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2.5 # 3bf

15.5/20

2.5.3b ~~f~~ 5/5

3. For each o.B. β, β' for $P_2(\mathbb{R})$, find the change of coordinates matrix Q that changes β' to β.

b) $\beta = \{1, x, x^2\}$.
$\beta' = \{a_2x^2 + a_1x + a_0,\ b_2x^2 + b_1x + b_0,\ c_2x^2 + c_1x + c_0\}$.

$$\begin{aligned}\beta_1' &= a_0(1) + a_1(x) + a_2(x^2)\\ \beta_2' &= b_0(1) + b_1(x) + b_2(x^2)\\ \beta_3' &= c_0(1) + c_1(x) + c_2(x^2)\end{aligned} \Rightarrow Q = \begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$$

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LEM (Discussed in class, but not derived :)):
Let $B = ([\beta_1]_\alpha\ [\beta_2]_\alpha\ \dots\ [\beta_n]_\alpha)$,
$B' = ([\beta_1']_\alpha\ [\beta_2']_\alpha\ \dots\ [\beta_n']_\alpha)$,
where α is the s.o.B. for $P_2(\mathbb{R})$.
Then $B_j' = [\beta_j']_\alpha = \sum_{i=1}^{n} Q_{ij}[\beta_i]_\alpha = BQ_j \Rightarrow B' = BQ$,
so $Q = B^{-1}B'$.

f) $\beta = \{2x^2 - x + 1,\ x^2 + 3x - 2,\ -x^2 + 2x + 1\}$
$\beta' = \{9x - 9,\ x^2 + 21x - 2,\ 3x^2 + 5x + 2\}$
$\alpha = \{1, x, x^2\}$

Use Lemma above. $B = \begin{pmatrix} 1 & -2 & 1 \\ -1 & 3 & 2 \\ 2 & 1 & -1 \end{pmatrix}$, $B' = \begin{pmatrix} -9 & -2 & 2 \\ 9 & 21 & 5 \\ 0 & 1 & 3 \end{pmatrix}$

To solve, transform $(B \mid B') \to (I_n \mid Q)$ using EROs.

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 1 & -9 & -2 & 2 \\ -1 & 3 & 2 & 9 & 21 & 5 \\ 2 & 1 & -1 & 0 & 1 & 3 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & -9 & -2 & 2 \\ 0 & 1 & 3 & 0 & 19 & 7 \\ 0 & 5 & -3 & 18 & 5 & -1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 7 & -9 & 36 & 16 \\ 0 & 1 & 3 & 0 & 19 & 7 \\ 0 & 0 & -18 & 18 & -90 & -36 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 7 & -9 & 36 & 16 \\ 0 & 1 & 3 & 0 & 19 & 7 \\ 0 & 0 & 1 & -1 & 5 & 2 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 2 \\ 0 & 1 & 0 & 3 & 4 & 1 \\ 0 & 0 & 1 & -1 & 5 & 2 \end{array}\right)$$

$$\Rightarrow Q = \begin{pmatrix} -2 & 1 & 2 \\ 3 & 4 & 1 \\ -1 & 5 & 2 \end{pmatrix}.$$

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2.5 # 6 b d

6. For each matrix A and O.B. β, find $[L_A]_\beta$. Also, find an invertible matrix Q s.t. $[L_A] = Q^{-1}AQ$.

b.) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

$L_A(\beta_1) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+2 \\ 2+1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$L_A(\beta_2) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1-2 \\ 2-1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\Rightarrow [L_A]_\beta = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$

By (THM 2.23 COR 1), $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

d) $A = \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix}$, $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

$A(\beta_1) = \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 13+1-8 \\ 1+13-8 \\ 4+4-20 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ -12 \end{pmatrix} \Rightarrow 6\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

$A(\beta_2) = \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 13-1 \\ 1-13 \\ 4-4 \end{pmatrix} = \begin{pmatrix} 12 \\ -12 \\ 0 \end{pmatrix} = 12\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$A(\beta_3) = \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 13+1+4 \\ 1+13+4 \\ 4+4+10 \end{pmatrix} = \begin{pmatrix} 18 \\ 18 \\ 18 \end{pmatrix} = 18\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\Rightarrow [L_A]_\beta = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 18 \end{pmatrix}$,

by (THM 2.23 COR 1) again,

$Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 16 \\ 7 \\ -3 \end{pmatrix}$

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2.5 #8

8) Let $T \in \mathcal{L}(V, W)$, V, W finite-dim. v.s. Let β, β' be o.B's for V, and let γ, γ' be o.B's for W. Then $[T]_{\beta'}^{\gamma'} = P^{-1}[T]_\beta^\gamma Q$, where Q is the matrix that changes β' coordinates to β coordinates, P is the matrix that converts γ' coordinates to γ coordinates

PF: If P is the matrix that coverts γ' to γ coordinates, then $P = [1_W]_{\gamma'}^{\gamma}$. Note that $1_W = (1_W)^{-1}$. By (THM 2.18), $[1_W]_\gamma^{\gamma'} = ([(1_W)^{-1}]_{\gamma'}^{\gamma})^{-1} = P^{-1}$; i.e., P^{-1} is the matrix to covert from γ' to γ coordinates. Similarly, $Q = [1_V]_{\beta'}^{\beta}$. This means the original expression on the right can be rewritten:

$$\left([1_W]_\gamma^{\gamma'}[T]_\beta^\gamma\right)[1_V]_{\beta'}^{\beta} = [1_W T]_\beta^{\gamma'}[1_V]_{\beta'}^{\beta} = [T]_\beta^{\gamma'}[1_V]_{\beta'}^{\beta}$$

$$= [T 1_V]_{\beta'}^{\gamma'} = [T]_{\beta'}^{\gamma'}$$

(by (THM 2.11)).

13. Let V be a finite-dim. v.s. over F, let $\beta = \{x_1, x_2, \ldots, x_n\}$ be a basis for V. Let Q be an $n \times n$ invertible matrix with entries from F. Define:

$$x_j' = \sum_{i=1}^{n} Q_{ij} x_i, \quad 1 \leq j \leq n,$$

and set $\beta' = \{x_1', x_2', \ldots, x_n'\}$.

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7
-3

a) Prove that β' is a basis for V.
$\text{card}(\beta') = \text{card}(\beta)$, so it is sufficient to show that β' is linearly ind. to show that it is a basis for V (by THM 1.10 (COR 2)).

Assume some lin. comb. of β' equal to 0,

$$\sum_{j=1}^{n} a_j x'_j = 0 \quad , \quad a_j \in F \ \forall \ 1 \le j \le n.$$

Then: $$\sum_{j=1}^{n} a_j\left(\sum_{i=1}^{n} Q_{ij} x_i\right) = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_j Q_{ij}\right) x_i = 0.$$

Since β lin. ind, $\sum_{j=1}^{n} a_j Q_{ij} = 0 \quad \forall 1 \le i \le n.$

Thus, $$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} a_j Q_{1j} \\ \sum_{j=1}^{n} a_j Q_{2j} \\ \vdots \\ \sum_{j=1}^{n} a_j Q_{nj} \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} & \cdots & Q_{1n} \\ Q_{21} & Q_{22} & \cdots & Q_{2n} \\ \vdots & & \ddots & \vdots \\ Q_{n1} & Q_{n2} & \cdots & Q_{nn} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

$\Rightarrow \quad 0 = QA$, where A is the $M_{n\times1}(F)$ vector of a_j's, and $\bar{0}$ is the $M_{n\times1}(F)$ zero vector.

Since Q is invertible, $$Q^{-1} 0 = (Q^{-1}Q)A$$
$$0 = I_n A$$
$$0 = A$$

$\Rightarrow a_1 = a_2 = \cdots = a_j = 0$, thus the coefficients are all zero, thus β' is lin. ind., thus β' is a basis.

<u>CLAIM</u>: Q is the change of coordinate matrix changing β' coordinates into β-coordinates.

<u>PF</u>: This is given trivially by the summation:

$$x'_j = \sum_{i=1}^{n} Q_{ij} x_i \, ,$$

which is equivalent to saying x'_j's coordinate vector in β is Q_j.

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3.1 #5, 9, 12

5. Prove that E is an elementary matrix IFF E^t is.

For any E formed by an elementary row operation, it's transpose was formed by the elementary column operation on the corresponding column(s), and vice versa.

Type I: The transpose of an elementary matrix formed by swapping two rows of I_n is the (elementary) matrix formed by swapping the corresponding columns in I_n, which is elementary.

Type II: Scaling a row i of I_n to form an elementary matrix scales E_{ii}, which is the same (elementary) matrix formed by scaling the ith column of I_n, so $E = E^t$ is elementary.

Type III: The transpose of an elementary matrix formed by adding a scaled factor of row i to row j of I_n is the matrix formed by adding the ith column to the jth column of I_n, which is elementary.

To prove the converse of this IFF statement, $(E^t)^t = E$, and the same arguments above can be used.

9. Prove that any elementary row (column) operation of type 1 can be obtained by a succession of three elementary row ops. of type 3 followed by one elementary row op. of type 2.

PF: Let A_{ik}, A_{jk} represent the kth element in the ith and jth row (column) vectors of a matrix A, and let $A_{ik} = c$, $A_{jk} = d$, $c, d \in F$, initially. Do the following elementary ops:

	initial				swapped
A_{ik}	c	c	$-d$	$-d$	d
A_{jk}	d	$c+d$	$c+d$	c	c

type III: add row (col) i to j. | type III: Subtract row (col) j from i | type III: add row (col) i to j | type II: Scale row (col) i by (-1).

This sequence of steps swaps corresponding pairs of values A_{ik} and A_{jk} from the rows (columns) i, j, thus effectively performing a type II op.

12. Let A be an $m \times n$ matrix. Prove that there exists a sequence of elementary row operations of Type I and III that transforms A into an upper triangular matrix.

PF: An algorithm will be defined (very similar to Gaussian elimination) to transform a matrix to upper triangular.

① Let $i = 1$.

② Examine the ith column of A. If $\exists\, j > i$, s.t. $A_{ji} \neq 0$, continue to step ③. Otherwise, this column is correct; all entries below the main diagonal entry are zero. Increment i and repeat step ② until $i = \min\{m, n\}$.

③ Perform a type I elementary row op. on rows i and j (from step 2). Note that this maintains the property that all entries below the main diagonal entry are zero for any previous column i', since $i, j > i'$, and thus $A_i = A_j = 0$. Now, $A_{ii} \neq 0$.

④ For each row $k > i$, perform a type III op. on it, by adding the ith row scaled by $\left(-\frac{A_{ki}}{A_{ii}}\right)$. For every $i' < i$, $A_{ki'} - \frac{A_{ki}}{A_{ii}}(A_{ii'}) = 0 - \frac{A_{ki}}{A_{ii}}(0) = 0$, so the property that all entries below the main diagonal entry remain zero. Also, $A_{ki} - \frac{A_{ki}}{A_{ii}}(A_{ii}) = A_{ki} - A_{ki} = 0$, so all entries in the current column below the main diagonal become zero.

⑤ If $i < \min\{m, n\}$, increment i and repeat step ②.

This algorithm terminates after a finite number of steps ($i \leq \min\{m, n\}$, and up to $m - i + 1$ elementary row ops per column), and each iteration ensures that all entries below the main diagonal become zero, without ruining this property on earlier columns, thus the resulting matrix is upper triangular.

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3.2 #5eg

5. For each of these matrices, compute the rank and the inverse if it exists.

e) $$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 3 & 3 & 1 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \\ 0 & 3 & 3 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{array}\right)$$

By (THM 3.6), $\text{rank}(A) = 3$. By the method of finding inverses, A^{-1} exists and $A^{-1} = \begin{pmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}$

g) $$\left(\begin{array}{cccc|cccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 5 & 5 & 1 & 0 & 1 & 0 & 0 \\ -2 & -3 & 0 & 3 & 0 & 0 & 1 & 0 \\ 3 & 4 & -2 & -3 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 2 & 0 & 1 & 0 \\ 0 & -2 & -5 & -3 & -3 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|cccc} 1 & 0 & -5 & -2 & 5 & -2 & 0 & 0 \\ 0 & 1 & 3 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 4 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -7 & 2 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -12 & -15 & 3 & -5 & 0 \\ 0 & 1 & 0 & 7 & 10 & -2 & 3 & 0 \\ 0 & 0 & 1 & -2 & -4 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -3 & 1 & 1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 51 & 15 & 7 & 12 \\ 0 & 1 & 0 & 0 & 31 & -9 & -4 & -7 \\ 0 & 0 & 1 & 0 & -10 & 3 & 1 & 2 \\ 0 & 0 & 0 & 1 & -3 & 1 & 1 & 1 \end{array}\right)$$

Thus by (THM 3.6), $\text{rank}(A) = 4$, and the matrix is invertible, and

$$A^{-1} = \begin{pmatrix} -51 & 15 & 7 & 12 \\ 31 & -9 & -4 & -7 \\ -10 & 3 & 1 & 2 \\ -3 & 1 & 1 & 1 \end{pmatrix}.$$

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3.2 # 6 af

6. For each of the following linear transformations T, determine whether T is invertible, and compute T^{-1} if it exists.

a) $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$, $T(f(x)) = f''(x) + 2f'(x) - f(x)$

$\beta = \{1, x, x^2\}$

$T(1) = 0 + 0 - 1$

$T(x) = 0 + 2 - x$ $\Rightarrow [T]_\beta = \begin{pmatrix} -1 & 2 & 2 \\ 0 & -1 & 4 \\ 0 & 0 & -1 \end{pmatrix}$

$T(x^2) = 2 + 4x - x^2$

By (Thm 2.18), $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$. So we invert $[T]_\beta$ to find the matrix representation of the inverse transformation.

$$\left(\begin{array}{ccc|ccc} -1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -1 & 4 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & -2 & -2 & -1 & 0 & 0 \\ 0 & 1 & -4 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & -10 & -1 & -2 & 0 \\ 0 & 1 & -4 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2 & -10 \\ 0 & 1 & 0 & 0 & -1 & -4 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right)$$

So $[T^{-1}]_\beta = \begin{pmatrix} -1 & -2 & -10 \\ 0 & -1 & -4 \\ 0 & 0 & -1 \end{pmatrix}$, so T is invertible (because no zero rows, $\operatorname{rank}(T) = n = 3$).

For some polynomial $f(x) = a_0 + a_1 x + a_2 x^2$, $[f]_\beta = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$,

and $[T^{-1}(f)]_\beta = [T^{-1}]_\beta [f]_\beta = \begin{pmatrix} -1 & -2 & -10 \\ 0 & -1 & -4 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -a_0 - 2a_1 - 10a_2 \\ -a_1 - 4a_2 \\ -a_2 \end{pmatrix}$

Thus $T^{-1}(a_0 + a_1 x + a_2 x^2) = (-a_0 - 2a_1 - 10a_2) + (-a_1 - 4a_2)x + (-a_2)x^2$

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$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

6f. $T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}^4$, $T(A) = (tr(A), tr(A^{-1}), tr(EA), tr(AE))$.

Let $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then:

$$tr(B) = tr\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 1 - 1 = 0$$
$$tr(B^t) = tr\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 1 - 1 = 0$$
$$tr(EB) = tr\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 0 + 0 = 0$$
$$tr(BE) = tr\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 0 + 0 = 0$$

$$\Rightarrow T(B) = (0, 0, 0, 0) = 0 \in \mathbb{R}^4$$

Since $B \neq 0$, $B \in N(T)$, then $N(T) \neq \{0\}$, so T is not 1-1 and not invertible.

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3.2 # 14, 15

14. Let $T, U \in \mathcal{L}(V, W)$.

a) Prove $R(T+U) \subseteq R(T) + R(U)$.

PF: $\forall x \in R(T+U)$, $\exists y \in V$ s.t. $x = (T+U)(v) = T(v) + U(v)$. Since $T(v) \in R(T)$, $U(v) \in R(U)$, $x \in \{x_1 + x_2 : x_1 \in R(T), x_2 \in R(U)\}$ $= R(T) + R(U)$. Thus $R(T+U) \subseteq R(T) + R(U)$.

b) Prove if W is finite-dimensional, then $\text{rank}(T+U) \leq \text{rank}(T) + \text{rank}(U)$.

PF:
$$\begin{aligned}
\text{rank}(T) + \text{rank}(U) &= \dim(R(T)) + \dim(R(U)) \\
&\geq \dim(R(T)) + \dim(R(U)) - \dim(R(T) \cap R(U)) \\
&= \dim(R(T) + R(U)) \quad \text{(by THM 1.11)} \\
&\geq \dim(R(T+U)) \quad \text{(by part a)} \\
&= \text{rank}(T+U)
\end{aligned}$$

c) Deduce from (b) that $\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$ for any $m \times n$ matrices A, B.

PF: $\text{rank}(A+B) \underset{\text{(by def rank of matrix)}}{=} \text{rank}(L_{A+B}) \underset{\text{(by THM 2.15)}}{=} \text{rank}(L_A + L_B)$,

and $\text{rank}(A) = \text{rank}(L_A)$, $\text{rank}(B) = \text{rank}(L_B)$.

Since $\text{rank}(L_A + L_B) \leq \text{rank}(L_A) + \text{rank}(L_B)$ from part a, then substitute ranks of matrices to get:

$$\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$$

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15 Suppose A, B are matrices with n rows. Prove that $M(A|B) = (MA|MB)$ for any $m \times n$ matrix M.

PF: Let $A \in M_{n \times p}(F)$, $B \in M_{n \times q}(F)$. It can be shown that corresponding columns of $M(A|B)$ and $(MA|MB)$ are equal.

Fix some column j of $M(A|B)$, $1 \leq j \leq p+q$. Then:

$$
\begin{aligned}
(M(A|B))_j &= M((A|B)_j) && \text{(THM 2.13)} \\
&= M \begin{cases} A_j, & 1 \leq j \leq p \\ B_{j-p}, & p+1 \leq j \leq p+q \end{cases} && \text{(DEF. Aux. Matrix)} \\
&= \begin{cases} M(A_j), & 1 \leq j \leq p \\ M(B_{j-p}), & p+1 \leq j \leq p+q \end{cases} \\
&= \begin{cases} (MA)_j, & 1 \leq j \leq p \\ (MB)_{j-p}, & p+1 \leq j \leq p+q \end{cases} && \text{(THM 2.13)} \\
&= (MA|MB)_j && \text{(DEF aux matrix)}
\end{aligned}
$$

Since the corresponding columns of $M(A|B)$ and $(MA|MB)$ are equal, and since both matrices have the same dimension (both $\in M_{m \times (p+q)}(F)$), they are equal.