Alg-7

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G is a group, $a, b \in G$

Prove that $(aba^{-1})^n = ab^n a^{-1}$ for $n \in \mathbb{N}$

Either $A(n)$: statement that $(aba^{-1})^n = ab^n a^{-1}$

or $S = \{n \in \mathbb{N} : (aba^{-1})^n = ab^n a^{-1}\}$

Show that $A(1)$ is true,

$A(n)$ true $\Rightarrow$ $A(n+1)$ is true

or else show

$1 \in S$, $n \in S \Rightarrow n+1 \in S$

then $A(n)$ is true for $\forall n \in \mathbb{N}$

or $S = \mathbb{N}$

$(aba^{-1})^1 = aba^{-1}$

$(aba^{-1})^n = ab^n a^{-1}$

$(aba^{-1})^{n+1} = \ldots$

Not enough:

Ex $A \in GL(n, \mathbb{K})$, $a \in \mathbb{K}^n$, Ex $A \in GL(n, \cancel{\mathbb{R}} K)$, $a \in K$

group under product — group under addition

I_n ok. — $e = 0$

not in general

$f_{A,a} : K^n \longrightarrow K^n$ defined by

$f_{A,a}(x) = Ax + a \in K^n$

$Aff(n, K) = \{ f_{A,a} : a \in K^n \ A \in GL(n,K) \}$

$(Aff(n,K), \circ)$ is a group

$f_{A,a} \circ f_{B,b} = f_{C,c}$ for some $C \in GL(n,K)$ and $c \in K$.

Homomorphism, Isomorphisms and Automorphisms

1- $\mathbb{R}_{>0} = (0, \infty)$. Then $(\mathbb{R}_{>0}, \cdot)$ is a group

Define $\ln : \mathbb{R}_{>0} \longrightarrow \mathbb{R}$ by

$\ln(x) = \int^x \frac{1}{t}\, dt$,

... >0

$$\ln(x) = \int_1^x \frac{1}{t}\,dt,$$

$$\ln xy = \ln x + \ln y$$

then $\ln$ is taking to product in $\mathbb{R}_{>}$ to sum in $\mathbb{R}$

2 $\exp: (\mathbb{R},+) \to (\mathbb{R}_{>0},\cdot)$

$$\exp(x+y) = e^{x+y} = e^x \cdot e^y$$

3 $\det: GL(n, K) \to K^* = K - \{0\}$ (K = $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$)

$$\det(A) \in K^*.$$

$$\det(AB) = \det(A)\det(B)$$

Def. G & G' are groups, $f: G \to G'$ is a map. f is a group homo-morphism iff $\underbrace{f(g_1 g_2)}_{\text{product in } G} = \underbrace{f(g_1)f(g_2)}_{\text{product in } G'}$ $\forall g_1, g_2 \in G$

Ex $(\mathbb{R},+), (\mathbb{C}^*,\cdot)$ are groups

$f: \mathbb{R} \to \mathbb{C}^*$ by $f(t) = e^{it} = \cos t + i\sin t$

Know $e^{it} \neq 0$

$i(s+t)$ ... is it ... $f(s)f(t)$

Know e ...

$$f(s+t) = e^{i(s+t)} = e^{is} \cdot e^{it} = f(s) f(t)$$

Seen $\ln : \mathbb{R}_{>} \to \mathbb{R}$ and $\exp : \mathbb{R} \to \mathbb{R}_{>}$ are homo.

$\det : M_{n \times n}(\mathbb{R}) \to \mathbb{R}$ is not a homo: $\det(A+B) \neq \det(A) + \det B$ for some $A, B \in M_{n\times n}(\mathbb{R})$

group under addition.

Ex (i) $GL(n, K) \xrightarrow{f_1} Aff(n, K)$

$f_1(A) = f_{A,0}$, $f_{A,a} : K^n \to K^n$ defined by $f_{A,a}(x) = Ax + a$

is a homo

(ii) $f_2 : (K^n, +) \to (Aff(n, K), \circ)$ defined by $f_2(a) = f_{I_n, a}$ is a homo

Ex $C_{\mathbb{R}}[a,b]$ = the set of real valued and functions on $[a,b]$

... $C[a,b]$

Ex $C_{\mathbb{R}}[a,b]$ - [illegible] and functions [illegible]

$f, g \in C_{\mathbb{R}}[a,b]$ $\quad f+g \in C_{\mathbb{R}}[a,b]$

then $(C_{\mathbb{R}}[a,b], +)$ is an abelian group $e = 0$, $0(t) = 0 \ \forall t \in [a,b]$

$\int_a^b : C_{\mathbb{R}}[a,b] \to \mathbb{R}$ by

$$\left(\int_a^b\right)(f) = \int_a^b f(t)\,dt .$$

is a homo

Ex $\mathcal{D}(I) = \{ f : I \to \mathbb{R}$ diff on I, an open interval of $\mathbb{R}$

$f, g \in \mathcal{D}(I)$

$\Rightarrow f+g \in \mathcal{D}(I)$

$D : \mathcal{D}(I) \to \mathcal{F}(I, \mathbb{R}) = \{ g : I \to \mathbb{R}$ functions $\}$

$D(f) = f'$, D is a homo

$H_1, \ldots, H_n$ are groups

[illegible] group.

$H_1, \dots, H_n$ are g...

$H = H_1 \times H_2 \times \dots \times H_n$, group.

$f : G \longrightarrow H = H_1 \times \dots \times H_n$

$f = (f_1 \dots f_n)$, where $f_i : G \to H_i$

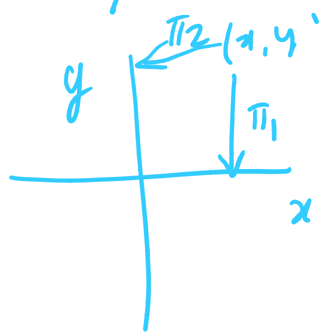
are homo $\Rightarrow$ f is homo.

$\pi_i : H_1 \times \dots \times H_n \longrightarrow H_i$

$\pi_i(h_1 \dots h_i \dots h_n) = h_i$, ith coordinate map

is a homo

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[dr, "f_i = \pi_i \circ f"'] & H = H_1 \times \dots \times H_n \arrow[d, "\pi_i"] \\ & H_i \end{tikzcd}$$



$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

Thm. G & G' are groups, $f : G \to G'$ homo
Then

(1) $f(e) = e'$, e' is id of G' and e is id of G

(2) $f(a^k) = f(a)^k$ for $k \in \mathbb{Z}$

(3) $g : G' \to G''$ homo of groups

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[dr] & G' \arrow[d, "g"] \\ & \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r] \arrow[dr, "g \circ f"'] & {} \arrow[d, "g"] \\ & G'' \end{tikzcd}$$

then $g \circ f : G \to G''$ is a homo

Proof (1) $f(e) = f(ee) = f(e) f(e)$

By uniqueness of e' we have $f(e) = e'$

(2) $k = 0$, it is (1)

$S = \{ n \in \mathbb{N} : f(a^n) = f(a)^n \}$

$1 \in S$ as $f(a) = f(a)$

Suppose $n \in S$. Now to show $n+1 \in S$: $f(a^{n+1}) = f(a)^{n+1}$

Now $f(a^{n+1}) = f(a^n a) = f(a^n) f(a)$ as f is homo

$= f(a)^n f(a)$ as $n \in S$

$= f(a)^{n+1} \Rightarrow n+1 \in S$

By induction $S = \mathbb{N}$

To show $f(a^{-1}) = f(a)^{-1}$. $a \in S$

Show $f(a^{-1}) = f(a)^{-1}$. $a \in S$

$e' = f(e) = f(a a^{-1}) = f(a) f(a^{-1})$ as f is homo

$= f(a^{-1}) f(a)$

$\Rightarrow f(a)^{-1} = f(a^{-1})$

$$\boxed{\begin{array}{l}\text{Then show } f(a^k) = f(a)^k \quad \forall k \in -\mathbb{N} \\ \text{where } -\mathbb{N} = \{-1, -2, -3, \ldots\} \\ ?\end{array}}$$

Def: $G \xrightarrow{f} G'$ groups, f is homo

$\ker f = \{a \in G : f(a) = e'\}$

is called the kernel of f.

Prop: $\ker f$ is a subgroup of G

$\ker f \leq G$

Proof (1) $e \in \ker f$

(2) $a, b \in \ker f \Rightarrow ab \in \ker f$

(2) $a, b \in \ker f \Rightarrow ab \in \ker f$

(3) $a \in \ker f \Rightarrow a^{-1} \in \ker f$.

Cor $a \in \ker f, \Rightarrow xax^{-1} \in \ker f$
$\forall x \in G$

Proof $a \in \ker f \Rightarrow f(a) = e'$

For $x \in G$, $f(xax^{-1}) = f(x) f(a) f(x)^{-1}$
$= f(x) e' f(x)^{-1}$
$= e'$

FACT: $G \xrightarrow{f} G'$ homo

$\ker f = \{e\} \iff f$ is injective