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Cplx Analysis
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§72 #3, 6, 7

3.) Find the Taylor series representation for:

$$\frac{1}{z} = \frac{1}{2+(z-2)} = \frac{1}{2} \cdot \frac{1}{1+\frac{z-2}{2}}$$

about $z_0 = 2$. Then, by differentiating termwise, show that:

$$\frac{1}{z^2} = \frac{1}{4}\sum_{n=0}^{\infty} (-1)^n (n+1)\left(\frac{z-2}{2}\right)^n, \quad (|z-2|<2)$$

$$\frac{1}{2} \cdot \frac{1}{1+\frac{z-2}{2}} = \frac{1}{2} \cdot \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-2}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n (z-2)^n}{2^{n+1}}$$

$$\left(\left|-\left(\frac{z-2}{2}\right)\right| \le 1 \Rightarrow |z-2|<2\right).$$

differentiating:

$$\frac{d}{dz}\frac{1}{z} = -\frac{1}{z^2} = \frac{d}{dz}\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(z-2)^n = \sum_{n=0}^{\infty} \frac{d}{dz}\frac{(-1)^n}{2^{n+1}}(z-2)^n$$

term by term differentiating allowed

$$\Rightarrow \frac{1}{z^2} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} n (z-2)^{n-1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{2^{n+1}} n (z-2)^{n-1}$$

let $m = n-1$; $$= \sum_{m=-1}^{\infty} \frac{(-1)^m}{2^{m+2}} (m+1)(z-2)^m$$

($m=-1$ term $= 0$, so can start @ 1)

$$= \frac{1}{4}\sum_{m=0}^{\infty} (-1)^m (m+1)\left(\frac{z-2}{2}\right)^m$$

§72 #6,7

6) In the w plane, integrate the Taylor series representation

$$\frac{1}{w} = \sum_{n=0}^{\infty} (-1)^n (w-1)^n \qquad (|w-1| < 1)$$

along a contour interior to its circle of convergence from $w=1$ to $w=z$ to obtain the representation:

$$\mathrm{Log}\, z = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (z-1)^n .$$

allowed to integrate power series representation termwise and get the correct result (§71 result):

$$\int \frac{1}{w} = \sum_{n=0}^{\infty} \int (-1)^n (w-1)^n = \sum_{n=0}^{\infty} (-1)^n \frac{(w-1)^{n+1}}{n+1}$$

$$\text{let } m = n+1 = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(w-1)^m}{m} .$$

$$\int_1^z \frac{1}{w} dw = \mathrm{Log}\, w \Big|_1^z = \underbrace{\mathrm{Log}\, z - \cancel{\mathrm{Log}\, 1} = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(z-1)^m}{m}}_{\text{desired result.}}$$

(Note that this only is true when entire contour lies in circle of convergence)

7). Show that if $f(z) = \frac{\mathrm{Log}\, z}{z-1}$ when $z \neq 1$ and $f(1) = 1$, then $f$ is analytic throughout the domain $0 < |z| < \infty$, $-\pi < \mathrm{Arg}\, z < \pi$.

We already know that this function is analytic throughout the specified domain $\setminus \{1\}$, since the numerator is analytic throughout this domain and the denominator is entire and has a zero at $z=1$. We can use the Taylor series expansion for Log z in a deleted nbd of $z=1$ to make an analytic continuation of $f$ @ 1:

$$f(z) = \underbrace{\frac{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(z-1)^n}{n}}{z-1}}_{\text{expression}} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(z-1)^{n-1}}{n} = \sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^n}{n+1}$$

$$(0 < |z-1| < 1)$$

Since this ~~function~~ expression is clearly analytic (only positive powers of $(z-1)$) @ $z=1$ and matches the value of $f$ in a deleted nbd of 1, if we assign $f(1) = \sum_{n=0}^{\infty} (-1)^n \frac{(1-1)^n}{n+1} = 1$, it also becomes analytic @ 1.

§§73 # 1, 5.

1.) Show that $\dfrac{e^z}{z(z^2+1)} = \dfrac{1}{z} + 1 - \dfrac{1}{2}z - \dfrac{5}{6}z^2 + \dots \quad (0<|z|<1)$

$$f(z) = \frac{1}{z} \cdot e^z \cdot \frac{1}{1-(-z^2)} = \frac{1}{z}\left(1+z+\frac{z^2}{2}+\frac{z^3}{6}+\dots\right)\left(1-z^2+z^4-z^6+\dots\right)$$

$$= \frac{1}{z}\left(\begin{array}{llllllll} 1 & & -z^2 & & +z^4 & & -z^6 & +\dots \\ & +z & & -z^3 & & +z^5 & & -z^7+\dots \\ & & +\frac{z^2}{2} & & -\frac{z^4}{2} & & +\frac{z^6}{2} & +\dots \\ & & & +\frac{z^3}{6} & & -\frac{z^5}{6} & & +\dots \end{array}\right)$$

$$= \frac{1}{z}\left(1+z-\frac{z^2}{2}-\frac{5z^3}{6}+\dots\right) = \frac{1}{z}+1-\frac{z}{2}-\frac{5z^2}{6}+\dots$$

5.) Note how the expansion:

$$\frac{1}{z^2 \sinh z} = \frac{1}{z^3} - \frac{1}{6}\cdot\frac{1}{z} + \frac{7}{360}z + \dots \quad (0<|z|<\pi)$$

follows from the example in §73. Show that:

$\displaystyle\int_C \frac{dz}{z^2 \sinh z} = -\frac{\pi i}{3}$, C is the P.O. circle $|z|=1$.

$z^2$ has a zero @ $z=0$,

$\sinh z$ has zeros @ $z = n\pi i$, $n \in \mathbb{Z}$ (§38).

Thus $f$ is analytic in the ~~annulus~~ punctured disk $(0<|z-0|<\pi)$,

So there is a (unique) Laurent series representation of

$f(z) = \displaystyle\sum_{n=-\infty}^{\infty} c_n (z-0)^n$, where C is a ~~[illegible]~~ positively oriented circle encircling the origin and completely lying in the punctured disk (which is true by hypothesis), and $c_n = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{(z-0)^{n+1}}\,dz$.

Thus $\displaystyle\int_C \frac{dz}{z^2 \sinh z} = \int_C \frac{1}{(z^2\sinh z)\, z^{n+1}}\,dz = \int_C \frac{f(z)}{z^{-1+1}}\,dz$

$= 2\pi i\, c_{-1} = 2\pi i\left(-\frac{1}{6}\right) = \dfrac{-\pi i}{3}.$

coefficient of the $\frac{1}{z}$ term

§77 # 1a,b, 2a,c, 5.

1) Find the residue @ $z=0$ of:

a) $\frac{1}{z+z^2} = \frac{1}{z}\cdot\frac{1}{1+z} = \frac{1}{z}\left(\frac{1}{1-(-z)}\right)$

punctured disk centered @0

$(0<|z-0|<1)$

$= \frac{1}{z}\left(1 - z + z^2 - z^3 + \dots\right) = \frac{1}{z} - 1 + z - \dots$

$\text{Res}_{z=0} f(z) = 1$

b) $z\cos\left(\frac{1}{z}\right) = z\left(1 - \frac{\left(\frac{1}{z}\right)^2}{2!} + \frac{\left(\frac{1}{z}\right)^4}{4!} - \dots\right)$ $(0<|z-0|<\infty)$

$= z - \frac{1}{2z} + \frac{1}{24z} - \dots$

in punctured disk centered at 0.

$\text{Res}_{z=0} f(z) = -\frac{1}{2}$

2a) Evaluate the integral of each of these fns. around the P.O. circle $|z|=3$ using Residue thm.

$f(z) = \frac{\exp(-z)}{z^2}$ one singularity of $f$, at $z=0$, enclosed in C.

$\int_C f(z)\,dz = 2\pi i \sum_{\text{enclosed sing's}} \text{Res} f(z) = 2\pi i \text{Res}_{z=0} f(z)$ (*)

Posc AOIC except at $z=0$ residue thm

$f(z) = \frac{1}{z^2}\left(1 - z + \frac{z^2}{2} - \frac{z^3}{6} + \dots\right) = \frac{1}{z^2} - \frac{1}{z} + \frac{1}{2} - \dots$ $(0<|z-0|<\infty)$

$\text{Res}_{z=0} f(z) = -1$

plugging this into (*), we get:

$\int_C f(z)\,dz = -2\pi i.$

§77 #2c, 5.

2c). $f(z) = z^2 \exp\left(\frac{1}{z}\right)$, $C$ = (same curve as before).

only one singularity of $f$ (enclosed within $C$) @ $z=0$.

$$f(z) = z^2\left(1 + \frac{1}{z} + \frac{\left(\frac{1}{z}\right)^2}{2} + \frac{\left(\frac{1}{z}\right)^3}{6} + \dots\right) \qquad (0<|z-0|<\infty)$$

$$= z^2 + z + \frac{1}{2} + \frac{1}{6z} + \dots$$

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{6}$$

residue thm

$$\int_C f(z)\,dz = 2\pi i \sum_{\text{sing's}} \operatorname{Res} f(z) = 2\pi i\left(\frac{1}{6}\right) = \frac{\pi i}{3}.$$

POSCC

AOIC except @ one interior singularity

5.) Let $C$ denote the P.O. circle $|z|=1$. Show that $\int_C \exp\left(z+\frac{1}{z}\right)dz = 2\pi i \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}$.

a) expand using the Maclaurin series for $\exp z$

$$\int_C \exp\left(z+\frac{1}{z}\right)dz = \int_C \exp z \exp\frac{1}{z}\,dz = \int_C \sum_{n=0}^{\infty} \frac{z^n}{n!} \exp\frac{1}{z}\,dz$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\int_C z^n \exp\left(\frac{1}{z}\right)dz. \quad ⊛$$

integration can be taken termwise, §71

b) use residue thm. Let $f(z) = z^n \exp\left(\frac{1}{z}\right)$

$$\int_C \underbrace{z^n \exp\left(\frac{1}{z}\right)}_{\text{AOIC except of a single singular point, } z=0} dz = 2\pi i \sum_{\text{enclosed sing's}} \operatorname{Res} f(z) = 2\pi i \operatorname*{Res}_{z=0} f(z).$$

POSCC

$f(z) = z^n \sum_{m=0}^{\infty} \left(\frac{1}{z}\right)^m \cdot \frac{1}{m!} = \sum_{m=0}^{\infty} \frac{z^{n-m}}{m!}$. $\operatorname*{Res}_{z=0}$ = coefficient of $z^{-1}$

term, i.e. when $n = m-1$ = $\frac{1}{(n+1)!}$ $\Rightarrow \int_C f(z)\,dz = \frac{2\pi i}{(n+1)!}$

Plugging into ⊛, $\int_C \exp\left(z+\frac{1}{z}\right)dz = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{2\pi i}{(n+1)!}\right) = 2\pi i \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}$

§79 #1a, b, 2, 4.

1) Write the principal part of the function at its isolated singular point and determine its type:

a) $f(z) = z\exp\left(\frac{1}{z}\right)$ isolated singular pt @ $z=0$.

$= z\left(\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n \cdot \frac{1}{n!}\right) = z\left(1 + \frac{1}{z} + \frac{1}{2z^2} + \dots\right) \quad (0<|z-0|<\infty)$

$= z + 1 + \underbrace{\frac{1}{2z} + \frac{1}{6z^2} + \frac{1}{24z^3} + \dots}_{\text{principal part}}$

principal part has infinitely-many terms, so essential singularity.

b). $f(z) = \frac{z^2}{1+z}$ . isolated singular pt. @ $z=-1$, $(0<|z+1|<\infty)$

$= \frac{((z+1)-1)^2}{z+1} = \left((z+1)^2 - 2(z+1) + 1\right) \cdot \frac{1}{z+1}$

$= (z+1) - 2 + \underbrace{\frac{1}{z+1}}_{\text{principal part}} \Rightarrow$ simple pole @ $z=-1$.

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2.) Show that the singular part of each of the following fns is a pole. Determine the order $m$ of that pole and corresponding residue $B$,

a) $f(z) = \frac{1-\cosh z}{z^3}$ isolated sing. @ $z=0$.

$= \frac{1}{z^3}\left(\cancel{1} - \left(\cancel{1} + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \dots\right)\right) \quad (0<|z-0|<\infty)$

$= \underbrace{-\frac{1}{2z}}_{\text{P.P.}} - \frac{z}{4!} - \frac{z^3}{6!} - \frac{z^5}{8!} - \dots$

$B = \operatorname{Res}_{z=0} f(z) = -\frac{1}{2}$ , order $= 1$.

§79 # 2, 4.

2b). $f(z) = \dfrac{1-\exp(2z)}{z^4}$ isolated sing. pt. @ $z=0$.

$= \frac{1}{z^4}\left(\cancel{1} - \left(\cancel{1} + (2z) + \frac{(2z)^2}{2!} + \frac{(2z)^3}{3!} + \cdots\right)\right)$ $(0<|z-0|<\infty)$

$= \underbrace{-\frac{2}{z^3} - \frac{4}{2z^2} - \frac{8}{6z}}_{p.p.} - \frac{16}{24} - \cdots$

$m=3$ $B = \operatorname{Res}_{z=0} f(z) = -\frac{4}{3}$

2c). $f(z) = \dfrac{\exp(2z)}{(z-1)^2}$ isolated sing. pt. @ $z=1$.

$= \frac{1}{(z-1)^2}\exp(2(z-1)+2)$ $(0<|z-1|<\infty)$

$= e^2\left(\frac{1}{(z-1)^2}\right)\left(1 + 2(z-1) + \frac{2^2(z-1)^2}{2!} + \frac{2^3(z-1)^3}{3!} + \cdots\right)$

$= \underbrace{\frac{e^2}{(z-1)^2} + \frac{2e^2}{z-1}}_{p.p.} + 2e^2 + \cdots$

$B = \operatorname{Res}_{z=1} f(z) = 2e^2$, $m=2$.

4.) write the function $f(z) = \dfrac{8a^3z^2}{(z^2+a^2)^3}$ $(a>0)$

as $f(z) = \dfrac{\phi(z)}{(z-ai)^3}$ where $\phi(z) = \dfrac{8a^3z^2}{(z+ai)^3}$

Point out why $\phi(z)$ has a Taylor series representation about $z=ai$, then use it to show that the principal part of $f$ at that point is:

$$\frac{\phi''(ai)/2}{z-ai} + \frac{\phi'(ai)}{(z-ai)^2} + \frac{\phi(ai)}{(z-ai)^3} = \frac{-i/2}{z-ai} - \frac{\frac{a}{2}}{(z-ai)^2} - \frac{a^2 i}{(z-ai)^3}$$

§79 #4.

4, cont'd.) $\phi(z) = \frac{8a^3 z^2}{(z+ai)^3}$ clearly has its only (isolated) singular point @ $z=-ai$, so it is analytic in some nbd of $z=ai$, so it must have a Taylor series expansion at $z=ai$:

$$\phi(z) = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(ai)}{n!}(z-ai)^n$$

Thus $f = (z-ai)^{-3} \sum_{n=0}^{\infty} \frac{\phi^{(n)}(ai)}{n!}(z-ai)^n$

$$= \sum_{n=0}^{\infty} \frac{\phi^{(n)}(ai)}{n!}(z-ai)^{n-3}$$

$$\circledast \quad = \underbrace{\frac{\phi(ai)}{(z-ai)^3} + \frac{\phi'(ai)}{(z-ai)^2} + \frac{\phi''(ai)}{2(z-ai)}}_{\text{p.p. of } f \text{ @ } z=ai} + \frac{\phi'''(ai)}{6}$$

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to show that this equals the other side of the inequality, just have to calculate $\phi(ai)$, $\phi'(ai)$, $\phi''(ai)$.

$$\phi(ai) = \frac{8a^3(ai)^2}{(2ai)^3} = \frac{8a^5 i^2}{2^3 a^3 i^3} = -a^2 i$$

$$\phi'(z) = 8a^3\left(\frac{(z+ai)^3(2z) - z^2(3(z+ai)^2)}{(z+ai)^6}\right)$$

$$= 8a^3\left(\frac{2z(z+ai) - 3z^2}{(z+ai)^4}\right)$$

$$\phi'(ai) = 8a^3\left(\frac{2ai(2ai) - 3(ai)^2}{(2ai)^5}\right) = 8a^3\left(\frac{4(ai)^2 - 3(ai)^2}{32(ai)^5}\right) = -\frac{a}{2}$$

$$\phi''(z) = 8a^3\left(\frac{(z+ai)^4(\cancel{4z} + 2ai - 6z) - (2z(z+ai) - 3z^2)(4(z+ai)^3)}{(z+ai)^8}\right)$$

$$\phi''(ai) = 8a^3\left(\frac{\cancel{(2ai)^4(4ai + 2ai - 6ai)}^{=0} - (2ai(2ai) - 3(ai)^2)(4(2ai)^3)}{(2ai)^8}\right)$$

$$= -\frac{2^5 a^8 i^5}{2^8 a^8 i^8} = -i^{-3} = -i$$

Substituting into $\circledast$, we get,

$$\text{P.P. } f \text{ @ } (z=2) = \frac{-a^2 i}{(z-ai)^3} - \frac{a/2}{(z-ai)^2} - \frac{i/2}{z-ai}$$

§81 # 1b–d, 2b, c, 4, 6.

$\boxed{\text{Henceforth call thm. in §80 the pole-residue thm.}}$

1) Show that any singular point in each fn. below is a pole. Determine the order of each pole, and find the corresponding residue $B$.

b.) $f(z) = \dfrac{z^2+2}{z-1}$ (only) singular point: $z = 1$.

$f(z) = \dfrac{\phi(z)}{(z-1)^1}$, $\phi(z) = z^2+2$

$\phi(1) = 1+2 = 3 \neq 0$

$\Downarrow$ $\phi$ AANZ @ $z = 1$.

By pole-residue thm, $f$ has simple pole @ $z = 1$, and $B = \operatorname{Res}_{z=1} f(z) = \phi(1) = 3$.

c) $f(z) = \left(\dfrac{z}{2z+1}\right)^3$ only singular point: $z = -\frac{1}{2}$

$f(z) = \dfrac{\phi(z)}{(z+\frac{1}{2})^3}$, $\phi(z) = \left(\dfrac{z}{2}\right)^3$

$\phi(-\frac{1}{2}) = \left(\dfrac{-1}{2}\right)^3 \cdot \frac{1}{2} = \frac{-1}{16}$

$\Downarrow$ $\phi$ AANZ @ $z = \frac{-1}{2}$

By pole-residue thm, $f$ has triple pole @ $z = \frac{-1}{2}$, and $\operatorname{Res}_{z=-\frac{1}{2}} f(z) = \dfrac{\phi''(z_0)}{2!} = \dfrac{\frac{3}{4}z_0}{2} = \dfrac{-3}{16}$

d) $f(z) = \dfrac{e^z}{z^2+\pi^2}$. $f$ has isolated singularities @ $z = \pm\pi i$.

$\phi_1 = \dfrac{e^z}{z-\pi i}$, $\phi_2 = \dfrac{e^z}{z+\pi i}$.

$f(z) = \dfrac{\phi_1(z)}{z+\pi i} = \dfrac{\phi_2(z)}{z-\pi i}$.

$\begin{cases} \phi_1(-\pi i) = \dfrac{e^{-\pi i}}{-2\pi i} = \dfrac{1}{2\pi i} \\ \phi_2(\pi i) = \dfrac{e^{\pi i}}{2\pi i} = -\dfrac{1}{2\pi i} \end{cases}$

$\Rightarrow$ $\phi_1$ AANZ @ $z = -\pi i$, $\phi_2$ AANZ @ $z = \pi i$

Thus $f$ has a pole of order 1 at both $z = \pm\pi i$,

and $\operatorname{Res}_{z=\pi i} f(z) = \phi_2(\pi i) = \dfrac{-1}{2\pi i}$

and $\operatorname{Res}_{z=-\pi i} f(z) = \phi_1(-\pi i) = \dfrac{1}{2\pi i}$

§81 #2b, c, 4, 6,

2). Show that:

b) $\underset{z=i}{Res}\ \dfrac{Log\, z}{(z^2+1)^2} = \dfrac{\pi+2i}{8}$

$f(z) = \dfrac{Log\, z}{(z^2+1)^2} = \dfrac{\frac{Log\, z}{(z+i)^2}}{(z-i)^2} = \dfrac{\phi(z)}{(z-i)^2}$, $\phi(z) = \dfrac{Log\, z}{(z+i)^2}$

$\phi(i) = \dfrac{\cancel{\ln(1)} + i(\frac{\pi}{2})}{(2i)^2} \neq 0 \Rightarrow \phi$ AANZ @ $z=i$

$\phi'(z) = \dfrac{(z+i)^2(\frac{1}{z}) - Log\, z\,(2(z+i))}{(z+i)^4}$

$\phi'(i) = \dfrac{(2i)^2(\frac{1}{i}) - (\cancel{\ln(1)} + i(\frac{\pi}{2}))(2(2i))}{(2i)^4}$

$= \dfrac{4i - i(\frac{\pi}{2})(4i)}{2^4} = \dfrac{\pi+2i}{8}.$

By pole-residue thm, $f$ has double pole @ $z=i$, and $\underset{z=i}{Res}\ f(z) = \dfrac{\phi'(i)}{1!} = \dfrac{\pi+2i}{8}$

c) $\underset{z=i}{Res}\ \dfrac{z^{\frac{1}{2}}}{(z^2+1)^2} = \dfrac{1-i}{8\sqrt{2}}$ $\quad (|z|>0,\ 0<\arg z<2\pi)$

$f(z) = \dfrac{z^{\frac{1}{2}}}{(z^2+1)^2} = \dfrac{\frac{z^{\frac{1}{2}}}{(z+i)^2}}{(z-i)^2} = \dfrac{\phi(z)}{(z-i)^2}$, $\phi(z) = \dfrac{z^{\frac{1}{2}}}{(z+i)^2}$

$\phi(i) = \dfrac{i^{\frac{1}{2}}}{(2i)^2} = \dfrac{e^{(\frac{1}{2}\log i)}}{4i^2} = \dfrac{e^{\frac{1}{2}(\frac{\pi i}{2})}}{-4} = -\dfrac{e^{\frac{i\pi}{4}}}{4} \neq 0.$

$\Downarrow$ $\phi$ AANZ @ $z=i$.

By pole-residue thm, $f$ has double pole @ $z=i$, and

$\underset{z=i}{Res}\ f(z) = \dfrac{\phi'(i)}{1!}$

$\phi'(z) = \dfrac{(2i)^2(\frac{1}{2}z^{-\frac{1}{2}}) - z^{\frac{1}{2}}(2(z+i))}{(z+i)^4}$

$\phi'(i) = \dfrac{(2i)^2(\frac{1}{2}i^{-\frac{1}{2}}) - i^{\frac{1}{2}}(2(2i))}{(2i)^4}$

2c, cont'd.

$$\left( \begin{array}{l} i^{-\frac{1}{2}} = e^{-\frac{1}{2}\log i} = e^{-\frac{1}{2}\left(i\frac{\pi}{2}\right)} = e^{-i\frac{\pi}{4}} \\ i^{\frac{1}{2}} = e^{i\frac{\pi}{4}} \end{array} \right)$$

$$\phi'(i) = \frac{(-4)\left(\frac{1}{2}\right)e^{-\frac{\pi}{4}i} - 4ie^{\frac{\pi}{4}i}}{16}$$

$$= \frac{-2e^{-\frac{\pi}{4}i} - 4e^{\frac{3\pi}{4}i}}{16} = \frac{-2e^{-\frac{\pi}{4}i} - 4\left(-e^{-\frac{\pi}{4}i}\right)}{16}$$

$$= \frac{e^{-\frac{\pi}{4}i}}{8} = \frac{1-i}{8\sqrt{2}} = \operatorname*{Res}_{z=i} f(z).$$

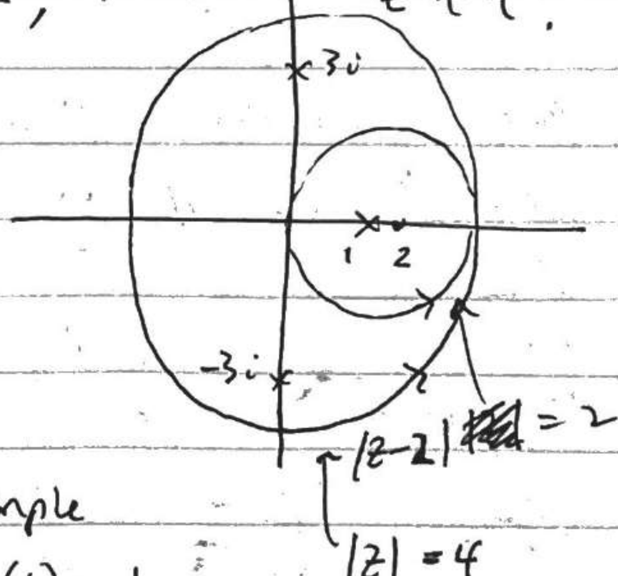
4.) Find the value of the integral

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}\,dz$$

a) where $C$ = p.o. circle $|z-2|=2$.

$$f(z) = \frac{3z^3+2}{(z-1)(z^2+9)} = \frac{\phi(z)}{z-1}, \quad \phi(z) = \frac{3z^2+2}{z^2+9}.$$

only singularity enclosed within $C$ is at $z=1$




$$\phi(1) = \frac{3+2}{1+9} = \frac{1}{2} \neq 0$$

$\Downarrow$

$\phi$ AANZ @ $z=1$.

By pole-residue thm, $f$ has simple pole @ $z=1$, and $\operatorname*{Res}_{z=1} f(z) = \phi(1) = \frac{1}{2}$.

residue thm

Thus $\int_C f(z)\,dz = 2\pi i\left(\frac{1}{2}\right) = \pi i$.

posCC

AOIC, except at simple pole $z=1$

§81 # 4b, 6.

4b). $|z| = 4$, P.O.

$f(z) = \dfrac{3z^3+2}{(z-1)(z^2+9)}$. 3 enclosed isolated singularities @ $z = 1, \pm 3i$.

$$f(z) = \frac{\dfrac{3z^3+2}{(z-1)(z+3i)}}{z-3i} = \frac{\phi_1(z)}{z-3i} \longrightarrow \phi_1 \text{ AANZ } @ z = 3i$$

$$= \frac{\dfrac{3z^3+2}{(z-1)(z-3i)}}{z+3i} = \frac{\phi_2(z)}{z+3i} \longrightarrow \phi_2 \text{ AANZ } @ z = -3i.$$

By pole-residue thm, $f$ has a simple pole @ $z = \pm 3i$,

and $\operatorname*{Res}_{z=3i} f(z) = \phi_1(3i) = \dfrac{3(3i)^3+2}{(3i-1)(3i+3i)} = \dfrac{3\cdot 27(-i)+2}{(3i-1)(6i)}$

$$= \frac{2-81i}{-18-6i}$$

and $\operatorname*{Res}_{z=-3i} f(z) = \phi_2(-3i) = \dfrac{3(-3i)^3+2}{(-3i-1)(-3i-3i)} = \dfrac{81i+2}{-18+6i}$.

From part a, $\operatorname*{Res}_{z=1} f(z) = \frac{1}{2}$.

$$\int_C \underbrace{f(z)}\, dz \overset{\text{residue thm}}{=} 2\pi i\left(\frac{1}{2} + \frac{2-81i}{-18-6i} + \frac{2+81i}{-18+6i}\right)$$

↑ POSCC ↑ AOLC except @ 3 isolated sing's

$$= 2\pi i\left(\frac{1}{2} + \frac{(2-81i)(-18+6i)+(2+81i)(-18-6i)}{18^2+6^2}\right)$$

$$= 2\pi i\left(\frac{1}{2} + \frac{-36+12i+1458i+486-36-12i-1458i+486}{324+36}\right)$$

$$= 2\pi i\left(\frac{1}{2} + \frac{900}{360}\right) = 2\pi i\left(\frac{1}{2}+\frac{5}{2}\right) = 6\pi i.$$

§81 #6

6). Evaluate the integral $\int_C \frac{\cosh \pi z}{z(z^2+1)}\,dz$ where $C$ is the P.O. circle $|z|=2$.

let $f(z)=$ integrand. There are singularities @ $z=0, \pm i$. all lie within $C$.

Singularity @ $z=0$:

$$f(z)=\frac{\phi(z)}{z}, \quad \phi(z)=\frac{\cosh \pi z}{z^2+1}, \quad \phi(0)=\frac{\cosh 0}{1}=1\neq 0$$

$\Downarrow$

$\phi$ AANZ @ $z=0$.

By pole-residue thm, simple pole @ $z=0$,

$\underset{z=0}{\text{Res}}\, f(z)=\phi(0)=1$.

Singularity @ $z=i$. $f(z)=\frac{\phi(z)}{z-i}$, $\phi(z)=\frac{\cosh \pi z}{z(z+i)}$.

$$\phi(i)=\frac{\cosh \pi i}{i(2i)}$$

$$=\frac{e^{\pi i}+e^{-\pi i}}{2(-2)}=\frac{-1-1}{-4}=\frac{1}{2}.\neq 0 \Rightarrow \phi \text{ AANZ } @ z=i.$$

By pole-residue thm, $\underset{z=i}{\text{Res}}\, f(z)=\frac{1}{2}$.

Singularity @ $z=-i$. $f(z)=\frac{\phi(z)}{z+i}$, $\phi(z)=\frac{\cosh \pi z}{z(z-i)}$.

$$\phi(-i)=\frac{\cosh(-\pi i)}{-i(-2i)}=\frac{\cosh \pi i}{-2i}=\frac{1}{2}=0$$ (Same calculation as above)

$\Downarrow$

$\phi$ AANZ @ $z=-i$.

By pole-residue thm, $\underset{z=-i}{\text{Res}}\, f(z)=\phi(-i)=\frac{1}{2}$.

$$\int_C \underbrace{f(z)}\,dz = 2\pi i\left(1+\frac{1}{2}+\frac{1}{2}\right)=4\pi i.$$

POSCC

AOIC except at finitely many isolated sing's

residue thm

§83 # 1, 3, 5, 6, 8.

$\boxed{\text{Henceforth call the thm 2 in §83 the simple pole-residue thm.}}$

1). Show that the point $z=0$ is a simple pole of the fn:
$f(z) = \csc z = \dfrac{1}{\sin z}$ and that the residue is 1.

$f(z) = \dfrac{p(z)}{q(z)}$, $p(z)=1$, $q(z) = \sin z$, $p, q$ entire
$p(0)=1$, $q(0) = \sin(0) = 0$,
$q'(0) = \cos(0) = 1$

By the simple pole-residue thm, $f$ has a simple pole @ $z=0$ and $\operatorname{Res}_{z=0} f(z) = \dfrac{p(0)}{q'(0)} = \dfrac{1}{1} = 1$.

3.) Show that

a) $\operatorname{Res}_{z=\frac{i\pi}{2}} \dfrac{\sinh z}{z^2\cosh z} = -\dfrac{4}{\pi^2}$.

$f(z) = \dfrac{p(z)}{q(z)}$, $\left.\begin{array}{l} p(z) = \sinh z, \\ q(z) = z^2\cosh z \end{array}\right\}$ Both entire fns.

$p\left(\frac{i\pi}{2}\right) = \dfrac{e^{\frac{i\pi}{2}} - e^{-\frac{i\pi}{2}}}{2} = \dfrac{2i}{2} = i \neq 0$.

$q\left(\frac{i\pi}{2}\right) = \left(\frac{i\pi}{2}\right)^2 \dfrac{e^{\frac{i\pi}{2}} + e^{-\frac{i\pi}{2}}}{2} = \left(\frac{i\pi}{2}\right)^2\left(\frac{i-i}{2}\right) = 0$

$q'\left(\frac{i\pi}{2}\right) = 2\left(\frac{i\pi}{2}\right)\cosh\left(\frac{i\pi}{2}\right) + \left(\frac{i\pi}{2}\right)^2 \sinh\left(\frac{i\pi}{2}\right)$

$= \cancel{2\left(\frac{i\pi}{2}\right)(0)} + \left(\frac{i\pi}{2}\right)^2 (i) = -\dfrac{i\pi^2}{4}$.

By simple pole-residue thm, $f$ has simple pole @ $z = \frac{i\pi}{2}$,
and $\operatorname{Res}_{z=\frac{i\pi}{2}} f(z) = \dfrac{p\left(\frac{i\pi}{2}\right)}{q'\left(\frac{i\pi}{2}\right)} = \dfrac{i}{-\frac{i\pi^2}{4}} = -\dfrac{4}{\pi^2}$.

b) $\operatorname{Res}_{z=\pi i} \dfrac{\exp(zt)}{\sinh z} + \operatorname{Res}_{z=-\pi i} \dfrac{\exp(zt)}{\sinh z} = -2\cos(\pi t)$.

$p(z) = \exp(zt)$, $q(z) = \sinh z$, $p, q$ are entire,
$z_{0_1} = \pi i$, $z_{0_2} = -\pi i$, $q'(z) = \cosh z$.

$p(z_{0_1}) = \exp(\pi i t) \neq 0$.
$q(z_{0_1}) = \sinh(\pi i) = \dfrac{e^{\pi i} - e^{-\pi i}}{2} = \dfrac{-1-(-1)}{2} = 0$.
$q'(z_{0_1}) = \cosh(\pi i) = \dfrac{e^{\pi i} + e^{-\pi i}}{2} = \dfrac{-1-1}{2} = -1$.

§83 #3b, 5, 6, 8.

3b, cont'd.) $p(z_{0_2}) = \exp(-\pi i t) \neq 0.$

$q(z_{0_2}) = \sinh(-\pi i) = -\sinh(\pi i) = 0.$

$q'(z_{0_2}) = \cosh(-\pi i) = \cosh(\pi i) = -1.$

by simple pole-residue thm, $f$ has simple poles @ $\pm\pi i$,

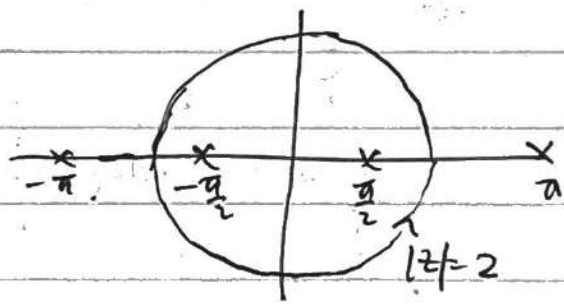
and $\displaystyle \operatorname*{Res}_{z=\pi i} f(z) = \frac{p(\pi i)}{q'(\pi i)} = \frac{e^{\pi i t}}{-1}$

and $\displaystyle \operatorname*{Res}_{z=-\pi i} f(z) = \frac{p(-\pi i)}{q(-\pi i)} = \frac{e^{-\pi i t}}{-1}$

so $\displaystyle \operatorname*{Res}_{z=\pi i} f(z) + \operatorname*{Res}_{z=-\pi i} f(z) = -2\left(\frac{e^{\pi i t} + e^{-\pi i t}}{2}\right) = -2\cos\pi t$

5.) Let $C$ denote the p.o. circle $|z|=2$, and evaluate.

a) $\displaystyle \int_C \tan z \, dz.$




$\cos z$ has zeros @ (only)

$z = (2n+1)\frac{\pi}{2}$, $n \in \mathbb{Z}$

(from §38), thus, only two

enclosed singularities @ $z = \pm\frac{\pi}{2}$.

$\displaystyle f(z) = \frac{p(z)}{q(z)} = \frac{\sin z}{\cos z}$, ~~[illegible]~~ $p = \sin z$, $q = \cos z$, both entire

$p(\frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1$

$q(\frac{\pi}{2}) = \cos(\frac{\pi}{2}) = 0.$ $\quad q'(\frac{\pi}{2}) = -\sin(\frac{\pi}{2}) = -1.$

$p(-\frac{\pi}{2}) = \sin(-\frac{\pi}{2}) = -1.$

$q(-\frac{\pi}{2}) = \cos(-\frac{\pi}{2}) = 0.$ $\quad q'(-\frac{\pi}{2}) = -\sin(-\frac{\pi}{2}) = 1.$

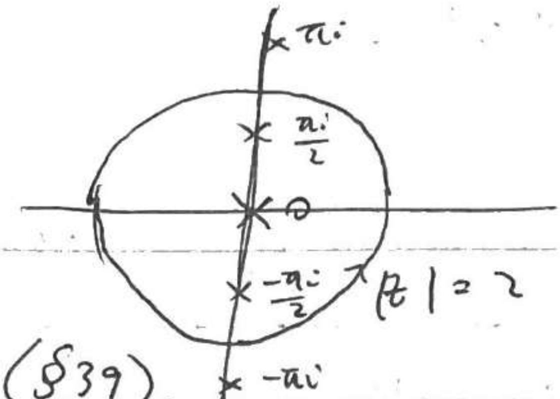
by simple pole residue thm, $f$ has simple poles @ $z = \pm\frac{\pi}{2}$,

and $\displaystyle \operatorname*{Res}_{z=\frac{\pi}{2}} f(z) = \frac{p(\frac{\pi}{2})}{q'(\frac{\pi}{2})} = -1,$

$\displaystyle \operatorname*{Res}_{z=-\frac{\pi}{2}} f(z) = \frac{p(-\frac{\pi}{2})}{q'(-\frac{\pi}{2})} = -1,$

$\displaystyle \underbrace{\int_C}_{\text{pos cc}} \underbrace{f(z)\,dz}_{\text{ADIC except two interior points}} \underset{\text{residue thm}}{=} 2\pi i(-1 + (-1)) = -4\pi i$

§83 #5b, 6, 8.




5b) $\int_C \frac{dz}{\sinh 2z}$

let $f(z)$ be integrand.

$\sinh z$ only has zeros @ $z = n\pi i$ (§39),

so $f(z)$ has sing's @ $z = \frac{n\pi i}{2}$, only enclosed singularities are @ $z = 0, \pm \frac{\pi i}{2}$.

$f(z) = \frac{p(z)}{q(z)}$, $p(z) = 1$, $q(z) = \sinh 2z$, $p, q$ entire,

$z_{0_1} = 0$, $z_{0_2} = \frac{\pi i}{2}$, $z_{0_3} = -\frac{\pi i}{2}$, $q'(z) = 2\cosh 2z$.

@ $z_{0_1}$: $p(z_{0_1}) = 1 \neq 0$,

$q(z_{0_1}) = \sinh 0 = 0$, $q'(z_{0_1}) = 2\cosh(0) = 2$.

@ $z_{0_2}$: $p(z_{0_2}) = 1 \neq 0$

$q(z_{0_2}) = \sinh(\pi i) = 0$, $q'(z_{0_2}) = 2\cosh \pi i = 2(-1) = -2$

@ $z_{0_3}$: $p(z_{0_3}) = 1 \neq 0$

$q(z_{0_3}) = \sinh(-\pi i) = 0$, $q'(z_{0_3}) = 2\cosh(-\pi i) = -2$.

by simple pole residue thm, $f$ has simple poles @ these singularities, and: $\operatorname{Res}_{z=0} f(z) = \frac{p(0)}{q'(0)} = \frac{1}{2}$

$\operatorname{Res}_{z=\frac{\pi i}{2}} f(z) = \frac{p(\frac{\pi i}{2})}{q'(\frac{\pi i}{2})} = -\frac{1}{2}$, $\operatorname{Res}_{z=-\frac{\pi i}{2}} = \frac{p(-\frac{\pi i}{2})}{q'(-\frac{\pi i}{2})} = -\frac{1}{2}$.

$$\underset{\text{POSCC}}{\int_C} \underbrace{f(z)}_{\text{AOLC except at 3 enclosed points}} dz \overset{\text{residue thm}}{=} 2\pi i \left(\cancel{\tfrac{1}{2} - \tfrac{1}{2}} - \tfrac{1}{2}\right) = -\pi i.$$

---

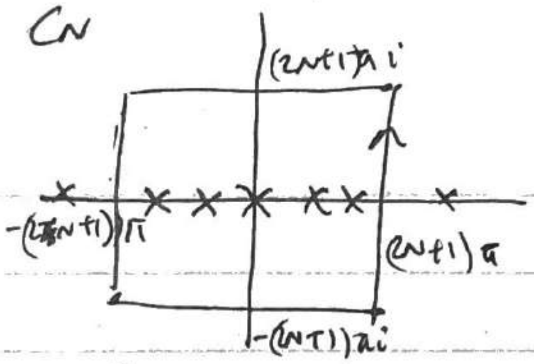
6.) Let $C_N$ denote the P.O. boundary of the square whose edges live along the lines $x = \pm\left(N + \frac{1}{2}\right)\pi$, $y = \pm\left(N + \frac{1}{2}\right)\pi$,

where $N$ is a positive integer. Show that

$$\int_{C_N} \frac{dz}{z^2 \sin z} = 2\pi i \left[\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}\right]$$

and then show that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$ in the limit as $N \to \infty$.

§83 #6, 8.




case when $N=2$.

6, cont'd. Let integrand $= f(z)$.

denominator $= z^2 \sin z$.

$z^2$ has a zero @ $z=0$

$\sin z$ has zeros @ $z = n\pi$ (§38).

Thus $f$ has $(2n+1)$ enclosed zeros

at $-N\pi, -(N-1)\pi, \ldots, (N-1)\pi, N\pi$.

$$f(z) = \frac{p(z)}{q(z)}, \quad p(z) = 1, \quad q(z) = z^2 \sin z, \quad p, q \text{ entire}.$$

$$p(n\pi) = 1 \neq 0, \quad q(n\pi) = (n\pi)^2 \sin(n\pi) = (n\pi)^2(0) = 0,$$

$$q'(n\pi) = 2(n\pi)\sin(n\pi) + (n\pi)^2 \cos n\pi$$

$$= \cancel{2(n\pi)(0)} + (n\pi)^2(-1)^n \neq 0 \text{ if } n \neq 0.$$

simple pole residue thm applies for nonzero singularities,

at which $\displaystyle\operatorname*{Res}_{\substack{z=n\pi,\\ n\neq 0}} f(z) = \frac{p(n\pi)}{q'(n\pi)} = \frac{1}{(n\pi)^2(-1)^n}$.

for sing @ $z=0$: use long div.

$$f(z) = \frac{1}{z^2 \sin z} = \frac{1}{z^2\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right)}$$

$$= \begin{array}{r|l} & z^{-3} + \frac{1}{3!}z^{-1} + \cdots \\ \hline z^3 - \frac{z^5}{3!} + \frac{z^7}{5!} - \cdots & 1 \\ & 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \\ \hline & \frac{z^2}{3!} - \frac{z^4}{5!} - \cdots \\ & \frac{z^2}{3!} - \cdots \\ \hline \end{array}$$

$$\operatorname*{Res}_{z=0} f(z) = b_1 = \frac{1}{6}.$$

$$\int_C \underbrace{f(z)}_{} \, dz \overset{\text{residue thm}}{=} 2\pi i \left[ \frac{1}{6} + \sum_{\substack{n=-N \\ n\neq 0}}^{N} \frac{1}{n^2\pi^2(-1)^n} \right]$$

POSCC — ADIC except @ finitely many interior pt's

$$= 2\pi i \left[ \frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2} \right].$$

§83 #6, 8.

6, cont'd.) From §47 #8, $\left|\int_{C_N} \frac{dz}{z^2 \sin z}\right| \le \frac{16}{(2N+1)\pi A}$,

so $\lim_{N\to\infty} \left|\int_{C_N} \frac{dz}{z^2 \sin z}\right| \le \lim_{N\to\infty} \frac{16}{(2N+1)\pi A} = 0$

$\Rightarrow \lim_{N\to\infty} \int_{C_N} \frac{dz}{z^2 \sin z} = 0$.

Thus $\lim_{N\to\infty}\left[\int_{C_N} f(z)\,dz\right] = 0 = \lim_{N\to\infty}\left[2\pi i\left(\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}\right)\right]$

$\Rightarrow -\frac{1}{6} = \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$.

---

8.) Consider the function $f(z) = \frac{1}{(q(z))^2}$ where $q$ is analytic at $z_0$, and $q(z_0) = 0$, $q'(z_0) \neq 0$. Show that $z_0$ is a pole of order $m = 2$ of $f$, with residue $B_0 = -\frac{q''(z_0)}{(q'(z_0))^3}$

---

By definition, $z_0$ is a pole of order 1 of $q$, so $q = (z - z_0)g(z)$, where $g$ is AANZ @ $z_0$.

Thus $f(z) = \frac{\phi(z)}{(z-z_0)^2}$, where $\phi(z) = \frac{1}{(g(z))^2}$, and $\phi(z)$ is clearly AANZ @ $z_0$. (since $g$ is AANZ @ $z_0$).

Thus by the pole-residue thm, $f$ has a pole of order 2 @ $z_0$, and $\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi'(z_0)}{1!} = \frac{-2g'(z_0)}{(g(z_0))^3}$. (*)

---

$q'(z) = \frac{d}{dz}((z-z_0)g(z)) = g(z) + (z-z_0)g'(z)$.

(**) $q'(z_0) = g(z_0) + \cancel{(z_0-z_0)g'(z_0)}$

$q''(z) = \frac{d}{dz}(g(z) + (z-z_0)g'(z)) = g'(z) + g'(z) + (z-z_0)g''(z)$

(***) $q''(z_0) = 2g'(z_0) + \cancel{(z_0-z_0)g''(z_0)}$

Substituting (**) and (***) into (*), we get the desired result.

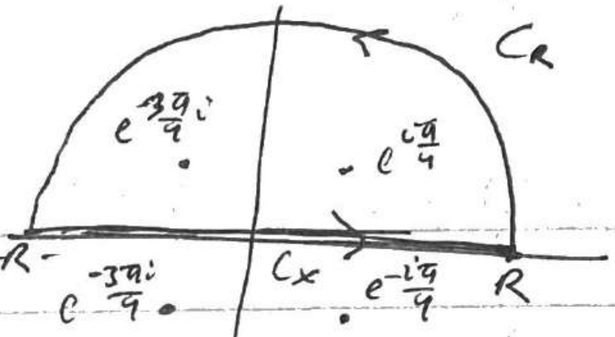
$B_0 = \frac{-2g'(z_0)}{(g(z_0))^3} = \frac{-q''(z_0)}{(q(z_0))^3}$.

§86 #3, 6, 9.

Show:

3). $\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$.

let $f(z) = \frac{1}{z^4+1}$.




f has four singularities @ $z = (-1)^{\frac{1}{4}}$, two lie above the x-axis - enclose them with a semicircular path such as that on the right.

let $C_x$ be the flat part of the contour, $C_R$ be the curved part, $C = C_x + C_R$ is the SCC of integration, $(R > 1)$.

(*)

$$\underset{\substack{\uparrow \\ \text{SCC}}}{\int_C} \underbrace{f(z)}_{\substack{\uparrow \\ \text{ACIC except for} \\ \text{two internal points}}} dz \underset{\substack{\uparrow \\ \text{residue thm}}}{=} 2\pi i \sum_{\substack{\text{enclosed} \\ \text{sing's}}} \text{Res} f(z) = \int_{-R}^{R} \frac{dx}{x^4+1} + \int_{C_R} f(z)\,dz$$

---

$f(z) = \frac{p(z)}{q(z)}$, $p = 1$, $q(z) = z^4+1$, $p, q$ entire.
let $z_{01} = e^{\frac{i\pi}{4}}$, $z_{02} = e^{\frac{3i\pi}{4}}$, $q'(z) = 4z^3$.

$p(z_{01}) = p(z_{02}) = 1$.

$q(z_{01}) = q(z_{02}) = 0$.

$q'(z_{01}) = 4z_{01}^3$, $q'(z_{02}) = 4z_{02}^3$.

By simple pole-residue thm, $f$ has simple poles @ $z = z_{01}, z_{02}$, and $\underset{z=z_{01}}{\text{Res}} f(z) = \frac{p(z_{01})}{q'(z_{01})} = \frac{1}{4z_{01}^3} = \frac{z_{01}}{4z_{01}^4} = -\frac{1}{4} z_{01}$

Similarly, $\underset{z=z_{02}}{\text{Res}} f(z) = -\frac{1}{4} z_{02}$.

and $\sum_{\substack{\text{enclosed} \\ \text{sing's}}} \text{Res} f(z) = -\frac{1}{4}(z_{01} + z_{02}) = -\frac{1}{4}\left(\cancel{\frac{1}{\sqrt{2}}} + \frac{1}{\sqrt{2}}i + \cancel{-\frac{1}{\sqrt{2}}} + \frac{1}{\sqrt{2}}i\right)$

(**) $= -\frac{1}{4}\left(\frac{2}{\sqrt{2}}i\right) = -\frac{i}{2\sqrt{2}}$

---

For any point on $C_R$, $|f(z)| = \left|\frac{1}{z^4+1}\right| = \frac{1}{|z^4+1|} \underset{\substack{\uparrow \\ \Delta 2}}{\leq} \frac{1}{\left||z^4| - |1|\right|}$

$= \frac{1}{|z|^4 - 1} = \frac{1}{R^4-1} = M_R$.

§8.6 #3, 6, 9.

3, cont'd). length of $C_R = \pi R, = L_R$

By ML-inequality, $\left|\int_{C_R} f(z)\,dz\right| \le M_R L_R = \pi R \cdot \frac{1}{R^4-1}$

$$\lim_{R\to\infty}\left|\int_{C_R} f(z)\,dz\right| \le \lim_{R\to\infty} \frac{\pi R}{R^4-1} = 0.$$

$$\Rightarrow \lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0. \quad ***$$

Taking the limit of * as $R\to\infty$, and substituting in ** and ***

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{x^4+1} = 2\pi i\left(-\frac{i}{2\sqrt{2}}\right) - \cancel{\lim_{R\to\infty}\int_{C_R} f(z)\,dz}^{\,\to 0}$$

$$\int_{-\infty}^{\infty}\frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}.$$

Since $\frac{1}{x^4+1}$ is even (quotient of two even polynomials),

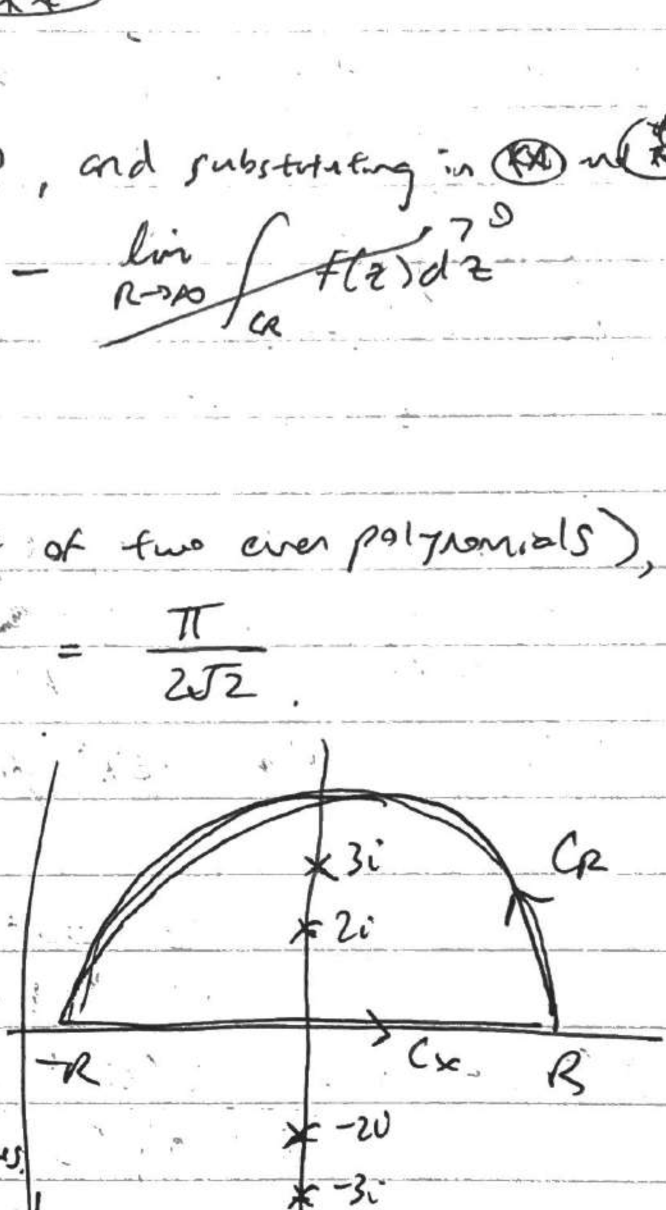
$$\int_0^{\infty}\frac{dx}{x^4+1} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}.$$

6). $\displaystyle\int_0^{\infty}\frac{x^2\,dx}{(x^2+9)(x^2+4)^2} = \frac{\pi}{200},$

$$f(z) = \frac{z^2}{(z^2+9)(z^2+4)^2}.$$

f has singularities @ $z = \pm 3i, \pm 2i$.

C encloses only the positive-$i$ values.




$$\int_C f(z)\,dz = 2\pi i \sum \text{Res } f(z)$$

↑ PosCC; ↑ AOIC except @ 2 enclosed singular pts; ↑ residue thm; enclosed sing's.

again, $C = C_x + C_R$, $R > 3$

$$= \int_{-R}^{R}\frac{x^2\,dx}{(x^2+9)(x^2+4)^2} + \int_{C_R} f(z)\,dz. \quad *$$

§8.6 #6, 9.

6, cont'd.) $\cdot$ Singularity @ $z = 3i$:

$$f = \frac{\frac{z^2}{(z+3i)(z^2+4)^2}}{z-3i} = \frac{\phi(z)}{z-3i}, \quad \phi(z) = \frac{z^2}{(z+3i)(z^2+4)^2}.$$

$$\phi(3i) = \frac{(3i)^2}{(3i+3i)(3i^2+4)^2} = \frac{3i}{2(\cancel{3i})(-9+4)^2} = \frac{3i}{50} \neq 0.$$

$\Downarrow$

$\phi$ AANZ @ $z = 3i$

By pole-residue thm, $f$ has simple pole @ $z = 3i$,

and $\operatorname{Res}_{z=3i} f(z) = \phi(3i) = \frac{3i}{50}$.

Singularity @ $z = 2i$.

$$f = \frac{\frac{z^2}{(z^2+9)(z+2i)^2}}{(z-2i)^2} = \frac{\phi(z)}{(z-2i)^2}, \quad \phi(z) = \frac{z^2}{z^4+4iz^3+5z^2+36iz-36}$$

$\downarrow$

AANZ @ $z = 2i$.

By pole-residue thm, $f$ has double pole @ $z = 2i$,

and $\operatorname{Res}_{z=2i} f(z) = \frac{\phi'(2i)}{1!} = \phi'(2i)$.

$$\phi'(z) = \frac{(z^4+4iz^3+5z^2+36iz-36)(2z) - z^2(4z^3+12iz^2+10z+36i)}{((z^2+9)(z+2i)^2)^2}$$

$$= \frac{-2z^5-4iz^4+36iz^2-72z}{((z^2+9)(z+2i)^2)^2}$$

$$\phi'(2i) = \frac{-2(2i)^5-4i(2i)^4+36i(2i)^2-72(2i)}{(((2i)^2+9)(2i+2i)^2)^2}$$

$= \ldots$ ~~algebra~~ arithmetic $\ldots = \frac{-13i}{200} = \operatorname{Res}_{z=2i} f(z)$.

$$\sum_{\text{sing's}} \operatorname{Res} f(z) = \frac{3i}{50} - \frac{13i}{200} = \frac{-i}{200} \quad (**).$$

On $C_R$: $|f(z)| = \left|\frac{z^2}{(z^2+9)(z^2+4)^2}\right| = \frac{|z|^2}{|z^2+9|\,|z^2+4|^2}$

$$\underset{\substack{\uparrow \\ \Delta_2 \text{ twice}}}{\leq} \frac{R^2}{(R^2-9)(R^2-4)^2} = M.$$

length of $C_R = \pi R = L$.

§86 #6, 9

6, cont'd.) By ML inequality: $\left| \int_{C_R} f(z)\,dz \right| \le ML = \frac{\pi R^3}{(R^2-9)(R^2-4)^2}$

$$\lim_{R\to\infty} \left| \int_{C_R} f(z)\,dz \right| = \lim_{R\to\infty} \frac{\pi R^3}{(R^2-9)(R^2-4)^2} = 0.$$

$$\Rightarrow \lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0. \quad (\ast\ast\ast).$$

Take limit of $(\ast)$ as $R\to\infty$ and substitute in $(\ast\ast)$ and $(\ast\ast\ast)$.

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{x^2\,dx}{(x^2+9)(x^2+4)^2} = 2\pi i\left(\frac{-i}{200}\right) - \cancel{\lim_{R\to\infty} \int_{C_R} f(z)\,dz}^{\,0}$$

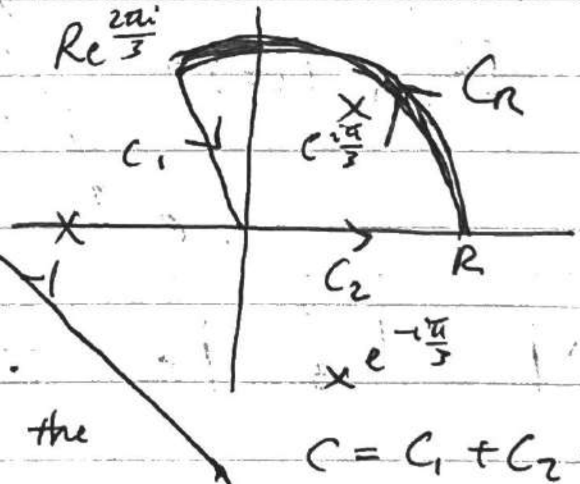
$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+9)(x^2+4)^2} = \frac{\pi}{100}.$$

Since integrand is even (quotient of two even polynomials):

$$\int_0^{\infty} \frac{x^2\,dx}{(x^2+9)(x^2+4)^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+9)(x^2+4)^2} = \frac{\pi}{200}.$$

9). Use a residue and the contour below, where $R > 1$, to establish the integration formula:

$$\int_0^{\infty} \frac{dx}{x^3+1} = \frac{2\pi}{3\sqrt{3}}.$$




$f(z) = \frac{1}{z^3+1}$

has 3 singularities: $z = (-1)^{\frac{1}{3}}$.

Only one is enclosed within the loop: $z = e^{\frac{i\pi}{3}}$. As before:

$C = C_1 + C_2 + C_R$, $R > 1$

$$\underset{\uparrow\ \text{POSCC}}{\int_C} \underbrace{f(z)}_{\text{AOIC except at one singularity inside } C}\,dz \underset{\text{residue thm}}{=} 2\pi i \sum_{\text{enclosed sing.}} \operatorname{Res} f(z)$$

$$(\ast) \quad = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_R} f(z)\,dz.$$

§86 #9.

9, cont'd.) $C_1$: $z(r) = re^{\frac{2\pi i}{3}}$, $0 \le r \le R$

$z'(r) = e^{\frac{2\pi i}{3}} dr$.

$$\int_{C_1} f(z)\,dz = \int_R^0 \frac{1}{(re^{\frac{2\pi i}{3}})^3+1} e^{\frac{2\pi i}{3}} dr$$

$$= -e^{\frac{2\pi i}{3}} \int_0^R \frac{1}{r^3 \underbrace{(e^{2\pi i})}_{=1} + 1} dr = -e^{\frac{2\pi i}{3}} \int_0^R \frac{dx}{x^3+1}.$$

Also, it is clear that:

$$\int_{C_2} f(z)\,dz = \int_0^R \frac{dx}{x^3+1}, \quad \text{so:}$$

$$\int_{C_1} f(z)dz + \int_{C_2} f(z)\,dz = \left(1 - e^{\frac{2\pi i}{3}}\right) \int_0^R \frac{dx}{x^3+1}. \quad (**)$$

We have to find the residue @ $z = e^{\frac{i\pi}{3}}$.

$f(z) = \frac{p(z)}{q(z)}$, $p(z) = 1$, $q(z) = z^3+1$, $p, z$ ative,

$p(e^{\frac{i\pi}{3}}) = 1$, $q(e^{\frac{i\pi}{3}}) = -1+1 = 0$, $q'(e^{\frac{i\pi}{3}}) = 3(e^{\frac{i\pi}{3}})^2$

By simple-pole-residue thm, $f$ has a simple pole @ $z = e^{\frac{i\pi}{3}}$, and

$$\operatorname*{Res}_{z=e^{\frac{i\pi}{3}}} f(z) = \frac{p(e^{\frac{i\pi}{3}})}{q'(e^{\frac{i\pi}{3}})} = \frac{1}{3(e^{\frac{i\pi}{3}})^2} = \frac{e^{\frac{i\pi}{3}}}{3(e^{\frac{i\pi}{3}})^3}$$

$$= -\frac{1}{3} e^{\frac{i\pi}{3}}. \quad (***)$$

on $C_R$:

$$|f(z)| = \left|\frac{1}{z^3+1}\right| = \frac{1}{|z^3+1|} \underset{\Delta_2}{\le} \frac{1}{|z|^3-1} = \frac{1}{R^3-1} = M$$

length of $C_R = \frac{2}{3}\pi R$. By ML-inequality:

$$\left|\int_{C_R} f(z)\,dz\right| \le ML = \frac{\frac{2}{3}\pi R}{R^3-1}$$

$$\lim_{R\to\infty} \left|\int_{C_R} f(z)\,dz\right| \le \lim_{R\to\infty} \frac{\frac{2}{3}\pi R}{R^3-1} = 0.$$

$$\Rightarrow \lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0. \quad (****)$$

§86 #9.

9, cont'd.)

Combining ✱, ✱✱, ✱✱✱, and ✱✱✱✱ and taking the limit as $R \to \infty$:

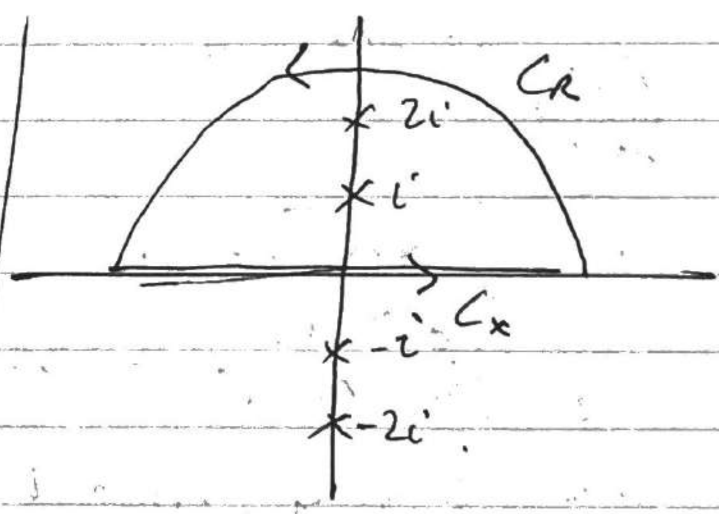
$$\frac{\lim_{R\to\infty}\left(1-e^{\frac{2\pi i}{3}}\right)\int_0^R \frac{dx}{x^3+1}}{1-e^{\frac{2\pi i}{3}}} = \frac{2\pi i\left(-\frac{1}{3}e^{\frac{i\pi}{3}}\right)}{1-e^{\frac{2\pi i}{3}}} - \cancel{\lim_{R\to\infty}\int_{C_R} f(z)\,dz}^{\,0}$$

$$\int_0^\infty \frac{dx}{x^3+1} = \frac{-\frac{2}{3}\pi i\, e^{\frac{i\pi}{3}}}{1-e^{\frac{2\pi i}{3}}}\cdot\left(\frac{1-e^{-\frac{2\pi i}{3}}}{1-e^{-\frac{2\pi i}{3}}}\right)$$

$$= \frac{-\frac{2}{3}\pi i\left(e^{\frac{i\pi}{3}}-e^{-\frac{i\pi}{3}}\right)}{\left(1-\left(-\frac{1}{2}\right)\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{\frac{4\pi}{3}\left(\frac{e^{\frac{i\pi}{3}}-e^{-\frac{i\pi}{3}}}{2i}\right)}{3} = \frac{4\pi}{9}\sin\frac{\pi}{3} = \frac{\cancel{4}^{\,2}\pi}{\cancel{9}_{\,3\sqrt{3}}}\cdot\frac{\cancel{\sqrt{3}}}{\cancel{2}} = \frac{2\pi}{3\sqrt{3}}$$

§88 #6. $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin x\,dx}{(x^2+1)(x^2+4)}$.




let $f(z) = \dfrac{ze^{iz}}{(z^2+1)(z^2+4)}$

two enclosed singularities: @ $z = i, 2i$.

$C = C_x + C_R$

$R > 2$.

As before:

$$\int_C \underbrace{f(z)}_{\text{A.O. i/C except two interior singular pts}}\,dz \overset{\text{residue thm}}{=} 2\pi i\sum_{\text{sing's}}\operatorname{Res} f(z)$$

↑ POSCCC

$$= \int_{-R}^{R}\frac{xe^{ix}\,dx}{(x^2+1)(x^2+4)} + \int_{C_R} f(z)\,dz. \quad ✱$$

§88 #6.

6, cont'd.)

$f(z) = \frac{p(z)}{q(z)}$, $p(z) = ze^{iz}$

$q(z) = z^4 + 5z^2 + 4. = (z^2+1)(z^2+4).$

$p, q$ entire ; $q'(z) = 4z^3 + 10z.$

$z_{0_1} = i,\ z_{0_2} = 2i.$

$p(i) = ie^{i^2} = ie^{-1} \neq 0.$

$p(2i) = 2ie^{(2i)^2} = 2ie^{-2} \neq 0.$

$q(i) = 0,\ q(2i) = 0.$

$q'(i) = 6i,\ q'(2i) = -12i.$

By simple-pole-residue thm, $f$ has simple poles @ $z = i, 2i$, and

$$\operatorname*{Res}_{z=i} f(z) = \frac{p(i)}{q'(i)} = \frac{\cancel{i}e^{-1}}{6\cancel{i}} = \frac{1}{6e},$$

$$\operatorname*{Res}_{z=2i} f(z) = \frac{p(2i)}{q'(2i)} = \frac{2\cancel{i}e^{-2}}{-12\cancel{i}} = \frac{-1}{6e^2}$$

$$\sum_{\substack{\text{enclosed}\\ \text{sing's}}} \operatorname{Res} f(z) = \frac{1}{6e} - \frac{1}{6e^2} = \frac{e-1}{6e^2} \quad (**).$$

Plugging (**) into (*):

$$\int_{-R}^{R} \frac{xe^{ix}\,dx}{(x^2+1)(x^2+4)} = 2\pi i\left(\frac{e-1}{6e^2}\right) - \int_{C_R} f(z)\,dz$$

taking imaginary parts of both sides:

$$\int_{-R}^{R} \frac{x\sin x\,dx}{(x^2+1)(x^2+4)} = \frac{2\pi}{6e^2}(e-1) - \operatorname{Im}\int_{C_R} f(z)\,dz. \quad (***)$$

$g(z) = \frac{z^2}{(z^2+1)(z^2+4)}$ is analytic in the upper half plane, exterior to the circle $|z| = 2$.

$$|g(z)| = \left|\frac{z^2}{(z^2+1)(z^2+4)}\right| = \frac{|z^2|}{|z^2+1||z^2+4|} \underset{\Delta_2 \text{ and some simplification.}}{\leq} \frac{R^2}{(R^2-1)(R^2-4)} = M_R

## §88 #6.

6, cont'd.

$$\lim_{R\to\infty} M_R = \lim_{R\to\infty} \frac{R^2}{(R^2-1)(R^2-4)} = 0.$$

By Jordan's thm: $\lim_{R\to\infty}\int_{C_R} g(z)e^{iz}\,dz = \lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0.$

$$\Rightarrow \lim_{R\to\infty}\left(\operatorname{Im}\int_{C_R} f(z)\,dz\right) \le \lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0$$

$$\Rightarrow \lim_{R\to\infty}\left[\operatorname{Im}\int_{C_R} f(z)\,dz\right] = 0. \quad (****)$$

Plugging in (****) into (***) and taking the limit as $R\to\infty$:

$$\lim_{R\to\infty}\int_{-R}^{R} \frac{x\sin x\,dx}{(x^2+1)(x^2+4)} = \frac{2\pi}{6e^2}(e-1) - \cancel{\lim_{R\to\infty}\left[\operatorname{Im}\int_{C_R} f(z)\,dz\right]}$$

$$\int_{-\infty}^{\infty} \frac{x\sin x\,dx}{(x^2+1)(x^2+4)} = \frac{\pi(e-1)}{3e^2}.$$

## §92 #1.

1). Show that: $\int_0^{2\pi}\frac{d\theta}{5+4\sin\theta} = \frac{2\pi}{3}.$

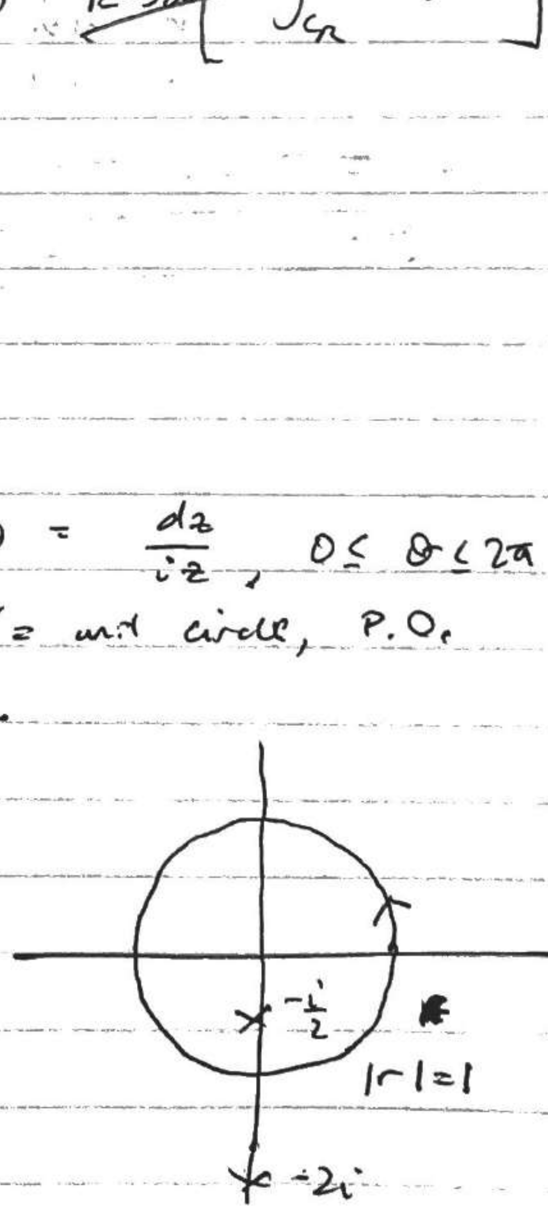
let $\sin\theta = \frac{z-z^{-1}}{2i}$, $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$, $0\le\theta<2\pi$

C = unit circle, P.O.

$$\int_0^{2\pi}\frac{d\theta}{5+4\sin\theta} = \int_C \frac{dz}{iz\left(5+4\left(\frac{z-z^{-1}}{2i}\right)\right)}$$

$$= \int_C \frac{dz}{5iz+2z^2-2}$$

$$= \int_C \frac{2\,dz}{(z+\frac{i}{2})(z+2i)}$$




let $f(z)$ be the integrand.

$f$ has only one singularity enclosed within $C$, $z = -\frac{i}{2}$.

## §92 #1

1, cont'd

$$f(z) = \frac{p(z)}{q(z)}, \qquad \left.\begin{array}{l} p(z) = 1, \\ q(z) = 2z^2 + 5iz - 2 \end{array}\right\} \text{entire functions}$$

$p\left(-\frac{i}{2}\right) = 1 \neq 0$ | $q'(z) = 4z + 5i$

$q\left(-\frac{i}{2}\right) = 0$

$q'\left(-\frac{i}{2}\right) = 3i.$

By simple pole residue thm, $\operatorname*{Res}_{z=-\frac{i}{2}} f(z) = \dfrac{p\left(\frac{-i}{2}\right)}{q'\left(\frac{-i}{2}\right)}$

$= \dfrac{1}{3i}$

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residue thm

$$\int_C \underbrace{f(z)}_{\text{AOIC except at one interior singular pt}} dz = 2\pi i \sum_{\text{sing's}} \operatorname{Res} \frac{f(z)}{2z} = 2\pi \cancel{i}\left(\frac{1}{3\cancel{i}}\right)$$

$$= \frac{2\pi}{3}.$$

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