Alg-16

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$H \underset{c}{\sim} K$ iff $K = xHx^{-1}$ for some $x \in G$.

$S = \{H : H \leq G\}$

$\underset{c}{\sim}$ on S by $H \underset{c}{\sim} K$ iff $K = xHx^{-1}$ for some $x \in G$.

1 - $\underset{c}{\sim}$ is an equivalence rel on S

2 $[H]_c = N_H$ (false) X

$[H]_c = \{xHx^{-1} : x \in G\}$

called corbit of H in S

Action of a group on a set

G — gp, S is a set

Group rep. if $S = V$, a v.s.

Ex. #7 P40 (Later)

#27 P54: G is a group

$S = \{H : H \leq G\}$, the set of all subgroups of G

for each $x \in G$, define

$c_x : S \to S$ by

$c_x(H) = xHx^{-1} \leq G$ as $H \in S$

$c_x c_y = c_{xy}$: $c_x c_y(H)$

$= c_x(yHy^{-1})$

$= x(yHy^{-1})x^{-1}$

$= (xy)H(y^{-1}x^{-1})$

$= xy\,H\,(xy)^{-1}$

$= c_{xy}(H)$

$= xy$

$= C_{xy}(H)$

$\Rightarrow \quad C_x : S \to S$ is a bijection

$C_x \circ C_y = C_{xy} = I_S$ iff

$= C_{xy=e}$

$C_x \circ C_{x^{-1}} = I_S = C_{x^{-1}} \circ C_x$

So for each $x \in S$, $C_x \in \mathrm{Perm}(S)$

$G \xrightarrow{\pi} \mathrm{Perm}(S)$

$x \longmapsto \pi(x) = C_x$

Then π is a homo.

$\pi(xy) = C_{xy} = C_x C_y = \pi(x)\pi(y)$

Ex. $S = G$. For each x,

$C_x : G \to G$ by $C_x(y) = xyx^{-1}$.

$\pi : \quad \in \mathrm{Perm}(S)$

x

Then $c_x \in Perm(S)$

$$G \xrightarrow{\pi} Perm(S)$$
$$x \longmapsto \pi(x) = c_x \quad \text{is a homo.}$$

$$c_x \in Aut(G)$$

$$G \xrightarrow{\pi} Aut(G) \leq Perm(S)$$

$\ker \pi$, where $\pi : G \to Perm(S)$

$$S = \{H : H \leq G\}$$

$$x \in \ker \pi \iff \pi(x) = e_{Perm(S)}$$
$$\|$$
$$c_x$$

$$c_x(H) = I_S(H) = H$$

$$xHx^{-1} = H$$

$x \in \ker \pi \iff xHx^{-1} = H \quad \forall H.$

$$\ker \pi = \{ x \in G : \begin{array}{l} xHx^{-1} = H \\ \forall H \end{array} \}$$

$=$

2^{nd} example $\quad x \in \ker \pi \iff$

$$C_x = \pi(x) = I_S \qquad S = G$$

$$C_x(y) = I_S(y)$$

$$xyx^{-1} = y$$

$$xy = yx \quad \forall y \in S = G$$

$$\ker \pi = \{ x \in G : xy = yx \quad \forall y \in G \}$$

$$= Z(G).$$

3 $\quad S$ is any set, $G = \mathrm{Perm}(S)$

$\sigma \in G. \quad G \times S \xrightarrow{\varphi} S$ by

$$\varphi(\sigma, s) = \sigma(s)$$

$$\varphi(\sigma, s) = \sigma(s)$$

$$\varphi(\sigma, s) = \varphi_\sigma(s) = \sigma(s)$$

$\sigma \in G$, $\varphi_\sigma : S \to S$

by $\varphi_\sigma(s) = \sigma(s)$

$\varphi_\sigma \in \mathrm{Perm}(S)$

$$\varphi_{\sigma\tau} = \varphi_\sigma \circ \varphi_\tau .$$

$$G \xrightarrow{\pi} \mathrm{Perm}(S)$$

by $\pi(\sigma) = \varphi_\sigma$

Then π is a homo

$\sigma \in \ker \pi \iff \pi(\sigma) = \underset{S}{I} \in \mathrm{Perm}(S)$ id.

$\pi(\sigma)(s) = s \quad \forall s \in S$

$$\overset{\shortparallel}{\sigma(s)} = s \qquad \forall s \in S$$

$$\sigma = I_S$$

π is injective homo.

G acts on S means there is a homo $\pi : G \longrightarrow Perm(S)$.

$$\pi(x) = \pi_x \in Perm(S)$$

$$\pi_x : S \longrightarrow S$$

$\pi_x(s) = xs \in S$, for each $x \in G$.

$$\pi(xy) = \pi(x)\pi(y)$$

$$\overset{\shortparallel}{\pi_{xy}} = \pi_x \pi_y$$

and $\pi(e) = \pi_e = I_S$

$$\pi_e(s) = I_S(s) \qquad s \in S$$

$\forall s \in S.$

$e \quad s$

$es = s \quad \forall s \in S.$

π induces a map from

$$G \times S \xrightarrow{\varphi} S$$

$$\varphi(x,s) = \pi(x)(s) = \pi_x(s)$$

$$\varphi(xy,s) = \pi_{xy}(s) = \pi_x(\underbrace{\pi_y(s)}_{\in S})$$

$$= \varphi(x, \widehat{\varphi(y,s)}).$$

$$\varphi(xy,s) = \varphi(x, \varphi(y,s)) \quad \forall x,y \in G \;\&\; s \in S$$

and $\varphi(e,s) = \pi_e(s) = s \quad \forall s$

$G \xrightarrow{\pi} \mathrm{Perm}(S)$ homo

$\Rightarrow \varphi : G \times S \to S$ has

1- $\varphi(xy,s) = \varphi(x, \varphi(y,s))$

2- $\varphi(e,s) = s \quad \forall x,y \in G \quad \forall s \in S.$

Conversely if $\varphi : G \times S \to G$

satisfies (1) $\varphi(xy, s) = \varphi(x, \varphi(y,s))$

(2) $\varphi(e, s) = s \quad \forall x, y \in G$

$\forall s \in S$

then there is a homo

$G \xrightarrow{\pi} \mathrm{Perm}(S)$ given by

$\pi(x) \in \mathrm{Perm}(S)$

$\pi(x)(s) = \varphi(x, s)$

$\pi(xy)(s) = \varphi(xy, s) = \varphi(x, \varphi(y,x))$

$= \pi(x)(\pi(y)(s))$

$= (\pi(x)\pi(y))(s)$

$\forall s$

$\Rightarrow \pi(xy) = \pi(x)\pi(y)$

Notation $\pi_x(s) = xs$

where $\pi : G \to \mathrm{Perm}(S)$ homo where $\pi: G \to Perm(S)$ homo

$\varphi: G \times S \to S$ satisfying

1- $\varphi(x, \varphi(y,s)) = \varphi(xy, s)$

2- $\varphi(e,s) = s$

then we write $\varphi(x,s) = xs \in S$

& (1) means $(xy)s = x(ys)$

(2) $es = s \quad \forall s \in S$

where e is the identity of G.

Ex G is a group $S = \mathcal{P}(G)$ the set of all subsets of G.

G acts on all subsets by translation

$G \xrightarrow{\pi} Perm(S)$ by left

$\pi(x)(A) = xA = T_x(A)$ - translation of A by x.

$T: S \to S$ is a bijection

$\Pi_x : S \to S$

$\Pi_x \Pi_y = \Pi_{xy}$ & $\Pi_e = I_S$

$\Pi_x^{-1} = \Pi_{x^{-1}}$

$\Pi : G \longrightarrow \mathrm{Perm}(S)$

$\Pi(x)(A) = x\underline{A}$, $A \in S$

is a homo.

Ex. G acts on itself by translation

$S = G$

$T_x : S \to S$ by $T_x(a) = xa$

left translation of a by x.

$T_x \circ T_y = T_{xy}$

$T_e = I_S =$

$$G \xrightarrow{T} \mathrm{Perm}(S)$$

$x \longmapsto T(x) = T_x$ is a hmo.

$x \in \ker T \iff T(x) = I_S$, identity of Perm(S)

$T_x(a) = I_S(a) \quad \forall a \in S$

$xa = a \quad \forall a \in S$

$\Rightarrow x = e$

$\ker T = \{e\}$, T is injective

T is faithful.

Ex Fix $H \leq G$. $S = G/H$, the set of all left cosets of H in G.

For $x \in G$, define a map $T_x : G \to S$ by

$T_x(aH) = (xa)H \in S$

$$T_x(aH) = (xa)H \in \quad >$$

Then $T_x(T_y(aH)) = T_x(yaH)$

$$= x(ya)H$$
$$= xy(aH)$$
$$= T_{xy}(aH)$$

$$\Rightarrow \quad T_x T_y = T_{xy}$$

$$\& \quad T_e = I_S$$

So $T_x^{-1} = T_{x^{-1}}$ i.e. $T_x \in Perm(S)$

$$G \xrightarrow{T} Perm(\underbrace{G/H}_{\overset{S}{\shortparallel}})$$

$$x \longrightarrow T(x) = T_x \in Perm(G/H)$$

is a homo

$$x \in \ker T \iff T_x = T(x) = e_{Perm(G/H)}$$

$$T_x(aH) = aH \quad \forall aH \in G/H$$

$$\Rightarrow xaH = aH \ : \ \forall a \in G$$

$$a = e \Rightarrow xeH = eH$$

$$xH = H$$

$$\Rightarrow x \in H$$

$\ker T = H$.

Ex V is a v.s over K.

$G = GL(V) =$ the set of invertible linear maps on V

$S = V$

G acts on V

$$G \xrightarrow{\pi} \mathrm{Perm}(V)$$

$$A \longmapsto \pi(A) = \pi_A : V \to V$$

$A \longmapsto \Pi(A) = \Pi_A$

$\Pi_A(v) = Av$

$\Pi_A \Pi_B = \Pi_{AB}$

Π is a homo

$A \in \ker \Pi \iff \Pi(A) = I_V$

$Av = v \quad \forall v \in V.$

$\ker \Pi = \{I_V\}$

#7 P40 $G \leq \mathrm{Perm}(S)$

For $s, t \in S$, define $s \underset{G}{\sim} t$

iff $\exists\, \sigma \in G$ s.t. $\sigma(s) = t$.

then (1) $\underset{G}{\sim}$ is an equivalence rel. in S

— $s \underset{G}{\sim} s$ as $I_S \in G$.

$s \sim t \iff \sigma(s) = t$ for ... $\in G$

$- \; s \underset{G}{\sim} t \iff \sigma(s) = t$ for some $\sigma \in G$

$$\Downarrow$$

$$\sigma^{-1} \in G$$

$$\iff t = \sigma^{-1}(s), \sigma^{-1} \in G$$

ie. $t \underset{G}{\sim} s$.

$$s \underset{G}{\sim} t, \; t \underset{G}{\sim} u \implies s \underset{G}{\sim} u$$

$\sigma(s) = t, \; \tau(t) = u$ for some $\sigma, \tau \in G$

$$\Downarrow$$

$$\sigma\tau \in G$$

$$\tau\sigma \in G$$

and $u = \tau(t) = \tau(\sigma(s))$

$$= (\tau \circ \sigma)(s)$$

$\implies s \underset{G}{\sim} u$, as $\tau\sigma \in G$.

) For $s \in S$, let $G_s = \{\sigma \in G : \sigma(s) = s\}$

2. For $s \in S$, let $G_s = \{\sigma \in G : \sigma(s) = s\}$

Then $G_s \leq G$.

called the isotropy or stabilizer subgroup of G

Proof. $\sigma \in G_s \Rightarrow \sigma(s) = s \Leftrightarrow s = \sigma^{-1}(s)$

$\Rightarrow \sigma^{-1} \in G$.

$I_S \in G_s$ as $I_S(s) = s$

$\sigma, \tau \in G_s \Rightarrow \sigma(s) = s$
$\tau(s) = s$

$(\sigma\tau)(s) = \sigma(\tau(s)) = \sigma(s) = s$

$\Rightarrow \sigma\tau \in G_s$

(3) $s \underset{G}{\sim} t$ $\therefore$ $\tau(s) = t \Longleftrightarrow G_t = \tau G_s \tau^{-1}$

$\Rightarrow$ Assume $\tau(s) = t$ for some $\tau \in G$

$\sigma \in G_t \Longleftrightarrow \sigma(t) = t$: To

$\sigma \in G_t \iff \underline{\sigma(t) = t}$: To

show $\sigma \in \tau G_s \tau^{-1}$

$\iff \sigma = \tau \alpha \tau^{-1}$ for

some $\alpha \in G_s$

$\alpha(s) = s$

$\tau(s) = t = \sigma(t)$

$s = \tau^{-1}(\sigma(t)) = (\tau^{-1}\sigma\tau)(s)$

$\implies \tau^{-1}\sigma\tau \in G_s$

$\alpha = \tau^{-1}\sigma\tau$

$\impliedby \sigma = \tau\alpha\tau^{-1} \in \tau G_s \tau^{-1}$

4. $s \in S$, put $G s = \{\sigma(s) : \sigma \in G\}$

$\subseteq S$

called the orbit of s in S

isotropy subgroup of G

orbit Gs is subset of S.

Here $G \leq Perm(S)$

More generally let S be a G-set

i.e. G acts on S or $\pi : G \to Perm(S)$ is a homo

(1) For $s \in S$, $G_s = \{x \in G : xs = s\}$

$= \{x \in G : \pi_x(s) = s\}$

Then G_s is a subgroup of G called a stabilizer or isotropy subgroup of G.

(2) For $s \in S$, let

$Gs = O(s) = \{xs : x \in G\}$

$= \{\pi_x(s) : x \in G\}$

called the orbit of s in S

Ex $H \leq G$. $S = G/H = \{aH : a \in G\}$

the set of all left cosets of H in G.

$x \in G$

$$T_x : S \longrightarrow S \text{ by}$$

$$T_x(aH) = (xa)H$$

$$T_x T_y = T_{xy} \;\&\; T_e = I_S$$

$$G \xrightarrow{\;T\;} \mathrm{Perm}(\overset{G/H)}{\overset{\|}{S}})$$

$$x \longmapsto T(x) = T_x \quad \text{is a}$$

homo

G acts on the left cosets of H in G by left translations

$$H \in S$$

isotropy group $\longrightarrow$ $G_H = \{x \in G : T_x(H) = H\}$

$$xH = H$$

group H $xH = H$

$= H$

$s = aH,$ $G_s = \{x \in G : T_x(s) = s\}$

$T_x(aH) = aH$

$xaH = aH$

$x \in \ker T \iff T_x(aH) = aH \quad \forall a \in G$

$xa\cancel{H}aH$

$\ker T = H$

Ex G acts on itself by conjugation

$G \xrightarrow{T} \mathrm{Perm}(G)$, homo

$x \longmapsto C_x, \ T(x)(a) = C_x(a) = axa^{-1}$

$x \in \ker T \iff T(x) = C_x = I_G$

$C_x(a) = a \quad \forall a \in G.$

$xax^{-1} = a \quad \forall a \in G$

$\ker T = Z(G)$.

Let $a \in S$, fixed

$$G_a = \{x \in G : T_x(a) = a\}$$
$$= \{x \in G : xax^{-1} = a\}$$
$$= \{x \in G : xa = ax\}$$
$$= C(a)$$ — conjugate class of a.

Ex S = The set of all subgroups of G
$$= \{H : H \leq G\}$$

$$G \xrightarrow{T} \mathrm{Perm}(S)$$ by

$$T(x) \equiv T_x \in \mathrm{Perm}(S)$$

where $T_x(H) = xHx^{-1} \in S$

$$x \in \operatorname{Ker} T \iff \underset{\substack{\| \\ T_x}}{T(x)} = I_S$$

$$T_x(H) = H \quad \forall H$$
$$\iff xHx^{-1} = H \Big\}$$

$$H \in S = \{H : H \le G\}$$

$$G_H = \{x \in G : T_x(H) = H\}$$
$$= \{x \in G : xHx^{-1} = H\} = N_H.$$

Prop G acts on S, $s \in S$
$$t \in Gs \iff Gs = Gt$$

Proof $t \in Gs \iff t = xs$ for some $x \in G$ where $Gs = \{xs : x \in G\}$
$$= \{T_x(s) : x \in G\}$$

$u \in Gs \implies u = ys$

$= y(x^{-1}t) = (yx^{-1})t \in Gt.$

$\implies u \in Gt$

& conversely

Proof G acts on S & $s \in S$.

If x & y are in the same left coset of G_s then $xs = ys$.

Proof. $H = G_s$, $x, \& y \in aH$ for some $a \in G$. $\implies$

$\underbrace{x = yk}$ for some $k \in H = G_s$

$x = ah_1$ $\quad h_1, h_2 \in H = G_s$

$y = ah_2$

$yh_2^{-1} = a$, $h_2^{-1} \in G_s$

$\implies x = ah_1 = (yh_2^{-1})h_1$

$$\Rightarrow x = a h_1 = (y h_2^{-1}) h_1$$
$$= yk, \quad k = h_2^{-1} h_1 \in G_s$$

Now $xs = (yk)s = y(ks)$
$= ys$ as $k \in G_s$