} 1, & n=0 \\ 0, & n\neq 0 \end{cases}$

if and only if $X(\omega)$ satisfies

$$\sum_{m=-\infty}^{\infty} X\left(\omega + \frac{2\pi m}{T}\right) = T$$

indep. of ω

Proof.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

So

$$x(nT) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega nT}\, d\omega$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\pi/T}^{\pi/T} X\left(\omega + \frac{2\pi m}{T}\right) e^{j\omega nT}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} \sum_{m=-\infty}^{\infty} X\left(\omega + \frac{2\pi m}{T}\right) e^{j\omega n T} \, d\omega$$

let

$$Z(\omega) = \sum_{m=-\infty}^{\infty} X\left(\omega + \frac{2\pi m}{T}\right)$$

$$x(nT) = \int_{-\pi/T}^{\pi/T} Z(\omega) e^{j\omega n T} \, d\omega$$

Want to show $x(nT) = \begin{cases} 1, & n=0 \\ 0, & n \neq 0 \end{cases} \Longleftrightarrow Z(\omega) = T$

$Z(\omega)$ is clearly $\frac{2\pi}{T}$ periodic, so we can use Fourier series

$$Z(\omega) = \sum_{k=-\infty}^{\infty} z_k e^{j\omega k T}, \quad z_k = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} Z(\omega) e^{-j\omega k T} \, d\omega$$

from earlier $\quad x(nT) = \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} Z(\omega) e^{j\omega n T} \, d\omega$

So $\quad T x(-nT) = \frac{T}{2\pi} \int^{\pi/T} Z(\omega) e^{-j\omega n T} \, d\omega$

So $\quad TX(-nT) = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} Z(\omega) e^{-j\omega nT} d\omega$

$= z_n$

i.e. $\quad Tx(-nT) = z_n$

So for $\quad X(nT) = \begin{cases} 1, & n=0 \\ 0, & \text{else} \end{cases}$, need $\quad z_n = \begin{cases} T, & n=0 \\ 0, & n \neq 0 \end{cases}$

that means O-ISI $\Leftrightarrow$ $Z(\omega) = \sum_{n=-\infty}^{\infty} z_n e^{j\omega nT} = T$ //

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What does this look like?

suppose $|C(\omega)| = \begin{cases} 1, & |\omega| \le 2\pi W \\ 0, & \text{else} \end{cases}$

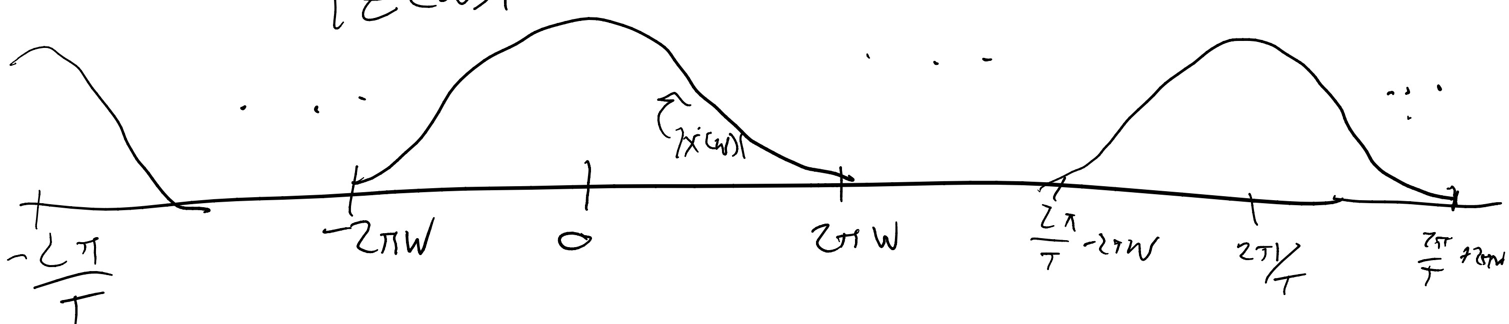
then $X(\omega) = 0 \;\; \forall |\omega| > 2\pi W$ as well

$$Z(\omega) = \sum_{m=-\infty}^{\infty} X\left(\omega + \frac{2\pi m}{T}\right)$$

three cases

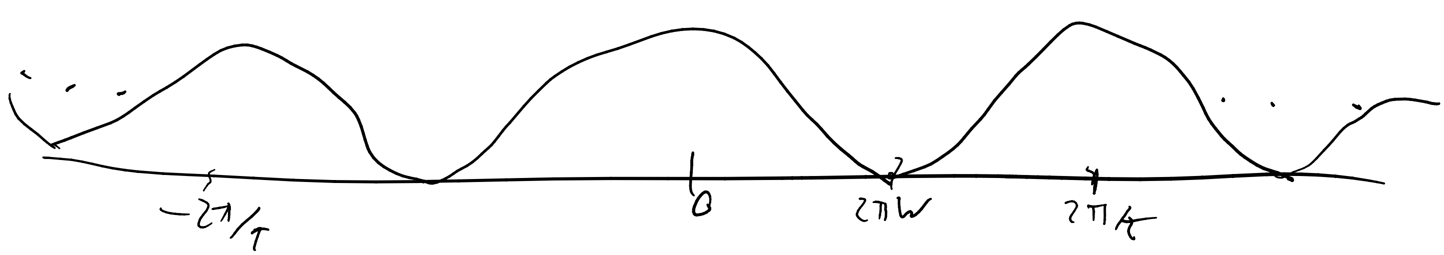
1. $T < \frac{1}{2W}$ no overlap

$|Z(\omega)|$



Clearly $Z(\omega)$ not constant in this case

2. $T = 1/2W$



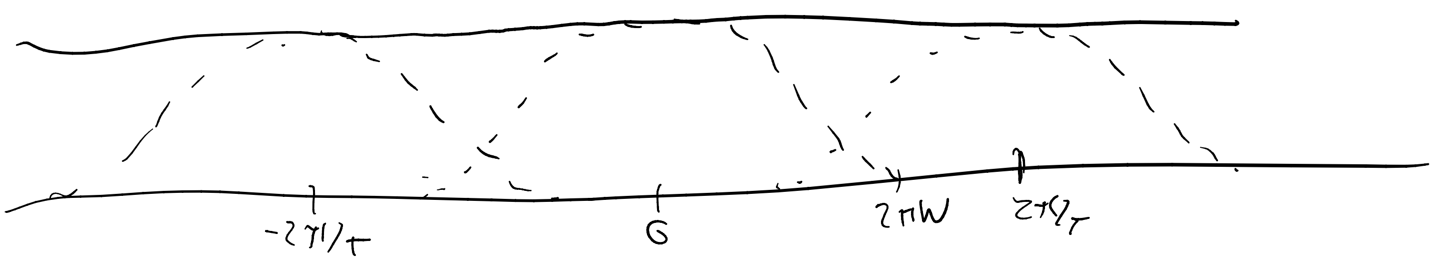
$X(\omega) = \{ T, |\omega| \le 2\pi W$

$Z(\omega)$ can only be constant if $X(\omega) = \begin{cases} T, & |\omega| \le 2\pi W \\ 0, & \text{else} \end{cases}$

true if $x(t) \propto \text{sinc}(t)$

noncausal, and infinite
not realizable

3. $T > \frac{1}{2W}$



must add up to a constant

there are many such signals but must design so that this is true

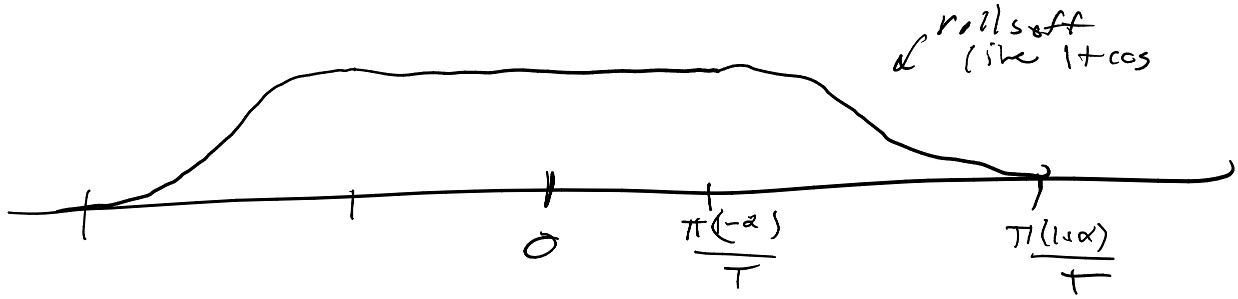
Pulse Shaping

Designing/chosing $p(t)$ so that $Z(\omega) = T \;\; \forall \omega$

$\rightarrow$ so that no ISI occurs

Raised Cosine is a common choice

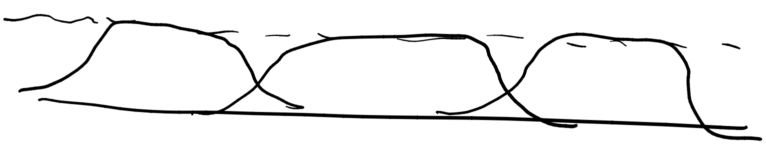
$$X_{rc}(\omega) = \begin{cases} T, & 0 \le |\omega| \le \pi(1-\alpha)/T \\ \frac{T}{2}\left(1+\cos\left(\frac{T}{2\alpha}\left(|\omega| - \pi\frac{1-\alpha}{T}\right)\right)\right), & \pi\frac{1-\alpha}{T} \le |\omega| \le \pi\frac{1+\alpha}{T} \\ 0, & |\omega| > \pi(1+\alpha)/T \end{cases}$$



$\alpha \equiv$ "rolloff factor" $0 \le \alpha \le 1$

larger α is, the larger the "rolloff region" is

at $\omega_s = \pi/T$



we need Nyq. zero ISI

Can show: $x_{rc}(t) = \text{sinc}\left(\frac{\pi t}{T}\right) \frac{\cos(\pi\alpha t/T)}{1-4\alpha^2 t^2/T^2}$

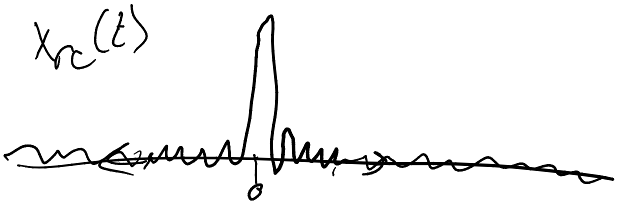
issue: compact in freq $\rightarrow$ infinite in time

(dies off way faster than sinc
"less infinite" than sinc)

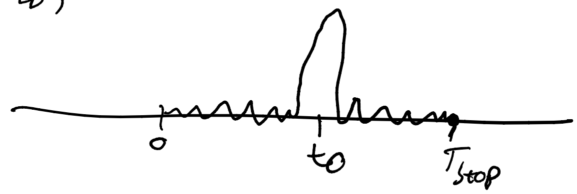
technically unrealizable

Also $\rightarrow$ [illegible] realize have to truncate also → noncausal, so to realize, have to truncate and shift in time by t_0

$x_{rc}(t)$



$\tilde{x}_{rc}(t-t_0)$



X is received filter * pulse * chanel

$x(t) = x_{rc}(t)$

if $\boxed{P(\omega) = \sqrt{X_{rc}(\omega)}\, e^{-j\omega t_0}}$

↑ how we pulseshape

match filter: $\sqrt{X_{rc}(\omega)}\, e^{j\omega t_0}$

Other ways of dealing w/ ISI

1. Allow it, and sample less frequently

   (assumption: x_{n-m} largest n near m

   So less affected if you only sample every ~ 2 timesteps)

2. Allow ISI but design pulse so that you know what ISI looks like, call controlled ...

2. Allow ISI but design pulse so that you know what ISI looks like, subtract it out

3. Allow ISI, statistically learn the form of either channel / ISI difference eqn.

Information Theory

Tuesday, October 27, 2020 5:32 PM

I talk about "information" a lot in this class already
— qualitatively

How do we quantify information?

- How important is any 1 bit? Is this bit more important than any other bit?
- Is there redundancy in my encoding scheme? (is that necessarily bad?)
- Can I send the same information in fewer bits

"I am laughing out loud" 22 chars

"lol" 3 chars

"" 1 UTF8 character

Developing Mathematical Information

1) Unlikely events contain more information than likelier ones

→ Information of an event is a monotonically decreasing function of the prob. of that event: $I = I(p)$ monotonically decreasing.

2) Two independent events should have information which is the sum of the information in each individual event.

if A, B indep, the $P(A \cap B) = P(A)P(B) = P_A P_B$

$$I(P_a P_b) = I(P_a) + I(P_B)$$

only the logarithm has this property

3) Information is positive

$$\boxed{I(x) = -\log x}$$

where x is a probability

or, in event notation for event A

$$\boxed{I[A] = -\log P[A]}$$

This log can be in any base. If we chose base

2: we say information has units bits

e: " " nats

easy conversion by $\log_a x = \frac{\log_b x}{\log_b a}$