Jonathan Lam

Circuit Analysis Exam 2

Professor: Brian Frost

Fall 2019

1 Theoretical Questions (3 points each)

1. Write a brief argument using only Kirchoff's laws for the equivalent capacitance formula for N capacitors in parallel.

2. Write a brief argument using only Kirchoff's laws for the equivalent inductance formula for N inductors in parallel.

3. Suppose I have a complex linear two-terminal circuit with open circuit output voltage V. I load it with a resistor with resistance R, and measure a voltage V_0 across the resistor. What is the Thevenin resistance?

4. Consider a source-free RC circuit with $v(0) = V_0$. Derive the voltage as a function of time for $t > 0$.

5. As C grows large, what device does a capacitor behave like over short time intervals? Provide a brief explanation.

6. Consider the circuit in Figure 1. The voltage source has two clear components – a DC and AC component. Draw the two circuits corresponding to a superposition-based analysis of this circuit using a DC/AC breakdown of the voltage source.

7. Solve for the voltage across the resistor in the DC component circuit from the previous question. What role does the DC component play in the solution to the entire system?

8. Suppose a circuit contains an inductor which is known to have a periodic voltage across it. What will happen (over time) to the inductor current if the average voltage over a period is greater than 0? Equal to 0? Less than 0?

9. Describe the principal of maximum power transfer for two-terminal linear circuits – what is the maximum power transfered, and to what object is it transfered?

10. Provide one "pro" and one "con" about each of overdamped, underdamped and critically damped responses. That is, give me one reason each response is desirable and one reason each is not.

11. Consider an LC tank with initial current 0 and initial voltage V_0. Derive the frequency at which the device resonates. Will the system, theoretically, resonate forever? Realistically?

2 Computational Questions (33 points total)

12. (14 points) Consider the circuit in Figure 2. Find a differential equation which determines the current through inductor L_2, and come up with parameter regions in which the circuit is underdamped, critically damped and overdamped.

13. (14 points) Consider the circuit in Figure 3. The terminals across which V_{out} is defined are, naturally, the output terminals of this circuit. Using the matrix methods discussed in class, provide two matrix equations which will allow me to find V_{TH} and R_{TH}. Show me not just these equations, but how, once solved, I can find these Thevenin equivalent values.

14. (5 points) Consider the circuit in Figure 3 again. Draw the circuits corresponding to a superposition analyisis of this circuit and explain how I would use superposition to solve for the voltage across R_3 if the circuit is not loaded. Just explain – no need to write any matrix equations.

3 Expository Example (34 points total)

15. (3 points) In this problem we will discuss the buck converter – a DC-to-DC converter circuit. Imagine a circuit such as that shown in Figure 4, where the switch is open or closed dependent on a periodic function of time $f(t)$. Assume the period of the function is T. For $t \in (0, T_{ON})$, the switch is closed, and for $t \in (T_{ON}, T)$ the switch is open. Draw the current through the resistor as a function of time over two periods and label the maximum current with a value.

16. (3 points) The average power absorbed by a circuit element is defined by

$$P = \frac{1}{T}\int_0^T p(t)\,dt,$$

where $p(t)$ is the instantaneous power absorbed by the element. For the scenario in the previous question, find the average power consumed by the resistor. How does it differ from the average power absorbed by a resistor if the switch had been closed for all time? Write your answer in terms of the *duty cycle* $D = T_{ON}/T$.

17. (4 points) Now consider the significantly more complex circuit in Figure 5. This circuit is called a *buck converter.* The switch S1 will operate just as the switch in the previous problems, and switch S2 will behave in exactly the opposite way – that is, switch S2 is open for $t \in (0, T_{ON})$ and closed for $t \in (T_{ON}, T)$, and also periodic with period T. Look at this circuit! Let v_L be the voltage acros the inductor, and v_C be the voltage across the capacitor. Draw at least two periods of $v_C + v_L$. (hint: this requires no math)

18. (8 points) Consider now only the period from 0 to T. Draw the two circuits corresponding to the "on" time period and the "off" time period for switch S1. Each of these circuits are second order, but given that the time periods over which they are defined are finite, they will never reach their steady state conditions. Still, what are their steady state conditions? How do their initial and final (meaning at $t = 0$, T_{ON} and T) conditions relate?

19. (8 points) The intended operation of a buck converter is that the capacitor voltage v_C is approximately constant at a voltage V_0. Assume that is in fact the case. Given this fact, find and draw the voltage across the inductor in time for one period. Given some initial current I_0, find and sketch the current in time as well over one period.

20. (2 points) From your math and your drawing it should be easy to tell the amount that I changes during the "on" time period and "off" time period – call these values Δi_{ON} and Δi_{OFF}. If I tell you the inductor current is *also* periodic with period T (which it must be), what is the relationship between Δi_{ON} and Δi_{OFF}?

21. (3 points) Using this relationship, tell me what the relationship between V_0 and V is in terms of D and circuit component values only.

22. (3 points) Amazingly, this is independent of some parameters. This is principally due to our assumption that the output voltage is approximately constant. What does this assumption correspond to in terms of circuit component values?

Jonathan Lam

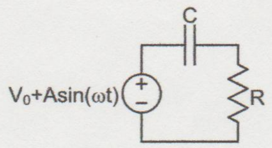


Figure 1: A sourced RC circuit.

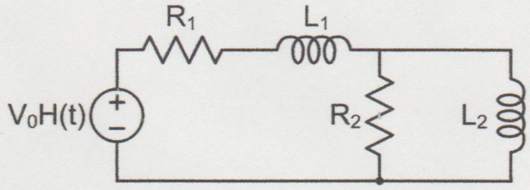


Figure 2: A second order circuit.

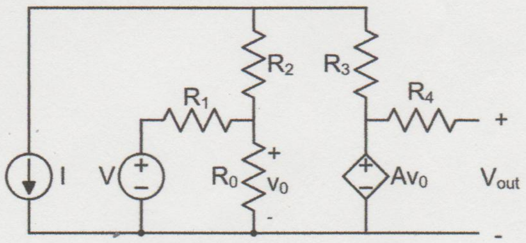


Figure 3: Linear two-terminal circuit.

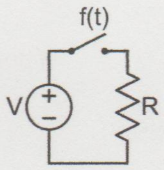


Figure 4: Resistive network with timed switching.

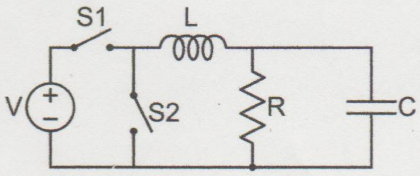
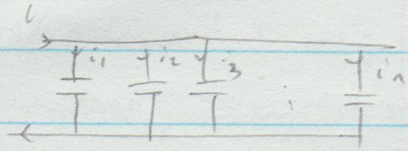


Figure 5: The buck converter.

(1)



$$i = i_1 + i_2 + \dots + i_n$$

all same voltage b/c same nodes

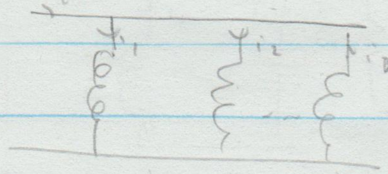
$$= C_1\frac{dv}{dt} + C_2\frac{dv}{dt} + \dots + C_n\frac{dv}{dt}$$

$$= (C_1 + C_2 + \dots + C_n)\frac{dv}{dt}$$

$$= (C_{eq})\frac{dv}{dt} \Rightarrow C_{eq} = C_1 + C_2 + \dots + C_n$$

(2)

-1



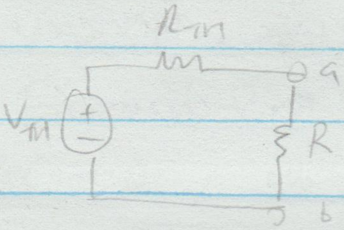
$$i = i_1 + i_2 + \dots + i_n$$

$$= \frac{1}{L_1}\int v\,dt + \dots + \frac{1}{L_n}\int v\,dt$$

$$= \left(\frac{1}{L_1} + \frac{1}{L_2} + \dots + \frac{1}{L_n}\right)\int v\,dt$$

$$= \left(\frac{1}{L_{eq}}\right)\int v\,dt \Rightarrow L_{eq} = \frac{1}{\frac{1}{L_1} + \frac{1}{L_2} + \dots + \frac{1}{L_n}}$$

(3)



$$V_{Th} = V$$

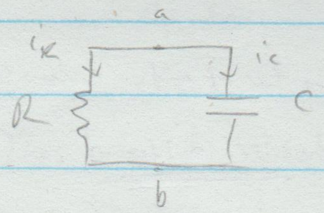
$$V_{ab} = V_0$$

voltage divider: $V\frac{R}{R+R_{Th}} = V_0$

$$\frac{V_0}{V} = \frac{R}{R+R_{Th}} \Rightarrow (R+R_{Th}) = \frac{RV}{V_0}$$

$$\Rightarrow R_{Th} = R\left(\frac{V}{V_0} - 1\right)$$ ✓

(4)



$$i_R + i_C = 0 \quad (KCL)$$

$$\frac{V_{ab}}{R} + C\int_0^t \frac{dV}{dt} = 0$$

$$\frac{dV}{dt} + \frac{1}{RC}V = 0 \Rightarrow \frac{dV}{dt} = -\frac{1}{RC}V \Rightarrow \frac{dV}{V} = -\frac{1}{RC}t$$

$$\Rightarrow \int_0^t \frac{dV}{V} = \int_0^t -\frac{1}{RC}t \Rightarrow \ln V = -\frac{t}{RC} + C \Rightarrow V = Ce^{-\frac{t}{RC}}$$

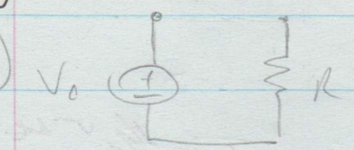
$$V(0) = Ce^0 = C \Rightarrow C = V_0$$

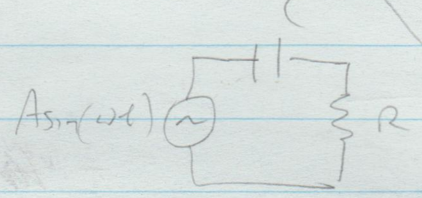
$$\Rightarrow V(t) = V_0e^{-\frac{t}{RC}}, \quad \text{for } t > 0$$

$\boxed{-1}$

-1 (5) as C grows, its time constant grows, means slower recharge / charge times. It acts more like an open circuit over short time intervals b/c so little current flowing through it (in other words, since $V(t) = \frac{1}{C}\int i\,dt$, i small or C large means smaller changes in V for same time interval)

~ only kinda more like battery

(6) 



✓

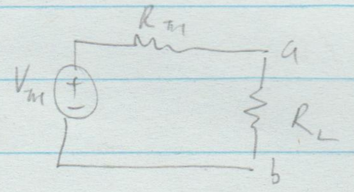
✓ (7) voltage across resistor from DC component is 0, b/c open circuit from capacitor. This increases the voltage difference across the capacitor, but doesn't contribute to the current

(8) $i = \frac{1}{L}\int v\,dt \Rightarrow i_{avg} = \frac{1}{L}\int v_{avg}\,dt.$

-0.5 inductor current will increase if average voltage > 0 ✓

avg " " " stay the same " " " = 0

" " " decrease " " " < 0. ✓

✓ (9) 

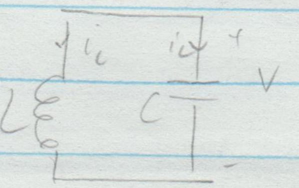
In a linear two-terminal circuit, the maximum power transferred over a load (i.e., power dissipated over R_L) is $\frac{V_{TH}^2}{4R_{TH}}$, and it is transferred if $R_{TH} = R_L$

$\boxed{-2.5}$

13.5

(10)	Con	Pro
overdamped:	slower than C.D. to get to S.S.	practical; reaches S.S. and easier to produce than C.D.
critically damped:	hard to achieve	reaches S.S. fastest
underdamped:	oscillatory behavior undesirable, doesn't die out quickly	can provide useful / interesting circuits like LC tank w/ oscillations.

(11)



$i_L(0) = 0, \quad v(0) = V_0$

KCL: $i_L + i_C = 0$

$$\frac{1}{L}\int v\,dt + C\frac{dv}{dt} = 0.$$

$\frac{1}{L}v + C\frac{d^2v}{dt^2} = 0.$

$\frac{d^2v}{dt^2} + \frac{1}{LC}v = 0.$

$\alpha = 0, \quad \omega_0 = \frac{1}{\sqrt{LC}}; \Rightarrow$ underdamped,

$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \omega_0 = \frac{1}{\sqrt{LC}}$, which is the resonating frequency

Theoretically, the complete response is in the form,

$v(t) = V_0 + e^{-\alpha t}(A_1\cos(\omega_d t) + A_2\sin(\omega_d t))$,

and since $\alpha = 0$,

$v(t) = V_0 + A_1\cos(\omega_d t) + A_2\sin(\omega_d t)$

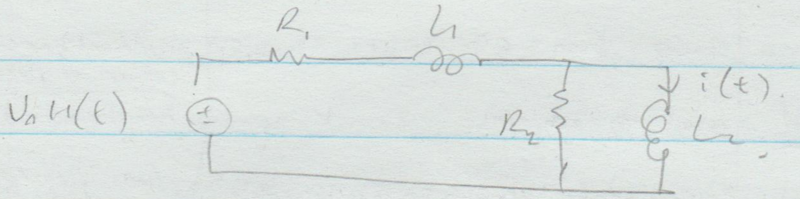
This should theoretically oscillate forever; however, there is always some resistance in a non-ideal LC tank (or any circuit) that leads to damping (not resonating forever).

-2.5

theor: $\frac{30.5}{33}$ good!

$\boxed{-2.5}$

(12)



Assuming S.S reached before $t<0$, $i(t) = v(t)$ must be zero, b/c no sources and any current would die out (be damped) through the resistors.

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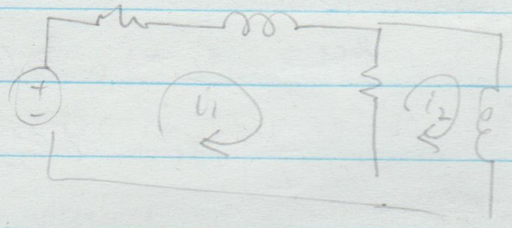
after $t \geq 0$:



i_2 is current through inductor L_2.

KVL: $-V_0 + i_1 R_1 + L_1 \frac{di_1}{dt} + (i_1 - i_2) R_2 = 0$

KVL: $(i_2 - i_1) R_2 + L_2 \frac{di_2}{dt} = 0$

$i_2 R_2 + L_2 \frac{di_2}{dt} = i_1 R_2$

$\Rightarrow \quad i_1 = i_2 + \frac{L_2}{R_2}\left(\frac{di_2}{dt}\right)$

plug back into first KVL

$$V_0 = \left(i_2 + \frac{L_2}{R_2}\left(\frac{di_2}{dt}\right)\right)R_1 + L_1 \frac{d}{dt}\left(i_2 + \frac{L_2}{R_2}\left(\frac{di_2}{dt}\right)\right)$$

$$+\left(\left(\cancel{i_2} + \frac{L_2}{\cancel{R_2}}\left(\frac{di_2}{dt}\right)\right) - \cancel{i_2}\right)\cancel{R_2}$$

$$V_0 = R_1 i_2 + \left(\frac{L_2}{R_1 R_2} + L_2\right)\left(\frac{di_2}{dt}\right) + L_1\left(\frac{di_2}{dt} + \frac{L_2}{R_2}\frac{d^2 i_2}{dt^2}\right)$$

$$V_0 = i_2 + \left(L_1 + L_2 + \frac{L_2}{R_1 R_2}\right)\frac{di_2}{dt} + \frac{L_1 L_2}{R_2}\frac{d^2 i_2}{dt^2}$$

consider transient response, ignore V_0

$$V_0 = \frac{d^2 i}{dt^2} + \left(\frac{R_2}{L_2} + \frac{R_2}{L_1} + \frac{1}{R_1 L_1}\right)\frac{di_2}{dt} + \left(\frac{R_2}{L_1 L_2}\right) i_2$$

$$\alpha = \frac{1}{2\left(\frac{R_2}{L_2} + \frac{R_2}{L_1} + \frac{1}{R_1 L_1}\right)}, \quad \omega_0 = \sqrt{\frac{L_1 L_2}{R_2}}$$

(for quadratic characteristic eqn $x^2 + Ax + B = 0$, $\alpha = \frac{1}{2A}$, $\omega_0 = \frac{1}{\sqrt{B}}$)

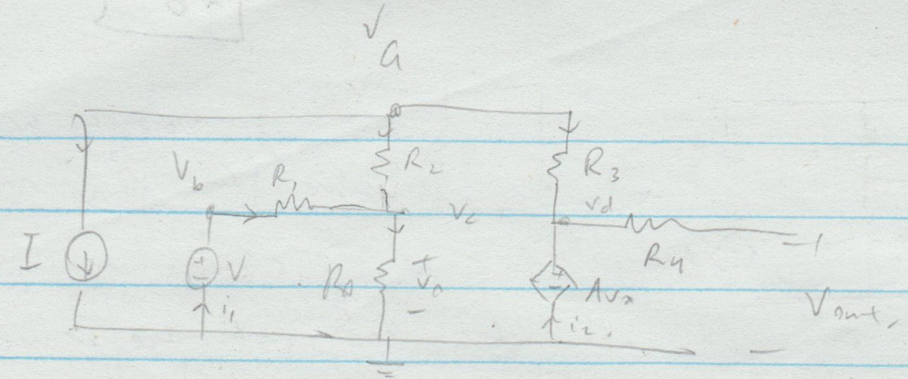
when $\alpha > \omega_0$, overdamped,

$\alpha = \omega_0$, critically damped,

$\alpha < \omega_0$, underdamped.

$\boxed{-2.5}$

(13)



V_{th}: find o.c. voltage V_{out}

MNA:

a: $I + (V_a - V_c)G_2 + (V_a - V_d)G_3 = 0$.

b: $i_1 = (V_b - V_c)G_1$ ✓

c: $(V_b - V_c)G_1 + (V_a - V_c)G_2 = V_c G_0$ ✓

d: $(V_a - V_d)G_3 = -i_2$, way you drew it -.5

$V = v_b$ ✓

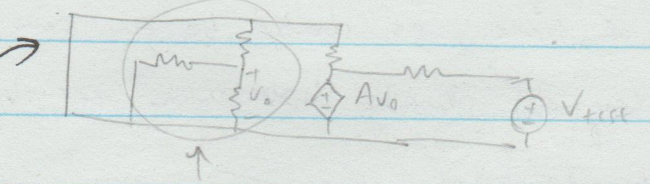
$Av_c = V_d$ ✓

$$\begin{pmatrix} G_2+G_3 & 0 & -G_2 & -G_3 & 0 & 0 \\ 0 & G_1 & -G_1 & 0 & -1 & 0 \\ G_2 & G_1 & (-G_0-G_1-G_2) & 0 & 0 & 0 \\ G_3 & 0 & 0 & -G_3 & 0 & -1 \\ 0 & 1 & \cancel{1} & 0 & 0 & 0 \\ 0 & 0 & -A & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} V_a \\ V_b \\ V_c \\ V_d \\ i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ V \\ 0 \end{pmatrix}$$

✓ $\boxed{V_d = V_{out} = V_{TH}}$ so solve for V_d.

To find R_{TH}: turn off indep. sources, use test voltage.

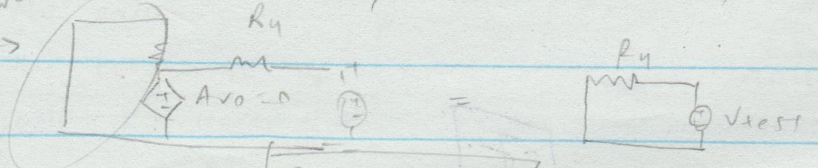
O.C.!



-3

no current flowing through here, so this acts like an o.c., and $V_0 = 0$, so $Av_0 = 0$.

no current flowing through here

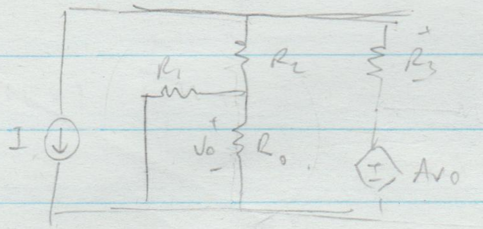


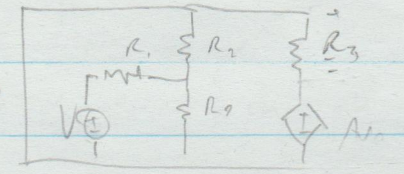
$\boxed{R_{eq} = R_4}$ (don't need matrix equations)

$\boxed{-6}$

$\boxed{-6}$

14.



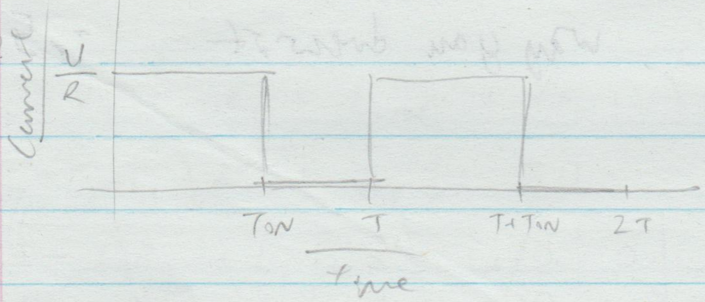


B.L.!

Solve for voltage across R_3 normally (using circuit analysis methods) for each of the circuits on the left, and add the two calculated voltages to get the voltage across R_3 in the original circuit (works b/c of linearity) -1

Comp: $\frac{28.5}{33}$

15



16 $P_{avg} = \frac{1}{2T}\int_0^{2T} p(t)\,dt = \frac{1}{2T}\int_0^{2T} i^2(t)\,R\,dt.$

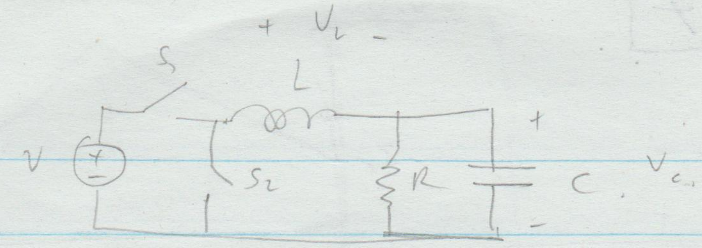
$$= \frac{2}{2T}\int_0^{T_{ON}} \left(\frac{V}{R}\right)^2 R\,dt = \frac{\cancel{2}}{\cancel{2}T}\frac{V^2}{R}\int_0^{T_{ON}} dt = \frac{V^2 T_{ON}}{RT} = \frac{T_{ON}}{T}\left(\frac{V^2}{R}\right)$$

$$= D\frac{V^2}{R}$$

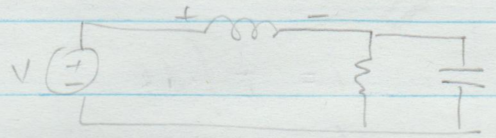
This is independent of time, so should be the same value throughout time (as long as starting and ending at the same part (phase?) of the period).

$\boxed{-7}$

(17)

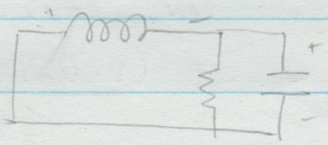


alternating between:



$V_L + V_c = V$

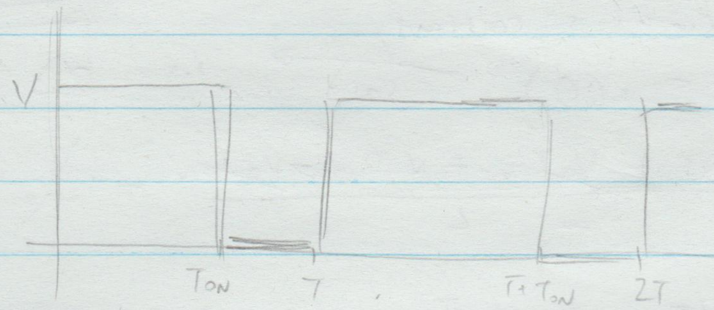
and:



parallel LRC,

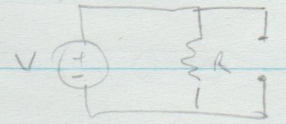
$V_L = -V_c$,

so $V_L + V_c = 0$



(18) Sketches.

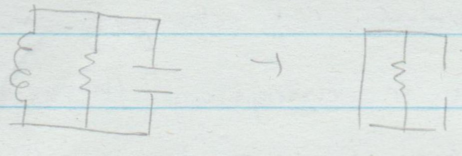
when S_1 closed, SS:



$V_L = 0$, $V_c = V$

$i_L = \frac{V}{R}$, $i_c = 0$.

when S_2 closed SS:



(nothing flowing, no voltage)

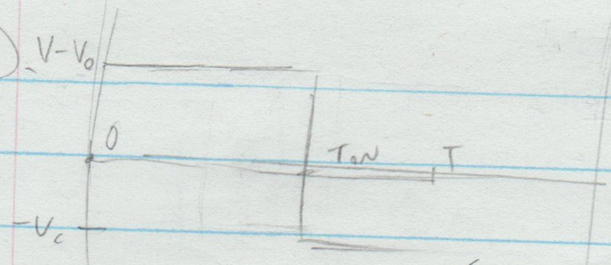
$V_L = 0$ $V_c = 0$

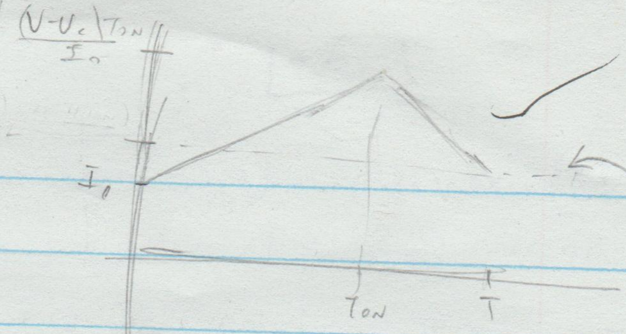
$i_L = 0$ $i_c = 0$.

At each "switching point" ($t = 0$, T_{on}, T, $T + T_{on}$, $2T$, ...), the conditions of the two circuits must match, since the circuit switches instantaneously between these two circuits, and with both of these, the inductor and capacitor prevent an instantaneous change in current and voltage conditions (so conditions must be same to ensure continuity)

$\boxed{-7}$

(19)





$i = \frac{1}{L}\int v\,dt$

(integrate one on the left).

at T_{ON}, $i = \frac{1}{L}\int_0^{T_{ON}} (V-V_0)\,dt + I_0$

$= \frac{(V-V_0)T_{ON}}{L} + I_0$

at T, $i = \frac{1}{L}\int_{T_{ON}}^{T} -V_0\,dt + I_{T_{ON}}$

$= -\frac{V_0(T-T_{ON})}{L} + \frac{(V-V_0)T_{ON}}{L} + I_0$

(20) if inductor current is constant,

(21) then $i(T) = i(0)$, $\xrightarrow{\text{so:}}$ i.e., $\boxed{\Delta i_{ON} = \Delta i_{off}}$

$$\cancel{I_0} + \frac{-V_0T + \cancel{V_0T_{ON}} + VT_{ON} - \cancel{V_0T_{ON}}}{L} = \cancel{I_0}$$

$\Rightarrow$ $VT_{ON} = V_0T$

$\frac{T_{ON}}{T} = \frac{V_0}{V}$

so $\frac{V_0}{V} = D$. (no other circuit components necessary)

(22) Capacitor value not changing much means that it takes a long time to charge. This can be a combination of L, R, and/or C being large. (i.e., large time constants)

$\boxed{\text{exp: } \frac{34}{34}}$ Wow!

$\boxed{\text{total: } \frac{93}{100}}$

Would've been nearly perfect if you realized ① → !