Alg-12

01 March 2021 12:33

Alg-12

Prob set 11,

#2 $\varphi_2 : \mathrm{Aff}(n, K) \to K^n$

defined by $\varphi_2(f_{A,a}) = a$

is not a hmo

(1) $\mathrm{Aff}(n,K)$ is a group under composition ($\mathrm{Aff}(n,K)$ $\left(\mathrm{Aff}(n,K) \cong GL(n,K) \ltimes H\right.$

where $H = \{f_{I,a} : a \in K^n\}$

the subgroup of all translations of $x \in K^n$ by $a \in K^n$

(2) $H \trianglelefteq \mathrm{Aff}(n,K)$

$f_{I,a}(x) = x + a$

(3) $GL(n,K) \hookrightarrow \mathrm{Aff}(n,K)$

$A \longmapsto f_{A,0} : K^n \to K^n$

is "linear" map.

$f_{A,0}(\alpha x + y) = \alpha f_{A,0}(x) + f_{A,0}(y)$

$\forall x, y \in K^n, \forall \alpha \in K$

Notation $\mathbb{Z}_n$ is any cyclic group of order n

$\mathbb{Z}_n = \langle a \rangle$, $|a| = n$.

Thm $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$

$$\mathbb{Z}/n\mathbb{Z} = \{[r]_n : 0 \le r \le n-1\}$$

Proof $f : \mathbb{Z} \longrightarrow \mathbb{Z}_n = \langle a \rangle$

by $f(m) = a^m$ is a homo

& f is surjective

$m \in \ker f \Rightarrow e = e_{\mathbb{Z}_n} = f(a) = a^m$

$\Rightarrow n \mid m \Rightarrow m = kn$.

$\ker f \subseteq n\mathbb{Z}$. Now we show $n\mathbb{Z} \subseteq \ker f$. $x \in n\mathbb{Z}$

show $n\mathbb{Z} \subseteq \ker f$. $x \in n\mathbb{Z}$

$x = nr,\ r \in \mathbb{Z}$ $\quad f(x) = f(nr)$

$$= a^{nr} = (a^n)^r$$
$$= e$$

1st iso thm $\Rightarrow$

$$\mathbb{Z}/\ker f \simeq f(\mathbb{Z}) = \mathbb{Z}_n$$
$$\parallel$$
$$\mathbb{Z}/n\mathbb{Z}$$

So $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$

$\mathbb{Z}_2 = \{[0]_2, [1]\}$, binary number system.

& is a "field"

FACT: $|G| \leq 5$, G is a group

$\Rightarrow$ G is abelian

We show $|G| < \infty$, G is a group

& $|G| = p$, prime, then G is

... & hence abelian

cyclic & hence abelian

$a \in G$, $a \neq e$ as $p > 1$

$H := \langle a \rangle \leq G$, $|H| \mid |G| = p$

by Lagrange's thm

$|H| = p$ or 1.

$\Rightarrow H = G = \langle a \rangle$

Ex $f : G \xrightarrow{hm} G'$, $G = \langle a \rangle$

$\Rightarrow f(G) \leq G'$ & cyclic

$\langle a \rangle = G \Rightarrow \langle f(a) \rangle = f(G)$

f is homo or not $\langle f(a) \rangle \subseteq f(G)$

f is homo $\Rightarrow f(G) \subseteq \langle f(a) \rangle$

& $\langle a \rangle = G$

---

① $N \trianglelefteq G \Rightarrow G/N$ is a group

① $N \trianglelefteq G \longrightarrow G/N$

② $G \xrightarrow{\pi} G/N$
$g \longmapsto gN$ is a homo

③ $\ker \pi = N$

1st iso. Thm $f : G \longrightarrow G'$ is homo
then $G/\ker f \cong f(G)$

Thm (Third iso). Let $K \subseteq H \subseteq G$ be normal subgroup of G

Then (1) K is normal in H

(2) The map $\varphi : G/K \longrightarrow G/H$ defined by $\varphi(gK) = gH$ is a homo & $\ker \varphi = H/K$.

$= \{hK : h \in H\}$ is a subgroup of G/K

(Third iso thm) (3) $\dfrac{G/K}{H/K} \cong \varphi(G/K) = G/H$.

Proof. (1) $K \leq H$ and K is normal H

Given that K & H are subgroup of G & $K \subseteq H$. $\Rightarrow$ K is a subgroup of H. Know $K \trianglelefteq G$

$\Rightarrow gKg^{-1} \subseteq K \quad \forall g \in G$. In particular

$$gKg^{-1} = K \quad \forall g \in H.$$

$$\Rightarrow K \trianglelefteq H$$

(2) $\varphi(gK) = gH$

$g_1 K = g_2 K \Rightarrow g_2^{-1} g_1 \in K \subseteq H \Rightarrow g_2 H = g_1 H$

$\varphi(g_1 K) = \varphi(g_2 K)$, φ is well defined.

φ is homo: $\varphi((g_1 K)(g_2 K))$

$$= \varphi(g_1 g_2 K)$$

$$= g_1 g_2 H$$

$$= (g_1 H)(g_2 H)$$

$$= \varphi(g_1 K)\varphi(g_2 K)$$

$\Rightarrow \varphi$ is homo

$\varphi(x) = e = H$

Suppose $x \in \ker(\bar{\varphi}|_{G/K})$ & $\varphi(x) = e_{G/H} = H$

$x = gK \iff H = \varphi(x) = \varphi(gK) = gH$.

$\iff g \in H$.

$x = gK, \ g \in H$

$\ker \varphi = \{gK : g \in H\} = H/K$.

(3) is just 1st iso thm with

$\varphi(G/K) = G/H$.

$\subseteq$. Let $y \in G/H$

to find $x \in G/K$ s.t.

$\varphi(x) = y$. Now

$y = gH$ for some $g \in G$.

Take $x = gK \Rightarrow \varphi(x) = \varphi(gK) \overset{\text{def}}{=} gH = y$.

So φ is surjective $= y$.

Then $$\left. {G/K} \middle/ {H/K} \right. \cong G/H = \varphi\left(G/K\right)$$
$\quad \underset{\text{ker}\,\varphi}{\shortparallel}$

Ex $n\mathbb{Z} \subseteq m\mathbb{Z} \iff m \mid n$.

are subgroup of $\mathbb{Z}$, an abelian group. So $n\mathbb{Z} \trianglelefteq m\mathbb{Z}$.

$$m\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_d, \quad d = \frac{n}{m}$$

$$\mathbb{Z}_d = \left\{ [r]_d : 0 \le r \le d-1 \right\}$$

Proof Define a map $m\mathbb{Z} \to \mathbb{Z}_d = \langle a \rangle$

$x \in m\mathbb{Z}$ $\quad f(mk) = a^{mk}$. is a homo

$x = mk$, $k \in \mathbb{Z}$

$$f(mk_1 + mk_2) = f(m(k_1+k_2)) = a^{m(k_1+k_2)} = a^{mk_1+mk_2}$$

$= a^{mk_1} a^{mk_2}$

$= f(mk_1) . f(mk_2)$

$\Rightarrow f$ is homo

$b \in \mathbb{Z}_d = \langle a \rangle$, $a^d = e$

$b = a^r$ for some $r \in \mathbb{Z}$

To find $x \in m\mathbb{Z}$
s.t. $f(x) = b = a^r$ (?)

So f is surjective

Hence by 1st iso thm

$$m\mathbb{Z}/_{\ker f} \simeq f(m\mathbb{Z}) = \mathbb{Z}_d$$

$$m\mathbb{Z}/_{n\mathbb{Z}}$$

$$\left| m\mathbb{Z}/_{n\mathbb{Z}} \right| = d = \frac{m}{n}$$

Thm. Let G be a group,
$H \leq G$ and $N \trianglelefteq G$

Then

(1) HN is a subgroup of G

(2) $H \cap N \trianglelefteq$ of H.

(3) The map $\varphi : H \longrightarrow G/N$ given by $\varphi(h) = hN$ is a homo with $\ker \varphi = H \cap N$

(4) $H/H \cap N \simeq f(H) = HN/N$

3rd iso Thm.

Cor G is abelian, H, N are subgroups. Then

$$H/H \cap N \simeq H+N/N$$