Alg-12

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$\varphi_1 : Aff(n, \bar{K}) \longrightarrow GL(n, K)$

defined by $\varphi_1(f_{A,a}) = A$

is a homo & $\ker \varphi_1 = ?$

~~Here~~ $(Aff(n,K), \circ)$, $(GL(n,K), \cdot)$

$\varphi_1(f_{A,a} \cdot f_{B,b}) \overset{No}{=} \varphi_1(f_{A,a}) \circ \varphi_1(f_{B,a})$

$\varphi' \; GL(n,K) \longrightarrow Aff(n,K)$

$\varphi(A) = f_{A,0}$

embedding

$\varphi_1 : Aff(n,K) \longrightarrow GL(n,K)$

$\varphi_1(f_{A,a}) = A$ is <u>homorphic</u> map

or homorphism.

K^n, $K^{\infty} = K^{\mathbb{N}}$

$= \{ f : \mathbb{N} \to K \}$

$\mathcal{E}_0$

$$(f+g)(n) = f(n)+g(n)$$

$$(\alpha f)(n) = \alpha f(n), \quad \alpha \in K$$

$$GL(K^\infty) = \{A : K^\infty \to K^\infty \text{ linear \& invertible}\}$$

$f \in K^\infty$, $T_f : K^\infty \to K^\infty$

by $T_f(g) = g + f$

$$Aff(K^\infty) = \{\varphi_{A,a}\}$$

$\varphi_{A,a} : K^\infty \to K^\infty$ s.t.

$$\varphi_{A,a}(f) = Af + a.$$

Rigid Motions$(n, \mathbb{R})$

$$= \{ f : \mathbb{R}^n \to \mathbb{R}^n \text{ s.t. } \|f(x)-f(y)\|_2 = \|x-y\|_2 \}$$

$$= \|x-y\|_2 \}$$

$$\|x\|_2 = \sqrt{x \cdot x}$$

$$f = f_{A,a} \quad , \quad A \in O(n)$$

$$A^t A = I_n = A A^t$$

2nd isomorphism Thm for groups

1st iso $\quad G \xrightarrow{f} G'$ is homo

$$G/\ker f \simeq f(G)$$

3rd iso. $\quad K \trianglelefteq G \;\&\; H \trianglelefteq G$

and $K \subseteq H \subseteq G$

$$\frac{G/K}{H/K} \simeq G/H$$

$$G/K \xrightarrow{f} G/H$$

$f(gK) = gH$ · is welldefined homo

$$\ker f = H/K = \{hK : h \in H\}$$

$$\trianglelefteq G/K$$

3rd iso is useful for

3rd iso is useful for solvable groups

2nd iso Thm

G is a group

$H \le G$, $N \trianglelefteq G$. Then

(1) HN is a subgroup of G

(2) $H \cap N$ is normal subgroup of N

(3) The map $f : H \to G/N$ defined by $f(h) = hN$ is a homomorphism

$\ker f = H \cap N$

and $H/H\cap N \simeq f(H) = HN/N$

Cor G is abelian. $H, N \le G$.

then $H/H\cap N \simeq H+N/N$

Proof (1) HN is a subgroup

$a, b \in HN \Rightarrow ab^{-1} \in HN$

$\left.\begin{array}{l} a = h_1 n_1 \\ b = h_2 n_2 \end{array}\right\} \Rightarrow ab^{-1} = h_1 \underbrace{n_1 n_2^{-1} h_2^{-1}}\in HN$?

$h_i \in H, n_i \in N$

$= h_1 h_2^{-1} h_2 n_1 n_2^{-1} h_2^{-1}$

$h_1 h_2^{-1} \in H$ as $H \le G$ &

Now $h_1 h_2^{-1} \in H$ as $H \leq G$ & $h_1, h_2 \in H$,

$n_1 n_2^{-1} \in N$ as $N \leq G$,

$h_2 n_1 n_2^{-1} h_2^{-1} \in N$ as $N \trianglelefteq G$. $(gNg^{-1} \subseteq N)$

$$\Rightarrow h_1 n_1 n_2^{-1} h_2^{-1} = (h_1 h_2^{-1}) h_2 (n_1 n_2^{-1}) h_2^{-1} \in HN$$

So HN is a subgroup of G.

Note H & K are subgroups of G then HK need not be a subgroup of G

$G = S_3$, $H = \langle \alpha \rangle$, $K = \langle \beta \rangle$

Is HK is a subgroup?

Yes as $K \trianglelefteq G$.

Define $f : H \to G/N$ by $f(h) = hN$ is a homomorphism:

$$f(h_1 h_2) = h_1 h_2 N = (h_1 N)(h_2 N) = f(h_1) f(h_2)$$

$h \in \ker f \iff \underset{\substack{\| \\ N}}{e_{G/N}} = f(h) = hN$

Thus $h \in \ker f \iff$

$N = hN$

$\iff h \in N$

So $\ker f = H \cap N$ — & hence normal in H.

By 1st iso $H/_{H\cap N} \simeq f(H)$

Want to show $f(H) = HN/_N$

Suppose $x \in f(H)$, $\exists\, h \in H$ s.t. $x = f(h) = hN \in HN/_N$

$= heN$

conversely, let $y \in HN/_N$

$y = gN$ for some $g \in HN$

$\Rightarrow g = hn$ for some $h \in H$ $n \in N$

So $y = gN = hnN = hN$ as $nN = N$

$= f(h)$.

$f(H) = HN/_N$.

$\simeq HN/$

$f(H) = HN/N$

Thus $H/_{H \cap N} \simeq HN/N$

Def. $Z(G) = \{g \in G : ag = ga \ \forall a \in G\}$ is called the center of G, a term coined by J. A. Jeguervic in 1904

center $\equiv$ Zentrum. in German

Thm (1) $Z(G) \leq G$ and

(2) $Z(G) \trianglelefteq G$.

Proof (1) $g_1, g_2 \in Z(G) \Rightarrow g_2 a = a g_2 \ \forall a \in G$

$\Rightarrow a g_2^{-1} = g_2^{-1} a \ \forall a \in G$

$\Rightarrow g_2^{-1} \in Z(G)$

Now $(g_1 g_2^{-1}) a = g_1 (g_2^{-1} a)$

$= g_1 (a g_2^{-1})$

$= (g_1 a) g_2^{-1}$

$= a (g_1 g_2^{-1}) \ \forall a \in G$

$\Rightarrow g_1 g_2^{-1} \in Z(G)$

$\Rightarrow g_1 g_2^{-1} \in Z(G)$

$\Rightarrow Z(G) \leq G.$

$g \in G,\ x \in Z(G).$

$g x g^{-1} \in Z(G)$

$\quad \| $

$\quad x \in Z(G)$

so $Z(G) \trianglelefteq G.$

$G \xrightarrow{\varphi} Aut(G)$

$\varphi(a) = c_a \ ,\ c_a(x) = a x a^{-1},\ c_a \in Aut(G)$

φ is homomorphism, $\varphi(G) = Inn(G)$
$= \{c_a : a \in G\}$

$\ker \varphi = Z(G)$

$G/Z(G) \simeq Inn(G)$

$\qquad\qquad\qquad \| $

$\qquad\qquad\qquad I(G).$

$\uparrow$

Gallian

Def. Let G be a group &
$S = \{ x y x^{-1} y^{-1} : x, y \in G \}$

$S = \{ xyx^{-1}y^{-1} : x, y \in G \}$

$G^c = \langle S \rangle$ called the commutator subgroup of G

(The commutator subgroup was introduced by G. A. Miller in 1898)

Thm (1) $G^c \trianglelefteq G$

(2) G/G^c is abelian

(3) $N \trianglelefteq G$ and G/N is abelian $\Rightarrow G^c \subseteq N$

(4) $H \leq G$ and $G^c \subseteq H$, then H is normal in G

Proof. $aba^{-1}b^{-1} \in S \subseteq G^c$, $x \in G$

$x(aba^{-1}b^{-1})x^{-1} \overset{?}{=} cdc^{-1}d^{-1}$ for some $c, d \in G$

$$= \underset{c}{(xax^{-1})}\underset{d}{(xbx^{-1})}\underbrace{(xa^{-1}x^{-1})}_{c^{-1}}\underbrace{(xb^{-1}x^{-1})}_{d^{-1}} \in S$$

$\Rightarrow G^c$ is normal in G.

(2) Know G/G^c is a group

$(xG^c)(yG^c) \overset{?}{=} (yG^c)(xG^c)$

$$(xG^c)(yG^c) = (yG^c)(xG^c)$$

$$\iff xyG^c = yxG^c$$

$$\iff xy(yx)^{-1} \in G^c$$

$$\iff xyx^{-1}y^{-1} \in G^c$$

hence $(xG^c)(yG^c)$
$= (yG^c)(xG^c)$

(3) $N \trianglelefteq G$. G/N is abelian.

To show $G^c \subseteq N$

We show $S = \{aba^{-1}b^{-1} : a, b \in G\}$
$\subseteq N$

$\Rightarrow G^c \subseteq N$.

For $a,b \in G$, $(aN)(bN) = (bN)(aN)$

$abN = baN$ $\qquad aH = bH$

$\Downarrow\Uparrow$

$\iff ab(ba)^{-1} \in N$. $\qquad ab^{-1} \in H$

$\Downarrow\Uparrow$

$\iff aba^{-1}b^{-1} \in N$ $\qquad ba^{-1} \in H$.

$\Rightarrow S \subseteq N$.

$\Rightarrow$ $\supseteq$ N $ba\ldots$

$aba^{-1}b^{-1}$, $a,b \in G$.

(4) Given $H \leq G$, $G^c \subseteq H$

Then H is normal in G

To show $gHg^{-1} \subseteq H$.

$ghg^{-1} \in gHg^{-1}$. Now $ghg^{-1}h^{-1} \in \underbrace{G^c}_{\subseteq H}$

So $ghg^{-1} = \underbrace{(ghg^{-1}h^{-1})}_{\in G^c \subseteq H} \overset{\in H}{h} \in H$ $\forall g \in G$

$\Rightarrow gHg^{-1} \subseteq H$ $\forall g \in G$.

Thus $H \trianglelefteq G$.

3^{rd} iso is important in solvable groups.

Def G is a group G is a solvable group iff there is a finite sequence $\{H_j\}_{j=0}^{r}$ of subgroups of G

s.t. (1) $\{e\} = H_r \subseteq H_{r-1} \subseteq \ldots H_j \subseteq H_{j-1} \subseteq \ldots \subseteq H_1 \subseteq H_0 = G$

(2) $H_j \trianglelefteq H_{j-1}$ for $j = 1 \ldots r$.

(2) $H_j \trianglelefteq H_{j-1}$ for $j = 1 \ldots r \ldots$

(3) H_{j-1}/H_j is abelian.

Thm (Feit-Thompson) 1963 : Every group odd order is solvable.

Ex! S_n is not solvable for $n \geq 5$

2 Kaplansky : $Aut(S_n)$.

3 $Aut(\mathbb{Z}_n) \simeq U_n = \{ [r]_n : \exists s \in \mathbb{Z}$ s.t. $[r]_n [s]_n = [1] \}$

$|U_n| = \varphi(n)$ - Euler's φ function

totient function

Some facts about solvable groups
(See: Herstein P116: #10-#13 sols in P252)

Thm (1) G is a group, $K \trianglelefteq G$, Assume K & G/K are solvable
then G is solvable

(2) G is solvable & $H \leq G$
Then H is solvable.

(3) G is solvable $f: G \to G'$ homomorphism

Then H is solvable

(3) G is solvable, $f: G \to G'$ homomorphism then $f(G) \leq G'$ is solvable subgroup.

(4) $H \leq G$, $N \trianglelefteq G$, both are solvable then HN is a solvable subgroup

(5) G is a group $\boxed{G^{(1)} = G^c}$

Define $G^{(k)}$ = commutator subgroup of $G^{(k-1)}$, $k > 1$

$\left(f^{(n)} = \frac{d^n f}{dx^n} \right. \qquad (G^c)^c \equiv G^{(2)}$

Then

(i) $G^{(k)} \trianglelefteq G$ for each k

(ii) G is solvable $\iff G^{(l)} = \{e\}$ for some $l \geq 1$ integer.

$l = 1 \quad G^{(1)} = \{e\}$

$aba^{-1}b^{-1} \in G^{(1)} = G^c = \{e\}$

$aba^{-1}b^{-1} = e$

$\Rightarrow ab = ba \quad \therefore G$ is abelian.

FACT. Every abelian group is solvable

$G = H_0 \supseteq H_1 = \{e\} = G^c = G^{(1)}$

$G = \langle a \rangle$, a cyclic group generated

$a \in G$. $f: \mathbb{Z} \longrightarrow G$ by

$f(m) = a^m$, is homomorphism, and surjective.

(1) $\ker f = \{0\}$

period or order of a is infinite

$\mathbb{Z} \simeq G$.

$b \in G$. to find $r \in \mathbb{Z}$ s.t. $f(r) = b$.

$b \in G = \langle a \rangle$

$b = a^r$ for some $r \in \mathbb{Z}$

$= f(r)$

(2) $\ker f = H \leq \mathbb{Z}$, $H \neq \{0\}$.

$= d\mathbb{Z}$. for some $1 < d \in \mathbb{N}$

$f(m) = e \quad \forall\, m \in \mathbb{Z}$

Then f is called a trivial homomorphism.

$d = |a| =$ order of a

and by 1st iso we have

$$\mathbb{Z}/d\mathbb{Z} \simeq G$$

$$|\mathbb{Z}/d\mathbb{Z}| = d$$

$$\mathbb{Z}/d\mathbb{Z} = \{ r + d\mathbb{Z} : 0 \le r \le d-1 \}$$

$$(r + d\mathbb{Z}) + s + d\mathbb{Z} \overset{\text{def}}{=} r + s + d\mathbb{Z}$$

$GL(n, K)$ is a group

$K = \mathbb{R}$ or $\mathbb{Q}$ $\quad (GL(n,K), \cdot) \xrightarrow{\det} (K^*, \cdot)$ homo

$$\ker \det = \{ A \in GL(n,K) : \det(A) = 1 \}$$

$\ker\det = SL(n,K)$ is normal

$$GL(n,K) / SL(n,K) \simeq \det(GL(n,K)) = K^*$$

$a \neq 0 \in K$, find

$A \in GL(n,K)$, $\det(A) = a$

$$\det \underbrace{\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}}_{A} = a.$$

$$GL(n,K) \big/ SL(n,K)$$

$$= \{ A\,SL(n,K),\ A \in GL(n,K) \}$$

Thm. Let $G_j = \langle a_j \rangle$, $|G_j| = d$ for $j = 1, 2$. Then there is a unique isomorphism $g: G_1 \longrightarrow G_2$ with $g(a_1) = a_2$.

Proof.