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9/19/19

PSET 2

(ASSUMPTION OF VARIOUS PROPERTIES OF exp, ln FUNCTIONS)

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1. Let $V = \mathbb{R}^+ = (0, +\infty)$, $F = \mathbb{R}$.

<u>DEF</u>: Given $x, y \in V$, $x +_v y := xy$ (real multiplication)

<u>DEF</u>: Given $a \in F$, $x \in V$, $a \cdot_v x := x^a = \exp(a \ln(x))$ (real exponentiation)

Show that $(V, +_v, \cdot_v)$ is a v.s. over $\mathbb{R}$.

<u>CLAIM</u>: $+_v$ is binary over V.

<u>PROOF</u>: $\forall x, y \in V$, $x, y \in \mathbb{R}$, and $xy \in \mathbb{R}$ (since $\cdot$ is binary over $\mathbb{R}$). Also, since $x, y > 0$, $xy > 0$ (two positive reals multiply to a positive real), so $xy \in \mathbb{R}^+$. Then $+_v$ maps $V \times V \to V$ and thus is binary over V.

<u>CLAIM</u>: $\cdot_v$ is external binary on V with field F.

<u>PROOF</u>: $\forall a \in F$, $x \in V$. $a \cdot_v x = \exp(a \ln(x))$. $\ln(x)$ exists and $\in \mathbb{R}$ $\forall x \in \mathbb{R}^+ = V$, so $a \ln(x) \in \mathbb{R}$ (since $\cdot$ is binary over $\mathbb{R}$). $\forall c \in \mathbb{R}$, $\exp(c) \in \mathbb{R}^+$ (property of exp), thus $\exp(a \ln(x)) \in \mathbb{R}^+ = V$, i.e., $\cdot_v$ maps $F \times V \to V$ and thus is external binary on V with field F.

VS1) <u>CLAIM</u>: $\forall x, y \in V$, $x +_v y = y +_v x$.

<u>PROOF</u>: $\forall x, y \in V$,

$$x +_v y = xy = yx = y +_v x$$

commutativity of $\cdot$ over $\mathbb{R}$

VS2) <u>CLAIM</u>: $\forall x, y, z \in V$, $x +_v (y +_v z) = (x +_v y) +_v z$

<u>PROOF</u>: $\forall x, y, z \in V$,

$$x +_v (y +_v z) = x +_v (yz) = x(yz) = (xy)z = (xy) +_v z = (x +_v y) +_v z$$

associativity of $\cdot$ over $\mathbb{R}$.

VS3) <u>CLAIM</u>: $\exists\, 0_v \in V$ s.t. $\forall x \in V$, $0_v +_v x = x$

<u>PROOF</u>: Let $0_v = 1_{(\mathbb{R})}$. Then, $\forall x \in V$,

$$0_v +_v x = 1x = x$$

(F3) on $\mathbb{R}$

VS4) CLAIM: $\forall x \in V$, $\exists y \in V$ s.t. $x +_v y = 0_v$.

PROOF: Let $x \in V = \mathbb{R}^+$. Then $\exists\, y = \frac{1}{x}$. Since $x \neq 0 \in \mathbb{R}$, $y \in \mathbb{R}$, and since $1, x > 0$, $y = \frac{1}{x} > 0 \in \mathbb{R}^+ = V$, and:

$$x +_v y = x\left(\frac{1}{x}\right) = 1_{(\mathbb{R})} = 0_v$$

VS5) CLAIM: $\forall x \in V$, $1 \cdot_v x = x$

PROOF: $\forall x \in V$:

$$\begin{aligned} 1 \cdot_v x &= \exp(1 \cdot \ln x) \\ &= \exp(\ln x) && (\text{F3 of } \mathbb{R}) \\ &= x && (\exp = \ln^{-1}) \end{aligned}$$

VS6) CLAIM: $\forall a, b \in F$, $\forall x \in V$, $(ab) \cdot_v x = a \cdot_v (b \cdot_v x)$

PROOF: $\forall a, b \in F$, $\forall x \in V$,

$$\begin{aligned} (ab) \cdot_v x &= \exp(ab \ln(x)) = \exp(a(b \ln x)) \\ &= \exp(a \ln(\exp(b \ln x))) && (\exp = \ln^{-1}) \\ &= \exp(a \ln(b \cdot_v x)) \\ &= a \cdot_v (b \cdot_v x) \end{aligned}$$

VS7) CLAIM: $\forall a \in F$, $x, y \in V$, $a \cdot_v (x +_v y) = a \cdot_v x +_v a \cdot_v y$.

PROOF: $\forall a \in F$, $x, y \in V$,

$$\begin{aligned} a \cdot_v (x +_v y) &= a \cdot_v (xy) = \exp(a \ln(xy)) \\ &= \exp(a(\ln x + \ln y)) && (\text{prop. of } \ln) \\ &= \exp(a \ln x + a \ln y) \\ &= \exp(a \ln x) \cdot \exp(a \ln y) && (\text{prop. of } \exp) \\ &= (a \cdot_v x) \cdot (a \cdot_v y) = a \cdot_v x +_v a \cdot_v y. \end{aligned}$$

VS8) CLAIM: $\forall a, b \in F$, $x \in V$, $(a+b) \cdot_v x = a \cdot_v x +_v b \cdot_v x$.

PROOF: $\forall a, b \in F$, $x \in V$,

$$\begin{aligned} (a+b) \cdot_v x &= \exp((a+b)\ln(x)) = \exp(a \ln x + b \ln x) \\ &= \exp(a \ln(x)) \cdot \exp(b \ln(x)) && (\text{prop. of } \exp) \\ &= (a \cdot_v x) \cdot (b \cdot_v x) = a \cdot_v x +_v b \cdot_v x \end{aligned}$$

Since $+_v$ is binary on V, $\cdot_v$ is external binary on V with field F, and V satisfies (VS1-8), $(V, +_v, \cdot_v)$ is a vector space.

PSET 2, cont'd

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2. DEF: $C(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous on } \mathbb{R} \}$

DEF: $C^{(n)}(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{R} \mid f^{(n)} \text{ is continuous on } \mathbb{R} \}$

a. Show that $C(\mathbb{R})$ is a v.s. over $\mathbb{R}$.

CLAIM: $C(\mathbb{R})$ SUBSP. $\mathcal{F}(\mathbb{R}, \mathbb{R})$

The following proofs will prove that $C(\mathbb{R})$ SUBSP. $\mathcal{F}(\mathbb{R}, \mathbb{R})$ using (THM 1.3), the necessary conditions for a subspace.

CLAIM: $C(\mathbb{R}) \subseteq \mathcal{F}(\mathbb{R}, \mathbb{R})$

PROOF: $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is the set of all functions $f : \mathbb{R} \to \mathbb{R}$ over the domain $\mathbb{R}$. $C(\mathbb{R})$ is a set of some functions $f : \mathbb{R} \to \mathbb{R}$ that exist (and are continuous) over the domain $\mathbb{R}$, thus $C(\mathbb{R}) \subseteq \mathcal{F}(\mathbb{R}, \mathbb{R})$.

DEF: $z := z(x) = 0 \quad \forall x \in \mathbb{R}$

(Note that this is the additive identity for $\mathcal{F}(\mathbb{R}, \mathbb{R})$, since $\forall f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$, $f + z = f(x) + z(x) = f(x) + 0 = f(x) = f$).

CLAIM: $z \in C(\mathbb{R})$.

PROOF: $z(a) = \lim_{x \to a} z(x) = 0 \quad \forall a \in \mathbb{R}$, so z cts. over $\mathbb{R}$ and $z \in C(\mathbb{R})$

CLAIM: $\forall f, g \in C(\mathbb{R})$, $f + g \in C(\mathbb{R})$.

PROOF: $\forall a \in \mathbb{R}$,

$$f(a) = \lim_{x \to a} f(x), \quad g(a) = \lim_{x \to a} g(x), \quad \text{(DEF continuity)}$$

$$f + g = f(a) + g(a) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

$$= \lim_{x \to a} (f(x) + g(x)) \quad \text{(linearity of lim)}$$

Therefore $f+g$ is continuous on $\mathbb{R}$, so $f + g \in C(\mathbb{R})$.

CLAIM: $\forall a \in \mathbb{R}$, $\forall f \in C(\mathbb{R})$, $af \in C(\mathbb{R})$

PROOF: $\forall b \in \mathbb{R}$, $f(b) = \lim_{x \to b} f(x)$, (DEF continuity)

$$af = af(b) = a \lim_{x \to b} f(x) = \lim_{x \to b} af(x) \quad \text{(linearity of lim)}$$

Therefore af is continuous on $\mathbb{R}$, so $af \in C(\mathbb{R})$

By THM 1.3, $C(\mathbb{R})$ SUBSP. $\mathcal{F}(\mathbb{R})$. Thus $C(\mathbb{R})$ is a v.s.

b. Show that $C^{(n)}(\mathbb{R})$ SUBSP. $C(\mathbb{R})$ $\forall n \in \mathbb{Z}^+$.

CLAIM: $C^{(n)}(\mathbb{R}) \subseteq C(\mathbb{R})$, $\forall n \in \mathbb{Z}^+$

PROOF: $\forall f \in C^{(n)}(\mathbb{R})$, $\forall n \in \mathbb{Z}^+$, f is n times differentiable $(n > 0)$.
Any differentiable function is continuous over its domain,
thus $f \in C(\mathbb{R})$ and $C^{(n)}(\mathbb{R}) \subseteq C(\mathbb{R})$.

CLAIM: $z \in C^{(n)}(\mathbb{R})$, $\forall n \in \mathbb{Z}^+$ (using same (DEF z))

PROOF: $\forall n \in \mathbb{Z}^+$, $z^{(n)} = z^{(n)}(x) = 0 = z \in C(\mathbb{R})$ $\forall x \in \mathbb{R}$.

CLAIM: $\forall f, g \in C^{(n)}(\mathbb{R})$, $\forall n \in \mathbb{Z}^+$, $f + g \in C^{(n)}(\mathbb{R})$

PROOF: $\forall f, g \in C^{(n)}(\mathbb{R})$, $\forall n \in \mathbb{Z}^+$, $\forall a \in \mathbb{R}$,

$$f^{(n)}(a) = \lim_{x \to a} f^{(n)}(x),$$

$$g^{(n)}(a) = \lim_{x \to a} g^{(n)}(x), \qquad \text{(DEF continuity)}$$

$$\frac{d^n}{dx^n}(f+g) = \frac{d^n}{dx^n}\left(f(a) + g(a)\right) \qquad \left(\text{linearity of } \frac{d}{dx}\right)$$

$$= f^{(n)}(a) + g^{(n)}(a) = \lim_{x \to a} f^{(n)}(x) + \lim_{x \to a} g^{(n)}(x)$$

$$= \lim_{x \to a}\left(f^{(n)}(x) + g^{(n)}(x)\right) \qquad \text{(linearity of lim)}$$

$$= \lim_{x \to a}\left(f^{(n)} + g^{(n)}\right) = \lim_{x \to a}\left(\frac{d^n}{dx^n}(f+g)\right) \qquad \left(\text{linearity of } \frac{d}{dx}\right)$$

Thus $f+g$ has a continuous nth-derivative, so $f+g \in C^{(n)}(\mathbb{R})$.

CLAIM: $\forall f \in C^{(n)}(\mathbb{R})$, $\forall a \in \mathbb{R}$, $\forall n \in \mathbb{Z}^+$, $af \in C^{(n)}(\mathbb{R})$

PROOF: $\forall f \in C^{(n)}(\mathbb{R})$, $\forall a, b \in \mathbb{R}$,

$$f^{(n)}(b) = \lim_{x \to b} f^{(n)}(x), \qquad \text{(DEF continuity)}$$

$$\frac{d^n}{dx^n}(af) = a\frac{d^n}{dx^n} f = af^{(n)}(b)$$

$$= a \lim_{x \to b} f^{(n)}(x) = \lim_{x \to b}\left(af^{(n)}(x)\right)$$

$$= \lim_{x \to b}\left(\frac{d^n}{dx^n}(af)\right)$$

Thus af has a continuous nth-derivative, so $af \in C^{(n)}(\mathbb{R})$

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2b, (cont'd.) By THM 1.3, $C^{(n)}(\mathbb{R})$ subsp. $C(\mathbb{R})$ $\forall n \in \mathbb{Z}^+$.

2c. Show $j > i \Rightarrow C^{(j)}(\mathbb{R})$ subsp. $C^{(i)}(\mathbb{R})$, $j, i \in \mathbb{Z}^+$.

CLAIM: $C^{(j)}(\mathbb{R}) \subseteq C^{(i)}(\mathbb{R})$

PROOF: $\forall f \in C^{(j)}(\mathbb{R})$, $f^{(j)}$ is continuous. In general, for $\forall k \in \mathbb{Z}^+$, if $f^{(k+1)}$ is continuous, $f^{(k)}$ is continuous. By induction, $f^{(j)}, f^{(j-1)}, \ldots f^{(1)}$ are continuous. Since $i < j$, $f^{(i)}$ is continuous. Thus $f \in C^{(i)}(\mathbb{R})$, so $C^{(j)}(\mathbb{R}) \subseteq C^{(i)}(\mathbb{R})$.

Since $C^{(n)}(\mathbb{R})$ is a v.s. (proved in (2b)), then $C^{(j)}(\mathbb{R})$, $C^{(i)}(\mathbb{R})$ are v.s. over $\mathbb{R}$. Since $C^{(j)}(\mathbb{R}) \subseteq C^{(i)}(\mathbb{R})$, $C^{(j)}(\mathbb{R})$ subsp. $C^{(i)}(\mathbb{R})$.

PSET 2 #3

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3. Let $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$,
$E = \{f \in V \mid f(-x) = f(x) \quad \forall x \in \mathbb{R}\}$.
$O = \{f \in V \mid f(-x) = -f(x) \quad \forall x \in \mathbb{R}\}$

a. Show E and O are proper subspaces of V.

CLAIM: E, O are proper subsets of V ($E, O \subseteq V$, $E, O \neq V$).
PROOF: $\forall f \in E$, $\forall g \in O$, by (DEF E) and (DEF O), $f \in V$, $g \in V$, thus $E, O \subseteq V$.

$\forall h \neq 0 \in E$, $i \neq 0 \in O$. Then, $h \notin O$ ($h(-x) = h(x) \neq -h(x)$) and $i \notin E$ ($i(-x) = -i(x) \neq i(x)$). Since $h \in V$, $i \in V$, $E, O \neq V$.

CLAIM: $z \in E, O$. (using same (DEF z) from problem 2)
(Note: z is the 0 vector of V.)
PROOF: $\forall x$, $z = z(x) = 0 = z(-x)$, so $z \in E$,
$z = z(x) = 0 = (-1) \cdot 0 = -z(x)$, so $z \in O$.

CLAIM: $\forall f, g \in E$, $f + g \in E$,
$\forall h, i \in O$, $h + i \in O$
PROOF: $\forall f, g \in E$, $\forall x \in \mathbb{R}$,
$f+g = (f+g)(x) = f(x) + g(x) = f(-x) + g(-x) = (f+g)(-x)$
$\therefore f+g \in E$.

$\forall h, i \in O$, $h + i \in O$.
$(h+i)(-x) = h(-x) + i(-x) = -h(x) - i(x)$
$= -1(h(x) + i(x)) = -(h+i)(x)$
$\therefore h+i \in O$.

CLAIM: $\forall a \in \mathbb{R}, \forall f \in E, \quad af \in E,$

$\forall b \in \mathbb{R}, \forall g \in O, \quad bg \in O.$

PROOF: $\forall a \in \mathbb{R}, \forall f \in E,$

$$(af)(x) = af(x) = af(-x) = (af)(-x)$$

$\therefore af \in E.$

$\forall b \in \mathbb{R}, \forall g \in O,$

$$(bg)(x) = bg(-x) = b(-g(x))$$
$$= -bg(x) = -(bg)(x)$$

$\therefore bg \in O.$

By (THM 1.3), E, O subsp. V. Since E, O are proper subsets of V, E, O proper subsp. V.

b. Argue that $\text{span}(E) = E$, $\text{span}(O) = O$.

LEM 1: $\forall V$ v.s., $\text{span}(V) = V$.

PROOF: $V \subseteq V$, so $\text{span}(V) \subseteq V$ (THM 1.5)

Also, $\forall x \in V$, x can be written as a linear combination of vectors in V, i.e., $1 \cdot x = x$. Thus $V \subseteq \text{span}(V)$.

Since $\text{span}(V) \subseteq V$, $V \subseteq \text{span}(V)$, $V = \text{span}(V)$.

CLAIM: $\text{span}(E) = E$, $\text{span}(O) = O$.

PROOF: E and O are both v.s. (proved in part a).

(LEM 1) states that the span of any v.s. is itself, so this result is immediate.

PSET 2, P3, cont'd.

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3c) Show that $E \cup O$ generates V.

CLAIM: $V \subseteq \text{span}(E \cup O)$
PROOF: $\forall f \in V$, let $f_e = f_e(x) = \frac{f(x) + f(-x)}{2}$,

$f_o = f_o(x) = \frac{f(x) - f(-x)}{2}$.

Since $f_e(-x) = \frac{f(-x) + f(x)}{2} = \frac{f(x) + f(-x)}{2} = f_e(x)$,
and $f_e : \mathbb{R} \to \mathbb{R}$, $f_e \in E$.

Since $f_o(-x) = \frac{f(-x) - f(x)}{2} = -\left(\frac{f(x) - f(-x)}{2}\right) = -f_o(x)$,
and $f_o : \mathbb{R} \to \mathbb{R}$, $f_o \in O$.

Then $\forall g \in V$, g can be expressed as the linear combination of g_e, g_o:
$$1 \cdot g_e + 1 \cdot g_o = g_e(x) + g_o(x) = \frac{g(x) + g(-x)}{2} + \frac{g(x) - g(-x)}{2}$$
$$= \left(\frac{g(x)}{2} + \frac{g(x)}{2}\right) + \left(\frac{g(-x) - g(-x)}{2}\right) = g(x) = g.$$
Since $g_e \in E$, $g_o \in O$, $g_e, g_o \in E \cup O$, and $V \subseteq \text{span}(E \cup O)$.

CLAIM: $E \cup O$ generates V.
PROOF: Since $E, O \subseteq V$, $E \cup O \subseteq V$. Then, by (THM 1.5), $\text{span}(E \cup O) \subseteq V$. Since $V \subseteq \text{span}(E \cup O)$, $V = \text{span}(E \cup O)$ $\therefore$ $E \cup O$ generates V.

1.2 # 8, 9, 21

CLAIM:

8. $\forall$ v.s. V, show $(a+b)(x+y) = ax + ay + bx + by$ $\forall x, y \in F$, $\forall a, b \in F$.

PROOF: $(a+b)(x+y)$

$= (a+b)x + (a+b)y$ (VS 7)

$= ax + bx + ay + by$ (VS 8)

$= ax + ay + bx + by$ (VS 1)

9. Prove corollaries 1, 2 of (THM 1.1), (THM 1.2 c).

(THM 1.1 Cor. 1) CLAIM: The vector 0 described in (VS3) is unique.

PROOF: Let 0_1, 0_2 satisfy additive identity property described in (VS3). Then, $\forall x \in V$,

$0_1 + x = x = 0_2 + x$

By (THM 1.2), $0_1 = 0_2$ $\therefore$ the additive identity for a v.s. is unique.

(THM 1.1 Cor. 2) CLAIM: The vector y described in (VS4) is unique.

PROOF: Let y_1, y_2 satisfy additive inverse property described in (VS4) $\forall x \in V$. Then,

$x + y_1 = 0 = x + y_2$

By (THM 1.1), $y_1 = y_2$ $\therefore$ the additive inverse $\forall x \in V$ is unique.

(THM 1.2c) CLAIM: $a0 = 0$ $\forall a \in F$ in any v.s. over F.

PROOF: $\forall a \in F$,

$a0 + a0 \underset{\text{by (VS7)}}{=} a(0+0) \underset{\text{(VS3)}}{=} a0 \underset{\text{(VS3)}}{=} a0 + 0$

By (THM 1.1), $a0 = 0$.

PSET 2

1.2 # 21

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21. Let V, W be v.s. over field F. Let $Z = \{(v,w) : v \in V, w \in W\}$. Prove that Z v.s. over F w/ the operations:
$(v_1, w_1) + (v_2, w_2) := (v_1 + v_2, w_1 + w_2)$,
$c(v_1, w_1) := (cv_1, cw_1)$

CLAIM: $+$ binary over Z.
PROOF: $\forall z_1 = (v_1, w_1), z_2 = (v_2, w_2) \in Z$. Then $v_1, v_2 \in V$, $w_1, w_2 \in W$, and by binary-ness of vector addition, $v_1 + v_2 \in V$, $w_1 + w_2 \in W$. $\therefore$ $z_1 + z_2 \in Z$, so $+$ maps $Z \times Z \to Z$ and is thus binary.

CLAIM: $\cdot$ is external binary over Z w/ field F.
PROOF: $\forall c \in F$, $\forall z = (v, w)$. Since v, w are v.s. over field F, scalar multiplication by F is external binary, i.e., $cv \in V$, $cw \in W$. $\therefore$ $cz \in Z$, so $\cdot$ maps $F \times Z \to Z$, and is therefore external binary over F.

VS1) CLAIM: $\forall z_1, z_2 \in Z$, $z_1 + z_2 = z_2 + z_1$.
PROOF: $\forall z_1 = (v_1, w_1), z_2 = (v_2, w_2) \in Z$,
$$z_1 + z_2 = (v_1 + v_2, w_1 + w_2) \underset{\text{(VS 1 for } W, V)}{=} (v_2 + v_1, w_2 + w_1) = z_2 + z_1$$

VS2) CLAIM: $\forall z_1, z_2, z_3 \in Z$, $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$.
PROOF: $\forall z_1 = (v_1, w_1), z_2 = (v_2, w_2), z_3 = (v_3, w_3)$,
$$\begin{aligned} z_1 + (z_2 + z_3) &= (v_1, w_1) + (v_2 + v_3, w_1 + w_2) \\ &= (v_1 + (v_2 + v_3), w_1 + (w_2 + w_3)) \\ &= ((v_1 + v_2) + v_3, (w_1 + w_2) + w_3) \qquad \text{(VS 2 for } V, W) \\ &= (v_1 + v_2, w_1 + w_2) + (v_3, w_3) \\ &= (z_1 + z_2) + z_3 \end{aligned}$$

1.2 # 21, cont'd.

VS 3) CLAIM: $\exists\, 0_Z \in Z$ s.t. $\forall z \in Z$, $0_Z + z = z$.

PROOF: $\exists$ additive identities $0_V \in V$, $0_W \in W$ (VS 3 for V, W).

Let $0_Z = (0_V, 0_W)$. Then, $\forall z = (v, w) \in Z$,

$$0_Z + z = (0_V + v,\ 0_W + w) \quad \text{(VS3 for } V, W\text{)}$$
$$= (v, w) = z.$$

VS 4) CLAIM: $\forall z_1 \in Z$, $\exists z_2 \in Z$ s.t. $z_1 + z_2 = 0_Z$.

PROOF: $\forall z_1 = (v, w) \in Z$,

v has an additive inverse $(-v)$, (VS4 for v)

w has an additive inverse $(-w)$, (VS4 for w)

Let $z_2 = (-v, -w)$, then:

$$z_1 + z_2 = (v + (-v),\ w + (-w))$$
$$= (0_V, 0_W) = 0_Z. \quad \text{(VS4 for } V, W\text{)}$$

VS5) CLAIM: $\forall z \in V$, $1z = z$.

PROOF: $\forall z = (v, w) \in V$, $1z = (1v, 1w) \underset{\text{(VS5 for } V, W\text{)}}{=} (v, w) = V$

VS 6) CLAIM: $\forall a, b \in F$, $\forall z \in Z$, $(a \cdot b) z = a \cdot (b \cdot z)$

PROOF: $\forall a, b \in F$, $\forall z = (v, w) \in Z$,

$$(a \cdot b) z = ((a \cdot b) v, (a \cdot b) w) \underset{\text{(VS6 for } V, W\text{)}}{=} (a \cdot (bv), a \cdot (bw)) = a(bv, bw) = a(bz)$$

VS7) CLAIM: $\forall a \in F$, $\forall z_1, z_2 \in Z$, $a(z_1 + z_2) = az_1 + az_2$

PROOF: $\forall a \in F$, $\forall z_1 = (v_1, w_1)$, $z_2 = (v_2, w_2) \in Z$,

$$a(z_1 + z_2) = a(v_1 + v_2,\ w_1 + w_2) = (a(v_1 + v_2),\ a(w_1 + w_2))$$
$$\underset{\text{(VS7 for } V, W\text{)}}{=} (av_1 + av_2,\ aw_1 + aw_2) = (av_1, aw_1) + (av_2, aw_2) = az_1 + az_2$$

VS8) CLAIM: $\forall a, b \in F$, $\forall z \in Z$, $(a + b) z = az + bz$.

PROOF: $\forall a, b \in F$, $\forall z = (v, w) \in Z$,

$$(a+b) z = ((a+b)v,\ (a+b)w) \underset{\text{(VS8 for } V, W\text{)}}{=} (av + bv,\ aw + bw)$$
$$= (av, aw) + (bv, bw) = az + bz.$$

Since $+$ binary on Z, $\cdot$ external binary on Z, and (VS 1-8) satisfied, Z is a V.S over F with the addition and multiplication defined here.

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1.3 #10, 22, 23, 25.

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10. Prove that $W_1 = \{(a_1, a_2, \ldots, a_n) \in F^n : a_1 + a_2 + \ldots + a_n = 0\}$ is a subspace, but $W_2 = \{(a_1, a_2, \ldots, a_n) \in F^n : a_1 + a_2 + \ldots + a_n = 1\}$ is not.

a. Prove W_1 is a subspace:

Note: the zero vector for F^n is known to be $\underbrace{(0, 0, 0, \ldots, 0)}_{n}$.
Denote zero vector for F^n 0_n ($n \in \mathbb{Z}^+$).

CLAIM: $0_n \in W_1$.
PROOF: In 0_n, $a_1 = a_2 = \ldots = a_n = 0$, so $a_1 + a_2 + \ldots + a_n = 0$.
Thus $0_n \in W_1$.

CLAIM: $\forall$ $x = (a_1, a_2, \ldots, a_n)$, $y = (b_1, b_2, \ldots, b_n) \in W_1$, then $x + y \in W_1$.
PROOF: $x + y = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$. Summing its terms:
$(a_1 + b_1) + (a_2 + b_2) + \ldots + (a_n + b_n)$
$= (a_1 + a_2 + \ldots + a_n) + (b_1 + b_2 + \ldots + b_n)$ (F1, F_2 for F)
$= 0 + 0 = 0$.
Thus $x + y \in W_1$

CLAIM: $\forall c \in F$, $\forall x = (a_1, a_2, \ldots, a_n) \in W_1$, $cx \in W_1$.
PROOF: $cx = (ca_1, ca_2, \ldots, ca_n)$. Summing the terms:
$ca_1 + ca_2 + \ldots + ca_n$
$= c(a_1 + a_2 + \ldots + a_n) = c(0) = 0$.
Thus $cx \in W_1$.

By (THM 1.3), W_1 satisfies the necessary conditions to be SUBSP. F^n.

b) CLAIM: W_2 not SUBSP. F^n.
PROOF: For 0_n, $a_1 = a_2 = \ldots = a_n = 0$. Thus, $a_1 + a_2 + \ldots + a_n = 0 \neq 1$, so $0_n \notin W_2$. By (THM 1.3), W_2 is not SUBSP. F^n because it doesn't contain the zero vector of F^n.

1.3 # 22, 23, 25

22. Let F_1, F_2 be fields.

<u>DEF</u>: $E := \{ f : F_1 \to F_2 : f(-t) = f(t) \}$.

<u>DEF</u>: $O := \{ O : F_1 \to F_2 : f(-t) = -f(t) \}$.

<u>DEF</u>: an element of E is called <u>even</u>, and an element of O is called <u>odd</u>

Prove E, O SUBSP. $\mathcal{F}(F_1, F_2)$

<u>CLAIM</u>: $E, O \subseteq \mathcal{F}(F_1, F_2)$

This is immediate from the definitions of $\mathcal{F}(F_1, F_2)$, and E, O.

<u>DEF</u>. Let $z_2 := z_2(x_1) = 0_2$, $x_1 \in F_1$, $0_2 \in F_2$ and is the zero element in F_2. This is the zero vector of $\mathcal{F}(F_1, F_2)$ (since addition of any element of $\mathcal{F}(F_1, F_2)$ with z_2 results in an addition with the additive identity in F_2).

<u>CLAIM</u>: $z_2 \in E, O$.

<u>PROOF</u>: $\forall a \in F_1$, $z_2(-a) = 0 = z_2(a)$, so $z_2 \in E$.

$\forall a \in F_1$, $z_2(-a) = 0 = (-1)(0) = -z(a)$, so $z_2 \in O$.

<u>CLAIM</u>: $\forall f, g \in E$, $f + g \in E$.

$\forall h, i \in O$, $h + i \in O$.

<u>PROOF</u>: $\forall f, g \in E$, $\forall a \in F_1$

$(f+g)(-a) = f(-a) + g(-a) = f(a) + g(a) = (f+g)(a)$,

so $f+g \in E$.

$(h+i)(-a) = h(-a) + g(-a) = -h(a) - g(a) = -1(h(a) + g(a))$

$= -(h+g)(a)$, so $h+i \in O$.

<u>CLAIM</u>: $\forall c \in F_2$, $\forall f \in E$, $\forall g \in O$, $cf \in F$, $cg \in O$.

<u>PROOF</u>: $\forall a \in F_1$, $(cf)(-a) = cf(-a) = cf(a) = (cf)(a)$, so $cf \in E$,

$(cg)(-a) = cg(-a) = c(-g(a)) = -(cg)(a)$, so $cg \in O$.

BY (THM 1.3), the necessary conditions are met to prove E, O SUBSP. $\mathcal{F}(F_1, F_2)$

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1.3 # 23, 25

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23. Let W_1, W_2 SUBSP. v.s. V.

a) Prove that $W_1 + W_2$ SUBSP. V and contains both W_1 and W_2.

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Let $W = W_1 + W_2$ (i.e., $W = \{x + y : x \in W_1, y \in W_2\}$

CLAIM: $W \in V$? You mean $W \subseteq V$.

PROOF: $\forall w = x + y \in V$, $x \in W_1 \subseteq V$, $y \in W_2 \subset V$.

Since + is binary over vector addition $x + y \in V$.

$\therefore W \in V$

CLAIM: $0_V \in W$.

PROOF: All subspaces contain the zero vector of their superspace.

Thus $0_V \in W_1, W_2$, and $\exists x \in W = 0_V + 0_V = 0_V$.

CLAIM: $\forall x, y \in W$, $x + y \in W$.

PROOF: $\forall x = x_1 + x_2$, $y = y_1 + y_2 \in W$, $x_1, y_1 \in W_1$, $x_2, y_2 \in W_2$. Then $x + y = x_1 + x_2 + y_1 + y_2$
$= (x_1 + y_1) + (x_2 + y_2)$. By binaryness of + over v.s., $x_1 + y_1 \in W_1$, $x_2 + y_2 \in W_2$, thus $x + y \in W$

CLAIM: $\forall a \in F$, $\forall x \in W$, $ax \in W$.

PROOF: $\forall a \in F$, $\forall x = x_1 + x_2 \in W$, $x_1 \in W_1$, $x_2 \in W_2$,

$ax = a(x_1 + x_2) \xrightarrow{\text{(VS7 for V)}} = ax_1 + ax_2$

Scalar multiplication is external binary over a v.s., so $ax_1 \in W_1$, $ax_2 \in W_2$, so $ax \in W$.

By (THM 1.3), W satisfies the conditions to be a SUBSP. of V.

1.3 # 23, cont'd.

CLAIM: W contains both W_1, W_2.

PROOF: Since $0_V \in W_2$, $\forall x \in W_1$, $\exists w \in W = x + 0_V = x$

Similarly, since $0_V \in W_1$, $\forall y \in W_2$,

$\exists w = 0_V + y = y$. Thus $W_1, W_2 \subseteq W$.

b. Prove that any subspace Z of V that contains both W_1 and W_2 must also contain W.

CLAIM: Z contains $W_1, W_2 \Rightarrow Z$ contains W

PROOF: $\forall x \in W_1$, $\forall y \in W_2$, $x + y \in Z$ because $+$ is binary over vector addition. Therefore, the set of $\{x + y : x \in W_1, y \in W_2\} = W \subseteq Z$.

25. Let W_1 denote set of all polynomials in $P(F)$ s.t. in the representation $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, we have $a_i = 0$ when i even. Also define W_2 set of all polynomials in $P(F)$ s.t. in the representation: $g(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0$, we have $b_i = 0$ whenever i is odd. Prove $P(F) = W_1 \oplus W_2$

To show that $P(F)$ is the direct sum of W_1, W_2, need to show that: (by (DEF direct sum)):

1) W_1, W_2 subsp $P(F)$
2) $W_1 \cap W_2 = \{0_P\}$
3) $W_1 + W_2 = V$.

25.1) DEF: Let $0_P := 0 + 0x + 0x^2 + \dots$ be the zero vector of $P(F)$.

CLAIM: $0_P \in W_1, W_2$.

PROOF: 0_P is expressible as a polynomial with all coefficients $a_i = 0$. Then $a_i = 0$ when i even, so $0_P \in W_1$, and $a_i = 0$ when i odd, so $0_P \in W_2$.

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1.3 # 25, cont'd.

CLAIM: $\forall w, x \in W_1$, $w + x \in W_1$,
$\forall y, z \in W_2$, $y + z \in W_2$

PROOF: $\forall w = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \in W_1$,
$x = b_m x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0 \in W_1$,
$w + x = (a_j + b_j)x^j + (a_{j-1} + b_{j-1})^{j-1} + \ldots + (a_1 + b_1)x + (a_0 + b_0)$,
$\forall i$ even, $(a_i + b_i) = 0 + 0 = 0$, $\therefore w + x \in W_1$.

$\forall y = c_n x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0 \in W_2$,
$\forall z = d_m x^m + d_{m-1} x^{m-1} + \ldots + d_1 x + d_0 \in W_2$,
$y + z = (c_j + d_j)x^j + (c_{j-1} + d_{j-1})^{j-1} + \ldots + (c_1 + d_1)x + (c_0 + d_0)$,
$\forall i$ odd, $(c_i + b_i) = 0 + 0 = 0$, $\therefore y + z \in W_2$.

CLAIM: $\forall c \in F$, $\forall x \in W_1$, $\forall y \in W_2$, $ax \in W_1$, $ay \in W_2$.

PROOF: $\forall x = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \in W_1$.
Then $cx = (ca_n)x^n + (ca_{n-1})x^{n-1} + \ldots + (ca_1)x + (ca_0)$.
$\forall i$ even, $ca_i = c(0) = 0$, $\therefore cx \in W_1$.

$\forall y \in b_m x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0 \in W_2$,
Then $cy = (cb_m)x^m + (b_{m-1})x^{m-1} + \ldots + (cb_1)x + (cb_0)$.
$\forall i$ odd, $cb_i = c(0) = 0$, $\therefore cb \in W_2$.

By (THM 1.3), W_1, W_2 SUBSP $P(F)$.

25. 2) CLAIM: $W_1 \cap W_2 = \{0_p\}$

PROOF: Since W_1, W_2 SUBSP. $P(F)$, $0_p \in W_1, W_2$.
Next, choose $x = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \neq 0_p \in P(F)$.
Then, $\forall i$ s.t. $a_i \neq 0$, i must be odd or even. If i even, $x \notin W_2$. If i even, $x \notin W_1$.
Thus, $\forall x \neq 0_p$, $x \notin W_1 \cap W_2$.

$\therefore W_1 \cap W_2 = \{0_p\}$.

25.3) Show $W_1 + W_2 = V$.

<u>CLAIM:</u> $W_1 + W_2 \subseteq P(F)$

<u>PROOF:</u> $W_1 + W_2 = \{x + y : x \in W_1, y \in W_2\}$. Since $W_1, W_2 \subseteq$
$\forall x \in W_1 \subseteq P(F), \forall y \in W_2 \subseteq$, $x+y \in P(F)$, thus.
$W_1 + W_2 \subseteq P(F)$

<u>CLAIM:</u> $P(F) \subseteq W_1 + W_2$

<u>PROOF:</u> $\forall x \in P(F)$, x can be represented as (for some odd n)

$$\begin{aligned}
x &= a_n x^n + b_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_3 x^3 + b_2 x^2 + a_1 x + b_0 \\
&= (a_n + 0) x^n + (0 + b_{n-1}) x^{n-1} + (a_{n-2} + 0) x^{n-2} + \dots \\
&\quad + (a_3 + 0) x^3 + (0 + b_2) x^2 + (a_1 + 0) x + (0 + b_0) \\
&= (a_n x^n + 0 x^{n-1} + a_{n-2} x^{n-2} + \dots + a_3 x^3 + 0 x^2 + a_1 x + 0) \\
&\quad + (0 x^n + b_{n-1} x^{n-1} + 0 x^{n-2} + \dots + 0 x^3 + b_2 x^2 + 0 x + b_0).
\end{aligned}$$

Thus x can be expressed as $y + z$, $y \in W_1$, $z \in W_2$,
so $P(F) \subseteq W_1 + W_2$.

Since $W_1 + W_2 \subseteq P(F)$, $P(F) \subseteq W_1 + W_2$, so $W_1 + W_2 = P(F)$.

Since the three conditions for a direct sum are satisfied,
$W_1 \oplus W_2 = P(F)$.

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1.4 # 2ae, 4ab, 13, 17.

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Solve the linear systems using Gaussian elimination.

2a) $2x_1 - 2x_2 - 3x_3 = -2$
$3x_1 - 3x_2 - 2x_3 + 5x_4 = 7$
$x_1 - x_2 - 2x_3 - x_4 = -3$

$x_1 - x_2 - 2x_3 - x_4 = -3$
$0 + 0 + x_3 + 2x_4 = 4$
$0 + 0 + 4x_3 + 8x_4 = 16$

$x_1 - x_2 - 2x_3 - x_4 = -3$
$0 + 0 + x_3 + 2x_4 = 4$
$0 + 0 + \cancel{x_3} + \cancel{2x_4} = \cancel{4} \Rightarrow 0 + 0 + 0 + 0 = 0.$

Let $x_3 = t$. Then $t + 2x_4 = 4 \Rightarrow x_4 = 2 - \frac{t}{2}$.

Let $x_1 = s$. Then $2s - 2x_2 - 3t = -2 \Rightarrow x_2 = 1 + s - \frac{3}{2}t$.
Then a solution is $(x_1, x_2, x_3, x_4) = (s, 1+s-\frac{3}{2}t, t, 2-\frac{t}{2})$.
$\forall t, s \in \mathbb{R}$.

c) $x_1 + 2x_2 - 4x_3 - x_4 + x_5 = 7$
$-x_1 + 10x_3 - 3x_4 - 4x_5 = -16$
$2x_1 + 5x_2 - 5x_3 - 4x_4 - x_5 = 2$
$4x_1 + 11x_2 - 7x_3 - 10x_4 - 2x_5 = 7.$

$x_1 + 2x_2 - 4x_3 - x_4 + x_5 = 7$
$2x_2 \quad 6x_3 - 4x_4 - 3x_5 = -9$
$x_2 + 3x_3 - 2x_4 - 3x_5 = -12$
$3x_2 + 9x_3 - 6x_4 - 6x_5 = -21$

$x_1 + 2x_2 - 4x_3 - x_4 + x_5 = 7$
$x_2 + 3x_3 - 2x_4 - 3x_5 = -12$
$6x_3 - 4x_4 - 3x_5 = -9$
$3x_5 = 15 \Rightarrow x_5 = 5$

Let $x_3 = t$. Then $6t - 4x_4 - 3(5) = -9$

$\Rightarrow x_4 = \dfrac{6t + 9 - 15}{4} = \frac{3}{2}t - \frac{3}{2}$

$x_2 + 3t - 2\left(\frac{3}{2}t - \frac{3}{2}\right) - 3(5) = -12$

$x_2 = \cancel{-12} + \cancel{15} - \cancel{3t} + \cancel{3t} \cancel{-3} = 0$

$x_1 + \cancel{2(0)} - 4t - \left(\frac{3}{2}t - \frac{3}{2}\right) + 5 = 7$

$x_1 = 7 + 4t + \frac{3}{2}t - \frac{3}{2} - 5$

$= \frac{1}{2} + \frac{11}{2}t$

Thus a solution is $(x_1, x_2, x_3, x_4, x_5) = \left(\frac{1}{2} + \frac{11}{2}t,\ 0,\ \frac{3}{2}t - \frac{3}{2},\ t,\ 5\right)$

$\forall\, t \in \mathbb{R}$.

4. Determine if first polynomial can be expressed as a lin. comb. of other two.

4 a. $x^3 - 3x + 5 = a(x^3 + 2x^2 - x + 1) + b(x^3 + 3x^2 - 1)$

$$\begin{cases} a + b = 1 \\ 2a + 3b = 0 \\ -a = -3 \quad \Rightarrow \quad a = 3,\ a + b = 1 \Rightarrow b = 1 - 3 = -2 \\ a - b = 5. \end{cases}$$

Check for consistency:

$3 + (-2) = 1,\quad 2(3) + 3(-2) = 0,\quad -3 = -3,\quad 3 - (-2) = 5.$ ✓

Yes. (solution: $(a, b) = (1, -2)$).

b. $4x^3 + 2x^2 - 6 = a(x^3 - 2x^2 + 4x + 1) + b(3x^3 - 6x^2 + x + 4)$

$$\begin{cases} a + 3b = 4 \\ \cancel{-2a - 6b = 2} \\ 4a + b = 0 \\ a + 4b = -6 \end{cases}$$

multiply by -2

Subtract: $b = -10$.

$4a + (-10) = 0 \Rightarrow a = \frac{5}{2}$.

$a + 4b = \frac{5}{2} + 4(-10) = \frac{5}{2} - 40 = \frac{-85}{2} \neq -6.$

This system is not consistent, thus, the first polynomial cannot be expressed as a linear comb. of the other two.

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1.4 # 13, 17.

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13. Show that if $S_1, S_2 \subseteq$ v.s. V s.t. $S_1 \subseteq S_2$, then span(S_1) $\subseteq$ span(S_2). In particular, if $S_1 \subseteq S_2$ and span(S_1) $= V$, deduce that span(S_2) $= V$.

Can be made even more efficient.

CLAIM: If $S_1, S_2 \subseteq$ v.s. V, $S_1 \subseteq S_2$, then span(S_1) $\subseteq$ span(S_2)

PROOF: For any linear combination $x \in$ span(S),
$x = a_1x_1 + a_2x_2 + \dots + a_nx_n$, then $a_1, a_2, \dots, a_n \in F$, $x_1, x_2, \dots, x_n \in S_1 \subseteq S_2$. Since all $x_i \in S_2$, this is also a linear combination in S_2, so span(S_1) $\subseteq$ span(S_2).

CLAIM: If $S_1 \subseteq S_2$, span(S_1) $= V$, then span(S_2) $= V$.

PROOF: By the above proof, span(S_1) $= V \subseteq$ span(S_2)
By (THM 1.5), span(S_2) $\subseteq V$. Therefore, span(S_2) $= V$.

17. Let W subsp v.s. V. Under what conditions are there only a finite # of distinct subsets S of W s.t. S generates W?

CLAIM: If W finite, then there are only a finite # of distinct generating subsets.

PROOF: If W finite, then there are only a finite # of all subsets (finite power set), so there can only be a finite # of generating subsets.

LEMMA: If W is an infinite v.s., $\forall x \in W$, let $S = W \setminus \{x\}$. Then, S is a generating set of W.

PROOF: Choose an $x \in W$.
Case 1: $x = 0$ $\forall y \in S$, $0 \cdot y = 0$, so $x \in$ span(S).
Case 2: $x \neq 0$. $\forall y \in S$, $\exists (x-y) \in S$, since $x, y \in W$, $x - y \in W$, and $x - y \neq x \in S$. Then $x \in$ span(S).

Thus $x \in$ span(S). Also, $\forall z \in S$, $z = 1 \cdot z \Rightarrow z \in$ span(S).
Thus $W = S \cup \{x\} \subseteq$ span(S). By (THM 1.5) span(S) $\subseteq W$, thus span(S) $= W$ and S generates W.

CLAIM: If W infinite, there are infinitely many generating subsets of W.

PROOF: By (LEM 2), any subset $S = W \setminus \{x\}$, $\forall x \in W$ is a generating subset of W. Since W is infinite, there are an infinite # of generating subsets.

Since W finite $\Rightarrow$ W has finitely many generating subsets, and W infinite $\Rightarrow$ W has infinitely many generating subsets, this implies the tautology:

W finite $\Leftrightarrow$ W has a finite # of generating subsets.