Jonathan Lam
Professor Topper
Physical Chemistry
CH160
2 / 28 / 19

Chapters 2-4 Outline
Textbook Reading Notes

2.3: A Working Definition of Temperature

- Isotherm: range of values where $PV = T$, for some constant T

2.6: The Ideal Gas Equation

- Ideal gas law: $PV = nRT = \overline{R}T$
    - Proportionality of P-T, V-T, P-n, V-n, and inverse proportionality of P-V

2.7: Dalton's Law of Partial Pressures

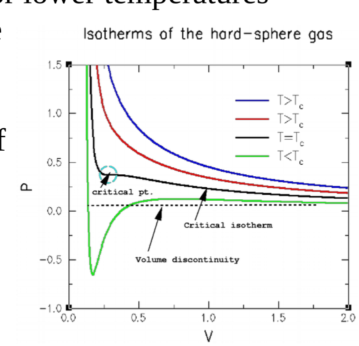
- Law of partial pressures: $P_{total} = \sum_{r=1}^{i} P_r$

2.8 Real Gases

- Compressibility factor: $Z = \frac{PV}{\bar{R}T}$
    - For ideal gases, $Z = 1$
    - For real gases: $\lim_{P \to 0} Z = 1$, $\lim_{P \to \infty} Z > 1$, b/c of non-negligible volume of particles
- Van der Waals equation: $(P + a\bar{V}^{-2})(\bar{V} - b) = RT$, $P = \frac{RT}{\bar{V} - b} - aV^{-2}$,
    - a is related to IMFs, b is related to volume of particles
    - a proportional to speed and number of collisions, both of which are inversely proportional to the molar volume
- Virial equation of state: $Z = 1 + \frac{B}{\bar{V}} + \frac{C}{\bar{V}^2} + \frac{D}{\bar{V}^3} + \cdots = 1 + B'P + C'P^2 + D'P^3 + \cdots$
    - Uses empirical data (aot VdW, which tries to account for physical properties)
    - Really only need first and second terms for reasonable accuracy, size of following terms is negligible for practical purposes

2.9: Condensation of Gases and the Critical State

- At high temperatures, P-V curves are roughly hyperbolic, not so for lower temperatures
- At lower temperatures, when compressing a gas (moving rtl on the P-V graph), pressure rises until a certain point, then change in volume without change in pressure (condensation), then rapid increase in pressure for small decreases in volume (compression of a liquid).



- Critical constants of a substance: P_c, T_c, V_c
    - T_c is the isotherm at which the gas cannot condense anymore
    - P_c and V_c are the points on the T_c isotherm at which the isotherm is tangent to the condensation curve
    - Critical constants are often used to estimate VdW constants:

        - $a = \frac{27R^2T_c^2}{64P_c}, b = \frac{RT_c}{8P_c}$

<u>3.1: The Kinetic Theory of Gases Model</u>

- Assumptions made:
  - Gases are many particles, size negligible compared to distances b/t them.
  - Molecules have mass but negligible volume.
  - Molecules are constantly in random motion.
  - Collisions between molecules are completely elastic.
  - There are no IMFs b/t particles.

<u>3.2: Pressure of a Gas</u>

- $P_{gas} = \frac{Nm\bar{v^2}}{3V} = \frac{2N}{3V}\bar{E}_{trans}$

  Derivation:

  $\Delta_{collision}(mv_x) = 2mv_x$

  $f_{collision} = \frac{2l}{v_x}$

  $P_{gas} = \frac{F}{A} = \frac{Nma}{l^2} = \frac{Nmv_x^2 f}{l^2} = (2mv_x)\left(\frac{2l}{v_x}\right) = \frac{Nmv_x^2}{l^3} = \frac{Nmv_x^2}{V}$

  $\bar{v^2} = \bar{v_x^2} + \bar{v_y^2} + \bar{v_z^2} = 3\bar{v_x^2}, v_x \approx v_y \approx v_z$

  $P_{gas} = \frac{Nm\bar{v^2}}{3V} = \frac{2N}{3V}\bar{E}_{trans}$

  (this derivation assumes constant volume, will be used in constant volume heat capacity)
- $\bar{E}_{trans} = \frac{3}{2}k_BT$
- $v_{rms} = \sqrt{\bar{v^2}} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3k_BT}{m}}$

<u>3.8: Equipartition of Energy</u>

- Each molecule has 3N DOFs
  - $\bar{E}_{trans} = \frac{3}{2}RT, \frac{1}{2}RT$ for each of three translational DOFs
  - For monatomic gases, only translational energy; for polyatomic gases, also rotational and vibrational energies
    - For linear molecules, 2 rotational DOFs with $\frac{1}{2}RT$ energy each; for nonlinear molecules, 3 rotational DOFs with $\frac{1}{2}RT$ energy each
    - For linear molecules, 3N-5 rotational DOFs with RT energy each ($\frac{1}{2}RT$ for both a kinetic and potential energy term); for nonlinear molecules, 3N-6 vibrational DOFs with RT energy each
  - Usually vibrational energy ignored at normal temperatures and pressures
  - $\bar{U} = \frac{3}{2}RT + \frac{1}{2}RT(\text{rotational DOFs}) + RT(\text{vibrational DOFs})$
  - $\bar{U}_{monatomic} = \frac{3}{2}RT$
  - $\bar{U}_{diatomic} = \frac{5}{2}RT$ (ignoring vibrational energy)

- $\bar{C}_V = \left(\frac{\partial \bar{U}}{\partial T}\right)_V$
    - Heat capacity is dependent on temperature, but can treat it like a constant in a small range of temperatures.

4.1: Work and Heat

- $w = -P_{ex}\Delta V$
  Derivation:
  $P_{ex} = \frac{mg}{A}$
  $w = -mg\Delta h =_{ex} \Delta h = -P_{ex}(V_2 - V_1)$
- For a reversible process: $w = -nRT \ln \frac{V_2}{V_1} = -nRT \ln \frac{P_1}{P_2}$
    - Reversible work is the maximum amount of work possible for this change in pressure and volume, but it is impossible to achieve in real life; reversibility is a measure of efficiency.
- Work is not a state function.
- Heat is also not a state function.

4.2: The First Law of Thermodynamics

- $dU = đq + đw$

4.3: Enthalpy

- In constant pressure: $\Delta U = q_p - P\Delta V \implies q_p = \Delta(U + PV)$
- Definition: $H = q_p = U + PV$
    - State function, because U, P, V are all state variables
    - When under constant pressure, is equal to q_p
- $\Delta H = \Delta U + P\Delta V + V\Delta P + \Delta P\Delta V$
    - Last term is not always non-negligible
    - If at constant pressure, then $\Delta H = \Delta U + P\Delta V$, $\Delta U = \Delta H - P\Delta V$
- ΔU doesn't have to equal ΔH for every equation
    - In chemical reactions involving evolution of gases (or reduction of moles of gases), the difference is $\Delta U - \Delta H = -P\Delta V = -RT\Delta n$, where n is the change in moles of gas, because some expansion work is done by gases and lowers the internal energy of the gas

4.4: A Closer Look at Heat Capacities

- $\bar{C}_V = \left(\frac{\partial \bar{U}}{\partial T}\right)_V \Rightarrow d\bar{U} = \bar{C}_V dT \Rightarrow \Delta U = n\bar{C}_V \Delta T$
- $\bar{C}_P = \left(\frac{đq}{dT}\right)_P = \left(\frac{\partial \bar{H}}{\partial T}\right)_P \Rightarrow d\bar{H} = \bar{C}_P dT \Rightarrow \Delta H = n\bar{C}_P \Delta T$
    - While these are written in terms of the heat capacities at constant volume and pressure, it works for any process because U is a state function
    - Here, heat capacities are treated independently of temperature, which is usually an acceptable approximation
- For an ideal gas, $\bar{C}_P - \bar{C}_V = R$
  Derivation:
  $H = U + PV = U + nRT$
  $dH = dU + nRdT$
  $n\bar{C}_P dT = n\bar{C}_V dT + nRdT$

4.5: Gas Expansion

- Isothermal expansion when T constant
  - $dU = dH = 0$, $q = -w$
  - Reversible and irreversible work calculated in section 4.1
- Adiabatic expansion when no heat exchange
  - $q = 0$, $dU = w$
  -

PHYSICAL CHEMISTRY
Chapter 5: The Second Law of Thermodynamics

5.1. Spontaneous Processes

- The reverse of a spontaneous process won't happen under the same conditions
- Viewing events with the random movement of many molecules makes certain events very unlikely to occur (i.e., these events would require the synchronization of a huge number of molecules)
- Cannot base spontaneity on a system losing energy (although this is common for spontaneous events), but rather entropy

Significant examples:

- Ball bouncing and losing thermal energy with each bounce; reverse process (gaining thermal energy from ground and rising up) cannot be spontaneous (intuitively)

5.2. Entropy

- Statistical definition of entropy: entropy is proportional to the number of molecules N, and probability is related to an exponential function of N, so let entropy $S \propto \ln W$, S is entropy, W is probability
    - Let the proportionality constant be k_B (Boltzmann's constant), so $S = k_B \ln W$
    - Because S is a state function, the path doesn't matter
    - For calculating changes in entropy: $\Delta S = S_2 - S_1 = k_B \ln \dfrac{W_2}{W_1}$
    - For a gas, $W = (CV_1)^N$ (approximately volume raised to the Nth number of molecules), so $\Delta S = k_B \ln \left(\dfrac{V_2}{V_1}\right)^N = nR \ln \dfrac{V_2}{V_1}$
        - Only for isothermal expansion, because entropy is affected by temperature
- Thermodynamic definition of entropy: $\Delta S = \dfrac{q_{rev}}{T}$ (or $dS = \dfrac{dq_{rev}}{T}$)
    - Derivation: For an isothermal, reversible expansion: $q = -w; q_{rev} = nRT \ln \dfrac{V_2}{V_1} \Rightarrow \dfrac{q_rev}{T} = nR \ln \dfrac{V_2}{V_1} = \Delta S$
    - Note that this equation works for all substances (not only gases)
    - Note that this equation requires the reversible q (even though $\Delta S_{rev} = \Delta S_{irrev}$ because state function, this calculation requires q_{rev})

Significant examples:

- Likelihood of finding N molecules in half a container vs. the whole container

5.3. The Carnot Heat Engine

- A heat engine converts heat to mechanical work (e.g., steam engines and steam-powered electricity generators)
- Carnot heat engine is an idealized heat engine of one mole of gas with four steps:
    - Four steps:
        - Isothermal, reversible expansion from V_1 to V_2 (at T_2)
        - Adiabatic, reversible expansion from V_2 to V_3 ($T_2 \rightarrow T_1$)
        - Isothermal, reversible compression from V_3 to V_4 (at T_1)
        - Adiabatic, reversible compression from V_4 to V_1 ($T_1 \rightarrow T_2$)
    - Properties:
        - $\Delta U(\text{cycle}) = 0$

- $q(\text{cycle}) = RT_2 \ln \frac{V_2}{V_1} + RT_1 \ln \frac{V_4}{V_3}$
  - $q_2 = RT_2 \ln \frac{V_2}{V_1}$ (heat absorbed from heat reservoir, positive value)
  - $q_1 = RT_1 \ln \frac{V_4}{V_3} = -RT_1 \ln \frac{V_2}{V_1}$ (heat discharged to cold reservoir, negative value)
- $w(\text{cycle}) = -RT_2 \ln \frac{V_2}{V_1} + \bar{C}_V(T_1 - T_2) - RT_1 \ln \frac{V_4}{V_3} + \bar{C}_V(T_2 - T_1) = -RT_2 \ln \frac{V_2}{V_1} - RT_1 \ln \frac{V_4}{V_3}$
  - From volume equation below, $w(\text{cycle}) = -R(T_2 - T_1) \ln \frac{V_2}{V_1}$
- $\frac{V_2}{V_1} = \frac{V_3}{V_4}$
  - Derivation: $P_1V_1 = P_2V_2$; $P_3V_3 = P_4V_4$; $P_2V_2Y^\gamma = P_3V_3^\gamma$; $P_1V_1^\gamma = P_4V_4^\gamma$; $\frac{P_2V_2^\gamma}{P_1V_1^\gamma} = \frac{P_3V_3^\gamma}{P_4V_4^\gamma} \Rightarrow \frac{P_2V_2}{P_1V_1} \times \left(\frac{V_2}{V_1}\right)^{\gamma-1} = \frac{P_3V_3}{P_4V_4} \times \left(\frac{V_3}{V_4}\right)^{\gamma-1} \Rightarrow \left(\frac{V_2}{V_1}\right)^{\gamma-1} = \left(\frac{V_3}{V_4}\right)^{\gamma-1}$

- Thermodynamic efficiency is the ratio of work done by heat engine to heat absorbed by engine ($\text{efficiency} = \frac{\text{net work done by heat engine}}{\text{heat absorbed by engine}} = \frac{|w|}{q}$)
  - For Carnot engine, $\text{efficiency} = \frac{R(T_2 - T_1) \ln \frac{V_2}{V_1}}{RT_2 \ln \frac{V_2}{V_1}} = \frac{T_2 - T_1}{T_2} = 1 - \frac{T_1}{T_2}$
    - Cannot be totally efficient; efficiency maximized by maximizing difference between T_1 and T_2
- For any cyclic process, $\sum_i \frac{q_i}{T_i} = \sum_i \Delta S_i = 0$
- Refrigerators, air conditioners, and heat pumps reverse the flow of spontaneous heat flow by applying work: $-q_2 = q_1 + w$
  - Coefficient of performance (COP) is measure of refrigerator or air conditioner's performance: $\text{COP} = \frac{q_1}{w} = \frac{T_1}{T_2 - T_1}$; gives the maximum COP value because it uses reversible heat value
  - COP for heat pumps is $\frac{T_2}{T_2 - T_1}$; more efficient than an electric heater

5.4. The Second Law of Thermodynamics

- Total change in entropy for a reversible process, $\Delta S_{surr} = 0$, $q_{surr} = -q_{sys}$
  - For an irreversible process, $\Delta S_{univ} \geq 0$

5.5. Entropy Changes

- Entropy change for reversible, isothermal expansion of an ideal gas is $nR \ln \frac{V_2}{V_1}$
- $\Delta_{mix}S = n_AR \ln \frac{V_A + V_B}{V_A} + n_BR \ln \frac{V_A + V_B}{V_B}$ for mixing of two gases (can be thought of as two gas expansions)
  - $\Delta_{mix}S = -R(n_A \ln x_A + n_B \ln x_B)$ (greater than 0, so spontaneous)
- $\Delta_{fus}S = \frac{\Delta_{fus}H}{T_f}$ (because enthalpy of fusion is equal to reversible heat)
  - Trouton's rule is an empirical observation that many liquids have a similar entropy of vaporization (because of similar structures of most liquids and gases)

- $\Delta S = C_p \ln \frac{T_2}{T_1}$ when heating

CH160 TEST 2 OUTLINE
JONATHAN LAM

Three Laws of Thermodynamics
1. Energy is always conserved.
2. The entropy of the universe never decreases.
3. The entropy of a crystalline structure at absolute zero has zero entropy.

Energy and Enthalpy
$\Delta H = q_p$ at constant pressure (e.g., coffee-cup calorimetry)
$\Delta U = q_v$ at constant volume (e.g., bomb calorimetry)
Difference between ΔH and ΔU is:
$\Delta H - \Delta U = \Delta(PV) = RT\Delta n$, at constant T, Δn is change in number of moles of gas

Calculating properties for reactions given properties of components
$\Delta H = \sum H(\text{products}) - \sum H(\text{reactants})$
$\Delta H = \sum BE(\text{reactants}) - \sum BE(\text{products})$ (notice sign reversal here)
$\Delta S = \sum S(\text{products}) - \sum S(\text{reactants})$
$\Delta G = \sum G(\text{products}) - \sum G(\text{reactants})$

For a non-standard temperature, $\Delta H = \Delta C_p \Delta T$

Entropy calculations
$S = k_B \ln W$ (W is number of countable microstates)
$S = k_B \ln 1 = 0$ for crystalline structure with no residual entropy at absolute zero
$S = \dfrac{q_{rev}}{T}$
$\Delta S_{univ} = \Delta S_{sys} + \Delta S_{surr} \geq 0$ (inequality for irreversible, equality for reversible)
$\Delta_{exp} S = nR \ln \dfrac{V_2}{V_1}$
$\Delta_{mix} S = -R\left(n_A \ln x_A + n_B \ln x_B\right)$
$\Delta_{vap} S = \dfrac{\Delta H}{T}$
$\Delta_{heating} S = n\bar{C_p} \ln \dfrac{T_2}{T_1}$
$\Delta S_{cycle} = 0$

Carnot Heat Engine
$\Delta S = 0$
$q = q_2 + q_1 = nR(T_2 - T_1) \ln \dfrac{V_2}{V_1}$
$w = -nR(T_2 - T_1) \ln \dfrac{V_2}{V_1}$
efficiency $= \dfrac{|w|}{q_2} = 1 - \dfrac{T_1}{T_2}$
For a reverse heat engine, $COP = \dfrac{q_1}{w} = \dfrac{T_1}{T_2 - T_1}$

For a heat pump, $COP = \frac{q_1}{w} = \frac{T_2}{T_2 - T_1}$

Main energy equations (definitions)
$G = H - TS$ (constant pressure, temperature)
$A = U - TS$ (constant volume, temperature)
$H = U + PV$ (constant pressure)
$U = H - PV$ (constant volume)

Process spontaneous when $\Delta G < 0$ (constant temperature and pressure) or $\Delta A < 0$ (constant temperature and volume); these are equivalent to saying $S_{univ} \leq 0$. If $\Delta G < -10\text{kJ}$, mostly products; if $\Delta G > 10\text{kJ}$, mostly reactants. Process can change spontaneity, namely:

		$\Delta_r H$	
		+	-
$\Delta_r S$	+	At high temperatures	Always
	-	Never	At low temperatures

For processes that do change spontaneity, turning point can be calculated by setting $\Delta G = 0 \Rightarrow T = \frac{\Delta H}{\Delta S}$.
Process endothermic when $\Delta H > 0$

Important total derivatives of main energy equations
$dG = VdP - SdT$
$dA = -PdV - SdT$
$dH = TdS + VdP$
$dU = TdS - PdV$

Basic derivatives of main energy equations
$\left(\frac{\partial G}{\partial P}\right)_T = V,\ \left(\frac{\partial G}{\partial T}\right)_P = -S, \ldots$
e.g., at constant temperature and changing pressure, ΔG is:
$\Delta G = nRT \ln \frac{P_2}{P_1}$ for an ideal gas
$\Delta G = V\Delta P$ for a solid/liquid

Maxwell relations
$$\left(\frac{\partial V}{\partial T}\right)_P = -\left(\frac{\partial S}{\partial P}\right)_T,\ \left(\frac{\partial P}{\partial T}\right)_V = \left(\frac{\partial S}{\partial V}\right)_T,\ \left(\frac{\partial T}{\partial P}\right)_S = \left(\frac{\partial V}{\partial S}\right)_P,\ \left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial P}{\partial S}\right)_V$$

Derivations of meanings of Helmholtz and Gibbs Free Energies
$dA = dU - TdS = dq_{rev} + dw_{rev} - dq_{rev} = dw_{rev}$ (at constant T)
$dG = dH - TdS - SdT = (dU + PdV + VdP) - TdS - SdT$
$= (dq_{rev} + dw_{rev}) + PdV + VdP - TdS - SdT$
$= TdS - PdV + w_{oth} + PdV + VdP - TdS - SdT = VdP - SdT + w_{oth}$
$= w_{oth}$ at constant (P, T)

Gibbs-Helmholtz equation

Relates temperature dependence of Gibbs energy change to the enthalpy change

$$\left(\frac{\partial\left(\frac{\Delta G}{T}\right)}{\partial T}\right)_P = -\frac{\Delta H}{T^2}$$

GAL PROGRAMMING ASSIGNMENT WRITEUP
PROF. KIRTMAN
ECE 251 — COMP. ARCH.
JONATHAN LAM

PART 1

Part I (combinational logic) was done with basic discrete logic:

```
Y0 = !A;
Y1 = A & B;
Y2 = A $ B;
```

PART 2

I did Part 2 (the 3-bit up/down counter) three different ways. For all of the methods, the output enable was asynchronous (`CNT.oe = OE;`), and the other pins all made synchronous changes.

1. Using a finite state machine (`LAMHW1-FSM.*`)

Using the `SEQUENCE` command and some conditionals, the three counter bits could easily be made to advance, decrease, reset to zero, or stay the same.

```
FIELD CNT = [Z2..0];
CNT.sp = 'b'0;
CNT.ar = 'b'0;
SEQUENCE CNT {
    $REPEAT i = [0..7]
        PRESENT S{i}
            IF EN & CLR NEXT 'b'0;
            IF EN & !CLR & DIR NEXT S{(i + 1) % 8};
            IF EN & !CLR & !DIR NEXT S{(i + 7) % 8};
            IF !EN NEXT S{i};
            DEFAULT NEXT S{i};
    $REPEND
}
```

I realized that the variables for the counter could not be part of the same indexed variable as the variables for the combinational logic output (i.e., the indexed `Y0..Y5` variable), so the counter variables are labelled `Z0..Z2` for this program.

2. Using a truth table (`LAMHW1-TT.*`)

Rather than using conditionals, this approach mapped six inputs (3 counter variables and `EN`, `DIR`, `CLR`) to the D inputs of the three counter macrocells using a single truth table. Using a truth table throws a warning if indexed variables are used, so I renamed `Y3..Y5` to `Y3O`, `Y4O`, and `Y5O`.

The order of the variables here is important to make the truth table work with the `$REPEAT` macro and binary literals, which needed the `EN` input to be the MSB and the `DIR` and `CLR` to be the LSBs.

```
$REPEAT i = [3..5]
    PIN {i + 14} = Y{i}O;
    Y{i}O.sp = 'b'0;
```

```
    Y{i}O.ar = 'b'0;
    Y{i}O.oe = OE;
$REPEND

FIELD input = [EN, Y5O, Y4O, Y3O, DIR, CLR];
FIELD output = [Y5O.d, Y4O.d, Y3O.d];
TABLE input=>output {
    'b'1XXXX1=>'b'0;                    /* EN=1, CLR=1 */
    $REPEAT i = [0..7]
        'b'{i}XX=>'b'{i};               /* EN=0 */
        'b'{i + 8}10=>'b'{(i + 1) % 8}; /* EN=1, CLR=0, DIR=1 */
        'b'{i + 8}00=>'b'{(i + 7) % 8}; /* EN=1, CLR=0, DIR=0 */
    $REPEND
}
```

3. Using discrete logic (`LAMHW1-LOGIC.*`)

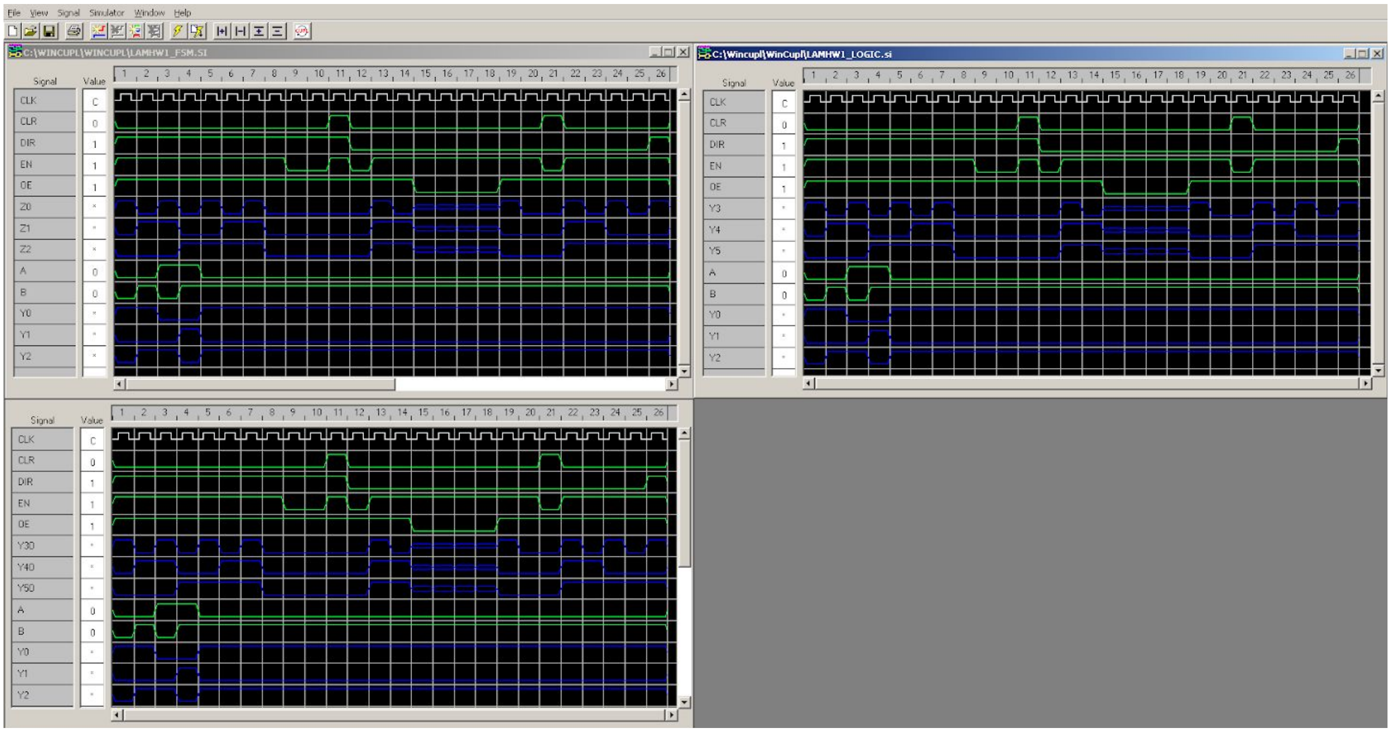
This actually turned out to be the quickest solution because of the simple logic and because I spent much more time learning to use more complex structures for the other two approaches before I tried this one. The logic follows a mostly intuitive path and is not minimized here in the code. (The only minimization I did was a K-map for `DIR`, `Y3`, `Y4`, `Y5` to get the innermost expression.)

```
FIELD CNT = [Y3..Y5];
CNT.sp = 'b'0;
CNT.ar = 'b'0;

Y5.d = (!EN & Y5)
     # (EN & !CLR & ((DIR & !(Y5 $ Y4 $ Y3) # !DIR & Y5) $ !(Y4 # Y3)));

Y4.d = (!EN & Y4)
     # (EN & !CLR & (!DIR $ Y4 $ Y3));

Y3.d = (!EN & Y3)
     # (EN & !CLR & !Y3);
```

TESTING AND RESULTS

Most of the testing was done using WinSim. The provided `*.si` files use the same order and vectors to show that all of these work the same, and for consistency to show that all of them provide the same (correct) output. A screenshot of the three simulations in WinSim is shown in Figure 1.

Figure 1. WinSim Simulations
(From top right, CCW: LOGIC, FSM, TT)



Once the simulation produced the correct results, writing the program to the Chipmaster and implementing it was straightforward and without bugs. (I only tested TT and FSM versions on the physical circuit, but having the same results in the simulation makes it likely that the LOGIC version will also produce a working circuit.) A schematic of the implementation is shown in Figure 2, and the specific implementation is shown in Figure 3. The project can be seen working in the video.

NOTES

A 555-timer was used to generate a low-frequency astable clock signal. A 7-segment display (M74 1318) and the 4511 7-segment driver were added to display the output of the counter for easier viewing of the counter's value.

When setting OE high, the 7-segment display shows “0,” but the counter bits should be high-impedance as per the simulation. (This is probably the default behavior of the 4511 when its inputs are not driven.)

In the video and in the annotated diagram, the GND wire is actually connected to pin 11 (an input), but still worked normally. When connecting GND to the correct input, the circuit behaved the same.

Figure 2. Schematic Diagram of Implementation

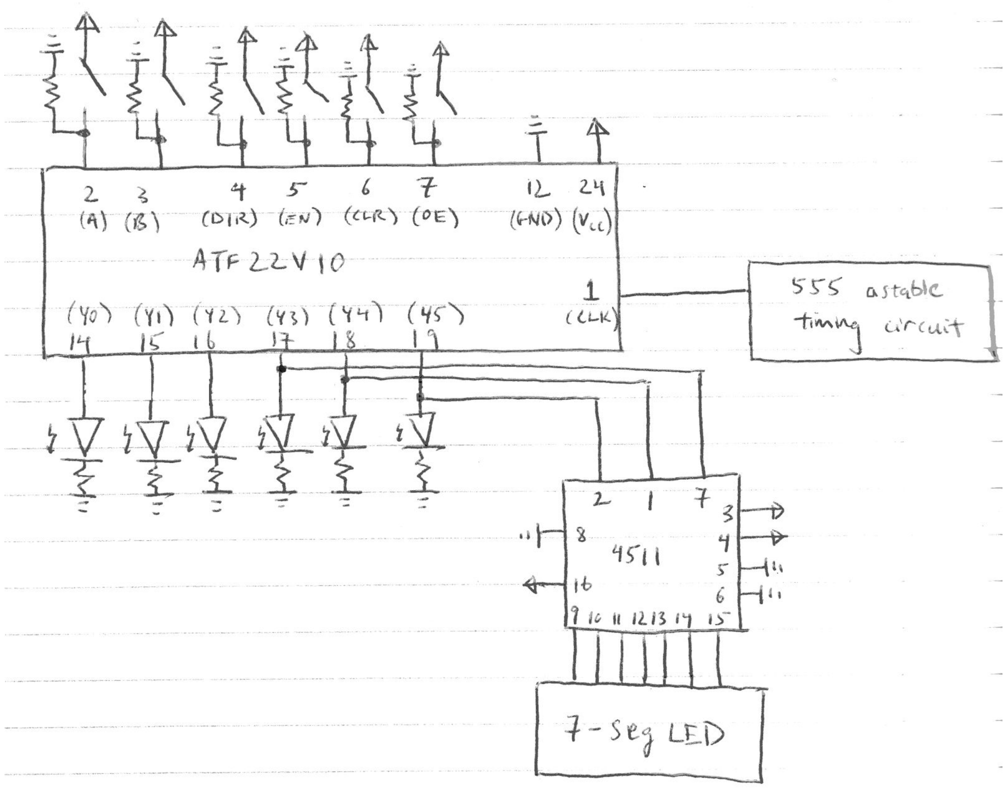
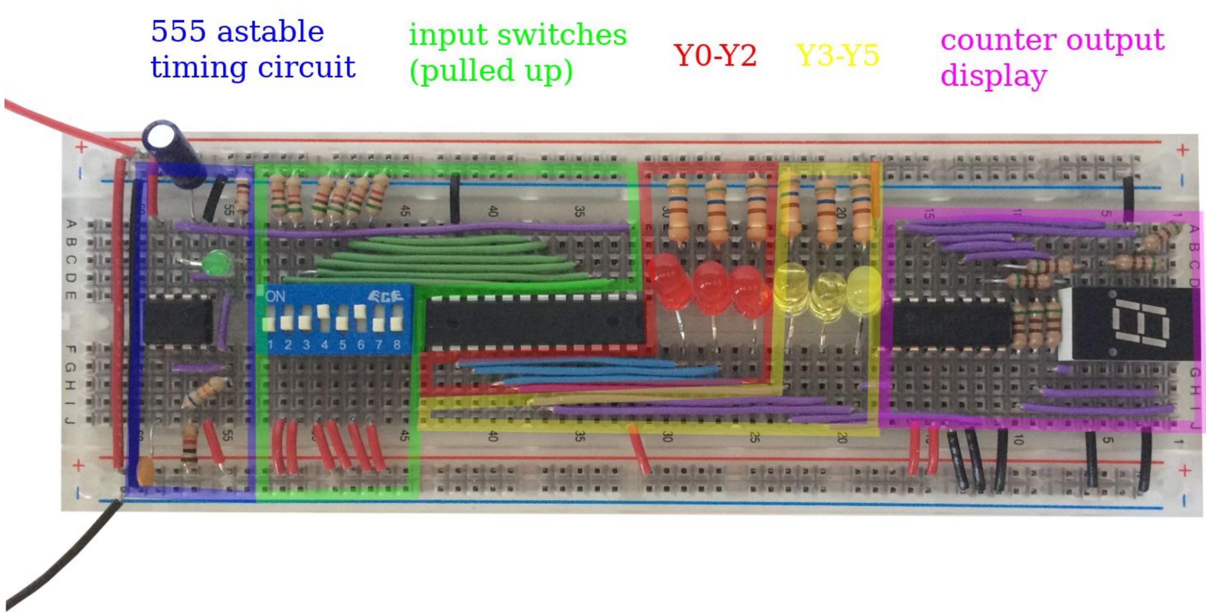


Figure 3. Annotated Image of Implementation



Jonathan Lam
Prof. Kirtman
ECE 251
Computer Architecture
4 / 25 / 19

Homework 2: Pic Programming

Implementation of given demo program

The connections necessary for the implementation of the given program were a high signal to master clear (MCLR), power and ground to VDD and VSS, a 4MHz oscillator and two capacitors on the oscillator inputs, and eight output LEDs from the PORTC (RC0..7) pins. Figures 1 and 2 are images of the (full) implementation.

The timing of the program was verified using an oscilloscope. The 4MHz clock was running at 4.0MHz as expected, and the frequency of the LSB was 1.992Hz (i.e., one HIGH-LOW period every ~0.5s, or a count every ~0.25s). This was verified with the debugging stopwatch tool in the MPLAB IDE, which counted 251006 instruction cycles (1μs per instruction cycle) per count. (Specifically, L3 takes 10μs per iteration, L2 takes 100*L3+4=1004μs per iteration, and L3 takes 250*L2+6=251006μs per iteration, leading to a 1000000/251006=3.984Hz counting signal, or 1.992Hz “signal” for the LSB.) No attempt was made to fix the slight offset.

Implementation of two control switches

First, the lowermost two TRISB bits were set high to make RB0 and RB1 inputs. (PORTB was chosen because had a similarly easy setup as PORTC.) RB0 was used as the start/stop input, and RB1 was used as the count by two input.

To implement start/stop behavior, the final two nop commands in L3 were replaced with a check for the start/stop input bit. If the start/stop bit (RB0) was not set, execution would continue and the code would goto L1, restarting the iteration count. (As long as RB0 stayed low, execution would always hit this goto and restart the iteration count before COUNT would update.) If the start/stop bit was set, then the goto statement would be skipped. Because the btfss command used to iterate this conditional behavior takes two instruction cycles on a skip and two nops were removed, timing was preserved.

To implement count by two behavior, btfsc was used to increment COUNT a second time if RB1 was set. If RB1 was not set, then the second increment would be skipped. This adds an additional (trivial) 2μs to each count timer.

Refactoring delay into subroutine

The L2 and L3 inner loops, which were used simply to iterate through a certain number of nop commands, were moved into a subroutine called DELAY. Every iteration of L2 runs for approximately 1ms (1004μs), so the number of iterations of L2, which is read from the W register on the start of the DELAY subroutine, is roughly the number of milliseconds of delay. To recreate the same 0.25s delay, I manually set .250 into the W register before calling the DELAY subroutine.

Figure 1: Annotated photo of implementation

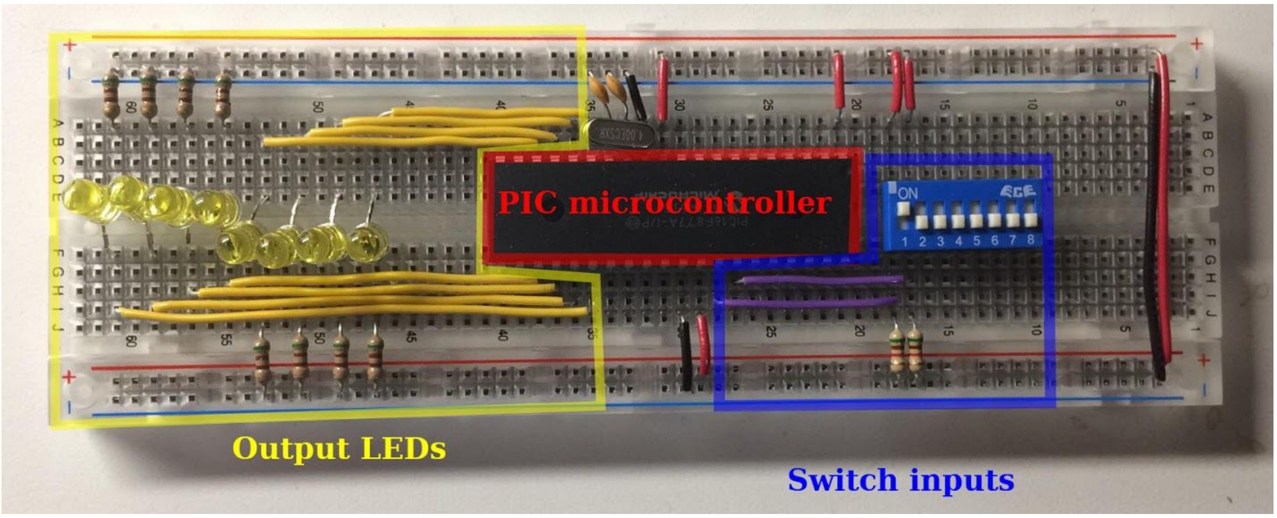
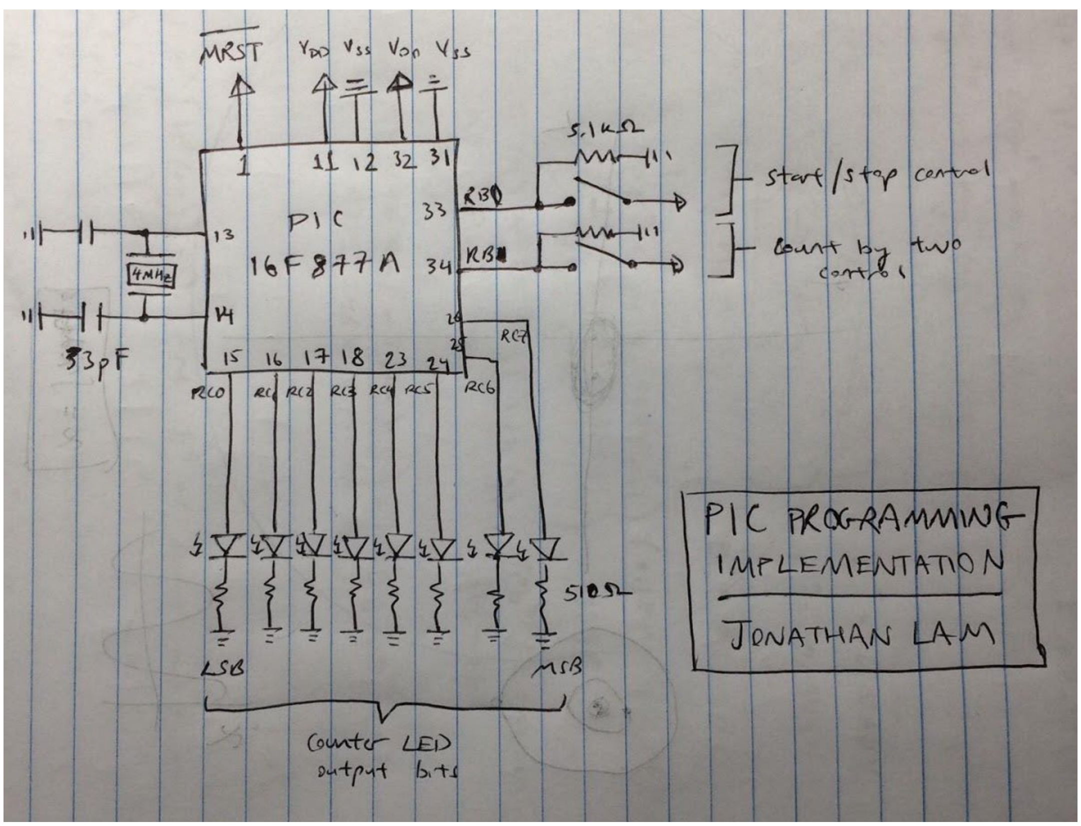


Figure 2: Schematic of implementation



Comments on the Revisions to the Utopia Paper

The first draft of this essay turned out pretty fluid, so I focused on mechanics rather than the main ideas. This was sparked by our discussion of the aspect of clarity in the discussion of the claim. My original claim was this wordy bumble:

> The story of More's Utopia is limited to fiction by the fact that it was created as a thought experiment useful only in solving a specific set of societal evils from the perspective of a the specific contemporary setting, and was not designed to self-correct or adapt in the case of disruptive internal or external stimuli.

There are a lot of unnecessarily wordy ("by the fact that it was created as …") or redundant parts ("specific … specific," "internal or external") in this sentence, so I reduced it down to the following neater sentence, which also got more specific to the overall claim of the paper (about rigidity):

> More's Utopian nation is limited to fiction by the rigidity in its goal of solving a specific set of contemporary social evils, a goal which overlooks the necessity and ability to adapt to disruptive stimuli.

Most of the (few) revisions continued in this manner. Another example is the final sentence, which I felt didn't do the conclusion justice.

Looking back at this essay, I noticed how little was actually original thought. Sure, this essay flowed well, but it was really just some of the dogmas of Marx, More, and the American Dream thrown in from a modern-biased view, and I'm afraid that limited how useful or interesting this paper turned out to be. (I.e., It feels like I'm arguing *their* views, not *my own*.) I didn't actively revise based on this, but I think the significance and originality is definitely something that'll be important to focus on when crafting future papers. And if you see any of this on future papers, please point them out to me. Thanks!

Jonathan Lam

p.s. I found it interesting and fitting to keep the contemporary (relevant?) 16C form "innumerous." Neither has the meaning changed in the modern equivalent nor has it become unrecognizable.

Jonathan Lam
Professor Germano
HSS2-K
20 February 2019

A Past Ideal: The Modern Irrelevance of More's Utopia
A Practical Social Consideration

When discussing the coming summer, my friend and I discovered a mutual dislike for each other's plans. For he was interested in doing independent research, and I a corporate internship. His argument was that he "[didn't] want to become a cog in a machine." Research, on the other hand, would be more fulfilling because it would be easier to make a recognizable impact on his own. My answer was, yes, you lose some independence working in a company, but there's a greater movement towards a greater goal; in the end, the amount of individual impact is comparable. Plus, companies (large technology businesses) tend to have a particular youthful culture that makes the work much more welcoming.

We agreed to disagree. What we had in common, however, was the ambition to find challenging work and make an impact on the world. Is that a bad thing? No— it's a movement towards innovation, and forging a vocational identity and economic stability for ourselves. But the second effect of that statement (the collection of wealth) is in disagreement with the nation of Utopia in Thomas More's novel of the same name. The idea of private ownership is intentionally made entirely foreign to the Utopians: people can contribute to (and receive their fair part from) the greater good without greed. Citizens work fewer hours, are healthy and strong from adequate nutrition and exercise, and never have the mental burden of worrying about personal or societal peril, whether financially or politically. The case of my friend and I wouldn't exist in Utopia— we wouldn't need to worry about finding summer opportunities because a secure future working at a utilitarian craft is guaranteed.

What's the catch? Utopia isn't real. It was never real, and it never will be, at least not at the scale of the fictional nation set in More's novel. Perhaps denying its existence due to lack of historical pretext or personal experience falls into the logical fallacy More is accused of by Hythloday (More 48), but there are a number of factors that deny its plausibility and usefulness in a real world. Utopia acts as a model society to those around it, yet avoids complexities such as internal factions or foreign interaction, solidifying the society's place place in fairytales, propaganda, and thought experiments. And Utopia's ability to impose societal goals on its citizens in the form of individual laws is the last nail in the coffin, stripping away any personal willpower and individuality. This is the outline of a society outwardly righteous and prosperous, but lacking in the flexibility to respond to attacks to its ideology. More's Utopian nation is limited to fiction by the rigidity in its goal of solving a specific set of contemporary social evils, a goal which overlooks the necessity and ability to adapt to disruptive stimuli.

Before any discussion of the Utopian society, Hythloday entertains More's character with stories of his travels in (and general disgust with) European societies in Book 1 of Utopia, namely:

1. A societal, political, and religious model where flattery towards authority and acting weaselous was considered a legitimate way of rising up the ranks.
2. Wealthy people were known to monopolize the farmland by means of "enclosure," buying the rights to land for exclusive use and destroying the land with herds of voracious sheep.
3. The poorly-designed penal system, especially with regard to the death penalty for theft. (He argues that this would make would-be thieves murderers, because the penalty of murder is equal to that of theft and a witness is destroyed in the process.)

Hythloday presents a logical argument of cause and effect, especially between the evils of enclosure and the theft that results from the idleness of innumerous farmers who lose their jobs. These problems can be condensed into the observation that "one herdsman or shepherd is sufficient to graze livestock on ground that would require many hands to cultivate and grow crops" (23). The great problem is that land that can be shared— and profited from— by *many* farmers can be bought and destroyed by *one* wealthy person, which in turn is a consequence of a society ruled by the wealthy.

This is followed by Book 2, which is a description of a society in which none of these problems occur. Their solutions are not addressed as straightforwardly as the problems themselves, but several aspects of Utopian rules discourage these misconducts. More's tone is superficially neutral and informative, but the story is told with a blatant lack of fault in any part of the Utopian society. Everything in the Utopian system is, to Hythloday and More, a logical consequence of some Utopian rule; and this supposedly sound logic never loses battles, never allows for economic collapse, and never causes discontent among the people over a continuous two millennia. This bias reinforces the impossibility of Utopian society, and therefore care should be taken to not foolishly accept every apparently logical conclusion drawn by the Utopians without critical consideration.

If Utopian principles are considered only from the context of the problems in Europe they aim to solve, it seems that the system is highly successful. The Utopians do away with personal wealth and possessions, and allow households the security of being able to take what they need from the warehouses, so there is no incentive for thievery. Besides a small hierarchy of elected officials, the general populace of Utopia live in equity, without wealth and with equal job opportunity, no one having power over another nor able to sabotage another's finances. Of course, bribery fails without money or possessions of value, and the decentralized religion eliminates the possibility of citizens exploiting a religion for power.

This naturally fits into communism, defined by the OED as "a theory that advocates the abolition of private ownership, all property being vested in the community, and the organization of labour for the common benefit of all members; a system of social organization in which this theory is put into practice." Many of the proposed benefits of a communist system are discussed, such as the anticipated growth of productivity and selflessness of the people, but the opposite effects are observed when the theory is implemented on the scale of nations (as observed in soviet Russia and communist China in the 20th century). The fact is that humans, like any animals, respond optimistically to positive feedback; when the economic return of a person's work is not representative of the labor invested (as is the case in a pure communist society), people naturally work subproductively. Of course, having written the book three centuries before the publication of the Communist Manifesto and implementation of communist societies, More understandably had no knowledge of the implications of a real implementation.

But suppose that communism could work, as communist-like practices have been implemented well on a smaller scale. A modern metaphor is a medical practice, a small group of doctors in the same specialty, in which the revenue is split evenly among the doctors. In a small and educated society, the members can all fully understand and agree to the advantages of equally-distributed payout. Utopia is tiny compared to real communist nations such as China or Russia, where size limited its efficacy; suppose that its citizens could all be thoroughly educated in the benefits of their system. Even then, there are still some glaring aspects of Utopian society that would make it non-ideal.

The greatest mistake of the Utopian system is its rigidity. One component of this is the very fact that many of its residents believe so strongly in the perfection of the system. If a system is ideal, why would there ever need to be any change? (Modern engineers may disagree, but the relative difficulty of sixteenth-century life disallowed much of the creative freedom that modern people take for granted.) And even if there is a need or want for improvement, the system of equity somewhat dampens an intellectual's ability to innovate. Lack of a meritocracy forces all scholars into an equal standing, rather than in the natural managerial hierarchy based on experience in the field. While it might be useful to mix professionals and junior workers in many trades for training purposes, high-tech advancements require carefully-organized teams of the brightest people willing to devote many hours each day into it. In Utopian society, there is an interesting "work law" that explicitly lays out over two-thirds of the time of the day, not including meal times and periods of preparation or transition:

> "Dividing the day and night into twenty-four equal hours, they devote only six to work, ... After lunch they take two hours of rest in the afternoon... Sleep takes up eight hours. ... After dinner they devote one hour to recreation" (61-62).

The logic behind the work law is the observation that work in Europe is very unevenly distributed. More cites the idleness given by several significant groups, such as the nonworking spouse, the religious leaders, the noblemen, and the beggars; if all had been put to work, then "it takes far fewer than you thought to produce everything that mortals use" (63). This may be a blessing to workers who want to end work early by putting a minimum work requirement, but it also limits workaholics by setting an upper limit on their work time. Even more harmful about this work law is that it measures time, rather than merit; even with the supervision of the syphogrants, there will always be a large proportion of idle people with the facade of engagement. For, why should a person work harder if his output is measured only in an easily collectible unit of "hours" rather than some physical quantity? If the syphogrant asks why the yield is so low, it is simple to claim disability and still receive equal benefits (here again lies the reality of communism). More also suggests when using the phrase "produce everything that mortals use" that adequate production of goods is the end goal of people's work; complacency arises because striving to invent or innovate is considered overwork that would hurt citizens. Utopia's geniuses and leaders under high-pressure situations may never be realized. The struggle that my friend and I are feeling to compete with millions of other youth to find a decent job is absent in Utopian society. No Utopian will know what it means for a homeless, single-parent to work three jobs, almost twenty hours a day, to barely survive. Therefore no Utopian will never know a certain hardship: the lifelong fight for a better life.

Utopia also has one fundamentally static construct that makes innovation difficult: farming. Farming is Utopia's greatest defense against idleness, because it productively employs a large number of its citizens. Any innovation that greatly improves the efficiency of farming (e.g., modern tractors and crop dusters) would corrupt Utopian society; rephrasing the aforementioned synopsis of Hythloday's strife, *one* machine operator *is sufficient to* harvest many crops *that would require many hands to cultivate and grow crops*. Technology reintroduces idleness just as wealthy land enclosure does, resembling the personal-versus-shared land disparity. This is still problematic in the modern US: machinery and cheap labor in other nations are, in the minds of some economic philosophies, stealing jobs from domestic workers. The problem would be more dire in Utopia because of the less-diversified economy and because of how important it is for everyone to have a job. The ideal of constantly-occupied citizens clashes with technological change that threatens jobs— they cannot coexist.

Natural disasters, clashing internal factions, an arms race, disease or blight, the rise of a globalized economy. More's portrayal of Utopia is an ancient society that has been scarred by none of these factors, cunning or the general health and spirit of the people to the rescue. The Utopians are bred to subsist using farming techniques that have not changed for millennia, and have never had to bring themselves to extremes for survival. In an emergency, the Utopians could not be depended on to think or fight or innovate like their lives depended on it, simply because there was never any need for any of them to try to think or fight or innovate so extraordinarily when it would not garner any more benefits than any subpar man.

Fast-forward half a millennium to consider the United States' modern solutions to the original problems. The punishment for theft has been reduced from a death penalty to jail time and occasionally community service (the latter reminiscent of Utopia); a republican system gives political leaders power through a fair election; and a capitalistic meritocracy with some semblance of equality replaces Utopia's communism. The difference between US equality and Utopian equity is subtle but significant: while equity keeps everyone at the same level of wealth and status, equality puts nobody at an initial advantage, and people's reputations are allowed to grow by merit of their achievements. The latter ideology has allowed the US to survive many human rights conflicts, natural disasters, political and legal disagreements, and world wars, and continues to do so as the world changes at an exponential rate.

Credit must be given where it is due: More's society addresses problematic contemporary systems, with a simple, logically-operated hypothesis that would be revolutionary centuries later. But the specific set of rules in Utopia binds its citizens too closely to this model of mediocre non-idleness, and the result is a society that generally restricts innovation and would react poorly to worldly change. More chooses the idea of a ideal society as one that solves social problems and presents a rigid ideal. But real world problems evolve, and genuinely ideal societies cannot stay ideal without parallel progress.

Works Cited

"communism, n." *OED Online*, Oxford University Press, December 2018, www.oed.com/view/Entry/37325. Accessed 3 February 2019.

More, Thomas. *Utopia*. Translated by Clarence H. Miller. New Haven: Yale University Press, 2001.

IGNORE THIS BRAINSTORM / OUTLINE

~~The society imposes societal goals on the individual, and (necessary) individual needs are suppressed~~
~~From the outside, itt seems like a model society, but only b/c insides are barely~~ **sbsisting**
(also they are willing to eliminate things for their bad uses even when they can be sometimes useful — taking things to extremes)

~~Before any discussion about Utopia, Hythloday begins about current problems in europe~~
- ~~Theft and puitive system~~
- ~~People trying to flatter authority: all want to gain power~~
- ~~Animal capitalism: rich dominate the industry~~

~~Utopia seems to be a society built aroudn solving these~~

Like american society @ its inception: for freedom, no taxes; ideal in theory but then complexities: some founding principles wrong
~~ADAPTATION~~
~~As opposed to Utopia, which seems to have the opposite of change~~
~~Why is this a problem? Society is stable after all~~
- ~~But society is only stable ww/ no external or internal stimuli— cannot handle any change~~

~~More on CHANGE~~
- ~~No want to change: no meritocracy~~
- ~~No want to change: things are ideal already~~
- ~~No ability to change: rules are limiting it (and some of the guiding factors of life are based around the fact that low-tech actions are typically honest and necessary)~~
- ~~I.e., stagnation is a part of their life~~

~~Stable and "ideal", but only in stable circumstances~~
- ~~I.e., also holding back from foreign intervention unless where necessary~~

Inflexibility may hurt individuals and society alike
- No pleasures — less creativity
- No money (gold, diamond) ⇒ no physical use of those
- Honesty is a character trait they want society to eminate, but they are also forcing it on their citizens; honesty is not something you tell something to do, nor is something honest if you are told to do it
- Turnt mental guidlines into hard and fast rules

~~People don't know how to fight/work for their lives, because everything is taken for granted — brittle~~

Conclusion
Bad QM — only built for a specific use case, not flexible enough to self-correct or adapt in other solutions
Good at what it was meant for in ideal conditions but that's about it

PROMPT

HSS2 K&L Professor Germano

Paper #1 Utopia

Thomas More's Utopia presents a strangely neutral vision of a strangely perfect society.

As we look closely at this social vision, it becomes more and more difficult to explain its contradictions. How are we meant to read its weirdly unfamiliar details?

In a five-page paper, use your close reading skills to make a claim about idealized social vision and its limits.

To do this, you'll want to focus on two or three episodes or passages and use them to think about how Utopia combines prescription (or at least a thought experiment) and critique of European sociopolitics.

If you want, you can bring in material and references beyond Utopia, but your paper has to demonstrate your close reading of passages from More.

Five numbered pages, double spaced, stapled (not clipped or crunched), printed on one side of the sheet only.

Name, date, section HSS2K or HSS2L on page 1.

Due in class Monday February 4.

Theft (& other punishment): no death policy, people are put to work (a system sometimes put into practice today)

War: still use mercenaries to fight war, also use bribery -- not very different from regular war

Idea of gems and currency --- make it seem like these things can be abandoned (76), but they are actually very valuable (e.g., gold as anticorrosive conductor and diamond as abrasive agent)

No overall societal purpose, and also problems with scaling up human guidelines on how to be to be hard societal rules

—

IGNORE THIS DRAFT 1

When discussing our plans for the coming summer, my friend and I discovered that we find a mutual disagreement for the other's thoughts. For he was interested in doing independent research (or teaching), and I a corporate internship.

His argument was that he "[didn't] want to become a cog in a machine." Research, on the other hand, would be much more fulfilling, because it would be easier to make an impact on his own. My answer to that was that, sure, you lose some independence working in a company, but there's a greater movement towards a greater goal; in the end, the amount of individual impact (the societal impact of a larger product per employee) would be similar. Plus, (technology) companies tend to have a particular youthful culture that makes the work much more welcoming.

We agreed to disagree. What we had in common, however, was the ambition to make an impact on the world. Is that a bad thing? No— it's a movement towards innovation, and for making a living for ourselves. Better that than a mediocre life of keeping the status quo and receiving a mediocre income. But the second effect of that statement (the collection of wealth) is in disagreement with the nation of Utopia in Thomas More's novel of the same name. The idea of harboring personal wealth or possessions is a completely foreign idea to the Utopians, but to the great benefit of society: people can contribute to (and receive their part from) the greater good— communism at its best. Likewise, personal pleasures and the military are regulated so that they are most efficiently put to the benefit of society. People work fewer hours, are healthy and strong from adequate nutrition and farming, and never have the mental burden of worrying about personal or societal peril, either financially or politically. Unlike my friend and I, things are already the best they can be, and people are afraid to change the system.

What is wrong about this model? For one, it doesn't exist, nor has it ever; but to use this argument is abusing the power of history. Rather, the society fails because it imposes societal goals on individuals and individual goals on society, which is completely inappropriate to ask of either. Perhaps in the smaller, un-globalized European societies of More's time, a Utopia-like nation could exist and function close to his description, but the larger the difference in scale between the individual and the society (i.e., the larger the target community), the more the society is doomed to fail.

The Utopian nation is described as having existed for almost two millennia, and their technology has barely changed since its conception. Farming is still understood as an honest and necessary task, so any able person is expected to partake in it. The houses are arranged as they were for thousands of years, in orderly rows, with minimal improvements (only in repair and maintenance, hardly in rebuilding) since the mud huts of ancient times. It is a society so stable that it seems to defy change.

What, then is the goal of Utopia? The citizens go about their lives in the same clothing material as they have for so many centuries, no more or less comfortable. Their magistrates are sent around the world to establish justice, and their armies are sent to establish peace for allies. It is a world where many western nations are led by corrupt governments, such as those Raphael Hythloday describe, and Utopia is just one out of the ordinary. But for such a remarkable nation, there is also remarkable exclusion. Even More (his character in the novel), a worldly person, is completely unaware of the nation in the world, so presumably it is completely unheard of among the general populace. They are known to trade so that they can acquire gold (for military expenses) and iron, but have little other interaction with the world. Their interactions with others are minimal. It is an isolated bubble of content and comfortable wealth and respect amongst all its citizens, a place where everyone is equal and contributes fairly. The nation seems to exist only to set a good example to others— not to play any part in global politics or economics. Perhaps their goal is to subsist, and set a good example of humble, ungreedy survival— like a rock in a turbulent stream.

While Hythloday claims there to be few laws, there is a lively set of standards— they guide society a little differently than rules or laws— that the Utopians follow. There are guidelines on what citizens should be doing at any given time.

Talk about gold

Works Cited

… Utopia

Indecent Particulars

Cover Letter to the Second Draft

Your comments from the first draft were very helpful. I started my revisions simply by reading over my discussion of *Gulliver's Travels* and trying to make sense of it (kind of like a reverse outline). What I realized was that I needed to narrow down some ideas so that I could work with them, so I created the distinction between two describable physical features— *defining physical characteristics* and *universal bodily actions*— and emphasized that the subject of this essay concerns the latter more than the former (for the different outcomes are argued to be greatly influenced by the differences of the level of focus in the vulgar bodily actions), and that the former is suppressed in both texts (also meaningful).

Specifically, I tried to pick apart the two apologies more, shifted around the idea of appearance in the essay to fit more closely in the conversation about bodily actions, and edited the claim in the introduction to match.

I did less work on the section on Cavendish. Unfortunately it is still quite separate from the Swift analysis, but I did try to improve the connection between the non-physical (i.e., the spirits and the worlds of their creation) and the physical characteristics.

Jonathan Lam

Jonathan Lam

Professor Germano

HSS2-K

17 April 2019

Indecent Particulars

> " … I was able to turn upon my Right, and to easy myself with making Water; which I very plentifully did, to the great astonishment of the People, who conjecturing by my Motions what I was going to do, immediately opened to the right and left on that side to avoid the Torrent which fell with such noise and violence from me" (Swift 27).

Jonathan Swift doesn't hold back from providing a wealth of crude imagery in *Gulliver's Travels* of the more indelicate aspects of human existence that often are overlooked in fantastical plotlines. In an adventure encompassing humans and other intelligent beings of all sizes and philosophical capacities, this detail is marginal at most, but that is far from the case in Swift's writing. In the similarly imaginative work *The Blazing World* by Margaret Cavendish, there is a separation between spiritual adventure and human unpleasantries. Both protagonists live with inhabitants of unfamiliar nations and discover imperfections in human society, but with opposite outcomes: Gulliver discovers a hatred for mankind and himself, while the Empress notices the flaws in human society but is content with her new knowledge. The difference is largely a result of the discrepancy between the two protagonists in the focus on ordinary bodily functions, even very normal activities like excretion; putting an emphasis on these inherently undesirable actions that unify human existence causes such the violent rejection of humankind that Gulliver experiences, and the Empress's blissful ignorance of the same actions allows her to appreciate her new knowledge.

The discussion of the nature of "human" bodies in fantasy worlds of intelligent beings with human bodies, unintelligent beings with human bodies, intelligent humanoid beings, and other life with a human-or-greater intellectual capacity warrants a brief discussion on what should be considered human. It would be underwhelming to treat only the Europeans (and other real-worldly citizens) of the two novels as human, and therefore "human" will refer loosely to the bipedal animal species of the Blazing World, the tiny Lilliputians and the large Brobdingnagians, and both the Houyhnhnms and their slaves the Yahoos. It would be unfair to exclude any of these when the subject texts are weakly based on literal meaning; any of these foreign beings with mental or physical likeness to humans are good sources of comparison to humans in the unlike sense.

Instead of describing defining characteristics of his body, Gulliver spends considerable effort to describe the *astonishment of the People* and the *violence* of the *Torrent* of his urine. Repeatedly. He apologizes after the second urine scene, explaining that he "would not have dwelt so long upon a Circumstance, that perhaps at first sight may appear not very momentous, if I had not thought it necessary to justify my Character" (Swift 30). The *momentous* aspect of peeing is not explicitly given, but the experience is described to be one of great urgency and shame. These emotions must be quite influential to him, since he takes a takes further opportunities to describe his urine. Gulliver decides to urinate on the palace to put out a fire without apology to the reader for its impact on the plot is apparent; but when a fourth passage regarding Gulliver's bodily discharge is mentioned, this time in Brobdingnag, another apology is affixed. The apology is a note on the usefulness of the discussion of "the like Particulars, which however insignificant they may appear to grovelling vulgar Minds, yet will certainly help a Philosopher" (Swift 89).

At first glance, this second apology may appear to be in same the flavor of the frequent promises Gulliver makes on his earnestness directly to the reader— in which only a credence trust is imbued— except that it makes a different kind of promise. Rather than asking for trust, this apology asks for a less literal ("vulgar") interpretation and more of an interpretive view (that which will "help a Philosopher") of his physical actions and the associated consequences. It's a conspicuous *LOOK HERE* billboard. Swift simultaneously almost completely neglects to address non-universally-human aspects of Gulliver, such as physical characteristics. The best you can do is infer his exact skin tone, eye color, height, body build, voice, and other personal features; the only physical descriptions are offhand results of storytelling, such as the comparison with the Yahoos (which only confirm several features being more human than ape) and the description of his new clothing obtained from each location. These too are universal characteristics of humankind. Unlike urination, however, these are *defining* characteristics of a person, which classify people in descriptions; urination and other basic human actions cannot define an individual. Either category contributes to the whole person, but Swift chooses to focus completely on the latter.

If the basic (external) human actions had to be listed, the list is short. Eating, Sleeping. Urinating and defecating. Gulliver is on an adventure! Food is diverse based on the societies and their resources, much like his clothes; it is a part of an adventure. But the other basic needs are universal to his adventures. Sleep is periodically discussed, and specifics are not discussed here; and the latter need can be *momentous* as Gulliver states in the first apology.

Momentous because it gives the Lilliputians a natural human action to blame (in the case of the royal palace fire), momentus because it is the most vulgar and nonconventional detail in other novels, and momentous because it shows that Gulliver feels a lasting shame when he can't urinate without a conventional bathroom. The feeling of shame is mirrored into a feeling of hatred for the Yahoos, who also have the same natural urges to discharge waste, and excrete on Gulliver when provoked. These actions by Gulliver and the Yahoos leave a negative impression of the human race— more exactly, the human body.

Cavendish takes a different approach and leaves the physical human elements, both in the defining physical description (which Swift ignores) but also in universal bodily action, to the imagination of the reader. Before becoming the Empress, the young lady protagonist is described to be astoundingly beautiful, surviving "only, by the light of her beauty, the heat of her youth, and protection of the gods" (Cavendish 126). Her acquaintance the Duchess of Newcastle is likewise not described except for her honesty; neither are the bear-men or other men-like beings described other than by name of their associated animal, and by their bipedal-ness.

No further account of the inhabitants of the Blazing World nor the ordinary humans are given. The only other visual clue is that Cavendish is the flesh-and-blood Duchess of Newcastle, but even that doesn't provide much insight into how her disembodied soul might appear to the Empress.

The Empress spends a great deal of time talking with the most wise *immaterial spirits*, with which she discusses the materiality of things. Naturally, because they lack a body, it is the ultimate experiment in separation of body and mind. The spirits use some form of "corporeal bodies" to move, but the exact nature of these bodies is not emphasized. The spirits repeat to the Empress that they themselves are not made of matter; that "spirits, being incorporeal, have no motion but from our corporeal vehicles, so that [they] move by the help of [their] bodies, and not the bodies by the help of us; for pure spirits are immovable" (Cavendish 168); and therefore they can have only a "supernatural knowledge" (Cavendish 169). Neither can they speak without a body, so the nature of the conversation between the two is also vague.

The distinction between *spirits* (like the immaterial spirits) and *souls* (of people) is unclear, for they seem to have the same ability to inhabit bodies and exist body-less, but differ in that immaterial spirits do not originate from a person nor have gender; for the sake of this claim, the difference isn't very important, because they are similar enough in their body-less-ness. Later, when the souls of the Empress and the Duchess leave to visit the Duchess's land, a spirit is left in charge of the Empress's body.

The curious Empress then asks the Duchess how the latter's body was kept when her soul was away attending the Empress, to which the Duchess replied that it was governed by "her sensitive and rational corporeal motions" (Cavendish 190). By this absurd state of existence— a body existing without a soul— Cavendish implies that the body is, as it is to the immaterial spirits, only a convenient *vehicle* that a person is free to take leave from. It also implies that the natural actions of the body— eating, sleeping, and excretion— can be carried out autonomously, or at least without the mind's supervision.

Not only can the souls leave to places off to places in the real world, as the Empress does with the Duchess, but a person can also live in a world of their own creation, free of bodily wants. So proceeds the

Duchess, who creates various worlds in an attempt to be the ruler of one, and the Empress in a similar manner (Cavendish 188). The spirits recommend this over attempting to conquer any of the real realms, which nature makes infinitely numerous and various, for "by creating a world within yourself, you may enjoy all both in whole and in parts, without control or opposition, and may make what world you please, and alter it when you please, and enjoy as much pleasure and delight as a world can afford you" (Cavendish 186). The world the Empress makes may be considered a metaphor for the world Cavendish makes, for it is an imaginary world, *without control or opposition*, with whatever power to *alter* it, and for the purpose of offering *pleasure and delight*. And a notable piece of that bliss is the lack of attention paid to ordinary bodily actions, and instead to hypothetical discourse. To be specific, there is only one mention of food for the Empress, and no mention of sleep or bodily excretion at all except through the vague term, *corporeal motions*.

Take into consideration two more case studies: the Laputians and the animal-men. The animal-men are the professionals: scientists, mathematicians, and orators, but not once are they spoken to outside of the knowledge of their field, except for the brief moment after the Empress was rescued. On the contrary, the Laputians are a civilization of mathematical and astronomical geniuses (analogues to Cavendish's spider-men and bird-men), but Swift disrupts their image twice. First, the greatest thinkers are lost in thought and require loud "bladders" used to draw attention away from speculation and to the real world. As these bladders are used on walks when a Laputian "is so wrapped up in Cogitation, that he is in manifest danger of falling down every Precipice" (Swift 148), it is safe to say that even the basic human functions cannot be carried out independently. (The image of a grown, wise Laputian requiring attending to in the bathroom like an untrained baby is a disturbing one.) Secondly, the imagery of the experiment with recycling excrement in food (168), or the episode of the dog dying of explosive diarrhea (170), are likewise unappealing and continue the motif of human excrement. Had these geniuses not displayed to Gulliver so openly these experiments, he may have held as high an esteem for them as the Empress did her respective experts.

It seems a little far-fetched to take two nontrivial texts, Swift’s and Cavendish’s, and claim that this single factor plays such a large role. However, the two texts are remarkably similar, and few details contrast so oppositely. Both tales include the magical passage to unexplored worlds of intelligent creatures never seen by man; both the Empress and Gulliver are clever and likeable enough to learn the language of and earn a high rank among the foreign society; and both endeavor to learn more about their world by assimilating the perspectives and knowledge of the beings they encounter. Both texts also trivialize (in the form of neglect) the value of physical appearance, and Cavendish even imagines two body-dissociated forms, where appearance and natural urges no longer exist. The difference is that the Empress neglects to mention any rude moments, which are inevitable for the human body, while Gulliver isn’t afraid to mention them. While it may seem like insignificant, the universality of these indecent particulars for human beings and the frequency at which they occur in *Gulliver’s Travels* make it a crucial factor in his disdain towards human beings.

Works Cited

Swift, Jonathan. *Gulliver's Travels*. Penguin Books, 2001.

Cavendish, Margaret. *The Blazing World and Other Writings*. Penguin Books, 2004.

Inner Worlds and Isolated Humans

~~In their respective adventures, the Empress of Margaret Cavendish's *The Blazing World* and Gulliver of Jonathan Swift's *Gulliver's Travels* encounter a great number of intelligent species, with which they converse. The nature of both societies either encourage the protagonist to seek internal adventures within their soulless body, or cause him to~~

~~Bodies are infinitely various and numerous, good for studying~~
~~Bodies are immutable, we cannot change who we are~~
~~Bodies can be studied and operated independently of souls~~
~~Experience adventures without using your body~~

~~INTRODUCTION~~

~~The discussion of the nature of "human" bodies in fantasy worlds of intelligent humans, unintelligent humans, intelligent humanoid beings, and other life with a human-or-greater intelligence warrants an agreement on what should be considered a human. The answer, given the context of these allegorical, fable-like adventures, is that any of these beings can be interpreted to be human, and to view their (sometimes outrightly ridiculous) physical features symbolically. (It is not Cavendish's goal to convince the reader that the Blazing World exists, nor is it Swift's goal to relate true stories from the islands of Lilliput, nor are the authors asserting that bear-men or horses are, in some simultaneous realm, equally as talented as humans.) Limiting the analysis to the protagonists and the few moments with the other ordinary humans would be underwhelming and failing to capture many important, human moments from both novels.~~

THE SOUL, AND THE IMMATERIAL

~~Before she becomes the Empress, the young lady is described to be astoundingly beautiful, enough so that the light of her beauty is enough to keep her alive. Her acquaintance the Duchess of Newcastle is not described except for her honesty. Neither are the bear-men or other men-like beings described other than by name, and by the fact that they are all bipedal (which strengthens the claim that they are representative of humans).~~

~~Before she becomes the Empress, the young lady survives "only, by the light of her beauty, the heat of her youth, and protection of the gods" (Cavendish 126); this is the only description provided of her physical appearance. Nor was any description made of the Emperor, nor her acquaintance Duchess of Newcastle; the latter being both a reference to the Margaret Cavendish, the real-life Duchess of Newcastle, and a disembodied soul, makes the image very confusing. Does she see a person, a holographic ghost, or nothing at all?~~

The same applies to the immaterial spirits which whom the Empress converses, for they are body-less: they use "bodies" to move, but the nature of these bodies is not clear. The spirits emphasize that they themselves are not made of matter; that "spirits, being incorporeal, have no motion but from our corporeal vehicles, so that [they] move by the help of [their] bodies, and not the bodies by the help of us; for pure spirits are immovable" (Cavendish 168); and therefore they can have only a "supernatural knowledge" (Cavendish 169). Neither can they speak without a body, so the nature of the conversation between the two is also vague.

The spirits then assert that a spirit can control a body "as well … as man can arm himself with a gauntlet of steel" (Cavendish 180), after which the Empress calls the Duchess to be her scribe. (Whose body the Duchess inhabits, whether the Duchess's or someone else's, is not clear either.) Here, Cavendish fails to distinguish between *spirits* (like the immaterial spirits) and *souls* (of flesh-and-blood people), for they seem to have the same ability to inhabit bodies. Later, when the Empress and the Duchess spiritually (soul-ly, if the difference matters) leave to visit the Duchess's land, a spirit was left in charge of the Empress's body.

The curious Empress then asks the Duchess how the latter's body was kept when her soul was away attending the Empress, to which the Duchess replied that it was governed by "her sensitive and rational corporeal motions" (Cavendish 190). By this absurd state of existence— a body existing without a soul, controlled either by an external spirit or *corporeal motions*— Cavendish implies that the body is, as it is to the immaterial spirits, only a convenient *vehicle*, and something that a person is free to take leave from.

And not only can the souls leave to places off to places in the real world, as the Empress does with the Duchess, but a person can also live in a world of their own creation, free of bodily wants. So proceeds the Duchess, who creates various worlds of her own in an attempt to be the ruler of one in the theme of Thales, Pythagoras, Epicures, Aristotle, Descartes, and of her own imagination, and the Empress in a similar manner (Cavendish 188). The spirits recommend this against attempting to conquer any of the real realms, of which nature makes infinitely numerous and various, for "by creating a world within yourself, you may enjoy all both in whole and in parts, without control or opposition, and may make what world you please, and alter it when you please, and enjoy as much pleasure and delight as a world can afford you" (Cavendish 186).

This point is hugely impactful, because it can encompass all artistic enterprises. To take a literal example, Cavendish places herself, the Duchess of Newcastle, into *The Blazing World*, as a powerful and respected advisor; and her character then recurses the behavior by creating philosophical worlds within herself. Or, the Empress's entry into the Blazing World may be some worldbuilding practice of her own; in both of their cases, they do as the spirits predict: *enjoy* their creation, find a blissful lack of *control or opposition*, and have the ability of *alter*ing it in whatever manner they please. This worldbuilding gives the same kind of *disembodied* freedom that makes Cavendish's stories so appealing, much as it does with other fantasies, or even in the modern equivalent of worldbuilding games like Minecraft. The pure fantastical element of Cavendish's writing, from the Utopian harmony of all anthropod species to the elements contrary to current scientific knowledge (and no more than speculative in contemporary science), as well as the fact that most of the Empress's adventures are carried out through discourse with other intellectuals rather than physical endeavors, gives the impression that the world's greatest, most unrestricted adventures are spiritual. The weaker implication is that the human body can still function without a mind by simple *rational*, *corporeal* motions, so it is possible for a person to truly drift away in their thoughts and let their body run in autopilot mode.

~~A GREATER PHYSICAL~~

~~Swift introduces a different view through Gulliver's travels. While a great part of Gulliver's interactions with the other species involve dialogue, the adventures are significantly more physically-oriented. In Lilliput alone, he gets tied up and poked with tiny non-poisoned arrows, urinates on the royal palace to save it from fiery destruction, and drags along an entire miniature fleet by hand.~~

~~The second disagreement with the immaterial spirits' prediction of inner worlds is that, while Gulliver is saved and is put into high regard by the people of honor in all four adventures by his wit and good-nature, there is a turning point in every adventure that evicts him from the island and back to England. This is unsurprising because of the contemporary gender stereotypes: Cavendish was an early (seventeenth-century) female novelist to take advantage of literature as a means of adventure, being restricted physically; but Swift was a male of a similar time and was more at leisure to engage in these physical activities.~~

~~Like the Empress, there is no mention of Gulliver's appearance: the only clues given are in his comparisons with the Yahoos, which have longer nails and darker skin than he does. However, there is detail about changes in his clothing, but only as it relates to weathering, passage of time, and the materials at hand: a little logistical plot detail.~~

~~A fair description of the members of each society Gulliver encounters is given, but there isn't much to comment on these literal descriptions, because they mimic well-known images (albeit perhaps scaled). The Lilliputians are small humans; the Brobdingnagians are large humans; the Houyhnhnms are horses; the Yahoos are (more primitive) humans. There is nothing really surprising to any of this: these societies all formed and act similarly to humankind, and various important people were able to have intelligent conversations with Gulliver. Actually, Gulliver having understood for a large part the motives of the ruling classes, even those that wished to evict him, these societies are all politically-aligned to those of Europe. The only thing setting Gulliver apart from the Lilliputians and the Brobdingnagians is size, for otherwise they are perfectly compatible.~~

~~The Houyhnhnms are similar to Cavendish's animal-men (bear-men, fish-men, bird-men, etc.). The animal-men have the ability to observe the world and debate just as humans do, the only difference being in their~~

~~physical capabilities (e.g., the bird-men with flight, and the worm-men of underground travel). The Houyhnhnms possess the ability to tame Yahoos and even do surprisingly nimble tasks like thread a needle.~~

~~While the animal-men from the Blazing World were part of Cavendish's inner adventure, the fear caused by difference in physical bodies cause Gulliver to be thrown out of Lilliput and Houyhnhnm land.~~

CONCLUSION

Brainstorm about the body

Quotes from Blazing World

“No, answered they; but, on the ontrary, natural material bodies gie spirits motion; for we spirits, being incorporeal, have no motion but from our corporeal vehicles, so that we move by the help of our boies, and not the bodies by the help of us; for pure spirits are immovable” (168)
“No, sai they; nor could we have any bodily sense, but only knowledge” (168)
“Not a natural, answere they, but a supernatural knowledge, which is a far better knowledge that n a natural.” (169)

Men and abnormal cruelty 176

“We wonder, proceeded the spritis, that you desire to be Empress of a terrestrial world., whenas you can create tour self a celestial world if you plese. What, said the Empress, can any mortal be a creator? Ys, answered the spirits; for every human creature can create an immaterial world fully inhavbited by immaterial creatures ,and populous of immaterial subjects, suchas we are, and all this within the compass of the head or scull” 185

Quotes from Gulliver’s travels

Physical bodies: the small men, the large men, the philosophical men, the truthful men
Bodies are a way to lead ppl to realize stuff when it is out of line with their knowledge

Bodies are just a medium for which transportation can be carried out
Limitations in the physical world create more creative other worlds
The immaterial spirits seem to be the most wise, and then the fish-men, and the other men, and then the hoo-mans
- Ghosts also tend to be wise

In gulliver’s travels, all of the different species act humanoid and are somewhat alike in their intelligences and governments, but the horses (being less dextrous) are of the best minds

Bodies are the works of nature, and therefore infinitely various, wondrous, and immutable. People can study spirits

“She had never heard of a medicine that could renew old age, and render it beautiful, vigorous and strong: nor would she have so easily believed it, has it been a medicine prepared by art; for she knew that art, being nature’s changeling, was not able to produce such a powerful effect, but being that the gum did grow naturally, she did not so much scruple at it; for she knew that nature’s works are so various and wonderful, that no particular creature is able to trace her ways.” (157)

Drugs and altering artificial effects, not altering their inherent, proper and particular natures

Genderbent — not a thing? Women are a thing

The chief method of obtaining knowledge was by the means of studying other bodies — i.e., the physical attributes of other things (i.e., sensible things)

Correlation between level of observation, intelligence, and dexterity? What about agility? What about the circumstances from which they came?

Yahoos and humans are physically the same, so there has to be some explanation of why they are so different — they have the same physical capabilities, for the most part

The world is for studying but our bodies are untouchable

Physical travelling, labor are important

Chaos theory in the creation of these societies

Mind-palace, immaterial spirits and fortune

What does it mean to have opposable thumbs

Importance of language, of speech — physical/symbolic communication and understanding

Is intelligence the only factor that saved these ppl?

    Gulliver was saved by his wit and goodness of his heart
    Empress was saved by her beauty, wit, religion
    These were both the best of people, utopian ppl in a utopian world

THESIS: The human body is a sort of imperfect, but immutable thing that is infinitely various and curious, and highly dependent on the environment from which it was formed.

But Gulliver's and Cavendish's worlds were both worlds created allegorically and not really meant to represent real ppl — but the *separation of societies*, and the *ignorance of these societies wrt each other*, is something very apparent in both novels

The difference in the bodies represent the vast differences in different societies and the resistance and ignorance they have of each other and the change necessary to accommodate another

INTRODUCTION w/ above thesis

BP1: Definition of human beings from the different things

Jonathan Lam

Professor Germano

HSS2-K

6 May 2019

Languages have Layers

There are several stages to the learning of a language. I discussed in a previous essay how a person advances in their study of a language ("The Language Addict"), growing from a babyish, unappreciative stage, to the average, "take-for-granted" perspective, and finally to a cynical, unappreciative stage for the characters in Shakespeare's *The Tempest*. In that essay, there was a discussion of people of all range of language proficiency, but for now the focus is on the most ignored one from *The Tempest*: that of Caliban, the man-beast.

Caliban's Italian-learning adventure is not described in depth in the play, given only by the vague knowledge of his being meticulously tutored by Prospero and Miranda. In the real world, Caliban's image closely resembles that of a slave, forced to be in servitude and learn the language of his captives. We get from Olaudah Equiano's *The Interesting Narrative* what was missing from Shakespeare's: the roller-coaster of emotions that accompany the journey. Is it love for the new language? Hate for the forced captivity and labor? Unlike Caliban, Equiano is fairly generous in his appreciation for new language. Once some of Jean Jacques Rousseau's philosophy about the origin of languages is considered, there are many similarities between Equiano's acquisition of a language and society's acquisition of language. Equiano gives an insight into some ways a person may naturally respond to learning new parts of a new language, namely: the joy of learning the idea of a word before the words; the frustration of learning the words before the idea; and the general confusion and lack of hope when trying to rationalize naming procedures.

Especially at the beginning of his captivity, Equiano is met with a sense of curiosity that allows him to overcome his (nontrivial) fear. This is observed both when he has the acquaintance of people who speak similar languages to him, and, as is the case more and more frequently, when he is only around strangers. An instance of the former is when he "ask[s] how the vessel could go? they told me they could not tell; but that there were cloths put upon the masts by the help of the ropes I saw, and then the vessel went on; and the white men had some spell or magic they put in the water when they liked in order to stop the vessel" (Equiano 57). The words he uses here are very different than those that he uses later, after he gains experience sailing. It is not only that he is young and that he hasn't yet seen the technologies that are at play here (i.e., the words "sail" and "anchor" or perhaps even the idea of a large waterborne vessel, the "ship"), but it's likely that these kinds of words don't even exist in his native language, because even the idea of oversea travel and the technology to start and stop a boat using wind and water forces is unknown. Thus, when he learns these words ("sail," "anchor," and "ship"), he is not learning only the *word*, but the *idea* itself.

When Rousseau discusses the invention of language, he postulates the opposite direction: that "general ideas can be introduced into the mind only with the help of words, and the understanding grasps them only by means of sentences" (Rousseau 24); but Equiano grasps the ideas that were so unknown to him by the profound human power of observation (seeing the sails and hearing from others what their purpose is) and using prior understanding ("magic" is a sort of understanding), without the help of their given words. If we were to interpret Rousseau's definition less strictly, so that the "help of words" does not have to be the correct term for a phrase, and rather the ability to use a circumlocutory phrase to describe an object (which in turn helps a person to understand the correct term when they do learn it), then it seems like *general ideas* and *words* are a sort of circular term because words are required to discuss the general idea behind other words, forming an interdependence rather than a one-way cause-effect relationship.

For Equiano, this means the source of numerous miracles along his sometimes-miserable journey. After this first discovery of the ship, there are numerous other moments. An exemplar moment is that in which he discovered two prominent aspects of the European world (emphasis added):

> "One morning, when I got upon deck, I saw it covered all over with the snow that fell over-night: as I had never seen any thing of the kind before, I thought it was salt; so I immediately ran down to the mate, and desired him, as well as I could, to come and see how somebody in the night had thrown salt all over the deck. He, knowing what it was, desire me to bring it down to him: accordingly I took up a handful of it, which I found very cold indeed; and when I brought it to him he desired me to taste it. I did so, and I was surprised beyond measure. I then asked him what it was? he told me it was *snow: but I cold not in any wise understand him*. He asked me if we had no such thing in my country? and I told him, No. I then asked him the use of it, and who made; he told me a great man in the heavens, called *God: but here again I was to all intents and purposes at a loss to understand him*." (Equiano 67)

Here we see both orders of words and the idea behind it: once again putting ideas before words (seeing the cold salt and afterwards associating it with "snow"), but also learning about the great entity called God. Then, God still had no meaning to him, as words do not inherently give meaning.

But this is only fair, as God is the first important word from European culture that is not something he would realize and immediately assign a word to. If God could be summarized into one sentence, He would be the abstraction of a whole church of morals resting in a person's head, and Rousseau acknowledges that "abstractions are irksome and rather unnatural operations of the mind" (24); but God cannot even be summarized as a single abstraction, but perhaps an abstraction of an abstraction of this church, or some more deeply-nested entity that lays in the collective minds of billions of people. What is God, or religion, but the agreement of many like-minded people decided to form into a single word to express? And since religions were born, how many biblical stories, how many religious miracles, how many arguments and beliefs have been sprung up in the name of that *God*, that word that has since become so riddled with connotations and controversies that it is no longer any single idea?

In other words, while Equiano can describe (by means of circumlocution) the sails ("cloths") and snow (in appearance like salt, in touch cold), there is no possibility that he could stumble upon the idea of *God*, in an encompassing sense in any setting after its establishment. A person raised in complete ignorance of God will never discover such a large, yet particular, idea as God and his rules, because there are no physical manifestations which can kindle the same image; and, indeed, even as the idea is unified, there cannot be total (dis)agreement amongst any two people about God's nature, as opposed to something quite corporeal like snow or salt. The difference between the two is that while the latter is something pure and invariant derived from nature, it doesn't require a word to keep the idea together. On the other hand, "God" doesn't have meaning without a society to maintain its complexity.

Another part of language to take into mind are names, especially those for humans and other intimate objects. Rousseau actually points this out as the first part of speech of language, because they can be mapped one-to-one to other objects; they are a subset of the non-abstract items. The inherent problem is that "each object was given a particular name, without regard to genus and species, … and all individual things presented themselves to their minds in isolation" (24). This is the opposite of abstracting, for since every object is assigned a different word, a different *name*, without regard to classification or to the names other people give to it. But they are still a representation of a real-life object in the same form as the more abstract ideas, so it can be thought of as the lowest level of abstraction of an object into the realm of language.

What is then fascinating about names is that people can easily disagree on them. Equiano discovers the ephemerality of his own name, for "in this place I was called Jacob; but on board the African snow I was called Michael" (63), and later on Pascal's ship "my captain and master called me *Gustavus Vasa*. I told him as well as I could that I would be called Jacob; but he said I should not, and still called me Gustavus" (64)[1]. His given name is lost to the forceful bullying of the Europeans, and a name is placed upon him, as if he were some object he himself were foreign to, much like the snow or cloth, and it were obvious. As if his name wasn't the correct one for himself, nor was any of his other past names bulldozed over his given one.

Equiano tends to take all these forms of new language with grace. He encounters his name changing; he observes in world certain new phenomena and learns their translations; and he is introduced to abstract ideas and absorbs them, becoming a devout Christian. But from his detailed chronology, the parts of this language learning that cause frustration and grief can be separated from the parts that are most pleasant: namely, the latter is when a person is allowed to embrace some representation of the idea with the word, immediately and delightfully finding

[1] "What's in a name? That which we call a rose \ By any other name would smell as sweet." (Shakespeare, II. ii. 1-2)

new context and meaning — here, learning a new word is merely adding to current knowledge, and not having foreign ideas or language in particular be maliciously forced.

This analysis can be extended to other situations which involve foreign language-learning, and contempt or love for the assimilation. Consider again Caliban, and perhaps an straightforward reason for his anger is discovered. He spends his whole life couped up on a tiny island, and, while taught by well-educated people, finds great dissatisfaction in the feeling of being forced upon him a very abstract, foreign representation with no apparent worth. Unlike Equiano, who learns many words through his first interaction with them, thus fusing a word with experience and the joy of first encounter, Caliban is stuck with the frustration of attempting to rationalize an abstract without the same kind of experience. It makes sense that the motivation for a tool like words be learnt at least before the learning of the word, or else it's an endeavor in vain.

Works Cited

Equiano, Olaudah, 1745-1797. *The Interesting Narrative and Other Writings*. New York :Penguin Books, 1995. Print.

Lam, Jonathan. "The Language Addict." N.p., 2018. Print.

Rousseau, Jean-Jacques, Alan Ritter, and Julia C. Bondanella. *Rousseau's Political Writings: New Translations, Interpretive Notes, Backgrounds, Commentaries*. New York: W.W. Norton, 1988. Print.

Shakespeare, William. *The Tempest*. Edited by Peter Holland, Penguin Putnam, 1999.

THESIS TIME!!!
What the heck is this
I am so doomed

Talk about how language can change someone

Rousseau and the formation of language
Something about the discovery of language

“The laguages of the different nations did not totally differ, nor were they so opious as those of the Europeans, particularly the English” (51)

“The language of these people resembled ours so nearly, that we understood each other perfectly. They had also the very same customs as we. There were likewise slaves daily sported with our darts and bows and rrows, as I had been to do at home. In this resemblance to my ormer happy state I p[assed about two months … “ (53)

~~“I uasked how the vessel could go? they told me they could not tell; but that there were cloths put upon the masts by the help of the ropes I saw, and then the vessel went on; and the white men had some spell or magic they put in the water when they liked in order to stop the vessel” (57)~~

“They at last took notice of my surprise; and one of them, willing to increase it, as well as to gratify my uriosity, made me look through it … This heightened my wonder: and I was now more persuaded thn ever that I was in another world, and that every thing about me was magic” (59) — first technologies

“I did not know what this could menan; and indeed I thought these people were full of nothing but magical arts” (60)

“I now totally lost the small remains of comfort I had enjoyed in conversing with my countrymen … I was now exceedingly miserable, and thought myself worse off than any of the rest o my companions; for they could talk to each other, but I had no person to speak tot that I could understand” (62)

“I was much astonished and shocked at this contrivance, which I afterwards learned was called the iron muzzle” (63)

“In this place I was called Jacob; but on board the African snow I was called Michael” (63)

“By this time, however, I could smatter a little imperfect English; and I wanted to know as well as I could where we were going” (64)

“While I was on board this ship, my captain and master named me *Gustavus Vasa*. I told him as well as I could that I would be called Jacob; but he said I should not, and still called me Gustavus; and when I refused to answer to my new name, which at first I did, it gained me many a cuff; so at length I submitted, and by which I have been known ever since” (64) — putting a name to that which is known is undesirable

~~“One morning, when I got upon deck, I saw it covered all over with the snow that fell over-night: as I had never seen any thing of the kind before, I thought it was salt; so I immediately ran down to the mate, and desired him, as well as I could, to come and see how somebody in the night had thrown salt all over the deck. He, knowing what it was, desired e to bring it down to him: accordingly i took up a handful of it, which I ffound very cold indeed; and when I brought it to him he desired me to taste it. I did os, and I was surpsied beyond measure. I then asked him what it was? he told me it was snow: but I cold not in any wise understand him. He asked me if we had no such thing in my~~

~~country? and I told him, No. I then asked him the use of it, and who made; he told me a great man in the heavens, called God: but here again I ws to all intents and purposes at a loss to understand him" (67)~~

"Its prodigious height, and its form, resembling a sugar loaf, filled me with wonder" (73)

"I could now speak English tolerably well, and I perfectly understood every thing that was said. I now not only felt myself quite easy with these new countrymen, but relished their society and manners. I no longer looked upon them as spirits, but as men superior to us; and therefore I had the stronger desire to resemble them; to imbibe their spirit, and imitate their manners" (78)

"Upon this Captain Doran said I talked too much English, and if I did not behave myself well, and be quiet, he had a method on board to make me" (94)

Joseph Clipson (121)

Two white men (159)

"Obstacles to the origin of languages" (22)

"The first … is to imagine how languages could have been necessary, for since men had no dealings with each other, and n need of any, it is imposible to understand the necessity for inventing languages or even how they became pssible, if they were not indispensable" (22)
-> domestic intercourse

"Observe further that since the child has to explain all his needs and, consequently, has ore to say to the mother than the mother to the child, it must be the child who makes the greatest efforts to invent language, and that the language he uses must be largely of his own making; this would bringginto being as many languages as there are individuals to speak them; the wandering, vagabond life, which gives no idiom the gtime to become consistent, contributes to this multiplication" (22)

… art of speaking as communication

"Man's first language, the most universal, the most energetic, and the only one he needed before it became necessary to persuade an assmebly of men, is the cry of nature" (23)
-> not very useful in ordinary life

"It must be assumed that the irst words men used had, in their mind, a much broader meaning than do those used in languages that are already formed, and that, ignorant of the division of discourse into its contituent parts, they at first gave each word the meaning of a whole sentence. …" "abstractions are irksome and rather unnatural operations of the mind" (24)

"At first, each object was given a particular name, without regard to genus and species, which these first founders were not in a position to distinguish, and all individual things presented themselves to their minds in isolation, as they are in the spectacle of nature" (24)

~~"Moreover, general ideas can be introduced into the mind only with the help of words, and the understanding grasps them only by means of sentences. This is one of the reasons why animals can neither formulate such ideas nor ever acquire the perfectibility that depends upon them" (24)~~

"Whatever the origin of language may be, it is easy to see from the lack of care nature has taken to bring men together through mutual needs or to facilitate their use of speech how little it has prepared them to be sociable, and how little it has contributed to all they have done to establish social bonds" (25)

"He had in instinct alone all that he needed to live in the state of nature; he has in cultivated reason only what he needs to live in society" (26)

Words to describe it before the things created, or things before words to describe it

Wonder in creating languages from a (more ignorant?) state, but less from the perspective of someone who already knows the language

In *The Interesting Narrative*, a man named Olaudah Equiano, or Gustavus Vassa, recounts the adventure that spans a lifetime

On the Government's Right to Property

Cover Letter to the Second Draft

The first draft turned out better in terms of the developing and significance of claim than originally thought when I had submitted it, so the revisions are mostly mechanical.

When Pam Newton came and talked about revision strategies, I noticed a number of ways my language could be improved, most notably with filler language and precision of diction. My word choice was especially rambly in this paper.

Some examples of substitutions were: "… is a very important tool" → "critical," "much better … definition" → "preferable," "be categorized into one of these three categories" → "reclassified," "the new owner of the estate" → "inheritor," "same items for sale at market" → "face value," "never a great option" → "last-ditch," etc. A lot of these phrases really shouldn't be interchangeable, so I think replacing the vagueness with these more powerful, connotative phrases has a noticeable change on how the essay reads. The essay does read a little formal (it must, in order to generate a claim about such rhetorical works), so adding these terms and changing some passive clauses to active ones livens up the writing.

As usual, the marked errors (such as changing the encyclopedia source to the original source), as well as some other overlooked bloops, from the first draft were corrected.

Jonathan Lam

Jonathan Lam

Professor Germano

HSS2-K

3 April 2019

On the Government's Right to Property

When a teacher goes to the store or supermarket and buys some food, it belongs to them. They spent the money. If a farmer plants seeds and reaps the harvest, the harvest belongs to them. John Locke discusses the acquisition of *property* by means of expending effort in his philosophical work *Second Treatise of Government*, which rationalizes the gain of property by these actions. But both teacher and farmer exist under some government, giving rise rise to numerous other questions: how much belongs to the government? What does the government pay when its citizens obtain property?

Fortunately, Locke performs as secretary to the group of eight lords proprietors in charge of the Carolina charter in America, and writes *The Fundamental Constitutions of Carolina*, an earlier document which cast a glance past philosophy into what was intended to be a working government, reflecting in part the lords proprietors' views. When the laws of the constitution are considered in light of the definition of his later work, it shows that government sometimes restricts acquisition of property as a means of avoiding arguments between the people (as opposed to the direct, illegal intervention), for which *record-keeping* is a critical tool. This record-keeping naturally leads people to manipulate their power by manipulating what is kept on record.

Locke maintains two definitions of *property* throughout *Second Treatise of Government*. The first, and more theoretical, declares that:

> "… every man has a *property* in his own *person*: this no body has any right to but himself. The *labour* of his body, and the *work* of his hands, we may say, are properly his. Whatsoever then he removes out of the state that nature hath provided, and left it in, he hath mixed his *labour*

with, and joined to it something that is his own, and thereby makes it his *property*" (V, §. 27.).

This statement provides a man ownership of *his body*, *his labor,* and *anything affected by his body or labor* as this idea of *his property*. Locke then spells out two further aspects of *property*:

> "... no man but he can have a right to what that is once joined to, at least where there is enough, and as good, left in common for others" (V, §. 27.).

The second statement is more difficult than the first: who has the right to call an item their *property* when multiple people invest their labor into that same item? And who is to judge whether or not there is enough left of the resources common to mankind, before anyone is stopped from claiming an unclaimed item as their *property*? Is there such a thing as shared property? Who is to enforce this kind of a rule that reads more like a moral code than a law; i.e., what happens in the case that a person's property is modified, but without knowledge by the owning party?

By disregarding this kind of logistical question, Locke leaves much of the specific implementation of *property* and its characteristics up to the specific implementation of the government. While it is important to achieve a solid understanding of Locke-ian *property* for this paper's claim, it is not necessary to address all of the faults in this definition. Luckily, he provides a second description of *property* when he discusses those things which all governments must achieve, one of which is, "Preservation of their Lives, Liberties, and Estates, which I call by the general Name, *Property*" (IX. §. 123.). This is the preferable definition, because "Lives, Liberties, and Estates" is a working definition: identifiable and quantifiable. While this does not include "Whatsoever ... he hath mixed his *labour* with ... and thereby makes it his *property*," as in the first definition, the fruits of his labor can practically be reclassified into one of these three categories. The categories align more closely with the modern connotation of "property," which in general fits best under "Estate" (as Americans can generally take for granted lives and liberties as an inherent truth and not explicitly as property), as well as the subject of many laws of *The Fundamental Constitutions of Carolina*.

The rules Locke scribes for the Carolina colony provide further insight on how property (i.e., the “proprietorship” of the lords proprietors) may be of legal importance. A significant number of laws address the estates of the lords and officials, especially their obtainment and size. These estates are originally provided by a royal charter by the English government and can be inherited, such that the inheritor "shall be obliged to take the name and arms of that proprietor whom he succeeds; which from thenceforth shall be the name and arms of his family and their posterity” (§. 7.), which indicates two significant aspects about property: that property can be given from one person to another, like currency, and not by a derivation from nature via labor; and that property can have its own properties, i.e., “name and arms,” associated with it. In the latter aspect, property can be thought to “mark” its owner with its inherent value; this value can be thought of as the sum total of the labors of all of the previous owners invested into that land (as far as when the land was first gifted sans labor by the charter). This is not an uncommon idea today, since a keepsake from a grandfather or land in the family for generations is vastly more valuable its face value. By extension, a person who inherits property with a long history of owners is imbued with a similar level of value, especially so for royalty.

The laws governing land ownership of the lords are very explicit. A province is divided into counties, which is subdivided into signiories, baronies, and precincts (which are further divided into colonies) (§. 3), and the land in a manor is meticulously restricted to within three and twelve thousand acres of land (§. 17), in which the manor’s lord is allowed to bestow a limited fraction of his estates upon others for a limited duration of time (§. 18). Inheritance of an estate is explicitly allowed, in which the inheritor inherits also the name and arms of the property.

Besides physical property, Locke also includes “rights” as a form of property. In a modern context, it sounds peculiar because rights are not like the monetary or real-estate property that can be quantified and transferred; but it is properly property because effort is put into using (usually in speech as “exercising our rights”) and maintaining them (“defending our rights”). People possess rights just as they do property, but, being an immaterial construct of legislature, it must come from an infinitely generous reservoir in nature, such that always is there “enough ... left in common for others,” as per Locke’s first definition, such that it can be obtained by anyone.

It might seem abstruse then that rights can be governed, but a prominent example exists in *The Fundamental Constitutions of Carolina*: regulation of the right to religion. The Constitutions decree that “No man shall be permitted to be a freeman of Carolina, or to have any estate or habitation within it, that doth not acknowledge a God” (§. 95), which limits people to belief in a God. How is this enforced? Decrees 98 and 99 state that all religious subscriptions, along with the date and terms of admittance, must be kept on public record.

Why does the system of rules pertaining to material and immaterial property have to be so complex? In both cases, property is being both restricted and protected: land is distributed in a predetermined fashion, preventing officials from losing land and from expanding; and religion is regulated by mandating a form of religion.

This doesn’t disagree with Locke’s vision of a government. He states that “The great and *chief end* ... of men’s ... putting themselves under a government, is the *preservation of their property”* (IX. §. 124.), by means of an established law and justice systems. At the same time, he states that legislature “*cannot take* from any man any part of his *property* without his own consent” (XI, §. 138.) because of the government’s goal.

While both of these statements seem to make sense independently of one another, and it seems hypocritical to adhere to one principle and break the other, it’s not always possible. For instance, what is the government to do in the case of an economic monopoly, or in the case of “enclosure” in sixteenth century European agriculture, when power legitimately, albeit not ethically, comes into the hands of a few? There is not “enough ... left in common for others,” but that may be for causes unrelated to the monopoly, so it would be unlawful for government to try to take away that property. The poor people and the government are then at a deadlock: neither party is legally able to obtain property that has already been claimed, without the consent of the proprietor. The only quasi-lawful way for dire situations, in which the monopolizing party is deemed to be harming others by restricting the common good, is for a state of war to be declared and an authorization of force. Of course, this is the last-ditch option and should be avoided when possible.

While Locke always argues that government should protect existing property, nothing in his philosophy prevents the government from preventing the gain of property. The complicated set of rules about land ownership exist to prevent the government from active intervention by this means of passive intervention-- a lord is lord only over what he is allotted, and not allowed to expand his manor indefinitely nor bestow it upon others in whatever manner he chooses. Though religion doesn’t have to be restricted like property does, a person’s religious subscription is regulated to be somewhat conformist (adhering to a monotheistic belief) and singular.

By means of restriction, people are still allowed to (or are forced to) own property, but only in a way that the government allows for. However, for the sake of the government, it is not always legal to obtain more property, such as obtaining more land or subscribing to multiple beliefs, for fear of the consequences that may arise. In other words, this kind of a social contract doesn’t allow the government to strip people of property, but it does allow for the government to prevent acquisition of property (to protect the first interest).

What is especially important to allowing restriction, as is important to all legal principles, is its documentation. The restrictions are written in law, but it’s also important that the citizens know where the property lies, as a sort of public testament to the law. Religion is written into a public record. Transactions and ownership of physical property, such as estates, are presumably also recorded in some official record. If it isn’t, the way that property and proprietor mark each other with “name and arms,” at least for significant inheritances, make it very clear who owns what and from where it came.

Public record-keeping inherently is a controversial idea. There may emerge a class of people who try to hide all of their property, or disguise it so that the regulations have the least effect. There are a number of possible causes, which pervade modern society: avoidance of record keeping for maximum personal benefit, such as tax evaders; suspicion of government or of fellow citizens; and the want of a private life without public probing. Conversely, another faction may seek to exploit this visibility to promote personal motives, or to protect personal assets by public documentation. While the government can only regulate its citizens' property of which it has knowledge, it can only be an estimate of what the people truly own (personal, private, and secret possessions outside of legal reach).

Much of the restriction comes from the specific implementation of government in *The Fundamental Constitutions of Carolina*, and ultimately its government and feudal system were never fully realized, even after multiple revisions (Bell). But it is far closer to implementation than the philosophy in *Second Treatise of Government*, and more recent history, such as the trust-busting of the turn of the twentieth century, show that it can be beneficial to restrain what a person can own.

Works Cited

Bell, John L. "Fundamental Constitutions." *Encyclopedia of North Carolina*, University of North Carolina Press, 2006, https://www.ncpedia.org/fundamental-constitutions.

Locke, John, and C B. Macpherson. *Second Treatise of Government*. Indianapolis, Ind: Hackett Pub. Co, 1980. Print.

"The Fundamental Constitutions of Carolina : March 1, 1669." *The Avalon Project*. Yale Law School. http://avalon.law.yale.edu/17th_century/nc05.asp (accessed March 10, 2019).

Jonathan Lam
Prof. Schrubshik
MA223
Vector Calculus
3 / 1 / 19

Chapter 15 Outline

Vector Calculus Test 1 Review

15.1: Multiple Integrals

- Partition a rectangular region R into small n rectangles in the x and y direction. Each subrectangle R_i has dimensions $(\Delta x, \Delta y)$ and area $\Delta A = \Delta x \Delta y$. Then, sum of values is $S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A$
- limit of S_n is the double integral: $\lim_{\|P\|\to 0} S_n = \iint_R f(x,y)dA = \iint_R f(x,y)dxdy$
- value of double integral can be interpreted as volume under surface $z = f(x,y)$
- Fubini's theorem states that double integrals can be evaluated as iterated integrals (in either order) for rectangle $R : x \in [a,b], y \in [c,d]$
    - $\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$

15.2: Double Integrals over General Regions

- Fubini's theorem for functional limits (in one dimension):
    - For $R : x \in [a,b], y \in [g_1(x), g_2(x)]$: $\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$
    - For $R : x \in [h_1(y), h_2(y)], y \in [c,d]$: $\iint_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$
- Sometimes it's easier to evaluate double integrals when order of integration is reversed: to do this, draw the region and express the limits as a function of the other variable.
- Properties of double integrals: scalars can be taken out, sum/difference is distributive over integration, domination of one integral over another if the first's integrand is greater for every value in the domain than the second's integrand, and additivity of integrals into the union of their regions if their regions are mutually exclusive.

15.3: Area by Double Integration

- The area of a closed, bounded plane region R is $\iint_R dA$ (literally sum of the area differentials)
- Average value of f over R: $\frac{\text{total value}}{\text{total area}} = \frac{\iint_R f(x,y)dA}{\iint_R dA}$

15.4: Double Integrals in Polar Form

- Partition a polar region into sectors of angle $\Delta\theta$, and then partition sectors into subregions of length Δr. Each subregion can be approximated with an area of length $r\Delta\theta$ and width Δr. Thus, $S_n = \sum_{k=1}^{n} f(x_k, y_k) r\Delta\theta$.
- $\lim_{\|P\|\to 0} S_n = \iint_R f(r,\theta)dA = \int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta) r dr d\theta$ for $R: \theta \in [\theta_1, \theta_2], r \in [r_1(\theta), r_2(\theta)]$. By Fubini's theorem, if θ is bounded by functions of r (unlikely but possible), the limits of integration can be switched. (And if both r and θ are bounded by constants, the order of integration doesn't matter.)
- Remember rules/tricks of changing polar equations to Cartesian ones:
  - $r^2 = x^2 + y^2$
  - $x = r\cos\theta$, $y = r\sin\theta$
  - $\tan\theta = \frac{y}{x}$
- Polar equations can be used to solve equations that are given in Cartesian form but contain some of the above equations, e.g., $\iint_R e^{x^2+y^2} dA$.

15.5: Triple Integrals in Rectangular Coordinates

- Partition a solid into small rectangular prisms of dimensions Δx, Δy, and Δz. The volume of each subspace is $\Delta V = \Delta x \Delta y \Delta z$. For a function $f(x,y,z)$ defined over the space D, $S_n = \sum_{k=1}^{n} f(x,y,z)\Delta V$.
- Limit of S_n is the triple integral $\iiint_D f(x,y,z)dV = \iiint_D f(x,y,z)dxdydz$ and can be evaluated with an iterated integral.
- Volume of a space D: $\iiint_D dA$
- Using functional limits: $\int_{x=a}^{x=b}\int_{y=g_1(x)}^{y=g_2(x)}\int_{z=h_1(x,y)}^{z=h_2(x,y)} f(x,y,z)dzdydx$
  - i.e., successive limits of integration can be functions that depend on the variables of outer integrals (same as double integral with Fubini's theorem)
- Average value of $f(x,y,z)$ over D: $\frac{\text{total value}}{\text{total volume}} = \frac{\iiint_D f(x,y,z)dV}{\iiint_D dV}$
- Triple integrals have same properties as double integrals.

15.6: Moments and Centers of Mass

- Mass and moments analogous to one- and two-dimensional counterparts:
  - Mass: $M = \iiint_D \sigma(x,y,z)dm$, $\sigma(x,y,z)$ is the density function
  - Moment along x-axis: $M_{yz} = \iiint_D x\sigma(x,y,z)dm$, analogous for moments along y (M_{xz}) and z (M_{xy}) axes

- Center of mass in x direction: $\bar{x} = \dfrac{M_{yz}}{M}$, analogous for COM in y and z directions
- Moment of inertia in one dimension: $I = \displaystyle\int r^2 dm$, $r = |x - \bar{x}|$
    - $KE_{shaft} = \dfrac{1}{2} I\omega^2$
    - For an object D, moment of inertia about L is $I_L = \displaystyle\iiint_D r^2 dm = \iiint_D r^2 \sigma(x,y,z) dV$, r is distance of a point $P(x,y,z)$from L
    - For two-dimensional plate, "polar moment" (moment about the origin) is $I_0 = \displaystyle\iint_R (x^2+y^2)\delta = I_x + I_y$

<u>15.7: Triple Integrals in Cylindrical Coordinates</u>

- A point in space can be represented in cylindrical coordinates $P(r, \theta, z)$:
    - r is length of projection of $\overrightarrow{OP}$ on the xy-plane
    - θ is the angle between the projection of $\overrightarrow{OP}$ on the xy-plane and the x-axis
    - z is the height of P (same as Cartesian coordinates)
- Constant parameter interpretations for function $f(r, \theta, z)$:
    - Constant r means f lies on a circular cylinder
    - Constant θ means f lies on a plane parallel to the z-axis
    - Constant z means f lies on plane parallel to the xy-plane
- For integral of space over cylindrical coordinates, partition first by angle θ, then by radius, then by height. Each subspace D_k is approximately a rectangular prism with dimensions $(r\Delta\theta, \Delta r, \Delta z)$ and volume $V_k = r\Delta r\Delta\theta\Delta z$. The integral is $\displaystyle\iiint_D f(r,\theta,z) r dr d\theta dz$.

<u>15.7: Triple Integrals in Spherical Coordinates</u>

- A point in space can be represented in spherical coordinates $P(\rho, \theta, \phi)$:
    - ρ is $\left\|\overrightarrow{OP}\right\|$
    - θ is same as cylindrical coordinates
    - ϕ is angle between $\overrightarrow{OP}$ and the z-axis
- Useful equations for spherical coordinates:
    - $r = \rho \sin\phi$
    - $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$
    - $z = \rho\cos\phi$
    - $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2}$
- Constant parameter interpretations for function $f(\rho, \theta, \phi)$:
    - Constant ρ means f lies on a sphere
    - Constant θ means f lies on a plane
    - Constant ϕ means f lies on a cone
- For integral of space over spherical coordinates, partition first by angle θ, then by radius, then by ϕ . Each subspace D_k is approximately a rectangular prism with dimensions $(\rho\Delta\theta, \rho\Delta\phi, \Delta\rho)$ and volume $V_k = \rho^2\Delta\rho\Delta\theta\Delta\phi$. The integral is $\displaystyle\iiint_D f(\rho,\theta,\phi)\rho^2 d\rho d\theta d\phi$.

LINE AND SURFACE INTEGRALS
MA113 TEST 2 EQUATION SHEET/OUTLINE

16.1. Line Integrals

- If a curve C is smooth for $a \leq t \leq b$, then line integral over C exists
- To evaluate a line integral given as a parametric function of x, y, z
  $\int_C f(x,y,z)ds = \int_a^b f(g(t),h(t),k(t))|\vec{v}(t)|dt$
- Applications for objects defined along a curve:
  - Mass: $\int_C \delta ds$
  - First moments and COM: $M_{yz} = \int_C x\delta ds$, $\bar{x} = \frac{M_{yz}}{M}$, same with other moments and COM
  - Moments of inertia: $M_x = \int_C (y^2+z^2)\delta ds$, same with other moments of inertia
  - For a line integral on a plane (flat), line integral may be interpreted as the area of the “wall” created along the curve with a height $f(t)$, where $f(t)$ is the integrand
- If piecewise smooth function curve made of finite smooth curves, line integral over entire curve is equal to the sum of the line integrals of the curves
- Value of the line integral may be path-dependent

16.2. Vector Fields and Line Integrals: Work, Circulation, and Flux

- A <u>vector field</u> is a function that assigns a vector to each point on its domain, e.g., $\vec{F}(x,y,z) = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$
  - Continuous if component functions continuous, differentiable if component functions differentiable
- <u>Gradient field</u> is field of gradient vectors, shows direction of greatest increase of f
  - i.e., $\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$
- Line integral of a curve in a vector field has the integrand being the scalar tangential component of F along C, or $\vec{F}\cdot\vec{T} = \vec{F}\cdot\frac{d\vec{r}}{ds}$, so $\int_C \vec{F}\cdot\vec{T}ds = \int_C \vec{F}\cdot\frac{d\vec{r}}{ds}ds = \int_C \vec{F}\cdot d\vec{r}$
  - To evaluate a line integral of a $\vec{F}(x,y,z)$ along $\vec{r}(t)$, express $\vec{F}$ in terms of t (i.e., $\vec{F}(x,y,z) \to \vec{F}(\vec{r}(t))$ by substituting functional components of $\vec{r}$ into functional components of $\vec{F}$) find $\frac{d\vec{r}}{dt}$, and evaluate $\int_C \vec{F}(\vec{r}(t))\cdot\frac{d\vec{r}}{dt}dt$
- If vector field $\vec{F}$ only has single component, then can express line integral wrt one coordinate
  - Define line integral of a function wrt one coordinate: $\int_C M(x,y,z)dx \equiv \int_C \vec{F}\cdot d\vec{r}$, where $\vec{F}$ only contains the x component function M.
  - $\int_C M(x,y,z)dx + \int_C N(x,y,z)dy + \int_C P(x,y,z)dz = \int_C Mdx + Ndy + Pdz$ (i.e., sum of component line integrals is the total line integral)
- Applications of line integrals:
  - $W = \int_C \vec{F}\cdot\vec{T}ds$ (work is a regular line integral, $\vec{F}$ is force vector field
  - Flow $= \int_C \vec{F}\cdot\vec{T}ds$, $\vec{F}$ is velocity vector field (usually of a fluid)

    - If the curve is closed (starts and ends in the same place, called circulation
  - Flux $= \int_C \vec{F}\cdot\vec{n}ds$, C is a simple (non-overlapping) closed curve, $\vec{F}$ is some field (fluid's velocity field, electric field, magnetic field), $\vec{n}$ is outward-pointing normal vector
    - Be careful about the signs: if curve traveling counterclockwise, then $\vec{n} = \vec{T}\times\hat{k}$, else switch order of vectors
    - For a closed curve counter-clockwise in the x-y plane, flux is $\oint_C Mdy - Ndx$

Summary of ways to indicate a line integral:

TABLE 16.2 Different ways to write the work integral for $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ over the curve $C: \mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$, $a \le t \le b$

$\mathbf{W} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$	The definition
$= \int_C \mathbf{F}\cdot d\mathbf{r}$	Vector differential form
$= \int_a^b \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt$	Parametric vector evaluation
$= \int_a^b \left(M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt}\right)dt$	Parametric scalar evaluation
$= \int_C M\,dx + N\,dy + P\,dz$	Scalar differential form

16.3. Path Independence, Conservative Fields, and Potential Functions

- $\vec{F}$ is a vector field defined in open region D in space; if for any two points A and B in D the line integral $\int_A^B \vec{F}\cdot d\vec{r}$ has the same value for any path, then the integral is path independent in D and $\vec{F}$ is conservative on D
  - Can represent integral with limits $\int_A^B$ instead of $\int_C$ to indicate path-independence
- If $\vec{F}$ is a vector field defined on D and $\vec{F} = \nabla f$ for some scalar function f on D, then f is called a potential function for $\vec{F}$
  - $\vec{F}$ is conservative $\Longleftrightarrow$ it is the gradient field of a potential function (see Theorem 2 below)
  - e.g., a gravitational potential is a scalar function whose gradient field is a gravitational field, same for electric potential, so gravitational and electric fields are conservative
- Assumptions necessary for conservative fields
  - Curves must be piecewise smooth (finitely-many smooth pieces connected end-to-end)
  - D is simply connected (every loop can be contracted to single point in D without ever leaving D)

- D is connected (two points in D can be connected without leaving D)
  - Note that simple-connectedness and connectedness do not imply one another *(why is this true? https://math.stackexchange.com/questions/729551/can-a-disconnected-set-be-simply-connected)*
- Theorem 1: Fundamental Theorem of Line Integrals (line integral analogue of FTC): Let C be a smooth curve joining the point A to the point B in the plane or in space and parametrized by $r(t)$. Let f be a differentiable function with a continuous gradient vector $F = \nabla f$ on a domain D containing C. Then $\int_A^B \vec{F} \cdot d\vec{r} = \int_A^B \nabla f \cdot d\vec{r} = f(B) - f(A)$
  - Proof: $\frac{df}{dt} = \nabla f \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt}$, which is equal to integrand of normal curve; then use FTC on this to show that $\int_C \vec{F} \cdot d\vec{r} = \int_A^B \frac{df}{dt} dt = f(B) - f(A)$
- Theorem 2: Conservative Fields are Gradient Fields: Let $\vec{F}(x, y, z)$ be a vector field whose components are continuous throughout open connected region D in space. Then $\vec{F}$ is conservative ⇔ $\vec{F}$ is a gradient field ∇f for a differentiable function f.
  - Proof: $\vec{F} = \nabla f \Rightarrow \vec{F}$ is conservative is easy to prove because of Theorem 1: the integral over a gradient field is only dependent on the endpoints (therefore conservative)
  - Proof: $\vec{F}$ is conservative $\Rightarrow \vec{F} = \nabla f$: Show that $\frac{\partial f}{\partial x} = M$, same with other components (see p.923)
- Theorem 3: Loop Property of Conservative Fields: Equivalency of the statements:
  - $\oint_C \vec{F} \cdot d\vec{r} = 0$ around every loop in D
  - $\vec{F}$ is conservative on D
  - (these in turn are equivalent to $\vec{F} = \nabla f$)
- Component Test for Conservative Fields: Let vector field $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$ on simply connected domain whose component functions have continuous first partial derivatives. Then $\vec{F}$ conservative IFF $\frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}$, $\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$, and $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
  - Proof that these equations work (but not why they imply conservative-ness): write $\vec{F} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$, then solve for $\frac{\partial N}{\partial z}$ and other partial derivatives
- Finding potential functions: Once it is known that a field is conservative, then $\frac{\partial f}{\partial x} = M$ (and same for other components); differentiate to get components of f (i.e., $f = \int M dx\hat{i} + \int N dy\hat{j} + \int P dz\hat{k}$)
- Exact differential forms: $Mdx + Ndy + Pdz$ is an expression in differential form. It is exact if it is the total differential of some scalar function f over domain D. A differential can be checked for exactness just like component test for conservative fields
  - If line integral over conservative field written in differential form $\int_C Mdx + Ndy + Pdz$, can compute using method above for conservative fields

16.4. Green's Theorem in the Plane

- The divergence (flux density) of a vector field $\vec{F} = M\hat{i} + N\hat{j}$ at (x, y) is $\text{div}\vec{F} = \dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y}$
  - Derivation: p. 932
  - Physical interpretation: similar to "expansion at a point": if larger vectors out than in, then positive divergence; if not, negative divergence; basically flux in an infinitesimal area (hence "flux density")
- The circulation density (k-component of curl, $\text{curl}\vec{F} \cdot \hat{k}$) of $\vec{F}$ at point (x, y) is $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$
  - k-component of the more general circulation field
  - Denotes spin (positive circulation density means counterclockwise) at a point
- Examples of divergence and circular density of certain vector fields:
  - Uniform expansion/compression: $\vec{F} = cx\hat{i} + cy\hat{j}$ constant divergence, no circulation density
  - Uniform rotation: $\vec{F} = -cy\vec{i} + cx\vec{j}$ 0 divergence, constant circulation density
  - Shearing flow: $\vec{F} = y\vec{i}$ 0 divergence, constant circulation density
  - Whirlpool: $\vec{F} = \dfrac{-y}{x^2 + y^2}i + \dfrac{x}{x^2 + y^2}\hat{j}$ 0 divergence, 0 circulation density
- Green's Theorem:
  - Theorem 4: Green's Theorem (Flux-Divergence or Normal Form): Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\vec{F}$ be a vector field in the plane, with components having continuous first partial derivatives in open region containing R. Then outward flux of $\vec{F}$ across C is: $\oint_C \vec{F} \cdot \vec{n} ds = \oint_C M dy - N dx = \iint_R \dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y} dx dy$, or the double integral of the divergence of the field over the region enclosed by the curve.
    - Makes sense – integrate "flux density" over a region to get flux
    - To remember this integral, think right side as "normal" integral of partials, left side as switch sign and multiply by $dxdy$
  - Theorem 5: Green's Theorem (Circulation-Curl or Tangential Form): (Same conditions as first part). Then ccw circulation is: $\oint_C \vec{F} \cdot \vec{T} ds = \oint_C N dy + M dx = \iint_R \dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} dx dy$
    - Integrate "circulation density" over a region to get circulation

    - To remember this integral, think of left side as “normal line integral”, think of right side as switch sign and divide by $dxdy$
  - Taking flux (Thm. 4) of $\vec{G_1} = N\hat{\imath} - M\hat{\jmath}$ gives circulation, taking circulation (Thm. 5) of $\vec{G_2} = -N\hat{\imath} + M\hat{\jmath}$ gives flux, so closely related; either can be used to solve some problems by interchanging M and N (see p. 938)
  - Proof on p. 939
  - Can be used on any plane with a simply connected region, and also for some non-simply connected regions if same orientation of curves (see p. 940)
- <u>Reverse Green’s Theorem to find area</u>: $\text{Area}R = \frac{1}{2}\oint xdy - ydx$
  - Derivation: $\text{Area} = \iint dydx = \iint \frac{1}{2} + \frac{1}{2}dydx = \oint \frac{1}{2}xdy - \frac{1}{2}ydx$ (or $\frac{\partial M}{\partial x} = \frac{1}{2}$, $\frac{\partial N}{\partial y} = \frac{1}{2}$)

16.5. Surfaces and Area

- Parametrization of a surface: $\vec{r}(u,v) = f(u,v)\hat{\imath} + g(u,v)\hat{\jmath} + h(u,v)\hat{k}$
- A parameterized surface $\vec{r}(u,v)$ is smooth if $\vec{r}_u$ and $\vec{r}_v$ are continuous and $\vec{r}_u \times \vec{r}_v$ are never 0 in the interior of the domain.
- The <u>area</u> of a smooth surface is $A = \iint_R |\vec{r}_u \times \vec{r}_v| dudv = \iint_R d\sigma$
- For an implicit surface $F(x,y,z) = c$ over closed and bounded region, assume $\nabla F \neq 0$, $\nabla F \cdot \vec{p} \neq 0$ ($\vec{p}$ is unit vector normal to plane “shadow,” so never folds back on itself), and smooth
  - $d\sigma = \frac{|\nabla F|}{|\nabla F \cdot \vec{p}|} dxdy$
  - <u>surface area of an implicit function</u> is $\iint_R d\sigma = \frac{|\nabla F|}{|\nabla F \cdot \vec{p}|} dA$, where $\vec{p}$ normal to R, $\nabla F \cdot \vec{p} \neq 0$
- $d\sigma = \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy$, for surface defined by $z = f(x,y)$

16.6. Surface Integrals

- The surface integral of a scalar function $f(x,y,z)$ is $\iint_S f(x,y,z)\, d\sigma$, where $d\sigma$ is one of the differential forms from 16.5

- Orientation of a surface
  - A smooth surface S is <u>orientable</u> or <u>two-sided</u> if it is possible to define a field $\vec{n}$ of unit normal vectors on S that varies continuously with position
    - Smooth closed surfaces are orientable
    - $\vec{n}$, by convention, points outwards from a closed surface
    - The Mobius band is not orientable
- $\text{Flux} = \iint_S \vec{F} \cdot \vec{n}\, d\sigma$
  - Can be positive or negative depending on orientation (not very important)
  - For a surface given parametrically, flux is $\iint_S \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} |\vec{r}_u \times \vec{r}_v|\, du\, dv = \iint_S \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\ du\, dv$
  - For a surface given implicitly, flux is: $\iint_S \vec{F} \cdot \frac{\nabla g}{|\nabla g|} \frac{|\nabla g|}{|\nabla g \cdot \vec{p}|} = \iint_S \vec{F} \cdot \frac{\nabla g}{|\nabla g \cdot \vec{p}|}\, dA$
- Applications of surface integrals
  - Same as 1D, 2D analogues, replace integral with $\iint_S$ and delta with $d\sigma$

16.7. Stokes' Theorem

- Let S be a piecewise smooth surface having a piecewise smooth boundary C. Let $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$ be a vector field whose components have continuous first partial derivatives on an open region containing S. Then the circulation of $\vec{F}$ around C in the direction counterclockwise wrt unit normal vector $\hat{n}$ is $\nabla \times \vec{F} \cdot \hat{n}$ over S.
  - $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n}\, d\sigma$
- If two different oriented surfaces have the same boundary, they have the same curl integral
- For a two-dimensional field (Green's Theorem):
  - $\oint_C \vec{F} \cdot d\vec{r} = \iint_R \nabla \times \vec{F} \cdot \hat{k}\, dA$ (i.e., the $\hat{k}$-component of curl)
    - $\nabla \times \vec{F} \cdot \hat{k}$ is the new circulation density, is equal to $\frac{1}{\text{Area}} \oint_C \vec{F} \cdot d\vec{r}$
- Circulation over a curve is the flux of the curl across a surface bounded by that curve, as long as the curves are traced in the same orientation (i.e., all curves have the surface to the left of them
- $\text{curl grad } f = \vec{0}$, or $\nabla \times \nabla f = \vec{0}$

- For a simply-connected open region D, $\nabla \times \vec{F} = \vec{0} \Rightarrow \oint_C \vec{F} \times d\vec{r} = 0 \Rightarrow$ field is conservative over D

16.8. Divergence Theorem

- Divergence Theorem: $\iint_S \vec{F} \cdot \hat{n}\, d\sigma = \iiint_D \nabla \cdot \vec{F}\, dV$
- For the field $\vec{F} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\rho^3}, \rho = \sqrt{x^2 + y^2 + z^2}$
  - For any region between two spherical shells, $\iiint_D \nabla \cdot \vec{F}\, dV = 0$
  - For any sphere, flux is 4π
- Gauss's Law: for any region encompassing the origin, $\iint_S \vec{E} \cdot \vec{n}\, d\sigma = \dfrac{q}{\epsilon_0}$ (p. 979)
- Continuity equation of hydrodynamics: $\nabla \cdot \vec{F} + \dfrac{\partial \delta}{\partial t} = 0$ (p. 979)
- Unifying Fundamental Theorem: the integral of a differential operator acting on a field over a region equals the sum of the field components appropriate to the operator over the boundary of the region

MA224: PROBABILITY TEST 1 OUTLINE
Chapter 1: Probability
Chapter 2: Discrete Distributions

1.1. Basic Concepts

- Random experiments: an experiment (action with multiple outcomes) for which the outcome cannot be predicted with certainty
- Sample space, universal set, or outcome space S: collection of all possible outcomes
  - Can be continuous or discrete (countable), infinite or finite
- Random variables: measurements on outcomes associated with random experiments
  - Usually denoted with capital letter
- Distribution of a random variable, or population: a description of the frequencies of different outcomes
  - Usually estimated through samples, collection of the observations that are obtained from repeated trials of the random experiment
  - Statistical inference: the process of making a conjecture about the distribution of a random variable based on a sample
- Probability of an outcome A, with frequency $f = \mathcal{N}(A)$ in n trials, is $P(A) = \dfrac{\mathcal{N}(A)}{n}$. This ratio is also called the relative frequency.
  - Frequency table, relative frequency table, histogram, or density histogram can be used to graphically/visually show frequencies of occurrences out of total trials
  - Relative frequencies can be unstable for small n, but tends to stabilize for a large n towards $P(A)$
- Probability mass function (p.m.f.) is a function that serves as a model for the probabilities of the outcomes of a random experiments
  - i.e., if random experiment repeated many times, it is expected that the relative frequency $\dfrac{\mathcal{N}(x_0)}{n} \to P(x = x_0) = f(x_0)$
  - Can construct a probability histogram, which should be close to the frequency histogram if n is large and the model is good
- Simpson's paradox: Relative frequencies are estimates towards probabilities, but you can't easily compare multiple groups of unlike relative frequencies to estimate a composite probability. (i.e., It's possible that to have two groups and two random experiments, in which one group has lower probabilities in both random experiments but a higher total probability than the other, based on the conditional probabilities of the two random experiments.)

1.2. Properties of Probability

- Event: Given an outcome space S, let $A \subset S$. Thus A is an event, a subset of S. When the outcome of the experiment is in A, then A has occurred.
- S and any event are sets, and follow set theory, and can be illustrated with Venn diagrams\
  - Null set: $\emptyset$
  - A subset of B: $A \subset B$
  - A union B: $A \cup B$
  - A intersection B: $A \cap B$
  - Complement of A: A'
  - $A_1, A_2, \ldots, A_k$ mutually exclusive events if $A_i \cap A_j = \emptyset, i \neq j$ (i.e., all events are disjoint sets)
  - $A_1, A_2, \ldots, A_k$ exhaustive events if $A_1 \cup A_2 \cup \cdots \cup A_k = S$

- Possible to have set of events that are both mutually exclusive and exhaustive
- Probability: a real-valued set function ($P : \{\mathbb{R}\} \rightarrow \mathbb{R}$) that assigns, to each event A in the sample space S the probability (real number) $P(A)$, which follows the following properties:
  - $P(A) \geq 0$
  - P(S)=1
  - If set of mutually exclusive events $A_1, A_2, \ldots, A_k$, then $P(A_1 \cup A_2 \cup \cdots \cup A_k) = P(A_1) + P(A_2) + \cdots + P(A_k)$ (for any countable number of events)
- Properties of probability (know how to prove each one, proofs are easy):
  - $P(A) = 1 - P(A')$
  - $P(\emptyset) = 0$
  - If $A \subseteq B$, then $P(A) \leq P(B)$
  - $P(A) \leq 1$
  - $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (for any event, not only mutually exclusive)
    - Can be extended to more elements: $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cup B \cup C)$
- If m equally likely, mutually-exclusive and exhaustive outcomes, then the probability of any of those outcomes is $\frac{1}{m}$
  - If h mutually-exclusive, equally likely outcomes in event A, then $P(A) = \frac{h}{m}$

1.3. Methods of Enumeration

- Multiplication principle: If event E_1 has n_1 possible outcomes, and for each of these possible outcomes, event E_2 has n_2 possible outcomes, then the composite experiment E_1E_2 has n_1n_2 possible outcomes.
- Permutation: each of the $n!$ possible arrangements of n different objects.
- Each of the ${}_nP_r$ arrangements is called a permutation of n objects taken r at a time (ordered sample of size r)
- Different sampling methods (n is number of distinct objects, r is sample size):
  - Without replacement, ordered: ${}_nP_r = \frac{n!}{r!}$
  - Without replacement, unordered: ${}_nC_r = \frac{n!}{r!(n-r)!} = \binom{n}{r}$
  - With replacement, ordered: n^r
  - With replacement, unordered: ${}_{n-1+r}C_r = \frac{(n-1+r)!}{r!(n-1)!}$
- Number of distinguishable permutations:
  - Same as choosing without replacement, unordered (${}_nC_r$)
  - For more than two distinguishable types, multinomial coefficient number of possibilities (i.e., for k distinguishable types such that $n_1 + n_2 + \cdots + n_k = n$, $\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!n_2!\ldots n_k!}$ possible outcomes)
- Note: $\binom{n}{r} = \binom{n}{n-r}$

1.4. Conditional Probability

- Conditional probability of event A given that event B has occurred, is: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$, given that $P(B) > 0$
  - Thus, $P(A \cap B) = P(B)P(A|B) = P(A)P(B|A)$ (think multiplication rule)
  - Note that conditional probability follows the axioms for a probability function

1.5. Independent Events

- A pair of events is independent (statistically independent, stochastically independent, independent in a probabilistic sense) if the occurrence of one does not change the probability of the occurrence of the other
  - i.e., $P(A) = P(A|B)$
  - Thus, because $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$, $P(A)P(B) = P(A \cap B)$ (special case of the multiplication rule)
- Theorem: If A and B are independent events, then so are (prove these as well, also easy proofs):
  - A and B'
  - A' and B
  - A' and B'
- Events A, B, and C are mutually independent IFF both conditions hold (can be extended to larger sets of events, where each pair, triplet, quartet, etc. satisfy the special multiplication rule):
  - A, B, and C are pairwise independent
  - $P(A \cap B \cap C) = P(A)P(B)P(C)$

1.6. Bayes's Theorem

- Consider a space that is partitioned into k mutually exclusive, exhaustive events $B_1, B_2, \ldots, B_k$ with known probabilities (prior probabilities), and a space A, such that $P(A|B_i), 1 \leq i \leq k$ is known
  - Once event A has occurred, the probability that the outcome was in event B_j is the posterior probability $P(B_j|A) = \dfrac{P(B_j \cap A)}{P(A)} = \dfrac{P(B_j)P(A|B_j)}{\sum\limits_{i=1}^{k} P(B_k)P(A|B_k)}, j = 1, 2, \ldots k$ (Bayes's Theorem)

2.1. Random Variables of the Discrete Type

- Given a random experiment with an outcome space S, a function X that maps every element of S to a real number is called a random variable ($X : S \to \mathbb{R}$)
  - The space of X is the set of real numbers $\{x : X(s) = x, s \in S\}$
  - In other words, X can be thought of as a numeric measurement (or designation) taken from a random experiment, a way of "mathematicalizing" an arbitrary outcome space by mapping it to the real number line
- Discrete (countable) types were mentioned in 1.1, this is the type dealt with in this chapter
- The p.m.f. of a discrete random variable X is a function $f(x) = P(X = x)$ with the following properties:
  - $f(x) > 0, x \in S$
  - $\sum\limits_{x \in S} f(x) = 1$

- $P(X \in A) = \sum_{x \in A} f(x)$, where $A \subseteq S$
- Hypergeometric distribution: when choosing n items from a collection of $N = N_1 + N_2$ objects, where N_1 and N_2 are the counts of the two distinguishable classes of objects, and the random variable X is the number of objects selected of type N_1, then the p.m.f. is:
$$f(x) = P(X = x) = \frac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N}{n}}, 0 \le x \le N_1, x \le n \le N$$

2.2. Mathematical Expectation

- Mathematical expectation, or expected value of function $u(X)$ random variable X of the discrete type with space S is $E(u(X)) = \sum_{x \in S} u(x)f(x)$
  - Since $u(X)$ is also a function mapping X to another value, it can be thought of as another random variable (will produce same result, just a different way to think about it)
- When it exists, the mathematical expectation E satisfies the properties (be able to prove these):
  - If c is a constant, then $E(c) = c$
  - If c is a constant and u is a function, then $E[cu(X)] = cE[u(X)]$
  - If c_1 and c_2 are constants and u_1 and u_2 are functions, then $E[c_1u_1(X) + c_2u_2(X)] = c_1E[u_1(X)] + c_2E[u_2(X)]$ (i.e., E is a linear operator)
- For a hypergeometric distribution, $E(X) = n\frac{N_1}{N} = np$ (makes sense)

2.3. The Mean, Variance, and Standard Deviation

- The mean of a random variable X is $\mu = E(X) = \sum_{x \in S} xf(x)$
  - This is the first moment about the origin
  - The first moment about the mean is $E(X - \mu) = 0$
- The variance of a random variable X is the second moment about the mean: $\sigma^2 = E((X-\mu)^2) = \sum_{x \in S}(x-\mu)^2 f(x)$
  - The standard deviation is $\sigma = (+)\sqrt{\sigma^2}$
  - Formula 2: $\sigma^2 = E(X^2 - 2\mu X + \mu^2) = E(X^2) - 2\mu E(X) + \mu^2 = E(X^2) - \mu^2$, because $\mu = E(X)$
  - Formula 3: $\sigma^2 = E(X(X-1)) + \mu - \mu^2$ (see derivation below under factorial moment)
- If random variable $Y = aX + b$ (linear mapping of random variable X) (be able to derive these and think about them intuitively):
  - $\mu_Y = a\mu_X + b$
  - $\sigma_Y^2 = a^2\sigma_X^2$ (or $\sigma_Y = a\sigma_X$)
- r^{th} moment of the distribution about b is $E((X-b)^r)$
- r^{th} factorial moment is $E(X(X-1)(X-2)\ldots(X-r+1))$
  - Using the second factorial moment, $E(X(X-1)) = E(X^2) - E(X)$. Thus, $\sigma^2 = E((X-\mu)^2) = E(X^2) - \mu^2 = E(X(X-1)) + \mu - \mu^2$. This is sometimes easier to calculate than using other formulas for variance
- Variance of a hypergeometric distribution is $\sigma^2 = npq\left(\frac{N-n}{N-1}\right)$

- If a random experiment is actually performed n times, the collection of outcomes is called a sample
  - The empirical distribution has a p.m.f. $f(x) = \frac{1}{n}$, sample mean $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, sample variance $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$ (sample mean and sample variance are used to estimate the population mean and standard deviation)

2.4. Bernoulli Trials and the Binomial Distribution

- Bernoulli experiment: random experiment with only two mutually-exclusive and exhaustive outcomes (usually success and failure)
  - Bernoulli trials: a sequence of Bernoulli experiments performed several independent times
  - Let $p = P(\text{success})$, $q = 1 - p$
  - A Bernoulli experiment (a single Bernoulli trial) has a Bernoulli distribution, with random variable X corresponding to success (1) or failure (0), with p.m.f. $f(x) = p^x q^{1-x}, x = 0, 1$; $\mu = p$; and $\sigma^2 = pq$
- When a sequence of n Bernoulli trials carried out, it has a binomial distribution, with random variable X indicating number of successes
  - p.m.f.: $f(x) = \binom{n}{x} p^x q^{n-q}, x = 0, 1, 2, \ldots n$; $\mu = np$; $\sigma^2 = npq$
  - Bernoulli distribution is special case of binomial distribution with $n = 1$
  - Binomial distribution be represented shorthand as $b(n, p)$
  - This can be used to approximate the hypergeometric distribution (similar, but without replacement) when n is large, since events are essentially independent of one another
- Cumulative distribution function (c.d.f.), or distribution function, of a random variable X, is a function such that $F(x) = P(X \leq x), -\infty < x < \infty$
  - $f(x) = F(x) - F(x-1)$
  - $P(X > x) = 1 - F(x)$

2.5. The Moment-Generating Function

- Let X be a random variable of the discrete type with p.m.f. $f(x)$ and space S. If there is a positive number h such that $E(e^{tX}) = \sum_{x \in S} e^{tx} f(x)$ exists and is finite for $-h < t < h$, then the function of t defined by $M(t) = E(e^{tX})$ is called the moment-generating function of X (m.g.f.)
  - $M(0) = \sum_{x \in S} f(x) = 1$
  - If the space of S is the set of mutually-exclusive, exhaustive events $\{b_1, b_2, b_3, \ldots\}$, then $M(t) = f(b_1)e^{b_1 t} + f(b_2)e^{b_2 t} + f(b_3)e^{b_3 t} + \cdots$; use this pattern to identify probabilities of the outcomes given the m.g.f., or v.v.
  - Note that the m.g.f. is unique to a distribution; if a m.g.f. exists, there exists one and only one distribution of probability associated with it
  - If moments are given (i.e., formula for $E(X^r), r = 1, 2, \ldots$), then usually can find closed-form m.g.f. using Maclauren series
- Differentiating the m.g.f. r times gives the r^{th} moment of the distribution (around 0) at $t = 0$
  - $M'(t) = \sum_{x \in S} x e^{xt} f(x) \Rightarrow M'(0) = E(X) = \mu$
  - $M''(t) = \sum_{x \in S} x^2 e^{xt} f(x) \Rightarrow M''(0) = E(X^2) \Rightarrow M''(0) - M'(0)^2 = \sigma^2$

- (and so on)
- Using these formulas can be used to easily find μ, σ^2 for the binomial distribution (with m.g.f. $M(t) = \left[(1-p) + pe^t\right]^n$

- Negative binomial distribution is a distribution where X is number of trials until r successes occur
  - p.m.f.: $f(x) = \binom{x-1}{r-1} p^r q^{x-r}, x = r, r+1, \dots$
  - m.g.f: $M(t) = \dfrac{(pe^t)^r}{[1-(1-p)e^t]^r}, (1-p)e^t < 1$ (don't have to memorize this, but this is used to find the μ, σ^2)
    - $\mu = \dfrac{r}{p}, \sigma = \dfrac{rq}{p^2}$
  - If $r = 1$, then called geometric distribution, p.m.f. $f(x) = pq^{x-1}, x = 1, 2, \dots$
    - The sum of the p.m.f. can easily be verified to equal 1 using the infinite sum of a geometric series formula
    - $P(X > k) = q^k$
    - c.d.f: $F(x) = P(X \leq x) = 1 - q^x$
    - $\mu = p^{-1}, \sigma^2 = \dfrac{q}{p^2}$

Summary of Counting Methods (from 1.3)

	With replacement	**Without replacement**
Ordered	n^r	${}_nP_r$
Unordered	${}_{n-1+r}C_r$	${}_nC_r$

Summary of Distributions

Name	**p.m.f.**	**m.g.f**	μ	σ^2	**Use**
Hypergeometric	$f(x) = \dfrac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N}{n}}$, $x = 0, 1, \dots N_1, x \leq n \leq N$	doesn't look fun on Wikipedia	np	$npq\left(\dfrac{N-n}{N-1}\right)$	How many of first type ("successes") when drawing w/o replacement
Binomial $b(n, p)$	$f(x) = \binom{n}{x} p^x q^{n-x}, x = 0, 1, \dots n$	$M(t) = \left[(1-p) + pe^t\right]^n$	np	npq	How many successes in n Bernoulli trials
Bernoulli (binomial, $n = 1$)	$f(x) = p^x q^{1-x}, x = 0, 1$	$M(t) = (1-p) + pe^t$	p	pq	Chance of success in 1 Bernoulli trial
Negative binomial	$f(x) = \binom{x-1}{r-1} p^r q^{x-r}$, $x = r, r+1, \dots, r \geq 1$	$M(t) = \dfrac{(pe^t)^r}{[1-(1-p)e^t]^r}$, $(1-p)e^t < 1$	$\dfrac{r}{p}$	$\dfrac{rq}{p^2}$	How many Bernoulli trials until r successes
Geometric (negative binomial,	$f(x) = pq^{x-1}, x = 1, 2, \dots$	$M(t) = \dfrac{pe^t}{1-(1-p)e^t}$, $(1-p)e^t < 1$	$\dfrac{1}{p}$	$\dfrac{q}{p^2}$	How many Bernoulli trials until first success

$r = 1)$					

MA224: Probability Test 2 Outline

Sections covered: 2.6, 3.1, 3.3-6, 4.1, some of 4.2, some of 5.1, 5.3-5.7.

2.6. The Poisson Distribution

- Def. 2.6.1. Let the number of changes that occur in a given continuous interval be counted. Then we have an <u>approximate Poisson distribution</u> with parameter $\lambda > 0$ if the conditions are satisfied:
  - The numbers of changes occurring in nonoverlapping intervals are independent
  - Probability of exactly one change occurring in short time interval h is λh
  - Probability of two or more changes in short time interval is 0
- <u>Poisson distribution</u> is binomial distribution $b\left(n, \dfrac{\lambda}{n}\right)$
  - Do n trials, so average is λ
  - $\lim\limits_{n\to\infty} f(x) = \dfrac{\lambda^x e^{-\lambda}}{x!}$
    - For a time interval t, distribution is $f(x) = \dfrac{(\lambda t)^x e^{-\lambda t}}{x!}$
  - $\mu = M'(0) = \lambda = \sigma^2$
- Can approximate binomial distribution with high n, p (and therefore λ) should be small
  - $\dfrac{(np)^x e^{-np}}{x!} \approx \begin{pmatrix} n \\ x \end{pmatrix} p^x (1-p)^{n-x} = b(n,p)$ (replace λ with np)
  - In turn can be used to approximate hypergeometric distribution with high n and small p (i.e., $N_1 = np$, $N_2 = n - np$)

3.1. Continuous-Type Data, 3.2. Exploratory Data Analysis

- (not useful)

3.3. Random Variables of the Continuous Type

- <u>Probability density function (pdf)</u> of a continuous random variable X is an integratable function such that $P(a < X < b) = \int_a^b f(x)\,dx$
  - $f(x) = 0$ when $x \notin S$
- <u>Cumulative density function</u> is $F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$
  - $P(X = b) = 0$
  - $P(a \le X \le b) = F(b) - F(a)$
- Same other definitions that were based on pdfs for discrete-type variables:
  - $\mu = E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$
  - $\sigma^2 = E\left[(X-\mu)^2\right] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$
  - $M(t) = E\left(e^{tx}\right) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx,\ -h < t < h$
- (From exercise 12) For function $R(t) = \ln M(t)$:
  - $\mu = R'(0)$
  - $\sigma^2 = R''(0)$
- Median found by using cdf: $F(x) = 0.5$

3.4. The Uniform and Exponential Distributions

- Uniform distribution:
  - $F(x) = \frac{x-a}{b-a},\ a \le x \le b$
  - $f(x) = \frac{1}{b-a},\ a \le x \le b$
  - $M(t) = \frac{e^{bt} - e^{at}}{t(b-a)}$
  - $\mu = \frac{a+b}{2}$
  - $\sigma^2 = \frac{(b-a)^2}{12}$
- Exponential distribution:
  - $F(w) = 1 - e^{-\lambda w} = 1 - e^{-\frac{w}{\theta}}$
    - $\Rightarrow P(X \ge x) = e^{-\frac{x}{\theta}}$
    - $\Rightarrow m\,(\text{median}) = \theta \ln 2$
  - $M(t) = \frac{1}{1-\theta t},\ t < \frac{1}{\theta}$
  - $f(w) = \lambda e^{-\lambda w} = \frac{1}{\theta} e^{-\frac{w}{\theta}},\ \theta = \frac{1}{\lambda},\ \theta > 0$
  - $\mu = \theta$, $\sigma^2 = \theta^2 \Rightarrow \sigma = \theta$
  - i.e., while λ is average changes per unit time, θ is average waiting time between changes (makes sense; should be inversely proportional)
  - “Failure rate is constant”: i.e., conditional probability over the same change in time is constant, i.e., $P(X > t_1 + t | X > t_1) = P(X > t)\ (= (P(X > t | X > 0)))$; has real-world application that its not worth replacing an object with constant failure rate
    - Similar to (discrete) geometric distribution; no other continuous distribution has this “forgetfulness” property

3.5. The Gamma and Chi-Square Distributions

- Gamma distribution:
  - Let W be the time until the α-th change occurs
  - $F(w) = \frac{\lambda(\lambda w)^{\alpha-1}}{(a-1)!} e^{-\lambda w}$
    - If $w < 0$, $F(w) = 0$, $F'(w) = 0$
  - Definition of gamma function, useful in writing the pdf of the distribution: $\Gamma(t) = \int_0^\infty y^{t-1} e^{-y}\, dy = (t-1)!,\ t > 0$
  - $f(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-\frac{x}{\theta}},\ 0 \le x < \infty$
  - $M(t) = \frac{1}{(1-\theta t)^\alpha},\ t < \frac{1}{\theta}$
  - $\mu = \alpha\theta$, $\sigma^2 = \alpha\theta^2$
- Chi-square distribution: Gamma distribution with $\theta = 2$, $\alpha = \frac{r}{2}$, r is a positive integer
  - $f(x) = \chi^2(r) = \frac{1}{\Gamma(\frac{r}{2}) 2^{\frac{r}{2}}} x^{\frac{r}{2}-1} e^{-\frac{x}{2}}$
  - $P(x \ge \chi^2_\alpha(r)) = \alpha$ (use lookup table for this)
  - $\mu = r,\ \sigma^2 = 2r$

- $M(t) = \dfrac{1}{(1-2t)^{-\frac{r}{2}}},\ t < \dfrac{1}{2}$
- Exponential distribution with $\mu = \theta = 2$ is $\chi^2(2)$

3.6. The Normal Distribution

- $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} \exp\left[-\dfrac{(x-\mu)^2}{2\sigma^2}\right]\ dx$
- $M(t) = \exp\left(\mu t + \dfrac{\sigma^2 t^2}{2}\right)$
- μ and σ^2 are given in the pdf
- $N(0,1)$ is called the standard normal distribution
- Sometimes want to find inverse of standard normal distribution function
  - I.e., normally find $P(X \leq x) = p$, but now find x given p (use lookup tables)
  - Also sometimes want to find $P(z \geq z_\alpha) = \alpha$ (find z_α, the upper percent)
  - $P(Z \leq -z_\alpha) = P(Z \geq z_\alpha) = \alpha$ (because of symmetry)
  - $z_{1-\alpha} = -z_\alpha$
  - Think of z_α as the z-score of the normal distribution where α of the distribution is greater than that z-scores
- Theorem 3.6-1. Standardizing a normal distribution. If X is $N(\mu, \sigma^2)$, then $Z = \dfrac{X-\mu}{\sigma} = N(0,1)$
  - (Basically taking z-score of distribution)
- Theorem 3.6-2. If the random variable X is $N(\mu, \sigma^2),\ \sigma^2 > 0$, then the random variable $V = \dfrac{(X-\mu)^2}{\sigma^2} = Z^2$ is $\chi^2(1)$.

4.1. Distributions of Two Random Variables

- Definition 4.1-1. Joint pmf $f(x,y) = P(X = x, Y = y)$
  - $\sum\sum_{(x,y)\in S} f(x,y) = 1$ (do a double summation instead of a single one)
- Marginal pmf of X is $f_1(x) = \sum_y f(x,y) = P(X = x),\ x \in S_1$
  - i.e., sum it over the other axis to add up all of its variability
  - X and Y are independent IFF $f(x,y) = f_1(x) f_2(x)$
    - Independence can only happen if the support is “rectangular”
- Mathematical expectation:
  - $\mu_x = E(u(x,y)) = E(x)$
  - $\sigma_x^2 = E(u(x,y)) = E((x-\mu)^2)$
  - μ_x and σ_x^2 can be calculated either from joint pmf or marginal pmf
- Continuous analogue with integrals
- Examples:
  - Hypergeometric pmf of multiple variables; marginal pmfs are also hypergeometric and are dependent
  - Binomial pmf to trinomial distribution pmf: marginal pmfs are also binomial and dependent

4.2. The Correlation Coefficient

- If $u(X_1, X_2) = (X_1 - \mu_1)(X_2 - \mu_2)$, $E(u(X_1, X_2)) = \sigma_{12}$ (covariance of X_1 and X_2)
- $\rho = \dfrac{\sigma_{12}}{\sigma_1 \sigma_2}$ is called the correlation coefficient (if the stdevs are positive)

- Need the joint pmf to compute (not a marginal one)
- $\sigma_{12} = E(X_1 X_2)$
- $E(X_1 X_2) = \mu_1 \mu_2 + \rho \sigma_1 \sigma_2$

5.1. Functions of One Random Variable

- Two methods to find the distribution of a function of a random variable (also a random variable)
  - Distribution function technique
    - Plug in the function into the cdf of the second variable
    - e.g., if X is a random variable, and $Y = u(X)$, express $P(Y \le y) = P(u(X) \le y) = P(X \le u^{-1}(y))$, and plug into the cdf of X, and then differentiate to get the pdf of Y
    - e.g., loggamma is the substitution of $Y = e^X$ if X is the gamma distribution, $g(y) = \dfrac{1}{\Gamma(\alpha)\theta^\alpha} \dfrac{(\ln y)^{\alpha-1}}{y^{1+\frac{1}{\theta}}}$
      - $\mu = \dfrac{1}{(1-\theta)^\alpha}$, $\sigma^2 = \dfrac{1}{(1-2\theta)^\alpha} - \dfrac{1}{(1-\theta)^{2\alpha}}$
    - e.g., Cauchy pdf (p. 217)
  - Change of variable technique: shortcut for distribution function technique (same methodology)
    - $g(y) = f\left[v(y)\right] \left|v'(y)\right|$

5.3. Several Independent Variables

- Dealing with the pmf resulting from repeated mutually-independent trials (i.e., each marginal pmf/pdf is the same)
- If all n distributions are the same, then called random sample of size n from that common distribution, and $g(x_1, x_2, \cdots, x_n) = f(x_1) f(x_2) \cdots f(x_n)$
- Theorem 5.3-1. Generalization of expected value (discrete form) $E(Y) = \sum_y y g(y) = \sum_{x_1} \sum_{x_2} \cdots \sum_{x_n} u(x_1, x_2, \cdots, x_n) f_1(x_1) f_2(x_2) \cdots f_n(x_n)$
- Theorem 5.3-2. If $Y = u_1(X_1) u_2(X_2) \cdots u_n(X_n)$, $E(Y) = E(u_1(X_1)) E(u_2(X_2)) \cdots E(u(X_n))$
- If $Y = \sum_{i=1}^{n} a_i X_i$, then $\mu_Y = \sum_{i=1}^{n} a_i \mu_i$ and $\sigma_Y^2 = \sum_{i=1}^{n} a_i^2 \sigma_i^2$
- For a random sample of n samples, where $\bar{X}$ is a function of the samples, $\mu_{\bar{X}} = \mu$, $\sigma_{\bar{X}}^2 = \dfrac{\sigma^2}{n}$

5.4. The Moment-Generating Function Technique

- Theorem 5.4-1. The moment-generating function $Y = \sum_{i=1}^{n} a_i X_i$ is $M_Y(t) = \prod_{i=1}^{n} M_{x_i}(a_i t)$
  - Corollary: for $Y = \sum X_i$, $M_Y(t) = [M(t)]^n$
  - Corollary: for $\bar{X} = \sum \frac{1}{n} X_i$, $M_{\bar{X}}(t) = \left[M\left(\dfrac{t}{n}\right)\right]^n$
- Examples:
  - For a bernoulli trial, $M(t) = q + pe^t$. For $Y = \sum X_i$, get $M_Y(t) = \left(q + pe^t\right)^n$ $(b(n, p))$
  - For a Chi-square trial, Y is sum of trials, get $\chi^2 \left(\sum_i r_i\right)$

- If samples of normal standard distributions ($N(0,1)$), and $W = \sum_{i=1}^{n} Z_i^2$, then $W = \chi^2(n)$ (using theorem 3.6-2)
- If samples $X_i = N(\mu_i, \sigma_i^2)$, then $W = \sum_{i=1}^{n} \frac{(X_i - \mu_i)^2}{\sigma_i^2} = \chi^2(n)$ (using theorem 3.6-1 and above corollary)

5.5. Random Functions Associated with Normal Distributions

- Theorem 5.5-1. If random samples X_i are independent normal distributions, then $Y = \sum_{i=1}^{n} c_i X_i$ has the distribution $N\left(\sum_{i=1}^{n} c_i \mu_i, \sum_{i=1}^{n} c_i^2 \sigma_i^2\right)$
  - Corollary: distribution of $\bar{X}$ is $N\left(\mu, \frac{\sigma^2}{n}\right)$

5.6. The Central Limit Theorem

- Theorem 5.6-1. Central Limit Theorem. The distribution of $\lim_{n\to\infty} W = \lim_{n\to\infty} \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma}$ is $N(0,1)$. (W is a function of $\bar{X}$, which is in turn a function of X).
  - $P(W \le w) \approx \int_{-\infty}^{w} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz = \phi(w)$
    - e.g., for $X = N(\mu, \sigma)$ approximation for $P(a < \bar{X} < b) \approx P\left(\frac{a-\mu}{\frac{\sigma}{\sqrt{n}}} < \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} < \frac{b-\mu}{\frac{\sigma}{\sqrt{n}}}\right)$
- CLT useful when distribution is symmetric, unimodal, and continuous
  - General rule of thumb is that it is a good approximation for large n ($n > 25$), but can be smaller if unimodal and symmetric
- For a single distribution of X, $W = \frac{X - \mu}{\sigma}$ can be used to create a Normal distribution to approximate X
  - $G(x) = F(\sigma x + \mu)$, $g(x) = \sigma f(\sigma x + \mu)$

5.7. Approximations for Discrete Distributions

- … Working here …

Differential Equations – MA240
Test 2 Outline

4.1. Preliminary Theory

- Theorem 4.1.1. Existence of solutions for IVPs: If the coefficient functions $a_n(x) \cdots a_0(x)$ and $g(x)$ are continuous throughout the interval and $a_n(x) \neq 0$ throughout the interval, then a solution $y(x)$ exists on the interval and is unique.
- Boundary value problems have no existence theorem.
- To solve nth-order ODE, need to solve associated homogeneous equation first
- Differential operator (including differential polynomial operator) is linear
  - Can write DEs as $L(y) = 0$ or $L(y) = g(x)$
- Theorem 4.1.2. Superposition principle – Homogeneous Equations: linear combination of homogeneous solutions is also a solution to the homogeneous ODE
  - Constant multiple of solution to homogeneous ODE also solution.
  - $y = 0$ is always a solution to a homogeneous ODE.
- Definition 4.1.1. Linear dependence/independence: A set of functions is linearly independent if there exists a set of constants $c_1 \cdots c_n$, not all 0, such that a linear combination of the functions with the constants, then linearly independent; otherwise, linearly dependent
- Definition 4.1.2. Wronskian: If each of the functions $f_1(x) \cdots f_n(x)$, determinant of functions and their derivatives (up to $n-1$th derivative) is called the Wronskian
- Theorem 4.1.3. Set of solutions is linearly independent on I IFF $W \neq 0$ for every x in the interval
- Definition 4.1.3. Any set of n linearly independent solutions of the homogeneous nth-order linear ODE on I is called a fundamental set of solutions
- Theorem 4.1.4. There exists a fundamental set of solutions for the homogeneous nth-order linear ODE on the interval I
- Theorem 4.1.5. The general solution of the linear homogeneous ODE is $y = c_1 y_1(x) + \cdots + c_n y_n(x)$
- Any solution of a linear ODE free of arbitrary parameters is called a particular solution
- General solution of a linear ODE is $y = y_c + y_p$
- Theorem 4.1.7. Superposition principle for nonhomogeneous linear ODEs: If $L(y_{p_i}) = g_i(x)$, and $y_p = y_{p_1} + \cdots + y_{p_n}$, then $L(y) = g_1(x) + \cdots + g_n(x)$

4.2. Reduction of Order

- If one solution to linear homogeneous ODE known, then second solution can be found by substituting $y_2(x) = u(x) y_1(x)$.
  - $y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx$
  - ***(know how to derive this one)***

4.3. Homogeneous Linear Equations with Constant Coefficients

- Auxiliary equation (in m)
- Three cases:
  - For distinct roots: $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$
  - For repeated roots: $y = c_1 e^{mx} + c_2 x e^{mx}$
  - For complex roots: $y = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x$

4.4. Undetermined Coefficients – Superposition Approach

- Approach to find y_p of a constant-coefficient non-homogeneous linear ODE with $g(x)$ being of constant, polynomial exponential, sin/cos, or product or linear combination of these forms
- Make sure to check the y_c to make sure there are no repeated terms; if there are, multiply by x
- For cosine and sine, make sure to have both cosine and sine terms in the y_p (because they generate one another as derivatives.)

4.6. Variation of Parameters

- No restrictions on finding y_p from the problem
- Based around finding $y_p(x) = u_1(x)y_1(x)$
- $u_i = \int \frac{W_i}{W} dx$
  - $y = -y_1 \int \frac{y_2(x)f(x)}{W} dx + y_2 \int \frac{y_1(x)f(x)}{W} dx$ for second-order
- Make sure ODE is in standard form before getting $f(x)$

4.7. Cauchy-Euler Equation

- $a_n x^n D^n y + a_{n-1} x^{n-1} D^{n-1} y + \cdots + a_0 y = 0$
  - Auxiliary equation $m(m-1)\cdots(m-n+1) + \cdots + 1 = 0$
- Three cases:
  - Distinct real roots: $y = c_1 x^{m_1 x} + c_2 x^{m_2 x}$
  - Repeated real root: $y = c_1 x^{mx} + c_2 x^{mx} \ln x$
  - Complex roots: $y = c_1 x^{\alpha x} \cos(\beta \ln x) + c_2 x^{\alpha x} \cos(\beta \ln x)$
- Reduction to constant-coefficients (generally easier to solve):
  - Substitution: $x = e^t$, $t = \ln x$
  - $\frac{dy}{dt} = \frac{1}{x}\frac{dy}{dx}, \frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right)$
  - Only works for solutions with $t > 0$; for negative solutions use $t = -x$

5.1. Linear Models: IVPs

- Free undamped motion: $m\ddot{x} + \omega^2 x = 0, \omega = \sqrt{\frac{k}{x}}$
  - This will yield a sinusoidal solution (constant coefficients, m has complex roots)
  - $x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$
  - Alternate form with amplitude: $x(t) = A \sin(\omega t + \phi)$, $A = \sqrt{c_1^2 + c_2^2}$, $\phi = \tan^{-1} \frac{c_1}{c_2}$
- Free damped motion: $m\ddot{x} + 2\lambda\dot{x} + \omega^2 x = 0$, $2\lambda = \frac{\beta}{m}$
  - $m = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$
  - Three cases:
    - $\lambda^2 - \omega^2 > 0$: overdamped
    - $\lambda^2 - \omega^2 = 0$: critically damped
    - $\lambda^2 - \omega^2 < 0$: underdamped
  - $x(t) = Ae^{-\lambda t} \sin(\sqrt{\omega^2 - \lambda^2} t + \phi)$
- Driven damped motion: $m\ddot{x} + 2\lambda\dot{x} + \omega^2 x = f(x)$
  - Complimentary function is transient term, particular solution is steady-state solution
  - If period of driven motion is same as period of object's motion, then resonance occurs
- Series circuit analogue to damping: $L\ddot{q} + R\dot{q} + \frac{1}{C} q = E(t)$
  - If $E(t) = 0$, electrical vibrations of the circuit are free

- If $R = 0$, then simple harmonic motion
- Discriminant is $R^2 - \frac{4L}{C}$
  - $R^2 - \frac{4L}{C} > 0$: overdamped
  - $R^2 - \frac{4L}{C} = 0$: critically damped
  - $R^2 - \frac{4L}{C} < 0$: underdamped
- Steady-state current analogous to steady-state solution of motion

5.2. Linear models: BVPs

- $EI = \frac{d^4y}{dx^4} = 0$ for beam deflection
  - For boundary points:
    - Embedded: $y = 0, \frac{dy}{dx} = 0$
    - Free: $\frac{d^2y}{dx^2} = 0, \frac{d^3y}{dx^3} = 0$
    - Simply supported (hinged): $y = 0, \frac{d^2y}{dx^2} = 0$
  - $y(x) = c_1 + c_2x + c_3x^2 + c_4x^3 + \frac{w_0}{24EI}x^4$
- Eigenvalues and eigenfunctions
  - Two-point BVP involving linear ODE with parameter λ: try to find values that lead to any non-trivial solutions
  - e.g., $\ddot{y} + \lambda y = 0$, $y(0) = 0$, $y(L) = 0$
    - for $\lambda \geq 0$, only trivial solution; for $\lambda < 0$, nontrivial solutions $\lambda = \left(\frac{n\pi}{L}\right)^2$; λ values that produce nontrivial solutions are called eigenvalues, and corresponding functions dependent on these eigenvalues are eigenfunctions (constant not important)

5.3. Nonlinear models

- E.g., nonlinear pendulum, estimate $\sin\theta \approx \theta$ in $l\frac{d^2\theta}{dt^2} = -g\sin\theta$
- E.g., catenary, $\frac{dy}{dx} = \frac{W}{T_1} = \frac{\rho s}{T_1}$, substitute $u = \frac{dy}{dx}$ (result is a hyperbolic cosine)
- ***E.g., rocket motion ******
- ***E.g., variable mass,*** $F = \frac{d}{dt}(mv)$***, determine formula for*** m***, do some substitutions***

6.1. Review of Series

- Ratio test to determine interval (and radius) of convergence
- Identity Property: If an infinite power theorem is equal to 0, then every coefficient c_n is 0
- A function is analytic at a point if it can be represented by a power series with $R > 0$ centered at that point (basically if differentiable)

- Sample Maclaurin series on right –>

Maclaurin Series	Interval of Convergence	
$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$	$(-\infty, \infty)$	
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$	$(-\infty, \infty)$	
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$	$(-\infty, \infty)$	
$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^{2n+1}$	$[-1, 1]$	(2)
$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n)!}x^{2n}$	$(-\infty, \infty)$	
$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}x^{2n+1}$	$(-\infty, \infty)$	
$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}x^n$	$(-1, 1]$	
$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$	$(-1, 1)$	

6.2. Solutions about ordinary points

- Definition 6.2.1. A point $x = x_0$ is an <u>ordinary point</u> of the DE (in standard form) if both $P(x)$ and $Q(x)$ are analytic at x_0. A non-ordinary point is <u>singular</u>.
- Theorem 6.2.1. If $x = x_0$ is an ordinary point of the DE, can find two linearly-independent power series solutions of the form $y = \sum_{n}^{\infty} c_n (x - x_0)^n$ that converges at least on $|x - x_0| < R$, R is the distance to the closest singular point
- Solving an ODE is "the method of undetermined series coefficients," with a <u>recurrence relation</u> and using the identity property; then collect and group terms at the end
- This can work with nonpolynomial coefficients as well, using series multiplication

6.3. Solutions about singular points

- A singular point $x = x_0$ is <u>regular</u> if $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2 Q(x)$ are both analytic at x_0. If not, <u>irregular</u>.
- Theorem 6.3.1. <u>Frobenius' Theorem</u>: If $x = x_0$ is a regular singular point of the DE, there exists at least one equation of the form $y = \sum_{n}^{\infty} c_n (x - x_0)^{n+r}$. The series will converge on at least $0 < x - x_0 < R$.
    - Need to find r before solving recurrence relation
    - No assurance of two linearly-independent solutions
- <u>Indicial equation</u> in r, roots are solutions for r
- Three cases:
    - If $r_1 \neq r_2$ and differ by non-integer, then two linearly-independent solutions of the regular form.
    - If $r_1 \neq r_2$ and differ by integer, then y_1 of regular form, $y_2 = cy_1(x)\ln x + (\text{regular form})$, c can be 0
    - If $r_1 = r_2$, $y_2 = y_1(x)\ln x + (\text{regular form})$ (analogous to that of Cauchy-Euler with repeated roots)

Random things from discussion questions

- $D^n x^{n-1} = 0$, $D^n x^n = n!$
- $\ddot{y} + k^2 y = 0 \Rightarrow y = c_1 \cos kx + c_2 \sin kx$
- $\ddot{y} - k^2 y = 0 \Rightarrow y = c_1 \cosh kx + c_2 \sinh kx$

Test 3 Outline
MA240 – Differential Equations

7.1. Definition of the Laplace Transform

- Definition 7.1.1.: Let f be a function defined for $t \geq 0$. Then $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$ is the Laplace Transform of f.
- $\mathcal{L}$ is a linear transform
- Theorem 7.1.1. Transforms of basic functions
  - $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$
  - $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$
  - $\mathcal{L}\{\sin kt\} = \dfrac{k}{s^2+k^2}$
  - $\mathcal{L}\{\cos kt\} = \dfrac{s}{s^2+k^2}$
  - $\mathcal{L}\{\sinh kt\} = \dfrac{k}{s^2-k^2}$
  - $\mathcal{L}\{\cosh kt\} = \dfrac{s}{s^2-k^2}$
- Theorem 7.1.2.: Sufficient conditions for existence: If f is piecewise continuous on $[0, \infty)$ and of exponential order, then the Laplace transform of f exists for $s > c$
  - f of exponential order if $|f(t)| \leq Me^{ct}$, for $t > T$, M, c, T constants
- Theorem 7.1.3. $\lim_{s \to \infty} F(s) = 0$, assuming $F(s)$ exists

7.2. Inverse Transforms and Transforms of Derivatives

- Factor functions with distinct linear factors using partial fraction decomposition
- Theorem 7.2.2. Transform of a derivative: If f and first $(n-1)$ derivatives PC and of exponential order, then $\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$

7.3., 7.4. Operational Properties of the Laplace Transform

- Theorem 7.3.1. Translation on s: $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$
- Theorem 7.3.2.: Translation on t: $\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$
  - Alternative form: $\mathcal{L}\{f(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}$ (only useful in forward direction)
  - To write a function using the unit step function, for each piecewise section $h(x)$ from a to b, add $h(t)(\mathcal{U}(t-a) - \mathcal{U}(t-b))$
  - $\mathcal{L}\{\mathcal{U}(t-a)\} = \dfrac{e^{-as}}{s}$
- Theorem 7.4.1. Derivatives of transforms: $\mathcal{L}\{t^n f(t)\} = (-1)^n \dfrac{d^n}{ds^n} F(s)$
  - Because this is a multiplicative rule like shifts on s, can use either to do $\mathcal{L}\{t^n e^{at}\}$
  - Methodology for inverse:
    - Integrate $F(s)$ n times until you get a function $G(s)$ that you can take the inverse Laplace transform of
    - $\mathcal{L}^{-1}\{F(s)\} = (-t)^n g(t)$
  - Inverse useful when powers of almost-usable form in the denominator (e.g., $\dfrac{s}{(s^2+16)^2}$)
- Theorem 7.4.2. Laplace of convolution: $\mathcal{L}\{f \circledast g\} = \mathcal{L}\{f\}\,\mathcal{L}\{g\} = F(s)G(s)$

- $f \circledast g = \int_0^t f(\tau)g(t-\tau)d\tau$
  - If $g(t) = 1$, then $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$ (Laplace of integral)
    - Useful in reverse form, can solve for integral when Laplace transform has a $\frac{1}{s}$ factor; then integrate $f(t)$
  - Useful for Volterra integral equations or interrodifferential equations
- Theorem 7.4.3. Transform of a periodic function: $\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$

7.5. The Dirac Delta Function

- Unit impulse: $\delta_a(t-t_0) = \begin{cases} 0, \ 0 \le t < t_0 - a \\ \frac{1}{2a}, \ t_0 - a \le t < t_0 + a \\ 0, \ t \ge t_0 + a \end{cases}$
- Dirac delta function: $\delta(t-t_0) = \lim_{a \to 0} \delta_a(t-t_0)$
- Theorem 7.5.1. $\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}, \ t_0 > 0$

7.6. Systems of Linear Differential Equations

- …

11.1. Orthogonal Functions

- Properties of the inner product (functional, complex analogue of dot product)
  - $(u,v) = (v,u)$ (commutativity)
  - $(ku,v) = k(u,v), \ k$ is a scalar (constants can be pulled out)
  - $(u,u) = 0$ if $u = 0$, $(u,u) > 0$ otherwise (positivity)
  - $(u+v,w) = (u,w) + (v,w)$ (distributivity of dot product over addition)
  - $||\phi(x))||^2 = \int_a^b \phi^2(x)dx$
    - Can normalize a function by dividing by its norm
- Definition of inner product of functions: $(f_1, f_2) = \int_a^b f_1(x)f_2(x)\,dx$
  - Definition of orthogonality of functions: $(f_1, f_2) = 0$
  - Note that the zero function is orthogonal to every function
  - A set of real-valued functions is an orthogonal set if every pair of functions in that set is orthogonal
    - An orthonormal set is an orthogonal set where $||\phi_n(x)|| = 1$
- Expressing vectors/functions in terms of orthogonal basis
  - Vector analogue: can use an orthogonal set of n vectors as a basis with which to express any n-space vector as a linear combination of them
    - To find coefficient of a basis vector, dot the entire expression with the basis vector and solve for the coefficient (which is also the projection): $c_n = \frac{\vec{u} \cdot \vec{v_n}}{||\vec{v_n}||^2}$
  - $f(x) = \sum_{n=0}^{\infty} \frac{(f, \phi_n)}{||\phi_n(x)||^2}\phi_n(x)$
    - This is called the orthogonal series expansion or the generalized Fourier series

- Definition of orthogonality with a weight function: ("orthogonal with respect to weight function $w(x)$) if $\int_a^b w(x)\phi_m(x)\phi_n(x)\,dx = 0,\ m \neq n$
  - In general, can include a weight function in an inner product – for our purposes, usually $w(x) = 1$

11.2. Fourier Series

- Definition: the Fourier series of a function f on the interval $(-p, p)$ is given by:
  - $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \sin\dfrac{n\pi x}{p} + b_n \sin\dfrac{n\pi x}{p}\right)$
  - $a_0 = \dfrac{1}{p}\int_{-p}^{p} f(x)\,dx$
  - $a_n = \dfrac{1}{p}\int_{-p}^{p} f(x)\cos\dfrac{n\pi x}{p}\,dx$
  - $b_n = \dfrac{1}{p}\int_{-p}^{p} f(x)\sin\dfrac{n\pi x}{p}\,dx$
- Definition: Piecewise continuous (PC) over a closed interval:
  - Has a finite number of jump discontinuities
  - f is continuous over each interval
- Convergence theorem for Fourier series: Let f and f' be PC on $[-p, p]$. For all x in $(-p, p)$, series converges at a point continuity. At a point of discontinuity the series converges to the average of the left- and right-hand limits
- $2p$ is the fundamental period of the sum; Fourier transform not only reflects function on $(-p, p)$ but also the periodic extension of f outside the interval.
  - At $x = p + 2n,\ n \in \mathbb{Z}$, converges to $\dfrac{f(+p-) + f(-p+)}{2}$ (average of left-hand limit of $x = p$ and right-hand limit of $x = -p$)

11.3. Fourier Cosine and Sine Series

- Even function can be represented with only a_0 term and a_n (cosine) terms
- Odd function can be represented with only b_n (sine terms)
  - Will converge to 0 at $x = -p, 0, p$
- Gibbs phenomenon (not covered in our class): overshooting of curve at a discontinuity; overshooting stays almost constant (doesn't go away) when $n \to \infty$, but width gets narrower
- Half-range extensions: for a function defined only over $(0, L)$:
  - 1. Can reflect the graph about the y-axis, now even
    - Choose $p = L$, now $b_n = 0$, $a_0 = \dfrac{2}{p}\int_0^L f(x)\,dx$, $a_n = \dfrac{2}{p}\int_0^p f(x)\cos\left(\dfrac{n\pi x}{p}\right)dx$; period is $2p$
  - 2. Can rotate the graph about the origin, now odd
    - Choose $p = L$, now $a_0 = a_n = 0$, $b_n = \dfrac{2}{p}\int_0^p f(x)\sin\left(\dfrac{n\pi x}{p}\right)dx$, period is $2p$
  - 3. Repeat function by defining $f(x + L) = f(x)$ on $(-L, 0)$

    - Choose $p = \dfrac{L}{2}$ and also integrate over $(0, L)$; $a_0 = \dfrac{2}{L}\displaystyle\int_0^L f(x)\,dx$, $a_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\left(\dfrac{2n\pi x}{L}\right)\,dx$, same with b_n (procedure works out to be the same as doing even and odd half-range extensions), period is L
- Fourier series can be used as a solution to a DE where solution is periodic
  - Can use half-range extensions if only positive/negative domain known
  - Assume solution in the form $\displaystyle\sum_{n=1}^{\infty} B_n \sin\left(\dfrac{n\pi x}{p}\right)$, match coefficients

12.1. Separable Partial Differential Equations

- A partial differential equation (pde) given by the a function $u(x, y)$ and can any first or second partial derivatives of u
- Focus is on finding particular solutions to pdes (more useful in real-life applications)
- Method of separation of variables:
  - Write: $u(x, y) = X(x)Y(y)$ and substitute into original pde
  - Separate variables: now have some expression like $\dfrac{X'}{X} = \dfrac{Y'}{Y} = -\lambda$
    - Equal to some constant (λ is the separation coefficient) because the ratios are functions of two different variables; for them to be equal must be equal to (the same) constants
  - Rewrite as linear equations, and solve. The λ will lead to an eigenvalue problem. Solve for eigenfunctions of one variable using BVPs, and plug those into the second equation.
- General solution is the sum of all nontrivial component solutions (superposition principle)
- Classifying pdes:
  - hyperbolic if $B^2 - 4AC > 0$
  - parabolic if $B^2 - 4AC = 0$
  - elliptic if $B^2 - 4AC < 0$

12.2., 12.3., 12.4., 12.5. Classical PDEs

- Heat equation: $ku_{xx} = u_t,\ k > 0$
- One-dimensional wave equation: $a^2 u_{xx} = u_{tt}$
- Two-dimensional Laplace's equation: $u_{xx} + u_{yy} = 0$
- Boundary conditions (can specify any of these at a boundary):
  - Dirichlet condition: u
  - Neumann condition: u_x
  - Dirichlet condition: $u_x + hu$

PHYSICS 112 MIDTERM REVIEW
Wolf Sections A & C, Spring 2019

Document Metanotes:
- The sections are topics and may not correspond to a chapter
- Using 8th edition Halliday & Resnick (H&R) problem numbers
- For all equations, don't have to memorize them, but know how to derive quickly
- LaTeX generated using plugin Auto-LaTeX Equations
- Any errors were unintentional

Capital T Truths
- $\sum \vec{F}_{net} = m\vec{a}$
- Newton's third law
- Conservation of total energy
  - Sometimes conservation of mechanical energy
- Conservation of linear momentum

General Problem-Solving Skills
- Draw FBDs for each system/object and extra diagrams as necessary
- Write Newton's second law for each system/object
- Label systems/objects (and, for conservation of energy and linear momentum, also label start/end times)
- Explain in words when necessary
- Don't do extra algebra unless necessary
- Box important equations
- (Quickly) take limits to check answers

Chapter 3: Vectors
- Understand vector operations (addition, subtraction, dot, cross)
  - Dot product is "a measure of parallelness," involved with projections; max when vectors parallel, 0 when orthogonal
  - Cross product is "a measure of perpendicularity"; max when vectors orthogonal, 0 when parallel
    - Know right-hand rule
- Know how to convert between Cartesian and magnitude-angle notation
- Understand how to break up vectors into its parts
  - Especially for ramp questions: understand how to break up vectors parallel/perpendicular to incline into vectors in the x/y directions and vv
- Understand rotation of a coordinate axes by some angle (analogous to translation by some vector)
- Understand spherical coordinates ($P(\rho, \theta, \phi)$)
- $\vec{x}$, $\vec{v}$, $\vec{t}$, $\vec{F}$, and $\vec{p}$ are all vector quantities (i.e., Newton's second law is a vector quantity, and write it so unless analyzing motion in 1D)
- Significant problems:
  - Ramps
  - Torque and significance of direction of torque vector

Chapters 2 & 4: One- and Two-Dimensional (Constant Acceleration) Kinematics
- $\vec{v} = \frac{d\vec{x}}{dt}$, speed $= |\vec{v}|$, $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{x}}{dt^2}$

  - (Don't get confused between speed and velocity! Speed is rate of change in distance, velocity is rate of change of displacement)
- Five basic constant-a kinematics eqs:
  - $v = v_0 + at$ $(a,\ t,\ v_0,\ v)$
  - $\Delta x = \frac{1}{2}at^2 + v_o t$ $(\Delta x,\ a,\ t,\ v_0)$
  - $\Delta x = -\frac{1}{2}at^2 + vt$ $(\Delta x,\ a,\ t,\ v)$
  - $\Delta x = \frac{1}{2}(v_0 + v)t$ $(\Delta x,\ t,\ v_0,\ v)$
  - $v^2 - v_0^2 = 2a\Delta x$ $(\Delta x,\ a,\ v_0,\ v)$
  - Know how to derive each equation, and make sure to choose correct equation for the question at hand
- If acceleration is not constant, integrate to get v and/or x (don't use above equations!)
- Know how to interpret a-t, v-t, and x-t graphs for particle's v, a, and x
- Projectile motion:
  - Only net acceleration is gravity acting downwards
  - y-t relation: $\Delta y = -\frac{1}{2}gt^2 + v_0 \sin\theta t$
  - x-t relation: $\Delta x = v_0 \cos\theta$
  - y-x relation: $\Delta y = (-\frac{g}{2(v_0 \cos\theta)^2})\Delta x^2 + (\tan\theta)\Delta x$
  - range (to starting height): $\Delta x = \frac{v_0^2 \sin 2\theta}{g}$
  - maximum height: $\Delta y = \frac{v_0^2}{2g}$
- Solution setup:
  - Is it a one-dimensional (displacement vs. time) or two dimensional (y vs. x) question?
  - Is it a constant acceleration kinematics question?
    - Determine which 4 variables are relevant (typically 3 known, 1 known) for kinematics problems, and choose the right equation
    - Would using the y-x parabolic equation be easier than 1D kinematics questions?
- Significant problems:
  - Section A Quiz 1: Incline problem H&R Ch4 #43
  - Section C Quiz 1
  - Train problem: H&R Ch2 #42
  - Stair Problem

Chapter 4: Uniform Circular Motion (The Centripetal Force)

- Centripetal acceleration: $a = \frac{v^2}{r}$ (understand both derivations)
  - This equation also applies to instantaneous acceleration for non-circular (i.e., elliptical) orbits
- Period: $T = \frac{2\pi r}{v}$
- The centripetal force is not a fundamental force, but rather the sum of fundamental forces pointing towards the center of rotation that cause an object to move in uniform circular motion (around a circle at constant velocity)
- Centripetal acceleration/force always aimed directly inwards, velocity always aimed tangentially to circle (perpendicular to acceleration)
- Significant problems:
  - Section C Quiz 2: H&R Ch6 #59

<u>Chapter 4: Relative Motion in Inertial Reference Frames</u>

- Let P refer to some object, A refer to first reference frame, and B refer to second reference frame
- Kinematics (always write subscripts to avoid confusion!):
  - $\vec{x}_{\frac{P}{A}} = \vec{x}_{\frac{B}{A}} + \vec{x}_{\frac{P}{B}}$
  - $\vec{v}_{\frac{P}{A}} = \vec{v}_{\frac{B}{A}} + \vec{v}_{\frac{P}{B}}$
  - $\vec{a}_{\frac{P}{A}} = \vec{a}_{\frac{P}{B}}$
- Significant problems:
  - Boat moving relative to water: Ch4 #82
  - Center of mass reference frames

<u>(Wolf's Class): Relative Motion in Accelerated Reference Frames</u>

- Let P refer to some object, A be a good (inertial) reference frame, and B be a bad (noninertial) reference frame
- You can write Newton's laws *only with a good frame*: $\sum \vec{F}_{net} = ma_{\frac{P}{A}}$
- Because $\vec{a}_{\frac{P}{A}} = \vec{a}_{\frac{B}{A}} + \vec{a}_{\frac{P}{B}}$, $\sum \vec{F}_{net} = m(\vec{a}_{\frac{B}{A}} + \vec{a}_{\frac{P}{B}})$
- Significant Problems:
  - Accelerated Reference Frames Packet (moodle)
  - Quiz 4

<u>Chapters 5 & 6: Newton's Second Law and Forces</u>

- A force is something that causes an object to move; a "push" or "pull"; something that can accelerate mass
  - Thus mass can be thought of as a coefficient of "resistance to accelerate" — the higher the mass, the lower the acceleration with the same force
- Newtonian mechanics breaks down at very high speeds due to special relativity and very small scales due to quantum mechanics
- Newton's Second Law: $\sum \vec{F}_{net} = m\vec{a}$
  - Only works in inertial reference frames
  - The forces considered are only external forces acting *on* the object; internal forces cancel in pairs (Newton's third law)
- Solution Setup:
  - **Always** draw FBDs for each object
  - **Always** write Newton's second law (vectorially) for each relevant object in relevant directions
  - Determine the object/system(s). Define them or circle them if it's not clear from the question
    - **Always** consider one object at a time, and only consider external forces acting on that object.
    - If multiple objects are moving together, you can treat them as one object for simpler problems
  - Choose axes that are easy to work with (e.g., ramp problems: tangential-perpendicular or x-y axes?).
- Significant Problems:
  - (See next section)

<u>Chapters 5 & 6 (& 8): Specific Forces</u>

- Gravity
  - $F_g = -mg$, positive direction is upwards
  - $F_g = -\frac{GMm}{R}$, for distances further from the surface of the Earth (but not closer to the center of the Earth)
- Normal
  - Always comes in pairs
  - Remember that pulleys exert normal force on the object they are attached to
- Tension
  - Pulleys redirect tension of ropes
  - Tension is same throughout a single rope
    - If masses attached tightly to some point between the ends of a rope, then the rope is segmented into two ropes that don't have to have the same tension
  - Rods can either pull or push (either tension or normal force) — in problem setup you can assume either pull or push on both objects, and if sign of solved force is negative, just change your assumption to the other
- Friction
  - If net force on object (on axis of friction) is less than $f_{s_{max}} = \mu_s N$, then static friction keeps it in place and $f_s = \text{net force on axis of friction}$ (i.e., static friction matches net force)
    - Remember that if object not moving, **static friction is usually not equal to** $f_{s_{max}}$ (e.g., if no net force, no static friction)
  - Otherwise, friction is $f_k = \mu_k N$
- Spring (Ch8)
  - $F = -kx$ (Hooke's Law), positive direction is stretched string, x is change in length of spring from equilibrium position
- Solution Setup:
  - "Conservation of Rope Length" principle (for a single rope)
  - Before using friction, determine if static or kinetic (one way is to assume static friction, check necessary frictional force to keep object(s) from sliding, and compare to $f_{s_{max}}$)
  - To determine direction of friction, can try problem first w/o friction ("turn off friction"), or can arbitrarily choose direction of friction and infer direction from sign of solved frictional force
- Significant Problems:
  - Friction problems: will it slide? Or determining static friction
  - Quiz #3: listing all possible scenarios involving friction (don't forget "no movement" is an option!)
  - Pulleys and/or ramps and/or friction
    - Section A Quiz #2: monkey problem Ch5 #57
  - Gravity (exact form) / electromagnetic force between two uniform rods

<u>Chapters 7 & 8: Work, Energy, and Power</u>

- Ch7 interpretation of energy: all energy is energy of movement
  - "Better bookkeeping"
- Ch8 interpretation of energy: there exists potential energy ("energy of configuration")
  - Potential energy of a force F (i.e., gravity, spring PE) is $U_F = -W_F$ (negative work by the force to get it to its "configuration")

- In a potential energy diagram, there must be a total mechanical energy. The difference between the total mechanical energy and the potential energy is the kinetic energy; the kinetic energy must be positive, so the object cannot exist in regions of negative potential energy.
  - $F = -\frac{dU}{dx}$ — **the force at a particular position is the negative of the slope**. This means that particle always has a force directed towards a lower potential energy (think like chemistry, lower potential energy means more stable); can think of the potential energy diagram like hills and the object like a skateboard, which keeps getting "pulled" towards the lowest energy points on the potential energy diagram, oscillating between the bounding turning points

- Energy is always conserved in an isolated system (Law of Conservation of Energy)
  - However, mechanical energy is not always conserved: also thermal energy "lost" and internal energies (e.g., chemical) that are part of the system's entire energy (i.e., $E_{total} = E_{mec} + E_{th} + E_{int}$, $\Delta E_{total_{isolated}} = 0$)
  - Using this fact, you can look at two distinct moments in time and solve for energies if you know E_{total} or E_{mec} is conserved, but be sure to define the start and end time
  - If system is not isolated (if external force pierces the "bubble"), then the amount of work done by the external force is equal to the change in total energy of the system
- Mechanical energy (potential + kinetic energies, not thermal or internal energies) is not always conserved, but if it is conserved problems usually become easier
  - In general: $E_{mec} = U + K$, $\Delta E_{mec} = \Delta U + W$
  - Mechanical energy conserved if no friction, sound, heat loss, or permanent deformation of material (nonconservative forces)
    - Static friction is conservative
    - If kinetic friction present, you can still do calculations with energy with the general conservation of energy formula
  - If E_{mec} conserved: $E_{mec_1} = E_{mec_2}$, $\Delta E_{mec} = 0 \Rightarrow \Delta U = -W \Rightarrow F = -\frac{dU}{dx}$
- Work is force applied over a distance, which corresponds to a change in energy of a system. Different ways to think about work:
  - $W = F \times x$
  - $W = \vec{F} \cdot \vec{x} = \vec{F}\cos(\theta)x$
  - $W = \int \vec{F} \cdot dx$ = area under integral of F-x curve
  - $W = \Delta K$ (Ch7 perspective, all energy is kinetic energy)
  - $W = \Delta E_{total}$ (Ch8 perspective, work done on system is the change in energy of the system (kinetic and potential energies included))
  - $W = -dU$ if mechanical energy is conserved
  - $W_{net} = \sum W$
- Work done by specific forces, and potential energies of those forces:
  - $W_g = -mg\Delta h$, $U_g = mg\Delta h$
  - $W_k = -\frac{1}{2}kx^2$, $U_k = \frac{1}{2}kx^2$
- Kinetic energy is $K = \frac{1}{2}mv^2$
- Power is rate of change of energy
  - $P = \frac{dW}{dt} = \vec{F} \cdot \vec{v}$ (Ch7, all energy is kinetic energy)
  - $P = \frac{dE}{dt}$
- Solution setup:

  - **Always** clearly define (using words is best) the start and end moments of analysis.
  - Check: is mechanical energy conserved? I.e., was any work done on the system? Was any heat lost?
- Significant problems:
  - Understanding potential energy diagrams at specific total mechanical energy levels
    - Gravitational potential energy versus radius
    - 6-12 potential energy curve
    - Arbitrary potential energy curves (and quantum tunneling?)
  - Pendulum vs. height: Ch7 #65
  - Work of block pulled at an angle: Ch7 #42
  - Section A Quiz 4: Work to lift spring onto table (variable force, either using integral or COM): Ch8 #32
  - Conveyor belt problem: Ch8 #66, moodle

<u>Chapter 9: Linear Momentum and Center of Mass</u>

- Center of mass (COM): $\bar{x} = \frac{1}{M}\int x dm$ (a weighted average of the mass weighted by distance)
  - If constant density: $\bar{x} = \frac{1}{V}\int x dV$
  - Choose reference point that makes calculations easy
- If not a point mass or spherically/linearly symmetric, center of mass must be calculated using that formula
  - Center of mass of linearly symmetric object is at line of symmetry
  - Center of mass of spherically symmetric object is at center
  - Because COM calculation is linear, if center of mass of two objects known, COM is the weighted average of those two objects (i.e., can use COM on objects if COMs of entire objects known)
  - Doesn't matter what reference point is chosen, COM is still the same
- Trick: if uniform density object with an air/vacuum bubble, can treat vacuum like negative mass and use the COM formula with the COMs of the object and the bubble
- Can treat object like all external forces acting on center of mass: $\sum \vec{F}_{net} = M\vec{a}_{COM}$
- Linear momentum defined to be $\vec{p} = m\vec{v}$
  - Thus $\sum F_{net} = \frac{d\vec{p}}{dt}$ (another way to write Newton's third law)
  - Thus $\vec{p}_f = \vec{p}_i + \Delta p = \vec{p}_i + \int \vec{F}_{net} dt$
- Impulse is defined as the change in linear momentum (analogous to work being the change in energy): $\vec{J} = \Delta\vec{p} = \int \vec{F}_{net} dt$
  - If no/small impulse, then linear momentum is conserved; an approximate is okay
  - Generally, for collisions linear momentum is conserved (especially for hard/bouncy objects, because then dt is small and the impulse is small): elastic collision
    - **Conservation of linear momentum does not imply conservation of (kinetic) energy, nor vv**
- Center of mass reference frame
  - For an elastic collision, velocities/linear momenta of objects relative to COM reverse (useful for solving for velocities without using energy formulas)
  - What are the symmetries of this object? (for COM)
  - Is linear momentum conserved? Is it approximately conserved? I.e., is there an impulse on the system? Is it large or can it be approximated away?

    - In questions where both kinetic energy and linear momentum are conserved (elastic collisions), may need to use both conservation of linear momentum and conservation of energy formulas to solve (e.g., Section C Quiz 5)
    - Does the problem need to be broken up into multiple phases? (i.e., is linear momentum conserved in part of the collision/process like in the ballistic pendulum problem?)
- Significant Problems
    - Section C Quiz 5: (involves both conservation of energy and conservation of linear momentum) Ch9 #69
    - Person, dog on boat walking to opposite sides
    - Rocket problem: moodle
    - Astronaut game / cat-like thing on a sled: Ch9 #136
    - Ballistic pendulum: Ch9 Sample problem 9-9

PH112
Quiz 5 Review
Review Session 2/27/19

Chapter 7: Work and Kinetic Energy
$W = F * \Delta x$
$= F \cdot dx$
$= F\cos(\theta) * x$
$= \int F(x)\,dx$
= area under F-x curve
$= \Delta K$ (for chapter 7 perspective, no U)
$= \Delta E_{total}$ (external work on system for chapter 8 perspective)
$= -dU$ if E_{mec} is conserved)
$= \Sigma W$
$W_g = -mg\Delta h$ = integral of $-mg$
$W_k = -1/2kx^2$ = integral of $-kx$
$K = 1/2mv^2$
$P = dW/dt$
$= F \cdot v$

Chapter 8: Potential Energy
$U = -F \cdot dx$
= negative of work done by a force to get to a particular configuration
$U_g = mgh$
$U_s = 1/2kx^2$
Force is conservative if total work along a closed path is 0; these are path-independent
- Else nonconservative
Energy is always conserved
- Mechanical energy is not always conserved, but when it is then $E_{mec\_2} = E_{mec\_1}$
$E_{mec} = U + K$
$\Delta E_{mec} = \Delta U + W$ (here work is an "internal work" equal to ΔK, not work done on the system)
- if E_{mec} conserved, $\Delta U = -W$
  - then $dU = -W = -Fdx \Rightarrow F = -dU/dx$
  - useful b/c intermediate states do not have to be considered
  - conservation of E_{mec} if no deformation of material, heat loss, (kinetic) friction
$E_{total} = E_{mec} + E_{th} + E_{int}$
$\Delta E_{total} = \Delta E_{mec} + \Delta E_{th} + \Delta E_{int}$ (most general form)
- pretty much ignore E_{int}, E_{th} is applicable if there is friction/heat
- potential energy diagrams
  - turning points and equilibrium points (neutral, stable, unstable)
$W = \Delta E_{total}$ (work done on the system)

Chapter 9: Linear momentum & COM
$COM = 1/M * \int r\,dm$
- or, for constant density: $COM = 1/P * \int r\,dP$
Choose a good reference pt for COM for easy calculation
Take advantage of symmetries (spherical, linear)
$F_{net} = M * a_{com}$

$= dp/dv$
- Net force can be thought of as concentrated at center of mass -- internal forces have no effect

$p = M * v_{com}$ (momentum of system)

$J = \int F(t)\, dt$ (impulse on system)

$= \Delta p$

Make sure to think about when energy and linear momentum are conserved. They are not mutually inclusive/exclusive!
- If both are conserved, you may need to end up using both
- If only one is conserved, make sure you do not use the wrong one.

<u>Problems from class</u>

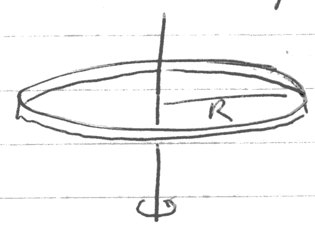
Set up center of mass of two sticks
Spring and a dome question
Rockets
Two boats w/ coal
Boat with dog and person on both ends, switch sides
Conveyor belt
Ballistic pendulum

MOMENTS OF INERTIA

Note: If you extrude a shape along the axis, Moment of inertia about that axis is the same as the non-extruded shape.

For a point mass: $I = mr^2$ (by definition)

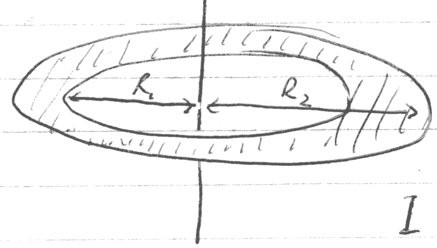
For a thin hoop about central axis
(or a thin cylinder about central axis (extruded thin hoop))



$$I = MR^2$$

(because every particle is R away from the center,
so $I = m_1R^2 + m_2R^2 + m_3R^2 + \ldots = MR^2$)

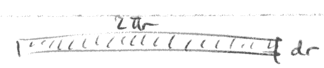
For an annular cylinder about central axis (extruded version)
or an annular ring:



Break it up into rings (hoops) from radius $= R_1$ to radius $= R_2$:

$$I = \int_{R_1}^{R_2} dI_{ring} = \int_{R_1}^{R_2} r^2\,dm$$

$$dm = (2\pi r)(dr)\sigma$$



each ring, unravelled

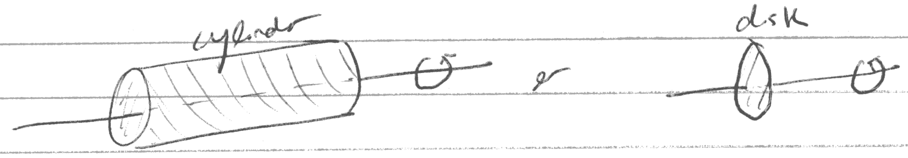
$$= \int_{R_1}^{R_2} r^2(2\pi r)\sigma\,dr = 2\pi\sigma \left.\frac{r^4}{4}\right|_{R_1}^{R_2}$$

$$= \frac{1}{2}\pi\sigma(R_2^4 - R_1^4) = \frac{1}{2}\cancel{\pi}\frac{M}{\cancel{\pi(R_2^2-R_1^2)}}\cancel{(R_2^2-R_1^2)}(R_2^2+R_1^2)$$

$$\sigma = \frac{M}{A} = \frac{M}{\pi(R_2^2-R_1^2)}$$

$$= \frac{1}{2}M(R_2^2+R_1^2)$$

For a solid disk or ~~cyl~~ cylinder (~~solid~~ extruded version) about central axis:



Think rings (hoops) from $r = 0$ to $r = R$:

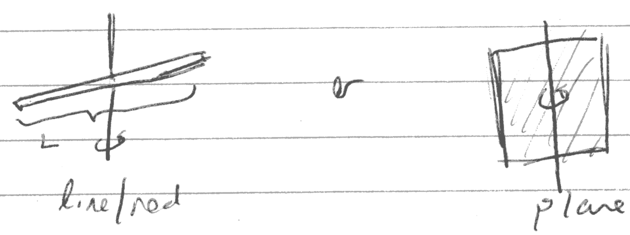
$$I = \int_0^R \underbrace{r^2\,dm}_{dI_{ring}} = 2\pi\sigma\int_0^R r^3\,dr = 2\pi\sigma\,\frac{r^4}{4}\Big|_0^R$$

$dm = (2\pi r)(dr)\sigma$

$$= \frac{\pi\sigma R^4}{2} = \frac{\cancel{\pi} R^{\cancel{4}2}}{2}\cdot\frac{M}{\cancel{\pi R^2}} = \frac{1}{2}MR^2$$

$\sigma = \frac{M}{A} = \frac{M}{\pi R^2}$

For a thin rod (basically a line) or a plane (extruded version) about a perpendicular line through its center



Integrate over length; each particle has $I = mr^2$

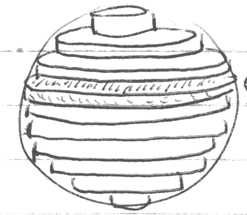
$$I = \int_{-\frac{L}{2}}^{\frac{L}{2}} mr^2 = \int_{-\frac{L}{2}}^{\frac{L}{2}} r^2\,dm = 2\sigma\int_0^{\frac{L}{2}} x^2\,dx = 2\sigma\,\frac{x^3}{3}\Big|_0^{\frac{L}{2}}$$

$r = x$ $dm = \sigma\,dx$ $\sigma = \frac{M}{L}$

$$= \frac{2M}{L}\,\frac{\left(\frac{L}{2}\right)^3}{3} = \frac{\cancel{2}ML^{\cancel{3}2}}{\cancel{8}4\cdot 3\cdot\cancel{L}} = \frac{1}{12}ML^2$$

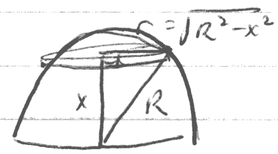
MOMENTS OF INERTIA, CONT'D.

Solid sphere through its center
(using I of a solid disk through its center)



each disk has $I = \frac{1}{2}Mr^2$
add up all the disks' I's

$$I = \int_{-R}^{R} dI = \cancel{2}\int_0^R \cancel{\tfrac{1}{2}}\, dm\, r^2 = \int_0^R \pi\sigma (R^2-x^2)^2\, dx$$



$$dm = (\pi r^2)(dx)\sigma = \pi(R^2-x^2)\sigma\, dx \qquad r^2 = (R^2-x^2)$$

$$= \pi\sigma \int_0^R R^4 - 2x^2R^2 + x^4\, dx = \pi\sigma\left(R^4 x - \frac{2R^2x^3}{3} + \frac{x^5}{5}\right)\Big|_0^R$$

$$\sigma = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3} = \frac{3M}{4\pi R^3}$$

$$= \frac{3\cancel{\pi}M}{4\cancel{\pi}R^3}\left(R^5 - \frac{2R^5}{3} + \frac{R^5}{5}\right) = \frac{3MR^{\cancel{5}\,2}}{4\cancel{R^3}}\left(\frac{15-10+3}{\cancel{15}\,5}\right)$$

$$= \frac{8MR^2}{4\cdot 5} = \frac{2}{5}MR^2$$

Solid sphere through its center (using triple cylindrical integrals).

$$I = \int_0^{2\pi} d\theta \int_0^R r \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} r^2\sigma\, dz\, dr$$

$$= 4\pi\sigma \int_0^R r^3 \int_0^{\sqrt{R^2-r^2}} dz\, dr = 4\pi\sigma \int_0^R r^3\sqrt{R^2-r^2}\, dr.$$

Let $r = R\sin\theta$, $dr = R\cos\theta\, d\theta$, $\theta = \sin^{-1}\left(\frac{r}{R}\right)$

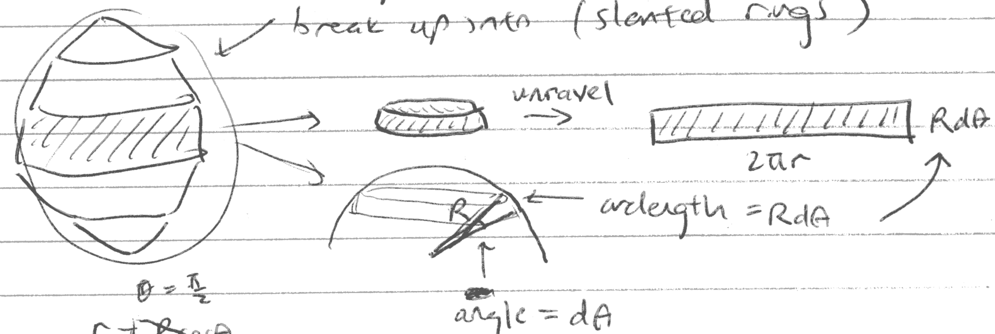
$$= 4\pi\sigma \int_{\sin^{-1}(0)}^{\sin^{-1}\left(\frac{R}{R}\right)} (R\sin\theta)^3 \sqrt{R^2 - R^2\sin^2\theta}\,(R\cos\theta)\, d\theta = 4\pi\sigma R^5 \int_0^{\frac{\pi}{2}} \sin^3\theta\cos^2\theta\, d\theta$$

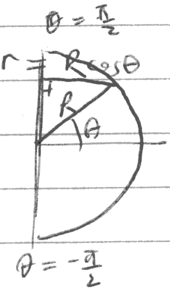
$$= 4\pi\sigma R^5 \int_0^{\frac{\pi}{2}} \cos^2\theta\sin\theta - \cos^4\theta\sin\theta\, d\theta = \frac{\cancel{4\pi}\cancel{R^5}\,R^2(3M)}{\cancel{4\pi}\cancel{R^5}}\left(-\frac{\cos^3\theta}{3} + \frac{\cos^5\theta}{5}\right)\Big|_0^{\frac{\pi}{2}}$$

$\sigma = \frac{3M}{4\pi R^3}$ (from top of page)

$$= 3MR^2\left((0+0) - \left(-\frac{1}{3} + \frac{1}{5}\right)\right) = \frac{2}{15}(3MR^2) = \frac{2}{5}MR^2$$

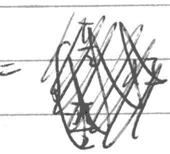
Hollow sphere (thin) through its center





$$dm = (2\pi r)(Rd\theta)\sigma = 2\pi R\cos\theta R\sigma d\theta$$
$$= 2\pi\sigma R^2\cos\theta\, d\theta$$

$$r^2 = (R\cos\theta)^2$$

$I =$  $= \int dI_{hoop} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2 dm = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} R^2\cos^2\theta\left(2\pi\sigma R^2\cos\theta\right)d\theta$

$$= 2\cdot R^4\cdot 2\pi\sigma\int_0^{\frac{\pi}{2}}\cos^3\theta\, d\theta = 4\pi\sigma R^4\int_0^{\frac{\pi}{2}}\cos\theta - \sin^2\theta\cos\theta\, d\theta$$

$$\sigma = \frac{M}{A} = \frac{M}{4\pi R^2}$$

$$\frac{\cancel{4\pi}R^{\cancel{4}\,2}\cdot M}{\cancel{4\pi R^2}}\left(\sin\theta - \frac{\sin^3\theta}{3}\right)\Bigg|_0^{\frac{\pi}{2}} = MR^2\left(1-\frac{1}{3}\right) = \frac{2}{3}MR^2$$

Hollow sphere through its center using a surface integral
(jk, haven't learnt these yet :/)

Hoop about any diameter:

Use perpendicular axis theorem.

I_z is known: MR^2 (I through center axis of hoop)

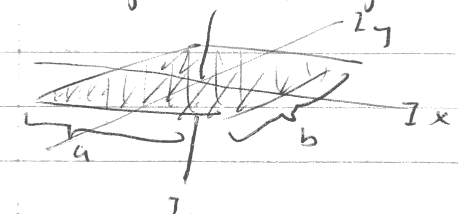
$$I_z = I_x + I_y \Rightarrow 2I_x = I_z \Rightarrow I_x = \frac{I_z}{2} = \frac{MR^2}{2}$$

$I_x = I_y$ by symmetry

MOMENTS OF INERTIA, CONT'D.

Slab about perpendicular axis through center,

Using perpendicular axis theorem and I of a plane through axis through its center,



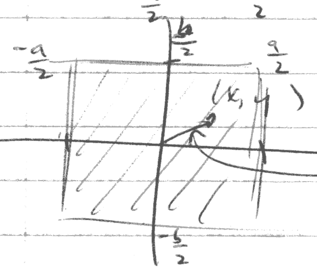
$I_x = \frac{1}{12} Ma^2$ (I_{plane} with $L = a$)

$I_y = \frac{1}{12} Mb^2$

$I_z = I_x + I_y = \frac{1}{12} M(a^2+b^2)$

Slab about perpendicular axis through center (using double integral)

$$I = \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \left(\sqrt{x^2+y^2}\right)^2 \sigma \, dy \, dx = 4\sigma \int_0^{\frac{a}{2}} \int_0^{\frac{b}{2}} (x^2+y^2) \cancel{\sigma} \, dy \, dx$$



$$= 4\sigma \int_0^{\frac{a}{2}} \left(x^2 y + \frac{y^3}{3}\right)\Big|_0^{\frac{b}{2}} dx$$

$$= 4\sigma \int_0^{\frac{a}{2}} \frac{x^2 b}{2} + \frac{b^3}{24} \, dx$$

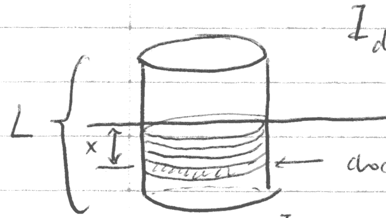
$$= 4\sigma\left(\frac{x^3 b}{6} + \frac{b^3 x}{24}\right)\Big|_0^{\frac{a}{2}} = \cancel{4}\left(\frac{a^3 b}{\cancel{48}_{12}} + \frac{b^3 a}{\cancel{48}_{12}}\right)$$

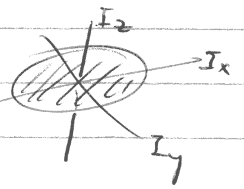
$$= \frac{1}{12} ab(a^2+b^2)\sigma = \frac{1}{12} \cancel{ab}(a^2+b^2)\frac{M}{\cancel{ab}} = \frac{1}{12} M(a^2+b^2)$$

$$\sigma = \frac{M}{A} = \frac{M}{ab}$$

Solid cylinder or disk (L=0) about central diameter

I_{disk} about its diameter: use perpendicular axis theorem,





$I_x = I_y$, $I_z = \frac{1}{2} MR^2$,

$\Rightarrow I_x = I_y = \frac{1}{4} MR^2$

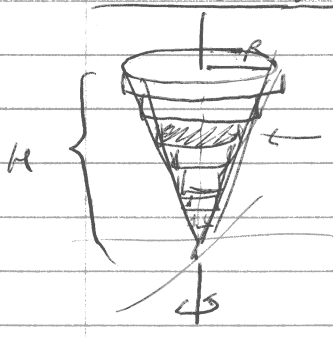
I_{disk} about its own axes is $\frac{1}{4} MR^2$. Using parallel axis theorem, I disk about center of cylinder is $\frac{1}{4} MR^2 + \cancel{M} Mx^2$, sum these up: $dm = \pi R^2 \sigma dx$

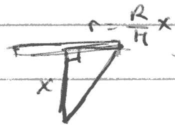
$$I = \int_{-\frac{L}{2}}^{\frac{L}{2}} I_{disk} = 2\int_0^{\frac{L}{2}} I_{disk} = 2\int_0^{\frac{L}{2}} \left(\frac{1}{4} R^2 + x^2\right) dm = 2\pi R^2 \sigma \int_0^{\frac{L}{2}} \left(\frac{1}{4} R^2 + x^2\right) dx$$

$$= 2\pi R^2 \sigma \left(\frac{1}{4} R^2 x + \frac{x^3}{3}\right)\Big|_0^{\frac{L}{2}} = \frac{2\cancel{\pi R^2} M}{\cancel{\pi R^2} L} = \frac{2M}{L}\left(\frac{R^2 L}{8} + \frac{L^3}{24}\right) = \frac{1}{4} MR^2 + \frac{1}{12} ML^2$$

$\sigma = \frac{M}{V} = \frac{M}{\pi R^2 L}$

Solid Cone about central axis (using disks)





$$I = \int_0^H dI_{disk} = \frac{1}{2}\int_0^H r^2 dm = \frac{\pi}{2}\sigma \int_0^H r^4 dx$$

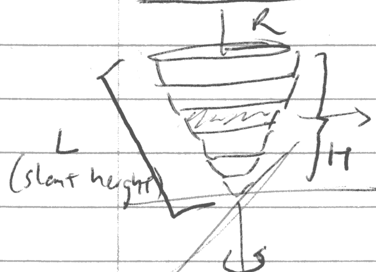
$$dm = (\pi r^2)(dx)\sigma$$

$$= \frac{\pi\sigma}{2}\int_0^H \left(\frac{R}{H}x\right)^4 dx = \frac{\pi\sigma R^4}{2H^4}\left(\frac{x^5}{5}\Big|_0^H\right)$$

$$\sigma = \frac{M}{V} = \frac{3M}{\pi R^2 H}$$

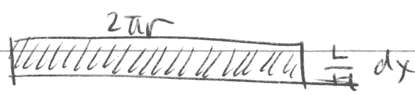
$$= \frac{\cancel{\pi} R^{\cancel{4}\,2}}{5\cdot 2\cancel{H^4}}\left(\frac{3M}{\cancel{\pi R^2 H}}\right)\cancel{H^5} = \frac{3}{10}MR^2$$

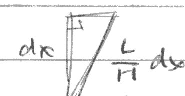
Hollow thin cone about central axis (using hoops/rings)



$$r = \frac{R}{H}x$$

unravel





$$dm = (2\pi r)\left(\frac{L}{H}dx\right)\sigma$$

$$I = \int_0^H r^2 dm = \int_0^H r^2 2\pi r\left(\frac{L}{H}\right)\sigma\, dx = 2\pi\sigma\left(\frac{L}{H}\right)\int_0^H r^3 dx$$

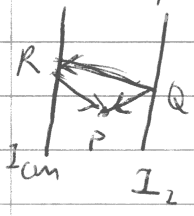
$$= 2\pi\sigma\left(\frac{L}{H}\right)\int_0^H \left(\frac{R}{H}\right)^3 x^3 dx = 2\pi\sigma\frac{LR^3}{H^4}\left(\frac{x^4}{4}\right)\Big|_0^H = \frac{\cancel{2\pi L}R^{\cancel{3}\,2}}{\cancel{H^4}}\cdot\frac{M}{\cancel{\pi R L}}\cdot\frac{\cancel{H^4}}{\cancel{4}\,2}$$

$$\sigma = \frac{M}{A} = \frac{M}{\pi R L}$$

$$= \frac{1}{2}MR^2$$

Remember ∥ and ⊥ axis theorems.

∥-axis proof: $\overrightarrow{QP} = \overrightarrow{QR} + \overrightarrow{RP}$



$$I_2 = \int (\overrightarrow{PQ})^2 dm = \int (\overrightarrow{QR} + \overrightarrow{RP})^2 dm$$

$$= \int (\overrightarrow{QR})^2 dm + 2\int (\overrightarrow{QR}\cdot\overrightarrow{RP})dm + \int (\overrightarrow{RP})^2 dm$$

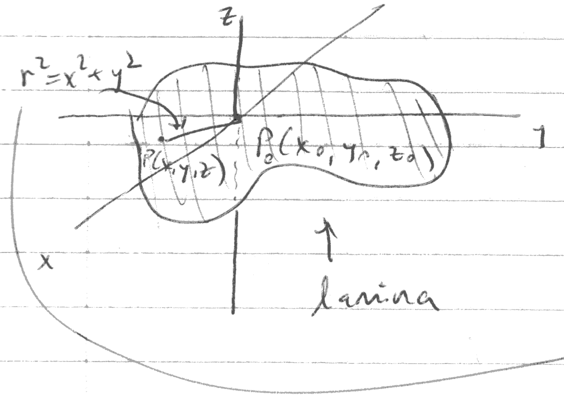
B/c distance b/t axes is fixed, let $\overrightarrow{QR}$ be constant h.

$$= h^2\int dm + \cancel{2h\int \overrightarrow{RP}\,dm}^{\,0} + \int (\overrightarrow{RP})^2 dm$$

$$= h^2 M + I_{com} \qquad \text{QED.}$$

MOMENT OF INERTIAS, CONT'D.

Proof of perpendicular-axis theorem.

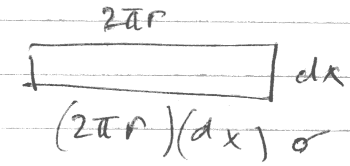


$$I_x = \int x^2\,dm$$
$$I_y = \int y^2\,dm$$
$$I_z = \int r^2\,dm = \int (x^2 + y^2)\,dm$$
$$= \int x^2\,dm + \int y^2\,dm$$
$$= I_x + I_y$$

(only works for laminas, and only for three perpendicular axes, two of which lie on the plane, which pass through the same point)

$$I = \int r^2\,dm.$$

$$dm = 2\pi r$$



$$I = \int_0^L r^3\, 2\pi\sigma\,dx = 2\pi\sigma \int_0^L \left(\frac{Rx}{L}\right)^3 dx$$

$$= 2\pi\sigma \left(\frac{R}{L}\right)^3 \left(\frac{x^4}{4}\right)\Big|_0^L = \frac{2\pi\sigma}{4_2} \frac{R^3}{L^3} L^4$$

$$= \frac{\pi\sigma R^3 L}{2}, \quad \sigma = \frac{M}{A} = \frac{M}{\pi R L}$$

$$= \frac{\pi \cancel{\sigma} R^{\cancel{3}2} \cancel{L}}{2} \frac{M}{\cancel{\pi R L}}$$

$$= \frac{1}{2} R^2 M$$

~~[illegible]~~ I of hollow cone integrating along slant height (and not regular height)

Derivation of Moment of Inertia of a Solid Cone
using Spherical Coordinates

$$\int_0^{2\pi} d\theta \int_0^{\arctan\left(\frac{R}{H}\right)} \int_0^{H\sec\phi} \left((\rho\sin\phi)^2 \sigma (\rho^2 \sin\phi)\right) d\rho\, d\phi$$

$$= 2\pi\sigma \int_0^{\arctan\left(\frac{R}{H}\right)} \sin^3(\phi) \int_0^{H\sec\phi} \rho^4\, d\rho\, d\phi$$

$$= \sigma \frac{2\pi H^5}{5} \int_0^{\arctan\left(\frac{R}{H}\right)} \sin^3(\phi) \sec^5(\phi)\, d\phi$$

$$= \sigma \frac{2\pi H^5}{5} \int_0^{\arctan\left(\frac{R}{H}\right)} \tan^3(\phi) \sec^2(\phi)\, d\phi$$

$$= \sigma \frac{2\pi H^5}{5} \left(\frac{\tan^4\phi}{4}\right)\Bigg|_0^{\arctan\left(\frac{R}{H}\right)} = \frac{\sigma\pi H^5}{10}\left(\frac{R}{H}\right)^4$$

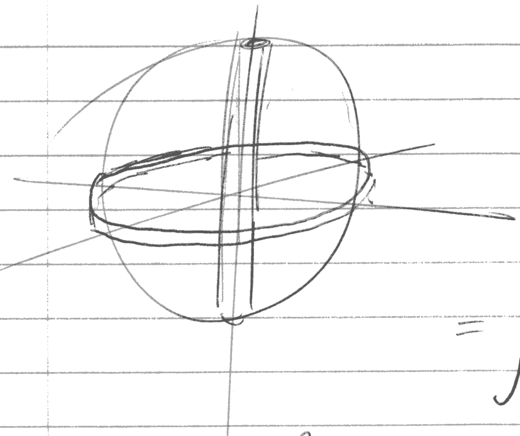
$$\sigma = \frac{M}{V} = \frac{3M}{\pi R^2 H}$$

$$= \frac{3M}{\cancel{\pi R^2 H}} \frac{\cancel{\pi H^5}}{10} \frac{R^{\cancel{4}\,2}}{\cancel{H^4}} = \frac{3}{10} M R^2$$

Add up thin cylinders to get hollow sphere

$dV = \overset{2\pi r}{\boxed{\qquad}}\, dr \qquad h$

$dm = \sigma 2\pi r h \, dr$



$$\int_0^R r^2 (2\pi r h \sigma)\, dr$$

$$h = 2\sqrt{R^2 - r^2}$$

$$= \int_0^R 4\pi r^3 \sqrt{R^2 - r^2}\, \sigma \, dr$$

$$= 4\pi\sigma \int_0^R r^3 \sqrt{R^2 - r^2}\, dr$$

let $r = R\sin\theta$,
$dr = R\cos\theta\, d\theta$
$\theta = \sin^{-1}\left(\frac{r}{R}\right)$

$$= 4\pi\sigma \int_0^{\frac{\pi}{2}} (R\sin\theta)^3 \sqrt{R^2 - (R\sin\theta)^2}\, R\cos\theta\, d\theta$$

$$= 4\pi\sigma R^5 \int_0^{\frac{\pi}{2}} \sin^3\theta \cos^2\theta \, d\theta$$

(this is the same as the triple spherical integrals),

also try adding up cylindrical shells to get solid sphere

$$\int_0^R \frac{2}{3} r^2 \, dm \qquad dm = 4\pi r^2 \, dr\, \sigma$$

$$= \frac{8}{3}\pi\sigma \int_0^R r^4 \, dr = \frac{8}{3}\pi\sigma \frac{R^5}{5} = \frac{8\pi\sigma R^5}{15}$$

$$\sigma = \frac{3M}{4\pi R^3} \qquad = \frac{8\pi R^5}{15} \cdot \frac{3M}{4\pi R^3} = \frac{2}{5}\pi R^3$$

$\cancel{\lim_{\Delta h \to 0} \frac{2}{5}M(R+\Delta h) - \frac{2}{5}M(R) \to 0}$

$\cancel{\frac{2}{5}M(R+\Delta h) - \frac{2}{5}MR = \frac{2}{5}M\left(\frac{R+\Delta h}{R}\right)^3 (R+\Delta h) - \frac{2}{5}MR}$

$\cancel{M = \frac{4}{3}\pi(R+\Delta h)^3 \sigma}$