Alg-18

Let G act on a set S, $\pi : G \to \mathrm{Perm}(S)$

For $s, t \in S$, define $s \underset{\pi}{\sim} t \iff$

$t = xs = \pi_x(s) = \pi(x)(s)$

Then $\underset{\pi}{\sim}$ is an equivalence rel on S

$[s] = \{t \in S : t \underset{\pi}{\sim} s\} = \mathcal{O}(s) = Gs$

Orbit of s

$[s] \subseteq S$

$S/_{\sim} = \{[s] : s \in S\} = \{Gs : s \in S\}$

$\equiv S/G.$

Thm $|Gs| = (G : G_s)$

isotropy or stabilizer of $s \in S$.

$G_s \leq G \quad G_s = \{x \in G : xs = s \;\; \pi_x(s) = s\}$

For $s \in S$ fixed

$f : G \to S$

$f(x) = xs$

f induces a map $\bar{f} : G/G_s \to Gs$

by $\bar{f}(xG_s) = xs = f(x)$

by $\bar{f}(xG_s) = xs = f(\cdots)$

$\bar{f}$ is bijective. Ef $t \in G_s$

Then $t = xs$ for some $x \in G$.

$\bar{f}(xG_s) = xs = t \in G_s \Rightarrow$

$\bar{f}$ is surjective.

$\bar{f}$ is injective:

suppose $\bar{f}(xG_s) = \bar{f}(yG_s)$

$$\Rightarrow xs = ys$$

$$\Rightarrow x^{-1}ys = s$$

$$\Rightarrow x^{-1}y \in G_s$$

$$\Rightarrow xG_s = yG_s$$

thus $\bar{f}$ is bijection

G_s & the set of left cosets G/G_s have the same number of elements i.e.

$$|G_s| = |G/G_s| = (G : G_s)$$

index

$$S = \bigcup_{s \in S} [s]_\pi$$

$$= \bigcup_{s \in S} G_s$$

$G_s \cap G_t = \phi$

$$\boxed{G s = G t \quad \text{or} \quad G s \cap G t = \phi \quad \text{for } s, t \in G}$$

Thm S is finite, then

$$|S| = \sum_{i=1}^{m} |G s_i| \quad \text{where}$$

$s_1, .. s_m$ s.t $\{G s_i\}_{i=1}^{m}$ are distinct

$$|S| = \sum_{i=1}^{m} (G : G_{s_i})$$

Thm G is a finite group acting on itself by conjugation

$$\left(\begin{array}{l} \pi : G \xrightarrow{\text{homo}} \mathrm{Perm}(G) \\ \pi(x)(a) = x a x^{-1}, \quad a \in G \end{array} \right.$$

$a \in S = G \qquad Ga = \{ s \in G : s \underset{c}{\sim} a \quad s = x a x^{-1} \text{ for } x \in G \}$

$= [a]_c = \{ x a x^{-1} : x \in G \}$

$= c(a)$ not a subgroup

is an orbit of $a \in S = G$

$$G_a = \{ x \in G : \underset{\underset{x a x^{-1}}{\shortparallel}}{x.a} = a \}$$

$= C(a)$ — centralizer

$= C(a)$ ... — centralizer of a in G

& is a subgroup of G

$|c(a)| = (G : C(a))$

Thm $H \leq G$. S = the set of all subgroups of G

$H \in S$, $K \in S$

$\pi : G \longrightarrow Perm(S)$

by $\pi(x) \in Perm(S)$

given by $\pi(x)(H) = x \cdot H$
$= xHx^{-1}$

π is a homo

G acts on the set of subgroups of by by conjugation.

$H \underset{c}{\sim} K \iff K = xHx^{-1}$

$[H]_c$ = orbit of H in S

$= G_H = \{x \cdot H : x \in G\}$

$= \{xHx^{-1} : x \in G\}$

conjugacy class of H.

$|[H]| = |G_H| = (G : G_H)$

$$|[H]_c| = |G_H| \overset{\text{orbit}}{} = (G : G_H)$$

$$G_H = \{x \in G : x.H = H\}$$

$$= \{x \in G : xHx^{-1} = H\}$$

$$= N_G(H) \text{ — normalizer of } H \text{ in } G$$

$$\leq G.$$

$$H \trianglelefteq N_G(H)$$

Suppose G is a finite group & acts on itself by conjugation.

$$G = S = \dot{\bigcup_{i=1}^{m}} Ga_i$$

$$\Rightarrow |G| = |S| = \sum_{i=1}^{m} |Ga_i|,$$

$$= \sum_{i=1}^{m} |c\ell(a_i)|$$

$$= \sum_i (G : G_{a_i})$$

$$= \sum_{i=1}^{m} (G : C(a_i))$$

Now $(G : C(a)) = 1 \iff a \in Z(G)$

$G/C(a) = \{ yC(a) : y \in G \}$

$C(a) = G \quad x \in G \Rightarrow xa = ax$

$x \in G, \ x \in C(a)$

$\iff a = xax^{-1}$

$\iff ax = xa \quad \forall x \in G$

$a \in Z(G).$

$Z(G) = \{ a \in G : (G : C(a)) = 1 \}$

let $a_1 \ldots a_r \in G : \quad (G : C(a_i)) > 1$
$i = 1 \ldots r$

$$|G| = \sum_{i=1}^{m} (G : C(a_{n_i}))$$

$$= |Z(G)| + \sum_{j=1}^{r} (G : C(a_i))$$

class eqn.

Cayley Thm (1854)

Every G is isomorphic to a subgroup of $\mathrm{Perm}(S)$ for some S.

Proof Take $S = G$. For each $x \in G$, define $T_x : G \to G$

by $T_x(a) = xa$,

claim $T_x \in \text{Perm}(G)$

$y \in G$ taking $a = x^{-1}y$

we get $T_x(a) = T_x(x^{-1}y)$

$= x(x^{-1}y)$

$= y$

So surjective.

Suppose $T_x(a) = T_x(b)$

$\Rightarrow xa = xb$

$\Rightarrow a = b$ by left cancellation law

Also $T_x \circ T_y = T_{xy}$ ... (1)

$T_e = I_S = I_G$ ... (2)

$T : G \to \text{Perm}(G) = \text{Perm}(S)$

$T(x) = T_x$ is a homo by (1)

$x \in \ker T \Leftrightarrow I_G = T(x) = T_x \Rightarrow$

$I(a) = T_x(a) \quad \forall a \in G.$

$I_G(a) = T_x(a) \quad \forall a \in G.$

$a = xa \quad \forall a \in G$

Tomj $a=e$

$\Rightarrow x = e$

$\ker T = \{e\}$. T is injective

homo $\quad T(G) \leq Perm(G)$

& by 1st iso thm.

$G \;=\; G/\ker T \simeq T(G) \leq Perm(G)$

G. K. Pedersen: C^*-algebras and their automorphism groups

Pascal

Thm Generalized Cayley's Thm. $H \leq G$. Then there is a homo $T: G \to Perm(G/H)$

& s.t. (1) $\ker T \subseteq H$

(2) $\ker T$ is the largest normal subgroup contained in H:

$N \trianglelefteq G$, $N \subseteq H$. then $N \subseteq \ker T$.

$N \subseteq \ker \,.$

Proof. For each $x \in G$, define $T_x : G/H \to G/H$

by $T_x(aH) = (xa)H \in G/H$

$T_x \circ T_y = T_{xy}$

$T_x \in Perm(G/H)$

$T_x(aH) = T_x(bH)$

$\Rightarrow (xa)H = (xb)H$

$\Rightarrow (xb)^{-1}(xa) \in H$

$\Rightarrow b^{-1}a \in H$

$\Rightarrow aH = bH$

T_x is injective

Suppose $z \in G/H$

$z = cH$, for some $c \in G$

$T_x(x^{-1}c\,H) = x(x^{-1}c)H$
$= cH$
$= z$

surjective.

The map $T : G \to Perm(G/H)$

defined by $T(x) = T_x$ is a homo

$T(xy) = T_{xy} = T_x \circ T_y = T(x)T(y)$

$T(x) = T_x$

$T(xy) = T_{xy} = T_x \circ T_y = T(x)T(y)$

$x \in \ker T \iff T_x = T(x) = I_{G/H}$ – id on G/H

$\iff T_x(aH) = I_{G/H}(aH)$

$\iff (xa)H = aH \quad \forall a$

Taking $a = e$, $xH = H \Rightarrow x \in H$

So $\ker T \subseteq H$.

Suppose $N \trianglelefteq G$ and $\underline{N \subseteq H}$.

To show $N \subseteq \ker T$

First observe that

$$\ker T = \bigcap_{a \in G} aHa^{-1} \subseteq H.$$

$x \in \ker T \iff T_x = T(x) = I_{G/H}$

$T_x(aH) = aH \quad \forall a \in G$

$\iff xaH = aH$

$\iff a^{-1}xaH = H$

$\Rightarrow a^{-1}xa \in H$

$\Rightarrow x \in aHa^{-1} \quad \forall a \in G$

$\Rightarrow x \in aHa^{-1} \ \forall a \in G$

$\Rightarrow x \in \bigcap_{a \in G} aHa^{-1}$

$\ker T = \bigcap_{a \in G} aHa^{-1}$

$N \subseteq H$ & $N \trianglelefteq$, Then

$N = aNa^{-1} \quad \forall a \in G$

$\subseteq aHa^{-1} \quad \forall a \in G$

$\Rightarrow N \subseteq \bigcap_{a \in G} aHa^{-1} = \ker T$

Thus $\ker T \trianglelefteq G$ &

$T : G \to P(G/H)$ homo

1st iso thm $G/\ker T \simeq T(G) \leq \mathrm{Perm}(G/H)$

Cor 1. $G \simeq$ a subgroup of $\mathrm{Perm}(S)$ for some S

(Cayley)

Take $H = \{e\}$ $\quad G/H = G = S$

$\ker T = \{e\} \Rightarrow G \simeq \mathrm{Im}(T)$

Cor 2. $|G| = n \Rightarrow G$ is isomorphic to a subgroup of S_n

Cor 2. $|G| = n \Longrightarrow G$ is isomorphic to a subgroup of S_n

Proof: Use cor 1

$$Perm(G) = S_n \quad \text{as}$$

$$G \longleftrightarrow \{1, \ldots n\}$$

$$\{x_1, \ldots x_n\} \xrightarrow{h} \{1, \ldots n\} = J_n$$

$$h(x_k) = k.$$

$$T_x : \begin{tikzcd} G \arrow[r, "T_x"] \arrow[d, "h"'] & G \arrow[d, "h"] \\ J_n \arrow[r, "\sigma_x"] & J_n \end{tikzcd}$$

$$\sigma_x = h^{-1} T_x h \in S_n$$

$$G \longrightarrow Perm(G) = S_n$$

is a homo with trivial kernel

G is iso to a subgroup of S_n

Cor 3. $H \leq G$. Suppose $(G:H) = m$

Then $\left| G/\ker T \right| \,\Big|\, m!$

where $T : G \longrightarrow Perm(G/H)$

$$T(x) = T_x, \quad T_x(aH) = xaH$$

Proof $|G/H| = m$

$$\mathrm{Perm}(G/H) = S_m$$

By $|G/\ker| = |\mathrm{Im}(T)| \mid |S_m|$

1st iso thm. Lagrange

$$|G/\ker T| \mid m!$$

Cor 1. $m = 2$, $H \leq G$, $(G:H) = 2$

then $H \trianglelefteq G$.

Cor 2. $\ker T \subseteq H$

$$(G : \ker T) = (G:H)(H:\ker T)$$
$$= 2$$

$$|G/\ker T| \mid 2!$$

$\Rightarrow (G : \ker T) = 2$ or 1.

$\Rightarrow (G : \ker T) = 2 = 2 \cdot (H : \ker T)$

$(H : \ker T) = 1$

$\Rightarrow H = \ker T$

Cor5. $H \le G$, $(G:H) = p$ prime
$|G| < \infty$ & p is the smallest
prime dividing $|G|$.
Then $H \trianglelefteq G$

Proof By the thm $T: G \to \mathrm{Perm}(G/H) = S_p$
we have $\ker T \subseteq H$ &
$(G:\ker T) = |G/\ker T| = |\mathrm{Im}(T)| \,\Big|\, p!$

$(G:\ker T) = (G:H)(H:\ker T)$
$= p(H:\ker T) \,\big|\, p!$

$(H:\ker T) \,\big|\, (p-1)!$

and $|G| = \dfrac{|G|}{|\ker T|} \cdot |\ker T|$
$= \dfrac{|G|}{|H|} \cdot \dfrac{|H|}{|\ker T|} \cdot |\ker T|$

$\Rightarrow (H:\ker T) = 1$
as p is the smallest
prime $p \,|\, |G|$

Structure of finite
abelian groups

Cyclic groups

G is cyclic $\Rightarrow$ subgroup is
cyclic
$\Rightarrow$ homo image is
cyclic

cyclic

$\Rightarrow$ homo image is cyclic

G is cyclic
- finite
- infinite

G_1, G_2 are cyclic

$|G_1| = |G_2| \Longrightarrow G_1 \simeq G_2$

$\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$

cyclic group of order n

$\boxed{\text{Frobenius – Stickelberger} \approx 1880}$