Jonathan Lam
Prof. Mintchev
MA 326
Lin. Alg
10/23/19

PSET 6

3.3 # 2d, 3d, 7ae

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3.3.10	5
3.4.14	5
4.2.30	

for real

2 Find the dimension of and a basis for the solution set.

d) $\begin{cases} 2x_1 + x_2 - x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \\ x_1 + 2x_2 - 2x_3 = 0 \end{cases} \to \left(\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 2 & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 1 & 2 & -2 & 0 \\ 2 & 1 & -1 & 0 \end{array}\right)$

$\to \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$x_3 = t$

Then $x_2 - t = 0 \Rightarrow x_2 = t$,
$x_1 = 0$, so the solution set is $t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$,
so a basis is $\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$.

3. Find all solns to the following systems.

d) $2x_1 + x_2 - x_3 = 5$
$x_1 - x_2 + x_3 = 1$
$x_1 + 2x_2 - 2x_3 = 4$

By inspection, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \to \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$ is a solution to this system

since $2(2) + 3 - 2 = 5$, $2 - 3 + 2 = 1$, $2 + 2(3) - 2(2) = 4$.
This is a particular solution, and the above exercise (2d) has the solution to the associated homogeneous equation, $t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. By (THM 3.9), the general solution is the set sum of these solutions, i.e., $\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $t \in \mathbb{R}$

7 Determine which of the following systems of lin. eqns. has a soln.

a) $\begin{cases} x_1 + x_2 - x_3 + 2x_4 = 2 \\ x_1 + x_2 + 2x_3 = 1 \\ 2x_1 + 2x_2 + x_3 + 2x_4 = 4 \end{cases} \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 2 \\ 1 & 1 & 2 & 0 & 1 \\ 2 & 2 & 1 & 2 & 4 \end{array}\right)$

$\rightarrow \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 2 \\ 0 & 0 & 3 & -2 & -1 \\ 0 & 0 & 3 & -2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 2 \\ 0 & 0 & 3 & -2 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$

By (THM), this situation ...

e) $\begin{cases} x_1 + 2x_2 - x_3 = 1 \\ 2x_1 + x_2 + 2x_3 = 3 \\ x_1 - 4x_2 + 7x_3 = 4 \end{cases} \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 1 & 2 & 3 \\ 1 & -4 & 7 & 4 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -3 & 4 & 1 \\ 0 & -6 & 8 & 3 \end{array}\right)$

$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -3 & 4 & 1 \\ 0 & -3 & 4 & \frac{3}{2} \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -3 & 4 & 1 \\ 0 & 0 & 0 & \frac{1}{2} \end{array}\right)$

By the same reasoning as in exercise (7a) above, this system is inconsistent (no solutions).

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Jonathan Lam
Prof. Matchev
MA 326
Lin. Alg.
10/24/19

3.3 # 9, 10.

9. Prove that the system of linear equations $Ax=b$ has a solution IFF $b \in R(L_A)$.

CLAIM: If $Ax=b$ has a solution, then $b \in R(L_A)$.
PF: If $Ax=b$ has a solution, $\exists\, x \in F^n$ s.t. $L_A(x)=b$.
Thus $b \in R(L_A)$.

CLAIM: If $b \in R(L_A)$, then $Ax=b$ is consistent.
PF: $b \in R(L_A) = \mathrm{Col}(A)$. Then b can be expressed as a linear combination of columns in A. Then $\mathrm{Col}(A|b) = \mathrm{Col}(A \cup \{b\}) = \mathrm{Col}(A)$, so $\mathrm{rank}(A|b) = \dim(\mathrm{Col}(A|b)) = \dim(\mathrm{Col}(A)) = \mathrm{rank}(A)$.
By (THM 3.11), $Ax=b$ is consistent.

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10. Prove or give a counterexample to the following statement: If the coefficient matrix of a system of m linear equations in n unknowns has rank m, then the system has a solution.

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This is true. PF: Let $Ax=b$ have a solution, $A \in M_{m\times n}(F)$, $\mathrm{rank}(A)=m$. $\mathrm{Rank}(A|b) \geq m$, since it is adding a column to the column space (which can't reduce rank). $\mathrm{Rank}(A|b) \leq m$, since rank is upper bounded by $\min(m,n)$, and $(A|b)$ has m rows. Thus $\mathrm{Rank}(A|b) = m = \mathrm{Rank}(A)$, thus by (THM 3.11) $Ax=b$ is consistent.

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3.4 #2dg, 7, 10, 14, 15.

Jonathan Lam
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2. Use Gaussian elimination to solve.

d) $x_1 - x_2 - 2x_3 + 3x_4 = -7$
$2x_1 - x_2 + 6x_3 + 6x_4 = -2$
$-2x_1 + x_2 - 4x_3 - 3x_4 = 0$
$3x_1 - 2x_2 + 9x_3 + 10x_4 = -5$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & -1 & -2 & 3 & -7 \\ 2 & -1 & 6 & 6 & -2 \\ -2 & 1 & -4 & -3 & 0 \\ 3 & -2 & 9 & 10 & -5 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & -1 & -2 & 3 & -7 \\ 0 & 1 & 10 & 0 & 12 \\ 0 & -1 & -8 & 3 & -14 \\ 0 & 1 & 15 & 1 & 16 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & -1 & -2 & 3 & -7 \\ 0 & 1 & 10 & 0 & 12 \\ 0 & 0 & 2 & 3 & -2 \\ 0 & 0 & 5 & 1 & 4 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 8 & 3 & 5 \\ 0 & 1 & 10 & 0 & 12 \\ 0 & 0 & 2 & 3 & -2 \\ 0 & 0 & 0 & \frac{-13}{2} & 9 \end{array}\right)$$

$$\frac{-13}{2} x_4 = 9 \Rightarrow x_4 = -\frac{9 \cdot 2}{13} = \frac{-18}{13}$$

$$2x_3 + 3\left(\frac{-18}{13}\right) = -2 \Rightarrow x_3 = \frac{-2 + 3\left(\frac{18}{13}\right)}{2}$$
$$= -1 + \frac{27}{13} = \frac{14}{13}$$

$$x_2 + 10\left(\frac{14}{13}\right) = 12 \Rightarrow x_2 = 12 - \frac{140}{13} = \frac{156}{13} - \frac{140}{13} = \frac{16}{13}$$

$$x_1 + 8\left(\frac{14}{13}\right) + 3\left(\frac{-18}{13}\right) = 5 \Rightarrow x_1 = 5 + \frac{54}{13} - \frac{112}{13}$$
$$= \frac{65}{13} + \frac{54}{13} - \frac{112}{13}$$
$$= \frac{119}{13} - \frac{112}{13} = \frac{7}{13}$$

Thus the solution is $\frac{1}{13}\begin{pmatrix} 7 \\ 16 \\ 14 \\ -18 \end{pmatrix}$.

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2g) $2x_1 - 2x_2 - x_3 + 6x_4 - 2x_5 = 1$

$x_1 - x_2 + x_3 + 2x_4 - x_5 = 2$

$4x_1 - 4x_2 + 5x_3 + 7x_4 - x_5 = 6$

$$\rightarrow \left(\begin{array}{ccccc|c} 2 & -2 & -1 & 6 & -2 & 1 \\ 1 & -1 & 1 & 2 & -1 & 2 \\ 4 & -4 & 5 & 7 & -1 & 6 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccccc|c} 1 & -1 & 1 & 2 & -1 & 2 \\ 0 & 0 & -3 & 2 & 0 & -3 \\ 0 & 0 & 1 & -1 & 3 & -2 \end{array}\right) \rightarrow \left(\begin{array}{ccccc|c} 1 & -1 & 1 & 2 & -1 & 2 \\ 0 & 0 & 1 & -1 & 3 & -2 \\ 0 & 0 & 0 & -1 & 9 & -9 \end{array}\right)$$

$$\begin{array}{ccccc} & \uparrow & & & \uparrow \\ & x_2 = s & & & x_5 = t \end{array}$$

$-x_4 + 9t = -9 \Rightarrow x_4 = 9t + 9$

$x_3 - (t+9) + 3t = -2 \Rightarrow x_3 = -2 - 3t + (9t+9) = 7 + 6t$

$x_1 - s\ t\ (7+6t) + 2(9t+9) - t = 2$

$\Rightarrow x_1 = 2 + s + t - (7+6t) - 2(9t+9)$

$= -23 + s - 23t$

So the solution is $\begin{pmatrix} -23 \\ 0 \\ 7 \\ 9 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -23 \\ 0 \\ 6 \\ 9 \\ 1 \end{pmatrix}$

Jonathan Lam
Prof. Mutchev
MA 326
Lin. Alg
10/23/19

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3.4 # 7, 13, 14, 15.

7. Find a subset of $\left\{ \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}, \begin{pmatrix} -8 \\ 12 \\ -4 \end{pmatrix}, \begin{pmatrix} 1 \\ 37 \\ -17 \end{pmatrix}, \begin{pmatrix} -3 \\ -5 \\ 8 \end{pmatrix} \right\}$.

that is a basis for $\mathbb{R}^3$.

This can be solved with an application of (THM 3.16), i.e., the columns in the RREF of the matrix of vectors that equal e_j are the corresponding columns to the basis of the column space of the original matrix (F^3).

$$\begin{pmatrix} 2 & 1 & -8 & 1 & -3 \\ -3 & 4 & 12 & 37 & -5 \\ 1 & -2 & -4 & -17 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -4 & -17 & 8 \\ 0 & 5 & 0 & 35 & -19 \\ 0 & -2 & 0 & -14 & -19 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -2 & -4 & -17 & 8 \\ 0 & 1 & 0 & 7 & \frac{-19}{5} \\ 0 & 1 & 0 & 7 & \frac{15}{2} \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -4 & -17 & 8 \\ 0 & 1 & 0 & 7 & \frac{-19}{5} \\ 0 & 0 & 0 & 0 & \frac{19}{2}+\frac{19}{5} \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -2 & -4 & -17 & 0 \\ 0 & 1 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -4 & -3 & 0 \\ 0 & 1 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Using (THM 3.16), $\left\{ \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}, \begin{pmatrix} -3 \\ -5 \\ 8 \end{pmatrix} \right\}$ is lin. ind.

and thus is a basis for $\mathbb{R}^3$.

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10 Let $V = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 - 2x_2 + 3x_3 - x_4 + 2x_5 = 0\}$

a) Show that $S = \{(0,1,1,1,0)\}$ is a lin. ind. subset of V.

PF: Since S is a singleton nonzero vector in V, it is linearly independent.

b) Extend S to a basis for V.

By inspection, $\left\{ \begin{pmatrix} 1\\0\\0\\1\\0 \end{pmatrix}, \begin{pmatrix} 2\\1\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} -3\\0\\1\\0\\0 \end{pmatrix}, \begin{pmatrix} 2\\0\\0\\0\\1 \end{pmatrix} \right\} \in V$, and are linearly ind.

Using the same method as in (exercise 9):

$$\begin{pmatrix} 0 & 1 & 2 & -3 & 2 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -3 & 2 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -3 & 2 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -3 & 2 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -3 & 3 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, $\left\{ \begin{pmatrix} 0\\1\\1\\1\\0 \end{pmatrix}, \begin{pmatrix} 1\\0\\0\\1\\0 \end{pmatrix}, \begin{pmatrix} 2\\1\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 2\\0\\0\\0\\1 \end{pmatrix} \right\}$ is lin. ind. by (THM 3.16c).

Since V is constrained by a homogeneous linear system of 1 equation in 5 variables, there are four free variables $\Rightarrow \dim(V) = 4$. Since cardinality of the set above has cardinality 4, it is a basis for V.

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3.4 # 14, 15

Jonathan Lam
Prof. Mutchew
MA 326
Lin. Alg.
10/2/19

Your argument can gain clarity by applying contrapositive reasoning throughout

14. If $(A|b)$ is in RREF, prove that A is also in RREF.

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To prove this, we need to show the three conditions in the definition of RREF are true for A. Let $(A|b) \in M_{m\times n}(F)$, $A \in M_{m\times(n-1)}(F)$

a) CLAIM: Any row containing a nonzero vector ? precedes any all-zero row.

PF: Assume that there is a all-zero row in A. Then the first $(n-1)$ elements of the corresponding row in $(A|b)$ must also be 0, and the last element may be either zero or nonzero. If it is zero, then by (c) of (def. RREF for $A|b$), this row is below any nonzero row in $(A|b)$ (and thus below any nonzero row in A). If the men element is nonzero, then it is a pivot element and thus must be below any row where the pivot occurs further left by (def. RREF (c)); thus it occurs below any nonzero row of A

b) CLAIM: The first nonzero entry in each row is the only nonzero entry in its column.

PF: The first nonzero entry for a row in A is also the first nonzero entry for its row in $(A|b)$. By (def RREF (b)), this pivot element in $(A|b)$ is the only element of its column, which is also the column for the pivot element in A.

c) CLAIM: The first nonzero entry in each row is 1 and it occurs in a column to the right of the first nonzero entry in the preceding row.

PF. For any pivot element in A, it is also the pivot element of $(A|b)$, and must be 1 and below and to the right of the pivot element in the preceding row, which is also the pivot element in the preceding row of A.

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15. Prove that the RREF of a matrix is unique.

Let $A \in M_{m\times n}(F)$. Let B be the RREF of A. Then, if $n > 1$, then $B = (B' \mid d)$, $B \in M_{m\times(n-1)}(F)$, $d \in M_{m\times 1}(F)$, and by (exercise 14), is in RREF. By induction, the set of submatrices $S = \{x \in M_{m\times i}(F) : x \text{ contains the first } i \text{ columns of } B\}$ are also in RREF. This proof will use induction, beginning on S_1 (the first column of S) and adding one column at a time.

Inductive base case: The RREF S_1 (first column of A) is uniquely defined by A.

PF: S_1 is row-equivalent to A_1. If $A_1 = 0 \in F^m$, then RREF of $A_1 = S_1 = 0 \in F^m$. If not, since S_1 is in RREF, the only other valid $M_{m\times 1}(F)$ RREF is e_1, thus S_1 must be e_1. Thus S_1 is uniquely defined by A_1.

Inductive hypothesis: If S_i is uniquely defined by A, then so is S_{i+1}.

PF: $S_{i+1} = (S_i \mid b)$, $b = B_{i+1} \in F^m$ ($i+1^{th}$ column of B).

If $\operatorname{rank}(S_{i+1}) = \operatorname{rank}(S_i) + 1 = k$, then e_{i+1} cannot exist as a column in S_i. However, by (THM 3.16 (b)), e_{i+1} must exist as a column in S_{i+1} if it is an RREF with rank $i+1$, so S_{i+1} must be $i+1$.

Else, $\operatorname{rank}(S_{i+1}) = \operatorname{rank}(S_i) = k$ (since $\operatorname{rank}(S_{i+1}) \geq \operatorname{rank}(S_i)$), and $b \in R(S_i)$. Then b is expressable as a linear combination over $\{e_1, e_2, \ldots, e_k\}$, which are vectors in S_i by (THM 3.16b). By (THM 3.16d), the coefficients to this linear combination are the same as the coefficients of the linear combiation over the corresponding columns of A (which are a basis) yielding A_{i+1}. Since A_{i+1} is determined by a unique linear combination over a basis, the coefficients to the linear combinations determining A_{i+1} and b are uniquely determined by A, thus b is unique. Since S_i, b uniquely determined by A, S_{i+1} is uniquely determined by A.