Jonathan Lam
Prof. Mintchev
MA 326
Lin. Alg.
9/9/19

PSET 1, P1

15/15

1	5
2	5
3	5
	15

5

Let $\mathbb{C} = \mathbb{R}^2$. Define $+_c, \cdot_c$ as follows:

<u>DEF:</u> $\forall (x_1, y_1), (x_2, y_2) \in \mathbb{C}$,

$$(x_1, y_1) +_c (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$
$$(x_1, y_1) \cdot_c (x_2, y_2) := (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2)$$

1.1. Prove $+_c, \cdot_c$ are binary on $\mathbb{C}$.

1 Look at (F4)!!

<u>CLAIM:</u> $+_c$ is binary on $\mathbb{C}$.

<u>PROOF:</u> $\forall c_1 = (x_1, y_1), c_2 = (x_2, y_2) \in \mathbb{C}$, $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Since $+$ is binary on $\mathbb{R}$, $x_1 + x_2, y_1 + y_2 \in \mathbb{R}$, $(x_1 + x_2, y_1 + y_2) = c_1 +_c c_2 \in \mathbb{C}$ $\therefore$ $+_c$ maps $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$, and is binary on $\mathbb{C}$.

<u>CLAIM:</u> $\cdot_c$ is binary on $\mathbb{C}$.

<u>PROOF:</u> $\forall c_1 = (x_1, y_1), c_2 = (x_2, y_2) \in \mathbb{C}$, $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Since $\cdot$, $+$ are binary on $\mathbb{R}$, $x_1x_2 - y_1y_2, x_1y_2 + y_1x_2 \in \mathbb{R}$, $(x_1x_2 - y_1y_2, x_1y_2 + y_1x_2) \in \mathbb{C}$ $\therefore$ $\cdot_c$ maps $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$, and is binary on $\mathbb{C}$.

The commutativity and associativity of the $+$ and $\cdot$ operators over the field $\mathbb{R}$ will be used implicitly in the following proofs.

1.2. Prove $(\mathbb{C}, +_c, \cdot_c)$ is a field.

F1). <u>CLAIM:</u> $\forall c_1, c_2 \in \mathbb{C}$, $c_1 +_c c_2 = c_2 +_c c_1$

<u>PROOF:</u> $\forall c_1 = (x_1, y_1), c_2 = (x_2, y_2) \in \mathbb{C}$,

$$c_1 +_c c_2 = (x_1 + x_2, y_1 + y_2) = (x_2 + x_1, y_2 + y_1) = c_2 +_c c_1$$

<u>CLAIM:</u> $\forall c_1, c_2 \in \mathbb{C}$, $c_1 \cdot_c c_2 = c_2 \cdot_c c_1$

<u>PROOF:</u> $\forall c_1 = (x_1, y_1), c_2 = (x_2, y_2) \in \mathbb{C}$,

$$\begin{aligned} c_1 \cdot_c c_2 &= (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2) \\ &= (x_2x_1 - y_2y_1, y_2x_1 + x_2y_1) \\ &= (x_2x_1 - y_2y_1, x_2y_1 + y_2x_1) = c_2 \cdot_c c_1 \end{aligned}$$

15/15

9/1

F2) CLAIM: $\forall c_1, c_2, c_3 \in \mathbb{C},\ c_1 +_c (c_2 +_c c_3) = (c_1 +_c c_2) +_c c_3$

PROOF: $\forall c_1 = (x_1, y_1),\ c_2 = (x_2, y_2),\ c_3 = (x_3, y_3)$,

$$c_1 +_c (c_2 +_c c_3) = (x_1, y_1) +_c (x_2 + x_3, y_2 + y_3)$$
$$= (x_1 + (x_2 + x_3), y_1 + (y_2 + y_3))$$
$$= ((x_1 + x_2) + x_3, (y_1 + y_2) + y_3)$$
$$= (x_1 + x_2, y_1 + y_2) +_c (x_3, y_3) = (c_1 +_c c_2) +_c c_3$$

CLAIM: $\forall c_1, c_2, c_3 \in \mathbb{C},\ c_1 \cdot_c (c_2 \cdot_c c_3) = (c_1 \cdot_c c_2) \cdot_c c_3$

PROOF: $\forall c_1 = (x_1, y_1),\ c_2 = (x_2, y_2),\ c_3 = (x_3, y_3)$,

$$c_1 \cdot_c (c_2 \cdot_c c_3) = (x_1, y_1) \cdot_c (x_2x_3 - y_2y_3,\ x_2y_3 + y_2x_3)$$
$$= (x_1(x_2x_3 - y_2y_3) - y_1(x_2y_3 + y_2x_3),\ x_1(x_2y_3 + y_2x_3) + y_1(x_2x_3 - y_2y_3))$$
$$= (x_1x_2x_3 - x_1y_2y_3 - y_1x_2y_3 - y_1y_2x_3,\ x_1x_2y_3 + x_1y_2x_3 + y_1x_2x_3 - y_1y_2y_3)$$
$$= ((x_1x_2 - y_1y_2)x_3 - (x_1y_2 + y_1x_2)y_3,\ (x_1x_2 - y_1y_2)y_3 + (x_1y_2 + y_1x_2)x_3)$$
$$= (x_1x_2 - y_1y_2,\ x_1y_2 + y_1x_2) \cdot_c (x_3, y_3) = (c_1 \cdot_c c_2) \cdot_c c_3$$

F3.) CLAIM: $\exists\ 0_c \in \mathbb{C}$ s.t. $\forall c \in \mathbb{C},\ 0_c +_c c = c$

PROOF: Let $0_c = (0, 0) \in \mathbb{C}$. $\forall c = (x, y) \in \mathbb{C}$,

$$0_c +_c c = (0 + x, 0 + y) = (x, y) = c.$$

CLAIM: $\exists\ 1_c \in \mathbb{C}$ s.t. $\forall c \in \mathbb{C},\ 1_c \cdot_c c = c$

PROOF: Let $1_c = (1, 0) \in \mathbb{C}$. $\forall c = (x, y) \in \mathbb{C}$,

$$1_c \cdot_c c = (1 \cdot x - 0 \cdot y,\ 1 \cdot y + 0 \cdot x) = (x - 0, y + 0) = (x, y) = c$$

F4.) CLAIM: $\exists\ c_1 \in \mathbb{C},\ \exists\ c_2 \in \mathbb{C}$ s.t. $c_1 +_c c_2 = 0_c$.

PROOF: $\forall c_1 = (x, y) \in \mathbb{C},\ \exists\ c_2 = (-x, -y) \in \mathbb{C}$, and

$$c_1 +_c c_2 = (x + (-x), y + (-y)) = (0, 0) = 0_c.$$

Note that $c_2 \in \mathbb{C}$ since $-x, -y \in \mathbb{R}$ by (F4) of $\mathbb{R}$.

CLAIM: $\exists\ c_1 \in \mathbb{C},\ c_1 \neq 0_c,\ \exists\ c_2 \in \mathbb{C}$ s.t. $c_1 \cdot_c c_2 = 1_c$.

PROOF: $\forall c_1 = (x, y) \in \mathbb{C},\ \exists\ c_2 = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right) \in \mathbb{C}$, and

$$c_1 \cdot_c c_2 = \left(x\left(\frac{x}{x^2+y^2}\right) - y\left(\frac{-y}{x^2+y^2}\right),\ x\left(\frac{-y}{x^2+y^2}\right) + \left(\frac{x}{x^2+y^2}\right)y\right)$$
$$= \left(\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2},\ \frac{-xy}{x^2+y^2} + \frac{xy}{x^2+y^2}\right) = \left(\frac{x^2+y^2}{x^2+y^2},\ \frac{-xy+xy}{x^2+y^2}\right)$$
$$= \left((x^2+y^2)(x^2+y^2)^{-1},\ 0 \cdot (x^2+y^2)^{-1}\right) = (1, 0) = 1_c.$$

Note that $c_2 \in \mathbb{C}$ unless $c_1 = (0, 0)$, in which case the multiplicative inverse of $0_{\mathbb{R}}$ is taken.

? You should emphasize $x^2+y^2 \neq 0$!

Jonathan Lam
Prof. Mintchev
MA326
Lin. Alg.
9/10/19

PSET 1, P1, cont'd.

F5) <u>CLAIM</u>: $\forall c_1, c_2, c_3 \in \mathbb{C},\ c_1 \cdot_c (c_2 +_c c_3) = c_1 \cdot_c c_2 +_c c_1 \cdot_c c_3$.

<u>PROOF</u>: $\forall c_1 = (x_1, y_1),\ c_2 = (x_2, y_2),\ c_3 = (x_3, y_3) \in \mathbb{C}$,

$$\begin{aligned} c_1 \cdot_c (c_2 +_c c_3) &= (x_1, y_1) \cdot_c (x_2 + x_3, y_2 + y_3) \\ &= (x_1(x_2+x_3) - y_1(y_2+y_3), x_1(y_2+y_3) + (x_2+x_3)y_1) \\ &= (x_1x_2 + x_1x_3 - y_1y_2 - y_1y_3,\ x_1y_2 + x_1y_3 + x_2y_1 + x_3y_1) \\ &= ((x_1x_2 - y_1y_2) + (x_1x_3 - y_1y_3), (x_1y_2 + x_2y_1) + (x_1y_3 + x_3y_1)) \\ &= (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1) +_c (x_1x_3 - y_1y_3, x_1y_3 + x_3y_1) \\ &= c_1 \cdot_c c_2 +_c c_1 \cdot_c c_3 \end{aligned}$$

since (F1–F5) are satisfied, $\cdot_c$, $+_c$ are binary, $\mathbb{C}$ is a field.

P2. Let $\mathbb{Q}[\sqrt{2}] = \{ a + b\sqrt{2} \mid a, b \in \mathbb{Q} \}$

2.1. 5 Show that $+, \cdot$ from $\mathbb{R}$ are binary on $\mathbb{Q}[\sqrt{2}]$.

<u>CLAIM</u>: $+$ from $\mathbb{R}$ is binary on $\mathbb{Q}[\sqrt{2}]$.

<u>PROOF</u>: $\forall q_1 = a_1 + b_1\sqrt{2},\ q_2 = a_2 + b_2\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$,

$$\begin{aligned} q_1 + q_2 &= (a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2}) \\ &= (a_1 + a_2) + (b_1 + b_2)\sqrt{2} \end{aligned}$$

Mind (F4)!

Since $a_1, b_1, a_2, b_2 \in \mathbb{Q}$ and $+$ is binary on $\mathbb{Q}$, $a_1 + a_2$, $b_1 + b_2 \in \mathbb{Q}$, $(a_1 + a_2) + (b_1 + b_2)\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$, thus $+$ maps $\mathbb{Q}[\sqrt{2}] \times \mathbb{Q}[\sqrt{2}] \to \mathbb{Q}[\sqrt{2}]$ and thus is binary on $\mathbb{Q}[\sqrt{2}]$.

<u>CLAIM</u>: $\forall q_1 = a_1 + b_1\sqrt{2},\ q_2 = a_2 + b_2\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$,

$$\begin{aligned} q_1 \cdot q_2 &= (a_1 + b_1\sqrt{2})(a_2 + b_2\sqrt{2}) = a_1a_2 + a_1b_2\sqrt{2} + b_1a_2\sqrt{2} + b_1b_2\sqrt{2}\sqrt{2} \\ &= a_1a_2 + b_1b_2\sqrt{2}\sqrt{2} + a_1b_2\sqrt{2} + b_1a_2\sqrt{2} = (a_1a_2 + 2b_1b_2) + (a_1b_2 + b_1a_2)\sqrt{2}. \end{aligned}$$

Since $a_1, b_1, a_2, b_2 \in \mathbb{Q}$ and $+, \cdot$ is binary on $\mathbb{Q}$, $(a_1a_2 + 2b_1b_2) + (a_1b_2 + b_1a_2)\sqrt{2} \in \mathbb{Q}\sqrt{2}$, thus $\cdot$ maps $\mathbb{Q}[\sqrt{2}] \times \mathbb{Q}[\sqrt{2}] \to \mathbb{Q}[\sqrt{2}]$ and thus is binary on $\mathbb{Q}[\sqrt{2}]$.

2.2. Show that $(\mathbb{Q}[\sqrt{2}], +, \cdot)$ is a field.

F1) $\forall q_1, q_2 \in \mathbb{Q}[\sqrt{2}],\ q_1 + q_2 = q_2 + q_1$

<u>PROOF</u>: $\forall q_1 = a_1 + b_1\sqrt{2},\ q_2 = a_2 + b_2\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$,

$$q_1 + q_2 = (a_1 + a_2) + (b_1 + b_2)\sqrt{2} = (a_2 + a_1) + (b_2 + b_1)\sqrt{2} = q_2 + q_1$$

9/1

CLAIM: $\forall q_1, q_2 \in \mathbb{Q}[\sqrt{2}]$, $q_1 \cdot q_2 = q_2 \cdot q_1$

PROOF: $\forall q_1 = a_1 + b_1\sqrt{2}$, $q_2 = a_2 + b_2\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$,

$$q_1 \cdot q_2 = (a_1a_2 + 2b_1b_2) + (a_1b_2 + b_1a_2)\sqrt{2}$$
$$= (a_2a_1 + 2b_2b_1) + (b_1a_2 + a_1b_2)\sqrt{2}$$
$$= (a_2a_1 + 2b_2b_1) + (a_2b_1 + b_2a_1)\sqrt{2} = q_2 \cdot q_1$$

F2) CLAIM: $\forall q_1, q_2, q_3 \in \mathbb{Q}[\sqrt{2}]$, $q_1 + (q_2 + q_3) = (q_1 + q_2) + q_3$

PROOF: $\forall q_1 = a_1 + b_1\sqrt{2}$, $q_2 = a_2 + b_2\sqrt{2}$, $q_3 = a_3 + b_3\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$,

$$q_1 + (q_2 + q_3) = (a_1 + b_1\sqrt{2}) + ((a_2 + a_3) + (b_2 + b_3)\sqrt{2})$$
$$= (a_1 + (a_2 + a_3)) + (b_1 + (b_2 + b_3))\sqrt{2}$$
$$= ((a_1 + a_2) + a_3) + ((b_1 + b_2) + b_3)\sqrt{2}$$
$$= ((a_1 + a_2) + (b_1 + b_2)\sqrt{2}) + (a_3 + b_3\sqrt{2})$$
$$= ((a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2})) + (a_3 + b_3\sqrt{2})$$
$$= (q_1 + q_2) + q_3$$

CLAIM: $\forall q_1, q_2, q_3 \in \mathbb{Q}[\sqrt{2}]$, $q_1 \cdot (q_2 \cdot q_3) = (q_1 \cdot q_2) \cdot q_3$

PROOF: $\forall q_1 = a_1 + b_1\sqrt{2}$, $q_2 = a_2 + b_2\sqrt{2}$, $q_3 = a_3 + b_3\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$,

$$q_1 \cdot (q_2 \cdot q_3) = (a_1 + b_1\sqrt{2}) \cdot ((a_2a_3 + 2b_2b_3) + (a_2b_3 + b_2a_3)\sqrt{2})$$
$$= (a_1(a_2a_3 + 2b_2b_3) + 2b_1(a_2b_3 + b_2a_3),$$
$$a_1(a_2b_3 + b_2a_3) + b_1(a_2a_3 + 2b_2b_3))$$
$$= (a_1a_2a_3 + 2a_1b_2b_3 + 2b_1a_2b_3 + 2b_1b_2a_3) +$$
$$(a_1a_2b_3 + a_1b_2a_3 + b_1a_2a_3 + 2b_1b_2b_3)\sqrt{2}$$
$$= ((a_1a_2 + 2b_1b_2)a_3 + 2(a_1b_2 + b_1a_2)b_3) +$$
$$((a_1a_2 + 2b_1b_2)b_3 + (a_1b_2 + b_1a_2)a_3)\sqrt{2}$$
$$= ((a_1a_2 + 2b_1b_2) + (a_1b_2 + b_1a_2)\sqrt{2}) \cdot (a_3 + b_3\sqrt{2})$$
$$= ((a_1 + b_1\sqrt{2}) \cdot (a_2 + b_2\sqrt{2})) \cdot (a_3 + b_3\sqrt{2})$$
$$= (q_1 \cdot q_2) \cdot q_3$$

PSET 1 P2, cont'd.

Jonathan Lam
Prof. Mintchev
Lin. Alg.
MA 326
9/11/19

F3) CLAIM: $\exists\, 0 \in \mathbb{Q}[\sqrt{2}]$ s.t. $\forall\, q \in \mathbb{Q}[\sqrt{2}]$, $q + 0 = q$

PROOF: Let $0 = 0 + 0\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$. $\forall q = a + b\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$,

$q + 0 = (a+0) + (b+0)\cdot\sqrt{2} = a + b\sqrt{2} = q$

CLAIM: $\exists\, 1 \in \mathbb{Q}[\sqrt{2}]$ s.t. $\forall\, q \in \mathbb{Q}[\sqrt{2}]$, $q \cdot 1 = q$.

PROOF: Let $1 = 1 + 0\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$. $\forall q = a + b\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$,

$q \cdot 1 = (a(1) + 2b(0)) + (a(0) + b(1))\sqrt{2}$
$= (a+0) + (0+b)\sqrt{2} = a + b\sqrt{2} = q$

F4) CLAIM: $\forall\, q_1 \in \mathbb{Q}[\sqrt{2}]$, $\exists\, q_2 \in \mathbb{Q}[\sqrt{2}]$ s.t. $q_1 + q_2 = 0_n$

PROOF: $\forall q_1 = a + b\sqrt{2}$. Let $q_2 = (-a) + (-b)\sqrt{2}$.

$q_1 + q_2 = (a + (-a)) + (b + (-b))\sqrt{2} = 0 + 0\sqrt{2} = 0_n$

CLAIM: $\forall\, q_1 \in \mathbb{Q}[\sqrt{2}]$, $q_1 \neq 0$, $\exists\, q_2 \in \mathbb{Q}[\sqrt{2}]$ s.t. $q_1 \cdot q_2 = 1_q$

PROOF: $\forall q_1 = a + b\sqrt{2}$, $q_1 \neq 0$. Let $q_2 = \left(\frac{a}{a^2-2b^2}\right) + \left(\frac{-b}{a^2-2b^2}\right)\sqrt{2}$.

$q_1 \cdot q_2 = \left(a\left(\frac{a}{a^2-2b^2}\right) + 2b\left(\frac{-b}{a^2-2b^2}\right)\right) + \left(a\left(\frac{-b}{a^2-2b^2}\right) + b\left(\frac{a}{a^2-2b^2}\right)\right)\sqrt{2}$
$= \left(\frac{a^2}{a^2-2b^2} + 2\frac{-b^2}{a^2-2b^2}\right) + \left(\frac{-ab}{a^2-2b^2} + \frac{ab}{a^2-2b^2}\right)\sqrt{2}$
$= \left(\frac{a^2-2b^2}{a^2-2b^2}\right) + \left(\frac{-ab+ab}{a^2-2b^2}\right)\sqrt{2} = 1 + 0\sqrt{2} = 1_q$

Note that $q_2 \in \mathbb{Q}[\sqrt{2}]$ exists unless $q_1 = 0$ (in which $(a^2-2b^2)^{-1} = 0^{-1}$ is not allowed). ? huh?

F5) CLAIM: $\forall\, q_1, q_2, q_3 \in \mathbb{Q}[\sqrt{2}]$, $q_1 \cdot (q_2 + q_3) = q_1 \cdot q_2 + q_1 \cdot q_3$.

PROOF: $\forall q_1 = a_1 + b_1\sqrt{2}$, $q_2 = a_2 + b_2\sqrt{2}$, $q_3 = a_3 + b_3\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$.

$q_1 \cdot (q_2 + q_3) = (a_1 + b_1\sqrt{2}) \cdot ((a_2 + a_3) + (b_2 + b_3)\sqrt{2})$
$= (a_1(a_2 + a_3) + b_1(b_2 + b_3)) + (a_1(b_2 + b_3) + (a_2 + a_3)b_1)\sqrt{2}$
$= (a_1a_2 + a_1a_3 + b_1b_2 + b_1b_3) + (a_1b_2 + a_1b_3 + a_2b_1 + a_3b_1)\sqrt{2}$
$= ((a_1a_2 + b_1b_2) + (a_1a_3 + b_1b_3)) + ((a_1b_2 + a_2b_1) + (a_1b_3 + a_3b_1))\sqrt{2}$
$= ((a_1a_2 + b_1b_2) + (a_1b_2 + a_2b_1)\sqrt{2}) + ((a_1a_3 + b_1b_3) + (a_1b_3 + a_3b_1))\sqrt{2}$
$= ((a_1 + b_1\sqrt{2}) \cdot (a_2 + b_2\sqrt{2})) + ((a_1 + b_1\sqrt{2}) \cdot (a_3 + b_3\sqrt{2}))$
$= q_1 \cdot q_2 + q_1 \cdot q_3$

Since $+, \cdot$ are binary on $\mathbb{Q}[\sqrt{2}]$, and (F1-F5) are satisfied, $(\mathbb{Q}[\sqrt{2}], +, \cdot)$ is a field.

Jonathan Lam
Prof. Mintchev
MA 326.
Lin. Alg.
9/10/19

PSET 1, P3.

DEF: Given $n \in \mathbb{Z}^+$, $\mathbb{Z}_n := \{0, 1, 2, \ldots, n-1\}$

DEF: Given $a \in \mathbb{Z}$, $n \in \mathbb{Z}^+$, $a \pmod{n} := r$ s.t. $a = m \cdot n + r$, $0 \le r < n$, $m, r \in \mathbb{Z}$. (This is a result of the division algorithm, which also states that m, r are uniquely defined for a particular a, n.)

DEF: $\forall\, a, b \in \mathbb{Z}_n$,
$a +_n b := (a+b) \pmod{n}$,
$a \cdot_n b := (a \cdot b) \pmod{n}$
(where $+$, $\cdot$ are addition and multiplication of the integers, which is known to be commutative, associative, and binary)

LEM 1: $\forall\, a \in \mathbb{Z}$, $n \in \mathbb{Z}^+$, $a \pmod{n} \in \mathbb{Z}_n$.
By (DEF $a \pmod{n}$) and the division algorithm, $a \pmod{n} = r$, where $r \in \mathbb{Z}$ and $0 \le r < n$, thus $a \pmod{n} = r \in \mathbb{Z}_n$.

CLAIM: $+_n$ is binary on $\mathbb{Z}_n$.
$\forall\, a, b \in \mathbb{Z}_n$, $a, b \in \mathbb{Z}$, and $a+b \in \mathbb{Z}$ since $+$ is binary on $\mathbb{Z}$. By (LEM 1), $((a+b) \in \mathbb{Z}) \pmod{n} \in \mathbb{Z}_n$, so $+_n$ maps $\mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_n$, and is thus binary on $\mathbb{Z}_n$.

CLAIM: $\cdot_n$ is binary on $\mathbb{Z}_n$.
$\forall\, a, b \in \mathbb{Z}_n$, $a, b \in \mathbb{Z}$, and $a \cdot b \in \mathbb{Z}$ since $\cdot$ is binary on $\mathbb{Z}$. By (LEM 1), $((a \cdot b) \in \mathbb{Z}) \pmod{n} \in \mathbb{Z}_n$, so $\cdot_n$ maps $\mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_n$, and is thus binary on $\mathbb{Z}_n$.

3.1. Prove: $n \in \mathbb{Z}^+$ is prime $\Rightarrow \mathbb{Z}_n$ is a field
(binary-ness of $+_n$, $\cdot_n$ already proven above, need to prove the 5 field axioms)

9/10

F1.) CLAIM: $\forall\ a, b \in \mathbb{Z}_n,\ a +_n b = b +_n a$.

PROOF: $\forall a, b \in \mathbb{Z}_n$,

$$a +_n b = (a+b) \pmod n = (b+a) \pmod n = b +_n a$$

CLAIM: $\forall\ a, b \in \mathbb{Z}_n,\ a \cdot_n b = b \cdot_n a$

PROOF: $\forall a, b \in \mathbb{Z}_n$,

$$a \cdot_n b = (a \cdot b) \pmod n = (b \cdot a) \pmod n = b \cdot_n a$$

LEM 2: $\forall\ a \in \mathbb{Z}_n,\ n \in \mathbb{Z}^+,\ a \pmod n = a$

PROOF: Since $a \in \mathbb{Z}_n$, $0 \le a < n$, so $0 \le a - 0 \cdot n = r < n$,

$\Rightarrow 0 \le a = r < n$, so by (DEF $a \pmod n$), $a \pmod n = r = a$.

LEM 3: $\forall a, b \in \mathbb{Z},\ n \in \mathbb{Z}^+,\ (a \pmod n \mathbin{\dot{+}} b \pmod n) \pmod n = (a \mathbin{\dot{+}} b) \pmod n$

($\dot{+}$ means this works both for $+$ and $\cdot$)

PROOF ($+$): Given $a, b \in \mathbb{Z}$, $n \in \mathbb{Z}^+$. By the division algorithm,

$\exists\ m_1, m_2, m_3, r_1, r_2, r_3 \in \mathbb{Z}$ s.t.

$0 \le a - m_1 \cdot n = r_1 = a \pmod n < n$,

$0 \le b - m_2 \cdot n = r_2 = b \pmod n < n$,

$0 \le (a \pmod n + b \pmod n) - m_3 n = (a \pmod n + b \pmod n) \pmod n$

$= a - m_1 \cdot n + b - m_2 \cdot n - m_3 \cdot n = (a+b) - (m_1 + m_2 + m_3) \cdot n = r_3 < n$.

Good

Since $a+b,\ m_1 + m_2 + m_3 \in \mathbb{Z}$, and by (DEF $(a+b) \pmod n$),

$(a \pmod n + b \pmod n) \pmod n = r_3 = (a+b) \pmod n$

PROOF ($\cdot$): Given $a, b \in \mathbb{Z}$, $n \in \mathbb{Z}^+$. By the division algorithm,

$\exists\ m_1, m_2, m_3, r_1, r_2, r_3 \in \mathbb{Z}$ s.t.

$0 \le a - m_1 \cdot n = r_1 = a \pmod n < n$,

$0 \le b - m_2 \cdot n = r_2 = b \pmod n < n$,

$0 \le (a \pmod n \cdot b \pmod n) - m_3 n = (a \pmod n \cdot b (\mathrm{mod})) \pmod n$

$= (a - m_1 \cdot n) \cdot (b - m_2 \cdot n) - m_3 \cdot n = ab - a m_2 n - b m_1 n + m_1 m_2 n - m_3 n$

$= (ab) - (a m_2 + b m_1 - m_1 m_2 + m_3) \cdot n = r_3 < n$

Since $a \cdot b,\ a \cdot m_2 + b \cdot m_1 - m_1 \cdot m_2 + m_3 \in \mathbb{Z}$, by (DEF $(a \cdot b) \pmod n$),

$(a \pmod n \cdot b \pmod n) \pmod n = r_3 = (a \cdot b) \pmod n$

Jonathan Lam

PSET 1, P3, cont'd.

Jonathan Lam
Prof. Mutchnik
MA 326
Lin. Alg.
9/11/19

F2) **CLAIM:** $\forall a, b, c \in \mathbb{Z}_n$, $a +_n (b +_n c) = (a +_n b) +_n c$

PROOF: $\forall a, b, c \in \mathbb{Z}_n$,

$$a +_n (b +_n c) = a +_n (b+c) \pmod n = (a + (b+c) \pmod n) \pmod n$$
$$= (a \pmod n + (b+c) \pmod n) \pmod n \quad \text{(LEM 2)}$$
$$= (a + (b+c)) \pmod n \quad \text{(LEM 3)}$$
$$= ((a+b) + c) \pmod n$$
$$= ((a+b) \pmod n + c \pmod n) \pmod n \quad \text{(LEM 3)}$$
$$= ((a+b) \pmod n + c) \pmod n \quad \text{(LEM 2)}$$
$$= (a+b) \pmod n +_n c = (a +_n b) +_n c$$

CLAIM: $\forall a, b, c \in \mathbb{Z}_n$, $a \cdot_n (b \cdot_n c) = (a \cdot_n b) \cdot_n c$

PROOF: $\forall a, b, c \in \mathbb{Z}_n$,

$$a \cdot_n (b \cdot_n c) = a \cdot_n (b \cdot c) \pmod n = (a \cdot (b \cdot c) \pmod n) \pmod n$$
$$= (a \pmod n \cdot (b \cdot c) \pmod n) \pmod n \quad \text{(LEM 2)}$$
$$= (a \cdot (b \cdot c)) \pmod n = ((a \cdot b) \cdot c) \pmod n \quad \text{(LEM 3)}$$
$$= ((a \cdot b) \pmod n \cdot c \pmod n) \pmod n \quad \text{(LEM 3)}$$
$$= ((a \cdot b) \pmod n \cdot c) \pmod n \quad \text{(LEM 2)}$$
$$= (a \cdot b) \pmod n \cdot_n c = (a \cdot_n b) \cdot_n c$$

F3) **CLAIM:** $\exists 0_n \in \mathbb{Z}_n$ s.t. $\forall a \in \mathbb{Z}_n$, $a +_n 0_n = a$.

PROOF: Let $0_n = 0_{\mathbb{Z}} \in \mathbb{Z}_n$. $\forall a \in \mathbb{Z}_n$,

$$a +_n 0_n = (a + 0) \pmod n = a \pmod n = a \quad \text{(LEM 2)}$$

CLAIM: $\exists 1_n \in \mathbb{Z}_n$ s.t. $\forall a \in \mathbb{Z}_n$, $a \cdot_n 1_n = a$.

PROOF: Let $1_n = 1_{\mathbb{Z}} \in \mathbb{Z}_n$. $\forall a \in \mathbb{Z}_n$,

$$a \cdot_n 1_n = (a \cdot 1) \pmod n = a \pmod n = a \quad \text{(LEM 2)}$$

F4). **CLAIM:** $\forall a \in \mathbb{Z}_n$, $\exists b \in \mathbb{Z}_n$ s.t. $a +_n b = 0_n$.

PROOF: Given $a \in \mathbb{Z}_n$, let $b = (n-a) \pmod n$.
By (LEM 1), $b \in \mathbb{Z}_n$.

$$a +_n b = (a + (n-a) \pmod n) \pmod n$$
$$= (a \pmod n + (n-a) \pmod n) \pmod n \quad \text{(LEM 2)}$$
$$= (a + (n-a)) \pmod n = (n + (a + -a)) \pmod n \quad \text{(LEM 3)}$$
$$= (n + 0) \pmod n = n \pmod n = 0.$$

9/11/19

(recall that n is prime is given)

CLAIM: $\forall a \in \mathbb{Z}_n$, $a \neq 0_n$, $\exists\ b \in \mathbb{Z}_n$ s.t. $a \cdot_n b = 1_n$.

PROOF: $a \equiv_n 0 \iff \exists\, m \in \mathbb{R}$ s.t. $a - m \cdot n = 0$

Since $a = a \pmod n = a - m \cdot n \neq 0$, $a \not\equiv_n 0$, $a \in \mathbb{Z}$, n prime, by Fermat's Little Theorem, $a^{n-1} \equiv_n 1 \Rightarrow a^{n-1} \pmod n = 1$.

Let $b = (a^{n-2}) \pmod n$. $b \in \mathbb{Z}_n$ by (LEM 1).

$$\begin{aligned} a \cdot_n b &= (a \cdot (a^{n-2}) \pmod n) \pmod n \\ &= (a \pmod n) \cdot (a^{n-2}) \pmod n) \pmod n && \text{(LEM 2)} \\ &= (a \cdot a^{n-2}) \pmod n && \text{(LEM 3)} \\ &= (a^{n-1}) \pmod n = 1 && \text{(FLT)}. \end{aligned}$$

Note that b exists and this identity holds for $a \neq 0$ (because of FLT).

F5) CLAIM: $\forall a, b, c \in \mathbb{Z}_n$, $a \cdot_n (b +_n c) = a \cdot_n b +_n a \cdot_n b$

PROOF: $a \cdot_n (b+c) \pmod n = (a \cdot (b+c) \pmod n) \pmod n$

$$\begin{aligned} &= (a \pmod n) \cdot (b+c) \pmod n) \pmod n && \text{(LEM 2)} \\ &= (a \cdot (b+c)) \pmod n = (a \cdot b + a \cdot c) \pmod n && \text{(LEM 3)} \\ &= ((a \cdot b) \pmod n + (a \cdot c) \pmod n) \pmod n && \text{(LEM 3)} \\ &= (a \cdot b) \pmod n +_n (a \cdot c) \pmod n = a \cdot_n b +_n a \cdot_n c. \end{aligned}$$

Since $+_n$, $\cdot_n$ binary and (F1–F5) are satisfied, $(\mathbb{C}, +_n, \cdot_n)$ is a field.

3.2. Prove: $\mathbb{Z}_n$ field $\Rightarrow$ n is prime.

LEM 4: Given a field F, if $a \cdot_F b = 0_F$, $a, b \in F$, then $a = 0_F$ or $b = 0_F$.

PROOF: If $a = 0$, proof complete. If not, then a^{-1} exists and $\in F$.

$a \cdot_F b = 0_F \Rightarrow (a^{-1}) \cdot_F a \cdot_F b = (a^{-1}) \cdot_F 0_F \Rightarrow 1_F \cdot b = 0_F \Rightarrow b = 0_F$.

CLAIM: If n is composite, then $\mathbb{Z}_n$ is not a field.

PROOF: If n composite, $\exists\ a > 0, b > 0 \in \mathbb{Z}_n$ s.t. $a \cdot b = n$, therefore $a \cdot_n b = (a \cdot b) \pmod n = n \pmod n = 0_n$.

Since $a \cdot_n b = 0$ and $a \neq 0$, $b \neq 0$, (LEM 4) (which is a property of all fields) does not hold, and therefore $\mathbb{Z}_n$ is not a field.

Jonathan Lam
Prof. Mintchev
MA 326
Lin. Alg.
9/19/19

PSET 2

(ASSUMPTION OF VARIOUS PROPERTIES OF exp, ln FUNCTIONS)

27/30

1.3.23 | 5
1.4.13 | 5
10

1. Let $V = \mathbb{R}^+ = (0, +\infty)$, $F = \mathbb{R}$.

DEF: Given $x, y \in V$, $x +_v y := xy$ (real multiplication)

DEF: Given $a \in F$, $x \in V$, $a \cdot_v x := x^a = \exp(a \ln(x))$ (real exponentiation)

Show that $(V, +_v, \cdot_v)$ is a v.s. over $\mathbb{R}$.

CLAIM: $+_v$ is binary over V.

PROOF: $\forall x, y \in V$, $x, y \in \mathbb{R}$, and $xy \in \mathbb{R}$ (since $\cdot$ is binary over $\mathbb{R}$). Also, since $x, y > 0$, $xy > 0$ (two positive reals multiply to a positive real), so $xy \in \mathbb{R}^+$. Then $+_v$ maps $V \times V \to V$ and thus is binary over V.

CLAIM: $\cdot_v$ is external binary on V with field F.

PROOF: $\forall a \in F$, $x \in V$. $a \cdot_v x = \exp(a \ln(x))$. $\ln(x)$ exists and $\in \mathbb{R}$ $\forall x \in \mathbb{R}^+ = V$, so $a \ln(x) \in \mathbb{R}$ (since $\cdot$ is binary over $\mathbb{R}$). $\forall c \in \mathbb{R}$, $\exp(c) \in \mathbb{R}^+$ (property of exp), thus $\exp(a \ln(x)) \in \mathbb{R}^+ = V$, i.e., $\cdot_v$ maps $F \times V \to V$ and thus is external binary on V with field F.

VS1) CLAIM: $\forall x, y \in V$, $x +_v y = y +_v x$.

PROOF: $\forall x, y \in V$,

$$x +_v y = \underbrace{xy = yx}_{\text{commutativity of } \cdot \text{ over } \mathbb{R}} = y +_v x$$

VS2) CLAIM: $\forall x, y, z \in V$, $x +_v (y +_v z) = (x +_v y) +_v z$

PROOF: $\forall x, y, z \in V$,

$$x +_v (y +_v z) = x +_v (yz) = \underbrace{x(yz) = (xy)z}_{\text{associativity of } \cdot \text{ over } \mathbb{R}} = (xy) +_v z = (x +_v y) +_v z$$

VS3) CLAIM: $\exists 0_v \in V$ s.t. $\forall x \in V$, $0_v +_v x = x$

PROOF: Let $0_v = 1_{(\mathbb{R})}$. Then, $\forall x \in V$,

$$0_v +_v x = \underbrace{1x = x}_{(F3) \text{ on } \mathbb{R}}$$

VS4) CLAIM: $\forall x \in V$, $\exists y \in V$ s.t. $x +_v y = 0_v$.

PROOF: Let $x \in V = \mathbb{R}^+$. Then $\exists\, y = \frac{1}{x}$. Since $x \neq 0 \in \mathbb{R}$, $y \in \mathbb{R}$, and since $1, x > 0$, $y = \frac{1}{x} > 0 \in \mathbb{R}^+ = V$, and:

$$x +_v y = x\left(\tfrac{1}{x}\right) = 1_{(\mathbb{R})} = 0_v$$

VS5) CLAIM: $\forall x \in V$, $1 \cdot_v x = x$

PROOF: $\forall x \in V$:

$$\begin{aligned} 1 \cdot_v x &= \exp(1 \cdot \ln x) \\ &= \exp(\ln x) && \text{(F3 of } \mathbb{R}) \\ &= x && (\exp = \ln^{-1}) \end{aligned}$$

VS 6) CLAIM: $\forall a, b \in F$, $\forall x \in V$, $(ab) \cdot_v x = a \cdot_v (b \cdot_v x)$

PROOF: $\forall a, b \in F$, $\forall x \in V$,

$$\begin{aligned} (ab) \cdot_v x &= \exp(ab \ln(x)) = \exp(a(b \ln x)) \\ &= \exp(a \ln(\exp(b \ln x))) && (\exp = \ln^{-1}) \\ &= \exp(a \ln(b \cdot_v x)) \\ &= a \cdot_v (b \cdot_v x) \end{aligned}$$

VS7) CLAIM: $\forall a \in F$, $x, y \in V$, $a \cdot_v (x +_v y) = a \cdot_v x +_v a \cdot_v y$.

PROOF: $\forall a \in F$, $x, y \in V$,

$$\begin{aligned} a \cdot_v (x +_v y) &= a \cdot_v (xy) = \exp(a \ln(xy)) \\ &= \exp(a(\ln x + \ln y)) && \text{(prop. of } \ln) \\ &= \exp(a \ln x + a \ln y) \\ &= \exp(a \ln x) \cdot \exp(a \ln y) && \text{(prop. of exp)} \\ &= (a \cdot_v x) \cdot (a \cdot_v y) = a \cdot_v x +_v a \cdot_v y. \end{aligned}$$

VS8) CLAIM: $\forall a, b \in F$, $x \in V$, $(a+b) \cdot_v x = a \cdot_v x +_v b \cdot_v x$.

PROOF: $\forall a, b \in F$, $x \in V$,

$$\begin{aligned} (a+b) \cdot_v x &= \exp((a+b)\ln(x)) = \exp(a \ln x + b \ln x) \\ &= \exp(a \ln(x)) \cdot \exp(b \ln(x)) && \text{(prop. of exp)} \\ &= (a \cdot_v x) \cdot (b \cdot_v x) = a \cdot_v x +_v b \cdot_v x \end{aligned}$$

Since $+_v$ is binary on V, $\cdot_v$ is external binary on V with field F, and V satisfies (VS1–8), $(V, +_v, \cdot_v)$ is a vector space.

Jonathan Lam
Prof. Mintchev
MA 326
Lin. Alg.
9/19/19

PSET 2, cont'd

2. DEF: $C(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous on } \mathbb{R} \}$

DEF: $C^{(n)}(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{R} \mid f^{(n)} \text{ is continuous on } \mathbb{R} \}$

a. Show that $C(\mathbb{R})$ is a v.s. over $\mathbb{R}$.

CLAIM: $C(\mathbb{R})$ SUBSP. $\mathcal{F}(\mathbb{R}, \mathbb{R})$

The following proofs will prove that $C(\mathbb{R})$ SUBSP. $\mathcal{F}(\mathbb{R}, \mathbb{R})$ using (THM 1.3), the necessary conditions for a subspace.

CLAIM: $C(\mathbb{R}) \subseteq \mathcal{F}(\mathbb{R}, \mathbb{R})$

PROOF. $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is the set of all functions $f : \mathbb{R} \to \mathbb{R}$ over the domain $\mathbb{R}$. $C(\mathbb{R})$ is a set of some functions $f : \mathbb{R} \to \mathbb{R}$ that exist (and are continuous) over the domain $\mathbb{R}$, thus $C(\mathbb{R}) \subseteq \mathcal{F}(\mathbb{R}, \mathbb{R})$.

DEF: $z := z(x) = 0 \quad \forall\, x \in \mathbb{R}$

(Note that this is the additive identity for $\mathcal{F}(\mathbb{R}, \mathbb{R})$, since $\forall\, f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$, $f + z = f(x) + z(x) = f(x) + 0 = f(x) = f$).

CLAIM: $z \in C(\mathbb{R})$.

PROOF: $z(a) = \lim_{x \to a} z(x) = 0 \quad \forall\, a \in \mathbb{R}$, so z cts. over $\mathbb{R}$ and $z \in C(\mathbb{R})$

CLAIM: $\forall\, f, g \in C(\mathbb{R})$, $f + g \in C(\mathbb{R})$.

PROOF: $\forall\, a \in \mathbb{R}$,

$f(a) = \lim_{x \to a} f(x)$, $g(a) = \lim_{x \to a} g(x)$, (DEF continuity)

$f + g = f(a) + g(a) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

$= \lim_{x \to a} (f(x) + g(x))$ (linearity of lim)

Therefore $f + g$ is continuous on $\mathbb{R}$, so $f + g \in C(\mathbb{R})$.

CLAIM: $\forall\, a \in \mathbb{R}$, $\forall\, f \in C(\mathbb{R})$, $af \in C(\mathbb{R})$

PROOF: $\forall\, b \in \mathbb{R}$, $f(b) = \lim_{x \to b} f(x)$, (DEF continuity)

$af = af(b) = a \lim_{x \to b} f(x) = \lim_{x \to b} af(x)$ (linearity of lim)

Therefore af is continuous on $\mathbb{R}$, so $af \in C(\mathbb{R})$

By THM 1.3, $C(\mathbb{R})$ SUBSP. $\mathcal{F}(\mathbb{R})$. Thus $C(\mathbb{R})$ is a v.s.

b. Show that $C^{(n)}(\mathbb{R})$ SUBSP. $C(\mathbb{R})$ $\forall n \in \mathbb{Z}^+$

CLAIM: $C^{(n)}(\mathbb{R}) \subseteq C(\mathbb{R})$, $\forall n \in \mathbb{Z}^+$
PROOF: $\forall f \in C^{(n)}(\mathbb{R}), \forall n \in \mathbb{Z}^+, f$ is n times differentiable $(n > 0)$.
Any differentiable function is continuous over its domain, thus $f \in C(\mathbb{R})$ and $C^{(n)}(\mathbb{R}) \subseteq C(\mathbb{R})$.

CLAIM: $z \in C^{(n)}(\mathbb{R})$, $\forall n \in \mathbb{Z}^+$ (using same (DEF z))
PROOF: $\forall n \in \mathbb{Z}^+$, $z^{(n)} = z^{(n)}(x) = 0 = z \in C(\mathbb{R})$ $\forall x \in \mathbb{R}$.

CLAIM: $\forall f, g \in C^{(n)}(\mathbb{R}), \forall n \in \mathbb{Z}^+, f+g \in C^{(n)}(\mathbb{R})$
PROOF: $\forall f, g \in C^{(n)}(\mathbb{R}), \forall n \in \mathbb{Z}^+, \forall a \in \mathbb{R}$,

$$f^{(n)}(a) = \lim_{x \to a} f^{(n)}(x),$$
$$g^{(n)}(a) = \lim_{x \to a} g^{(n)}(x), \qquad \text{(DEF continuity)}$$
$$\frac{d^n}{dx^n}(f+g) = \frac{d^n}{dx^n}(f(a)+g(a)) \qquad \left(\text{linearity of } \frac{d}{dx}\right)$$
$$= f^{(n)}(a) + g^{(n)}(a) = \lim_{x \to a} f^{(n)}(x) + \lim_{x \to a} g^{(n)}(x)$$
$$= \lim_{x \to a}\left(f^{(n)}(x) + g^{(n)}(x)\right) \qquad \text{(linearity of lim)}$$
$$= \lim_{x \to a}\left(f^{(n)} + g^{(n)}\right) = \lim_{x \to a}\left(\frac{d^n}{dx^n}(f+g)\right) \qquad \left(\text{linearity of } \frac{d}{dx}\right)$$

Thus $f+g$ has a continuous nth-derivative, so $f+g \in C^{(n)}(\mathbb{R})$.

CLAIM: $\forall f \in C^{(n)}(\mathbb{R}), \forall a \in \mathbb{R}, \forall n \in \mathbb{Z}^+$, $af \in C^{(n)}(\mathbb{R})$.
PROOF: $\forall f \in C^{(n)}(\mathbb{R}), \forall a, b \in \mathbb{R}$,

$$f^{(n)}(b) = \lim_{x \to b} f^{(n)}(x), \qquad \text{(DEF continuity)}$$
$$\frac{d^n}{dx^n}(af) = a\frac{d^n}{dx^n}f = af^{(n)}(b)$$
$$= a \lim_{x \to b} f^{(n)}(x) = \lim_{x \to b}\left(af^{(n)}(x)\right)$$
$$= \lim_{x \to b}\left(\frac{d^n}{dx^n}(af)\right)$$

Thus af has a continuous nth-derivative, so $af \in C^{(n)}(\mathbb{R})$

Jonathan Lam
Prof. Mintchev
MA 326
Lin Alg.
9/18/19

2b, (cont'd.) By THM 1.3, $C^{(n)}(\mathbb{R})$ SUBSP. $C(\mathbb{R})$ $\forall n \in \mathbb{Z}^+$.

2c. Show $j > i \Rightarrow C^{(j)}(\mathbb{R})$ SUBSP. $C^{(i)}(\mathbb{R})$, $j, i \in \mathbb{Z}^+$.

CLAIM: $C^{(j)}(\mathbb{R}) \subseteq C^{(i)}(\mathbb{R})$

PROOF: $\forall f \in C^{(j)}(\mathbb{R})$, $f^{(j)}$ is continuous. In general, for $\forall k \in \mathbb{Z}^+$, if $f^{(k+1)}$ is continuous, $f^{(k)}$ is continuous. By induction, $f^{(j)}, f^{(j-1)}, \ldots f^{(1)}$ are continuous. Since $i < j$, $f^{(i)}$ is continuous. Thus $f \in C^{(i)}(\mathbb{R})$, so $C^{(j)}(\mathbb{R}) \subseteq C^{(i)}(\mathbb{R})$.

Since $C^{(n)}(\mathbb{R})$ is a v.s. (proved in (2b)), then $C^{(j)}(\mathbb{R})$, $C^{(i)}(\mathbb{R})$ are v.s. over $\mathbb{R}$. Since $C^{(j)}(\mathbb{R}) \subseteq C^{(i)}(\mathbb{R})$, $C^{(j)}(\mathbb{R})$ SUBSP. $C^{(i)}(\mathbb{R})$.

PSET 2 #3

Jonathan Lam
Prof. Mintchev
MA 326
Lin. Alg.
9/19/19

3. Let $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$,
$E = \{f \in V \mid f(-x) = f(x) \quad \forall x \in \mathbb{R}\}$.
$O = \{f \in V \mid f(-x) = -f(x) \quad \forall x \in \mathbb{R}\}$

a. Show E and O are proper subspaces of V.

CLAIM: E, O are proper subsets of V ($E, O \subseteq V$, $E, O \neq V$).
PROOF: $\forall f \in E$, $\forall g \in O$, by (DEF E) and (DEF O), $f \in V$, $g \in V$, thus $E, O \subseteq V$.

$\forall h \neq 0 \in E$, $i \neq 0 \in O$. Then, $h \notin O$ ($h(-x) = h(x) \neq -h(x)$) and $i \notin E$ ($i(-x) = -i(x) \neq i(x)$). Since $h \in V$, $i \in V$, $E, O \neq V$.

CLAIM: $z \in E, O$. (using same (DEF z) from problem 2)
(Note: z is the 0 vector of V.)
PROOF: $\forall x$, $z = z(x) = 0 = z(-x)$, so $z \in E$,
$z = z(x) = 0 = (-1) \cdot 0 = -z(x)$, so $z \in O$.

CLAIM: $\forall f, g \in E$, $f + g \in E$,
$\forall h, i \in O$, $h + i \in O$
PROOF: $\forall f, g \in E$, $\forall x \in \mathbb{R}$,
$f+g = (f+g)(x) = f(x) + g(x) = f(-x) + g(-x) = (f+g)(-x)$
$\therefore f+g \in E$.

$\forall h, i \in O$, $h + i \in O$.
$(h+i)(-x) = h(-x) + i(-x) = -h(x) - i(x)$
$= -1(h(x) + i(x)) = -(h+i)(x)$
$\therefore h + i \in O$.

CLAIM: $\forall a \in \mathbb{R}, \forall f \in E, \quad af \in E,$
$\forall b \in \mathbb{R}, \forall g \in O, \quad bg \in O.$

PROOF: $\forall a \in \mathbb{R}, \forall f \in E,$

$$(af)(x) = af(x) = af(-x) = (af)(-x)$$

$\therefore \; af \in E.$

$\forall b \in \mathbb{R}, \forall g \in O,$

$$(bg)(-x) = bg(-x) = b(-g(x))$$
$$= -bg(x) = -(bg)(x)$$

$\therefore \; bg \in O.$

By (THM 1.3), E, O subsp. V. Since E, O are proper subsets of V, E, O proper subsp. V.

b. Argue that $\text{span}(E) = E$, $\text{span}(O) = O$.

LEM 1: $\forall V$ v.s., $\text{span}(V) = V$.

PROOF: $V \subseteq V$, so $\text{span}(V) \subseteq V$ (THM 1.5)

Also, $\forall x \in V$, x can be written as a linear combination of vectors in V, i.e., $1 \cdot x = x$. Thus $V \subseteq \text{span}(V)$.

Since $\text{span}(V) \subseteq V$, $V \subseteq \text{span}(V)$, $V = \text{span}(V)$.

CLAIM: $\text{span}(E) = E$, $\text{span}(O) = O$.

PROOF: E and O are both v.s. (proved in part a). (LEM 1) states that the span of any v.s. is itself, so this result is immediate.

PSET 2, P3, cont'd.

Jonathan Lam
Prof. Mintchev
MA 326
Lin. Alg.
9/19/19

3c) Show that $E \cup O$ generates V.

CLAIM: $V \subseteq \text{span}(E \cup O)$

PROOF: $\forall f \in V$, let $f_e = f_e(x) = \frac{f(x) + f(-x)}{2}$,

$f_o = f_o(x) = \frac{f(x) - f(-x)}{2}$.

Since $f_e(-x) = \frac{f(-x) + f(x)}{2} = \frac{f(x) + f(-x)}{2} = f_e(x)$,
and $f_e : \mathbb{R} \to \mathbb{R}$, $f_e \in E$.

Since $f_o(-x) = \frac{f(-x) - f(x)}{2} = -\left(\frac{f(x) - f(-x)}{2}\right) = -f_o(x)$,
and $f_o : \mathbb{R} \to \mathbb{R}$, $f_o \in O$.

Then $\forall g \in V$, g can be expressed as the linear combination of g_e, g_o:

$1 \cdot g_e + 1 \cdot g_o = g_e(x) + g_o(x) = \frac{g(x) + g(-x)}{2} + \frac{g(x) - g(-x)}{2}$
$= \left(\frac{g(x)}{2} + \frac{g(x)}{2}\right) + \left(\frac{g(x) - g(-x)}{2}\right) = g(x) = g$.

Since $g_e \in E$, $g_o \in O$, $g_e, g_o \in E \cup O$, and
$V \subseteq \text{span}(E \cup O)$.

CLAIM: $E \cup O$ generates V.

PROOF: Since $E, O \subseteq V$, $E \cup O \subseteq V$. Then, by (THM 1.5), $\text{span}(E \cup O) \subseteq V$. Since $V \subseteq \text{span}(E \cup O)$,
$V = \text{span}(E \cup O)$ $\therefore$ $E \cup O$ generates V.

1.2 # 8, 9, 21

CLAIM:

8. $\forall$ v.s. V, show $(a+b)(x+y) = ax + ay + bx + by$
$\forall x, y \in F$, $\forall a, b \in F$.

PROOF: $(a+b)(x+y)$

$= (a+b)x + (a+b)y$ (VS 7)

$= ax + bx + ay + by$ (VS 8)

$= ax + ay + bx + by$ (VS 1)

9. Prove corollaries 1, 2 of (THM 1.1), (THM 1.2 c).

(THM 1.1 Cor. 1) CLAIM: The vector 0 described in (VS3) is unique.
PROOF: Let 0_1, 0_2 satisfy additive identity property described in (VS3). Then, $\forall x \in V$,

$0_1 + x = x = 0_2 + x$

By (THM 1.2), $0_1 = 0_2$ $\therefore$ the additive identity for a v.s. is unique.

(THM 1.1 Cor. 2) CLAIM: The vector y described in (VS4) is unique.
PROOF: Let y_1, y_2 satisfy additive inverse property described in (VS4) $\forall x \in V$. Then,

$x + y_1 = 0 = x + y_2$

By (THM 1.1), $y_1 = y_2$ $\therefore$ the additive inverse $\forall x \in V$ is unique.

(THM 1.2c) CLAIM: $a0 = 0$ $\forall a \in F$ in any v.s. over F.
PROOF: $\forall a \in F$,

$a0 + a0 \underset{\text{by (VS7)}}{=} a(0+0) \underset{\text{(VS3)}}{=} a0 \underset{\text{(VS3)}}{=} a0 + 0$

By (THM 1.1), $a0 = 0$.

PSET 2

1.2 # 21

Jonathan Lam
Prof. Mintchev
Lin. Alg.
MA 326
9/18/19

21. Let V, W be v.s. over field F. Let $Z = \{(v,w) : v \in V, w \in W\}$. Prove that Z v.s. over F w/ the operations:
$(v_1, w_1) + (v_2, w_2) := (v_1 + v_2, w_1 + w_2)$,
$c(v_1, w_1) := (cv_1, cw_1)$

CLAIM: $+$ binary over Z.
PROOF: $\forall z_1 = (v_1, w_1), z_2 = (v_2, w_2) \in Z$. Then $v_1, v_2 \in V$, $w_1, w_2 \in W$, and by binary-ness of vector addition, $v_1 + v_2 \in V$, $w_1 + w_2 \in W$. $\therefore z_1 + z_2 \in Z$, so $+$ maps $Z \times Z \to Z$ and is thus binary.

CLAIM: $\cdot$ is external binary over Z w/ field F.
PROOF: $\forall c \in F$, $\forall z = (v, w)$. Since v, w are v.s. over field F, scalar multiplication by F is external binary, i.e., $cv \in V$, $cw \in W$. $\therefore cz \in Z$, so $\cdot$ maps $F \times Z \to Z$, and is therefore external binary over F.

VS1) CLAIM: $\forall z_1, z_2 \in Z$, $z_1 + z_2 = z_2 + z_1$.
PROOF: $\forall z_1 = (v_1, w_1), z_2 = (v_2, w_2) \in Z$,
$$z_1 + z_2 = (v_1 + v_2, w_1 + w_2) \underset{\text{(VS1 for } W, V)}{=} (v_2 + v_1, w_2 + w_1) = z_2 + z_1$$

VS2) CLAIM: $\forall z_1, z_2, z_3 \in Z$, $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$.
PROOF: $\forall z_1 = (v_1, w_1), z_2 = (v_2, w_2), z_3 = (v_3, w_3)$,
$$\begin{aligned} z_1 + (z_2 + z_3) &= (v_1, w_1) + (v_2 + v_3, w_1 + w_2) \\ &= (v_1 + (v_2 + v_3), w_1 + (w_2 + w_3)) \\ &= ((v_1 + v_2) + v_3, (w_1 + w_2) + w_3) \quad \text{(VS2 for } V, W) \\ &= (v_1 + v_2, w_1 + w_2) + (v_3, w_3) \\ &= (z_1 + z_2) + z_3 \end{aligned}$$

1.2 # 21, cont'd.

VS 3) CLAIM: $\exists\ 0_Z \in Z$ s.t. $\forall z \in Z$, $0_Z + z = z$.

PROOF: $\exists$ additive identities $0_V \in V$, $0_W \in W$ (VS3 for v, w).

Let $0_Z = (0_V, 0_W)$. Then, $\forall z = (v, w) \in Z$,

$$0_Z + z = (0_V + v,\ 0_W + w) \quad \text{(VS3 for v, w)}$$
$$= (v, w) = z.$$

VS 4) CLAIM: $\forall z_1 \in Z$, $\exists\ z_2 \in Z$ s.t. $z_1 + z_2 = 0_Z$.

PROOF: $\forall z_1 = (v, w) \in Z$,

v has an additive inverse $(-v)$, (VS4 for v)

w has an additive inverse $(-w)$, (VS4 for w)

Let $z_2 = (-v, -w)$, then:

$$z_1 + z_2 = (v + (-v),\ w + (-w))$$
$$= (0_V, 0_W) = 0_Z. \quad \text{(VS4 for v, w)}$$

VS5) CLAIM: $\forall z \in V$, $1z = z$.

PROOF: $\forall z = (v, w) \in V$, $1z = \underbrace{(1v, 1w) = (v, w)}_{\text{VS5 for v,w}} = v$

VS 6) CLAIM: $\forall a, b \in F$, $\forall z \in Z$, $(a \cdot b) z = a \cdot (b \cdot z)$

PROOF: $\forall a, b \in F$, $\forall z = (v, w) \in Z$,

$$(a \cdot b) z = \underbrace{((a \cdot b)v,\ (a \cdot b)w) =}_{\text{VS6 for v, w}} (a \cdot (bv),\ a \cdot (bw)) = a(bv, bw) = a(bz)$$

VS7) CLAIM: $\forall a \in F$, $\forall z_1, z_2 \in Z$, $a(z_1 + z_2) = az_1 + az_2$

PROOF: $\forall a \in F$, $\forall z_1 = (v_1, w_1)$, $z_2 = (v_2, w_2) \in Z$,

$$a(z_1 + z_2) = a(v_1 + v_2,\ w_1 + w_2) = (a(v_1 + v_2),\ a(w_1 + w_2))$$
$$\underbrace{= (av_1 + av_2,\ aw_1 + aw_2)}_{\text{VS7 for V, W}} = (av_1, aw_1) + (av_2, aw_2) = az_1 + az_2$$

VS8) CLAIM: $\forall a, b \in F$, $\forall z \in Z$, $(a + b) z = az + bz$.

PROOF: $\forall a, b \in F$, $\forall z = (v, w) \in Z$,

$$(a + b) z = \underbrace{((a+b)v,\ (a+b)w) =}_{\text{VS8 for v, w}} (av + bv,\ aw + bw)$$
$$= (av, aw) + (bv, bw) = az + bz.$$

Since + binary on Z, $\cdot$ external binary on Z, and (VS1-8) satisfied, Z is a V.S over F with the addition and multiplication defined here.

PSET 2

1.3 # 10, 22, 23, 25.

Jonathan Lam
Prof. Mintchev
MA 326
Lin. Alg.
9/19/19

10. Prove that $W_1 = \{(a_1, a_2, \ldots, a_n) \in F^n : a_1 + a_2 + \ldots + a_n = 0\}$ is a subspace, but $W_2 = \{(a_1, a_2, \ldots, a_n) \in F^n : a_1 + a_2 + \ldots + a_n = 1\}$ is not.

a. Prove: W_1 is a subspace:

Note: the zero vector for F^n is known to be $\underbrace{(0, 0, 0, \ldots, 0)}_{n}$.
Denote zero vector for F^n 0_n $(n \in \mathbb{Z}+)$.

CLAIM: $0_n \in W_1$.
PROOF: In 0_n, $a_1 = a_2 = \ldots = a_n = 0$, so $a_1 + a_2 + \ldots + a_n = 0$.
Thus $0_n \in W_1$.

CLAIM: $\forall\ x = (a_1, a_2, \ldots, a_n),\ y = (b_1, b_2, \ldots, b_n) \in W_1$, then $x + y \in W_1$.
PROOF: $x + y = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$. Summing its terms:
$(a_1 + b_1) + (a_2 + b_2) + \ldots + (a_n + b_n)$
$= (a_1 + a_2 + \ldots + a_n) + (b_1 + b_2 + \ldots + b_n)$ (F1, F2 for F)
$= 0 + 0 = 0$.
Thus $x + y \in W_1$

CLAIM: $\forall c \in F,\ \forall x = (a_1, a_2, \ldots, a_n) \in W_1,\ ax \in W_1$.
PROOF: $cx = (ca_1, ca_2, \ldots, ca_n)$. Summing the terms:
$ca_1 + ca_2 + \ldots + ca_n$
$= c(a_1 + a_2 + \ldots + a_n) = c(0) = 0$.
Thus $cx \in W_1$.

By (THM 1.3), W_1 satisfies the necessary conditions to be SUBSP. F^n.

b) CLAIM: W_2 not SUBSP. F^n.
PROOF: For 0_n, $a_1 = a_2 = \ldots = a_n = 0$. Thus, $a_1 + a_2 + \ldots + a_n = 0 \neq 1$, so $0_n \notin W_2$. By (THM 1.3), W_2 is not SUBSP. F^n because it doesn't contain the zero vector of F^n.

1.3 # 22, 23, 25

22. Let F_1, F_2 be fields.

DEF: $E := \{ f : F_1 \to F_2 : f(-t) = f(t) \}$.

DEF: $O := \{ O : F_1 \to F_2 : f(-t) = -f(t) \}$.

DEF: an element of E is called even, and an element of O is called odd

Prove E, O SUBSP. $\mathcal{F}(F_1, F_2)$

CLAIM: $E, O \subseteq \mathcal{F}(F_1, F_2)$

This is immediate from the definitions of $\mathcal{F}(F_1, F_2)$, and E, O.

DEF. Let $z_2 := z_2(x_1) = 0_2$, $x_1 \in F_1$, $0_2 \in F_2$ and is the zero element in F_2. This is the zero vector of $\mathcal{F}(F_1, F_2)$ (since addition of any element of $\mathcal{F}(F_1, F_2)$ with z_2 results in an addition with the additive identity in F_2).

CLAIM: $z_2 \in E, O$

PROOF: $\forall a \in F_1$, $z_2(-a) = 0 = z_2(a)$, so $z_2 \in E$.
$\forall a \in F_1$, $z_2(-a) = 0 = (-1)(0) = -z(a)$, so $z_2 \in O$.

CLAIM: $\forall f, g \in E$, $f + g \in E$.
$\forall h, i \in O$, $h + i \in O$.

PROOF: $\forall f, g \in E$, $\forall a \in F_1$
$(f+g)(-a) = f(-a) + g(-a) = f(a) + g(a) = (f+g)(a)$,
so $f+g \in E$.
$(h+i)(-a) = h(-a) + g(-a) = -h(a) - g(a) = -1(h(a) + g(a))$
$= -(h+g)(a)$, so $h+i \in O$.

CLAIM: $\forall c \in F_2$, $\forall f \in E$, $\forall g \in O$, $cf \in F$, $cg \in O$.

PROOF: $\forall a \in F_1$, $(cf)(-a) = cf(-a) = cf(a) = (cf)(a)$, so $cf \in E$,
$(cg)(-a) = cg(-a) = c(-g(a)) = -(cg)(a)$, so $cg \in O$.

BY (THM 1.3), the necessary conditions are met to prove E, O SUBSP. $\mathcal{F}(F_1, F_2)$

Jonathan Lam
Prof. Mintchev
MA 326
Lin. Alg
9/19/19

PSET 2

1.3 # 23, 25

23. Let W_1, W_2 SUBSP. v.s. V.

a) Prove that $W_1 + W_2$ SUBSP. V and contains both W_1 and W_2.

5

Let $W = W_1 + W_2$ (i.e., $W = \{x + y : x \in W_1, y \in W_2\}$

CLAIM: $W \in V$? You mean $W \subseteq V$.

PROOF: $\forall w = x + y \in V$, $x \in W_1 \subseteq V$, $y \in W_2 \subset V$.

Since $+$ is binary over vector addition $x + y \in V$.

$\therefore W \in V$

CLAIM: $0_V \in W$.

PROOF: All subspaces contain the zero vector of their superspace.

Thus $0_V \in W_1, W_2$, and $\exists x \in W = 0_V + 0_V = 0_V$.

CLAIM: $\forall x, y \in W$, $x + y \in W$.

PROOF: $\forall x = x_1 + x_2$, $y = y_1 + y_2 \in W$, $x_1, y_1 \in W_1$, $x_2, y_2 \in W_2$. Then $x + y = x_1 + x_2 + y_1 + y_2$

$= (x_1 + y_1) + (x_2 + y_2)$. By binaryness of $+$ over v.s., $x_1 + y_1 \in W_1$, $x_2 + y_2 \in W_2$, thus $x + y \in W$

CLAIM: $\forall a \in F$, $\forall x \in W$, $ax \in W$.

PROOF: $\forall a \in F$, $\forall x = x_1 + x_2 \in W$, $x_1 \in W_1$, $x_2 \in W_2$,

$ax = a(x_1 + x_2) \xrightarrow{\text{(VS7 for V)}} ax_1 + ax_2$

Scalar multiplication is external binary over a v.s., so $ax_1 \in W_1$, $ax_2 \in W_2$, so $ax \in W$.

By (THM 1.3), W satisfies the conditions to be a SUBSP. of V.

1.3 #23, cont'd.

CLAIM: W contains both W_1, W_2.

PROOF: Since $0_V \in W_2$, $\forall x \in W_1$, $\exists w \in W = x + 0_V = x$

Similarly, since $0_V \in W_1$, $\forall y \in W_2$,

$\exists w = 0_V + y = y$. Thus $W_1, W_2 \subseteq W$.

b. Prove that any subspace Z of V that contains both W_1 and W_2 must also contain W.

CLAIM: Z contains $W_1, W_2 \Rightarrow Z$ contains W

PROOF: $\forall x \in W_1$, $\forall y \in W_2$, $x + y \in Z$ because $+$ is binary over vector addition. Therefore, the set of $\{x + y : x \in W_1, y \in W_2\} = W \subseteq Z$.

25. Let W_1 denote set of all polynomials in $P(F)$ s.t. in the representation $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, we have $a_i = 0$ when i even. Also define W_2 set of all polynomials in $P(F)$ s.t. in the representation: $g(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0$, we have $b_i = 0$ whenever i is odd. Prove $P(F) = W_1 \oplus W_2$.

To show that $P(F)$ is the direct sum of W_1, W_2, need to show that: (by (DEF direct sum)):

1) W_1, W_2 subsp $P(F)$
2) $W_1 \cap W_2 = \{0_P\}$
3) $W_1 + W_2 = V$.

25.1) DEF: Let $0_P := 0 + 0x + 0x^2 + \dots$ be the zero vector of $P(F)$.

CLAIM: $0_P \in W_1, W_2$.

PROOF: 0_P is expressible as a polynomial with all coefficients $a_i = 0$. Then $a_i = 0$ when i even, so $0_P \in W_1$, and $a_i = 0$ when i odd, so $0_P \in W_2$.

PSET 2

Jonathan Lam
Prof. Mintchev
MA326
Lin. Alg.
7/19/19

1.3 # 25, cont'd.

CLAIM: $\forall\, w, x \in W_1$, $w + x \in W_1$,
$\forall\, y, z \in W_2$, $y + z \in W_2$

PROOF: $\forall\, w = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 \in W_1$,
$x = b_n x^n + b_{n-1} x^{n-1} + \dots + b_1 x + b_0 \in W_1$,
$w + x = (a_j + b_j)x^j + (a_{j-1} + b_{j-1})^{j-1} + \dots + (a_1 + b_1)x + (a_0 + b_0)$,
$\forall i$ even, $(a_i + b_i) = 0 + 0 = 0$, $\therefore w + x \in W_1$.

$\forall y = c_n x^n + c_{n-1}x^{n-1} + \dots + c_1 x + c_0 \in W_2$,
$\forall z = d_m x^m + d_{m-1}x^{m-1} + \dots + d_1 x + d_0 \in W_2$,
$y + z = (c_j + d_j)x^j + (c_{j-1} + d_{j-1})^{j-1} + \dots + (c_1 + d_1)x + (c_0 + d_0)$,
$\forall i$ odd, $(c_i + b_i) = 0 + 0 = 0$, $\therefore y + z \in W_2$.

CLAIM: $\forall c \in F$, $\forall x \in W_1$, $\forall y \in W_2$, $ax \in W_1$, $ay \in W_2$.
PROOF: $\forall x = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 \in W_1$.
Then $cx = (ca_n)x^n + (ca_{n-1})x^{n-1} + \dots + (ca_1)x + (ca_0)$.
$\forall i$ even, $ca_i = c(0) = 0$, $\therefore cx \in W_1$.

$\forall y \in b_m x^m + b_{m-1}x^{m-1} + \dots + b_1 x + b_0 \in W_2$,
Then $cy = (cb_m)x^m + (cb_{m-1})x^{m-1} + \dots + (cb_1)x + (cb_0)$.
$\forall i$ odd, $cb_i = c(0) = 0$, $\therefore cb \in W_2$.

By (THM 1.3), W_1, W_2 SUBSP $P(F)$.

25. 2) CLAIM: $W_1 \cap W_2 = \{0_p\}$
PROOF: Since W_1, W_2 SUBSP. $P(F)$, $0_p \in W_1, W_2$.
Next, choose $x = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 \neq 0_p \in P(F)$.
Then, $\forall i$ s.t. $a_i \neq 0$, i must be odd or even. If i even, $x \notin W_2$. If i even, $x \notin W_1$.
Thus, $\forall x \neq 0_p$, $x \notin W_1 \cap W_2$.

$\therefore W_1 \cap W_2 = \{0_p\}$.

25.3) Show $W_1 + W_2 = V$.

<u>CLAIM</u>: $W_1 + W_2 \subset P(F)$

<u>PROOF</u>: $W_1 + W_2 = \{x + y : x \in W_1, y \in W_2\}$. Since $W_1, W_2 \subseteq$
$\forall x \in W_1 \subseteq P(F), \forall y \in W_2 \subseteq$, $x + y \in P(F)$, thus.
$W_1 + W_2 \subseteq P(F)$.

<u>CLAIM</u>: $P(F) \subseteq W_1 + W_2$

<u>PROOF</u>: $\forall x \in P(F)$, x can be represented as (for some odd n)

$$x = a_n x^n + b_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_3 x^3 + b_2 x^2 + a_1 x + b_0.$$
$$= (a_n + 0) x^n + (0 + b_{n-1}) x^{n-1} + (a_{n-2} + 0) x^{n-2} + \dots$$
$$+ (a_3 + 0) x^3 + (0 + b_2) x^2 + (a_1 + 0) x + (0 + b_0)$$
$$= (a_n x^n + 0 x^{n-1} + a_{n-2} x^{n-2} + \dots + a_3 x^3 + 0 x^2 + a_1 x + 0)$$
$$+ (0 x^n + b_{n-1} x^{n-1} + 0 x^{n-2} + \dots + 0 x^3 + b_2 x^2 + 0 x + b_0).$$

Thus x can be expressed as $y + z$, $y \in W_1$, $z \in W_2$,
so $P(F) \subseteq W_1 + W_2$.

Since $W_1 + W_2 \subseteq P(F)$, $P(F) \subseteq W_1 + W_2$, so $W_1 + W_2 = P(F)$.

Since the three conditions for a direct sum are satisfied,
$W_1 \oplus W_2 = P(F)$.

PSET 2

Jonathan Lam
Prof. Mintchev
MA326
Lin. Alg.
9/19/19

1.4 # 2ae, 4ab, 13, 17.

Solve the linear systems using Gaussian elimination.

2a) $2x_1 - 2x_2 - 3x_3 = -2$

$3x_1 - 3x_2 - 2x_3 + 5x_4 = 7$

$x_1 - x_2 - 2x_3 - x_4 = -3$

$x_1 - x_2 - 2x_3 - x_4 = -3$

$0 + 0 + x_3 + 2x_4 = 4$

$0 + 0 + 4x_3 + 8x_4 = 16$

$x_1 - x_2 - 2x_3 - x_4 = -3$

$0 + 0 + x_3 + 2x_4 = 4$

$0 + 0 + \cancel{x_3} + \cancel{2x_4} = \cancel{4} \Rightarrow 0 + 0 + 0 + 0 = 0.$

Let $x_3 = t$. Then $t + 2x_4 = 4 \Rightarrow x_4 = 2 - \frac{t}{2}$.

Let $x_1 = s$. Then $2s - 2x_2 - 3t = -2 \Rightarrow x_2 = 1 + s - \frac{3}{2}t$.

Then a solution is $(x_1, x_2, x_3, x_4) = (s, 1 + s - \frac{3}{2}t, t, 2 - \frac{t}{2})$.

$\forall t, s \in \mathbb{R}$.

e) $x_1 + 2x_2 - 4x_3 - x_4 + x_5 = 7$

$-x_1 + 10x_3 - 3x_4 - 4x_5 = -16$

$2x_1 + 5x_2 - 5x_3 - 4x_4 - x_5 = 2$

$4x_1 + 11x_2 - 7x_3 - 10x_4 - 2x_5 = 7.$

$x_1 + 2x_2 - 4x_3 - x_4 + x_5 = 7$

$2x_2 \quad 6x_3 - 4x_4 - 3x_5 = -9$

$x_2 + 3x_3 - 2x_4 - 3x_5 = -12$

$3x_2 + 9x_3 - 6x_4 - 6x_5 = -21$

$x_1 + 2x_2 - 4x_3 - x_4 + x_5 = 7$

$x_2 + 3x_3 - 2x_4 - 3x_5 = -12$

$6x_3 - 4x_4 - 3x_5 = -9$

$3x_5 = 15 \Rightarrow x_5 = 5$

Let $x_3 = t$. Then $6t - 4x_4 - 3(5) = -9$

$\Rightarrow x_4 = \dfrac{6t + 9 - 15}{4} = \frac{3}{2}t - \frac{3}{2}$

$x_2 + 3t - 2\left(\frac{3}{2}t - \frac{3}{2}\right) - 3(5) = -12$

$x_2 = \cancel{-12} + \cancel{15} - \cancel{3t} + \cancel{3t} \cancel{-3} = 0$

$x_1 + \cancel{2(0)} - 4t - \left(\frac{3}{2}t - \frac{3}{2}\right) + 5 = 7$

$x_1 = 7 + 4t + \frac{3}{2}t - \frac{3}{2} - 5$

$= \frac{1}{2} + \frac{11}{2}t$

Thus a solution is $(x_1, x_2, x_3, x_4, x_5) = \left(\frac{1}{2} + \frac{11}{2}t,\ 0,\ \frac{3}{2}t - \frac{3}{2},\ t,\ 5\right)$

$\forall\, t \in \mathbb{R}$.

4. Determine if first polynomial can be expressed as a lin. comb. of other two.

a. $x^3 - 3x + 5 = a(x^3 + 2x^2 - x + 1) + b(x^3 + 3x^2 - 1)$

$$\begin{cases} a + b = 1 \\ 2a + 3b = 0 \\ -a = -3 \\ a - b = 5. \end{cases}$$

$-a = -3 \Rightarrow a = 3$, $a + b = 1 \Rightarrow b = 1 - 3 = -2$

Check for consistency:

$3 + (-2) = 1$, $2(3) + 3(-2) = 0$, $-3 = -3$, $3 - (-2) = 5$. ✓

Yes. (solution: $(a, b) = (1, -2)$).

b. $4x^3 + 2x^2 - 6 = a(x^3 - 2x^2 + 4x + 1) + b(3x^3 - 6x^2 + x + 4)$

$$\begin{cases} a + 3b = 4 \\ \cancel{-2a - 6b = 2} \\ 4a + b = 0 \\ a + 4b = -6 \end{cases}$$

multiply by -2

Subtract: $b = -10$.

$4a + (-10) = 0 \Rightarrow a = \frac{5}{2}$.

$a + 4b = \frac{5}{2} + 4(-10) = \frac{5}{2} - 40 = \frac{-85}{2} \neq -6$.

This system is not consistent, thus, the first polynomial cannot be expressed as a linear comb. of the other two.

PSET 2

1.4 # 13, 17.

Jonathan Lam
Prof. Mintchev
MA 326
Lin. Alg.
9/19/19

5

13. Show that if $S_1, S_2 \subseteq$ v.s. V s.t. $S_1 \subseteq S_2$, then $\text{span}(S_1) \subseteq \text{span}(S_2)$. In particular, if $S_1 \subseteq S_2$ and $\text{span}(S_1) = V$, deduce that $\text{span}(S_2) = V$.

<u>CLAIM</u>: If $S_1, S_2 \subseteq$ v.s. V, $S_1 \subseteq S_2$, then $\text{span}(S_1) \subseteq \text{span}(S_2)$

<u>PROOF</u>: For any linear combination $x \in \text{span}(S)$,
$x = a_1x_1 + a_2x_2 + \ldots + a_nx_n$, then $a_1, a_2, \ldots a_n \in F$, $x_1, x_2, \ldots, x_n \in S_1 \subseteq S_2$. Since all $x_i \in S_2$, this is also a linear combination in S_2, so $\text{span}(S_1) \subseteq \text{span}(S_2)$.

can be made even more efficient.

<u>CLAIM</u>: If $S_1 \subseteq S_2$, $\text{span}(S_1) = V$, then $\text{span}(S_2) = V$.

<u>PROOF</u>: By the above proof, $\text{span}(S_1) = V \subseteq \text{span}(S_2)$
By (THM 1.5), $\text{span}(S_2) \subseteq V$. Therefore, $\text{span}(S_2) = V$.

17. Let W subsp v.s. V. Under what conditions are there only a finite # of distinct subsets S of W s.t. S generates W?

<u>CLAIM</u>: If W finite, then there are only a finite # of distinct generating subsets.

<u>PROOF</u>: If W finite, then there are only a finite # of all subsets (finite power set), so there can only be a finite # of generating subsets.

<u>LEMMA</u>: If W is an infinite v.s., $\forall x \in W$, let $S = W \setminus \{x\}$. Then, S is a generating set of W.

<u>PROOF</u>: Choose an $x \in W$.

Case 1: $x = 0$ $\forall y \in S$, $0 \cdot y = 0$, so $x \in \text{span}(S)$.

Case 2: $x \neq 0$. $\forall y \in S$, $\exists (x-y) \in S$, since $x, y \in W$, $x - y \in W$, and $x - y \neq x \in S$. Then $x \in \text{span}(S)$.

Thus $x \in \text{span}(S)$. Also, $\forall z \in S$, $z = 1 \cdot z \Rightarrow z \in \text{span}(S)$.
Thus $W = S \cup \{x\} \subseteq \text{span}(S)$. By (THM 1.5) $\text{span}(S) \subseteq W$, thus $\text{span}(S) = W$ and S generates W.

9/19/19

CLAIM: If W infinite, there are infinitely many generating subsets of W.

PROOF: By (LEM 2), any subset $S = W \setminus \{x\}$, $\forall x \in W$ is a generating subset of W. Since W is infinite, there are an infinite # of generating subsets.

Since W finite $\Rightarrow$ W has finitely many generating subsets, and W infinite $\Rightarrow$ W has infinitely many generating subsets, this implies the tautology.

W finite $\Leftrightarrow$ W has a finite # of generating subsets.

Jonathan Lam
Prof. Mintchev
MA326
Lin. Alg.
9/26/19

PSET 3

1.5 # 2adf, 3, 5, 16

29/30

1.5.5 | 5/4
2.1.14 | 9

2. Determine whether the sets of vectors are linearly independent or dependent.

a) $\left\{\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix}\right\}$ in $M_{2\times 2}(\mathbb{R})$

linearly independent $\Leftrightarrow$ no nontrivial lin. combs. of vectors in the set, i.e., determine if system has nontrivial solutions.

$$a\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix} + b\begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$a(1) + b(-2) = 0$

$(a(-3) + b(6) = 0) \to$ divide by $(-3) \to a(1) + b(-2) = 0$

$(a(-2) + b(4) = 0) \to$ divide by $(-2) \to a(1) + b(-2) = 0$

$(a(4) + b(-8) = 0) \to$ divide by $4 \to a(1) + b(-2) = 0$

Now only 1 equation remaining with two variables. Any combination $(a, b) = (2t, t)\ \forall t \in \mathbb{R}$ is a solution, therefore there are nontrivial sol'ns to this lin comb $\Rightarrow$ dependent set.

d) $\{x^3 - x, 2x^3 + 4, -2x^3 + 3x^2 + 2x + 6\}$ in $P_3(\mathbb{R})$

same method as in (2a):

$a(x^3 - x) + b(2x^2 + 4) + c(-2x^3 + 3x^2 + 2x + 6) = 0 = 0x^3 + 0x^2 + 0x + 0$

polynomials equal $\Leftrightarrow$ coefficients match:

$$\begin{cases} a - 2c = 0 \to \text{multiply by } -1 \to -a + 2c = 0, \text{ same as } 3^{rd} \text{ eq.} \\ 2b + 3c = 0 \to \text{multiply by } 2 \to 4b + 6c = 0, \text{ same as } 4^{th} \text{ eq.} \\ -a + 2c = 0 \\ 4b + 6c = 0 \end{cases}$$

$$\begin{cases} -a + 2c = 0 \\ 2b + 3c = 0 \end{cases}$$

Let $b = t$, then $2t + 3c = 0 \Rightarrow c = \frac{-2}{3}t$

$-a + 2c = 0 \Rightarrow -a + 2\left(\frac{-2}{3}t\right) = 0 \to a = \frac{-4}{3}t$

This means $\left(-\frac{4}{3}t, t, -\frac{2}{3}t\right), \forall t \in \mathbb{R}$ is a sol'n. Since there are nontrivial lin. cons. that sum to 0, this set is linearly dependent.

9/2

2f.) using same method as in (2a), (2d):

$\{(1,-1,2), (2,0,1), (-1,2,-1)\}$ in $\mathbb{R}^3$.

$a(1,-1,2) + b(2,0,1) + c(-1,2,-1) = (0,0,0)$

$a + 2b - c = 0$

$-a \quad + 2c = 0$ add

$2a + b - c = 0$

$a + 2b - c = 0$

$2b + c = 0$

$-3b + c = 0$

$a + 2b - c = 0$

$2b + c = 0$

$\frac{5}{2}c = 0 \Rightarrow c = 0$

$2b + (0) = 0 \Rightarrow 2b = 0 \Rightarrow b = 0$

$a + 2(0) - (0) \Rightarrow a = 0$

The only solution to this system of lin. eqs. is $(a,b,c) = (0,0,0)$, thus there are only trivial soln's to the lin. comb $\Rightarrow$ the set is linearly independent.

3. **CLAIM**: the set

$$\left\{\begin{pmatrix}1&1\\0&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&0\\1&1\end{pmatrix}, \begin{pmatrix}1&0\\1&0\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\0&1\\0&1\end{pmatrix}\right\} \quad (\text{in } M_{2\times3}(F))$$

is linearly dependent.

PROOF:

Since $1\begin{pmatrix}1&1\\0&0\\0&0\end{pmatrix} + 1\begin{pmatrix}0&0\\1&1\\0&0\end{pmatrix} + 1\begin{pmatrix}0&0\\0&0\\1&1\end{pmatrix} + (-1)\begin{pmatrix}1&0\\1&0\\1&0\end{pmatrix} + (-1)\begin{pmatrix}0&1\\0&1\\0&1\end{pmatrix} = \begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix}$,

then there exists a nontrivial lin. comb. which sums to the zero vector in $M_{2\times3}(F)$, thus the set is linearly dependent.

PSET 3

1.5 # 5, 16

Jonathan Lam
Prof. Mintchev
MA 326
Lin. Alg.
9/26/19

probably better to index a single variable...

5. CLAIM: the set $S = \{1, x, x^2, \ldots, x^n\}$ is linearly independent in $P_n(F)$.

5

PROOF: The linear combination over elements in S:
$a(1) + b(x) + c(x^2) + \ldots + z x^n$ is only equal to the zero vector of $P_n = 0 + 0x + 0x^2 + \ldots + 0x^n$ if all the coefficients are equal, i.e., $a=0, b=0, c=0, \ldots, z=0$. Thus, the only linear combination over the elements in S that sum to 0 is the trivial lin. comb, so S is linearly independent.

16. Prove: A set S of vectors is linearly independent $\Leftrightarrow$ each finite subset of S is linearly independent.

CLAIM: $(\Rightarrow)$ If S is lin. ind., then each finite subset of S is also linearly ind.

PROOF: Let T be a subset of S. By (THM 1.6, COR. 1), since $T \subseteq S$ and S is lin. ind., T is also lin. ind. T includes all finite subsets of S, so all finite subsets of S are lin. ind.

CLAIM: $(\Leftarrow)$ If each finite subset of S is linearly independent, then S is also linearly independent.

PROOF: Proof by contrapositive: This claim is logically equivalent to proving S is linearly dependent $\Rightarrow$ not all finite subsets of S are linearly independent. Thus, assume S is lin dep. By (DEF lin. dep.), $\exists$ (finite) $T = \{u_1, u_2, u_3, \ldots, u_n\} \subseteq S$, $\{a_1, a_2, \ldots, a_n\} \in F$ s.t. $a_1 u_1 + a_2 u_2 + \ldots + a_n u_n$ yields the zero vector in the vector space. Since $T \subseteq T$, T is also linearly dep. by definition. Since T is a finite subset of S, $\exists$ linearly dep. subset of $S \Rightarrow$ not all subsets of S are linearly ind.

1.6 #20, 24, 29, 31.

Jonathan Lam
Prof. Mintchev
MA 326
Lin. Alg.
9/27/19

20. Let V be a v.s. with dimension n, and let S be subset of V, $\mathrm{span}(S) = V$.

a) CLAIM: There exists a subset of S that is a basis for V.

PROOF: Let $\beta = \{b_1, b_2, \ldots, b_n\}$ be a basis for V. Then $\beta_1 \subseteq V = \mathrm{span}(V)$, so each vector b_i, $1 \le i \le n$ can be represented as the lin. comb. $b_i = \sum_{j=1}^{k_i} a_{ij} s_{ij}$,

for some $a_{ij} \in F$, $s_{ij} \in S$, k_i finite (since lin. combos involve a finite # of vectors).

Let $T = \{ s_{ij} \in V : \forall i, 1 \le i \le n, 1 \le j \le k_i \}$, i.e., T is the set of all vectors from S in the linear combinations forming the basis vectors, and T is finite since n is finite, k_i is finite $\forall 1 \le i \le n$.

Since β generates V, and β can be represented as a lin. comb. of vectors over S, then $\forall v \in V$,

$$v = \sum_{i=1}^{n} c_i b_i = \sum_{i=1}^{n} c_i \sum_{j=1}^{k_i} a_{ij} s_{ij} = c_1 a_{11} s_{11} + c_1 a_{12} s_{12} + \cdots + c_1 a_{1k_1} s_{1k_1} + c_2 a_{21} s_{21} + \cdots + c_n a_{nk_n} s_{nk_n}$$

which is a linear combination of vectors over T. Thus V is generated by T, a finite subset of S. By (THM 1.9), there exists a subset of T that is a basis for V. Since $T \subseteq V$, that basis must also be in S.

b.) CLAIM: S contains at least n vectors

PROOF: By (THM 1.10, COR 1), all bases of a v.s. have the same cardinality, which is the dimension of the v.s., n. Since a basis is the subset of S, then S must contain at least as many vectors as the basis, i.e., n vectors.

Jonathan Lam
Prof. Mintchev
MA 326
Lin. Alg.
9/27/19

PSET 3

1.6 #24

24. Let $f(x)$ be a polynomial of degree n in $P_n(\mathbb{R})$. Prove that $\forall g(x) \in P_n(\mathbb{R})$, $\exists a_0, a_1, \ldots, a_n$ s.t.
$g(x) = a_0 f(x) + a_1 f'(x) + \ldots + a_n f^{(n)}(x)$

CLAIM: $\{f, f', \ldots, f^{(n)}\}$ is lin. ind.

PROOF: Let z be the zero function in $P(\mathbb{R})$, i.e., $z = 0 + 0x + 0x^2 + \ldots$ Assume a lin. comb. over $\{f, f', \ldots, f^{(n)}\}$: $c_0 f + c_1 f' + \ldots + c_n f^{(n)} = z$. By calculus, $\forall h \in P_i(\mathbb{R})$, $h' \in P_{i-1}(\mathbb{R})$, and degree$(h')$ < degree(h). Since f is the only function with degree n, it is the only polynomial with a nonzero x^n term; thus $c_0 = 0$ or else the result of the lin. comb. would have a nonzero x^n coefficient.

Now the lin. comb. over $\{f, f', \ldots, f^{(n)}\}$ is equivalent to $c_1 f' + c_2 f'' + \ldots + c_n f^{(n)} = z$. By the same reasoning as above, c_1 must be 0, or else the result of the lin. comb. would have a nonzero x^{n-1} term. By induction, $c_0 = c_1 = \ldots = c_n = 0$. Thus, $\{f, f', \ldots, f^{(n)}\}$ is lin. ind. //

CLAIM: Any $g \in P_n(\mathbb{R})$ can be expressed as a lin. comb over $\{f, f', \ldots, f^{(n)}\}$, $f \in P_n(\mathbb{R})$ w/ degree n.

PROOF. $\{f, f', \ldots, f^{(n)}\}$ was shown above to be lin. ind., and has cardinality $n+1$ (by inspection). $\dim(P_n(\mathbb{R})) = n+1$. By (THM 1.10 COR. 2), $\{f, f', \ldots, f^{(n)}\}$ is a basis for $P_n(\mathbb{R})$, and thus $g \in \text{span}(\{f, f', \ldots, f^{(n)}\})$.

PSET 3
1.6 # 29

Jonathan Lam
Prof. Matchen
MA 326
Lin. Alg.
10/2/19

29 a) Given that W_1, W_2 finite-dimensional subspaces of v.s. V, then $W_1 + W_2$ finite-dimensional, and $\dim(W_1+W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

Let $\dim(W_1) = n$, $\dim(W_2) = m$, $\dim(W_1 \cap W_2) = l$
Note that $W_1 \cap W_2$ subsp. W_1, W_2 (by THM. 1.4), so $\dim(W_1 \cap W_2) \le \dim(W_1), \dim(W_2)$ (by THM. 1.11).
Fix a basis β_l of $W_1 \cap W_2$, $\beta_l = \{u_1, u_2, \ldots, u_l\}$.
By (THM 1.10 COR 2), we can extend β_l (which is a linearly ind. subset of W_1) to a basis β_n for W_1, $\beta_n = \beta_l \cup \{v_1, v_2, \ldots, v_{n-l}\}$. The same argument applies to extend β_l to a basis β_m for W_2; $\beta_m = \beta_l \cup \{w_1, w_2, \ldots, w_{m-l}\}$
Finally, let $\beta = \beta_n \cup \beta_m = \{u_1, u_2, \ldots, u_l, v_1, v_2, \ldots, v_{n-l}, w_1, w_2, \ldots, w_{m-l}\}$.

CLAIM: β generates V.
PROOF: $\forall v \in V$, $v = v_1 + v_2$, $v_1 \in W_1$, $v_2 \in W_2$. Then:
$v_1 = \sum_{i=1}^{l} a_i u_i + \sum_{i=1}^{n-l} b_i v_i$,
$v_2 = \sum_{i=1}^{l} c_i u_i + \sum_{i=1}^{m-l} d_i w_i$
$v = \sum_{i=1}^{l} (a_i + c_i) u_i + \sum_{i=1}^{n-l} b_i v_i + \sum_{i=1}^{m-l} d_i v_i$
$\therefore v \in \operatorname{span}(\beta)$, so β generates V.

CLAIM: β is lin. ind.
PROOF: Assume $\sum_{i=1}^{l} a_i u_i + \sum_{i=1}^{n-l} b_i v_i + \sum_{i=1}^{m-l} c_i w_i = 0$.
Then, let $x = -\sum_{i=1}^{m-l} c_i w_i = \sum_{i=1}^{n} a_i u_i + \sum_{i=1}^{n-l} b_i v_i$.
Since x can be expressed as a lin. comb over w_i, then $x \in W_2$. Since x can be expressed as a lin. comb. over $\beta_n = \{u_1, u_2, \ldots, u_l, v_1, v_2, \ldots, v_{n-l}\}$, $x \in W_2$ as well, so $x \in W_1 \cap W_2$ and can be expressed as a lin. comb over β_l: $x = \sum_{i=1}^{l} d_i u_i$. Substituting this back into the original equation: $\sum_{i=1}^{l} (a_i + d_i) u_i + \sum_{i=1}^{n-l} b_i v_i = 0$. Since this is a lin. comb over the lin. ind. set β_n, $b_1 = b_2 = \ldots = b_{n-l} = 0$. Thus the original lin. comb. is equivalent to $\sum_{i=1}^{l} a_i u_i + \sum_{i=1}^{m-l} c_i w_i = 0$. Since this is a lin. comb. over the lin. ind. set β_m, $a_1 = a_2 = \ldots = a_l = c_1 = c_2 = \ldots = c_{m-l} = 0$. Thus all the coefficients must be 0, so β is lin. ind.

CLAIM: $\dim(W_1+W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

PROOF: By its construction, $\text{card}(\beta) = l + (n-l) + (m-l)$
$= n+m-l$, where $n = \dim(W_1)$, $m = \dim(W_2)$, and
$l = \dim(W_1 \cap W_2)$. Since β is a basis for $W_1+W_2 = V$,
$\dim(W_1+W_2) = \text{card}(\beta) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

b) Let W_1, W_2 be finite-dimensional subsp. of V, and let
$W_1 + W_2 = V$. Deduce that $V = W_1 \oplus W_2 \Leftrightarrow \dim(W_1) + \dim(W_2) =$
$\dim(V)$.

PF $(\Rightarrow)$: By (DEF $\oplus$), $W_1 \cap W_2 = \{0\}$. By (part a),
$\dim(V) = \dim(W_1) + \dim(W_2) - \dim(\{0\})$
$= \dim(W_1) + \dim(W_2) - 0 = \dim(W_1) + \dim(W_2)$.

PF $(\Leftarrow)$: $W_1 + W_2 = V$, and W_1, W_2 subsp. V by hyp. Also,
$$\underbrace{\dim(W_1) + \dim(W_2) = \dim(V)}_{\text{by hyp}} \quad \underbrace{\dim(V) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)}_{\text{by (part a)}}$$

Thus $\dim(W_1 \cap W_2) = 0 \Rightarrow W_1 \cap W_2 = \{0\}$. Thus
$W_1 \oplus W_2 = V$ by def.

PSET 3
1.6 #31

Jonathan Lam
Prof. Mintchev
MA326
Lin. Alg.
9/27/19

31) Let W_1, W_2 subsp. V, $\dim(W_1) = m$, $\dim(W_2) = n$, $m \geq n$.

a) CLAIM: $\dim(W_1 \cap W_2) \leq n$.
PROOF: $W_1 \cap W_2$ subsp. V (by THM 1.4), and since $W_1 \cap W_2 \subseteq W_1$, it is also a subsp. of W_1.
By (THM 1.11), $\dim(W_1 \cap W_2) \leq \dim(W_1) = n$.

b) CLAIM: $\dim(W_1 + W_2) \leq m + n$.
PROOF: By the result of (1.6 exercise #29),
$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$
$= m + n - \dim(W_1 \cap W_2)$. Since dimension of any v.s. is nonnegative, $\dim(W_1 + W_2) \leq m + n$.

Jonathan Lam
Prof. Mintchev
MA 326
Lin. Alg
9/27/19

PSET 3

2.1 # 14, 26

or a vector space?

14. Let V, W v.s., $T: V \to W$ linear.

a) CLAIM: T 1-1 $\iff$ T carries linearly independent subsets of V onto linearly independent subsets of W.

abuse of notation leads to utter confusion: is V a set or a vector

4

Very poor choices of notation

PROOF ($\Rightarrow$): Let T be 1-1, $v \subseteq V$ ind., $w = T(v)$.

Assume some lin. comb. over $w = 0$, i.e., $\sum_{i=0}^{n} a_i w_i = 0$, $0 \le n \le \text{card}(w)$

Then: $\sum_{i=0}^{n} a_i T(v_i) = T\left(\sum_{i=0}^{n} a_i v_i\right) = 0 = T(0)$ (by linearity of T).

Since T is 1-1, $\sum_{i=0}^{n} a_i v_i = 0$, and since this is a lin. comb over a lin. ind. set v, $a_1 = a_2 = \ldots = a_n = 0$.

Since these are also the coefficients to the lin. comb. over w yielding 0, w is also linearly independent.

PROOF ($\Leftarrow$): Proof by contrapositive statement: if T not 1-1, then $\exists$ a lin. ind. subset $\{x\}$ of V such that $\{T(x)\}$ is linearly dependent. Since T not 1-1, by (THM 2.4) $N(T) \neq \{0\}$, so $\exists\, v_0 \in V \neq 0$ s.t. $T(v_0) = 0$.

Since the singleton set of any nonzero vector is lin. ind., and any set containing the zero vector is lin. dep, then $\{v_0\}$ is ind., but $\{T(v_0)\} = \{0\}$ is dep.

Please conform to standards from textbook & lecture!

b) Suppose T 1-1, $S \subseteq V$.

CLAIM: S lin. ind. $\iff$ $T(S)$ lin. ind.

PROOF ($\Rightarrow$): This was proved in (part a).

PROOF ($\Leftarrow$): This proof similar to (part a proof $\Rightarrow$): Assume lin. comb. over S, i.e., $\sum_{i=1}^{n} a_i s_i = 0$, $1 \le n \le \text{card}(S)$. Take the transform, i.e., $T\left(\sum_{i=1}^{n} a_i s_i\right) = \sum_{i=1}^{n} a_i T(s_i) = T(0) = 0$. Since $T(S)$ lin. ind., all coefficients $a_1 = a_2 = \ldots = a_n = 0$, thus S must also be lin. ind.

9/27

c) suppose $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V and T is 1-1 and onto. Prove that $T(\beta) = \{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis for W.

PROOF: By (part b), since β lin. ind. and T 1-1, then $T(\beta)$ is lin. ind. By (THM 2.5), since T 1-1, $\text{rank}(T) = \dim(T) = \text{card}(\beta) = \text{card}(T(\beta))$. Since T is onto, $R(T) = W$; since $\text{card}(T(\beta)) = \text{rank}(T)$, by (THM 1.10 COR 2) $T(\beta)$ is a basis for $R(T) = W$.

PSET 3

2.1 #26

Jonathan Lam
Jonathan Lam
Prof. Mutcher
MA 326
Lin. Alg.
9/27/19

26. $T: V \to V$ is the projection on W_1 along W_2.
i.e., $W_1 \oplus W_2 = V$, $\forall v \in V$, $v = v_1 + v_2$, $v_1 \in W_1$, $v_2 \in W_2$,
$T(v) = T(v_1 + v_2) = v_1$

a) Prove that T is linear, and $W_1 = \{x \in V : T(x) = x\}$.

CLAIM: T linear.

PROOF: $\forall\ x = x_1 + x_2$, $y = y_1 + y_2$, $x, y \in V$, $x_1, y_1 \in W_1$, $x_2, y_2 \in W_2$, $\forall a \in F$, then:

$$T(ax+y) = T(a(x_1+x_2) + (y_1+y_2)) = T(ax_1 + ax_2 + y_1 + y_2)$$
$$= T((ax_1 + y_1) + (ax_2 + y_2))$$
$$= ax_1 + y_1 = aT(x) + T(y)$$

CLAIM: $W_1 = \{x \in V : T(x) = x\}$.

PROOF: (Proof by containment both ways).

$\forall x \in W_1$, then x also in V (since $W_1 \subseteq V$).
Since $0 \in W_2$, $x = x + 0 \in V$, and $T(x) = T(x+0) = x$.
Thus $x \in \{x \in V : T(x) = x\}$, so $W_1 \subseteq \{x \in V : T(x) = x\}$.
$\forall x \in \{x \in V : T(x) = x\}$, x lies in the codomain W_1 of T (since $T(x) = x$). Thus $x \in W_1$, so $\{x \in V : T(x) = x\} \subseteq W_1$.
By containment both ways, $W_1 = \{x \in V : T(x) = x\}$.

b) CLAIM: $W_1 = R(T)$

PROOF: Since the codomain of T is W_1, $R(T) \subseteq W_1$.
By (part a), $W_1 = \{x \in V : T(x) = x\} \subseteq R(T)$ (since W_1 is a subset of values of $T(x)$). By containment both ways, $W_1 = R(T)$

CLAIM: $W_2 = N(T)$

PROOF: Let $v \in V$. If $v \in W_2$, then since $0 \in W_1$, $T(v) = T(0 + v) = 0$, so $v \in N(T)$. If $v \notin W_2$, then $v = v_1 + v_2$, $v_1 \in W_1 \neq 0$, $v_2 \in W_2$, and $T(v) = T(v_1 + v_2) = v_1 \neq 0$, so $v \notin N(T)$. Thus $v \in N(T) \Leftrightarrow v \in W_2$, so $W_2 = N(T)$.

9/27/1[illegible]

c) Describe T if $W_1 = V$.
By (Part a), then $\forall x \in V$, $T(x) = x$, so $T = 1$ (identity transform).
Some properties of T: $\dim(W_1) = \text{rank}(T) = \dim(V)$ (by THM 1.11), and by (THM 2.5) and (THM 2.4), T is onto, 1-1, and $N(T) = W_2 = \{0\}$.

d) Describe T if W_1 is the zero subspace.
Then $\text{rank}(T) = 0$, and by the dimension theorem (THM 2.3), $\text{nullity}(T) = \dim(V)$. Since $W_2 = N(T)$ (by part b), by (THM 1.11), $N(T) = V$. This means that T is the zero transformation, i.e., $\forall x \in V$, $T(x) = 0$.

PSET 3

2.2 # 2(af), 5(bf), 11

Jonathan Lam
Prof. Mintchev
MA326
Lin. Alg.
9/27/19

2. Let β, γ be standard ordered bases for $\mathbb{R}^n$, $\mathbb{R}^m$, respectively. For each lin. transformation $T: \mathbb{R}^n \to \mathbb{R}^m$, compute $[T]_\beta^\gamma$.

a.) $T: \mathbb{R}^2 \to \mathbb{R}^3$, $T(a_1, a_2) = (2a_1 - a_2, 3a_1 + 4a_2, a_1)$.

$T(1,0) = (2,3,1) = 2(1,0,0) + 3(0,1,0) + 1(0,0,1)$

$T(0,1) = (-1,4,0) = -1(1,0,0) + 4(0,1,0) + 0(0,0,1)$

$\Rightarrow [T]_\beta^\gamma = \begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{pmatrix}$

f.) $T: \mathbb{R}^n \to \mathbb{R}^n$, $T(a_1, a_2, \ldots, a_n) = (a_n, a_{n-1}, \ldots, a_1)$

$T(1,0,0,\ldots,0,0) = (0,0,0,\ldots,0,1)$

$T(0,1,0,\ldots,0,0) = (0,0,0,\ldots,1,0)$

$T(0,0,1,\ldots,0,0) = (0,0,0,\ldots,1,0,0)$

$\vdots$

$T(0,0,\ldots,1,0) = (0,1,\ldots,0,0)$

$T(0,0,\ldots,0,1) = (1,0,\ldots,0,0)$.

$\Rightarrow [T]_\beta = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ \vdots & & & 1 & 0 \\ \vdots & & \iddots & & \vdots \\ 0 & 1 & & & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}$

(i.e., ones on counterdiagonal, 0's elsewhere).

5 b) $T: P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$, $T(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix}$

Compute $[T]_\beta^\alpha$. $\left(\beta = \{1, x, x^2\}, \alpha = \left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}\right)$

$T(1) = \begin{pmatrix}0&2\\0&0\end{pmatrix} = 0\begin{pmatrix}1&0\\0&0\end{pmatrix} + 2\begin{pmatrix}0&1\\0&0\end{pmatrix} + 0\begin{pmatrix}0&0\\1&0\end{pmatrix} + 0\begin{pmatrix}0&0\\0&1\end{pmatrix}$

$T(x) = \begin{pmatrix}1&2\\0&0\end{pmatrix} = 1\begin{pmatrix}1&0\\0&0\end{pmatrix} + 2\begin{pmatrix}0&1\\0&0\end{pmatrix} + 0\begin{pmatrix}0&0\\1&0\end{pmatrix} + 0\begin{pmatrix}0&0\\0&1\end{pmatrix}$

$T(x^2) = \begin{pmatrix}0&2\\0&2\end{pmatrix} = 0\begin{pmatrix}1&0\\0&0\end{pmatrix} + 2\begin{pmatrix}0&1\\0&0\end{pmatrix} + 0\begin{pmatrix}0&0\\1&0\end{pmatrix} + 2\begin{pmatrix}0&0\\0&1\end{pmatrix}$

$\Rightarrow [T]_\beta^\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

9/27/19

f) $f(x) = 3 - 6x + x^2$. Compute $[f(x)]_\beta$. $\left(\beta = \{1, x, x^2\}\right)$

$f(x) = 3(1) + (-6)(x) + (1)x^2$,

so $[f(x)]_\beta = \begin{pmatrix} 3 \\ -6 \\ 1 \end{pmatrix}$

11. Let V be v.s., $\dim(V) = n$, $T: V \to V$ linear.
Suppose W is T-invariant subsp. of V, $\dim(W) = k$.
Show that there is a basis β for V s.t. $[T]_\beta$ has the form:
$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, where A is $k \times k$ matrix, O is the $(n-k)\times k$ zero matrix

PROOF: Let $\beta' = \{u_1, u_2, \ldots, u_k\}$ be a basis for W.
Since W subsp. V, $\dim(W) \leq \dim(V)$, so $k \leq n$.
By Thm 1.10 (Cor 2), β' can be grown into a basis
$\beta = \{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_{n-k}\}$ for V.

Construct the matrix representation of T in the ordered basis β by finding the coordinate vector of each basis vector. Since W is T-invariant, $T(u_i) \in W$, $1 \leq i \leq k$, so
$T(u_i) = \sum_{i=1}^{k} a_i u_i + \sum_{i=1}^{k-n} (0) v_i$, and the coordinate vector is

$$\begin{array}{c} \left.\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right. \begin{array}{l} \left.\vphantom{\begin{matrix} a \\ a \\ \vdots \\ a \end{matrix}}\right\} k \\ \left.\vphantom{\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}}\right\} k-n \end{array} \end{array}$$

for all u_i (first k basis vectors), where a_i are arbitrary scalars

This will be the form of the first k column vectors (corresponding to β') in the matrix representation of T in the ordered basis β:

$$\begin{array}{c} k\left\{\vphantom{\begin{matrix} a \\ a \\ \vdots \\ a \end{matrix}}\right. \\ n-k\left\{\vphantom{\begin{matrix} c \\ \vdots \\ c \end{matrix}}\right. \end{array} \left(\begin{array}{cccc|ccc} a_{11} & a_{12} & \cdots & a_{1k} & b_{11} & \cdots & b_{1(n-k)} \\ a_{21} & a_{22} & & a_{2k} & b_{21} & & \\ \vdots & \vdots & & \vdots & \vdots & & \\ a_{k1} & a_{k2} & & a_{kk} & b_{k1} & \cdots & b_{k(n-k)} \\ \hline 0 & 0 & & 0 & c_{11} & \cdots & c_{1(n-k)} \\ \vdots & \vdots & \cdots & \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 & c_{(n-k)1} & \cdots & c_{(n-k)(n-k)} \end{array}\right) = \begin{pmatrix} A & B \\ O & C \end{pmatrix}$$

with the first k columns and the last $n-k$ columns,

where $A \in M_{k\times k}(F)$, $B \in M_{k\times(n-k)}(F)$, $C \in M_{(n-k)\times(n-k)}(F)$, and O is the $(n-k)\times k$ zero matrix. Elements in A, B, C are arbitrary.

PSET 4

2.3 # 13, 15

19/22

Jonathan Lan
Prof. Mintchev
MA 326
Lin. Alg
10/9/19

13. Let A, B be $n \times n$ matrices. Prove that $tr(AB) = tr(BA)$, $tr(A^t) = tr(A)$.

2.3.13 | 4.5/5
2.4.4 | 4
9.5

4.5

CLAIM: $tr(A^t) = tr(A)$.

PF: $tr(A^t) = \sum_{i=1}^{n} (A^t)_{ii} = \sum_{i=1}^{n} (A)_{ii} = tr(A)$

since $(A^t)_{ij} = (A)_{ji}$, $i=j$

what is this saying?

CLAIM: $tr(AB) = tr(BA)$.

PF: $tr(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} (A)_{ij}(B)_{ji}\right) = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} (B)_{ji}(A)_{ij}\right)$
$= \sum_{j=1}^{n} (BA)_{jj} = tr(BA)$

you are doing two things at once; you need to explain!

15. Let M, A be matrices s.t. MA is defined. If the jth column of A is a lin. comb. of a set of columns of A, prove that the jth column of MA is a lin. comb. of the corresponding columns of MA with the same corresponding coefficients.

PF: Let A_j indicate the jth column vector of A. Then
$A_j = \sum_{i=1}^{n} c_i A_i$, $c_i \in F$ (by hyp.). By (THM 2.13 a),
$(MA)_j = M(A_j) = M\left(\sum_{i=1}^{n} c_i A_i\right) = \sum_{i=1}^{n} c_i (MA_i) = \sum_{i=1}^{n} c_i (MA)_i$.

(By THM 2.13 a)

Thus $(MA)_j$ is a lin. comb. over the column vectors of MA with the same coefficients as A_j over the column vectors of A.

SET 4

2.3 #16.

Jonathan Lam
Prof. Mintchev
MA 326
Lin. Alg.
10/9/19

16. Let V be finite-dimensional v.s., $T \in \mathcal{L}(V)$.
a) If $\text{rank}(T) = \text{rank}(T^2)$, prove that $R(T) \cap N(T) = \{0\}$. Deduce that $V = R(T) \oplus N(T)$.

LEM 1 If $\text{rank}(T^i) = \text{rank}(T^{2i})$, $i \in \mathbb{Z}^+$, $R(T) \cap N(T) = \{0\}$.
PF: By (DIM THM), $\text{rank}(T^i) + \text{nullity}(T^i) = \dim(V) = \text{rank}(T^{2i}) + \text{nullity}(T^{2i})$. Since $\text{rank}(T^i) = \text{rank}(T^{2i})$, $\text{nullity}(T^i) = \text{nullity}(T^{2i})$. Since $N(T^i)$ subsp. $N(T^{2i})$ (since $\forall x \in N(T^i)$, $T^{2i}(x) = T^i(T^i(x)) = T^i(0) = 0$), by (THM 1.11) $N(T^i) = N(T^{2i})$.

Let $x \in R(T^i) \cap N(T^i)$. Then, $T^i(x) = 0$, and $\exists y \in V$ s.t. $T^i(y) = x$. Since $T^2(y) = T^i(T^i(x)) = 0$, $y \in N(T^{2i}) = N(T^i)$.
Thus $T(y) = 0 = x$ $\therefore$ $R(T^i) \cap N(T^i)$ consists only of the 0 vector. //

CLAIM: If $\text{rank}(T) = \text{rank}(T^2)$, $R(T) \cap N(T) = \{0\}$.
PF: use (LEM. 1) with $i = 1$.

CLAIM: $V = R(T) \oplus N(T)$.
PF: 1) $R(T), N(T)$ subsp. V by (THM 2.1)
2) $R(T) \cap N(T) = \{0\}$ by (LEMMA 1)
3) $R(T) + N(T)$ subsp. V, and $\dim(R(T) + N(T))$
$= \dim(R(T)) + \dim(N(T)) - \dim(R(T) \cap N(T))$
$= \text{rank}(T) + \text{nullity}(T) - 0 = \dim(V)$
(by (1.6 exercise #29 a), (DIM. THM)). Since $R(T) + N(T)$ subsp. V and $\dim(R(T) + N(T)) = \dim(V)$, $R(T) + N(T) = V$ by (THM 1.11).

By (DEF $\oplus$), $V = R(T) \oplus N(T)$.

10/9

b) Prove that $V = R(T^k) \oplus N(T^k)$ for some positive integer k.

Since $R(T^{i+1}) \subseteq R(T^i)$ $\forall\, i \in \mathbb{Z}^+$, $\operatorname{rank}(T^{i+1}) \leq \operatorname{rank}(T^i)$. Thus rank is non-increasing with increased iterations of applying T. Since $\operatorname{rank}(T^i)$ is lower bounded by 0, there is some (finite) integer k s.t. $\operatorname{rank}(T^k) = \operatorname{rank}(T^{k+1})$. $R(T^k)$ is thus T-invariant, so $\forall\, j > k$, $\operatorname{rank}(T^k) = \operatorname{rank}(T^j)$. Since $R(T^k) = R(T^{2k})$, by (LEM 1), $R(T^k) \cap N(T^k) = \{0\}$.

$V = R(T^k) \oplus N(T^k)$ is proved by definition analogously to in (part a). The identical conditions 1, 2, and 3 may be used, replacing T with T^k and using the new result above.

SET 4

2.3 #17.

Jonathan Lam
Prof. Mintchev
MA 326
Lin. Alg.
10/9/19

17. Let V be a v.s. Determine all linear transformations $T \in \mathcal{L}(V)$ s.t. $T = T^2$.

LEM 1 $R(T) = \{y : T(y) = y\}$.

PF: Let $x \in V$, and $R' = \{y : T(y) = y\}$.
If $T(x) = x$, then $x \in R'$.
If $T(x) = z$, $z \neq x \in V$, then $T(z) = T^2(x) = T(x) = z$
thus $z \in R'$.
Since all possible images of x lie in R', $R(T) \subseteq R'$.
Also, $R' \subseteq R(T)$ $\therefore$ $R(T) = R'(T) = \{y : T(y) = y\}$.

LEM 2 $V = R(T) \oplus N(T)$

PF: Let $x \in N(T) \cap R(T)$. Then $T(x) = 0$, and since $T(x) = x$, x must be 0. Thus $N(T) \cap R(T) = \{0\}$.
$T = T^2 \Rightarrow R(T) = R(T^2)$. By the previous exercise (2.3 #16 a), $V = R(T) \oplus N(T)$.

CLAIM: $\{T\} \subseteq \mathcal{L}(V)$ s.t. $T = T^2$ is the set of projections of V onto one its subspaces

PF: $\forall x \in V$, $x = T(x) + (x - T(x))$, where $T(x) \in R(T)$, $(x - T(x)) \in N(T)$ (since $T(x - T(x)) = T(x) - T^2(x) = T(x) - T(x) = 0$), and $T(x) = T(T(x) + T(x - T(x))) = T^2(x) + 0 = T(x) + 0$.
By (DEF projection), T is a projection onto $R(T)$ along $N(T)$.

Remarks / characterizations of V:

- T exists for every projection of V onto one of it's subspaces; i.e., whenever $\exists$ W, Z subsp. V s.t. $W \oplus Z = V$, $\exists T = T^2$ projecting V onto W
- By (LEM 1), $\forall x \in R(T)$, $y \notin R(T)$, $T(x) = x$, $T(y) = 0$. (In other words, T maps V to $R(T)$, and is the identity map within $R(T)$).

SET 4

2.4 # 2bef, 4

Prof. Mintchev
MA 326
Lin. Alg.
10/9/19

2. Determine whether T is invertible. Justify your answer.

b) $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) = (3a_1, -a_2, a_2, 4a_1)$.

Not invertible, since $\dim(\mathbb{R}^2) \neq \dim(\mathbb{R}^3)$, so $\mathbb{R}^2 \not\cong \mathbb{R}^3$ (THM 2.19) so no invertible lin. transforms from $\mathbb{R}^2$ to $\mathbb{R}^3$.

e) $T: M_{2\times2}(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + 2bx + (c+d)x^2$.

Not invertible. For some $f \neq 0 \in F$, $T\begin{pmatrix} 0 & 0 \\ f & -f \end{pmatrix} = 0 + 0x + (f-f)x^2$ $= 0 + 0x + 0x^2 = 0$, thus $N(T) \neq \{0\}$, so T not 1-1 (THM 2.4) and thus not invertible (since invertible transforms must be 1-1).

f) $T: M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix}$.

Invertible: let $U\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} b & a-b \\ c & d-c \end{pmatrix}$

Let $x = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in M_{2\times2}(\mathbb{R})$. Then:

$(UT)(x) = U\begin{pmatrix} e+f & e \\ g & g+h \end{pmatrix} = \begin{pmatrix} e & (e+f)-e \\ g & (g+h)-g \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} = x$

$(TU)(x) = T\begin{pmatrix} f & e-f \\ g & h-g \end{pmatrix} = \begin{pmatrix} f+(e-f) & f \\ g & g+h-g \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} = x$

Since $T \in \mathcal{L}(M_{2\times2}(\mathbb{R}))$, and $UT = TU = I_2$, T invertible.

4. Let A, B be $n\times n$ invertible matrices. Prove that (AB) is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

5

You should probably still have parentheses, since you are using associativity!

$(AB)(B^{-1}A^{-1}) = ABB^{-1}A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$

$(B^{-1}A^{-1})(AB) = B^{-1}A^{-1}AB = B^{-1}I_nB = B^{-1}B = I_n$

AB is invertible, and its inverse is $(B^{-1}A^{-1})$, since $(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I_n$.

PSET 4

2.4 # 17, 20.

Jonathan Lam
Prof. Matchel
MA 326
Lin. Alg.
10/9/19

17. Let V, W be finite-dim. v.s., $T \in \mathcal{L}(V, W)$ isomorphism. Let V_0 subsp. V.

a) Prove $T(V_0)$ subsp. W.

Since V_0 v.s., we can define $U: V_0 \to W$, $U(x) = T(x)$ $\forall x \in V_0$.
$T(V_0) = R(U)$, and $R(U)$ is a subsp. of W (THM 2.1)

b) Prove that $\dim(V_0) = \dim(T(V_0))$.

Let β_0 be a basis for V_0. Since T 1-1, $T(\beta_0)$ is also lin. ind, and it spans $T(V_0)$, so $T(\beta_0)$ is a basis for $T(V_0)$. $\dim(V_0) = \mathrm{card}(\beta_0) = \mathrm{card}(T(\beta_0)) = \dim(T(V_0))$.

20. Let $T \in \mathcal{L}(V, W)$ be a lin. transf. from n-dim. v.s. V to m-dim. v.s. W. Let β, γ be o.b.'s for V, W, respectively. Prove $\mathrm{rank}(T) = \mathrm{rank}(L_A)$, $\mathrm{nullity}(T) = \mathrm{nullity}(L_A)$, where $A = [T]_\beta^\gamma$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\varphi_\beta"'] & W \arrow[d, "\varphi_\gamma"] \\ F^n \arrow[r, "L_A"'] & F^m \end{tikzcd}$$

CLAIM: $\mathrm{rank}(T) = \mathrm{rank}(L_A)$.

PF: $\varphi_\beta, \varphi_\gamma$ are isomorphisms (THM 2.21), and $L_A \varphi_\beta = \varphi_\gamma T$. Since φ_β isomorphism, it is onto, so $R(\varphi_\beta) = \varphi_\beta(V) = F^n$. Then $R(L_A) = L_A(F^n) = L_A(\varphi_\beta(V))$
$= \varphi_\gamma(T(V)) = R(\varphi_\gamma T)$.

10/9/19

By exercise 17, $R(T) = T(V)$ subsp. W, and

$\dim(R(T)) = \dim(T(V)) = \dim(\varphi_\gamma(T(V))) = \dim(R(\varphi_\gamma T))$

$= \dim(R(L_A))$

$\Rightarrow \text{rank}(T) = \text{rank}(L_A)$.

CLAIM: $\text{nullity}(T) = \text{nullity}(L_A)$.

PF: Since $\varphi_\beta, \varphi_\gamma$ isomorphisms, $\dim(V) = \dim(F^n) = n$.

By (dim THM), $\text{rank}(L_A) + \text{nullity}(L_A) = \dim(F^n)$

$= \dim(V) = \text{rank}(T) + \text{nullity}(T)$, and since

$\text{rank}(T) = \text{rank}(L_A)$, then $\text{nullity}(L_A) = \text{nullity}(T)$

(by cancellation law of addition).

Jonathan Lam
Prof Mintchev
MA326
Lin. Alg
10/17/19

PSET 5

2.5 #3bf

15.5/20

2.5.3bf 5/5

3. For each o.B. β, β' for $P_2(\mathbb{R})$, find the change of coordinate matrix Q that changes β' to β.

b) $\beta = \{1, x, x^2\}$.
$\beta' = \{a_2x^2 + a_1x + a_0,\ b_2x^2 + b_1x + b_0,\ c_2x^2 + c_1x + c_0\}$.

$$\begin{aligned}\beta'_1 &= a_0(1) + a_1(x) + a_2(x^2)\\ \beta'_2 &= b_0(1) + b_1(x) + b_2(x^2)\\ \beta'_3 &= c_0(1) + c_1(x) + c_2(x^2)\end{aligned} \Rightarrow Q = \begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$$

5

LEM (Discussed in class, but not derived :):
Let $B = ([\beta_1]_\alpha\ [\beta_2]_\alpha\ \cdots\ [\beta_n]_\alpha)$,
$B' = ([\beta'_1]_\alpha\ [\beta'_2]_\alpha\ \cdots\ [\beta'_n]_\alpha)$,
where α is the s.o.B. for $P_2(\mathbb{R})$.
Then $B'_j = [\beta'_j]_\alpha = \sum_{i=1}^{n} Q_{ij}[\beta_i]_\alpha = BQ_j \Rightarrow B' = BQ$,
so $Q = B^{-1}B'$.

f.) $\beta = \{2x^2 - x + 1,\ x^2 + 3x - 2,\ -x^2 + 2x + 1\}$
$\beta' = \{9x - 9,\ x^2 + 21x - 2,\ 3x^2 + 5x + 2\}$
$\alpha = \{1, x, x^2\}$

Use Lemma above. $B = \begin{pmatrix} 1 & -2 & 1 \\ -1 & 3 & 2 \\ 2 & 1 & -1 \end{pmatrix}$, $B' = \begin{pmatrix} -9 & -2 & 2 \\ 9 & 21 & 5 \\ 0 & 1 & 3 \end{pmatrix}$

To solve, transform $(B \mid B') \to (I_n \mid Q)$ using EROs.

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 1 & -9 & -2 & 2 \\ -1 & 3 & 2 & 9 & 21 & 5 \\ 2 & 1 & -1 & 0 & 1 & 3 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & -9 & -2 & 2 \\ 0 & 1 & 3 & 0 & 19 & 7 \\ 0 & 5 & -3 & 18 & 5 & -1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 7 & -9 & 36 & 16 \\ 0 & 1 & 3 & 0 & 19 & 7 \\ 0 & 0 & -18 & 18 & -90 & -36 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 7 & -9 & 36 & 16 \\ 0 & 1 & 3 & 0 & 19 & 7 \\ 0 & 0 & 1 & -1 & 5 & 2 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 2 \\ 0 & 1 & 0 & 3 & 4 & 1 \\ 0 & 0 & 1 & -1 & 5 & 2 \end{array}\right)$$

$$\Rightarrow Q = \begin{pmatrix} -2 & 1 & 2 \\ 3 & 4 & 1 \\ -1 & 5 & 2 \end{pmatrix}.$$

Jonathan Lam
Prof Mintchev
MA326
Lin. Alg.
10/16/19

PSET 5

2.5 # 6 b d

6. For each matrix A and O.B. β, find $[L_A]_\beta$. Also, find an invertible matrix Q s.t. $[L_A] = Q^{-1}AQ$.

b.) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

$L_A(\beta_1) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+2 \\ 2+1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$L_A(\beta_2) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1-2 \\ 2-1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\Rightarrow [L_A]_\beta = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$

By (THM 2.23 COR 1), $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

d) $A = \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix}$, $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

$A(\beta_1) = \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 13+1-8 \\ 1+13-8 \\ 4+4-20 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ -12 \end{pmatrix} \Rightarrow 6 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

$A(\beta_2) = \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 13-1 \\ 1-13 \\ 4-4 \end{pmatrix} = \begin{pmatrix} 12 \\ -12 \\ 0 \end{pmatrix} = 12 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$A(\beta_3) = \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 13+1+4 \\ 1+13+4 \\ 4+4+10 \end{pmatrix} = \begin{pmatrix} 18 \\ 18 \\ 18 \end{pmatrix} = 18 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\Rightarrow [L_A]_\beta = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 18 \end{pmatrix}$,

by (THM 2.23 COR 1) again,

$Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{pmatrix}$.

$\begin{pmatrix} 16 \\ 7 \\ -3 \end{pmatrix}$

Jonathan Lam
Prof. Mintchev
MA326
Lin. Alg
10/16/19

PSET 5

2.5 #8

8) Let $T \in \mathcal{L}(V, W)$, V, W finite-dim. v.s. Let β, β' be o.B's for V, and let γ, γ' be o.B's for W. Then $[T]_{\beta'}^{\gamma'} = P^{-1}[T]_\beta^\gamma Q$, where Q is the matrix that changes β' coordinates to β coordinates, P is the matrix that converts γ' coordinates to γ coordinates.

PF: If P is the matrix that converts γ' to γ coordinates, then $P = [1_W]_{\gamma'}^{\gamma}$. Note that $1_W = (1_W)^{-1}$! By (THM 2.18), $[1_W]_\gamma^{\gamma'} = ([(1_W)^{-1}]_{\gamma'}^{\gamma})^{-1} = P^{-1}$; i.e., P^{-1} is the matrix to convert from γ' to γ coordinates. Similarly, $Q = [1_V]_{\beta'}^{\beta}$. This means the original expression on the right can be rewritten:

$$\left([1_W]_\gamma^{\gamma'}[T]_\beta^\gamma\right)[1_V]_{\beta'}^{\beta} = [1_W T]_\beta^{\gamma'}[1_V]_{\beta'}^{\beta} = [T]_\beta^{\gamma'}[1_V]_{\beta'}^{\beta}$$

$$= [T 1_V]_{\beta'}^{\gamma'} = [T]_{\beta'}^{\gamma'}$$

(by (THM 2.11)).

13. Let V be a finite-dim. v.s. over F, let $\beta = \{x_1, x_2, \ldots, x_n\}$ be a basis for V. Let Q be an $n \times n$ invertible matrix with entries from F. Define:

$$x_j' = \sum_{i=1}^{n} Q_{ij} x_i, \quad 1 \le j \le n,$$

and set $\beta' = \{x_1', x_2', \ldots, x_n'\}$.

a) Prove that β' is a basis for V.
card(β') = card(β), so it is sufficient to show that β' is linearly ind. to show that it is a basis for V (by THM 1.10 (COR 2)).

16
7
-3

10/1

Assume some lin. combs of β' equal to 0,

$$\sum_{j=1}^{n} a_j x'_j = 0 \quad , \quad a_j \in F \ \forall \ 1 \le j \le n.$$

Then: $$\sum_{j=1}^{n} a_j \left(\sum_{i=1}^{n} Q_{ij} x_i \right) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_j Q_{ij} \right) x_i = 0.$$

Since β lin. ind, $\sum_{j=1}^{n} a_j Q_{ij} = 0 \ \forall 1 \le i \le n.$

Thus, $$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} a_j Q_{1j} \\ \sum_{j=1}^{n} a_j Q_{2j} \\ \vdots \\ \sum_{j=1}^{n} a_j Q_{nj} \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} & \cdots & Q_{1n} \\ Q_{21} & Q_{22} & \cdots & Q_{2n} \\ \vdots & & \ddots & \vdots \\ Q_{n1} & Q_{n2} & \cdots & Q_{nn} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

$\Rightarrow \quad \bar{0} = QA$, where A is the $M_{n\times 1}(F)$ vector of a_j's, and $\bar{0}$ is the $M_{n\times 1}(F)$ zero vector.

Since Q is invertible, $$Q^{-1} 0 = (Q^{-1} Q) A$$
$$0 = I_n A$$
$$0 = A$$

$\Rightarrow a_1 = a_2 = \ldots = a_j = 0$, thus the coefficients are all zero, thus β' is lin. ind., thus β' is a basis.

<u>CLAIM</u>: Q is the change of coordinate matrix changing β' coordinates into β-coordinates.

<u>PF</u>: This is given trivially by the summation:

$$x'_j = \sum_{i=1}^{n} Q_{ij} x_i ,$$

which is equivalent to saying x'_j's coordinate vector in β is Q_j.

Jonathan Lam
Prof. Mitchel
MA 326
Lin. Alg
10/16/19

PSET 5

3.1 # 5, 9, 12

5. Prove that E is an elementary matrix IFF E^t is.

For any E formed by an elementary row operation, it's transpose was formed by the elementary column operation on the corresponding column(s), and vice versa.

Type I: The transpose of an elementary matrix formed by swapping two rows of I_n is the (elementary) matrix formed by swapping the corresponding columns in I_n, which is elementary.

Type II: Scaling a row i of I_n to form an elementary matrix scales E_{ii}, which is the same (elementary) matrix formed by scaling the ith column of I_n, so $E = E^t$ is elementary.

Type III: The transpose of an elementary matrix formed by adding a scaled factor of row i to row j of I_n is the matrix formed by adding the ith column to the jth column of I_n, which is elementary.

To prove the converse of this IFF statement, $(E^t)^t = E$, and the same arguments above can be used.

9. Prove that any elementary row (column) operation of type 1 can be obtained by a succession of three elementary row ops. of type 3 followed by one elementary row op. of type 2.

PF: Let A_{ik}, A_{jk} represent the kth element in the ith and jth row (column) vectors of a matrix A, and let $A_{ik} = c$, $A_{jk} = d$, $c, d \in F$, initially. Do the following elementary ops:

	initial				swapped
A_{ik}	c	c	$-d$	$-d$	d
A_{jk}	d	$c+d$	$c+d$	c	c

type III: add row (col) i to j. / type III: Subtract row (col) j from i / type III: add row (col) i to j / type II: scale row (col) i by (-1).

10/16

This sequence of steps swaps corresponding pairs of values A_{ik} and A_{jk} from the rows (columns) i, j, thus effectively performing a type II op.

12. Let A be an $m \times n$ matrix. Prove that there exists a sequence of elementary row operations of Type I and III that transforms A into an upper triangular matrix.

PF: An algorithm will be defined (very similar to Gaussian elimination) to transform a matrix to upper triangular.

① Let $i = 1$.
② Examine the ith column of A. If $\exists\, j > i$, s.t. $A_{ji} \neq 0$, continue to step ③. Otherwise, this column is correct; all entries below the main diagonal entry are zero. Increment i and repeat step ② until $i = \min\{m, n\}$.
③ (from step 2) perform a type I elementary row op. on rows i and j. Note that this maintains the property that all entries below the main diagonal entry are zero for any previous column i', since $i, j > i'$, and thus $A_i = A_j = 0$. Now, $A_{ii} \neq 0$.
④ For each row $k > i$, perform a type III op. on it, by adding the ith row scaled by $\left(-\frac{A_{ki}}{A_{ii}}\right)$. For every $i' < i$, $A_{ki'} - \frac{A_{ki}}{A_{ii}}(A_{ii'}) = 0 - \frac{A_{ki}}{A_{ii}}(0) = 0$, so the property that all entries below the main diagonal entry remain zero. Also, $A_{ki} - \frac{A_{ki}}{\cancel{A_{ii}}}(\cancel{A_{ii}}) = A_{ki} - A_{ki} = 0$, so all entries in the current column below the main diagonal become zero.
⑤ If $i < \min\{m, n\}$, increment i and repeat step ②.

This algorithm terminates after a finite number of steps ($i \leq \min\{m, n\}$, and up to $m - i + 1$ elementary row ops per column), and each iteration ensures that all entries below the main diagonal become zero, without ruining this property on earlier columns, thus the resulting matrix is upper triangular.

Jonathan Lam
Prof. Mintchev
MA 326
Lin. Alg
10/16/19

PSET 5.

3.2 #5eg

5. For each of these matrices, compute the rank and the inverse if it exists.

e)
$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 3 & 3 & 1 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \\ 0 & 3 & 3 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{array}\right)$$

By (THM 3.6), $\text{rank}(A) = 3$. By the method of finding inverses, A^{-1} exists and $A^{-1} = \begin{pmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}$

g)
$$\left(\begin{array}{cccc|cccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 5 & 5 & 1 & 0 & 1 & 0 & 0 \\ -2 & -3 & 0 & 3 & 0 & 0 & 1 & 0 \\ 3 & 4 & -2 & -3 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 2 & 0 & 1 & 0 \\ 0 & -2 & -5 & -3 & -3 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|cccc} 1 & 0 & -5 & -2 & 5 & -2 & 0 & 0 \\ 0 & 1 & 3 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 4 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -7 & 2 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -12 & -15 & 3 & -5 & 0 \\ 0 & 1 & 0 & 7 & 10 & -2 & 3 & 0 \\ 0 & 0 & 1 & -2 & -4 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -3 & 1 & 1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 51 & 15 & 7 & 12 \\ 0 & 1 & 0 & 0 & 31 & -9 & -4 & -7 \\ 0 & 0 & 1 & 0 & -10 & 3 & 1 & 2 \\ 0 & 0 & 0 & 1 & -3 & 1 & 1 & 1 \end{array}\right)$$

Thus by (THM 3.6), $\text{rank}(A) = 4$, and the matrix is invertible, and

$$A^{-1} = \begin{pmatrix} -51 & 15 & 7 & 12 \\ 31 & -9 & -4 & -7 \\ -10 & 3 & 1 & 2 \\ -3 & 1 & 1 & 1 \end{pmatrix}.$$

Jonathan Lam
Prof. Mintchev
MA 326
Lin. Alg.
10/16/19

PSET 5

3.2 # 6 af

6. For each of the following linear transformations T, determine whether T is invertible, and compute T^{-1} if it exists.

a) $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$, $T(f(x)) = f''(x) + 2f'(x) - f(x)$

$\beta = \{1, x, x^2\}$

$T(1) = 0 + 0 - 1$

$T(x) = 0 + 2 - x$ $\Rightarrow [T]_\beta = \begin{pmatrix} -1 & 2 & 2 \\ 0 & -1 & 4 \\ 0 & 0 & -1 \end{pmatrix}$

$T(x^2) = 2 + 4x - x^2$

By (THM 2.18), $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$. So we invert $[T]_\beta$ to find the matrix representation of the inverse transformation.

$$\left(\begin{array}{ccc|ccc} -1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -1 & 4 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & -2 & -2 & -1 & 0 & 0 \\ 0 & 1 & -4 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & -10 & -1 & -2 & 0 \\ 0 & 1 & -4 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2 & -10 \\ 0 & 1 & 0 & 0 & -1 & -4 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right)$$

So $[T^{-1}]_\beta = \begin{pmatrix} -1 & -2 & -10 \\ 0 & -1 & -4 \\ 0 & 0 & -1 \end{pmatrix}$, so T is invertible (because no zero rows, $\text{rank}(T) = n = 3$).

For some polynomial $f(x) = a_0 + a_1 x + a_2 x^2$, $[f]_\beta = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$,

and $[T^{-1}(f)]_\beta = [T^{-1}]_\beta [f]_\beta = \begin{pmatrix} -1 & -2 & -10 \\ 0 & -1 & -4 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -a_0 - 2a_1 - 10a_2 \\ -a_1 - 4a_2 \\ -a_2 \end{pmatrix}$

Thus $T^{-1}(a_0 + a_1 x + a_2 x^2) = (-a_0 - 2a_1 - 10a_2) + (-a_1 - 4a_2)x + (-a_2)x^2$

10/16

$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

6f. $T: M_{2\times 2}(\mathbb{R}) \to \mathbb{R}^4$, $T(A) = (tr(A), tr(A^t), tr(EA), tr(AE))$

Let $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then:

$tr(B) = tr\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 1 - 1 = 0$

$tr(B^t) = tr\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 1 - 1 = 0$

$tr(EB) = tr\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 0 + 0 = 0$

$tr(BE) = tr\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 0 + 0 = 0$

$\Rightarrow T(B) = (0, 0, 0, 0) = 0 \in \mathbb{R}^4$

Since $B \neq 0$, $B \in N(T)$, then $N(T) \neq \{0\}$,
so T is not 1-1 and not invertible.

Jonathan Lam
Prof Mintchev
MA326
Lin. Alg
10/16/19

PSET 5

3.2 # 14, 15

14. Let $T, U \in \mathcal{L}(V, W)$.

a) Prove $R(T+U) \subseteq R(T) + R(U)$.

PF. $\forall x \in R(T+U)$, $\exists$ $y \in V$ s.t. $x = (T+U)(v) = T(v) + U(v)$. Since $T(v) \in R(T)$, $U(v) \in R(U)$, $x \in \{x_1 + x_2 : x_1 \in R(T), x_2 \in R(U)\}$ $= R(T) + R(U)$. Thus $R(T+U) \subseteq R(T) + R(U)$.

b) Prove if W is finite-dimensional, then $\text{rank}(T+U) \leq \text{rank}(T) + \text{rank}(U)$.

PF: $\text{rank}(T) + \text{rank}(U) = \dim(R(T)) + \dim(R(U))$

$\geq \dim(R(T)) + \dim(R(U)) - \dim(R(T) \cap R(U))$

$= \dim(R(T) + R(U))$ (by THM 1.11)

$\geq \dim(R(T+U))$ (by part a)

$= \text{rank}(T+U)$

c) Deduce from (b) that $\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$ for any $m \times n$ matrices A, B.

PF: $\text{rank}(A+B) = \text{rank}(L_{A+B}) = \text{rank}(L_A + L_B)$
(by def rank of matrix) (by THM 2.15),

and $\text{rank}(A) = \text{rank}(L_A)$, $\text{rank}(B) = \text{rank}(L_B)$.

Since $\text{rank}(L_A + L_B) \leq \text{rank}(L_A) + \text{rank}(L_B)$ from part a, then substitute ranks of matrices to get:

$\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$.

10/16/19

15 Suppose A, B are matrices with n rows. Prove that $M(A|B) = (MA|MB)$ for any $m \times n$ matrix M.

PF: Let $A \in M_{n\times p}(F)$, $B \in M_{n \times q}(F)$. It can be shown that corresponding columns of $M(A|B)$ and $(MA|MB)$ are equal.

Fix some column j of $M(A|B)$, $1 \le j \le p+q$. Then:

$$(M(A|B))_j = M((A|B)_j) \quad \text{(THM 2.13)}$$

$$= M\begin{cases} A_j & , 1 \le j \le p \\ B_{j-p} & , p+1 \le j \le p+q \end{cases} \quad \text{(DEF. Aux. matrix)}$$

$$= \begin{cases} M(A_j) & , 1 \le j \le p \\ M(B_{j-p}) & , p+1 \le j \le p+q \end{cases}$$

$$= \begin{cases} (MA)_j & , 1 \le j \le p \\ (MB)_{j-p} & , p+1 \le j \le p+q \end{cases} \quad \text{(THM 2.13)}$$

$$= (MA|MB)_j \quad \text{(DEF aux matrix)}$$

Since the corresponding columns of $M(A|B)$ and $(MA|MB)$ are equal, and since both matrices have the same dimension (both $\in M_{m\times(p+q)}(F)$), they are equal.

Jonathan Lam
Prof. Mintchev
MA 326
Lin. Alg
10/23/19

PSET 6

3.3 # 2d, 3d, 7ae

17/20

3.3. 10	5
3.4. 14	5
4.2. 30	

for next

2. Find the dimension of and a basis for the solution set.

d) $$\begin{cases} 2x_1 + x_2 - x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \\ x_1 + 2x_2 - 2x_3 = 0 \end{cases} \rightarrow \left(\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 2 & -2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 1 & 2 & -2 & 0 \\ 2 & 1 & -1 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\uparrow$
$x_3 = t$

Then $x_2 - t = 0 \Rightarrow x_2 = t$,
$x_1 = 0$, so the solution set is $t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$,
so a basis is $\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$.

3. Find all sol'ns to the following systems.

d) $$\begin{aligned} 2x_1 + x_2 - x_3 &= 5 \\ x_1 - x_2 + x_3 &= 1 \\ x_1 + 2x_2 - 2x_3 &= 4 \end{aligned}$$

By inspection, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \rightarrow \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$ is a solution to this system

since $2(2) + 3 - 2 = 5$, $2 - 3 + 2 = 1$, $2 + 2(3) - 2(2) = 4$.
This is a particular solution, and the above exercise (2d) has the solution to the associated homogeneous equation, $t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. By (THM 3.9), the general solution is the set sum of these solutions, i.e., $\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $t \in \mathbb{R}$

10/

7 Determine which of the following systems of lin. eqns. has a soln.

a) $\begin{cases} x_1 + x_2 - x_3 + 2x_4 = 2 \\ x_1 + x_2 + 2x_3 = 1 \\ 2x_1 + 2x_2 + x_3 + 2x_4 = 4 \end{cases} \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 2 \\ 1 & 1 & 2 & 0 & 1 \\ 2 & 2 & 1 & 2 & 4 \end{array}\right)$

$\rightarrow \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 2 \\ 0 & 0 & 3 & -2 & -1 \\ 0 & 0 & 3 & -2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 2 \\ 0 & 0 & 3 & -2 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$

By (Thm), this situation ...

e) $\begin{cases} x_1 + 2x_2 - x_3 = 1 \\ 2x_1 + x_2 + 2x_3 = 3 \\ x_1 - 4x_2 + 7x_3 = 4 \end{cases} \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 1 & 2 & 3 \\ 1 & -4 & 7 & 4 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -3 & 4 & 1 \\ 0 & -6 & 8 & 3 \end{array}\right)$

$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -3 & 4 & 1 \\ 0 & -3 & 4 & \frac{3}{2} \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -3 & 4 & 1 \\ 0 & 0 & 0 & \frac{1}{2} \end{array}\right)$

By the same reasoning as in exercise (7a) above, this system is inconsistent (no solutions).

Jonathan Lam
Prof. Matchett
MA 326
Lin. Alg.
10/24/19

PSET 6

3.3 #9, 10.

9. Prove that the system of linear equations $Ax=b$ has a solution IFF $b \in R(L_A)$.

CLAIM: If $Ax=b$ has a solution, then $b \in R(L_A)$.
PF: If $Ax=b$ has a solution, $\exists x \in F^n$ s.t. $L_A(x)=b$. Thus $b \in R(L_A)$.

CLAIM: If $b \in R(L_A)$, then $Ax=b$ is consistent.
PF: $b \in R(L_A) = Col(A)$. Then b can be expressed as a linear combination of columns in A. Then $Col(A|b) = Col(A \cup \{b\}) = Col(A)$, so $rank(A|b) = dim(Col(A|b)) = dim(Col(A)) = rank(A)$. By (THM 3.11), $Ax=b$ is consistent.

---

10. Prove or give a counterexample to the following statement: If the coefficient matrix of a system of m linear equations in n unknowns has rank m, then the system has a solution.

5

This is true. PF: Let $Ax=b$ have a solution, $A \in M_{m \times n}(F)$, $rank(A)=m$. $Rank(A|b) \geq m$, since it is adding a column to the column space (which can't reduce rank). $Rank(A|b) \leq m$, since rank is upper bounded by $min(m,n)$, and $(A|b)$ has m rows. Thus $Rank(A|b) = m = Rank(A)$, thus by (THM 3.11) $Ax=b$ is consistent.

PSET 6

3.4 #2dg, 7, 10, 14, 15.

Jonathan Lam
Prof. Mintchev
MA 326
Lin. Alg.
10/23/19

2. Use Gaussian elimination to solve.

d)
$$\begin{aligned} x_1 - x_2 - 2x_3 + 3x_4 &= -7 \\ 2x_1 - x_2 + 6x_3 + 6x_4 &= -2 \\ -2x_1 + x_2 - 4x_3 - 3x_4 &= 0 \\ 3x_1 - 2x_2 + 9x_3 + 10x_4 &= -5 \end{aligned} \rightarrow \left(\begin{array}{cccc|c} 1 & -1 & -2 & 3 & -7 \\ 2 & -1 & 6 & 6 & -2 \\ -2 & 1 & -4 & -3 & 0 \\ 3 & -2 & 9 & 10 & -5 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & -1 & -2 & 3 & -7 \\ 0 & 1 & 10 & 0 & 12 \\ 0 & -1 & -8 & 3 & -14 \\ 0 & 1 & 15 & 1 & 16 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & -1 & -2 & 3 & -7 \\ 0 & 1 & 10 & 0 & 12 \\ 0 & 0 & 2 & 3 & -2 \\ 0 & 0 & 5 & 1 & 4 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 8 & 3 & 5 \\ 0 & 1 & 10 & 0 & 12 \\ 0 & 0 & 2 & 3 & -2 \\ 0 & 0 & 0 & -\frac{13}{2} & 9 \end{array}\right)$$

$$-\frac{13}{2}x_4 = 9 \Rightarrow x_4 = -\frac{9 \cdot 2}{13} = \frac{-18}{13}$$

$$2x_3 + 3\left(\frac{-18}{13}\right) = -2 \Rightarrow x_3 = \frac{-2 + 3\left(\frac{18}{13}\right)}{2}$$

$$= -1 + \frac{27}{13} = \frac{14}{13}$$

$$x_2 + 10\left(\frac{14}{13}\right) = 12 \Rightarrow x_2 = 12 - \frac{140}{13} = \frac{156}{13} - \frac{140}{13} = \frac{16}{13}$$

$$x_1 + 8\left(\frac{14}{13}\right) + 3\left(\frac{-18}{13}\right) = 5 \Rightarrow x_1 = 5 + \frac{54}{13} - \frac{112}{13}$$

$$= \frac{65}{13} + \frac{54}{13} - \frac{112}{13}$$

$$= \frac{119}{13} - \frac{112}{13} = \frac{7}{13}$$

Thus the solution is $\frac{1}{13}\begin{pmatrix} 7 \\ 16 \\ 14 \\ -18 \end{pmatrix}$.

2g) $2x_1 - 2x_2 - x_3 + 6x_4 - 2x_5 = 1$
$x_1 - x_2 + x_3 + 2x_4 - x_5 = 2$
$4x_1 - 4x_2 + 5x_3 + 7x_4 - x_5 = 6$

$$\rightarrow \left(\begin{array}{ccccc|c} 2 & -2 & -1 & 6 & -2 & 1 \\ 1 & -1 & 1 & 2 & -1 & 2 \\ 4 & -4 & 5 & 7 & -1 & 6 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccccc|c} 1 & -1 & 1 & 2 & -1 & 2 \\ 0 & 0 & -3 & 2 & 0 & -3 \\ 0 & 0 & 1 & -1 & 3 & -2 \end{array}\right) \rightarrow \left(\begin{array}{ccccc|c} 1 & -1 & 1 & 2 & -1 & 2 \\ 0 & 0 & 1 & -1 & 3 & -2 \\ 0 & 0 & 0 & -1 & 9 & -9 \end{array}\right)$$

$\uparrow$ $x_2 = s$ $\qquad$ $\uparrow$ $x_5 = t$

$-x_4 + 9t = -9 \Rightarrow x_4 = 9t + 9$

$x_3 - (t+9) + 3t = -2 \Rightarrow x_3 = -2 - 3t + (9t+9) = 7 + 6t$

$x_1 - s\ t\ (7+6t) + 2(9t+9) - t = 2$

$\Rightarrow x_1 = 2 + s + t - (7+6t) - 2(9t+9)$

$= -23 + s - 23t$

So the solution is $\begin{pmatrix} -23 \\ 0 \\ 7 \\ 9 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -23 \\ 0 \\ 6 \\ 9 \\ 1 \end{pmatrix}$

Jonathan Lam
Prof. Mutcher
MA 326
Lin. Alg
10/23/19

PSET 6

3.4 # 7, 13, 14, 15.

7. Find a subset of $\left\{\begin{pmatrix}2\\-3\\1\end{pmatrix}, \begin{pmatrix}1\\4\\-2\end{pmatrix}, \begin{pmatrix}-8\\12\\-4\end{pmatrix}, \begin{pmatrix}1\\37\\-17\end{pmatrix}, \begin{pmatrix}-3\\-5\\8\end{pmatrix}\right\}$.

that is a basis for $\mathbb{R}^3$.

This can be solved with an application of (THM 3.16), i.e., the columns in the RREF of the matrix of vectors that equal e_i are the corresponding columns to the basis of the column space of the original matrix (F^3).

$$\begin{pmatrix}2 & 1 & -8 & 1 & -3\\ -3 & 4 & 12 & 37 & -5\\ 1 & -2 & -4 & -17 & 8\end{pmatrix} \rightarrow \begin{pmatrix}1 & -2 & -4 & -17 & 8\\ 0 & 5 & 0 & 35 & -19\\ 0 & -2 & 0 & -14 & -19\end{pmatrix}$$

$$\rightarrow \begin{pmatrix}1 & -2 & -4 & -17 & 8\\ 0 & 1 & 0 & 7 & \frac{-19}{5}\\ 0 & 1 & 0 & 7 & \frac{19}{2}\end{pmatrix} \rightarrow \begin{pmatrix}1 & -2 & -4 & -17 & 8\\ 0 & 1 & 0 & 7 & -\frac{19}{5}\\ 0 & 0 & 0 & 0 & \frac{19}{2}+\frac{19}{5}\end{pmatrix}$$

$$\rightarrow \begin{pmatrix}1 & -2 & -4 & -17 & 0\\ 0 & 1 & 0 & 7 & 0\\ 0 & 0 & 0 & 0 & 1\end{pmatrix} \rightarrow \begin{pmatrix}1 & 0 & -4 & -3 & 0\\ 0 & 1 & 0 & 7 & 0\\ 0 & 0 & 0 & 0 & 1\end{pmatrix}$$

Using (THM 3.16), $\left\{\begin{pmatrix}2\\-3\\1\end{pmatrix}, \begin{pmatrix}1\\4\\-2\end{pmatrix}, \begin{pmatrix}-3\\-5\\8\end{pmatrix}\right\}$ is lin. ind.

and thus is a basis for $\mathbb{R}^3$.

10/2

10 Let $V = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 - 2x_2 + 3x_3 - x_4 + 2x_5 = 0\}$

a) Show that $S = \{(0,1,1,1,0)\}$ is a lin. ind. subset of V.

PF: Since S is a singleton nonzero vector in V, it is linearly independent.

b) Extend S to a basis for V.

By inspection, $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \in V$, and are linearly ind.

Using the same method as in (exercise 9):

$$\begin{pmatrix} 0 & 1 & 2 & -3 & 2 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -3 & 2 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -3 & 2 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -3 & 2 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -3 & 3 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, $\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is lin. ind. by (THM 3.16c).

Since V is constrained by a homogeneous linear system of 1 equation in 5 variables, there are four free variables $\Rightarrow \dim(V) = 4$. Since cardinality of the set above has cardinality 4, it is a basis for V.

PSET 6

3.4 # 14, 15

Jonathan Lam
Prof. Mintchev
MA 326
Lin. Alg.
10/2/19

Your argument can gain clarity by applying contrapositive reasoning throughout

5

14. If $(A|b)$ is in RREF, prove that A is also in RREF.

To prove this, we need to show the three conditions in the definition of RREF are true for A. Let $(A|b) \in M_{m\times n}(F)$, $A \in M_{m\times(n-1)}(F)$

a) CLAIM: Any row containing a nonzero vector ? precedes any all-zero row.

PF: Assume that there is a all-zero row in A. Then the first $(n-1)$ elements of the corresponding row in $(A|b)$ must also be 0, and the last element may be either zero or nonzero. If it is zero, then by (c) of (def. RREF for $A|b$), this row is below any nonzero row in $(A|b)$ (and thus below any nonzero row in A). If the mth element is nonzero, then it is a pivot element and thus must be below any row where the pivot occurs further left by (def. RREF (c)); thus it occurs below any nonzero row of A

b) CLAIM: The first nonzero entry in each row is the only nonzero entry in its column.

PF: The first nonzero entry for a row in A is also the first nonzero entry for its row in $(A|b)$. By (def RREF (b)), this pivot element in $(A|b)$ is the only element of its column, which is also the column for the pivot element in A.

c) CLAIM: The first nonzero entry in each row is 1 and it occurs in a column to the right of the first nonzero entry in the preceding row.

PF. For any pivot element in A, it is also the pivot element of $(A|b)$, and must be 1 and below and to the right of the pivot element in the preceding row, which is also the pivot element in the preceding row of A.

10/24/19

15. Prove that the RREF of a matrix is unique.

Let $A \in M_{m\times n}(F)$. Let B be the RREF of A. Then, if $n > 1$, then $B = (B' \mid d)$, $B' \in M_{m\times(n-1)}(F)$, $d \in M_{m\times 1}(F)$, and by (exercise 14), is in RREF. By induction, the set of submatrices $S = \{x \in M_{m\times i}(F) : x$ contains the first i columns of $B\}$ are also in RREF. This proof will use induction, beginning on S_1 (the first column of S) and adding one column at a time.

Inductive base case: The RREF S_1 is uniquely defined by A.

PF: S_1 is row-equivalent to A_1 (first column of A). If $A_1 = 0 \in F^m$, then RREF of $A_1 = S_1 = 0 \in F^m$. If not, since S_1 is in RREF, the only other valid $M_{m\times 1}(F)$ RREF is e_1, thus S_1 must be e_1. Thus S_1 is uniquely defined by A_1.

Inductive hypothesis: If S_i is uniquely defined by A, then so is S_{i+1}.

PF: $S_{i+1} = (S_i \mid b)$, $b = B_{i+1} \in F^m$ ($i+1^{th}$ column of B).

If rank(S_{i+1}) = rank$(S_i) + 1 = k$, then e_{i+1} cannot exist as a column in S_i. However, by (THM 3.16 (b)), e_{i+1} must exist as a column in S_{i+1} if it is an RREF with rank $i+1$, so S_{i+1} must be $_{i+1}$.

Else, rank(S_{i+1}) = rank$(S_i) = k$ (since rank$(S_{i+1}) \geq$ rank(S_i)), and $b \in R(S_i)$. Then b is expressable as a linear combination over $\{e_1, e_2, \ldots, e_k\}$, which are vectors in S_i by (THM 3.16b). By (THM 3.16d), the coefficients to this linear combination are the same as the coefficients of the linear combination over the corresponding columns of A (which are a basis) yielding A_{i+1}. Since A_{i+1} is determined by a unique linear combination over a basis, the coefficients to the linear combinations determining A_{i+1} and b are uniquely determined by A, thus b is unique. Since S_i, b uniquely determined by A, S_{i+1} is uniquely determined by A.

PSET 7

Jonathan Lam
Prof. Mintchev
MA326
Lin. Alg.
11/7/19

33/35

4.2.30 | 4.5
4.4.5 | 4
5.1.12 | 5
5.1.19 | 5
18.5

4.1 # 12, 22

12. The classical adjoint of a 2x2 matrix $A \in M_{2\times 2}(F)$ is the matrix:

$$C = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$$

a) CLAIM: $CA = AC = [\det(A)] I_2$

PF: $CA = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & A_{22}A_{12} - A_{12}A_{22} \\ -A_{11}A_{21} + A_{11}A_{21} & -A_{12}A_{21} + A_{11}A_{22} \end{pmatrix}$

$= \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & 0 \\ 0 & A_{11}A_{22} - A_{12}A_{21} \end{pmatrix} = (A_{11}A_{22} - A_{21}A_{12})\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = [\det(A)] I_2$

and $AC = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} = \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & -A_{11}A_{12} + A_{11}A_{12} \\ A_{21}A_{22} - A_{21}A_{22} & -A_{12}A_{21} + A_{11}A_{22} \end{pmatrix}$

$= \begin{pmatrix} A_{11}A_{22} - A_{21}A_{12} & 0 \\ 0 & A_{11}A_{22} - A_{12}A_{21} \end{pmatrix} = (A_{11}A_{22} - A_{21}A_{12})\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = [\det(A)] I_2$

b) CLAIM: $\det(C) = \det(A)$

PF: $\det(C) = C_{11}C_{22} - C_{12}C_{21} = (A_{22})(A_{11}) - (-A_{12})(-A_{21})$
$= A_{11}A_{22} - A_{12}A_{21} = \det(A)$

c) CLAIM: The classical adjoint of A^t is C^t.

PF: $A^t = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix}$

classical adjoint of $A^t = \begin{pmatrix} (A^t)_{22} & -(A^t)_{12} \\ -(A^t)_{21} & (A^t)_{11} \end{pmatrix} = \begin{pmatrix} A_{22} & -A_{21} \\ -A_{12} & A_{11} \end{pmatrix} = C^t$

d) CLAIM: If A inv., then $A^{-1} = [\det(A)]^{-1} C$

PF: This statement is proved in (THM 4.2).

11/

11. Let $\delta : M_{2\times 2}(F) \to F$ be a fn. with the following properties:

i) δ is a linear fn. of each row of the matrix when the other row is held fixed.

ii) If the two rows of $A \in M_{2\times 2}(F)$ are equal, then $\delta(A) = 0$.

iii) If I is the 2×2 identity matrix, then $\delta(I) = 1$.

Prove that $\delta(A) = \det(A)$ $\forall A \in M_{2\times 2}(F)$.

<u>LEMMA</u>: $\delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$

<u>PF</u>:
$$\delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \underset{\text{by (i)}}{=} \delta\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \cancel{(-1)\underbrace{\delta\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}}_{=0 \text{ by (ii)}}} \underset{\text{by (i)}}{=} (-1)\delta\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \cancel{\underbrace{\delta\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}}_{=0 \text{ by (ii)}}}$$

$$= (-1)\left(\underbrace{\delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{=1 \text{ by (iii)}} + \cancel{\underbrace{\delta\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}}_{=0 \text{ by (ii)}}}\right) = -1.$$

<u>CLAIM</u>: $\forall A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(F)$, $\delta(A) = ad - bc = \det(A)$.

<u>PF</u>:
$$\delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\delta\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} + b\delta\begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix}$$

$$= a\left(c\underbrace{\delta\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}}_{=0 \text{ by (ii)}} + d\underbrace{\delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{=1 \text{ by (iii)}}\right) + b\left(c\underbrace{\delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{=-1 \text{ by (LEMMA)}} + d\underbrace{\delta\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}}_{=0 \text{ by (ii)}}\right)$$

$$= a(0 + d) + b(c(-1) + 0)$$

$$= ad - bc$$

PSET 7

Jonathan Lam
Prof. Mintchev
MA 326
Lin. Alg.
11/7/19

4.2 # 28, 29, 30

28. Compute $\det(E_i)$ if E_i is an elementary matrix of type i

Case 1: Let $i=1$. Then E_1 is obtained by interchanging two rows of I, so by (THM 4.5) $\det(E_1) = -\det(I) = -1$.

Case 2: Let $i=2$. Then E_2 is obtained by multiplying one row j of I_n by some $k \in F$. Then that row is $e_j \in F^n$, and a cofactor expansion along that row of E_2 is $\sum_{i=1}^{n} (-1)^{i+j} E_{ij} \left|\tilde{E}_{ij}\right|$

$= (-1)^{j+j}(k)(1) = k$, since the only nonzero element of that row is the element of the diagonal, and $\tilde{E}_{ii} = I_{n-1} \Rightarrow \det(\tilde{E}_{ii}) = 1$.

Case 3: Let $i=3$. Then E_3 is determined by adding a multiple of one row of I to another. Then $\det(E_3) = \det(I)$ by (THM 4.6).

---

29. Prove that if E is an elementary matrix, then $\det(E^t) = \det(E)$.

LEM: Type 1 and Type 2 elementary matrices are symmetric.
PF: For type 1 elem. matrices, two rows of I_n are swapped; let these be rows j and k of I_n. Then $E_{jk} = E_{kj} = 1$, $E_{jj} = E_{kk} = 0$, and the other entries are all untouched; therefore E remains symmetric. Similarly, a type 2 elem. matrix is clearly symmetric, since it is a diagonal matrix.

Case 1: For type 1 and type 2 elementary matrices:
PF: By the Lemma, type 1 and type 2 elementary matrices E_i are symmetric, so $E_i = E_i^t$, and $\det(E_i) = \det(E_i^t)$.

Case 2: For type 3 elementary matrices.
PF: The transpose of a type 3 elementary matrix formed by adding k times row i to row j is also a type 3 elementary matrix formed by adding k times row j to row

i. Since E_3 and E_3^t are type 3 elementary matrices, by (exercise 28) $\det(E_3) = \det(E_3^t) = 1$.

---

30. Let the rows of $A \in M_{n\times n}(F)$ be $a_1, a_2, \ldots, a_n$, and let B be the matrix in which the rows are $a_n, a_{n-1}, \ldots, a_1$. Calculate $\det(B)$ in terms of $\det(A)$.

4.5

This should be done by induction.

if n even:

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n/2} \\ a_{n/2+1} \\ \vdots \\ a_{n-1} \\ a_n \end{pmatrix}$$

$n/2$ swaps

if n odd:

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{\frac{n-1}{2}} \\ a_{\frac{n+1}{2}} \\ a_{\frac{n+2}{2}} \\ \vdots \\ a_{n-1} \\ a_n \end{pmatrix} \leftarrow \text{no swap for central element}$$

$\frac{n-1}{2}$ swaps

There are $\lfloor n/2 \rfloor$ row interchanges. By (THM 4.5), each interchange flips the sign of the determinant. Thus,

$\det(B) = (-1)^{\lfloor n/2 \rfloor} \det(A)$.

PSET 7

Jonathan Lam
Prof. Mintchev
MA 326
Lin. Alg.
11/7/19

4.3 # 11, 12, 15

11. A matrix $M \in M_{n\times n}(\mathbb{C})$ is called skew-symmetric if $M^t = -M$. Prove that if M is skew-symmetric and n is odd, then M is not invertible. What happens if n is even?

CLAIM: If M skew symmetric, n odd, then n not invertible.

PF: By (THM 4.8), $\det(A) = \det(A^t)$. Thus

$$\det(M) = \det(M^t) = \det(-M) = (-1)^n \det(M).$$

(obtained by n row scaling by (-1)).

Thus $\det(M) = (-1)\det(M) \Rightarrow \det(M) = 0 \Rightarrow$ M is not invertible.

This restriction is not true if n is even, since the above restriction is $\det(M) = (-1)^n \det(M) \Rightarrow \det(M) = \det(M)$ if n is even. Both invertible and non-invertible matrices with even n exist, e.g. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (invertible) and $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (non invertible).

12. A matrix $Q \in M_{n\times n}(\mathbb{R})$ is called orthogonal if $QQ^t = I$. Prove that if Q is orthogonal, then $\det(Q) = \pm 1$.

PF: By (THM 4.8), $\det(Q) = \det(Q^t)$. By (THM 4.7),

$$\det(QQ^t) = \det(Q)\det(Q^t) = \det^2(Q) = \det(I) = 1$$

Thus $\det^2(Q) = 1$, so $\det(Q) = \pm 1$.

15. Prove that if $A, B \in M_{n\times n}(F)$ are similar, then $\det(A) = \det(B)$.

PF: By (DEF similarity), $\exists\ Q$ inv. $\in M_{n\times n}(F)$ s.t. $A = Q^{-1}BQ$. By (COR. to THM. 4.7), $\det(Q^{-1}) = (\det(Q))^{-1}$. By (THM 4.7), determinants of products of matrices are equivalent to products of the determinants, i.e., $\det(A) = \det(Q^{-1}BQ)$ $= \det(Q^{-1})\det(B)\det(Q) = \frac{1}{\det(Q)}\det(B)\det(Q) = \frac{\cancel{\det(Q)}}{\cancel{\det(Q)}}\det(B)$ $= \det(B)$

4.4 #5, 6

4

5. Suppose that $M \in M_{(m+n)\times(m+n)}(F)$ can be written in the form,

$M = \begin{pmatrix} A & B \\ 0 & I_m \end{pmatrix}$, where A is square. Prove that $\det(M) = \det(A)$.

PF: Denote the square matrix consisting of the first i rows and i columns of M as M_i. Calculate the determinant of M by the cofactor expansion along row $m+n$ (the last row).

✓8 Induction

The determinant is $\sum_{i=1}^{m+n} (-1)^{i+(m+n)} M_{(m+n)i} |\tilde{M}_{(m+n)i}|$. Since all but the last term are zeroes, this sum is equal to $(-1)^{(m+n)+(m+n)} M_{(m+n)(m+n)} |\tilde{M}_{(m+n)(m+n)}|$ $= (1)(1) |\tilde{M}_{(m+1)(m+1)}|$. Note that this is the determinant of $M_{(m+n-1)}$, so $\det(M_{m+n}) = \det(M_{m+n-1})$. This identical logic can be applied inductively to $M_{m+n-1}, M_{m+n-2}, \ldots, M_{n+1}$, since the determinant of each of these matrices may be evaluated as a cofactor expansion of their respective bottom rows, which consist of all zeroes with a leading 1 in the last column. Hence,

$\det(M) = \det(M_{m+n}) = \det(M_{m+n-1}) = \ldots = \det(M_{n+1}) = \det(M_n) = \det(A)$

---

6. Prove that if $M \in M_{(m+n)\times(m+n)}(F)$ can be written in the form $M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, where A and C are square matrices, then $\det(M) = \det(A) \cdot \det(C)$.

PF: Let A be an $n\times n$ matrix and C be an $m\times m$ matrix.

Let $F = \begin{pmatrix} I_n & 0 \\ 0 & C \end{pmatrix}$. The determinant of F may be obtained by doing the cofactor expansion along the first row, which evaluates to $\sum_{i=1}^{n} (-1)^{i+1} F_{1i} |\tilde{F}_{1i}| = (-1)^{1+1}(1)|\tilde{F}_{11}| = |\tilde{F}_{11}|$. This same logic can be recursively applied n times (by similar logic to that stated in (exercise 5, above) to obtain $\det(F) = \det(C)$.

Now, let $G = \begin{pmatrix} A & B \\ 0 & I_m \end{pmatrix}$. Multiplying F and G, we obtain:

$$(FG)_{ij} = \begin{cases} \sum_{k=1}^{n} (e_i)_k A_{kj} = A_{ij}, & i,j \le n \\ \sum_{k=1}^{n} (e_i)_k B_{k(j-n)} = B_{i(j-n)}, & i \le n,\ j > n \\ 0, & i > n,\ j \le n \\ \sum_{k=1}^{m} C_{(i-n)(k+n)} (e_k)_{(k+n)} = C_{(i-n)(j-n)}, & i, j > n \end{cases} \Rightarrow FG = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = M.$$

Then $\det(M) = \det(F)\det(G) = \det(C) \cdot \det(A)$

5.1 # 3cd, 4deg, 7, 12, 14, 15, 19, 22.

Jonathan Lam
Prof Matchev
MA 326
Lin Alg.
11/7/19.

3. For each of the following matrices $A \in M_{n\times n}(F)$:
i) Determine all eigenvalues of A.
ii) For each eigenvalue of A, find the set of e-vects corresponding to λ.
iii) If possible, find a basis for F^n consisting of eigenvalues of A.
iv) If successful in finding such a basis, determine an invertible matrix Q and diagonal matrix D st. $D = Q^{-1}AQ$.

c.) $A = \begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix}$, $F = \mathbb{C}$.

i) $\begin{vmatrix} i-t & 1 \\ 2 & -i-t \end{vmatrix} = (i-t)(-i-t) - 2 = 0$

$-i^2 + t^2 - 2 = t^2 - 1 = 0 \Rightarrow t = \pm 1.$ (e-vals)

ii) for $\lambda = 1$: $\left(\begin{array}{cc|c} i-1 & 1 & 0 \\ 2 & -i-1 & 0 \end{array}\right) \Rightarrow (i-1)x_1 = -x_2 \Rightarrow x_2 = (1-i)x_1$

so solution set $= t\begin{pmatrix} 1 \\ 1-i \end{pmatrix}$

for $\lambda = -1$: $\left(\begin{array}{cc|c} i+1 & 1 & 0 \\ 2 & -i+1 & 0 \end{array}\right) \Rightarrow (i+1)x_1 = -x_2 \Rightarrow x_2 = (-i-1)x_1$

So solution set $= t\begin{pmatrix} 1 \\ -i-1 \end{pmatrix}$

iii) $\beta = \left\{ \begin{pmatrix} 1 \\ 1-i \end{pmatrix}, \begin{pmatrix} 1 \\ -i-1 \end{pmatrix} \right\}$

iv) By (THM 5.1), $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

By (Cor to THM. 2.23), $Q = \begin{pmatrix} 1 & 1 \\ 1-i & -i-1 \end{pmatrix}$

d) $A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix}$, $F = \mathbb{R}$.

i) $\begin{vmatrix} 2-t & 0 & -1 \\ 4 & 1-t & -4 \\ 2 & 0 & -1-t \end{vmatrix} = (1-t)\begin{vmatrix} 2-t & -1 \\ 2 & -1-t \end{vmatrix} = (1-t)\left((2-t)(-1-t) - (-1)(2)\right)$

$= (1-t)(\cancel{-2} - t + t^2 \cancel{+2}) = t(1-t)(t-1) = -t(t-1)^2$

$\Rightarrow t = 0, 1$ (e-vals).

ii) For $\lambda = 0$:

$$\left(\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 4 & 1 & -4 & 0 \\ 2 & 0 & -1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 4 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

then let $x_3 = t$, $x_2 = 2t$, $x_1 = \frac{1}{2}t$ $\Rightarrow$ soln. set $= t\begin{pmatrix} \frac{1}{2} \\ 2 \\ 1 \end{pmatrix}$

For $\lambda = 1$:

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 4 & 0 & -4 & 0 \\ 2 & 0 & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \text{soln. set}$$

$\Rightarrow$ soln. set $= t\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

iii) β: $\left\{ \begin{pmatrix} \frac{1}{2} \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

iv). $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$Q = \begin{pmatrix} \frac{1}{2} & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

PSET 7

Jonathan Lam
Prof. Mintchev
MA326
Lin. Alg.
11/7/19

5.1 # 4 deg, 7, 12, 14, 15, 19, 22

4. For each linear operator T on V, find the eigenvalues of T and an ordered basis β for V s.t. $[T]_\beta$ is a diagonal matrix.

d). $V = P_1(\mathbb{R})$, $T(ax+b) = (-6a+2b)x + (-6a+b)$

$\beta = \{1, x\}$

$$[T]_\beta = \begin{pmatrix} 1 & -6 \\ 2 & -6 \end{pmatrix}$$

$$\begin{vmatrix} 1-t & -6 \\ 2 & -6-t \end{vmatrix} = (1-t)(-6-t) + 12 = 0$$
$$= -6 + 5t + t^2 + 12 = t^2 + 5t\ 6 = 0$$
$$\Rightarrow (t+3)(t+2) = 0 \Rightarrow t = -2, -3$$

Finding e-vects:

$$\left(\begin{array}{cc|c} 3 & -6 & 0 \\ 2 & -4 & 0 \end{array}\right) = \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \text{solns} = t\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 4 & -6 & 0 \\ 2 & -3 & 0 \end{array}\right) = \left(\begin{array}{cc|c} 2 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \text{solns} = t\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$\{\lambda\} = \{-2, -3\}$

basis of e-vects of $P_1(\mathbb{R}) = \{2+x,\ 3+2x\}$

---

e) $V = P_2(\mathbb{R})$, $T(f(x)) = xf'(x) + f(2)x + f(3)$

$\beta = \{1, x, x^2\}$

$$[T]_\beta = \begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\begin{vmatrix} 1-t & 3 & 9 \\ 1 & 3-t & 4 \\ 0 & 0 & 2-t \end{vmatrix} = (2-t)\begin{vmatrix} 1-t & 3 \\ 1 & 3-t \end{vmatrix}$$
$$= (2-t)\left((1-t)(3-t) - 3(1)\right)$$
$$= (2-t)(\cancel{3} - 4t + t^2 - \cancel{3}) = t(2-t)(t-4) = 0$$
$$\Rightarrow t = 0, 2, 4 \quad \text{(e-vals)}.$$

Finding e-vects:

$$\left(\begin{array}{ccc|c} 1 & 3 & 9 & 0 \\ 1 & 3 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right) = \left(\begin{array}{cccc} 1 & 3 & 9 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \text{solns.} = t\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -1 & 3 & 9 & 0 \\ 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 1 & 4 & 0 \\ 0 & 4 & 13 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \text{solns} = t\begin{pmatrix} 3 \\ 13 \\ -4 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -3 & 3 & 9 & 0 \\ 1 & -1 & 4 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \text{solns} = t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$\{\lambda\} = \{0, 2, 4\}$

basis of $P_2(\mathbb{R}) = \{3 - x,\ 3 + 13x - 4x^2,\ 1 + x\}$

---

9) $V = P_3(\mathbb{R})$, $T(f(x)) = xf'(x) + f''(x) - f(2)$

$\beta = \{1, x, x^2, x^3\}$

$$[T]_\beta = \begin{pmatrix} -1 & -2 & -2 & -8 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

finding e-vals: $\begin{vmatrix} -1-t & -2 & -2 & -8 \\ 0 & 1-t & 0 & 6 \\ 0 & 0 & 2-t & 0 \\ 0 & 0 & 0 & 3-t \end{vmatrix} = (-1-t)(1-t)(2-t)(3-t)$

$\Rightarrow$ evals $= -1, 1, 2, 3$

$$\left(\begin{array}{cccc|c} 0 & -2 & -2 & -8 & 0 \\ 0 & 2 & 0 & 6 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{array}\right) = \left(\begin{array}{cccc|c} 0 & 1 & 1 & 4 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) = \left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \text{soln. set} = t\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Jonathan Lam
Prof. Mintchev
MA 326
Lin. Alg.
11/7/19.

PSET 7

5.1 #4g, 7, 12, 14, 15, 19, 22.

(4g, cont'd.)

$$\left(\begin{array}{cccc|c} -2 & -2 & -2 & -8 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right) = \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \text{soln. set} = t\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} -3 & -2 & -2 & -8 & 0 \\ 0 & -1 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) = \left(\begin{array}{cccc|c} 3 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \text{soln. set} = t\begin{pmatrix} -2 \\ 0 \\ 3 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} -4 & -2 & -2 & -8 & 0 \\ 0 & -2 & 0 & 6 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{cccc|c} 2 & 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \text{soln. set} = t\begin{pmatrix} -\frac{7}{2} \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

$\{\lambda\} = \{-1, 1, 2, 3\}$

basis for $P_4(\mathbb{R}) = \{1,\ 1-x,\ -2+3x^2,\ -\frac{7}{2}+3x+x^3\}$

---

7. DEF. (determinant of a linear operator.) Let T be a linear operator on a finite-dim. v.s. V. We define the determinant of T, denoted $\det(T)$ as follows: choose any o.B. β for V, and define $\det(T) = \det([T]_\beta)$.

a) Prove that the preceding definition is independent of the choice of the choice of an o.B. for V. That is, if β and γ are two o.B.s for V, then $\det([T]_\beta) = \det([T]_\gamma)$.

PF: By (THM. 2.23), $[T]_\beta$ and $[T]_\gamma$ are similar (i.e., $[T]_\beta = Q^{-1}[T]_\gamma Q$, where Q is the change-of-basis matrix converting β to γ coordinates). Since Q is invertible, $\det(Q^{-1}) = (\det(Q))^{-1}$, and $\det([T]_\beta) = \det(Q^{-1}[T]_\gamma Q)$

$= \det(Q^{-1})\det([T]_\gamma)\det(Q) = \frac{\cancel{\det(Q)}}{\cancel{\det(Q)}}\det([T]_\gamma) = \det([T]_\gamma)$

b.) Prove that T invertible $\Leftrightarrow$ $\det(T) \neq 0$.

PF: Fix some basis β of V. T invertible $\Leftrightarrow$ $[T]_\beta$ inv $\Leftrightarrow$ $\det([T]_\beta) = \det(T) \neq 0$.

c). Prove that, if T invertible, then $\det(T^{-1}) = (\det(T))^{-1}$.

PF: Fix some basis β of V. $\det(T^{-1}) = \det([T^{-1}]_\beta) = \det([T]_\beta^{-1}) = (\det([T]_\beta))^{-1}$
$= (\det(T))^{-1}$.

d). Prove that if U is also a linear operator on V, then
$\det(TU) = \det(T) \cdot \det(U)$.

PF: Fix some basis β of V. $\det(TU) = \det([TU]_\beta) = \det([T]_\beta [U]_\beta)$
$= \det([T]_\beta) \det([U]_\beta) = \det(T) \cdot \det(U)$.

e). Prove that $\det(T - \lambda I_V) = \det([T]_\beta - \lambda I)$ for any scalar λ and any O.B. β for V.

PF: Fix a scalar λ and an O.B. β for V.
$\det(T - \lambda I_V) = \det([T - \lambda I_V]_\beta) = \det([T]_\beta - [\lambda I_V]_\beta)$
$= \det([T]_\beta - \lambda [I_V]_\beta) = \det([T]_\beta - \lambda I)$.

12. a) Prove that similar matrices have the same characteristic polynomial.

PF. Let A, B, and Q be $n \times n$ matrices, s.t. Q is invertible and $A = Q^{-1} B Q$ (i.e. A similar to B). Then
$A - \lambda I_n = Q^{-1} B Q - \lambda Q^{-1} Q = Q^{-1}(BQ - \lambda Q) = Q^{-1}(B - \lambda I_n) Q$.
Thus $f_A(t) = \det(A - tI_n) = \det(B - tI_n) = f_B(t)$
(exercise 4.3 #15)

b). Show that the definition of the characteristic polynomial of a linear operator over a finite-dim. v.s. V is independent of the choice of basis for V.

PF: Let $T \in \mathcal{L}(V)$, and β, γ be distinct bases for V. Then the characteristic polynomial of T w.r.t. β is $\det([T]_\beta - \lambda I)$, and the characteristic polynomial of T w.r.t. γ is $\det([T]_\gamma - \lambda I)$. Since $[T]_\beta = Q^{-1} [T]_\gamma Q$ (THM 2.23), $[T]_\beta$ is similar to $[T]_\gamma$, so the characteristic polynomials are equal by (part a).

PSET #7

Jonathan Lam
Prof. Mintchev
MA326
Lin. Alg
11/7/19

5.1 # 14, 15, 19, 22.

14. For any square matrix A, prove that A and A^t have the same characteristic polynomial (and hence the same e-vals).

PF: $f_{A^t}(t) = \det(A^t - tI_n) = \det((A - tI_n)^t) = \det(A - tI_n) = f_A(t)$

15. a) Let T be a lin. operator on a v.s. V, and let x be an e-vect of T corresponding to the e-val λ. For any positive integer m, prove that x is an e-vect of T^m corresponding to the e-val λ^m.

PF: (by induction) Base case: $T^1(x) = \lambda^1 x$.

Inductive hyp: assume $T^{n-1}(x) = \lambda^{n-1} x$, $n > 1$. Then $T^n(x) = \lambda^n x$.

Proof of inductive hyp: $T^n(x) = T(T^{n-1}(x)) = T(\lambda^{n-1} x) = \lambda^{n-1} T(x)$
$= \lambda^{n-1}\lambda x = \lambda^n x$. Thus $T^m(x) = \lambda^m x \ \forall m \in \mathbb{Z}^+$

b) Let $A \in M_{n\times n}(F)$, and let x be an e-vect of A corresponding to the e-val λ. For any positive integer m, prove that x is an e-vect of A^m corresponding to the e-val λ^m.

PF: (by induction) Base case: $A^1 x = \lambda^1 x$.

Inductive hyp: assume $A^{n-1}(x) = \lambda^{n-1} x$, $n > 1$. Then $A^n x = \lambda^n x$.

Proof of inductive hyp: $A^n x = A^1(A^{n-1} x) = A(\lambda^{n-1} x) = \lambda^{n-1}(Ax)$
$= \lambda^{n-1}\lambda x = \lambda^n x$. Thus $A^m(x) = \lambda^m x \ \forall m \in \mathbb{Z}^+$.

19. Let $A, B \in M_{n\times n}(F)$ and similar. Prove that there exists an n-dim v.s. V, a lin. op. T on V, and o.B.'s β and γ for V s.t. $A = [T]_\beta$, $B = [T]_\gamma$.

5

PF: Let $V = F^n$, β = s.o.B. of V, $T = L_A$, and $B = Q^{-1}AQ$.
Thus $[L_A]_\beta = A$. Also, Q is an inv. $n\times n$ matrix. By (exercise 13 from 2.5), $\exists\, \gamma$ o.B. of V s.t. $Q = [I_V]_\gamma^\beta$. Thus $B = Q^{-1}AQ$
$= [I_V]_\beta^\gamma [L_A]_\beta [I_V]_\gamma^\beta = [L_A]_\gamma$.

11/7/19

22. a) Let T be a lin. op. over a v.s. V over the field F, and let $g(t) = \sum_{n=0}^{\infty} a_n t^n \in P(F)$ be an arbitrary polynomial. Prove that if x is an e-vect of T corresponding to the e-val. λ, then $(g(T))(x) = g(\lambda)x$

PF: $(g(T))(x) = \left(\sum_{n=0}^{\infty} a_n T^n\right)(x) \underbrace{=}_{\text{linearity of } T} \sum_{n=0}^{\infty} a_n T^n(x) \underbrace{=}_{\text{(exercise 15a)}} \sum_{n=0}^{\infty} a_n \lambda^n x$
$= \left(\sum_{n=0}^{\infty} a_n \lambda^n\right) x = g(\lambda)x.$

b) Let $A \in M_{n\times n}(F)$, $g(t)$ as declared in (part a). Prove that if x is an e-vect of A with corresponding e-val λ, then $(g(A))x = g(\lambda)x$.

PF: $(g(A))x = \left(\sum_{n=0}^{\infty} a_n A^n\right)x \underbrace{=}_{\text{linearity of mat. mult.}} \sum_{n=0}^{\infty} a_n A^n x \underbrace{=}_{\text{(exercise 15b)}} \sum_{n=0}^{\infty} a_n \lambda^n x = \left(\sum_{n=0}^{\infty} a_n \lambda^n\right)x = g(\lambda)x$

c) Verify (part b) for $g(t) = 2t^2 - t + 1$, $x = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\lambda = 4$, $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$

$g(A) = 2\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}^2 - \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2\begin{pmatrix} 7 & 6 \\ 9 & 10 \end{pmatrix} + \begin{pmatrix} 0 & -2 \\ -3 & -1 \end{pmatrix} = \begin{pmatrix} 14 & 10 \\ 15 & 19 \end{pmatrix}$

$g(\lambda) = 2(4^2) - 4 + 1 = 29$

$(g(A))x = \begin{pmatrix} 14 & 10 \\ 15 & 19 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 58 \\ 87 \end{pmatrix} = 29\begin{pmatrix} 2 \\ 3 \end{pmatrix} = g(\lambda)x.$ ✓

Jonathan Lam
Prof. Mitchell
MA 326
Lin. Alg
11/12/19

PSET 8

5.2 # 2bd, 3ac, 7, 9, 11

E.6	5	
	5	
5.2.2(bd)	10	

23.5/25

2. For each of the following matrices $A \in M_{n\times n}(\mathbb{R})$, test A for diagonalizability, and if A is diagonalizable, find an invertible matrix Q and a diagonal matrix D s.t. $Q^{-1}AQ = D$.

b) $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ $\quad \begin{vmatrix} 1-t & 3 \\ 3 & 1-t \end{vmatrix} = (1-t)^2 - 9 = 1 - 2t + t^2 - 9$

5

$= t^2 - 2t - 8 = (t-4)(t+2) = 0 \Rightarrow \lambda = \{4, -2\}$.

Since 2 distinct e-vals where $n=2$, A is diagonalizable.

$\lambda = 4$ $\quad \left(\begin{array}{cc|c} -3 & 3 & 0 \\ 3 & -3 & 0 \end{array}\right) = \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow$ soln. set $= t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = -2$ $\quad \left(\begin{array}{cc|c} 3 & 3 & 0 \\ 3 & 3 & 0 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow$ soln. set $= t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$.

d) $A = \begin{pmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{pmatrix}$ $\quad \begin{vmatrix} 7-t & -4 & 0 \\ 8 & -5-t & 0 \\ 6 & -6 & 3-t \end{vmatrix} = (3-t)\begin{vmatrix} 7-t & -4 \\ 8 & -5-t \end{vmatrix}$

$= (3-t)\left((7-t)(-5-t) + 32\right) = (3-t)(-35 + 32 - 2t + t^2)$

$= (3-t)(t^2 - 2t - 3) = -(t-3)(t-3)(t+1) = -(t-3)^2(t+1) = 0$

$\Rightarrow \lambda = \{3, -1\}$.

Need to check if $d_3 = m_3 = 2$.

$\left(\begin{array}{ccc|c} 4 & -4 & 0 & 0 \\ 8 & -8 & 0 & 0 \\ 6 & -6 & 0 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$ soln. set $= t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$d_3 = m_3 = 2$ ✓ so A diagonalizable.

$\left(\begin{array}{ccc|c} 8 & -4 & 0 & 0 \\ 8 & -4 & 0 & 0 \\ 6 & -6 & 4 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ -3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$ soln. set $= t\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$

$Q = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 4 \\ 0 & 1 & 3 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

11/12

3. For each of the following linear operators T on a v.s. V, test T for diagonalizability, and if T is diagonalizable, find a basis β for V s.t. $[T]_\beta$ is a diagonal matrix.

a) $V = P_3(\mathbb{R})$, $T(f(x)) = f'(x) + f''(x)$. Let $\beta = \{1, x, x^2, x^3\}$

$$[T]_\beta = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad f_T(t) = \begin{vmatrix} -t & 1 & 2 & 0 \\ 0 & -t & 2 & 6 \\ 0 & 0 & -t & 3 \\ 0 & 0 & 0 & -t \end{vmatrix} = t^4$$

$M_0 = 4$; if $d_0 = 4$, then diagonalizable.

$$\left(\begin{array}{cccc|c} 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 6 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \text{soln. set} = t\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$d_0 = 1 \Rightarrow$ not diagonalizable

e) $V = \mathbb{C}^2$, $T(z, w) = (z + iw, iz + w)$. Let $F = \mathbb{C}$, $\beta = \{(1,0), (0,1)\}$.

$$[T]_\beta = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \quad f_T(t) = (1-t)^2 + 1 = t^2 - 2t + 2 = 0$$

$$\Rightarrow t = \frac{2 \pm \sqrt{4 - 4(2)}}{2} = 1 \pm \sqrt{1-2} = 1 \pm i.$$

T must be diagonalizable, since 2 e-vals

$\lambda = 1+i$ $\quad \left(\begin{array}{cc|c} -i & i & 0 \\ i & -i & 0 \end{array}\right) = \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \text{soln. set} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = 1-i$ $\quad \left(\begin{array}{cc|c} i & i & 0 \\ i & i & 0 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \text{soln. set} = t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\beta = \{(1, 1), (1, -1)\}$

(thus $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $D = \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}$)

Jonathan Lam
Prof. Mintchev
MA 326
Lin Alg
11/12/19

PSET 8

5.2 # 7, 9, 11

7. For $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$, find an expression for A^n where n is an arbitrary positive integer.

$D = Q^{-1}AQ \Rightarrow A = QDQ^{-1} \Rightarrow A^n = QD^nQ^{-1}$ (by example 7).

Finding a diagonal matrix representation D and matrix Q:

$$\begin{vmatrix} 1-t & 4 \\ 2 & 3-t \end{vmatrix} = (1-t)(3-t) - 8 = 3 - 4t + t^2 - 8$$
$$= t^2 - 4t - 5 = (t-5)(t+1) \Rightarrow \lambda = \{5, -1\}.$$
$\Rightarrow$ diagonalizable

$$\left(\begin{array}{cc|c} -4 & 4 & 0 \\ 2 & -2 & 0 \end{array}\right) = \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \text{soln. set} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$\left(\begin{array}{cc|c} 2 & 4 & 0 \\ 2 & 4 & 0 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \text{soln. set} = t\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\Rightarrow Q = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Q^{-1} = \frac{1}{1(1) - 2(1)}\begin{pmatrix} -1 & -2 \\ -1 & 1 \end{pmatrix} = \frac{1}{-3}\begin{pmatrix} -1 & -2 \\ -1 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$$

$$A^n = QD^nQ^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 5^n & 0 \\ 0 & (-1)^n \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$$
$$= \frac{1}{3}\begin{pmatrix} 5^n & 2(-1)^n \\ 5^n & -(-1)^n \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 5^n + 2(-1)^n & 2(5^n) - 2(-1)^n \\ 5^n - (-1)^n & 2(5^n) + (-1)^n \end{pmatrix}$$

---

9. Let T be the lin. op. on a finite-dim. v.s. V, and suppose there exists an O.B. β for V s.t. $[T]_\beta$ is an upper triangular matrix.

a) Prove that the characteristic polynomial for T splits.

b) State and prove the analogous result for matrices.

PF (a): The characteristic polynomial is $\det(T - tI_V) = \det([T]_\beta - tI)$. $[T]_\beta - tI$ is also upper triangular (b/c sum of upper-triangular matrices is upper triangular), and the determinant of an upper triangular matrix is the product of the terms along the diagonal. Thus:

$$f_T(t) = \prod_{i=1}^{n}\left(([T]_\beta)_{ii} - t\right) = \prod_{i=1}^{n}\left((-1)(t - ([T]_\beta)_{ii})\right) = (-1)^n \prod_{i=1}^{n}\left(t - ([T]_\beta)_{ii}\right)$$

11/13/1

b) Let $A \in M_{n\times n}(F)$, and let A be upper triangular. Prove that the characteristic polynomial for A splits.

PF: Let β = SOB. F^n. Then $A = [L_A]_\beta$ is upper triangular. By (5.1 exercise 7e), the characteristic polynomial of A equals the characteristic polynomial of L_A, which splits according to (part a).

11. Let A (call this B) be an $n\times n$ matrix that is similar to an upper triangular matrix, and has the distinct e-vals $\lambda_1, \lambda_2, \ldots \lambda_k$ w/ corresponding multiplicities $m_1, m_2, \ldots, m_k$. Prove the following.

a) $tr(A) = \sum_{i=1}^{k} m_i \lambda_i$

b) $\det(A) = (\lambda_1)^{m_1}(\lambda_2)^{m_2}\cdots(\lambda_k)^{m_k}$

LEM 1: trace of similar matrices are equal

PF: Let A, B be similar $M_{n\times n}(F)$ matrices. Then $\exists\, Q$ s.t. $A = Q^{-1}BQ$. By (Section 2.3 exercise 13), $tr((Q^{-1}B)Q) = tr(Q(Q^{-1}B)) = tr(IB) = tr(B) = tr(A)$

LEM 2: Using the assumptions from the exercise, all of the entries along the main diagonal of B are e-vals of A, and each value λ_i appears m_i times on B's main diagonal.

PF: Since B, A similar, they have the same characteristic polynomial and e-vals. Since $B-\lambda I$ is upper tri., $f_A(t) = f_B(t) = \prod_{i=1}^{n}(B_{ii}-t)$ (and unique factorization by Thm E.9). Thus $\{B_{ii}\}$, $1\le i\le n$ is the set of eigenvalues of B (and therefore A), and the multiplicity m_i is the number of times λ_i appears on the main diagonal.

PF (a): By (LEM 2), each e-val λ_i of A appears m_i times on B's diagonal, and by (LEM 1), $tr(A) = tr(B) = \underbrace{\lambda_1+\lambda_1+\ldots+\lambda_1}_{m_1} + \underbrace{\lambda_2+\ldots+\lambda_2}_{m_2} + \ldots + \underbrace{\lambda_k+\lambda_k+\ldots+\lambda_k}_{m_k} = \sum_{i=1}^{k}\lambda_i m_i$.

PF (b): By (LEM 2), each e-val λ_i of A appears m_i times on B's diagonal, and since determinants of similar matrices are equal, $\det(A) = \det(B) = \underbrace{\lambda_1\lambda_1\ldots\lambda_1}_{m_1}\underbrace{\lambda_2\lambda_2\ldots\lambda_2}_{m_2}\ldots\underbrace{\lambda_k\lambda_k\ldots\lambda_k}_{m_k} = \prod_{i=1}^{n}\lambda_i^{m_i}$.

Jonathan Lam
Prof. Mintchev
MA 326
Lin. Alg.
11/12/19

E. 3, 4, 5, 6, 7

THM. E.3. Let $f(x)$ be a polynomial w/ coefficients from a field F, and let T be a linear operator on a v.s. V over F. Then the following statements are true.
a) $f(T)$ is a linear operator on V.
b) If β is a finite OB of V and $A = [T]_\beta$, then $[f(T)]_\beta = f(A)$.

PF (a): Let $f(T) = a_0 I + a_1 T + \dots + a_n T^n$. Clearly, I and T^i $(i \geq 1)$ are linear operators on V, and the sum of linear operators on the same v.s. is a linear operator.

PF (b): Let f be as defined in part (a).
$[f(T)]_\beta = [a_0 I + a_1 T + \dots + a_n T^n]_\beta = [a_0 I]_\beta + [a_1 T]_\beta + \dots + [a_n T^n]_\beta$
$= a_0 [I]_\beta + a_1 [T]_\beta + \dots + a_n [T^n]_\beta = a_0 I + a_1 [T]_\beta + \dots + a_n [T^n]_\beta$
$= a_0 I + a_1 [T]_\beta + \dots + a_n [T]_\beta^n = f(A)$

THM E.4 Let T be a lin. op. over a v.s. V over a field F, and let A be a square matrix with entries from F. Then, for any polynomials $f_1(x)$ and $f_2(x)$ with coefficients from F:
a) $f_1(T) f_2(T) = f_2(T) f_1(T)$
b) $f_1(A) f_2(A) = f_2(A) f_1(A)$

PF (a): Let $f_1(t) = \sum_{i=0}^{n} a_i t^i$, $f_2(t) = \sum_{j=0}^{m} b_j t^j$, and $T^0 = I_V$.
Then $f_1(T) f_2(T) = \left(\sum_{i=0}^{n} a_i T^i\right)\left(\sum_{j=0}^{m} b_j T^j\right) = \sum_{i=0}^{n}\sum_{j=0}^{m} a_i b_j T^i T^j$
$= \left(\sum_{j=0}^{m} b_j T^j\right)\left(\sum_{i=0}^{n} a_i T^i\right) = f_2(T) f_1(T)$

PF (b): Let f_1, f_2 as defined in part (a), $A^0 = I$.
Then $f_1(A) f_2(A) = \left(\sum_{i=1}^{n} a_i A_i\right)\left(\sum_{j=0}^{m} b_j A^j\right) = \sum_{i=0}^{n}\sum_{j=0}^{m} a_i b_j A^i A^j$
$= \left(\sum_{j=0}^{m} b_j A^j\right)\left(\sum_{i=0}^{n} a_i A^i\right) = f_2(A) f_1(A)$.

11/12/1

THM E.5 Let T be a lin. op. on a v.s. V over a field F, and let A be an $n \times n$ matrix with entries from F. If $f_1(x)$ and $f_2(x)$ are prime polynomials with entries from F, then there exist polynomials $q_1(x)$ and $q_2(x)$ with entries from F s.t.

a) $q_1(T) f_1(T) + q_2(T) f_2(T) = I_V$

b) $q_1(A) f_1(A) + q_2(A) f_2(A) = I_n$

(THM. E.2) states that if $f_1(x), f_2(x) \in P(F)$ relatively prime $\exists\ q_1(x), q_2(x) \in P(F)$ s.t. $q_1(x) f_1(x) + q_2(x) f_2(x) = 1$, where $1 \in P(F)$ is the polynomial comprising of the multiplicative identity of the indeterminate variable.

For (part a), the indeterminate is $T \in \mathcal{L}(V)$, and the multiplicative identity is I_V; for (part b), the indeterminate is $A \in M_{n \times n}(F)$, and the multiplicative identity is I_n. Apply (THM E.2) to these indeterminates.

Jonathan Lam
Prof. Mintchev
MA 326
Lin. Alg
11/12/19

PSET 8

E.6, 7.

THM. E.6 Let $\phi(x)$ and $f(x)$ be polynomials. If $\phi(x)$ is irreducible and $\phi(x)$ does not divide $f(x)$, then $\phi(x)$ and $f(x)$ are relatively prime.

5

PF: . $\phi(x)$ irreducible $\Rightarrow$ it cannot be expressed as a product of polynomials with degree ≥ 1. Thus $\phi(x)$ may only be represented as the product of a scaled multiple of itself and a scalar (polynomial w/ degree 0). Since $\phi(x)$ doesn't divide $f(x)$, the only polynomials that may divide both $\phi(x)$ and $f(x)$ are polynomials of degree 0. Thus relatively prime.

you probably need to flesh this out!

THM E.7. Any two distinct irreducible monic polynomials are relatively prime.

PF: Two polynomials that are both distinct and monic may not be scalar multiples of one another. Thus, neither polynomial divides the other with the quotient polynomial being of degree 0. Since both are irreducible, no polynomial with positive degree divides either polynomial. Thus the only polynomials that can divide both polynomials have degree 0 $\Rightarrow$ the polynomials are relatively prime.

12 /12

6.6 ± 2.9

Ph213 – Section D

Quiz 1

159 not sure

Name (Print): Jonathan Lam Slot: 159

The E field produced by a uniform 1-D ring of charge of charge density λ and radius R has magnitude $k2\pi\lambda Rh/(h^2+R^2)^{3/2}$ at a point along the symmetry (z) axis of the ring, at height h above the plane of the ring. Assume λ is positive, so the field points away from the ring, along the z axis. This expression is a given, you do not need to prove it.

For each of the charge distribution geometries below, use rings as building blocks to **SET UP**, but ***DO NOT* SOLVE**, integrals for the E field produced by the charge distribution on the z axis.

No magic appearance of memorized formulas.

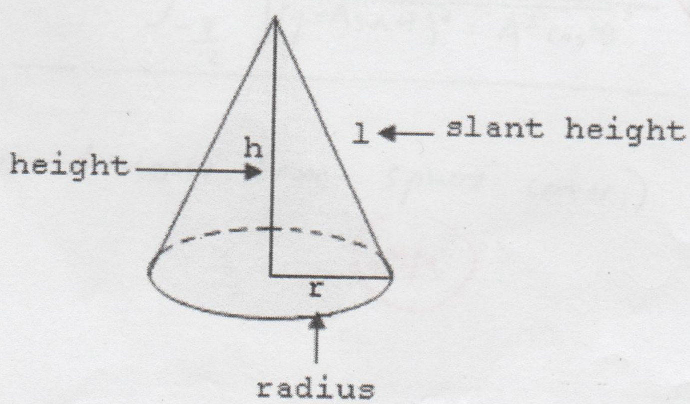
You must show all of the following:
* the proper transition from λ to σ, as we did in class
* a perfectly annotated sketch showing the entire charge distribution as well as the randomly placed building block that contributes its dE
* all steps necessary to write the integrand as a function of a single variable
* proper (explicit) limits of integration.

THEN STOP. DO NOT DO THE INTEGRALS. DO NOT TAKE LIMITS. (4 pts each)

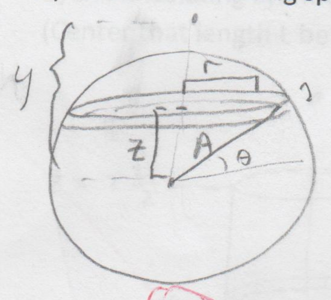
a) A thin conducting spherical shell of uniform charge density σ and radius A centered at the origin.

b) A thin insulating cylinder of uniform surface charge density σ, radius B and length L. (Center that length L between z = +L/2 and z = -L/2).

c) A thin insulating cone of uniform surface charge density σ, with dimensions as shown below. The charge is only on the "body" not on the circular base. If it helps, the surface area of the cone can be written as π r L or as π r $\sqrt{h^2+r^2}$. (L is the 'slant height' shown as "l" below.)



a) A thin conducting spherical shell of uniform σ and radius A centered at the origin.



FOR A RING:

$$E = \frac{k\, 2\pi \lambda R h}{\sqrt{h^2+R^2}^{\,3}}$$

$$\lambda = \frac{q}{2\pi R}$$

q is charge of ring

$$E = \frac{k\, \cancel{2\pi} \left(\frac{q}{\cancel{2\pi R}}\right) \cancel{R}\, h}{\sqrt{h^2+r^2}^{\,3}} = k \frac{q h}{\sqrt{h^2+r^2}^{\,3}}$$

FOR THE RING WITH HEIGHT $(A\,d\theta)$:

$$dq = \sigma(2\pi r)(A\,d\theta) = \sigma\, 2\pi A^2 \cos\theta\, d\theta \quad (+.5)$$

$r = A\cos\theta$ (+.5) $\qquad R = A$

$z = A\sin\theta$ (+.5)

$h = y - z = y - A\sin\theta$ (+.5)

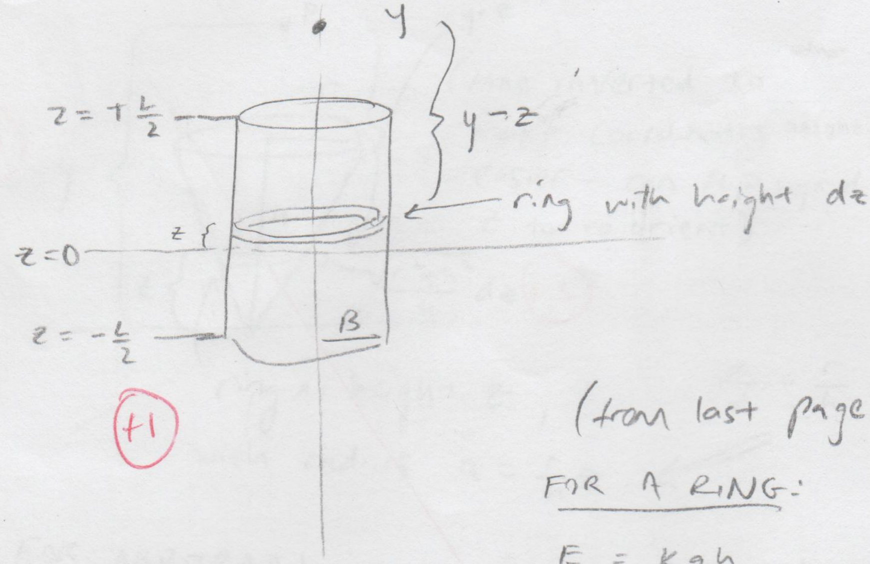
$$dE = \frac{k\,dq\,h}{\sqrt{h^2+R^2}^{\,3}} = \frac{k(\sigma 2\pi A^2\cos\theta)(y - A\sin\theta)}{\sqrt{(y-A\sin\theta)^2+(A\cos\theta)^2}^{\,3}}\,d\theta$$

$$\Rightarrow \boxed{E = k\sigma\, 2\pi A^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\theta\,(y - A\sin\theta)}{\sqrt{(y-A\sin\theta)^2 + A^2\cos^2\theta}^{\,3}}\,d\theta} \quad (+1)$$

(where y is distance from sphere center.)

+4/4

b) thin insulating cylinder of uniform surface charge density σ, radius B and length L. (Center that length L between z = +L/2 and z = -L/2).



+1

(from last page:)

FOR A RING:

$$E = \frac{k q h}{\sqrt{r^2+h^2}^{\,3}}$$

FOR A RING ON THIS CYLINDER W/ HEIGHT dz:

$dq = \sigma\, 2\pi B\, dz$ +1

$h = y - z$ +.5

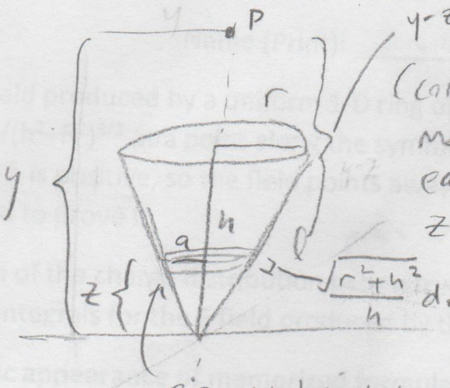
$r = B$

$$dE = k\frac{dq\, h}{\sqrt{r^2+h^2}^{\,3}} = k\frac{\sigma\, 2\pi B\, dz\,(y-z)}{\sqrt{B^2+(y-z)^2}^{\,\frac{3}{2}}}$$

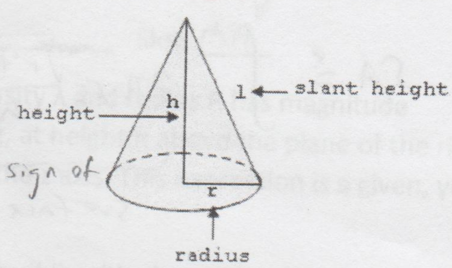
$$\Rightarrow \boxed{E = k\sigma 2\pi B \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{(y-z)\, dz}{\sqrt{B^2+(y-z)^2}^{\,\frac{3}{2}}}}$$ +.5 +1

+4/4

c) A thin insulating cone of uniform surface charge density σ, with dimensions as shown below. The charge is only on the "body" not on the circular base. If it helps, the surface area of the cone can be written as π r L or as π r $\sqrt{h^2 + r^2}$. (L is the 'slant height' shown as "l" below.)



(cone inverted to make coordinates easier – can flip sign of z to re orient)



ring at height z with radius $a = \frac{r}{h}z$.

$\frac{a}{z} = \frac{r}{h} \Rightarrow a = \frac{r}{h}z$

FOR ARBITRARY RING: $E = k\frac{qh}{\sqrt{r^2+h^2}^3}$ on z-axis.

HERE: Ring has height $\frac{\sqrt{r^2+h^2}}{h}dz$ (i.e., slant height) and radius a;

$dq = \sigma 2\pi a\left(\frac{\sqrt{r^2+h^2}}{h}\right)dz = \sigma \frac{2\pi r\sqrt{r^2+h^2}}{h^2} z\,dz$ +.5

$a = \frac{r}{h}z$ +.5

h = (distance from ring) = y − z +.5

r (ring radius) = $a = \frac{r}{h}z$

$dE = k\frac{dq\,h}{\sqrt{r^2+h^2}^3} = k\frac{\sigma 2\pi r\sqrt{r^2+h^2}\,z\,dz\,(y-z)}{h^2\sqrt{\left(\frac{r}{h}z\right)^2+(y-z)^2}^3}$

$$\Rightarrow \boxed{E = \frac{k\sigma 2\pi r\sqrt{r^2+h^2}}{h^2}\int_0^h \frac{z(y-z)\,dz}{\sqrt{\left(\frac{r}{h}z\right)^2+(y-z)^2}^3}}$$ +1

+4/4

10.5/14

4.9 ± 2.5

Ph213 – Section D Quiz 2

Name (Print): Jonathan Lam Slot: 159

1) There are two points in 3-D space, point A and point B. There are also various static charge distributions and/or external electric fields in various places. What is the ***meaning*** of $V_B - V_A$? Keep it simple (a minimum of jargon), but complete and accurate. (2)

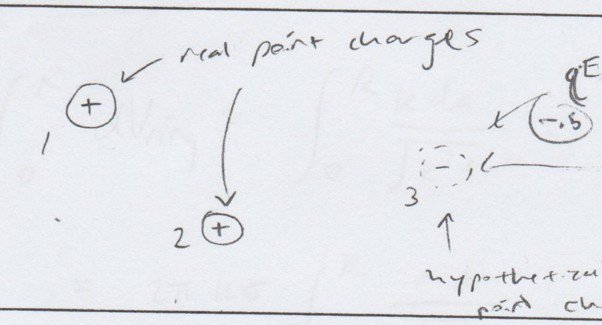
The amount of energy it takes to move one (hypothetical) coulomb of charge from point A to B through the existing charge distribution

+1/2

2) ***Why*** were we able to use Gauss' Law to find the E field above a randomly-shaped lump of metal, despite the total lack of symmetry of that lump? (3)

We were working at an altitude that was so small, so that the surface was always essentially normal to the field, which is the necessary case for the E-field analysis by Gauss' Law. +1/3

3) What is the relationship between EP and EPE as you complete a charge distribution by adding a final point charge to an existing charge distribution? (Include a simple sketch or two.) (3)



$q EP_3 = EPE_{123} - EPE_{12}$

$Vq + EPE_{12} = EPE_{123}$

∞

(-.5)

+2.5/3

i.e., EP can be thought of as the work applied to get an electrically charged system from one configuration to another, and therefore it is also the change in EPE (per coulomb) when a

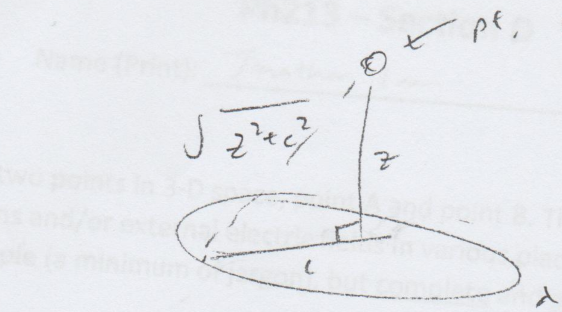
4) a) Trivially derive V (a distance z along the symmetry axis) for a 1-D ring of charge of density λ & radius c.

b) Using the result of part a, derive V (along that axis) for a 2-D disk of charge of density σ and radius R. Setup and solve. You must make a proper transition from λ to σ (as we did in E field calculations).

c) Take all reasonable limits of your answer and show that these results are "expected." Make the appropriate annotated sketch of course. (6)

3

a)



$$dq = \lambda\, ds \checkmark$$

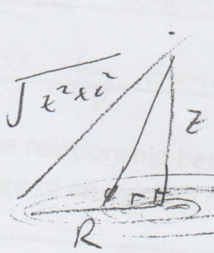
$$V = \int_0^{2\pi c} \frac{k\,dq}{r} = \frac{k}{\sqrt{z^2+c^2}} \int_0^{2\pi c} \lambda\, ds = \frac{k\lambda}{\sqrt{z^2+c^2}} (2\pi c) \checkmark$$

Then $q = 2\pi c \lambda$ ✓

so $V = \dfrac{kq}{\sqrt{z^2+c^2}}$ ✓ (+1/1)

---

b)



← each ring has charge $dq = \sigma\, 2\pi r\, dr$

$$V = \int_0^R dV_{ring} = \int_0^R \frac{k\,dq}{\sqrt{z^2+r^2}} = \int_0^R \frac{k\sigma\, 2\pi r\, dr}{\sqrt{z^2+r^2}}$$

$$= 2\pi k\sigma \int_0^R \frac{r\,dr}{\sqrt{z^2+r^2}}$$

let $u = z^2+r^2$
$du = 2r\,dr$

$$= \pi k\sigma \int_{z^2}^{z^2+R^2} u^{-\frac{1}{2}}\, du$$

$$= \pi k\sigma \left(2u^{\frac{1}{2}}\right)\Big|_{z^2}^{z^2+R^2} = 2\pi k\sigma\left(\sqrt{z^2+R^2} - z\right) \quad (+2/2)$$

$$= \frac{\sigma}{2\epsilon_0}\left(\sqrt{z^2+R^2} - z\right) \checkmark$$

Ph213 – Section D Quiz 3

13 /14

9.9 ± 3.0

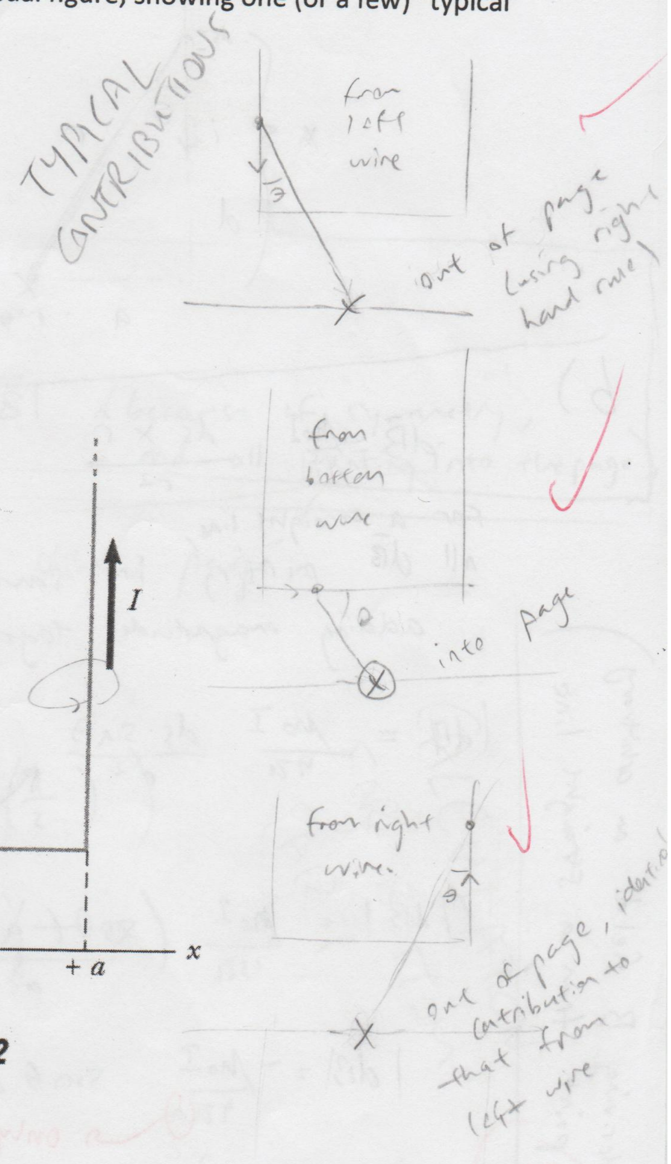
Name (Print): Jonathan Lam Slot: 159

Using Biot-Savart, find the magnetic field (a **vector**) at the origin in terms of I, a, d.
(As the dotted lines suggest, the wires parallel to the y axis extend to y = +infinity.)

a) Setup the relevant integral(s). You must include the usual figure, showing one (or a few) "typical" contributions to the total field. [7 pts]

b) Solve. [7 pts]

Show ALL work. Words are often helpful.[1]



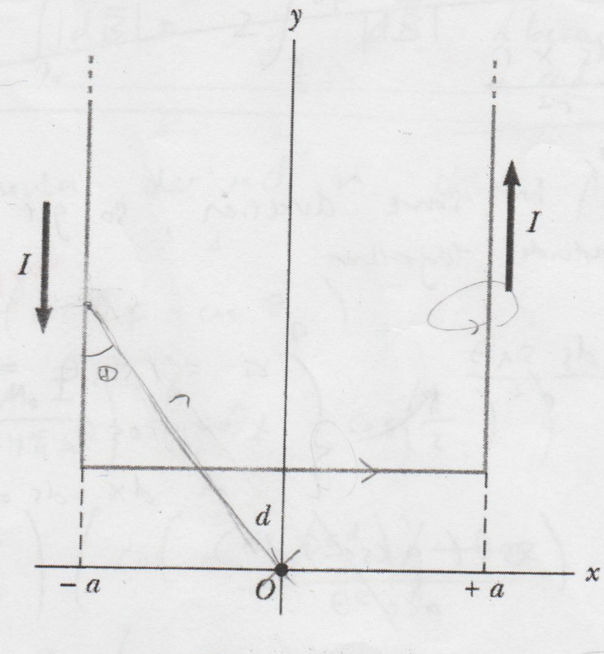


Figure P30.12

B field from left, right wires coming straight out of page, B field from bottom part going into page.

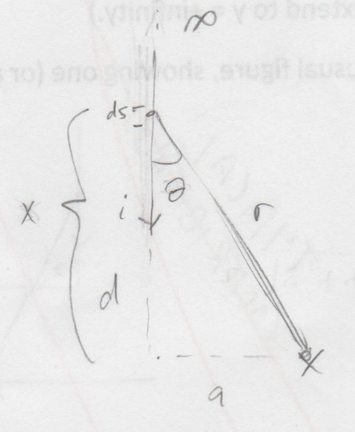
So we can do $|B_L| + |B_R| - |B_B| = 2|B_L| - |B_B|$

(because of symmetry b/t left and right wires)

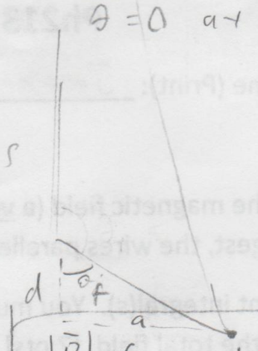
[1] No 'trick' is required to solve this problem, and you will have to do integral(s) regardless, but you may find it somewhat useful to think for a bit before setting up the integrals.

contribution

a) From left and right wires



$\bar{\theta} \neq 0$ at ∞



$\theta_f = \tan^{-1}\left(\frac{d}{a}\right)$

$d\bar{B}$ from Savart next

$|\bar{B}|$ from left wire $= \int_0^{\theta_f} |d\bar{B}|$

Out of the page

Equal to $\bar{B}$ contribution from right b/c of symmetry and b/c both are in the same direction

b)

$$d\bar{B} = \frac{\mu_0 I}{4\pi} \frac{d\bar{s} \times \hat{r}}{r^2}$$

For a straight line, all $d\bar{B}$ pointing in same direction, so get magnitude by adding magnitudes together

(deriving B field on arbitrary point from a straight line)

$$|d\bar{B}| = \frac{\mu_0 I}{4\pi} \frac{ds \sin\theta}{r^2}$$

$$\begin{cases} a = r\sin\theta \Rightarrow r = a\csc\theta \\ x = r\cos\theta = a\csc\theta\cot\theta = a\cot\theta \\ \Rightarrow dx = ds = -a\csc^2\theta \, d\theta \end{cases}$$

$$|d\bar{B}| = \frac{\mu_0 I}{4\pi} \frac{\sin\theta(-\cancel{a}\cancel{\csc^2\theta}\, d\theta)}{a^{\cancel{2}}\cancel{\csc^2\theta}}$$

$$|d\bar{B}| = -\frac{\mu_0 I}{4\pi a} \sin\theta \, d\theta$$

only for left and right sections.

(-.5)

$$\text{so } |\bar{B}| = \frac{\mu_0 I}{4\pi a}\int_{\theta_0}^{\theta_f} -\sin\theta \, d\theta = \frac{\mu_0 I}{4\pi a}\left(\cos\theta_f - \cos\theta_0\right)$$

θ_f should be $\tan^{-1}\left(\frac{d}{a}\right)$, otherwise you are effectively multiplying by a $\ominus$ sign.

for left and right, $\theta_f = \tan^{-1}\left(\frac{d}{a}\right)$, $\theta_0 = 0$

~~integrate from~~

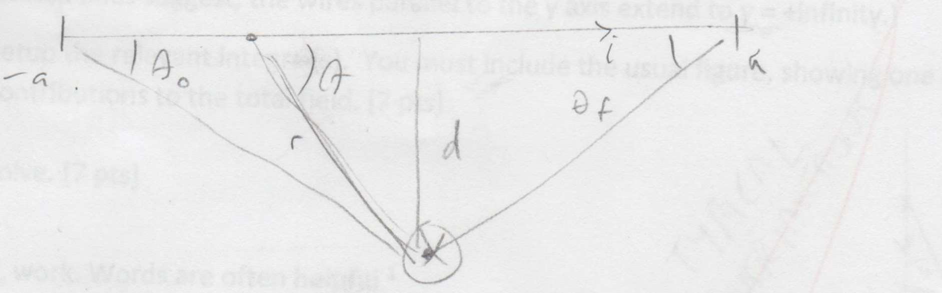
(-.5)

$$\text{so } \frac{\mu_0 I}{4\pi a}\left(\cos\left(\tan^{-1}\left(\frac{d}{a}\right)\right) - \cos(0)\right) = \frac{\mu_0 I (-1)}{4\pi a}\left(\frac{d}{\sqrt{a^2+d^2}} - 1\right)$$

from left and right wires each

From bottom part

$\theta_0 = \tan^{-1}\left(\frac{-d}{a}\right)$ $\qquad \theta_f = \tan^{-1}\left(\frac{d}{a}\right)$



overall contribution from bottom:

$$\boxed{|\bar{B}| = \int_{\theta_0}^{\theta_f} |d\bar{B}| = 2\int_{\frac{\pi}{2}}^{\theta_f} |d\bar{B}|}$$ (because of symmetry, and all pointing into the page)

b) using formula derived on other page:

$$|B| = \frac{\mu_0 I}{4\pi a}\left(\cos\theta_f - \cos\theta_0\right)$$ d now

$$= 2\left(\frac{\mu_0 I}{4\pi a}\right)\left(\cos\theta_f - \cos\left(\frac{\pi}{2}\right)\right)$$

$$= 2\left(\frac{\mu_0 I}{4\pi a}\right)\left(\cos\left(\tan^{-1}\left(\frac{-d}{a}\right)\right) - 0\right)$$

$$= \frac{\mu_0 I}{2\pi a}\left(\frac{a}{\sqrt{a^2+d^2}}\right)$$

d

CALCULATION FOR BOTTOM PART

+13/14

adding all contributions together:

wires on left and right

wire on bottom

$$|B_{origin}| = 2\left(\frac{\mu_0 I}{4\pi a}\left(\frac{d}{\sqrt{a^2+d^2}} - 1\right)\right) - \frac{\mu_0 I}{2\pi a}\left(\frac{a}{\sqrt{a^2+d^2}}\right)$$

$$= \boxed{\frac{\mu_0 I}{2\pi a}\left(\frac{d-a}{\sqrt{a^2+d^2}} - 1\right)}$$ (pointing out of the page)

✓

6 /14

6.9±4.1

Ph213 – Section D Quiz 4

Name (Print): Jonathan Lam Slot: 159

- **You have 15 minutes for this problem.**
- **I will not answer any questions about this problem.**
- **If you fail to specify (again!) a direction for B, you will receive a zero for this quiz, regardless of how much correct work you've done.**

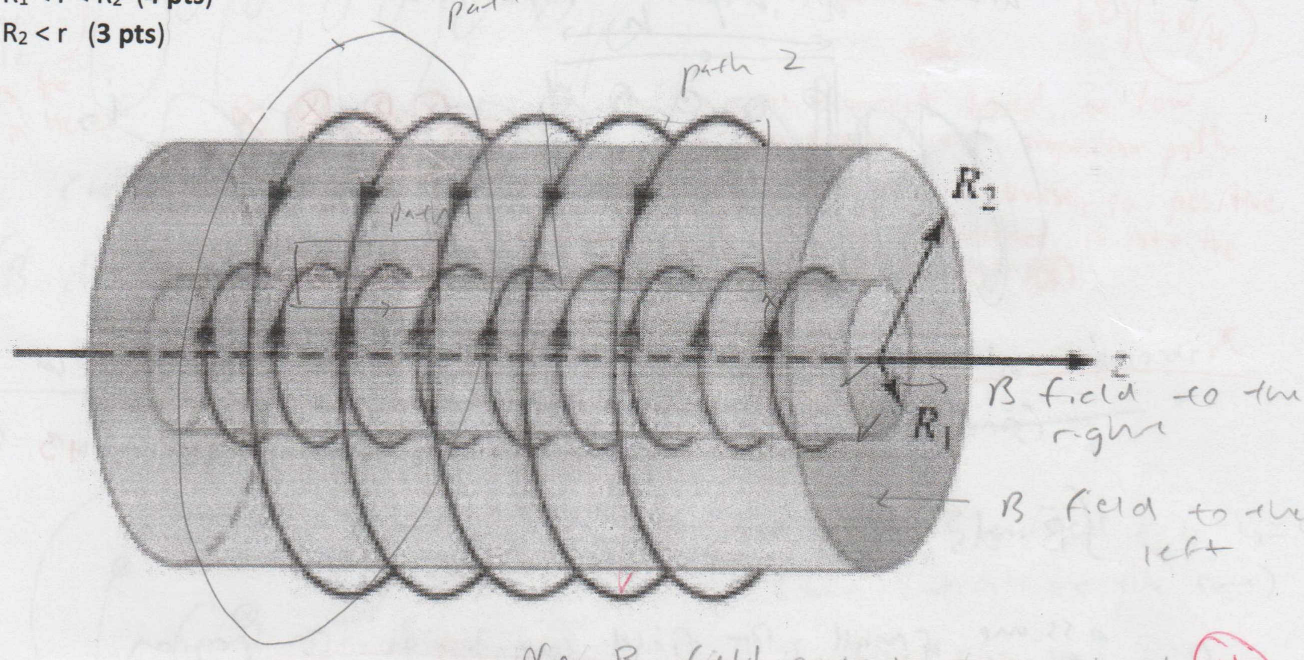
Two long solenoids are nested on the same (z) axis, as in the figure below. The inner solenoid has radius R_1 and n_1 turns per unit length. The outer solenoid has radius R_2 and n_2 turns per unit length. Each solenoid carries the same current I but the currents are flowing in opposite directions as indicated by the arrows.

You will be using Ampere's Law to find the total $\vec{B}$ field in 3 regions.

a) What approximations are **needed** to solve this problem with Ampere's Law? (Just concisely state the approximations. Don't write a book.) **(3 pts)**

b) Use Ampere's Law to find the total $\vec{B}$ field in the following three regions, showing your Amperian path[1] in each case. You can draw the three paths on the single diagram below, labelling them 1, 2, 3, for the three regions.

1) $r < R_1$ **(4 pts)**
2) $R_1 < r < R_2$ **(4 pts)**
3) $R_2 < r$ **(3 pts)**



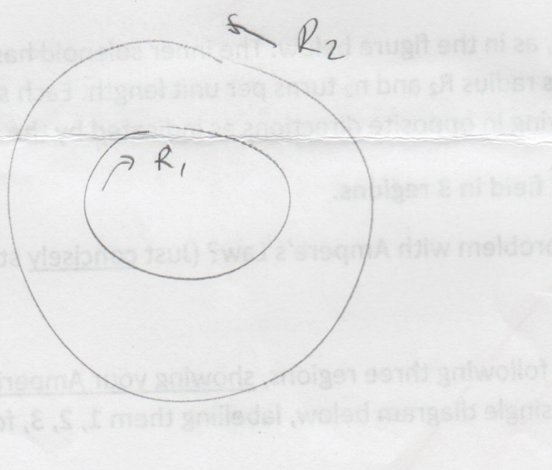
No B field outside outer solenoid. +1

[1] Yes, you can use a single Amperian path to find the field in each region.

a) - tightly wrapped rings, i.e., turn density is high, approximately uniform field radially

- solenoid is long, i.e., no fringing/end behavior

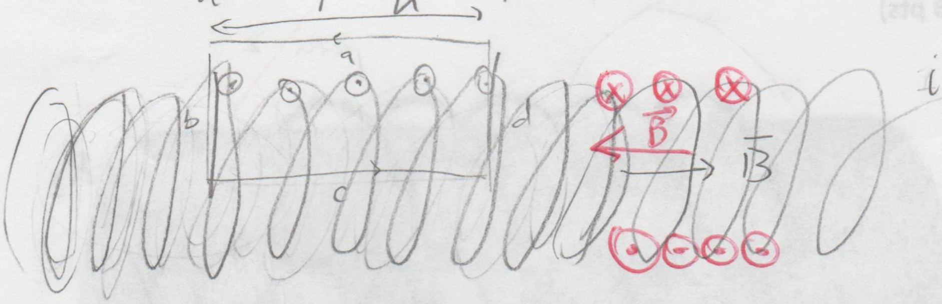
(+2) B field = 0 outside solenoid assumptions point given on first page. total +3/3 for part a)



a) +3/3

b) +3/11

b.1) draw an amperian loop inside

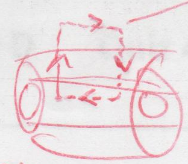


along edges b, d, no contribution to b-field (since cosine would equal zero.) +3

$$\int \vec{B} \cdot d\vec{s} = \mu_0 I$$

assume small B-field contribution to region outside this inner solenoid caused by this solenoid; thus most $\bar{B}$ field is along edge c.

what about B field from the other coil in the region outside the small solenoid.



This is the correct path. Or at least easier path to analyze.

$$\int \vec{B} \cdot d\vec{s} = \mu_0 I$$

all B pointing to the ~~right~~ left. in this diagram, so can sum magnitudes. Also, $\vec{B}$ field is parallel to the edge, so simple multiplication.

There is a B field outside the small solenoid

$$Bh = \mu_0 i n_1 h$$

b1) +0/4

$$B = \mu_0 i n_1 \text{, where } n \text{ is the turn density}$$ ✗

$$\left(n_1 = \frac{\# \text{ turns}}{\text{distance}} \right)$$

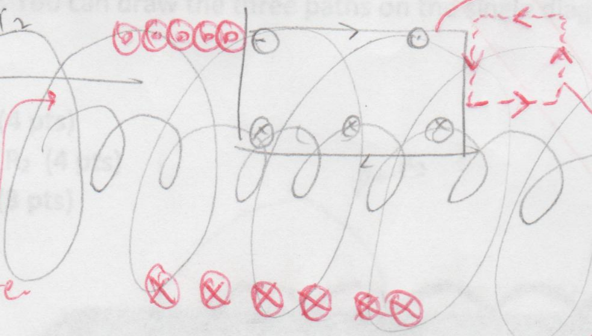
This is the B field inside the first solenoid,

I think you might have confused these. ⊗ = into page ⊙ = out of page.

b2) Assume contribution of i_2

B field to the left by right hand rule

B field is to the right in here.



this is the solution to part 1.

this is path for part 2

region in between i_1

b2) +0/4

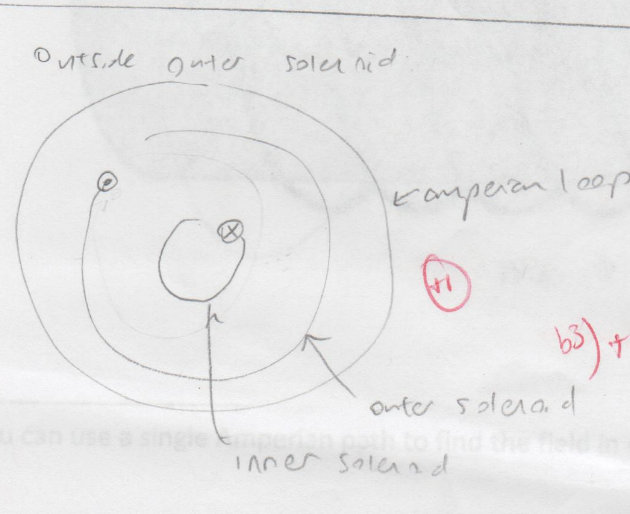
signs incorrect based on how you drew your amperian path. Clockwise, so positive I enclosed is into the page. (⊗)

$$\text{current enclosed} = i n_2 - i n_1 = i(n_2 - n_1)$$

$$\int \vec{B} \cdot d\vec{s} = \mu_0 I = \mu_0 i (n_2 - n_1) h$$

$$\vec{B} = \mu_0 i (n_2 - n_1) \text{ to the left between solenoids}$$ ✗

b3) outside outer solenoid



+1

b3) +3/3

net current enclosed $= i_1 - i_2 = 0$

(since currents are the same)

Thus $\int B \cdot ds = \mu_0 I = 0$

$$B \cdot 2\pi R = 0$$

+2

$B = 0$ ✓

outside both solenoids.

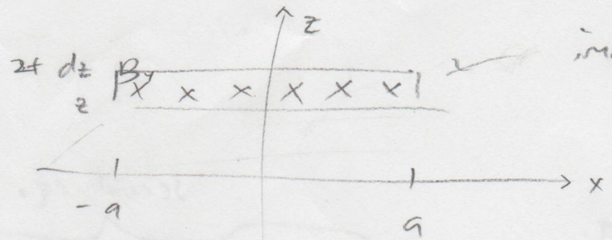
Ph213 – Section D **Quiz 5**

14 /14

Name (Print): Jonathan Lam Slot: 159

11.0 ± 3.9

Derive the EM wave equations for E & B. Assume propagation in the z direction. Assume the fields don't change except in the z direction (i.e., plane waves).



imaginary path

B field ⊥ to xz plane (B_y)

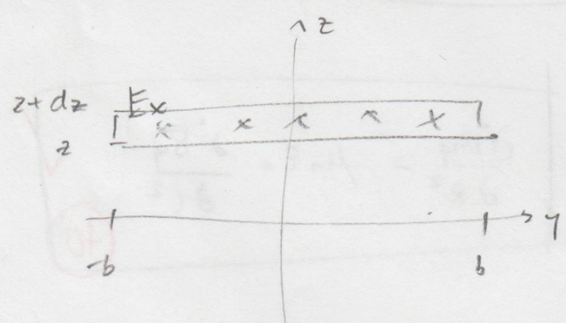
Faraday's Law $\oint \vec{E}_x \cdot d\vec{s} = \mathcal{E} = -\frac{d\Phi_B}{dt}$

dz contributions to ds are small, ignore

$$(E_{x_{z+dz}})(2a) - (E_{x_z})(2a) = -\frac{d}{dt}(B_y dA)$$

$$\frac{\partial E_x}{\partial z}(\cancel{dz\,2a}) = -\frac{d}{dt}(B_y \cdot \cancel{2a \cdot dz})$$

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}$$ ✓ (+5)



Ampere's Law

$$\oint \vec{B}_y \cdot d\vec{s} = \mu_0 \left(\cancel{I} + \varepsilon_0 \frac{d\Phi_E}{dt}\right)$$ ✓

real current = 0.

$$(B_{y_z})(2b) - (B_{y_{z+dz}})(2b) = \mu_0 \varepsilon_0 \left(\frac{d}{dt}(E\,dA)\right)$$

~~$B_y(z+dz) - B_y(z)$~~ ok

$$-\frac{\partial B_y}{\partial z}(\cancel{2b \cdot dz}) = \mu_0 \varepsilon_0 \left(\frac{\partial E_x}{\partial t} \cancel{2b\,dz}\right)$$

negative b/c of orientation. ok

$$\Rightarrow \frac{\partial B_y}{\partial z} = -\mu_0 \varepsilon_0 \frac{\partial E_x}{\partial t}$$ ✓ (+5)

From front side of page:

$$\begin{cases} \frac{\partial E_x}{\partial z} = \frac{-\partial B_y}{\partial t} \\ \frac{\partial B_y}{\partial z} = -\mu_0 \varepsilon_0 \frac{\partial E_x}{\partial t}. \end{cases}$$

Solving for $E(z,t)$:

differentiate:

$$\underbrace{\frac{\partial}{\partial z}\left(\frac{\partial E_x}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{-\partial B_y}{\partial t}\right)}_{\text{first eq.}} = \underbrace{-\left(\frac{\partial}{\partial t}\left(\frac{\partial B_y}{\partial t}\right)\right) = \mu_0 \varepsilon_0 \frac{\partial E_x}{\partial t}}_{\text{second eq.}}$$

equal

$$\Rightarrow \boxed{\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2}}$$

✓ +2

Solving for $B(z,t)$:

differentiate:

$$\underbrace{\frac{\partial}{\partial t}\left(\frac{\partial B_y}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\frac{\partial E_x}{\partial z}\right)}_{\text{first eq.}} = \underbrace{\frac{1}{\mu_0 \varepsilon_0}\left(\frac{\partial}{\partial z}\left(-\mu_0 \varepsilon_0 \frac{\partial E_x}{\partial t}\right)\right) = \frac{1}{\mu_0 \varepsilon_0}\frac{\partial}{\partial z}\left(\frac{\partial B_y}{\partial z}\right)}_{\text{second eq.}}$$

equal

$$\Rightarrow \frac{\partial^2 B_y}{\partial t^2} = \frac{1}{\mu_0 \varepsilon_0}\frac{\partial^2 B_y}{\partial z^2} \Rightarrow \boxed{\frac{\partial^2 B_y}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 B_y}{\partial t^2}}$$

✓ +2

Both of these fit the wave equation form,

And may be solved by sinusoidal equations of z and t such as:

$$E(z,t) = E_0 \cos(kz - \omega t)$$

$$B(z,t) = B_0 \cos(kz - \omega t)$$

Digital signature
Name: Jonathan Lam
Partner's name: Andrew Kim
Date: 12/09/19

1 Purpose

Optical diffraction is used to estimate the width of a human hair. This involved the calculation of the wavelength of a light source (a laser) by measuring attributes of an N-slit diffraction spectrum. This calculated wavelength is used to estimate the human hair's width by measuring attributes of a wide single-slit diffraction pattern.

2 Data

Note 1: The position of the front of the laser was measured, not the position of the diffraction grating, single slit, or hair. However, the objects or slits were put with a ring stand and clamp as close as possible to the laser (≈1 to 3mm) in front of the laser to reduce this error.

Note 2: The reported instrumental errors are those of a single reading, before any calculations; e.g., the instrumental error for the optical bench is not equal to the error for a length measurement (which involves two readings, and whose error is taken into account in the error propagation sections).

Note 3: (Table 5) and (Table 8) are identical, as the setup was not changed between these parts (except that the Vernier caliper was replaced with a hair). This also shows the flexibility of this setup to determine either wavelength or slit width, as well as the fascinating result that an object and a slit of the same thin finite width can generate the same diffraction pattern.

Note 4: The wavelength of the lasers used in this experiment have wavelengths known to be approximately 650nm, but the exact value is not known (i.e., there was no sticker on our laser indicating its wavelength),

2.1 Part A. N-slit diffraction pattern

Table 1: Bench measurements

Pos. screen (cm)	121.82
Pos. laser (cm)	105.03
Laser offset (cm)	1.06
Dist. laser to screen (cm)	15.73
Instrumental error (cm)	0.05

Table 2: Distances from center to intensity maxima

Diff. grating line density (lines/mm)	1000
Dist. center to left maximum (cm)	12.999
Dist. center to right maximum (cm)	13.548
Instrumental error (cm)	0.002

2.2 Part B. (Wide) single-slit diffraction pattern

Table 3: Measured distances from fourth minima to the left to other minima

a (cm)	$d_{-4,4}$ (cm)	$d_{-4,3}$ (cm)	$d_{-4,2}$ (cm)	$d_{-4,1}$ (cm)	$d_{-4,-1}$ (cm)	$d_{-4,-2}$ (cm)	$d_{-4,-3}$ (cm)
0.02	2.458	2.236	1.962	1.610	0.920	0.638	0.300
0.03	2.080	1.810	1.634	1.300	0.796	0.628	0.300
0.04	1.472	1.298	1.098	0.942	0.648	0.414	0.178

Table 4: Distances of minima to center of diffraction pattern (h)

a (cm)	h (cm)							
	$p = 4$		$p = 3$		$p = 2$		$p = 1$	
	left	right	left	right	left	right	left	right
0.02	1.283	1.265	0.971	0.965	0.697	0.627	0.345	0.345
0.03	1.032	1.048	0.762	0.748	0.586	0.420	0.252	0.252
0.04	0.677	0.795	0.503	0.617	0.303	0.381	0.147	0.147

Table 5: Distance from laser to screen

Pos. screen (cm)	124.70
Pos. laser (cm)	8.65
Laser offset (cm)	1.06
Dist. laser to screen (cm)	114.99
Instrumental error (cm)	0.05

2.3 Part C. Hair diffraction pattern

Table 6: Measured distances from fourth minima to the left to other minima

Owner	$d_{-4,4}$ (cm)	$d_{-4,3}$ (cm)	$d_{-4,2}$ (cm)	$d_{-4,1}$ (cm)	$d_{-4,-1}$ (cm)	$d_{-4,-2}$ (cm)	$d_{-4,-3}$ (cm)
Andrew	8.748	7.750	6.792	5.450	3.422	2.298	1.780
Jon	7.278	6.300	5.300	4.412	2.704	1.830	0.914

Table 7: Distances of minima to center of diffraction pattern (h)

Owner	h (cm)							
	$p = 4$		$p = 3$		$p = 2$		$p = 1$	
	left	right	left	right	left	right	left	right
Andrew	4.312	4.436	3.314	2.656	2.356	2.138	1.014	1.014
Jon	3.720	3.558	2.742	2.644	1.742	1.728	0.854	0.854

Table 8: Distance from laser to screen

Pos. screen (cm)	124.70
Pos. laser (cm)	8.65
Laser offset (cm)	1.06
Dist. laser to screen (cm)	114.99
Instrumental error (cm)	0.05

3 Explanation of errors

If the backplane holding the paper is not normal (i.e., perpendicular) to the incident rays, it may cause asymmetry and random error in the measurements (it will systematically increase the values of h on one side and decrease the values of h on the other). We didn't realize or attempt to quantify or correct this source of error. While there is a large deviation from the two intensity maxima from the center in Part A, it is less clear from Part B whether the backplane was tilted, as the measurements from one side are not notably systematically higher or lower than the other.

There is also the possibility of systematic error from errors with measuring offsets. We did not measure the offset of the paper screen from the center of the backplane (thus considering the paper to have zero offset from its holding device), but this may cause l to be systematically small. Similarly, we estimated the distance between the slit(s) or object and the paper as the distance between the tip of the laser and the paper, and placed the slit(s) or object very close to the tip of the laser, and this may cause l to be systematically large. However, these offsets (on the order of roughly 1-3mm) are deemed insignificant compared to the distance l (on the order of dozens of centimeters). To further mitigate this relative error, we made l large (15.73cm for Part A and 114.99cm for Parts B and C) by increasing the distance between the laser and the screen.

There is some random error introduced by marking the position of the intensity maxima (in Part A) or minima (in Parts B and C) on the paper. For either intensity extrema, it is difficult to pinpoint the exact position of the extrema, even if the bright or dark spots are fairly small (an estimate of this visual error may be ± 0.5mm). Furthermore, the intensity minima is not exactly at the center of the bright or dark regions because of each extrema's asymmetry (except the central peak); however, this is likely dwarfed by visual and instrumental uncertainties.

It was difficult to get correct measurements with the calipers, as we were measuring between penciled lines on paper and it is difficult to measure the perpendicular distance because of the nature of the calipers. This should add some small, unquantifiable random error to the caliper's instrumental uncertainty.

4 Calculations

4.1 Instrumental and length errors

Two measurement tools were used: the optical bench and a Vernier caliper. The instrumental error for lengths obtained using the Vernier caliper is 0.002cm. (See the note in the Explanation of errors section about additional error introduced when using the Vernier calipers.)

The instrumental error for the optical bench is 0.05cm. For lengths obtained using the optical bench, the error is 0.07cm, because lengths are obtained by the difference of two independent measurements with an instrumental error of 0.05cm each, i.e., for a length measurement s obtained using the optical bench,

$$s = s_2 - s_1$$

$$\delta s = \sqrt{\delta s_2^2 + \delta s_1^2} = \sqrt{0.05\text{cm}^2 + 0.05\text{cm}^2} = 0.07\text{cm}$$

4.2 Mean and standard deviation of the mean (STDOM)

The best for a variable x is calculated is given by the mean.

$$\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i$$

The error propagation for the mean, assuming values of x are independent, is given by the RMS of the errors.

$$\delta \bar{x} = \frac{1}{N} \sqrt{\sum_{i=1}^{N} \delta x_i^2}$$

The STDOM is another interpretation of the error of the mean and is given by the following equation.

$$\sigma_{\bar{x}} = \frac{1}{\sqrt{N}} \left(\frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x}^2) \right)$$

Where applicable (i.e., in Parts B and C), both the error propagation for the mean and STDOM are calculated, and the larger error is reported and/or used for aggregated error propagation calculations, i.e.:

$$x = \bar{x} \pm \max(\delta \bar{x}, \sigma_{\bar{x}})$$

4.3 Calculation of wavelength from diffraction grating

In section 7.1, we derive (Eq. 8) for the positions of the intensity maxima for a laser's light from an N-slit grating.

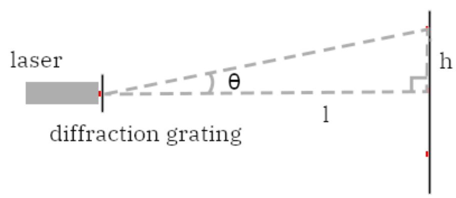
$$\sin\theta = \frac{m\lambda}{d}$$

where d is the slit density, and θ is the angle from the optical (horizontal) axis to the m-th intensity maximum on either side. In particular, since only one intensity maxima region was viewed on either side, we have two samples for the $m = 1$ case, with the same d value and differing θs. Rearranging the equation, we solve for λ.

$$\lambda = d \sin \theta$$

in the case where $m = 1$. $\sin \theta$ is calculated from the ratio of lengths rather than calculating the sine of a measured angle, as can be viewed in (Fig. 1).

Figure 1: Setup for estimation of wavelength from diffraction grating



Thus

$$\sin \theta = \frac{h}{\sqrt{l^2 + h^2}} \tag{1}$$

4.3.1 Calculation of l

The calculation of the distance l from laser tip (which is placed roughly a millimeter away from the diffraction grating) to screen is simply the difference of the position of the screen and laser positions, plus the offset of the tip of the laser from the center of the reading. In other words, let p_s be the screen position, p_l be the laser position, and o_l be the laser offset from the reading be from (Table 1); then

$$l = p_s - (p_l - o_l) \tag{2}$$

and thus

$$\delta l = \sqrt{\delta p_s^2 + \delta p_l^2 + \delta o_l^2}$$

Since p_s and p_l were estimated from the optical bench (instrumental uncertainty 0.05cm) and o_l was estimated with the Vernier caliper (instrumental uncertainty 0.002cm), the error δl is

$$\delta l = \sqrt{(0.05\text{cm})^2 + (0.05\text{cm})^2 + (0.002\text{cm})^2} = 0.07\text{cm} \tag{3}$$

4.3.2 Error propagation for wavelength from diffraction grating

We first calculate $\delta(\sin\theta)$, which will also be used in error calculations for subsequent sections.

$$\delta(\sin\theta) = \sqrt{\left(\frac{\partial(\sin\theta)}{\partial h}\delta h\right)^2 + \left(\frac{\partial(\sin\theta)}{\partial l}\delta l\right)^2} \tag{4}$$

$$= \sqrt{\left(\frac{l^2\delta h}{\sqrt{l^2+h^2}^3}\right)^2 + \left(\frac{hl\delta l}{\sqrt{l^2+h^2}^3}\right)^2} = \sqrt{\frac{(l^2\delta h)^2 + (hl\delta l)^2}{(l^2+h^2)^3}}$$

We don't perform an error propagation on d, so the error propagation for λ is straightforward.

$$\delta\lambda = d\delta(\sin\theta)$$

Another interpretation for uncertainty is half of the range of the two wavelengths.

$$\delta\lambda = \frac{|\lambda_1 - \lambda_2|}{2}$$

4.3.3 Sample calculations

Using (Table 1), we obtain l:

$$l = 121.82\text{cm} - (105.03\text{cm} + 1.06\text{cm}) = 15.73\text{cm}$$

Now we calculate $\sin\theta$ and λ, using the above calculated results and (Table 1).

$$\sin\theta = \frac{12.999\text{cm}}{\sqrt{(12.999\text{cm})^2 + (15.73\text{cm})^2}} = 0.637$$

$$\lambda = (0.001\text{mm})(0.637) = 637\text{nm}$$

Now we calculate error propagation. δh is the instrumental error for the Vernier caliper, and δl is as calculated in (Eq. 3).

$$\delta(\sin\theta) = \sqrt{\frac{((15.73\text{cm})^2(0.002\text{cm}))^2 + ((15.73\text{cm})(12.999\text{cm})(0.07\text{cm}))^2}{((15.73\text{cm})^2 + (12.999\text{cm})^2)^3}}$$

$$= 0.002$$

$$\delta\lambda = (0.001\text{mm})(0.002) = 2\text{nm}$$

The average value of the two calculated wavelengths and the uncertainty (half of the range) are displayed below.

$$\bar{\lambda} = \frac{637 - 653}{2} = 645\text{nm}$$

$$\delta\bar{\lambda} = \frac{653 - 637}{2} = 8\text{nm}$$

This value is much larger than the calculated error propagation uncertainties of $\bar{1}.7$nm and $\bar{1}.4$nm (respectively) of the two trials, which hints that there is some larger, unaccounted-for error in the error propagation (see explanation of errors section, particularly about the screen's slant).

4.4 Calculation of wavelength from single slit

In section 7.2, we derive (Eq. 10) for the positions of the intensity minima of a laser's light from a single finite-width slit.

$$\sin\theta = \frac{p\lambda}{a}$$

where a is the width of the slit and θ is the angle from the optical (horizontal) axis to the p-th minima. The equation is rearranged to solve for wavelength.

$$\lambda = \frac{a}{p}\sin\theta$$

where $\sin\theta$ is calculated as in (Eq. 1).

4.4.1 Calculation of l and h

l is calculated the same way as in (Eq. 2).

Measurements for h use measurements from (Table 3) to obtain the values in (Table 4). All length measurements were taken with respect to the fourth minimum on the left side (i.e., length from fourth to third minima, length from fourth to second minima, ..., length from fourth to fourth minima on opposite sides); therefore, seven length measurements were taken for each aperture width. The distance from the fourth minimum to the center was approximated as halfway between the first minima on each side of the center, and each value for h is the difference between the length of the fourth minimum and the desired minimum and the length between the fourth minimum and the center. For clarity, denote the minima on the left to have negative indices (i.e., minima -1, -2, -3, -4) and the minima on the right to have positive indices (i.e., minima 1, 2, 3, 4), denote the distance between the i-th and j-th minima as $d_{i,j}$, and call the center the minimum with index 0. Thus:

$$d_{-4,0} = \frac{d_{-4,-1} + d_{-4,1}}{2}$$

and

$$h_i = d_{i,0} = |d_{-4,i} - d_{-4,0}|$$

Since all lengths were measured with respect to minimum -4,

$$\delta d_{-4,i} = 0.002\text{cm} \;\forall i \in -4, -3, \ldots, 4$$

and thus

$$\delta h = \sqrt{(0.002\text{cm})^2 + \left(\frac{0.002\text{cm}}{2}\right)^2 + \left(\frac{0.002cm}{2}\right)^2} = 0.00\bar{2}4\text{cm} \tag{5}$$

4.4.2 Error propagation for calculation of wavelength from single slit

This calculation is very similar to that of the previous section, except now a is also a measured value with an uncertainty. Since measurement error in a affects the error of h, a and θ are dependent. Embedding the error expression for $\delta(\sin\theta)$ from (Eq. 4), we have

$$\delta\lambda = \left|\frac{\partial\lambda}{\partial a}\delta a\right| + \left|\frac{\partial\lambda}{\partial(\sin\theta)}\delta(\sin\theta)\right| = \left|\frac{(\sin\theta)\delta a}{p}\right| + \left|\frac{a\delta(\sin\theta)}{p}\right|$$

Since all of the terms will be positive, this simplifies to

$$\delta\lambda = p^{-1}((\sin\theta)\delta a + a\delta(\sin\theta)) \tag{6}$$

4.4.3 Sample calculation

The sample calculation shown is that of the fourth minimum on the right. We calculate h, l, and $\sin\theta$, and λ, for $a = 0.002$cm in that order.

$$h = \left|2.548\text{cm} - \frac{1.962\text{cm} + 1.610\text{cm}}{2}\right| = 1.283\text{cm}$$

$$l = 124.70\text{cm} - (8.65\text{cm} + 1.06\text{cm}) = 114.99\text{cm}$$

$$\sin\theta = \frac{1.283\text{cm}}{\sqrt{(1.283\text{cm})^2 + (114.99\text{cm})^2}} = 0.0112$$

$$\lambda = \frac{(0.02\text{cm})(0.0112)}{4} = 558\text{nm}$$

We calculate the error next. δh is given by (Eq. 5), and δl is given by (Eq. 3). δa is the instrumental error of the Vernier caliper.

$$\delta(\sin\theta) = \sqrt{\frac{((114.99\text{cm})^2(0.002\text{cm}))^2 + ((1.283\text{cm})(114.99\text{cm}))^2}{((1.283\text{cm})^2 + (114.99\text{cm})^2)}^3} = 0.0000\bar{2}24$$

$$\delta\lambda = \frac{(0.0112)(0.002\text{cm}) + (0.02\text{cm})(0.0000224)}{4} = \bar{5}6.9\text{nm}$$

The mean, standard deviation of the mean (STDOM), and error propagation for the mean is taken for each aperture size. For $a = 0.02$cm, the calculations are shown below.

$$\bar{\lambda}_{a=0.02\text{cm}} = \frac{558\text{nm} + 563\text{nm} + \cdots + 550.\text{nm}}{8} = 573\text{nm}$$

$$\sigma_{\bar{\lambda}_{a=0.02}\text{cm}} = \frac{\sqrt{(558\text{nm} - 573\text{nm})^2 + (563\text{nm} - 573\text{nm})^2 + \cdots + (550.\text{nm} - 573\text{nm})^2}}{(8-1)\sqrt{8}}$$
$$= \bar{8}.83\text{nm}$$
$$\delta_{\bar{\lambda}_{a=0.02}\text{cm}} = \frac{\sqrt{(57\text{nm})^2 + (58\text{nm})^2 + \cdots + (56\text{nm})^2}}{8} = \bar{2}1\text{nm}$$

Thus $\bar{\lambda}_{a=0.02\text{cm}} = 570 \pm 20\text{nm}$.

Next, the best value from all aperture sizes is aggregated by taking the mean of the calculated $\bar{\lambda}$ from each aperture size, and the error calculation is calculated likewise (calculate STDOM and error propagation for the mean and reporting the larger error). The calculations are similar to those shown above for each aperture size and thus omitted. The results for this section are summarized in (Table 10).

4.5 Calculation of width of human hair

Optical diffraction predicts that the diffraction pattern for a thin finite-width gap is the same as that produced by a thin object of the same finite width. (Eq. 10) is used like in the preceding section, and manipulated to solve for the a, the width of the object.

$$a = \frac{p\lambda}{(\sin\theta)}$$

where $\sin\theta$ is calculated as in (Eq. 1).

4.5.1 Error propagation for the width of human hair

The errors $\delta(\sin\theta)$ (calculated in (Eq. 4)) and $\delta\lambda$ (calculated in (Eq. 6)) are independent and thus added in quadrature.

$$\delta a = \sqrt{\left(\frac{\partial a}{\partial\lambda}\delta\lambda\right)^2 + \left(\frac{\partial a}{\partial(\sin\theta)}\delta(\sin\theta)\right)^2} = \sqrt{\left(\frac{p\delta\lambda}{(\sin\theta)}\right)^2 + \left(\frac{p\lambda\delta(\sin\theta)}{(\sin\theta)^2}\right)^2}$$
$$= \frac{p}{(\sin\theta)^2}\sqrt{((\sin\theta)\delta\lambda)^2 + (\lambda\delta(\sin\theta))^2}$$

4.5.2 Sample calculation

The sample calculation shown is that for Andrew's hair using the fourth minimum on the right. The process is very similar to the previous sample calculation, and it uses the best value for λ calculated in the previous section.

$$h = \left|8.748\text{cm} - \frac{5.450\text{cm} + 3.422\text{cm}}{2}\right| = 4.312\text{cm}$$
$$l = 124.70\text{cm} - (8.65\text{cm} + 1.06\text{cm}) = 114.99\text{cm}$$
$$\sin\theta = \frac{4.312\text{cm}}{\sqrt{(4.312\text{cm})^2 + (114.99\text{cm})^2}} = 0.0375$$

$$a = \frac{4(611\text{nm})}{0.0375} = 65.2\mu\text{m}$$

For the error propagation, δh and δl are known, as in the previous section. $\delta\lambda$ is the error for λ calculated in the previous section.

$$\delta(\sin\theta) = \sqrt{\frac{((114.99\text{cm})^2(0.002\text{cm}))^2 + ((114.99\text{cm})(4.312\text{cm})(0.07\text{cm}))^2}{((114.99\text{cm})^2 + (4.312\text{cm})^2)^3}} = 0.0000\bar{3}13$$

$$\delta a = \frac{4}{0.0375^2}\sqrt{(0.0375 * (14\text{nm}))^2 + ((611\text{nm}) * 0.0000237)} = 1.48\mu\text{m}$$

The mean, STDOM, and error propagation for the mean are identical calculations as those in the preceding section for each aperture size, so no sample calculations will be shown here. The results from this section are summarized in (Table 11).

5 Results

5.1 Part A

Table 9: Calculation of wavelength from N-slit diffraction

Sample 1 λ (nm)	Sample 2 λ (nm)	$\bar{\lambda}$ (nm)
637	653	645 ± 8

5.2 Part B

Table 10: Wavelength calculated from single-slit diffraction

a (cm)	$\bar{\lambda}$ (nm)	$\delta\bar{\lambda}$ (err. prop.) (nm)	$\delta\bar{\lambda}$ (STDOM) (nm)
0.02	570	20	9
0.03	660	20	20
0.04	$6\bar{0}0$	12	30

Reported $\bar{\lambda}$ (nm)	610 ± 14

5.3 Part C

Table 11: Width of human hair calculated by single-slit diffraction

Owner	$\bar{a}$ (μm)	$\delta\bar{a}$ (err. prop.) (μm)	$\delta\bar{a}$ (STDOM) (μm)	Reported $\bar{a}$ (μm)
Andrew	66.9	0.5	2	67 ± 2
Jon	79.8	0.7	0.9	79.8 ± 0.9

6 Conclusion

Reasonable values were obtained from Parts A and B for the wavelength of the laser, 645 ± 8nm and 610 ± 14nm, respectively. While the latter doesn't capture the expected approximate true wavelength of roughly 650nm within one standard deviation, the 6% error is small.

In Part B, the diameter of Andrew's and Jon's hairs were determined to be $67 \pm 2\mu$m and $79.8 \pm 0.2\mu$m, respectively. These are reasonable values; Brian Ley from *The Physics Factbook* estimates that most human hair falls in the range of 17 to 181μm[1]. The small standard deviation and the small error of the slit method in Part B indicates that this method has a high precision for very thin objects, most much higher than can be achieved with Vernier calipers, linear scales, or more coarse instruments (and perhaps similar to or better than micrometer calipers).

[1] https://hypertextbook.com/facts/1999/BrianLey.shtml

7 Answers to questions

7.1 Intensity maxima in N-slit diffraction

The intensity pattern of an N-slit pattern, with slits spaced d distance away on a distant screen is given by (Eq. 7), as defined in class.

$$I(\delta) = I_0 \left[\frac{\sin \frac{N\delta}{2}}{\sin \frac{\delta}{2}} \right]^2 \tag{7}$$

where $\delta = \frac{2\pi}{\lambda} d \sin\theta$ (the phase delay) and θ is the angle between the ray pointing to the position of interest on the screen and the central ray. While it seems plausible that the maxima lie elsewhere, the maxima lie in the most obvious positions: where the denominator of the fraction is zero. Thus $x = \pi m,\ m = \pm 1, \pm 2, \ldots.$

$$\frac{\delta}{2} = m\pi \Rightarrow \frac{2\pi}{\lambda} d \sin\theta = 2\pi m \Rightarrow \sin\theta = \frac{m\lambda}{d} \tag{8}$$

We may calculate the maximum intensity at the points. Let $x = \frac{\delta}{2}$. We use L'Hopital's rule twice to evaluate this limit.

$$I_{\text{max}} = \lim_{x\to 0} I_0 \left[\frac{\sin Nx}{\sin x} \right]^2 = I_0 \lim_{x\to 0} \frac{2N \sin(Nx)\cos(Nx)}{2\sin(x)\cos(x)} = I_0 \frac{N \sin(2Nx)}{\sin(2x)}$$

$$= I_0 N \lim_{x\to 0} \frac{2N \cos 2Nx}{2\cos 2x} = I_0 N^2 \frac{cos(0)}{cos(0)} = I_0 N^2$$

7.2 Intensity minima in single-slit diffraction

The intensity pattern of a (finite-width) single-slit diffraction, using the same variable conventions as in the previous section, is given by (Eq. 9).

$$I(\beta) = I_0 \left(\frac{\sin\beta}{\beta} \right)^2 \tag{9}$$

where $\beta = \frac{\pi}{\lambda} a \sin\theta$. This clearly has minima when $\sin\beta = 0$, or, equivalent, when $\beta = p\pi,\ p = \pm 1, \pm 2, \ldots$

$$\frac{\pi}{\lambda} a \sin\theta = p\pi \Rightarrow \sin\theta = \frac{p\lambda}{a} \tag{10}$$

1 Purpose

The laws of reflection and refraction are observed and quantified in this lab. The main result of the measurements and calculations is the index of reflection of glass and tap water. Two methods are used, and their outputs are compared to the standard values within error bars.

In Part A, the calculation of the index of reflection of glass and tap water are tackled through an application of Snell's Law called Pfund's method, which is based on diffuse scattering and the critical refraction angle along the glass-air and glass-liquid boundaries. A petri dish is used as the glass sample, a laser used as the light source, and a paper with markings as the diffuse scatterer and the source of markings to measure; its thickness is measured with a micrometer, and the diameter of the boundaries of the rings caused by the scattering is measured indirectly with a vernier caliper. This also brings into light many possible sources of error, such as the method for indirect measurement.

In Part B, Snell's law is used more directly by measuring angles and using the Snell's law relationship to find the index of refraction of the liquid. This method explores using geometric construction to correctly measure the angles, as well as a different method of error propagation due to the fact that the angles cannot be simply averaged like the other measurement types.

2 Data

Table 1: Petri Dish Thickness

Sample	Measured thickness (cm)	Corrected thickness (cm)
1	0.2369	0.2354
2	0.2651	0.2636
3	0.2460	0.2445
4	0.2568	0.2553
5	0.2355	0.2340
6	0.2240	0.2225

Zero offset (cm)	0.0015
Instrumental error (cm)	0.0005
Random error (cm)	0.006
Mean (cm)	0.243
Uncertainty for the mean (cm)	0.006

Table 2: Ring diameter without liquid

Sample	Diameter (cm)
1	0.772
2	0.750
3	0.756
4	0.785
5	0.806
6	0.750

Zero offset (cm)	0.000
Instrumental error (cm)	0.002
Random error (cm)	0.009
Mean (cm)	0.770
Uncertainty for the mean (cm)	0.009

Table 3: Ring diameter with liquid

Sample	Diameter (cm)
1	1.814
2	1.850
3	1.950
4	1.884
5	1.834
6	1.840

Zero offset (cm)	0.000
Instrumental error (cm)	0.002
Random error (cm)	0.02
Mean (cm)	1.86
Uncertainty for the mean (cm)	0.02

Table 4: Snell's law method measured incident and refracted angles

Sample	Incident angle (deg)	Refracted angle (deg)
1	43.0	29.7
2	44.0	30.1
3	22.0	15.2
4	21.9	15.3
5	53.0	36.1
6	56.5	35.5

3 Calculations

Note that the full precision of each measurement is kept the final result of each calculation, in which the results are rounded for brevity.

3.1 Instrumental error

The vernier caliper has markings to the nearest 0.002cm, and reading it involves choosing the closest matching margin. Since there is no approximation between the markings, the instrumental uncertainty is $\delta S = \pm 0.002$cm.

The micrometer has markings to the nearest 0.001cm. Reading it involves visual estimation between markings, so the instrumental error for a reading is $\delta S = \pm 0.0005 cm$. The micrometer had a nonzero zero-offset of 0.0015cm, so this value was subtracted from all of the measured values to yield the corrected values displayed in (Table 1).

The protractor has markings to the nearest 0.5°, so each angle reading has accuracy $\delta A_{read} = 0.25°$. Since each angle measurement is a function of two readings (the degree readings of the two rays that contain the angle), the error for a single angle measurement δA is calculated from the (independent) errors of the left and right readings.

$$\delta A = \sqrt{\delta A_{left}^2 + \delta A_{right}^2} = \sqrt{(0.25°)^2 + (0.25°)^2} = 0.354° = 0.4° \qquad (1)$$

3.2 Sample mean and random error

For each sample of length measurements (Part A), the measurement is recorded with both a "best value" and a specified random error. The "best value" is the mean $\bar{x}$, and the random error is reported as standard deviation of the mean (STDOM) σ_x.

For (Part B), calculating sample mean and standard deviation of the raw measured data was not applicable, as the angle values were not centered around a common point and did not have a common spread.

3.3 Calculation of n_g and error using Pfund's method

The calculation for the average index of refraction of glass, $\bar{n}_g$, is shown below in (2). This is derived in (Figure 2).

$$n_g = \frac{\sqrt{d^2 + 16t^2}}{d} \tag{2}$$

where d is the average diameter of the boundary between the gray and the bright ring, and t is the average thickness of the petri dish.

The error propagation for the calculation of n_g is shown below in (3). The values of d and t are dependent (i.e., a thicker petri dish would cause a change in ring diameter by the geometry of the problem; see (Figure 2)), and thus the absolute values of their errors are added. δd and δt are the larger of the random error and instrumental error for the respective metrics.

$$\begin{aligned}\frac{\partial n_g}{\partial d} &= \frac{-16t^2}{d^2\sqrt{d^2 + 16t^2}} \\ \frac{\partial n_g}{\partial t} &= -\frac{16t}{d\sqrt{d^2 + 16t^2}} \\ \delta n_g &= \left|\frac{\partial n_g}{\partial d}\delta d\right| + \left|\frac{\partial n_g}{\partial t}\delta t\right| \end{aligned} \tag{3}$$

Sample calculations

The calculation for n_g is shown below.

$$n_g = \frac{\sqrt{(0.770\text{cm})^2 + 16(0.243\text{cm})^2}}{0.770\text{cm}} = 1.61$$

The calculation for δn_g is shown below.

$$\delta n_g = \left|\frac{0.770\text{cm}^2 - 16(0.243\text{cm}^2)}{0.770\text{cm}^2\sqrt{0.770\text{cm}^2 + 16(0.243\text{cm})^2}}\, 0.009\text{cm}\right| + \left|\frac{16(0.243\text{cm})}{0.770\text{cm}\sqrt{0.770\text{cm}^2 + 16(0.243\text{cm})^2}}\, 0.006\text{cm}\right| = 0.0367$$

3.4 Calculation of n_l and error using Pfund's method

The calculation for the index of refraction of the liquid, n_l, is shown below in (4). This is derived from (Figure 3).

$$n_l = \frac{n_g\, d}{\sqrt{d^2 + 16t^2}} \tag{4}$$

where d is the diameter of the boundary between the gray and the bright stationary ring, and t is the thickness of the petri dish.

As before, d is dependent on t because of the geometry of the calculation. Since n_g is a function of t, it is also dependent on t. Thus, the absolute values of the errors are added. δd and δt are the larger of the random error and instrumental error for the respective metrics.

$$\frac{\partial n_l}{\partial n_g} = \frac{d}{\sqrt{d^2 + 16t^2}}$$
$$\frac{\partial n_l}{\partial d} = \frac{16 n_g t^2}{(d^2 + 16t^2)^{\frac{3}{2}}}$$
$$\frac{\partial n_l}{\partial t} = -\frac{16 n_g d t}{(d^2 + 16t^2)^{\frac{3}{2}}}$$
$$\delta n_l = \left|\frac{\partial n_l}{\partial n_g}\,\delta n_g\right| + \left|\frac{\partial n_l}{\partial d}\,\delta d\right| + \left|\frac{\partial n_l}{\partial t}\,\delta t\right| \tag{5}$$

Sample calculations

The calculation for n_l is shown below.

$$n_l = \frac{1.61(1.86\text{cm})}{\sqrt{1.86\text{cm}^2 + 16(0.243\text{cm}^2)}} = 1.431$$

The calculation for the error for n_l is shown below.

$$\delta n_l = \left|\frac{1.86\text{cm}}{\sqrt{1.86\text{cm}^2 + 16(0.243\text{cm}^2)}} 0.0367\text{cm}\right| + \left|\frac{16(1.61)(0.243\text{cm})^2}{\sqrt{1.86^2 + 16(0.243\text{cm})}^3} 0.02\text{cm}\right| + \left|-\frac{16(1.61)(1.86\text{cm}(0.243\text{cm}))}{\sqrt{1.86\text{cm}^2 + 16(0.243\text{cm})^2}^3} 0.006\text{cm}\right| = 0.0436$$

3.5 Calculation of n_l and error using Snell's law method

Snell's law is stated in (6).

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{6}$$

In the case of this lab, let medium 1 (n_a, θ_a) be air, and medium 2 (n_l, θ_l) be the liquid. Approximating $n_a \approx 1$, then the index of refraction of the liquid can be rewritten in terms of the two angles:

$$n_l = \frac{\sin\theta_a}{\sin\theta_l} \tag{7}$$

Both the mean and error propagation are more difficult to calculate here than with the two methods from (Part A), since the raw measurements (angle values) cannot be averaged, and the mean and error calculations calculated from the means and errors of the measurements. This is because the angles are not centered around the same value, and do not have a similar spread – they vary widely between different samples. Therefore, the index of refraction of the liquid and its error have to be calculated for each sample, and then this set of data must be appropriately aggregated.

The aggregation for the mean is fairly intuitive: average the calculated n_l for each sample.

$$\bar{n}_l = \frac{1}{6}\sum_{i=1}^{6} n_{l_i} \tag{8}$$

For the error calculation, the error of a single sample can be calculated as shown in (9). The error calculation sums the absolute values of the errors, since the two angle values are dependent on one another, by Snell's law.

$$\frac{\partial n_l}{\partial \theta_i} = \frac{\cos\theta_i}{\sin\theta_r}$$

$$\frac{\partial n_l}{\partial \theta_r} = \frac{\sin\theta_i \cos\theta_r}{\sin^2\theta_r}$$

$$\delta n_l = \left|\frac{\partial n_l}{\partial \theta_i}\delta\theta_i\right| + \left|\frac{\partial n_l}{\partial \theta_r}\delta\theta_r\right| \tag{9}$$

where $\delta\theta_1$ and $\delta\theta_2$ are the instrumental error for the protractor measurements (see (1)). Since the mean is a function of six independent calculations, the error of the mean can be calculated by summing the errors of each calculation in quadriture, as shown in (10).

$$\delta\bar{n}_l = \frac{1}{6}\sqrt{\sum_{i=1}^{6}\left(\frac{\partial \bar{n}_l}{\partial n_{l_i}}\delta n_{l_i}\right)^2} = \frac{1}{6}\sqrt{\sum_{i=1}^{6}\delta n_{l_i}^2} \tag{10}$$

For all error calculations in this section, the angles are necessarily converted to their respective radian equivalents: The trigonometric and differential relations of sin and cos are derived from the radian interpretations of angles. Furthermore, using degrees will leave the error calculation with the wrong units (in degrees, rather than being dimensionless as n_l is) and will change the value of the error. The conversion from radians to degrees is shown in (**??**), and the converted angle values from (Table 4) are shown in (Table 5).

$$\theta_{\text{rad}} = \theta_\circ \times \frac{\pi}{180^\circ} \tag{11}$$

Sample calculations

An sample calculation for n_l (for the first sample of (Table 4)) is shown below.

$$n_{l_1} = \frac{\sin 0.750}{\sin 0.518} = 1.38$$

The calculation for the error for this sample, δn_l, is shown below.

$$\delta n_{l_1} = \left|\frac{\cos 0.750}{\sin 0.518}0.006\right| + \left|\frac{\sin 0.750\cos 0.518}{\sin^2 0.518}0.006\right| = 0.0145$$

Table 5: Angle values, converted from degrees to radians

Sample	Incident angle (rad)	Refracted angle (rad)
1	0.750	0.518
2	0.768	0.525
3	0.384	0.265
4	0.382	0.267
5	0.925	0.630
6	0.986	0.620

This calculation is repeated once for each sample. The calculated index of refraction and error for each sample are shown in (Table 6).

Table 6: Index of refraction of liquid and error for each sample using Snell's law

Sample	n_l	δn_l
1	1.376	0.0240
2	1.385	0.0236
3	1.429	0.0543
4	1.414	0.0536
5	1.355	0.0178
6	1.436	0.0183

The mean index of refraction is a simple arithmetic mean of the n_l values from (Table 6).

$$\bar{n}_l = \frac{1}{6}(1.376 + 1.385 + 1.429 + 1.414 + 1.355 + 1.436) = 1.399$$

The error calculation for the mean is shown below.

$$\delta n_l = \frac{1}{6}\sqrt{0.0240^2 + 0.0236^2 + 0.0543^2 + 0.0536^2 + 0.0178^2 + 0.0183^2} = 0.0145$$

4 Results

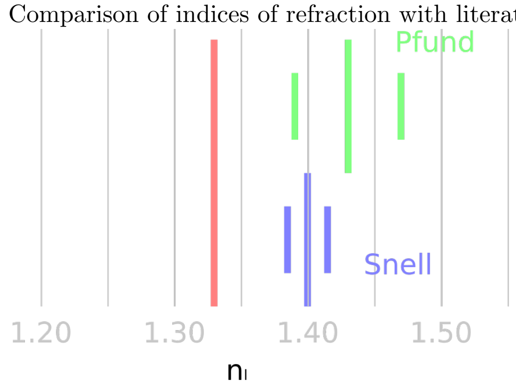
The calculated indices of refraction, along with their reported errors, are summarized in (Table 7), and a visual summary is provided in (Figure 1).

Table 7: Summary of indices of refraction

Material	Pfund's method	Snell's law method	Literature value
Liquid (tap water)	1.43 ± 0.04	1.40 ± 0.015	1.33[1]
Glass	1.61 ± 0.04	–	–

As can be seen in (Figure 1), neither the error of the mean for Pfund's method nor Snell's method captured the literature value.

Figure 1: Comparison of indices of refraction with literature value



[1]at 20°C, http://hyperphysics.phy-astr.gsu.edu/hbase/Tables/indrf.html

5 Conclusion

The calculated index of refraction using Pfund's method is 1.61 ± 0.04cm. While the exact composition of the petri dish is not known, the index of refraction of glasses is typically between 1.52 (crown glass) and 1.65 (heavy flint glass)[2], so it can be deduced that this calculated value is reasonable.

The calculated index of refraction of the liquid (tap water) was 1.43 ± 0.04 using Pfund's method, and 1.40 ± 0.015 using the Snell's law method. The literature value is 1.33. Neither of the calculated values' error intervals capture the literature value – however, the two calculated error intervals are small (0.04 and 0.015) and overlap, which may suggest that some part of the procedure caused a systematically high calculation for indices for refraction.

There were many sources of error, delineated below.

For Pfund's method, the index of refraction of air was approximated to be 1 to make calculations simpler. However, the literature value of air at STP is 1.00029[2]. Since, in the calculation for critical angle of the glass, $n_2 = \frac{n_a}{\sin\theta_c}$, this would cause the resulting answer to be systematically lower but a small factor ($\frac{1}{1.00029}$ of the value obtained using the literature value). This likely had a very small factor, judging by the fact that the results were mostly reported to only three significant figures.

The procedure for Pfund's method asked for many indirect measurements, and this was likely the source of the largest (random and overall) error in the results since the measurement tools are very precise, even if not quantifiable and thus not factored into the error propagation. The procedure asks to visually determine the diameter of a ring, by visually "copying" the ring onto a second sheet of paper, which was then measured by a Vernier caliper. It is not certain if this error is random or systematic, because it cannot be quantified – there is no way to check what the true diameter of the rings are. Adding to the difficulty of the visual determination of ring diameter is that sometimes the edge of the rings were blurry, and the paper was broken at some places, causing the reflection and the edge of the ring to be less strongly-defined in those sections.

In Pfund's method, the thickness of the bottom of the petri dish is measured. However, while the model assumes that the thickness is only of glass, the measurement obtained with the micrometer caliper also included a layer of paper, and in some places, tape. This makes the value of t systematically higher than the true value, which in turn makes n_g systematically higher (but has no effect on n_l, since it cancels out).

In the Snell's law procedure, there is much random error introduced by the procedure. Firstly, a circle had to be drawn around the petri dish such that the edge of the petri dish was right above the circle, but small deviations in pencil angle when drawing the circle could make the circle larger or smaller than the actual petri dish diameter. There is some random error in pin placement and the construction of the center of the circle (using perpendicular bisectors) that wasn't accounted for, because it doesn't quantifiably affect any measurement

[2] http://hyperphysics.phy-astr.gsu.edu/hbase/Tables/indrf.html

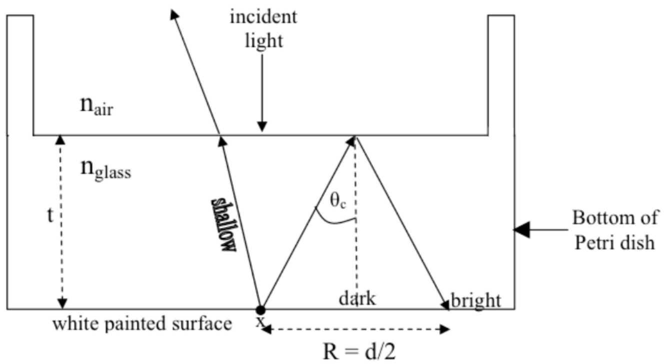
(but it does affect the angle measurements with the protractor).

For both methods, the tap water seemed to be a little cloudy (have some particulate matter). This may have introduced some systematic error in the true n_l value, making it deviate from the literature value. This may help explain the high precision but low accuracy (relative to the literature value) for both n_l calculations.

6 Answers to questions

6.1 Derivation of n_g formula for Pfund's method

Figure 2: Schematic of setup to determine n_g



We begin with Snell's law:

$$n_a \sin\theta_a = n_g \sin\theta_g$$

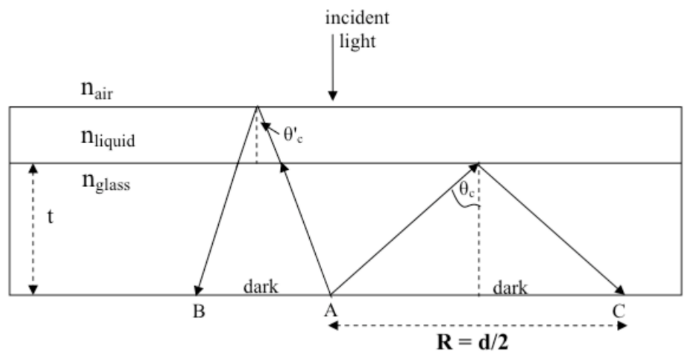
We assume that $n_a \approx 1$. At the critical angle in the glass θ_{g_c}, $\theta_a = 90°$. The Snell's law equation becomes

$$n_g = \frac{(1)\sin 90°}{\sin\theta_{g_c}} = \csc\theta_{g_c} = \frac{\sqrt{((d/2)/2)^2 + t^2}}{(d/2)/2} = \frac{\sqrt{d^2 + 16t^2}}{d}$$

(The calculation of csc comes from the right-triangle geometry in the schematic.)

6.2 Derivation of n_l formula for Pfund's method

Figure 3: Schematic of setup to determine n_l



The bright ring caused by the reflection on the left is caused by the internal reflection by the liquid-air boundary. We are interested instead on the reflection on the right, which uses the glass-liquid boundary. Again, we begin with Snell's law.

$$n_g \sin\theta_g = n_l \sin\theta_l$$

At the critical angle for this boundary from the glass n_{g_c}, we get $\theta_g = 90°$ Solving for n_l, we get:

$$n_l \sin 90° = n_g \sin\theta_{g_c}$$

The rest of the derivation follows similarly for that previously shown for n_g:

$$n_l = n_g \frac{(d/2)/2}{\sqrt{((d/2)/2)^2 + t^2}} = \frac{n_g d}{4\sqrt{(d/4)^2 + t^2}} = \frac{n_g d}{\sqrt{d^2 + 16t^2}}$$

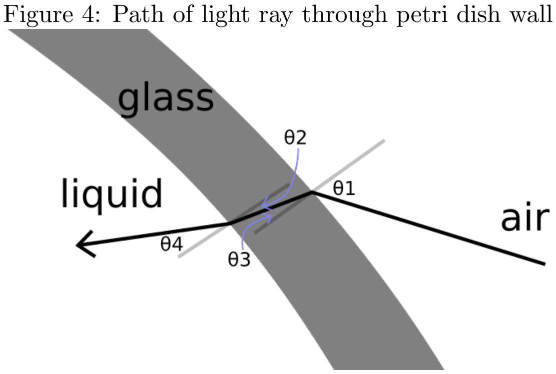
(This again uses the right-angle geometry of the schematic.)

6.3 Assumptions made by the Snell's law technique

The Snell's law technique approximated the refraction to be between air and liquid on the edge of the petri dish, assuming that the refraction caused by the petri dish glass is trivial. This approximation can be justified by the fact that the light beam passing through the petri dish passes through a thin layer of glass, with two nearly-parallel refracting surfaces and two refractions in opposite directions.

6.4 Derivation of n_l formula for Pfund's method

Figure 4: Path of light ray through petri dish wall



There are two Snell's law relevant to the setup in (Figure 4):

$$n_a \sin\theta_1 = n_g \sin\theta_2$$

$$n_g \sin\theta_3 = n_l \sin\theta_4$$

Note that, while the petri dish is curved, if you zoom in on a small region, the inner and outer edge are essentially parallel, thus $\theta_3 \approx \theta_4$, so

$$n_a \sin\theta_1 \approx n_l \sin\theta_4$$

This demonstrates that the glass's refraction is small and can be approximated away by assuming that the glass is thin and not overly curved. From this diagram it can also be intuited that the effects of this are that the two inner angles within the glass are slightly different and, since there are two points of refraction, the path of light may be offset a little bit from if there was only an air-liquid boundary. This approximation is reasonable by the justification above, and by the small error of the mean for the Snell's law calculations of 0.015.

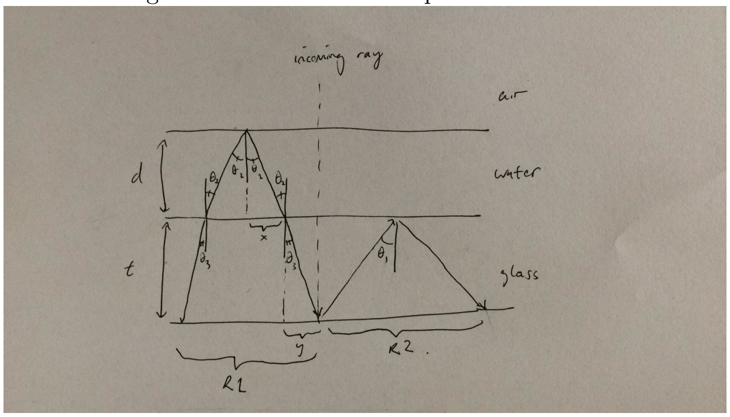
As with the Pfund's law method, the Snell's law technique also assumes that the liquid is transparent or translucent, and that $n_a = 0$. These are reasonable, given that the liquid in question is transparent and the index of refraction of air is known to be very close to 1.

6.5 Will Pfund's method work for liquids of all n?

Pfund's method used here to calculate n_l is not applicable for liquids of all n. It uses the property of total internal reflection of the glass, and total internal reflection only occurs if the liquid has a lower n than glass. In other words, the critical angle (only after which exists the phenomenon of total internal reflection) only exists if the refracted angle is shallower from the surface than the incident angle, which happens only if the index of refraction of the second medium is lower than that of the first.

6.6 Extra credit: depth of water

Figure 5: Schematic of setup to determine d



Let R_1 be the radius of the ring caused by the total internal reflection of water with air, R_2 be the radius of the ring caused by total internal reflection of glass with water (the quantity measured in Pfund's method for n_l).

For θ_2, which is the critical angle of water with air:

$$n_l \sin\theta_2 = n_a \sin 90^\circ = 1 \tag{12}$$

For θ_1, which is the refracted angle of the light ray from liquid to glass with incident angle θ_2:

$$n_g \sin\theta_1 = n_l \sin\theta_2 = 1 \tag{13}$$

By trigonometry:

$$\sin\theta_1 = \frac{y}{\sqrt{y^2+t^2}} \tag{14}$$

$$\sin\theta_2 = \frac{x}{\sqrt{x^2+d^2}} \tag{15}$$

By (13), (14):

$$n_g \frac{y}{\sqrt{y^2+t^2}} = 1$$

$$n_g^2 y^2 = y^2 + t^2$$

$$y = \frac{t}{\sqrt{n_g^2-1}} \tag{16}$$

Similarly, by (13), (15):

$$x = \frac{d}{\sqrt{n_l^2-1}} \tag{17}$$

By definition of x, y in the diagram, $2(x+y) = R_1$. Thus:

$$2\left(\frac{d}{\sqrt{n_l^2-1}} + \frac{t}{\sqrt{n_g^2-1}}\right) = R_2 \tag{18}$$

Solving for d:

$$d = \left(R_2 - \frac{2t}{\sqrt{n_g^2-1}}\right)\frac{\sqrt{n_l^2-1}}{2} \tag{19}$$

Note: All of the images were scanned at 1200dpi on the Epson scanners in the CUCC. The only modifications on them were cropping, inverting, and flipping.

Figure 1: Self-portrait, taken on the East side of 41 Cooper. Unfortunately, it's difficult not to squint and blink with the sun in my eyes, so I look pretty angry. Radi Farraj assisted with taking this photo.

The above picture is the best self-portrait with my face clearly visible. It was taken on a sunny day with the sun directly facing me with a 30 second exposure. Developing took (roughly) 60 seconds in the ordinary developer solution, 30 seconds in the stop bath, and 60 seconds in the fixer. (The times are not exact, and are approximated with counting by mouth.)

Shown below are some other portraits I took, along with the lessons learnt from each one. All of these solutions were developed in the same manner as the first.

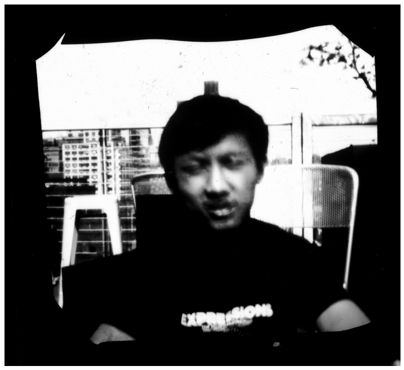


Figure 2: This is my first picture taken with this camera. There is a small light leak in the bottom right, which I fixed by adding more tape around the back edges. The fence and buildings are clearer than the foreground, so I moved the camera further away from the subject for future photos.

Figure 3: Coding! This was fun and the resolution is pretty high, but the photo is a little overexposed.

Figure 4: The second photo taken with this camera. After moving the camera further away, the subject is much clearer. It was likely not in the fixer long enough, because the photo quality greatly degraded after a day.

Figure 5: A picture of (Figure 4) from my smartphone, on the day it was taken before it was destroyed.

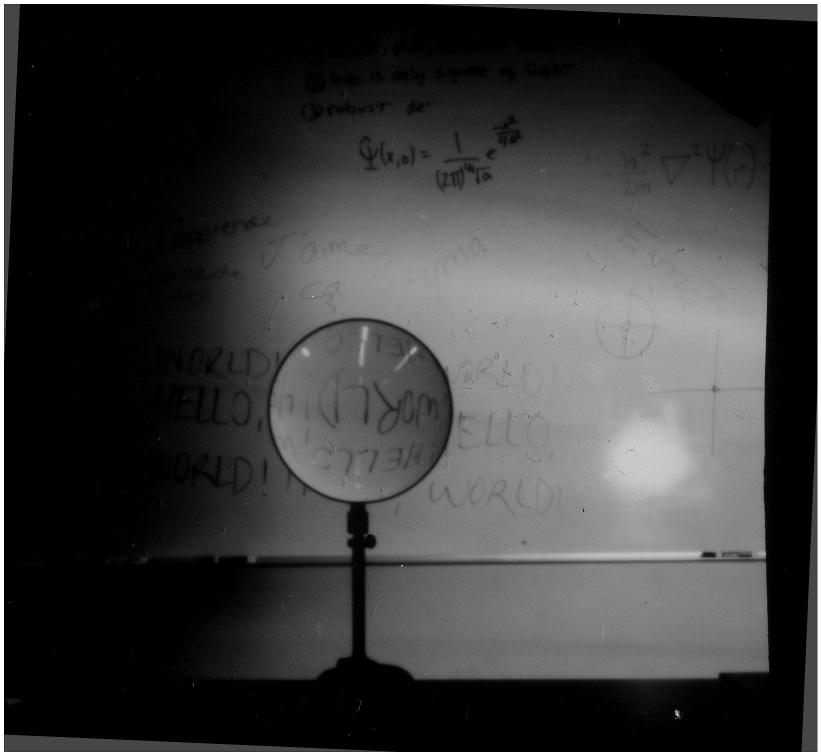


Figure 6: Photograph of refraction from a convex lens, causing the image behind it to be rotated 180^o. The bright spot on the whiteboard (and the overall gradient) is caused by the location of the special light used to decrease exposure time; the bright lines on the lens are reflections of the overhead lights.

This image was taken in the physics lab with one of the special floodlights illuminating the whiteboard (on the right side of the above photograph), of the large convex lens in front of the words "HELLO, WORLD!" written repeatedly on the whiteboard, some of which can be seen reading upside-down and right-to-left inside of the lens. The exposure time was (roughly) 20 minutes, and the developing time was the same as the portraits.

The physical phenomenon illustrated in this photograph is refraction of light along the glass-air barrier in a lens. This is a plano-convex lens where the whiteboard is farther from the camera than the focal point, so the image is flipped (both horizontally and vertically; i.e., rotated 180^o), and there is some curvilinear distortion. A quick schematic of what is happening is shown in (Figure 7).

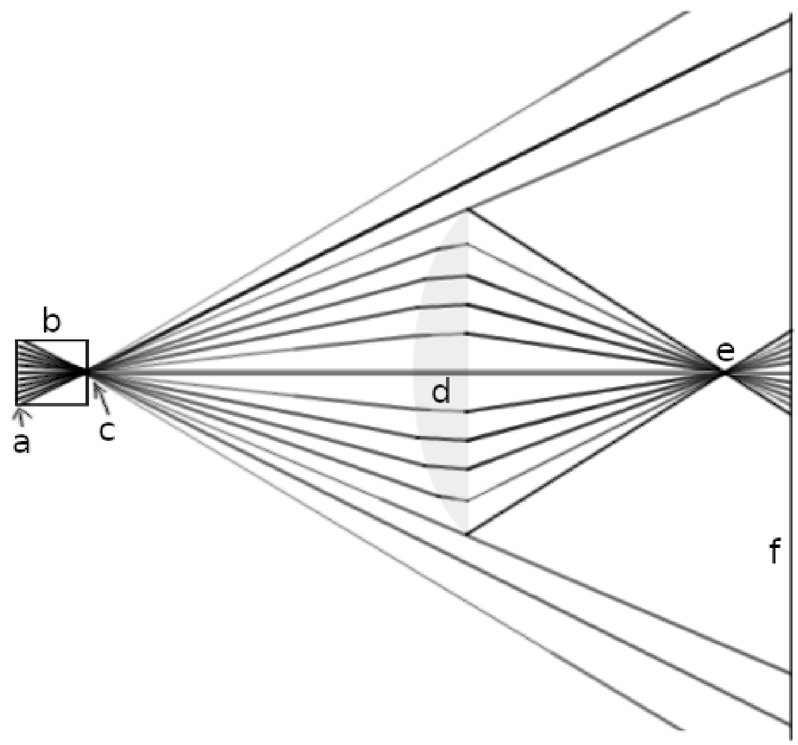


Figure 7: Schematic of the path of light rays through a convex lens. Image created in GIMP. (a) Photographic paper (film); (b) pinhole camera, light-proofed except for pinhole; (c) pinhole; (d) (plano-)convex lens; (e) focal point; (f) whiteboard (object).

As seen from this image, the light rays project an image linearly in the area outside of the lens. The light rays that intersect the lens all converge to its focal point at (e); since the whiteboard is farther than this point, the image inside of the lens appears inverted. Once the light enters the camera, all of it is inverted one more time, but this is applied to all of the light rays and doesn't change the image. At each intersection of the light between glass and air (and vice-versa), the light refracts according to Snell's Law, i.e., $n_1 \sin\theta_1 = n_2 \sin\theta_2$, where n_1 and $\sin\theta_1$ are the index of refraction and angle of incidence, respectively, for one material, and n_2 and $\sin\theta_2$ are the index of refraction and angle of refraction, respectively, for the second material.

Figure 8: Photograph of FedEx truck developed using caffenol. This was taken in front of the building on the left of the Muji store near Foundation, and was taken on top of a garbage can (on which you can see the reflection of the FedEx truck in the bottom part of the picture).

The process for photographing the truck was the same as for the other photos. It was a sunny day and the subject is in almost-direct sunlight (sun was SWW from the perspective of the camera), so the exposure was 45 seconds. Even with a few bikers and walkers walking by, this photo was surprisingly detailed and clear (you can even read "fedex.com" and "The World On Time," and barely "1.800.GoFedEx."

The recipe for the alternate developer, caffenol, was from `https://www.diyphotography.net/caffenol-processing-film-coffee-supermarket-ingredients`; this involved dissolving 60g washing soda, 16g Vitamin C supplement, and 40g instant coffee in 1L water. 12 minutes were allowed for the developing, followed by a quick rinse with tap water, and then the usual 30 seconds in stop bath, and 60 seconds in the fixer.

Here are the other notable photos taken over the period of this lab.

Figure 9: This image is a portrait that fell to the same circumstances as (Figure 4).

Figure 10: Like (Figure 5) for (Figure 4), this is an image from my smartphone of (Figure 9) before it was destroyed. You can see that it was a little underdeveloped, as it was a cloudy day and the exposure was 90 seconds.

I was experimenting multiple photos of the same image with color filters to be able to build a color image (by digital composition). Unfortunately, I only had cheap blue-red 3D glasses. Only the blue filter worked decently; the red was far too dark (see (Figure 12)), and my makeshift green filters (marker and plastic) yielded blurry images, so this experiment failed. It is also hard to tell that (Figure 11) used the blue filter at all.

Figure 11: Image of nearby buildings using the blue filter, 4.5 minute exposure.

Figure 12: Image of the same nearby buildings using the red filter, 30 minute exposure.

Digital signature
Name: Jonathan Lam (lam12)
Section: Yecko D
Date: 12/16/19

1 Purpose

Python (and the `matplotlib`, `numpy` libraries) is used to generate plots of wave interference and diffraction patterns to reinforce the results derived in lecture and viewed visually in Lab 5. The intensity pattern generated by N-slit interference, single-slit diffraction are shown, plotted using results derived in class. Additionally, the pattern generated by two finite-width slits (involving both interference and diffraction) is plotted, a result not derived in class.

2 Results

Python environment: Python 3.7.3, Anaconda, Inc. on Linux

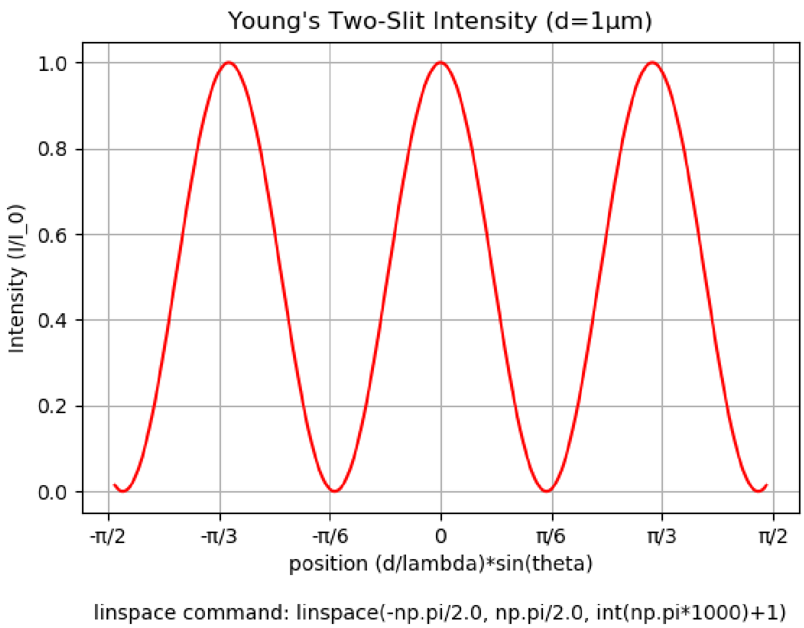


Figure 1: The intensity is clearly sinusoidally periodic w.r.t. δ, the phase difference.

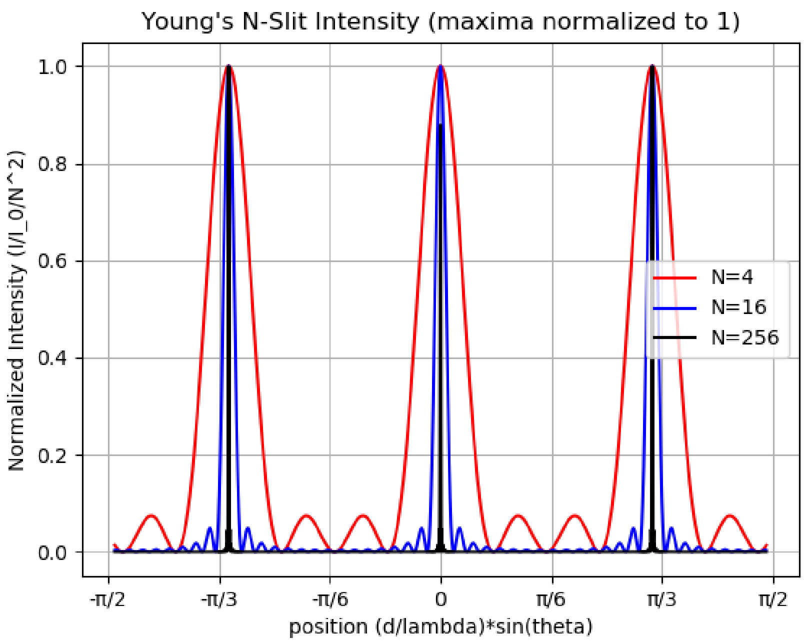


Figure 2: As before, the intensities are periodic w.r.t. δ, but there are proportionally more minor peaks w.r.t. N. Also, the overall intensity ratio I/I_0 is scaled proportional to N^2.

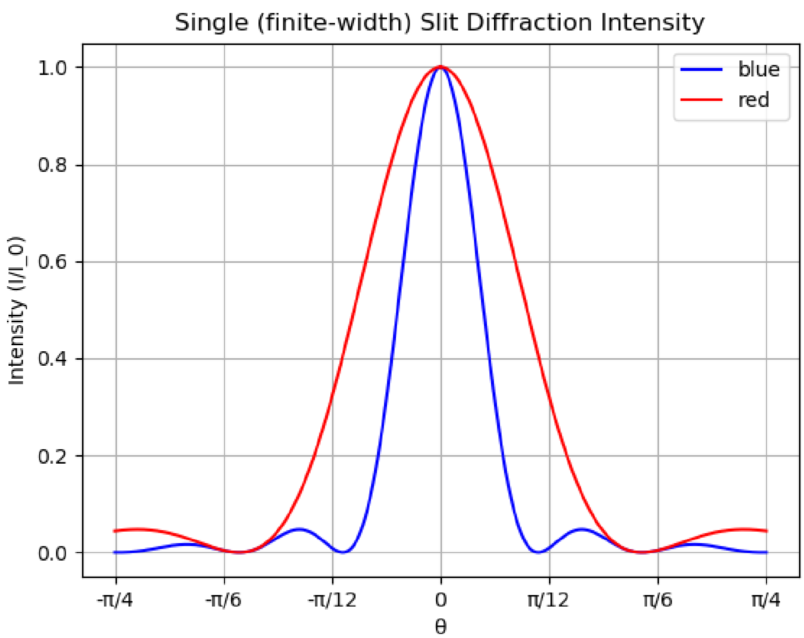


Figure 3: The two diffraction patterns for blue and red have the same overall shape, but the higher wavelength (red) produces a larger spread.

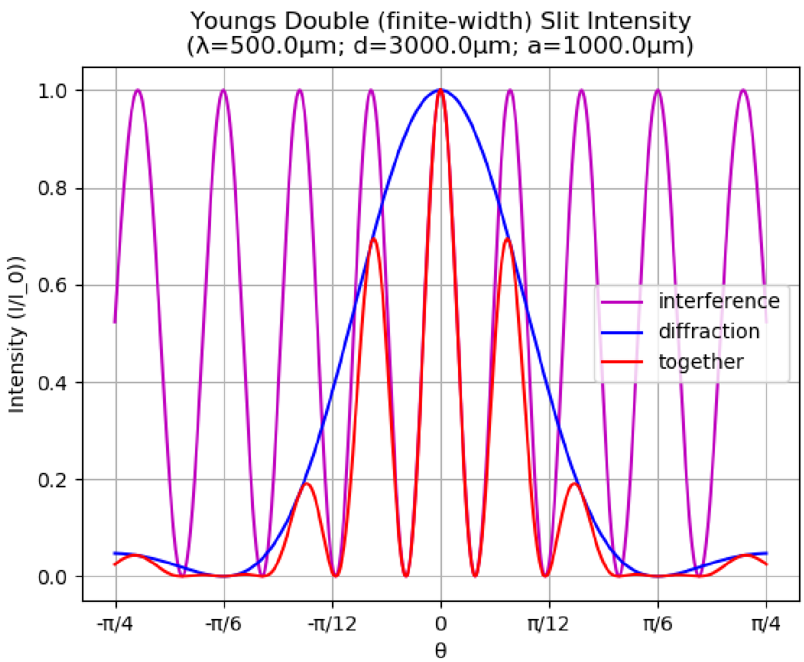


Figure 4: The interference pattern is modulated by the diffraction pattern of the finite-width slit. As explored in the Questions section, this modulation causes an unexpected peak at $m = 3$.

3 Conclusion

The plots produce the expected results. In particular, the two-slit intensity pattern shows the well-known periodic pattern with peaks of equal height (a uniform light-dark pattern, as observed in Lab 5). The N-slit intensity pattern shows peak maxima at the same locations, with peaks narrowing, number of small maxima increasing, and height of I/I_0 increasing (with large peak maxima at $I/I_0 = N^2$) as N increases. For the single finite-width slit, a higher wavelength produces a wider diffraction pattern, as expected. For two finite-width slits interfering and diffracting, we get an interference pattern with the same frequency as the thin-slit diffraction pattern, but with an intensity that is bounded above by the diffraction pattern of a single-wide slit. As discussed in (Section 4: Question), the interference pattern may have missing peaks near where the diffraction pattern has a minima.

4 Question

4.1 Why is the m=3 order missing in Figure 2?

Visually, it is clear that the thid peak lies under the minima for the diffraction curve. Since the interference/diffraction pattern caused by the two finite-width slits is the product of the interference and diffraction patterns, it is limited by the height of the height of the diffraction curve, which is 0 at $m = 3$ according to the following calculations.

$$\lambda = 500.0\mu\text{m}; d = 3000.0\mu\text{m}; a = 1000.0\mu\text{m}$$

Interference pattern height at $m = 3$:

$$\sin\theta = \frac{m\lambda}{d} \Rightarrow \sin\theta = \left(\frac{3 \cdot 500.0\mu\text{m}}{3000.0\mu\text{m}}\right) = 0.5000$$

$$(\theta = 0.5236\text{rad})$$

$$(I/I_0 = 1)$$

Diffraction pattern height at $\theta = \theta_3$.

$$I/I_0 = \left(\frac{\sin\frac{\pi a\sin\theta}{\lambda}}{\frac{\pi a\sin\theta}{\lambda}}\right)^2 = \left(\frac{\sin\frac{\pi(1000.0\mu\text{m})(0.5)}{500.0\mu\text{m}}}{\frac{\pi(1000.0\mu\text{m})(0.5)}{500.0\mu\text{m}}}\right)^2 = \left(\frac{\sin\pi}{\pi}\right)^2 = 0.0000$$

Combined pattern height at $\theta = \theta_3$.

$$\frac{I}{I_0} = 1 \cdot 0 = 0$$

1. Purpose

In Part A of the procedure, the diameter and thickness of a cylindrical disk were measured using a metric scale and a vernier caliper. Since the measuring tools have differing instrumental uncertainties, this allows the comparison of the effect of instrumental uncertainty on the volume calculated from these measurements.

In Part B of the procedure, the dimensions (length and width) and mass of a rectangular plate and a cylindrical disk were measured, and their densities were calculated and compared to each other. A micrometer was used for thickness measurements, a vernier caliper was used for other linear measurements, and a digital balance used for mass measurements. This is compared to the literature value of the material to determine the method's accuracy.

Part C involves measuring the lengths of two moderately-sized ($n = 20$) samples of small acrylic pieces. The report uses a hypothesis test to determine whether the two samples come from the same sample within a 95% confidence interval.

2. Data

2.1. Part A data

Table 1: Diameter of the cylindrical disk (metric ruler)

Right Reading (cm)	Left Reading (cm)	Length (cm)
4.84	1.00	3.84
6.84	3.00	3.84
7.91	4.10	3.81
12.95	9.15	3.80
9.38	5.45	3.93
9.95	6.13	3.82

Instrumental Error (cm)	0.04
Random Error (cm)	0.02
Diameter (cm)	3.84 ± 0.04

Table 2: Thickness of the cylindrical disk (metric ruler)

Right Reading (cm)	Left Reading (cm)	Length (cm)
6.00	5.70	0.31
5.61	5.30	0.31
11.10	10.80	0.30
8.61	8.29	0.32
7.60	7.30	0.30
7.20	6.90	0.30

Instrumental Error (cm)	0.04
Random Error (cm)	0.003
Diameter (cm)	0.30 ± 0.04

Table 3. Diameter of the cylindrical disk (vernier caliper)

Length (cm)	Corrected Length (cm)
3.850	3.850
3.844	3.844
3.854	3.854
3.854	3.854
3.864	3.864
3.852	3.852

Zero Error (cm)	0.000
Instrumental Error (cm)	0.002
Random Error (cm)	0.003
Diameter (cm)	3.853 ± 0.003

Table 4. Thickness of the cylindrical disk (vernier caliper)

Length (cm)	Corrected Length (cm)
0.324	0.324
0.320	0.320
0.324	0.324
0.330	0.330
0.324	0.324
0.327	0.327

Zero Error (cm)	0.000
Instrumental Error (cm)	0.002
Random Error (cm)	0.001
Diameter (cm)	0.325 ± 0.001

2.2. Part B data

Table 5. Length of the rectangular object (vernier caliper)

Length (cm)	Corrected Length (cm)
3.824	3.824
3.826	3.826
3.826	3.826
3.842	3.842
3.844	3.844
3.826	3.826

Zero Error (cm)	0.000
Instrumental Error (cm)	0.002
Random Error (cm)	0.004
Diameter (cm)	3.831 ± 0.004

Table 6. Width of the rectangular object (vernier caliper)

Length (cm)	Corrected Length (cm)
5.060	5.060
5.058	5.058
5.050	5.050
5.040	5.040
5.050	5.050
5.042	5.042

Zero Error (cm)	0.000
Instrumental Error (cm)	0.002
Random Error (cm)	0.003
Diameter (cm)	5.050 ± 0.003

Table 7. Thickness of the rectangular object (micrometer)

Length (cm)	Corrected Length (cm)
0.3111	0.3111
0.3165	0.3165
0.3151	0.3151
0.3092	0.3092
0.3154	0.3154
0.3156	0.3156

Zero Error (cm)	0.0000
Instrumental Error (cm)	0.0005
Random Error (cm)	0.0012
Diameter (cm)	0.314 ± 0.0012

Table 8. Thickness of the cylindrical disk (micrometer)

Length (cm)	Corrected Length (cm)
0.3239	0.3239
0.3172	0.3172
0.3302	0.3302
0.3179	0.3179
0.3251	0.3251
0.3185	0.3185

Zero Error (cm)	0.0000
Instrumental Error (cm)	0.0005
Random Error (cm)	0.002
Diameter (cm)	0.322 ± 0.002

Table 9. Mass of the cylindrical disk (digital balance)

Length (cm)	Corrected Length (cm)
9.9	9.9
9.9	9.9
9.9	9.9
9.9	9.9
9.9	9.9
9.9	9.9

Zero Error (cm)	0.00
Instrumental Error (cm)	0.05
Random Error (cm)	0.00
Diameter (cm)	9.9 ± 0.05

Table 10. Mass of the rectangular object (digital balance)

Length (cm)	Corrected Length (cm)
16.1	16.1
16.1	16.1
16.1	16.1
16.1	16.1
16.1	16.1
16.1	16.1

Zero Error (cm)	0.00
Instrumental Error (cm)	0.05
Random Error (cm)	0.00
Diameter (cm)	16.1 ± 0.05

2.3. Part C data

Table 11. Length of acrylic pieces in Bottle #4 (micrometer)

Length (cm)	Corrected Length (cm)
1.3875	1.3875
1.3972	1.3972
1.4040	1.4040
1.2975	1.2975
1.3560	1.3560
1.2930	1.2930
1.3415	1.3415
1.3980	1.3980
1.3440	1.3440
1.2875	1.2875
1.3450	1.3450
1.3962	1.3962
1.3445	1.3445
1.2914	1.2914
1.4395	1.4395
1.2945	1.2945
1.3920	1.3920
1.3421	1.3421
1.2940	1.2940
1.3130	1.3130

Zero Error (cm)	0.0000
Instrumental Error (cm)	0.0005
Bottle 4 STDEV (cm)	0.046943
Bottle 4 STDOM (cm)	0.019164
Bottle 4 length (cm)	1.35 ± 0.05

Table 12. Length of acrylic pieces in Bottle #5 (micrometer)

Length (cm)	Corrected Length (cm)
1.3458	1.3458
1.3500	1.3500
1.4320	1.4320
1.3391	1.3391
1.3597	1.3597
1.3575	1.3575
1.3971	1.3971
1.3380	1.3380
1.4410	1.4410
1.2318	1.2318
1.3750	1.3750
1.3470	1.3470
1.2718	1.2718
1.3371	1.3371
1.3252	1.3252
1.3746	1.3746
1.3641	1.3641
1.3472	1.3472
1.3781	1.3781
1.2295	1.2295

Zero Error (cm)	0.0000
Instrumental Error (cm)	0.0005
Bottle 5 STDEV (cm)	0.053855
Bottle 5 STDOM (cm)	0.021986
Bottle 5 length (cm)	1.35 ± 0.05

3. Calculation

3.1. Instrument errors

The metric ruler has markings to the nearest 0.05cm and involves visual estimation, so the instrumental error for a single reading is $\delta S_{read} = \pm 0.025\text{cm}$. A length measurement using the metric ruler is a function of two (independent) readings. Thus the error for a single length measurement δS is calculated from the errors of the left and right instrumental readings of the metric ruler:

$$\delta S = \sqrt{\delta S_{left}^2 + \delta S_{right}^2} = \sqrt{(0.025\text{cm})^2 + (0.025\text{cm})^2} = 0.0354\text{cm} = 0.04\text{cm}$$

The vernier caliper has markings to the nearest 0.002cm, and reading it involves choosing the closest matching margin. Since there is no approximation between the markings, the instrumental uncertainty is $\delta S = \pm 0.002\text{cm}$.

The micrometer has markings to the nearest 0.001cm. Reading it involves visual estimation between markings, so the instrumental error for a reading is $\delta S = \pm 0.0005\text{cm}$.

The digital balance reports data to 0.1g. This means that the instrumental error is $\delta S = \pm 0.05g$.

None of the instruments had a measurable zero error (i.e., the zero errors were all 0 to all significant figures), so no offset corrections were performed on measurements.

3.2. Sample mean and random error

For each sample of measurements, the measurement is recorded with both a “best value” and a specified random error. The “best value” is the mean $\bar{x}$, and the random error is reported as standard deviation of the mean (STDOM) $\sigma_{\bar{x}}$. These are discussed in (3.5. Part C calculations) with sample calculations.

3.3. Part A calculations

The volume of the cylindrical disk is calculated with the volume formula:

$$V_{disk} = \pi \left(\frac{d}{2}\right)^2 h$$

where d is the disk diameter and h is the cylinder height. The error propagation formula is:

$$\delta V = \sqrt{\left(\frac{\partial V}{\partial d}\delta d\right)^2 + \left(\frac{\partial V}{\partial h}\delta h\right)^2} = \sqrt{\left(\frac{\pi dh}{2}\delta d\right)^2 + \left(\frac{\pi d^2}{4}\delta h\right)^2}$$

where δd and δh are the larger of the instrumental and random error for diameter and height measurements, respectively. The 2-norm is used because the measurements are independent.

Sample calculations

For measurements using the metric ruler:

$$V_{disk} = \pi \left(\frac{3.84\text{cm}}{2}\right)^2 \cdot 0.30\text{cm} = 3.5\text{cm}^3$$
$$\delta V = \sqrt{\left(\frac{\pi \cdot 3.84\text{cm} \cdot 0.30\text{cm}}{2} \cdot 0.04\text{cm}\right)^2 + \left(\frac{\pi (3.84\text{cm})^2}{4} \cdot 0.04\text{cm}\right)^2} = 0.414\text{cm}^3$$

In this calculation, the instrumental error (0.04cm) is larger than the random error and therefore used for δd and δh. For measurements using the vernier caliper:

$$V_{disk} = \pi \left(\frac{3.853\text{cm}}{2}\right)^2 \cdot 0.325\text{cm} = 3.79\text{cm}^3$$
$$\delta V = \sqrt{\left(\frac{\pi \cdot 3.853\text{cm} \cdot 0.325\text{cm}}{2} \cdot 0.003\text{cm}\right)^2 + \left(\frac{\pi (3.853\text{cm})^2}{4} \cdot 0.002\text{cm}\right)^2} = 0.0239\text{cm}^3$$

Again, the larger of the instrumental and random errors are chosen; this is the common theme for error propagation and will be done without explanation in any following error propagation problems.

Change in uncertainty

Define the relative change of uncertainty be:

$$\text{rel. change uncertainty} = \frac{\text{new uncertainty} - \text{old uncertainty}}{\text{old uncertainty}} \times 100\%$$

Then, the relative change in uncertainty by switching from the metric ruler to the vernier caliper is:

$$\text{rel change } \% = \frac{0.002\text{cm} - 0.035\text{cm}}{0.035\text{cm}} \times 100\% = -94\%$$

since the error for a ruler measurement is 0.035cm and that of the vernier caliper is 0.002cm. The relative change in the volume calculation is:

$$\text{rel change } \% = \frac{0.0239cm - 0.414cm}{0.414cm} \times 100\% = -94.2\%$$

Interestingly, this is essentially equal to the relative change in uncertainty of the measuring instrument.

3.4. Part B calculations

The density of the rectangular object is calculated using the following formula:

$$\rho = \frac{m}{lwh}$$

where ρ is the calculated density, m is the measured mass, l, w, and h are measured length, width, and height, respectively. The error propagation for the density of the rectangular solid is:

$$\delta\rho = \sqrt{\left(\frac{\partial\rho}{\partial m}\delta m\right)^2 + \left(\frac{\partial\rho}{\partial l}\delta l\right)^2 + \left(\frac{\partial\rho}{\partial w}\delta w\right)^2 + \left(\frac{\partial\rho}{\partial h}\delta h\right)^2}$$
$$= \sqrt{\left(\frac{1}{lwh}\delta m\right)^2 + \left(-\frac{m}{l^2wh}\delta l\right)^2 + \left(-\frac{m}{lw^2h}\delta w\right)^2 + \left(-\frac{m}{lwh^2}\delta h\right)^2}$$

The density of the cylindrical disk is calculated using the following formula:

$$\rho = \frac{m}{\pi\left(\frac{d}{2}\right)^2 h}$$

where ρ is the calculated density, m is the measured mass, d is the measured diameter, and h is the measured thickness. The error propagation for the density of the cylindrical disk is:

$$\delta\rho = \sqrt{\left(\frac{\partial\rho}{\partial m}\delta m\right)^2 + \left(\frac{\partial\rho}{\partial d}\delta d\right)^2 + \left(\frac{\partial\rho}{\partial h}\delta h\right)^2}$$
$$= \sqrt{\left(\frac{4}{\pi d^2 h}\delta m\right)^2 + \left(-\frac{8m}{\pi d^3 h}\delta d\right)^2 + \left(-\frac{4m}{\pi d^2 h^2}\delta h\right)^2}$$

since the measurements for mass, diameter, and thickness are all independent.

Sample calculations

For the density of the rectangular object:

$$\rho = \frac{16.1\text{g}}{3.831\text{cm}\cdot 5.050\text{cm}\cdot 0.314\text{cm}} = 2.65\frac{\text{g}}{\text{cm}^3}$$

$$\delta\rho = \sqrt{\left(\frac{1}{3.831\text{cm}\cdot 5.050\text{cm}\cdot 0.314\text{cm}}\cdot 0.05g\right)^2 + \cdots + \left(\frac{16.1\text{g}}{3.831\text{cm}\cdot 5.050\text{cm}\cdot (0.314\text{cm})^2}\cdot 0.001\text{cm}\right)^2} = 0.07\frac{\text{g}}{\text{cm}^3}$$

For the density of the cylindrical disk:

$$\rho = \frac{9.9\text{g}}{\pi\left(\frac{3.853\text{cm}}{2}\right)^2 0.314\text{cm}} = 2.6\frac{\text{g}}{\text{cm}^3}$$

$$\delta\rho = \sqrt{\left(\frac{4}{\pi(3.853\text{cm})^2\cdot 0.3185\text{cm}}\cdot 0.05\text{g}\right)^2 + \cdots + \left(\frac{4\cdot 9.9\text{g}}{\pi(3.853\text{cm})^2\cdot(0.3185\text{cm})^2}\cdot 0.002\text{cm}\right)^2} = 0.03\frac{\text{g}}{\text{cm}^3}$$

Comparison with the literature value

The aluminum alloy composition of the cylindrical disk and rectangular object is known to be Alloy 6061, which has a density of 2.70g[1]. Both densities are close to (within $0.10\frac{\text{g}}{\text{cm}^3}$) but less than this literature value. For the rectangular object, it falls within one standard deviation of the mean; but for the cylindrical disk, it is roughly three standard deviations above the mean.

Because of their closeness to the literature value, the densities of the rectangular object and the cylindrical disk have a small percent error when compared to the disk.

$$\%\ \text{error} = \frac{|\text{empirical}-\text{literature}|}{\text{literature}} \times 100\%$$
$$\%\ \text{error}_{\text{rect}} = \frac{|2.65\frac{\text{g}}{\text{cm}^3}-2.70\frac{\text{g}}{\text{cm}^3}|}{2.70\frac{\text{g}}{\text{cm}^3}} \times 100\% = 1.9\%$$
$$\%\ \text{error}_{\text{cylinder}} = \frac{|2.6\frac{\text{g}}{\text{cm}^3}-2.70\frac{\text{g}}{\text{cm}^3}|}{2.70\frac{\text{g}}{\text{cm}^3}} \times 100\% = 4\%$$

A graphic of the comparison of the literature and calculated values is shown in (Results, Figure 1).

3.5. Part C calculations

For a sample x, the mean $\bar{x}$, the sample variance S_x^2, the standard deviation S_x, and the standard deviation of the mean (STDOM) $\sigma_{\bar{x}}$ are given by the following formulas:

$$\bar{x} = \frac{1}{N_x}\sum_{i=1}^{N_x} x_i$$
$$S_x^2 = \frac{1}{N_x-1}\sum_{i=1}^{N_x}(x_i-\bar{x})^2$$
$$S_x = \sqrt{S_x^2}$$
$$\sigma_{\bar{x}} = \frac{S_x}{\sqrt{N_x}}$$

Sample calculations

Sample calculations for sample bottle #4 (sample x) are shown below.

$$\bar{x} = \frac{1.3875\text{cm}+1.3972\text{cm}+\cdots1.3130\text{cm}}{20} = 1.3479\text{cm}$$
$$S_x^2 = \frac{(1.3875\text{cm}-1.3479\text{cm})^2+(1.3972\text{cm}-1.3479\text{cm})^2+\cdots(1.3130\text{cm}-1.3479\text{cm})^2}{19} = 0.0022036\text{cm}^2$$
$$S_x = \sqrt{0.022036\text{cm}^2} = 0.046943\text{cm}$$

---

[1] ASM Handbook, Volume 2: Properties and Selection: Nonferrous Alloys and Special-Purpose Materials ASM Handbook Committee, p 102 DOI: 10.1361/asmhba0001060

$$\sigma_{\bar{x}} = \frac{0.046943\text{cm}}{\sqrt{20}} = 0.010497\text{cm}$$

For sample bottle #5 (sample y), the respective values are (calculations not shown here):

$$\bar{y} = 1.3471\text{cm}$$
$$S_y^2 = 0.0029003\text{cm}^2$$
$$S_y = 0.053855\text{cm}$$
$$\sigma_{\bar{y}} = 0.012042\text{cm}$$

Two-sample t-test

See (Questions, 3) for a brief discussion on the shape of the sample distributions. The two-sample t-statistic is calculated as follows for two samples:

$$t = \frac{|\bar{x}-\bar{y}|}{\sqrt{\sigma_{\bar{x}}^2+\sigma_{\bar{y}}^2}}$$

For the two samples, #4 (x) and #5 (y), the t-statistic is calculated this way:

$$t = \frac{|1.3479\text{cm}-1.3471\text{cm}|}{\sqrt{(0.010497\text{cm})^2+(0.012042\text{cm})^2}} = 0.05009$$

Since $t << 1.96$, there is a strong suggestion the means of these two distributions is the same. See (Questions, 4) for a more complete discussion.

4. Results

Table 13. Calculated volumes and errors using metric ruler and vernier caliper

Measuring instrument	Instrumental Error (cm)	Reported Volume (cm^3)
metric ruler	0.04	3.5 ± 0.4
vernier caliper	0.002	3.79 ± 0.02

Table 14. Calculated densities and errors of the rectangular and cylindrical objects

Object	Density ($\frac{g}{cm^3}$)
cylindrical disk	2.65 0.07
rectangular object	2.7 0.03

Literature density ($\frac{g}{cm^3}$)	2.70

Figure 1 demonstrates that the calculated densities for the two samples are very close to the literature value. It also shows how the range for random error for the rectangular object captures the literature value, while the smaller random error for the cylindrical disk (which also has a lower mean) does not.

Figure 1. Measured densities relative to base density

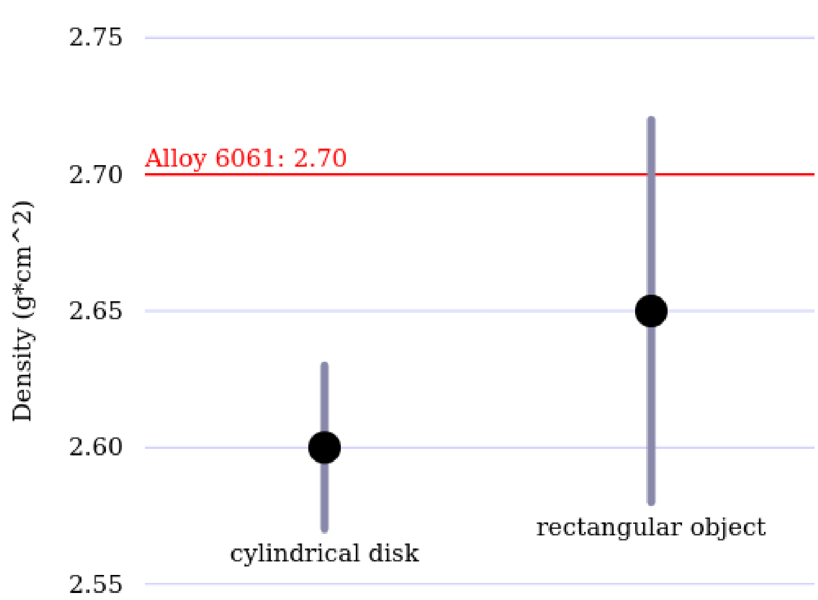


Table 15. Descriptive statistics of the samples of acrylic pieces

Container	Sample Mean (cm)	Sample Variance (cm)	Standard Deviation (cm)	Standard Deviation of the Mean (cm)
4	1.3479	0.0022036	0.0469428	0.010497
5	1.3471	0.00290038	0.0538552	0.012042

The calculated t-statistic for the two-sample t-test is 0.0528. Since $t < 1.96$, we fail to reject the null hypothesis at a 95% confidence interval.

The following two figures demonstrate. Their shapes are discussed in (Questions, 3), and their analysis is provided in (Questions, 4). It is apparent that the distributions are not perfectly normal (i.e., multiple peaks, some gaps in the distribution), but they are both not heavily skewed and the peak(s) are concentrated in the centers of the distribution. Also, the distributions have similar center, spread, and counts per bin. This suggests that the samples come from a similar parent population.

Figure 2. Histogram of length of acrylic pieces in bottle #4 (cm)

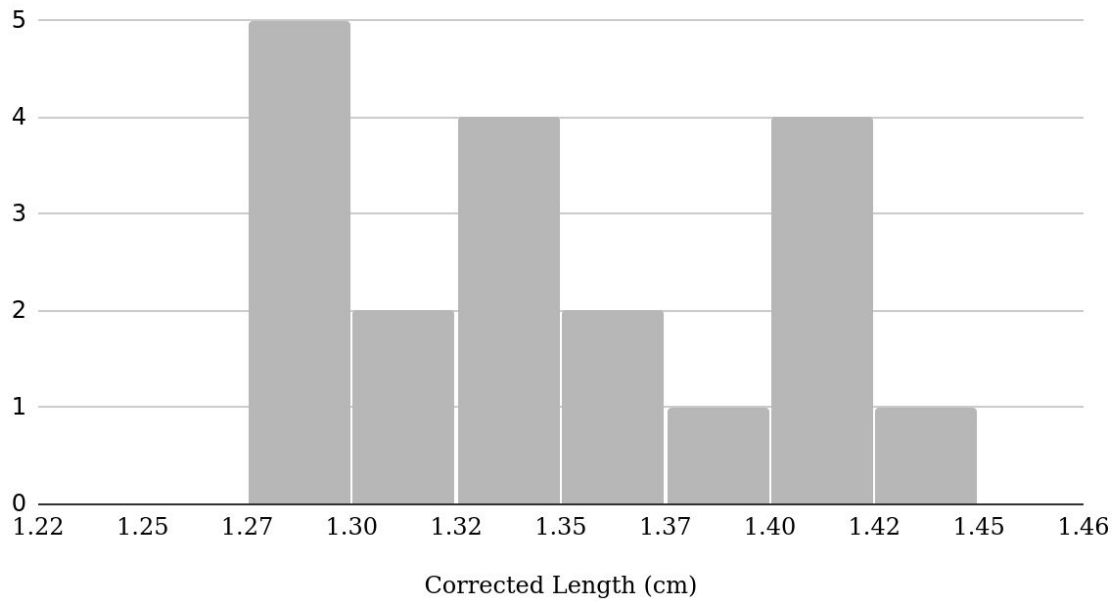
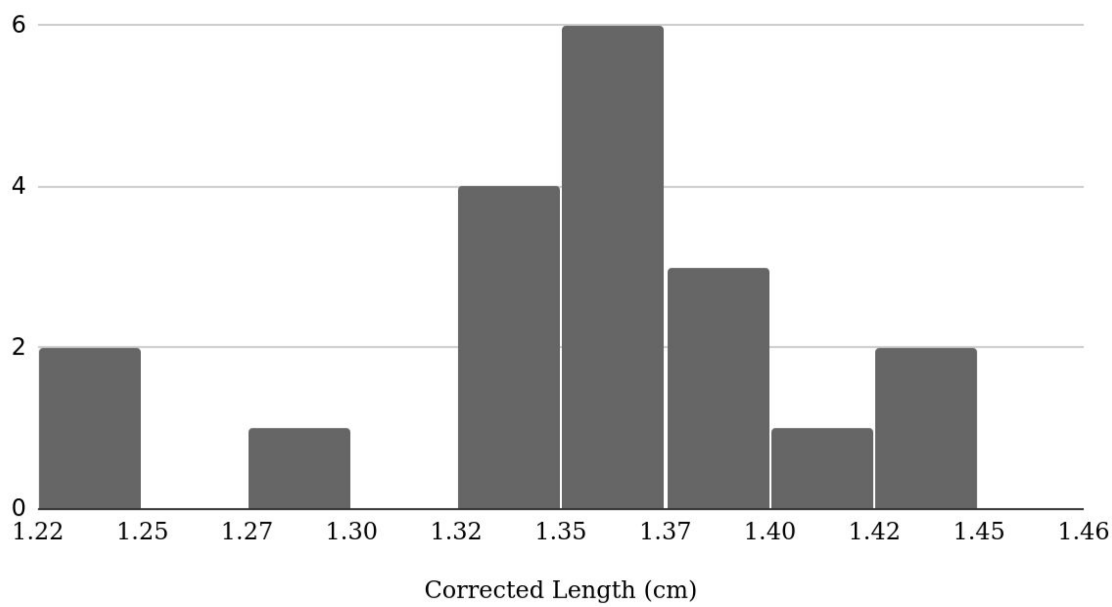


Figure 3. Histogram of length of acrylic pieces in bottle #5 (cm)



5. Conclusion

When calculating the volume from lengths and widths measured with the metric ruler and the vernier caliper in Part A, it was demonstrated that the lower random and instrumental errors calculated when using the vernier caliper generated a smaller error in the resulting volume by error propagation; a relative decrease of 94% uncertainty in the measuring instrument caused a relative decrease of 94.2% uncertainty in the volume calculation.

Densities calculated from the measurements of a digital balance, vernier caliper, and micrometer were close to the expected value. While the calculated value and random error for the cylindrical disk did not include the literature value for the known density of the alloy, its real difference from the density of the alloy was very small ($< 0.1 \frac{\mathrm{g}}{\mathrm{cm}^3}$), and the rectangular object's random error interval around the mean did capture the literature value.

Statistical measures were calculated on two larger samples of data, and a two-sample t-test for the difference of means produced a very small t value. As a result, the null hypothesis was not rejected, so there is no statistically significant difference between the means of the two samples.

A possible source of error was the quality of the geometries of the materials measured. The calculations assume that the geometries of the objects are a perfect cylinder (cylindrical disk) and perfect rectangular prisms (rectangular object and acrylic pieces), but this cannot be the case due to minor manufacturing errors. This was especially true of the acrylic piececs, of which some had very noticeable slants. It was also noted that there was a piece of paper taped to the center of the disk, which may make it slightly wider at the center and slightly more massive than if it were only the aluminum disk. Depending on the flaw in the geometry, each flaw may lead to systematically high, systematic low, or inconsistent (random) errors.

Another source of systematic error is the measurement of the diameter of the circular disk using the metric ruler. The method involved measuring the distance across a chort that is visually estimated to be a diameter without any sort of construction to verify it. Since the diameter is the longest chord, this will result in systematically low diameter measurements.

6. Questions

Question 1: Comparison of volume error values

The relative change in uncertainty (defined in (3.3, Change in uncertainty)) is -94% upon switching from the metric ruler to the vernier caliper. The relative change in uncertainty of the resulting volume calculation also is roughly -94%. A large decrease in uncertainty is expected, although the equal numerical value was unexpected — whether or not this is always the case is up to future experimentation and/or mathematical investigation.

Question 2: Agreement of density values

The calculated density values of $2.65\frac{\mathrm{g}}{\mathrm{cm}^3}$ for the rectangular prism and $2.6\frac{\mathrm{g}}{\mathrm{cm}^3}$ for the cylindrical disk were close to each other and the accepted literature value of $2.70\frac{\mathrm{g}}{\mathrm{cm}^3}$ (with 1.9% and 4% errors, respectively). It falls within the range of the random error for rectangular prism $\left(\pm 0.07\frac{\mathrm{g}}{\mathrm{cm}^3}\right)$; however, for the cylindrical disk, which has a lower mean that is coupled with a smaller error $\left(\pm 0.03\frac{\mathrm{g}}{\mathrm{cm}^3}\right)$, the literature value doesn't fall into its range of random error.

The fact that both errors were too low may imply a possible systematic error with measuring (see (Conclusion) for a more thorough discussion of this error). Nonetheless, all of these values are very close (within $0.1\frac{\mathrm{g}}{\mathrm{cm}^3}$), so the errors likely had a small effect.

Question 3: Shape of the sample histograms

Neither histogram of lengths of the sample is very normal-shaped. The sample for bottle #4 has a smaller spread than the sample for bottle #5. It is trimodal, but the peaks are closely packed near the center of the distribution. Bottle #5 is unimodal, but it has two gaps (which Bottle #4 doesn't have) and therefore two low outliers. Both distributions are mostly symmetric.

Even though neither distribution is perfectly unimodal and symmetric, they are mostly symmetric, do not have extremely skewed or distinct multimodal distributions, and are moderately-sized ($n = 20$). Thus, the distributions roughly satisfy the normality condition for the hypothesis test, and the two-sample t-test for the difference of means may be performed on this sample.

Question 4: Conclusions about the two-sample t-test

The t-statistic of 0.05009 is much less than the threshold $t < 1.96$ for a 5% significance level, which provides no evidence to support the alternative hypothesis that the difference of means is nonzero, and we fail to reject the null hypothesis that the difference in means is zero. This owes a large part to the means of both simulations only differing by less than a hundredth of a millimeter. The small STDOMs of both samples is also small ($\sigma \approx 0.01\mathrm{cm}$), which strengthens this claim by asserting that the means are already close to their true (population) values.

Jonathan Lam (lam12@cooper.edu); Partner: Andrew Kim

1 Purpose

Geometric optics provides simple relationships between objects and their images through a spherical or parabolic lens. In particular, the thin lens equation relates focal length, image distance, and object distance through a simple inverse-sum law. This lab explores the use of symmetry of the relationship between the image and object distance to focus an image with the source and screen a given distance apart at two distinct lens positions, and how this may be used with Bessel's method (a reformulation of the thin lens equation) to experimentally determine focal length of a lens.

Secondly, two lenses are placed apart at a distance nearly the sum of their focal lengths to explore the application of lenses in a refracting telescope, and the empirical angular magnification compared to the calculated value.

2 Data

Each lens had a two-character identifier scratched onto it. From now on, these will the lenses will be consistently referred to as lenses 1 and 2, respectively.

Table 1: Lens identifiers

Lens 1 identifier	Lens 2 identifier
F1	77

2.1 Variable reference

p_{so} (corrected) source position

p_{l1} (corrected) lens position 1

p_{l2} (corrected) lens position 2

p_{sc} (corrected) screen position

f calculated trial focal length

δf uncertainty for calculated focal length

$\bar{f}$ mean calculated focal length for lens

$\delta\bar{f}$ uncertainty for mean calculated focal length for lens

l_1 length of longer tape

l_2 length of shorter tape

m_θ empirical (angular) magnification

2.2 Part A Data

Table 2: Estimated lens focal lengths

Lens 1 estimated f (cm)	Lens 2 estimated f (cm)
5.50	28.00

Table 3: Positions of source, lens, and image for lens 1

Trial	p_{so} (cm)	p_{l1} (cm)	p_{l2} (cm)	p_{sc} (cm)	f (cm)	δf (cm)
1	77.85	84.58	113.71	120.04	5.519	0.0606
2	66.98	73.69	113.91	120.04	5.643	0.0657
3	96.61	105.80	111.62	120.04	5.496	0.0311
4	87.88	96.44	113.29	120.04	5.833	0.0487
5	73.52	80.29	113.82	120.04	5.588	0.0629
6	62.31	69.05	113.96	120.04	5.698	0.0673

Source distance offset (cm)	1.80
Lens distance offset (cm)	0.00
Screen distance offset (cm)	0.00
Instrumental error (cm)	0.05
$\bar{f}$ (cm)	5.6
$\delta\bar{f}$ (cm)	0.14

Table 4: Positions of source, lens, and image for lens 2

Trial	p_{so} (cm)	p_{l1} (cm)	p_{l2} (cm)	p_{sc} (cm)	f (cm)	δf (cm)
1	1.85	40.01	83.22	120.04	25.598	0.0383
2	10.12	51.92	79.62	120.04	25.735	0.0314
3	14.50	59.95	76.79	120.04	25.713	0.0261
4	8.29	49.00	79.98	120.04	25.790	0.0329
5	3.81	42.18	82.16	120.04	25.619	0.0370
6	17.88	64.70	72.10	120.04	25.406	0.0214

Source distance offset (cm)	1.80
Lens distance offset (cm)	0.00
Screen distance offset (cm)	0.00
Instrumental error (cm)	0.05
$\bar{f}$ (cm)	25.64
$\delta\bar{f}$ (cm)	0.08

2.3 Part B Data

Table 5: Expected angular magnification

Expected angular magnification (cm)	-4.6
Error for mean angular magnification (cm)	0.1

Table 6: Refracting telescope tape lengths

Trial	Distance between lenses (cm)	l_1 (cm)	l_2 (cm)	Trial m_θ
1	31.00	82.20	20.90	3.933
2	31.00	92.10	18.90	4.873
3	31.10	58.50	11.55	5.065

Mean angular magnification (cm)	-4.62
Error for mean angular magnification (cm)	0.04

3 Calculations

Note that the full precision of each measurement is kept until the final result of each calculation, in which the results are rounded to the proper number of significant digits.

3.1 Instrumental Error

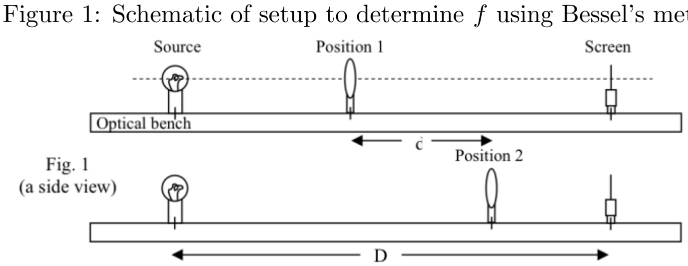
Measurements for (Part A) of this lab were taken with the linear scale on the optical bench, and measurements for (Part B) of this experiment (lengths of tape) were taken using the meter stick. Both instruments had markings to the nearest 0.1cm, so the instrumental uncertainty is $\delta S = \pm 0.05$cm for every reading. Distance measurements (e.g., l_1, l_2) are the function of two uncertainties and the error $\delta l_1 = \delta l_2 = 0.05\sqrt{2}$cm reflects this.

3.2 Sample mean and random error

For each position measurement p_x (i.e., p_{sc}, p_{so}, p_{l1}, p_{l2}), the mean $\bar{p}_x$ and standard error σ_{p_x} may not be calculated, as these measurement are not centered around a mean value; the only mean and standard deviation may be calculated for the focal lengths, as these should be centered around the true value of the focal length of the lens.

3.3 Calculation of the focal length using Bessel's method

Figure 1: Schematic of setup to determine f using Bessel's method



The focal length of the lenses is calculated using Bessel's method. Given the setup indicated in (Figure), Bessel's method uses (Equation 1) to solve for the focal length f of the lens, and is derived in (Section 6.1).

$$f = \frac{D^2 - d^2}{4D} \tag{1}$$

In short, the source and screen are set up more than four focal lengths away from each other on the optical bench, and their positions (p_{so} and p_{sc}, respectively) measured. There should be two positions that the lens may be placed (p_{l1} and

p_{l2}), such that the image of the source is focused on the screen. In this setup, we make the substitutions

$$D = |p_{sc} - p_{so}|$$
$$d = |p_{l2} - p_{l1}|$$

to obtain

$$f = \frac{|p_{sc} - p_{so}|^2 - |p_{l2} - p_{l1}|^2}{4|p_{sc} - p_{so}|}$$

Since the measurements for each trial are not centered around any particular value and may vary dramatically, this method may not be used to calculate average $\bar{f}$ using average values for each position measurement, but rather only the focal length based on the measurements for a single trial. The average calculated focal length may be found as a mean of the sample values:

$$\bar{f} = \frac{1}{n}\sum_{i=1}^{n} f_i$$

3.3.1 Error propagation

The measurements p_{sc} and p_{so} are independent, as the distance between source and image is arbitrarily chosen. However, the values for p_{l1}, p_{l2}, and D are dependent, as the arbitrary choice of D determines the two possible lens positions. This inspires a two-part error propagation calculation.

$$\delta D = \sqrt{\left(\frac{\partial D}{\partial p_{so}}\delta p_{so}\right)^2 + \left(\frac{\partial D}{\partial p_{sc}}\delta p_{sc}\right)^2}$$

$$\delta f = \left|\frac{\partial f}{\partial D}\delta D\right| + \left|\frac{\partial f}{\partial p_{l1}}\delta p_{l1}\right| + \left|\frac{\partial f}{\partial p_{l2}}\delta p_{l2}\right|$$

In this case, the partial derivatives are (signs are not important because the error is always positive):

$$\frac{\partial D}{\partial p_{so}} = \frac{\partial D}{\partial p_{sc}} = 1$$
$$\frac{\partial f}{\partial D} = \frac{D^2 + d^2}{4D^2}$$
$$\frac{\partial f}{\partial p_{l1}} = \frac{\partial f}{\partial p_{l2}} = \frac{d}{2D}$$

Making all substitutions, the error calculation for f is

$$\delta f = \left|\frac{D^2 + d^2}{4D^2}\sqrt{\delta p_{so}^2 + \delta p_{sc}^2}\right| + \left|\frac{d}{2D}\delta p_{l1}\right| + \left|\frac{d}{2D}\delta p_{l2}\right|$$

This is the error calculation for a single f_i calculation, where δp_x is the instrumental error for the p_x measurement. Note that since all positions were measured with the same instrument, $\delta p_x = \delta p$ are equal, that all terms inside the absolute value bars are positive, and the first term includes the f calculation, and thus the calculation may be simplified to

$$\delta f = \left(\left(f + \frac{d^2}{2D} \right) \sqrt{2} + d \right) \frac{\delta p}{D}$$

Since each sample focal length calculation f_i is independently calculated, the error for the mean $\delta \bar{f}$ is the RMS of the f_i errors, i.e.,

$$\delta \bar{f} = \sqrt{\sum_{i=1}^{n} (\delta f_i)^2}$$

3.3.2 Sample Bessel's law focal length and error propagation calculations

The calculation for the focal length for sample 1 using Bessel's law is shown below.

$$f_1 = \frac{(120.04\text{cm} - 77.85\text{cm})^2 + (113.71\text{cm} - 84.58\text{cm})^2}{4(120.04\text{cm} - 77.85\text{cm})} = 5.\bar{5}19\text{cm}$$

The calculation for the error for the focal length for sample 1 is shown below.

$$\delta f_1 = \left(\left(5.519\text{cm} + \frac{(113.71cm - 84.58\text{cm})^2}{4(120.04cm - 77.85\text{cm})} \right) \sqrt{2} + (113.71\text{cm} - 84.58\text{cm}) \right)$$
$$\times \frac{0.05\text{cm}}{120.04\text{cm} - 77.85\text{cm}} = 0.06\text{cm}$$

The average and average uncertainty are as calculated below.

$$\bar{f} = \frac{1}{6}(5.\bar{5}19\text{cm} + 5.\bar{6}43\text{cm} + 5.\bar{4}96\text{cm} + 5.\bar{8}33\text{cm} + 5.\bar{5}88\text{cm} + 5.\bar{6}98\text{cm}) = 5.\bar{6}30\text{cm}$$

$$\delta \bar{f} = \sqrt{(0.0\bar{6}06\text{cm})^2 + (0.0\bar{6}57\text{cm})^2 + \cdots (0.0\bar{6}73\text{cm})^2} = 0.14\text{cm}$$

3.4 Calculation of angular magnification using focal lengths

The expected angular magnification for the telescope is

$$m_\theta = -\frac{f_{obj}}{f_{eye}}$$

3.4.1 Error propagation

The focal lengths of the two lenses are independent of one another, so

$$\delta m_\theta = \sqrt{\left(\frac{\partial m_\theta}{\partial f_{obj}} \delta f_{obj} \right)^2 + \left(\frac{\delta m_\theta}{\delta f_{eye}} \delta f_{eye} \right)^2} = \sqrt{\left(\frac{1}{f_{eye}} \delta f_{obj} \right)^2 + \left(\frac{f_{obj}}{f_{eye}^2} \delta f_{eye} \right)^2}$$

where f_{obj}, f_{eye} are obtained in (Section 3.3.1).

3.4.2 Calculation for angular magnification and error propagation

$$m_\theta = -\frac{5.\bar{6}30\text{cm}}{25.6\bar{4}4\text{cm}} = -4.\bar{5}55$$

$$\delta m_\theta = \sqrt{\left(\frac{1}{5.630cm}0.0777cm\right)^2 + \left(\frac{25.644cm}{(5.630cm)^2}0.141cm\right)^2} = 0.11$$

3.5 Calculation of angular magnification using empirical tape lengths

The angular magnification for the telescope is the ratio of the lengths of tape (such that the shorter piece of tape viewed through the telescope appears as long as the longer piece of tape viewed without the telescope). Denote this $m_{\theta\text{emp}}$.

$$m_{\theta\text{emp}} = \frac{l_1}{l_2}$$

A mean is taken over the three trials for the empirical angular magnifications to get the mean empirical angular magnifications.

A percent error calculation is performed to check the closeness of this angular magnification from the angular magnification calculated using focal lengths.

$$\%\ \text{Err.} = \frac{|\bar{m}_{\theta\text{emp}} - m_\theta|}{m_\theta} \times 100\%$$

3.5.1 Error propagation for calculation of angular magnification using empirical tape lengths

Since this equation is of the same form as the previous calculation for angular magnification, the error propagation (for a single trial) is of the same form.

$$\delta m_{\theta\text{emp}} = \sqrt{\left(\frac{1}{l_2}\delta l_1\right)^2 + \left(\frac{l_1}{l_2^2}\delta l_2\right)^2}$$

Again, the error for the mean value is the RMS of the trial errors.

3.5.2 Sample calculation of empirical angular magnification and error propagation

For trial 1, the empirical angular magnification calculation is

$$m_{\theta\text{emp}_1} = \frac{82.20\text{cm}}{20.90\text{cm}} = 3.9\bar{3}3$$

Note that $\delta l_1 = \delta l_2 = 0.05\sqrt{2}\text{cm}$, since it is a distance measurement and not a single reading. The error for this trial is thus

$$\delta m_{\theta\text{emp}_1} = \sqrt{\left(\frac{1}{20.9\text{cm}}0.07\text{cm}\right)^2 + \left(\frac{82.2\text{cm}}{(20.9\text{cm})^2}0.07\text{cm}\right)^2} = 0.014$$

The mean empirical angular magnification is

$$\bar{m}_{\theta \mathrm{emp}} = \frac{1}{3}(3.933 + 4.873 + 5.065) = 4.6\bar{2}4$$

The error for the mean empirical angular magnification is

$$\delta \bar{m}_{\theta \mathrm{emp}} = \sqrt{0.0137^2 + 0.0186^2 + 0.0316^2} = 0.04$$

The percent error calculation is:

$$\% \text{ error} = \frac{|4.6\bar{2}4 - 4.\bar{5}55|}{4.\bar{5}55} = 1.5\%$$

4 Results summary

The results from Part A of the lab are summarized in (Table).

Table 7: Focal lengths summary

Lens	f (cm)
1	$5.6 \pm 0.14cm$
2	$25.64 \pm 0.08cm$

The results from Part B are summarized in (Table).

Table 8: Angular magnification summary

Method	Angular magnification
Using focal lengths from Part A	-4.6 ± 0.1
Using tape lengths from Part B	-4.62 ± 0.04

% error	1.5%

5 Error analysis

This section should be precluded by the fact that all measurements were performed at a certain configuration of the lens(es) with objects that are visually determined to be in focus or of equal length. There is inherent error in this, as the human eye is not especially precise with determining either of these.

5.1 Part A sources of error

In the original estimate of focal lengths, an estimation takes place assuming that the focal length is approximately the length of a lens from a screen that it aims to focus an image onto. The derivation for this estimation is present in Section 7.2, as well as an argument for why it is an overestimation – thus there is an inherent modeling error. Moreover, there is also large error introduced by the method used to measure this distance; e.g., if the meter stick is not held perfectly perpendicular to the ground, if the image is not precisely focused, or if the reading is not taken at the center of the lens (as was hard to do, as one of the lenses has a short focal length and thus was positioned very near the ground), then these are all sources of random error. The reasons why procedural precautions were not made is because this was only a rough estimate for the next stages. Thus no proper error analysis is made for this section.

There is an offset for each of the lens, screen, and source. To attempt to measure this offset, we held a straight edge (meter stick) flat on the surfaces of the source and screen, perpendicular to the optical bench, and attempted to read the difference between the surface's position and the optical bench's reading. For the lens and the screen (which is almost flat; a sheet of paper), we were unable to measure a discernible difference between the two values, and made the (reasonable) assumption that the error contributed by these offsets is very small compared to the distances between them (which would ultimately be used in all of the calculations). However, the source offset was significant, 1.80cm, and was added to each source measurement. These small offsets may add some systematic error to the calculations, but we performed the a correction to the best of our knowledge with only the source offsets. A better method to measure offsets would have been to use a more precise measuring tool, such as calipers.

There was a significant amount of light pollution from other groups' flashlights, which made it more difficult to determine precisely at what lens position the image was clearly focused, which may introduce some degree of random error.

5.2 Part B sources of error

The procedure failed to provide an exact measure for the distance between the two lenses (only dictating that the two lenses be slightly less than the sum of the two focal lengths apart). We experimented with lenses 31.0cm to 31.1cm away, with a large standard deviation of results. It is difficult to say whether this small difference in lens distance may have contributed a significant error, or if it was the inconsistency of the human eye to determine when the two pieces of tape were indeed the same length, one being viewed inside the telescope and one viewed outside. The latter is much more likely to be the overwhelmingly larger source of error, as there is no way to put both images side-to-side within the same viewing frame, which is how it is easiest to compare objects visually; instead, either the telescope has to be moved into view and overwriting the non-telescope image, or both eyes may be used simultaneously, but it is difficult to compare items when each eye is seeing a different image.

Another possible source of random error is the changing of viewing distance from the telescope wielder to the pieces of tape. This was not measured during the experiment, but there was a wide range of viewing distances used (in an attempt to make the visual comparison between the lengths easier), and its effect is not measured or known. Theoretically, this should make no difference to the outcome if the eye was accurate in its comparison, but it may likely have introduced differences in perception that lead to random error.

6 Conclusion

By using Bessel's method, the focal lengths of the eyepiece lens and the object lens were 5.6 ± 0.14cm and 25.64 ± 0.08cm. Although the true focal lengths of these lenses is unknown, the small standard deviations (on the order of 1mm) indicate that Bessel's method produces fairly reliable results. The initial estimated focal length for lens 1 (5.50cm) is captured within the error bars, but the estimated focal length for lens 2 (28.00cm) is not, but off by less than 10%. Given that the initial estimate is really a rough estimate, this error is acceptable.

This result is further strengthened by Part B's verification of the angular magnification of a telescope formed using the lenses. The expected angular magnification calculated using the focal lengths in Part A is -4.6 ± 0.1, while the average angular magnification calculated using tape lengths is -4.62 ± 0.08. yielding only a 1.5% error. While these error margins do not overlap, the magnification factors are very close (the difference is only 0.2) and the errors are very small, so this may be interpreted as both methods converging towards a similar, precise result.

There is, as expected for a method with large visual estimation and uncertainty, a large standard deviation between trials, and thus many trials are recommended to obtain a more precise result through this method.

7 Answers to questions

7.1 Derivation of Bessel's law

If the distance between the source and object is fixed, then the distance p between the lens and the object, and the distance i between the lens and the projected image is related by the thin lens equation

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{i} \tag{2}$$

Since the relationship between p and i is symmetric, if the lens focuses the image when it is at distance p from the source, it will also focus it at length i from the source. We have two other relations that can be determined from (Figure): firstly, since the images are real and the distance between the source and image is the sum of p and i,

$$p + i = D$$

Since the lens may focus an image at position p or position i from the source, the distance between the two focusing positions of the lenses is

$$p - i = d$$

Solving for f from (Equation 2), we get

$$f = \frac{1}{\frac{1}{p} + \frac{1}{i}} = \frac{pi}{p + i}$$

With some algebraic manipulation:

$$pi = \frac{(p^2 + 2pi + i^2) - (p^2 - 2pi + i^2)}{4} = \frac{(p + i)^2 - (p - i)^2}{4} = \frac{D^2 - d^2}{4}$$

Substituting in for pi, we get Bessel's equation for focal length

$$f = \frac{D^2 - d^2}{4D}$$

We can rearrange Bessel's equation into the following quadratic inequality, given that $D, d > 0$ ($D > 0$ clearly, and $d > 0$ for the separation of lens positions). This inequality demonstrates the reason for requiring that D is greater than four times the focal length.

$$D^2 - 4fD - d^2 = 0 \Rightarrow D^2 - 4fD > 0 \Rightarrow D(D - 4f) > 0 \Rightarrow D > 4f$$

(Bessel's equation also works in the degenerate case of $d = 0$, corresponding to the scenario in which $i = p$, in which $f = \frac{D^2 - 0}{4D} = \frac{D}{4}$, as expected; however, the experimental procedure for this lab expected two lens positions producing clear images to use the nontrivial form of Bessel's equation.)

7.2 Initial overestimation of focal length

To estimate focal length, a sharp image of an object far away was produced – for this experiment, an image of the overhead lights was produced on the floor. We used the approximation that the object was at infinite distance (i.e., $p = \infty$) to make the following approximation from the thin lens equation:

$$\frac{1}{f} = \lim_{p \to \infty} \left(\frac{1}{p} \right) + \frac{1}{i} \Rightarrow \frac{1}{f} = 0 + \frac{1}{i} \Rightarrow f \approx i$$

Of course, p is finite, especially considering the measurable distance to the ceiling lights, so $\frac{1}{p} = \epsilon > 0$. Thus, a more accurate representation using the thin lens equation is

$$\frac{1}{f} = \frac{1}{i} + \epsilon \Rightarrow f = \frac{1}{\frac{1}{i} + \epsilon} < i$$

Therefore i is an overestimation for the focal length. Since the ratio $\frac{p}{i}$ was very large and this approximation was only used to get a rough estimate of the focal length to be more accurately measured in the next part of the procedure, this approximation should not contribute any error to the final results of this lab.