Tuesday, September 29, 2020 5:56 PM

Angle Modulation: $u(t) = A_c \cos(w_c t + \varphi(t))$

PM: $\varphi(t) = K_p\, m(t)$

FM: $\varphi(t) = 2\pi K_f \int_{-\infty}^{t} m(\tau)\, d\tau$

$$f_i(t) = f_c + \frac{1}{2\pi} \frac{d}{dt} \varphi(t)$$

So FM modulates $f_i - f_c$

Tuesday, September 29, 2020 6:01 PM

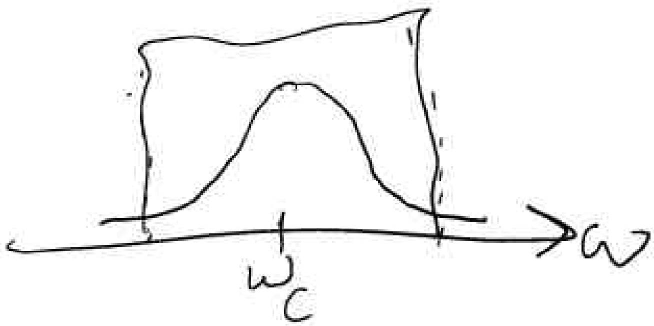
Tuesday, September 29, 2020 6:01 PM

Using the Bessel fncns — a tone

$\downarrow$ angle modulation

band-unlimited

must be approx bandlimited



FM $\Rightarrow$ band unlimited

$\downarrow$ BPF

band limited (distorted)

tone thru PM $\Rightarrow u(t) = \sum_{n=-\infty}^{\infty} A_c J_n(\beta) \cos((\omega_c + n\omega_0)t)$

$\uparrow$ decays in n

"Rule of Thumb" if we have mod. index β, freq. of tone f_0, the effective BW is $B = 2(\beta+1)f_0$ $\leftarrow$ both FM and PM

centered around f_c

centered around f_c

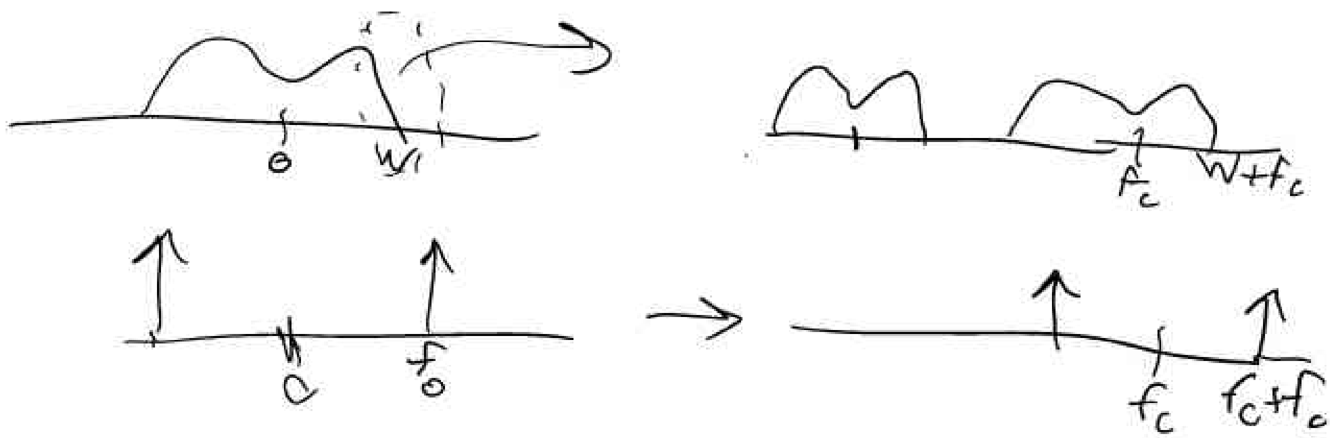
for a tone this gives us 98% of signal power

Carson's Rule

For message signal with BW W, the BW of the angle-modulated signal is approximately

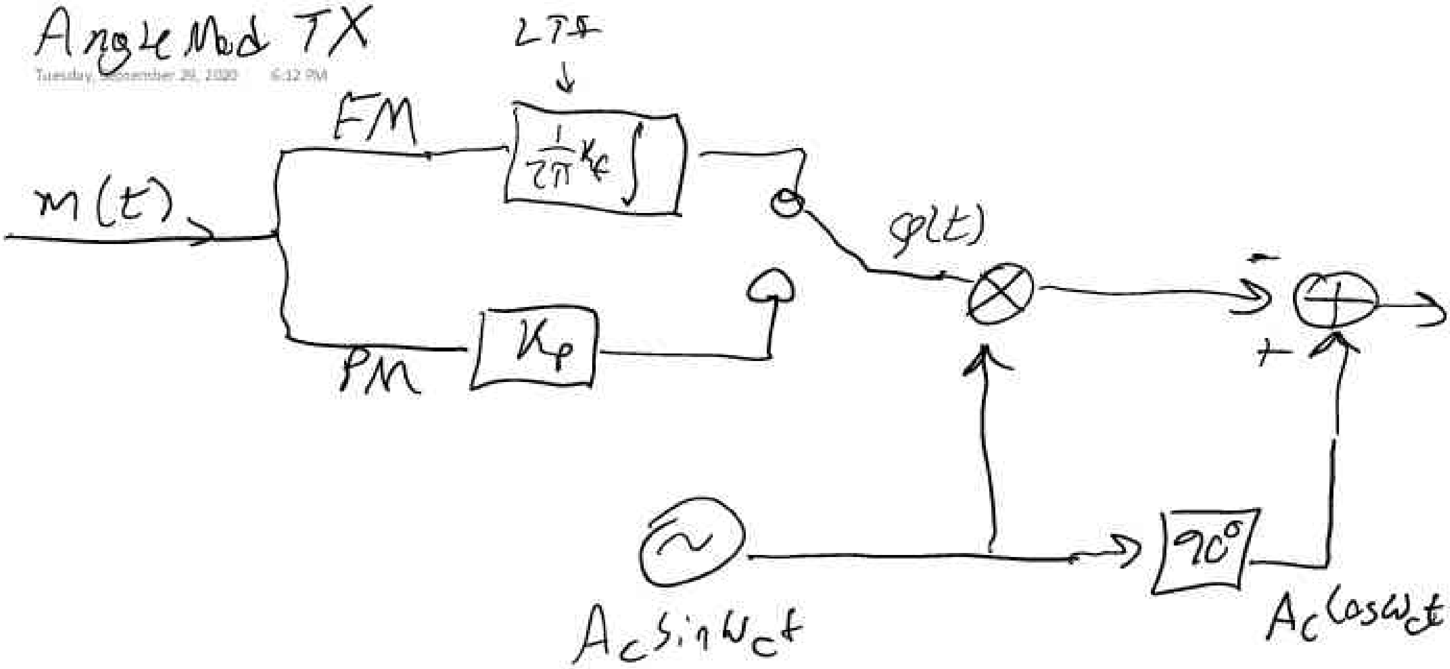
$$\boxed{B_c = 2(\beta+1)W}$$ centered about f_c

even in AM, had tone freq. $f_o \sim$ to W



Angle Mod TX

Tuesday, December 29, 2020 6:12 PM



$$\underbrace{-\varphi(t)A_c\sin\omega_c t + A_c\cos\omega_c t}$$

narrow band FM formula
derived using $\cos\theta \approx 1$
$\sin\theta \approx \theta$

Less narrow band signal

$$u(t) = A_c\cos(\omega_c t + \Phi(t))$$

pick an integer n s.t. $\Phi(t)/n$ is small

(relative to 2π
so that
$\sin\Phi/n \approx \Phi/n$)

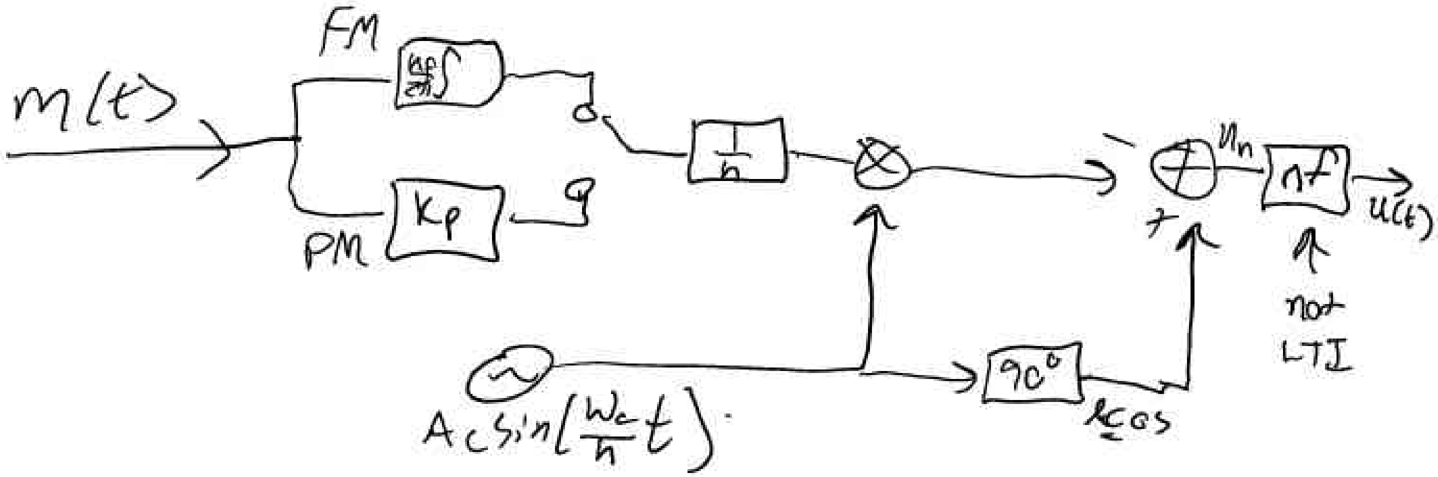
you can always do this so long as m bounded

then $u_n(t) = A_c \cos\left(\frac{\omega_c}{n}t + \frac{\Phi(t)}{n}\right)$ ← narrowband signal

Can generate these using the scheme above

And then use "frequency multiplier" by factor n

(find in lab)



$n=2$

$$\cos\omega_c t \longrightarrow \otimes \longrightarrow \frac{1+\cos 2\omega_c t}{2}$$

Demod

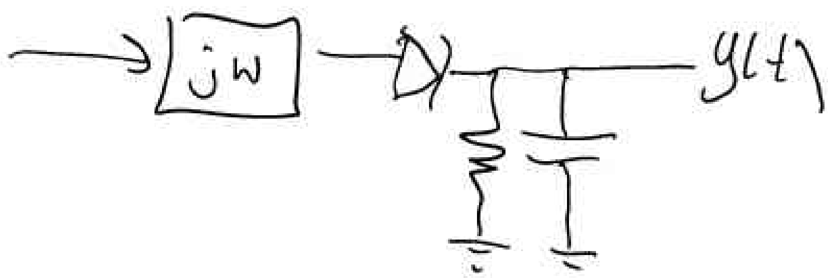
Tuesday, September 29, 2020 6:25 PM

$$v_o(t) = A_c\left(\omega_c + \varphi'(t)\right)(-\sin\omega_c t)$$

$$\underset{FM}{=} -\underbrace{A_c(\omega_c + m(t))\sin\omega_c t}$$

Conventional AM

FM demod



FM has very inexpensive demod

(no mixer)

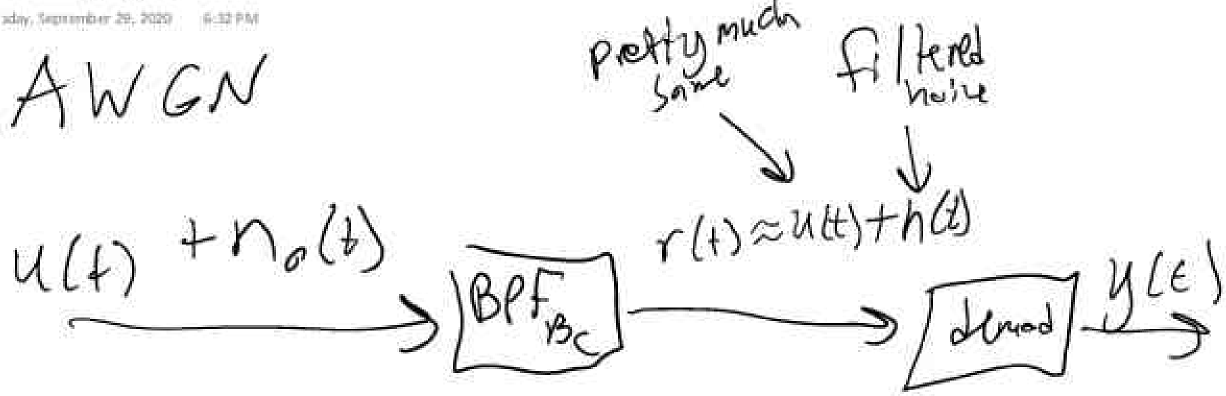
PM – backwards IQ demod (like for SSB)

(can do for FM too)

Noise

Tuesday, September 29, 2020 6:32 PM

AWGN



$$n(t) = n_I(t)\cos\omega_c t - n_Q(t)\sin\omega_c t$$

$$r(t) = u(t) + n(t)$$

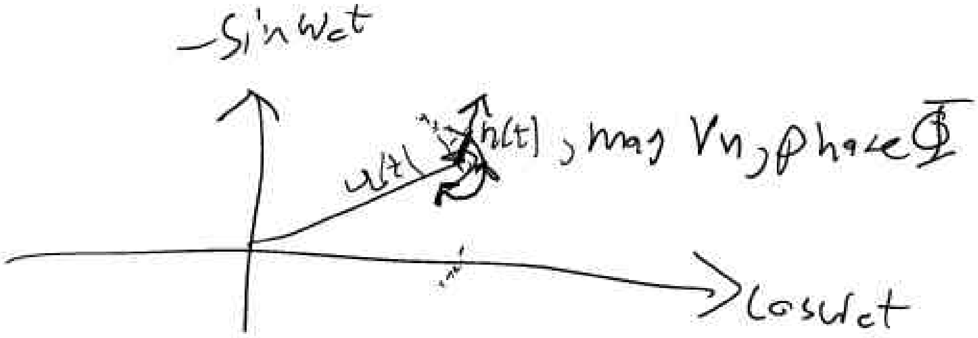
↓ polar form

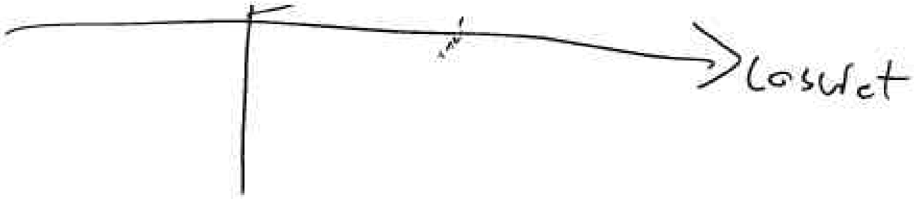
$$= u(t) + \sqrt{n_I^2(t) + n_Q^2(t)}\,\cos\left(\omega_c t + \arctan\left(\frac{n_Q(t)}{n_I(t)}\right)\right)$$

$$= u(t) + V_n(t)\cos(\omega_c t + \Phi_n(t))$$

↑

narrowly vibrating phasor





"drunken phasor"

our signal + gaussian
- distru phasor

Noise power smaller → σ ↓ and the radius of likely uncertainty ↓

distinct from AM in that the phase is as vulnerable as the amplitude

$$|r(t)| \approx A_c + V_n(t)\cos\left(\Phi_n(t) - \varphi(t)\right)$$

$$\angle r(t) \approx \varphi(t) + \arctan\left(\frac{V_n(t)\sin\left(\Phi_n - \varphi(t)\right)}{A_c + V_n(t)\cos\left(\Phi_n(t) - \varphi(t)\right)}\right)$$

taylor arctan x ≈ x

$$\approx \varphi(t) + \frac{V_n \sin\left(\Phi_n(t) - \varphi(t)\right)}{A_c + V_n(t)\cos\left(\Phi_n(t) - \varphi(t)\right)}$$

high SNR

$(\Phi_n(t) - \varphi(t))$

high SNR

$$V_n \ll A_c \approx \varphi(t) + \frac{V_n}{A_c} \sin(\Phi_n(t) - \varphi(t))$$

$$= |r(t)| \cos(\omega_c t + \angle r(t))$$

$$r(t) \approx \left(A_c + V_n(t) \cos(\Phi_n(t) - \varphi(t))\right)$$

$$\times \cos\left(\omega_c t + \varphi(t) + \frac{V_n(t)}{A_c} \sin(\Phi_c(t) - \varphi(t))\right)$$

demod $\rightarrow$ PM: get phase
FM: get f_i

$$y_{PM}(t) = \varphi(t) + \frac{V_n(t)}{A_c} \sin(\Phi_n(t) - \varphi(t))$$

$$\underset{\text{signal}}{\uparrow} + \underset{\text{noise}}{\uparrow} = Y_n = \frac{V_n}{A_c} \sin \Phi - \varphi$$

$$y_{FM}(t) = \frac{1}{2\pi} \frac{d}{dt} \left(\varphi + \frac{V_n}{A_c} \sin(\Phi_n - \varphi)\right)$$

$$= \underset{\uparrow}{k_f m(t)} + \underset{\uparrow}{\frac{1}{2\pi} Y_n'(t)}$$

signal + noise

To get y = signal + noise, used a high SNR assumption

functions of n_I, n_Q

$$Y_n = \frac{V_n}{A_c} \sin(\Phi - \varphi)$$

functions of the scheme

$A_c \nearrow$ decreases noise power

in AM, increase A_c, SNR $\nearrow$ because $\nearrow P_s$ but noise power is fixed

here $\nearrow A_c \rightarrow \searrow N$ but has no effect on S

but still $\nearrow A_c \rightarrow$ SNR $\nearrow$

Same effect thru different route

$$Y_n(t) = \frac{V_n(t)}{A_c} \sin\left(\Phi_n(t) - \varphi(t)\right)$$

$$= \frac{1}{A_c}\left(V_n(t) \sin \Phi_n \cos\varphi - V_n \cos\Phi_n \sin\varphi \right)$$

$$= \frac{1}{A_c}\left(n_Q(t) \cos\varphi - n_I(t) \sin\varphi \right)$$

we had a narrowband approx for FM/PM, can apply

$B_C >>> W$ ← BW of $m(t)$ So we can

↑ BW of noise

assume $\varphi(t) \approx \varphi$ approx constant compared to Φ_n

$$Y_n \approx \frac{n_Q(t)}{A_c} \cos\varphi - \frac{n_I(t)}{A_c} \sin\varphi$$

input noise (after filter) $I: n_I$, $Q: n_Q$, output $I: \frac{n_Q}{A_c}$, $Q: \frac{n_I}{A_c}$

$\uparrow A_c \;\searrow P_N$

only relationship w/ hae now

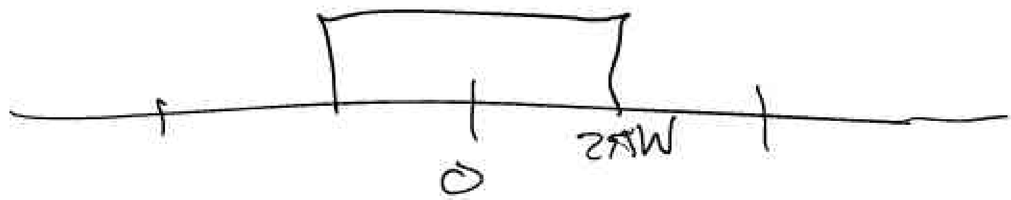
after demod, at bb

PSD

I/Q have same PSD as Y_n

$$S_{Y_n}(\omega) = S_{\frac{n_Q}{A_c}}(\omega) = S_{\frac{n_I}{A_c}}(\omega) = \begin{cases} N_0/A_c^2, & |\omega| \le 2\pi W \\ 0, & \text{else} \end{cases}$$

Noise PSD



For PM

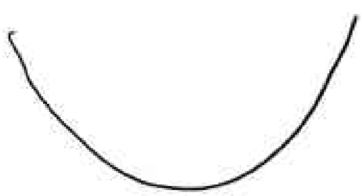
for FM,

noise is $\frac{1}{2\pi} Y_n'(t)$

$\downarrow$ PSD

$\left|\frac{1}{2\pi} j\omega\right|^2 S_{Y_n}(\omega)$

$\frac{1}{4\pi^2}\omega^2 S_{Y_n}(\omega)$

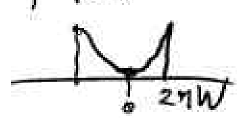


quadratic PSD

baseband case:

PM

FM



$$P_n = \frac{1}{2\pi}\int S_n(\omega)\,d\omega = \begin{cases} \frac{2N_0 W}{A_c^2}, & PM \\ \frac{2}{3}\frac{W^3 N_0}{A_c^2}, & FM \end{cases}$$

Signal power $$P_S = \begin{cases} K_p^2 P_m, & PM \\ K_f^2 P_m, & FM \end{cases}$$

$$\boxed{\begin{aligned} SNR_{PM} &= \frac{K_p^2 A_c^2}{2}\,\frac{P_m}{N_0 W} \\ SNR_{FM} &= \frac{3K_f^2 A_c^2}{2W^2}\,\frac{P_m}{N_0 W} \end{aligned}}$$

$\leftarrow$ better in narrowband case (as $W\uparrow$, $SNR\downarrow$ bad!)

$$\beta_f = \frac{K_f}{W}\max|m(t)|$$

$$\beta_p = K_p \max|m(t)|$$

$$P_R = A_c^2/2$$

$$SNR_{PM} = P_R\left(\frac{\beta_p}{\max(|m(t)|)}\right)^2 \frac{P_M}{N_0 W}$$

$$SNR_{PM} = P_R \left(\frac{\ldots}{\max(|m(t)|)} \right)^2 \frac{\ldots}{N_0 W}$$

$$SNR_{FM} = P_R \left(\frac{\beta_f}{\max(|m(t)|)} \right)^2 \frac{P_M}{N_0 W}$$

Same formula in terms of mod. index

$\nearrow \beta$ increases SNR

$\nearrow \beta$ also increases B_C by Carson

ex. Power/BW tradeoff

(issue, $\nearrow \beta$ too much, not narrowband any more, maybe approx falls apart...)

in FM, most noise is at higher freqs,

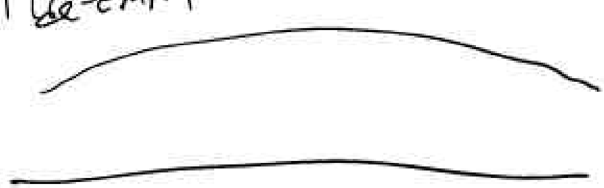
- Pre-emphasis filter amplifies high-freq before mod.

$H_{pre\text{-}emph}$

SNR ~ const

after demod

$H_{de\text{-}emph}$



de emphasis filter

- in FM, higher freq. see more noise

Wrap-up of Analog

Tuesday, September 29, 2020 7:24 PM

BW-efficiency: SSB-AM > DSB-AM >> FM / PM

VSB-AM

Power-efficiency: FM / PM >> AM > conventional AM

highest SNR

Ease of implementation: $\underbrace{\text{Conv. AM} > \text{FM}}_{\text{car radios!}}$ > everything else (I, Q demod, reed an ixer)

A → D

Tuesday, September 29, 2020 7:32 PM

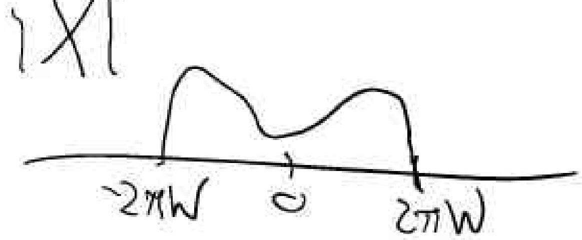
- Review of sampling theory / DTFT
- Quantization (discrete-time vs. digital)
- Coding Intro
- Intro to decision theory

→ dive into Modulation schemes

Tuesday, September 29, 2020 7:37 PM

Say $x(t)$ cts-time signal, $X(\Omega)=0$ for $|\Omega| \geq 2\pi W$

cts-time freq.



<u>Nyquist Sampling THM</u>: The sampled signal $x[n] = x(nT_s)$ can "perfectly reconstruct" the signal $x(t)$ so long as

$$\frac{1}{T_s} = \boxed{f_s \geq 2W}$$

$$\hookrightarrow x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\text{sinc}\left(\frac{t}{T_s} - n\right)$$

is the perfect reconstruction

Tuesday, September 29, 2020 7:42 PM

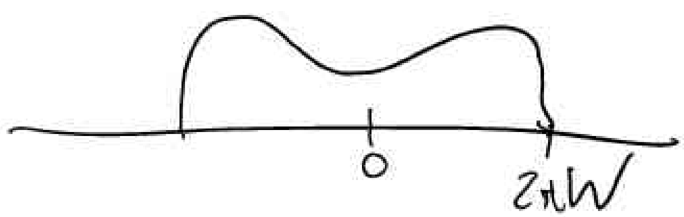
<u>Def</u> (DTFT)

$$X(\omega)=\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

1) $X(\omega)$ is 2π-periodic

2) $X:\mathbb{R}\to\mathbb{C}$

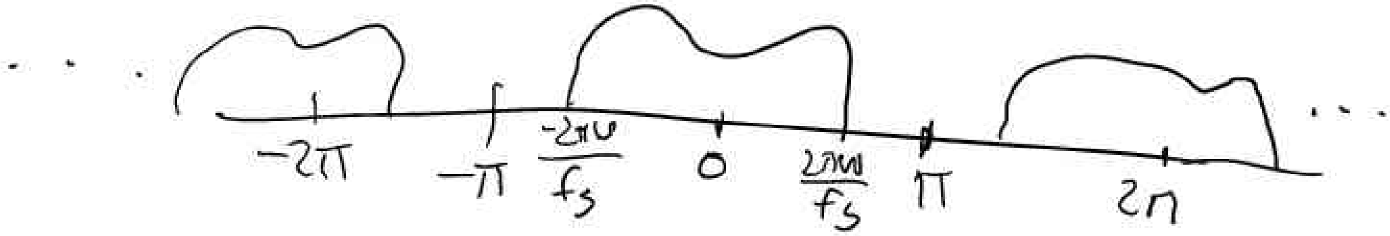
$X(\Omega)$



$\downarrow f_s > 2W$

$\boxed{\omega_d = 2\pi f / f_s = \Omega / f_s}$ ←

$X(\omega)$



if $f_s < 2W$, these overlap, and I get <u>aliasing</u>

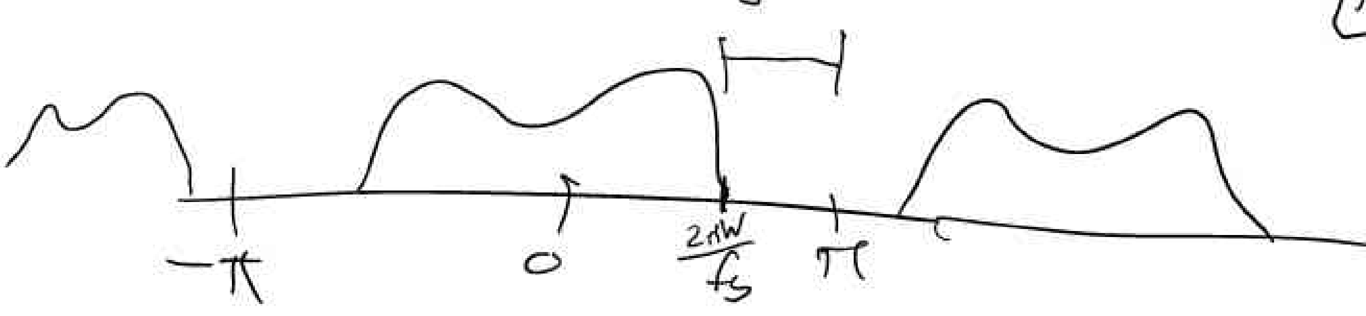
Tuesday, September 29, 2020 7:46 PM

1) <u>Nyquist Rate</u>. the slowest you can sample a bandlimited signal and still avoid aliasing ($f_s = 2W$)

2) <u>Nyquist Frequency</u>: If I sampled at f_s, what is the highest freq. of a signal that will <u>not</u> get aliased (cutoff of the anti-aliasing filter) ($f_c = f_s/2$)

3) <u>Guard Band</u>

guard band: in ω_d is $\pi - \frac{2\pi W}{f_s}$ or in f is $\boxed{f_s - 2W}$



Tuesday, September 29, 2020 7:49 PM

Quantization

Ex Scalar quantization

Represent my signal using N possible values
$\hat{x}_i$, $i=1,2,\dots,N$. Commonly $N=2^K$

Discrete-time signal with values in $\mathbb{R}$

partitioning $\mathbb{R}$ into N intervals, R_i, $i=1,\dots,N$

s.t. the quantization map is given by

$$Q(x[n]) = \hat{x}[n] = \hat{x}_i \text{ for } x[n] \in R_i$$

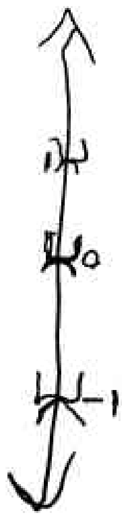
$N=4$

$R_1 = (-\infty, -1)$

$R_2 = [-1, 0)$

$R_3 = [0, 1)$

$R_4 = [1, \infty)$



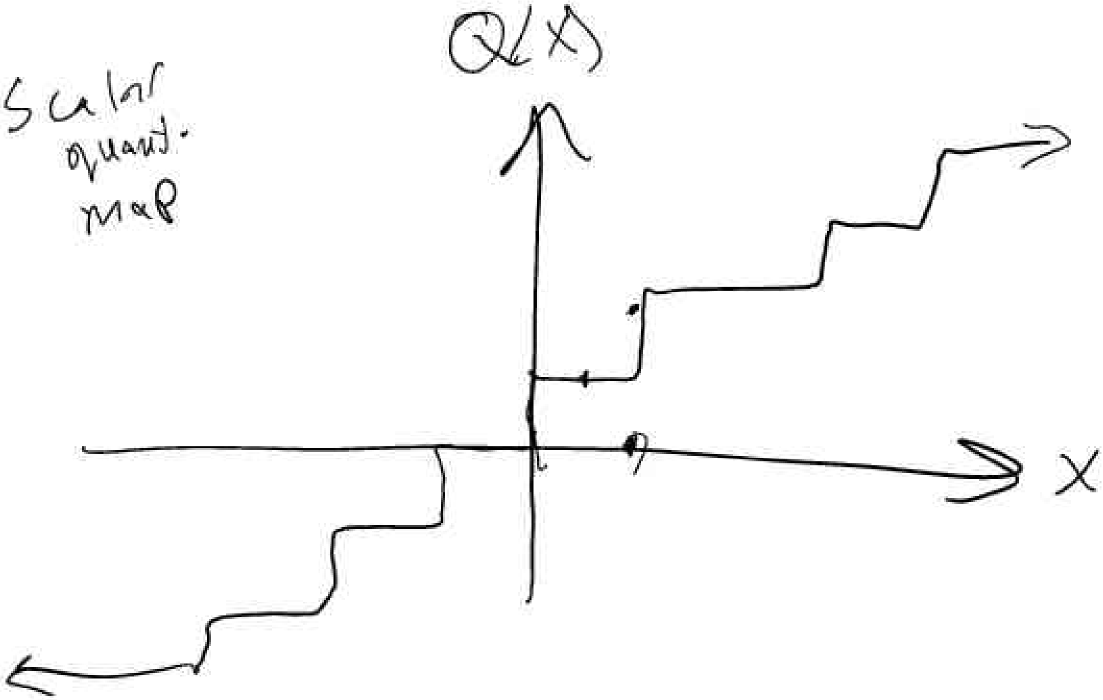
$\hat{x}_1 = -2$ $\hat{x}_3 = 0$

$\hat{x}_2 = -1$ $\hat{x}_4 = 1$

[illegible] $\hat{x}$ as 00 $\hat{x}$ as 10 ex. rep. $\hat{x}_1$ as 00 $\quad$ $\hat{x}_3$ as 10

$\hat{x}_2$ as 01 $\quad$ $\hat{x}_4$ as 11

Scalar quant. map



Q:

I have N-level quantization scheme, how many bits do I need to rep. 1 value

A: $\lceil \log_2 N \rceil$

we are wasteful when $N \neq 2^k$

Quantization → Distortion

→ Distortion
(quantization noise)

1) How does this affect the SNR?
2) How does this affect the spectrum? ← harder

Def Squared Error Distortion for quantizer Q and signal $x[n]$ is given by

$$\left(x[n] - Q(x[n])\right)^2 \equiv \tilde{x}^2$$

this varies with n

Def The average distortion (or mean square error) is given by

$$\boxed{D = E[\tilde{x}^2]}$$

Ex. $X(t)$ white Gaussian 0-mean

$$S_X(\omega) = \begin{cases} 2, & |\frac{\omega}{2\pi}| \leq 100\text{Hz} \\ 0, & \text{else} \end{cases}$$

Sample X at Nyquist ($f_s = 200$ Hz)

Let's say I use 8-level quantizer

$R_1 = (-\infty, -60)$, $R_2 = (-60, -40)$, ... $R_7 = (40, 60)$, $R_8 = (60, \infty)$

$\hat{x}_1 = -70$, $\hat{x}_2 = -50$, $\hat{x}_3 = -30$, ..., $\hat{x}_8 = 70$

Rate: 3 bits/sample (8 level)

200 samples/second

$\boxed{R = 600 \text{ bits/second}}$

$$D = E[X - Q(X)] = \int_{-\infty}^{\infty} (x - Q(x))^2 f_X(x)\,dx$$

$f_X(x)$ is pdf of X which has $S_X(\omega) = \begin{cases} 2, & |f|<100 \\ 0, & \text{else} \end{cases}$

we know $R_X(0) = \sigma^2 = E[X^2]$

$$\sigma^2 = R_X(\tau)\Big|_{\tau=0} = \int_{-\infty}^{\infty} \frac{1}{2\pi} S_X(\omega) e^{j\omega\tau} \, d\omega \Big|_{\tau=0}$$

$$= \frac{1}{2\pi} \int_{-2\pi(100)}^{2\pi(100)} 2 \, d\omega = 400$$

$$f_X(x) = \frac{1}{\sqrt{2\pi(400)}} e^{-x^2/800}$$

$$D = \int_{\mathbb{R}} (x - Q(x))^2 f_X(x)$$

$$= \sum_{i=1}^{8} \int_{R_i} (x - \hat{x}_i)^2 f_X(x) \, dx$$

closed form $\quad$ closed form

$\rightarrow$ MATLAB

$$\boxed{D \approx 33.38}$$

$\boxed{D \approx 33.38}$

mean-squared error = $\boxed{\text{quantization noise}}$

Tuesday, September 29, 2020 8:18 PM

<u>Def</u> For r.v. X, quantizer Q,
the signal-to-quantization noise ratio (<u>SQNR</u>)
is

$$SQNR = \frac{E[X^2]}{E[(X-Q(X))^2]}$$ ← power of a r.v.

Ex. $E[X^2] = 400$, $D = 33.38$, $SQNR \approx 11.98$

$$SQNR_{dB} = 10\log_{10}(11.98) = 10.78\,dB$$

Find the best quantizer

Optimal = greatest possible SQNR

or

for fixed power

lowest possible D

Uniform Case

Every region same size i.e. $R_i = (a_i, a_{i+1})$

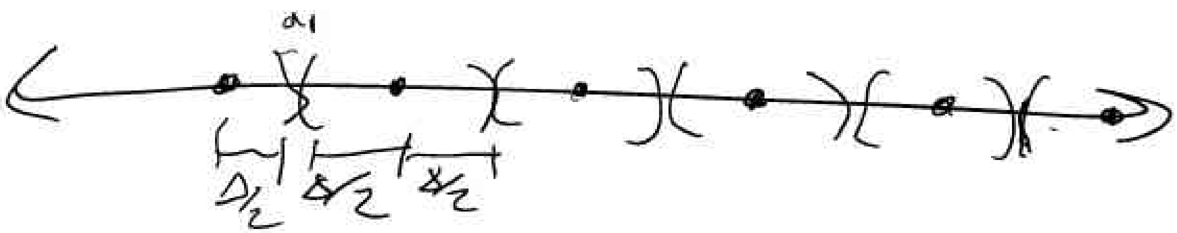
then $a_{i+1} - a_i = \Delta$

for $i = 1, \ldots, N-1$ because $R_1 = (-\infty, a_0)$

$R_N = (a_{N-1}, \infty)$

we choose $\hat{x}_i = a_i - \Delta/2$, for $i = 1, \ldots N-1$

$\hat{x}_N = a_{N-1} + \Delta/2$



$D = \int^{a_1} \ldots \int \ldots \int^{\infty}$

$$D = \int_{-\infty} (x-(a_1-\Delta/2))^2 f_X(x)\,dx + \int_{a_{N-1}} (x-(a_{N-1}+\Delta/2))^2 f_X(x)\,dx$$

$$+ \sum_{i=1}^{N-2} \int_{a_1+(i-1)\Delta}^{a_1+\Delta} (x-(a_1+(a_1-2)\Delta+\Delta/2))^2 f_X(x)\,dx$$

\# parameters $= 2$, a_1, Δ

if I know I want N-level quantizer

To find optimal quantizer,

minimize $D(a_1, \Delta)$ in $\mathbb{R}^2$

have a functional form, easy to search the 2-d space using SGD for example

<u>Non-uniform case</u> $R_i = (a_i, a_{i+1})$

$$D = \sum_i \int_{R_i} (x - \hat{x}_i)^2 f_x(x)\, dx$$

$a_1, \ldots, a_{N-1}$ are all parameters

$\hat{x}_1, \ldots, \hat{x}_N$ are all paraeters

$2N-1$ parameters

$$\frac{\partial D}{\partial a_i} = f_x(a_i)\left((a_i - \hat{x}_i)^2 - (a_i - \hat{x}_{i+1})^2\right) = 0$$

$$a_i = \frac{1}{2}(\hat{x}_i + \hat{x}_{i+1})$$

So analytically, the a_i are best chosen as the mid points,

can show analytically that $\frac{\partial D}{\partial \hat{x}_i} = 0 \rightarrow \hat{x}_i = \frac{\int_{a_{i-1}}^{a_i} x f_x(x)\,dx}{\int_{a_{i-1}}^{a_i} f_x(x)\,dx}$

<u>Centroid</u>

but still, this only <u>constrains</u> a $2N-1$ parameter optimization problem

$\rightarrow$ use SGD still, but constrained

- ·

→ use SGD still, but constrained to $a_i \to$ midpts of $\hat{x}$, and $\hat{x}$ being centroids of R_i

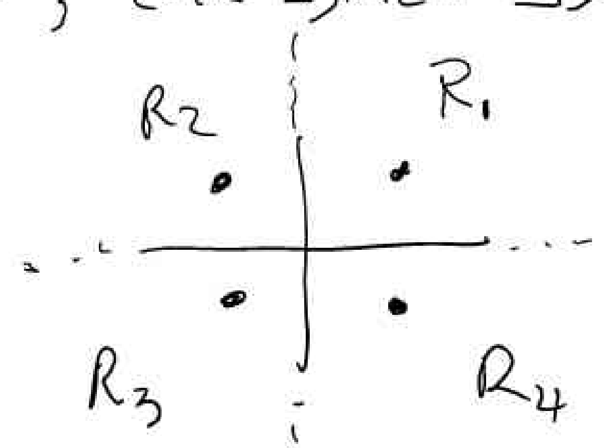
Optimal quantizer "findable" by SGD if X, N are given, easier in uniform case

Scalar quant: $x[n] \xrightarrow{Q} \hat{x}[n] \in \mathbb{R}$

<u>Vector quantization</u>: map multiple time instances to <u>one</u> quantized value

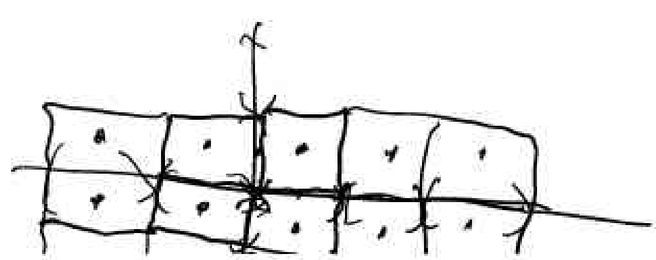
$$\underbrace{(x[n_1], \ldots, x[n_K])}_{\in \mathbb{R}^K} \xrightarrow{Q} \hat{x}_i \in \mathbb{R}^K$$

<u>Ex.</u> $K=2$, $(x[n], x[n+1]) \rightarrow \begin{cases} (1,1) \\ (-1,1) \\ (-1,-1) \\ (1,-1) \end{cases}$ if $x_n \gtrless 0$ and $x_{n+1} \lesseqgtr 0$

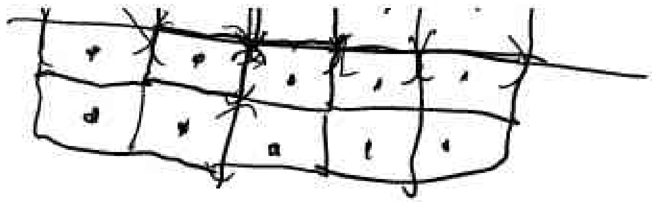


if I'm quantizing pairwise

uniform scalar $\longrightarrow$ what is uniform vector?



Cartesian Product of two uniform



of
two uniform
Scalar, i.e. rectangles

My decision regions can have arbitrary shape in arbitrary dimension — pretty crazy!

optimality - still use SGD in very high dim.

→ still, centroids of decision regions are optimal

Quantizing in $\mathbb{R}^n$ with K quant. regions
(K values)

$$\begin{pmatrix} x[n] \\ \vdots \\ x[n+m] \end{pmatrix} \to \hat{x}_i \in \mathbb{R}^m, \; i=1,\dots,K$$

m signal values → K quantized values

$\lceil \log_2 K \rceil$ bits required to rep. m signal vals

$$\boxed{R = \frac{\log_2 K}{m} \text{ bits/sample}}$$