TEST 2

Started: 3:54
Ended: 4:43

Jonathan Lam
Prof. Smyth
MA345
Cplx Analysis
4/6/20

1) $\log(1+i)$

$1+i = \sqrt{2}\, e^{i\frac{\pi}{4}}$

$= \ln|z| + i\arg z = \ln\sqrt{2} + i\left(\frac{\pi}{4} + 2\pi n\right),\ n \in \mathbb{Z}$

$= \frac{1}{2}\ln 2 + i\left(\frac{\pi}{4} + 2\pi n\right),\ n \in \mathbb{Z}.$

2) $(-1+i)^{2i}$ (P.V.).

$= \exp\left(2i \operatorname{Log}(-1+i)\right)$

$-1+i = \sqrt{2}\, e^{\frac{3\pi}{4} i}$

$= \exp\left(2i\left(\ln\sqrt{2} + i\left(\frac{3\pi}{4}\right)\right)\right)$

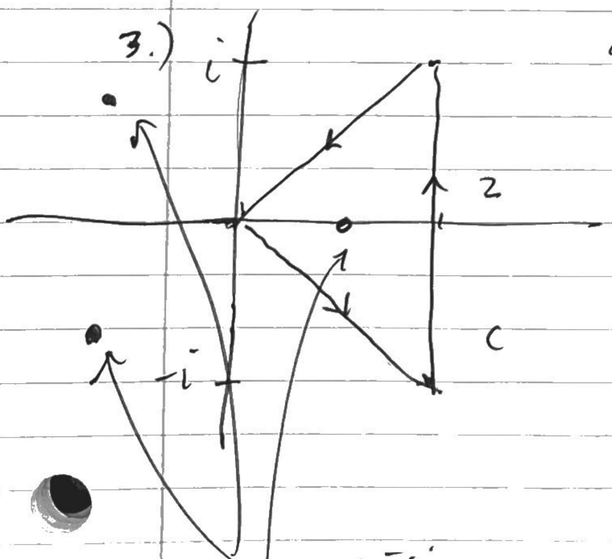
$= \exp\left(2i\left(\frac{1}{2}\right)\ln 2 + 2i^2\left(\frac{3\pi}{4}\right)\right)$

$= \exp\left(i\ln 2 - \frac{3\pi}{2}\right)$

$= \exp\left(-\frac{3\pi}{2}\right)\exp\left(i\ln 2\right)$

$= \exp\left(-\frac{3\pi}{2}\right) 2^i = \exp\left(-\frac{3\pi}{2}\right)\left(\cos(\ln 2) + i\sin(\ln 2)\right)$

3.)



singularities of integrand
(ones in LHP are singularities of f).

determine value of $\int_C \frac{z^2}{z^3-1}\,dz$

$z^3 - 1 = 0 \Rightarrow z = (1)^{\frac{1}{3}}$

$=$ (skipping algebra): $\left\{e^{\pm\frac{2\pi}{3}i},\ 1\right\}$

let $f(z) = \frac{z^2}{z^2+z+1}$

Its singularities are at $e^{\pm\frac{2\pi}{3}i}$, so it is AOI C

$$\int_C \frac{z^2}{z^3-1}\,dz =$$

f AOIC

CIF(0)

$$\int_C \frac{f(z)}{z-1}\,dz = 2\pi i\, f(1)$$

POSCC

1 is an interior point to C

$$= 2\pi i \left(\frac{1^2}{1^2+1+1}\right)$$

$$= 2\pi i \left(\frac{1}{3}\right)$$

$$= \frac{2}{3}\pi i.$$

4.) a) $C = |z| = 1$, PO.

a) Find $\int_C \bar{z} + z\, dz$

$z = e^{i\theta}$, $0 \le \theta \le 2\pi$

$z' = ie^{i\theta}$

$$= \int_0^{2\pi} \left(e^{-i\theta} + e^{i\theta}\right) ie^{i\theta}\, d\theta$$

$$= \int_0^{2\pi} i\left((1) + e^{2i\theta}\right) d\theta = i\int_0^{2\pi} d\theta + \int_0^{2\pi} ie^{2i\theta}\, d\theta$$

antiderivative is θ

antiderivative is $\frac{1}{2}e^{2i\theta}$

$$= i\theta \Big|_0^{2\pi} + \frac{1}{2}e^{2i\theta}\Big|_0^{2\pi}$$

$$= 2\pi i + \frac{1}{2}\left(e^{4i\pi} - e^0\right)$$

$$= 2\pi i + \cancel{\frac{1}{2}(1-1)} = 2\pi i$$

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4b). $\int_C e^{z^2} dz$.

composition of two entire functions (e^z and z^2), thus also entire.

thus $\int_C e^{z^2} dz = 0$.

SCC AOIC CG

5.) $C: |z| = 2$ (P.O.)

$$\int_C \frac{e^{5z}}{(z-1)^3} = \frac{2\pi i}{2!} \frac{d^2}{dz^2} e^{5z} \Big|_{z=1}$$

AOIC (entire, since composition of entire fns)

posCC

1 is an interior point to C

CIF(2)

$$\frac{d}{dz} e^{5z} = 5e^{5z}$$

$$\frac{d^2}{dz^2} e^{5z} = 25 e^{5z}$$

$$= \frac{\cancel{2}\pi i}{\cancel{2}} \cdot 25 e^{5(1)} = 25\pi i e^5.$$

6.) $z(t) = 4e^{i2t} + e^{i11t}$, $0 \le t \le 2\pi$.

find $\displaystyle\int_C \frac{dz}{z^2}$

$z(0) = 4e^{i(0)} + e^{i(0)} = 4+1=5$,
$z(2\pi) = 4e^{i(4\pi)} + e^{i(22\pi)} = 4+1=5$
$\Downarrow$
final pt. = initial pt $\Rightarrow$ CC.

antiderivative of $\frac{1}{z^2} = \frac{-2}{z^3}$, antiderivative exists everywhere.

antiderivative thm.

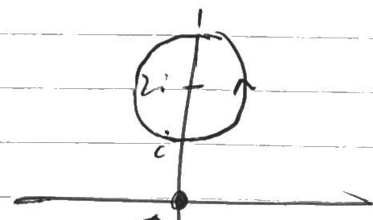
thus $\displaystyle\int_C \frac{dz}{z^2} = 0$

closed contour

antiderivative exists everywhere

7.) $C: |z-2i| = 1$ (P.O.)

Find $\displaystyle\int_C \frac{e^z}{z}\,dz$



Singularity of integrand

Let $f(z) = \frac{e^z}{z}$

f is quotient of two entire fns, so only singularities when denom. = 0.
Thus analytic in $\mathbb{C}\setminus\{0\}$.

AOIC

$\displaystyle\int_{C} f(z)\,dz = 0$

SCC

CG

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8.) $C_R: |z| = R$ (P.O.).

a) $\lim_{R\to\infty} \int_{C_R} \frac{1}{z^2+5z+i}\, dz$.

guess limit will be 0. use ML.

$|z^2+5z+i| \geq ||z^2| - |5z+i|| = |R^2 - |5z+i||$

(in limit as $R \to \infty$, $R^2 > 5z+i$, so we can remove outer absolute val. bars)

$= R^2 - |5z+i| \geq R^2 - ||5z| - |i|| = R^2 - |5|z| - 1|$

$= R^2 - |5R-1|$

(same reasoning here: $R > 1$ in limit as $R \to \infty$)

$= R^2 - 5R - 1$.

thus $\left| \frac{1}{z^2+5z+i} \right| \leq \frac{1}{R^2-5R-1} = M$

length of arc $= 2\pi R$.

By ML, $\left| \int_{C_R} \frac{1}{z^2+5z+i}\, dz \right| \leq ML = \frac{2\pi R}{R^2-5R-1}$.

$\lim_{R\to\infty} \left| \int_{C_R} \frac{1}{z^2+5z+i}\, dz \right| = \lim_{R\to\infty} \frac{2\pi R}{R^2-5R-1} = 0$

(formal evaluation of limit not shown here)

$\Rightarrow \lim_{R\to\infty} \int_{C_R} \frac{1}{z^2+5z+i} = 0$.

(if limit of modulus $\to 0$, then limit of value $\to 0$).

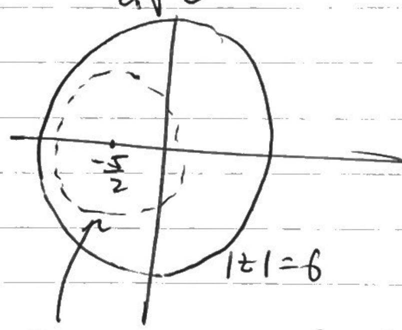
8b) $\displaystyle\int_{C_6} \frac{1}{z^2+5z+i}\,dz$

factor ~~denom~~ denom:

$$z = \frac{-5 \pm (5^2-4(1)(i))^{\frac{1}{2}}}{2(1)}$$

$$= \frac{-5 \pm (25-4i)^{\frac{1}{2}}}{2} = \pm z_s \text{ (singularities)}$$

$$\frac{\left(\sqrt{25^2+4^2}\right)^{\frac{1}{2}}}{2} \leq 3.5$$



two singularities lie somewhere ~~on inner circle~~ both within C. on this dotted circle.

(singularities lie on circle $\left|z+\frac{5}{2}\right| = \left|\frac{(25-4i)^{\frac{1}{2}}}{2}\right| \leq 6$)

$$= \int_{\substack{|z|=6 \\ \circlearrowleft}} \frac{1}{(z-z_s)(z+z_s)}\,dz$$

partial fractions:

$$\frac{1}{(z-z_s)(z+z_s)} = \frac{A}{z-z_s} + \frac{B}{z+z_s}$$

$A = \frac{1}{2}$

$B = -\frac{1}{2}$

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numerator AOIC

using partial fractions:

$$\int_C \frac{1}{(z-z_s)(z+z_s)}\,dz = \frac{1}{2}\left(\int_C \frac{1}{z-z_s}\,dz \cancel{z} - \int_C \frac{1}{z+z_s}\,dz\right)$$

interior point

posCC

$$\underset{\text{CIF(a) twice}}{=} \frac{1}{2}\left(2\pi i(1) - 2\pi i(1)\right) = 0.$$