Alg-10

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Thm $H \le G$. $a \in G$, $a \notin H$

Then H & aH have the same number of elements & write $|H| = |aH|$

Def. X & Y have the same cardinality or the same number of elements, written $|X| = |Y|$ or $\overline{\overline{X}} = \overline{\overline{Y}}$ iff

$\exists f : X \to Y$ bijection.

$$H \xrightarrow{f} aH$$
$$h \longmapsto ah \quad \text{is a bijection}$$

$ah_1 = ah_2 \overset{?}{\Rightarrow} h_1 = h_2$ i.e. $f(h_1) = f(h_2)$

$\Rightarrow h_1 = h_2 \ \forall h_1, h_2 \in H$.

injective.

For $\underline{y \in aH}$, to find $h \in H$ s.t.

$f(h) = y$. $y = ah$ for some $h \in H$

then $f(h) = ah = y \Rightarrow$ surjective.

Then ...

Thm : $H \leq G$, $|G| < \infty$, (G is finite)

Then

(1) $|G| = (G:H)|H|$

(2) $|H| \mid |G|$

(3) $|a| \mid |G|$ for each $a \in G$

(4) $K \leq H \leq G$, K & H are subgroups of G. Then

$$(G:K) = (G:H)(H:K)$$

Proof $|G| < \infty$, $\exists$ $a_1 \ldots a_r$ distinct

s.t $G = \dot{\bigcup}_{i=1}^{r} a_i H$ ($\cdot$ means $a_i H \cap a_j H = \phi$ for $i \neq j$)

$$|G| = \sum_{i=1}^{r} |a_i H|$$

$$= \sum_{i=1}^{r} |H| = r|H|.$$

$$r = (G:H)$$

$$\Rightarrow |G| = (G:H)|H|$$

$\Rightarrow |G| = (G:H)|H|$

(2) (3) follows from (1)

(3) $a \in G$, then $|\langle a \rangle| = |H|$ where $|a| =$

$H = \langle a \rangle$.

$$|a| = |H| \,\Big|\, |G|$$

(4) $|G| = (G:H)|H|$

$= (G:H)(H:K)|K|$ as $K \leq H$.

$\|$

$(G:K)|K|$

$$\Rightarrow \underline{(G:K) = (G:H)(H:K)}$$

Cor. Let G be a finite group of order n & $x \in G$. Then

$$x^n = e \quad (a^{|G|} = e \quad \forall a \in G)$$

Proof $|x| \,\Big|\, |G| \Rightarrow n = k|x|$, for $k \in \mathbb{N}$

$x^n = x^{k|x|} = (x^{|x|})^k = e^k$

Now $x^n = x^{k|x|} = \left(x^{|x|}\right)^k = e^k = e$

Ex S_n = the permutation group on $\{1, \dots, n\}$

$$|S_n| = n!$$

Proof. By induction $n = 1$,

$$|S_1| = 1.$$

Assume $|S_{n-1}| = (n-1)!$, $n \geq 2$

Let $H = \{\sigma \in S_n : \sigma(n) = n\}$

$$H \simeq S_{n-1}$$

$S_n \longrightarrow S_{n-1}$ by

$\sigma \longmapsto \tau_\sigma$ where

$\tau_\sigma : \{1, \dots, n-1\} \circlearrowleft$

defined by $\tau_\sigma(k) = \sigma(k)$

and $k \in \{1, 2, \dots, n-1\}$

homo and $k \in \{1, 2, \ldots n-1\}$

$\ker(\ ) = H$

Claim $(S_n : H) = n$

Then $|S_n| = (S_n : H)|H|$

$= n \cdot (n-1)!$

$= n!$

For each $i \in \{1, \ldots n\}$

$\tau_i(k) = k$ if $k \neq n$ or $k \neq i$

$\tau_i(n) = i, \quad \tau_i(i) = n.$

Then $\tau_i \in S_n$ & $\{\tau_i H\}_{i=1}^{n}$

are distinct.

$\tau_i H \neq \tau_j H \quad , \; i \neq j$

$\sigma \in H$, then $(\tau_i \sigma)(n) = \tau_i(n) = i$

and $(\tau_j \sigma)(n) = \tau_j(n) = j$

2nd proof of $|S_n| = n!$, later.

Thm $f: G \to G'$ homo & $H = \ker f$. Let $x' = f(x)$ for some $x \in G$. Then $f^{-1}(x') = xH$

Question $H \leq G$. Is $Hx = xH$ $\forall x \in G$.

No. $G = S_3 = \{I, \alpha, \beta, \alpha\beta, \beta\alpha, \beta^2\}$

$\alpha^2 = I \quad \beta^3 = I$

$\alpha(1) = 2, \ \alpha(2) = 1, \ \alpha(3) = 3$

$\beta(1) = 2, \ \beta(2) = 3 \quad \beta(3) = 1.$

$|\alpha| = 2, \ |\beta| = 3$

$(\alpha\beta)(3) = 2$

$(\beta\alpha)(3) = 1$

$H_2 = \langle\beta\rangle, \quad H_1 = \langle\alpha\rangle$

$|H_2| = 3, \quad |H_1| = 2, \quad (G : H_1) = 3$

$(G : H_2) = 2$

$xH_2 = H_2 x \quad \forall x \in G = S_3$

Thm $(G : H) = 2 \Rightarrow aH = Ha \quad \forall a \in G.$

Later

Back S_3, H_1 & H_2

$H_1 = \{I, \alpha\}$

$\beta H_1 = \{\beta, \beta\alpha\}$

$H_1\beta = \{\beta, \alpha\beta\}$. Is $H_1\beta = \beta H$?

$\boxed{\alpha\beta \neq \beta\alpha}$ so No

Fact: G is abelian. $H \leq G$

Then $aH = Ha \quad \forall a \in G.$

i.e. $aHa^{-1} = H \quad \forall a \in G.$

Def. Let G be a group. A subgroup H of G is a normal subgroup of G iff it satisfies the following equivalent conditions

Nor 1. $xHx^{-1} = H \quad \forall x \in G$

$(\iff xH = Hx)$

Nor 2. $H = \ker f$, for some f & G' where $f: G \to G'$ is homo.

Fact Nor 1 $\iff$ Nor 2

Now show only Nor 2 $\Rightarrow$ Nor 1.

Assume $H = \ker f$. Let $x \in G$

to show $xHx^{-1} = H$.

We show only $xHx^{-1} \subseteq H$

$\forall x \in H$. $(\subseteq xHx^{-1})$

Let $y \in xHx^{-1}$. Then $y = xhx^{-1}, h \in H$.

$y \in H$ iff. $f(y) \stackrel{?}{=} f(x)f(h)f(x)^{-1}$

as f is

$y \in H$ i.e. $f(y) = f(x)f(h)f(x)^{-1}$

as f is homo

$= f(x)\, e\, f(x)^{-1}$

$= f(x)f(x)^{-1} = e$

proving $y \in \ker f = H$

Now $\forall a \in G$, $aHa^{-1} \subseteq H$.

let $x \in G$ & $a = x^{-1}$, we have

$x^{-1}H(x^{-1})^{-1} = aHa^{-1} \subseteq H$

$x^{-1}Hx \subseteq H$.

$\Rightarrow H \subseteq xHx^{-1} \quad \forall x \in G$.