Alg-22

G is a group. H_1, H_2 are two subgroups of G

$G = H_1 \oplus H_2$ means

(1) $G = H_1 + H_2$ i.e. each $g \in G$, $\exists\, h_i \in H_i$ $i = 1, 2$ s.t. $g = h_1 + h_2$

(2) $H_1 \cap H_2 = \{e\}$

$G = H_1 \oplus H_2 \iff$ for each $g \in G$, $\exists!\, h_i \in H_i$ s.t. $g = h_1 + h_2$

$\Rightarrow$ $g = h_1 + h_2$

$h_1, \ldots, h_n \in H$

$\Rightarrow \quad g = h_1 + h_2$

$g = k_1 + k_2 \quad , \quad h_1, k_1 \in H_1$

$h_2, k_2 \in H_2$

$\Rightarrow \quad g - g = h_1 - k_1 + h_2 - k_2$

$h_1 - k_1 = -(h_2 - k_2) \in H_1 \cap H_2$

$= \{e\}$

$\Rightarrow h_1 = k_1, \; h_2 = k_2$

Write $G = H_1 H_2$ iff

$g = h_1 h_2$ & $H_1 \cap H_2 = \{e\}$

$\Leftrightarrow \quad g = h_1 h_2$ for unique $h_1 \in H_1, h_2 \in H_2$

$n = |A|$, A is abelian

$1 = (m, m')$ & $n = mm'$

then $A = A_m \oplus A_{m'}$

$\ldots A : ra = 0 \}$

Then $A = \{a \in A : ra = 0\}$

where $A_r = \{a \in A : ra = 0\}$

Thm (i) $n = \exp A$, $n = mm'$, $(m, m') = 1$

$A = A_m \oplus A_{m'}$

(2) $n = |A|$, $n = mm'$, $(m, m') = 1$

$A = A_m \oplus A_{m'}$

③ $n = \exp(A)$ then

$A = \bigoplus_{p|n} A(p)$ $A(p) = \{0\}$ if $p \nmid n$.

④ $n = \prod_{i=1}^{k} p_i^{r_i} = |A|$

Then $A = \bigoplus_{i=1}^{k} A_i$

$A_i = \{a \in A : p_i^{r_i} a = 0\}$

$|A_i| = p_i^{r_i}$ $i = 1 \cdots k$

2 $|A_i| = p_i^{r_i}$, $i = 1 \ldots k$

Proof (3) Assume $n = \exp(A) = \prod_{i=1}^{k} p_i^{r_i}$ where

$p_1 \ldots p_k$ are distinct primes dividing n, $r_i \geq 1$

To prove $A = \bigoplus_{i=1}^{k} A(p_i)$ we use induction on # of primes

If $k = 1$. then $|A| = n = p_1^{r_1}$

$A(p_1) = \{a \in A : |a| = p^m$ for some $m \geq 0\}$

then $A(p_1) \trianglelefteq A$ as A is abelian

↑ ok for A abelian

↑ subgroup for A abelian

$$a \in A \Rightarrow p_1^{r_1} a = 0$$

$|G| < +\infty$, $a \in G$

$|a| \mid |G|$

$\Rightarrow a^{|G|} = e$

$|G| a = 0$, $p_1^{r_1} a = 0$

$a \in A(p_1)$

Assume the theorem is true $\forall$ abelian group A

$|A| = \prod_{j=1}^{k-1} q_j^{s_i}$, $s_i \geq 1$ $q_1 \dots q_{k-1}$ dist. primes

$\Rightarrow A = \bigoplus_{i=1}^{k-1} A(q_i)$ — induction step.

///

$i=1$

step

Now $n = \prod_{i=1}^{k} p_i^{r_i}$ $\quad n = \exp(A)$

$= mm'$, $\quad m' = p_k^{r_k}$

$m = \prod_{i=1}^{k-1} p_i^{r_i}$

Then $(m, m') = 1$.

By (1) we have

$A = A_m \oplus A_{m'}$

$A_m = \{a \in A : ma = 0\}$

$\exp(A_m) = m$

$\Rightarrow m = \prod_{i=1}^{k-1} p_i^{r_i}$

By induction

$A_m = \bigoplus_{i=1}^{k-1} A(p_i)$

& $A = A_m \oplus A_{m'}$

$= \bigoplus_{k-1} A(p) \oplus A_{m'}$

$$= \bigoplus_{i=1}^{k-1} A(p_i) \oplus A_{m'}$$

$$A_{m'} = \{ a \in A : m'a = 0 \}$$

$$= \{ a \in A : p_k^{r_k} m = 0 \}$$

$$= A(p_k)$$

$$\Rightarrow A = \bigoplus_{i=1}^{k} A(p_i)$$

(4) $n = |A| = \prod_{i=1}^{k} p_i^{r_i}$

then $A = \bigoplus_{i=1}^{k} A(p_i)$ & $|A(p_i)| = p_i^{r_i}$

Lagrange's Thm says, $|G| < \infty$

$H \leq G \Rightarrow |H| \mid |G|$

Fix A_4, $6 \mid |A_4| = 12$

A_4 has no subgroup of order 6.

There is a partial converse of Lagrange's thm.

G is a group, $|G| < \infty$

$p^\alpha \mid |G| \Rightarrow \exists$ a subgroup H of G of order p^α, $\alpha \geq 1$

Sylow's Thm

Def G is a finite group. $H \leq G$ is a p-Sylow subgroup of G iff $p^n = |H|$ and $p^n \mid |G|$ and $p^{n+1} \nmid |G|$

H is a p-Sylow subgroup

H is a "

iff $|H|$ is the highest power of p dividing the order of G

$|G| = p^{r+1}$, H is a p-Sylow subgroup of G

then $|H| = p^{r+1}$, $H = G$.

and any subgroup H of order p^{α}, $0 \leq \alpha \leq r+1$ is not a p-Sylow subgroup

Suppose G is a finite p-group if $|G| = p^{r}$ for some $r \geq 0$

$H = G$ is the only

$H \leq G$ ...

p-Sylow subgroup

$|H| = p^\alpha$, $0 \leq \alpha < r$

H is a p-group

Thm (Cauchy 1.8.44)

(a) G is a finite abelian group & p is a prime dividing $|G|$ then $\exists\, a \in G$ of period p & hence a subgroup $\neq$ of order p ($H = \langle a \rangle$

(2) G is a finite group & $p \mid |G|$ where p is a prime. Then $\exists\, a \in G$ of period p & hence a subgroup $\neq$ of order p

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