Jonathan Lam
Prof. Mutchler
MA326
Lin. Alg.
11/7/19

PSET 7

33/35

4.1 # 12, 22

4.2.30 | 4
4.4.5 | 5
5.1.12 | 5
5.1.19 | 4.5
18.5

12. The classical adjoint of a 2x2 matrix $A \in M_{2\times 2}(F)$ is the matrix:

$$C = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$$

a) CLAIM: $CA = AC = [\det(A)]I_2$

PF: $CA = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & A_{22}A_{12} - A_{12}A_{22} \\ -A_{11}A_{21} + A_{11}A_{21} & -A_{12}A_{21} + A_{11}A_{22} \end{pmatrix}$

$= \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & 0 \\ 0 & A_{11}A_{22} - A_{12}A_{21} \end{pmatrix} = (A_{11}A_{22} - A_{21}A_{12})\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = [\det(A)]I_2$

and $AC = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} = \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} & -A_{11}A_{12} + A_{11}A_{12} \\ A_{21}A_{22} - A_{21}A_{22} & -A_{12}A_{21} + A_{11}A_{22} \end{pmatrix}$

$= \begin{pmatrix} A_{11}A_{22} - A_{21}A_{12} & 0 \\ 0 & A_{11}A_{22} - A_{12}A_{21} \end{pmatrix} = (A_{11}A_{22} - A_{21}A_{12})\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = [\det(A)]I_2$

b) CLAIM: $\det(C) = \det(A)$

PF: $\det(C) = C_{11}C_{22} - C_{12}C_{21} = (A_{22})(A_{11}) - (-A_{12})(-A_{21})$
$= A_{11}A_{22} - A_{12}A_{21} = \det(A)$

c) CLAIM: The classical adjoint of A^t is C^t.

PF: $A^t = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix}$

classical adjoint of $A^t = \begin{pmatrix} (A^t)_{22} & -(A^t)_{12} \\ -(A^t)_{21} & (A^t)_{11} \end{pmatrix} = \begin{pmatrix} A_{22} & -A_{21} \\ -A_{12} & A_{11} \end{pmatrix} = C^t$

d) CLAIM: If A inv., then $A^{-1} = [\det(A)]^{-1}C$

PF: This statement is proved in (THM 4.2).

11. Let $\delta : M_{2\times 2}(F) \to F$ be a fn. with the following properties:

i) δ is a linear fn. of each row of the matrix when the other row is held fixed.

ii) If the two rows of $A \in M_{2\times 2}(F)$ are equal, then $\delta(A) = 0$.

iii) If I is the 2x2 identity matrix, then $\delta(I) = 1$.

Prove that $\delta(A) = \det(A)$ $\forall A \in M_{2\times 2}(F)$.

<u>LEMMA</u>: $\delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$

<u>PF</u>:

$$\delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \underset{\text{by (i)}}{=} \delta\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \cancel{(-1)\underbrace{\delta\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}}_{=0 \text{ by (ii)}}} \underset{\text{by (i)}}{=} (-1)\delta\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \cancel{\underbrace{\delta\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}}_{=0 \text{ by (ii)}}}$$

$$= (-1)\Bigg(\underbrace{\delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{=1 \text{ by (iii)}} + \cancel{\underbrace{\delta\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}}_{=0 \text{ by (ii)}}}\Bigg) = -1.$$

<u>CLAIM</u>: $\forall A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(F)$, $\delta(A) = ad - bc = \det(A)$.

<u>PF</u>:

$$\delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\delta\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} + b\delta\begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix}$$

$$= a\Bigg(c\underbrace{\delta\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}}_{=0 \text{ by (ii)}} + d\underbrace{\delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{=1 \text{ by (iii)}}\Bigg) + b\Bigg(c\underbrace{\delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{=-1 \text{ by (LEMMA)}} + d\underbrace{\delta\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}}_{=0 \text{ by (ii)}}\Bigg)$$

$$= a(0 + d) + b(c(-1) + 0)$$

$$= ad - bc$$

PSET 7

Jonathan Lam
Prof. Mintchev
MA 326
Lin. Alg.
11/7/19

4.2 # 28, 29, 30

28. Compute $\det(E_i)$ if E_i is an elementary matrix of type i

Case 1: Let $i=1$. Then E_1 is obtained by interchanging two rows of I, so by (THM 4.5) $\det(E_1) = -\det(I) = -1$.

Case 2: Let $i=2$. Then E_2 is obtained by multiplying one row j of I_n by some $k \in F$. Then that row is $e_j \in F^n$, and a cofactor expansion along that row of E_2 is $\sum_{i=1}^{n} (-1)^{i+j} E_{ij} \left| \tilde{E}_{ij} \right|$

$= (-1)^{j+j}(k)(1) = k$, since the only nonzero element of that row is the element of the diagonal, and $\tilde{E}_{ii} = I_{n-1} \Rightarrow \det(\tilde{E}_{ii}) = 1$.

Case 3: Let $i=3$. Then E_3 is determined by adding a multiple of one row of I to another. Then $\det(E_3) = \det(I)$ by (THM 4.6).

29. Prove that if E is an elementary matrix, then $\det(E^t) = \det(E)$.

LEM: Type 1 and Type 2 elem. matrices are symmetric.
PF: For type 1 elem. matrices, two rows of I_n are swapped; let these be rows j and k of I_n. Then $E_{jk} = E_{kj} = 1$, $E_{jj} = E_{kk} = 0$, and the other entries are all untouched; therefore E remains symmetric. Similarly, a type 2 elem. matrix is clearly symmetric, since it is a diagonal matrix.

Case 1: For type 1 and type 2 elementary matrices:
PF: By the Lemma, Type 1 and type 2 elementary matrices E_i are symmetric, so $E_i = E_i^t$, and $\det(E_i) = \det(E_i^t)$.

Case 2: For type 3 elementary matrices.
PF: The transpose of a type 3 elementary matrix formed by adding k times row i to row j is also a type 3 elementary matrix formed by adding k times row j to row

i. Since E_3 and E_3^t are type 3 elementary matrices, by (exercise 28) $\det(E_3) = \det(E_3^t) = 1$.

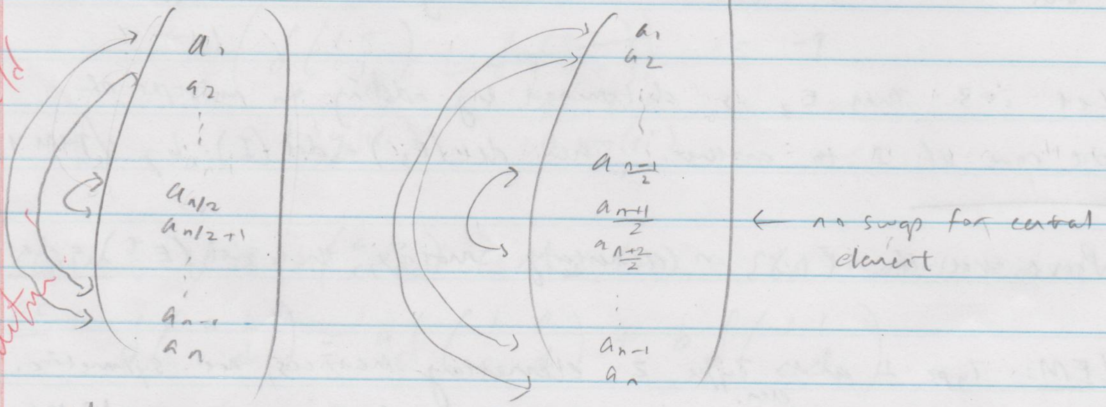
---

30. Let the rows of $A \in M_{n\times n}(F)$ be $a_1, a_2, \ldots, a_n$, and let B be the matrix in which the rows are $a_n, a_{n-1}, \ldots, a_1$. Calculate $\det(B)$ in terms of $\det(A)$.

4.5

This should be done by induction

if n even: if n odd:



$n/2$ swaps $\frac{n-1}{2}$ swaps

There are $\lfloor n/2 \rfloor$ row interchanges. By (THM 4.5), each interchange flips the sign of the determinant. Thus, $\det(B) = (-1)^{\lfloor n/2 \rfloor} \det(A)$.

Jonathan Lu
Prof. Mitchell
MA 326
Lin. Alg.
11/7/19

PSET 7

4.3 # 11, 12, 15

11. A matrix $M \in M_{n\times n}(\mathbb{C})$ is called skew-symmetric if $M^t = -M$. Prove that if M is skew-symmetric and n is odd, then M is not invertible. What happens if n is even?

CLAIM: If M skew symmetric, n odd, then n not invertible.

PF: By (THM. 4.8), $\det(A) = \det(A^t)$. Thus

$\det(M) = \det(M^t) = \det(-M) = (-1)^n \det(M)$.

(obtained by n row scaling by (-1)).

Thus $\det(M) = (-1)\det(M) \Rightarrow \det(M) = 0 \Rightarrow$ M is not invertible.

This restriction is not true if n is even, since the above restriction is $\det(M) = (-1)^n\det(M) \Rightarrow \det(M) = \det(M)$ if n is even. Both invertible and non-invertible matrices with even n exist, e.g. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (invertible) and $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (non invertible).

---

12. A matrix $Q \in M_{n\times n}(\mathbb{R})$ is called orthogonal if $QQ^t = I$. Prove that if Q is orthogonal, then $\det(Q) = \pm 1$.

PF: By (THM 4.8), $\det(Q) = \det(Q^t)$. By (THM 4.7), $\det(QQ^t) = \det(Q)\det(Q^t) = \det^2(Q) = \det(I) = 1$

Thus $\det^2(Q) = 1$, so $\det(Q) = \pm 1$.

---

15. Prove that if $A, B \in M_{n\times n}(F)$ are similar, then $\det(A) = \det(B)$.

PF: By (DEF invertibility), $\exists\, Q$ inv. $\in M_{n\times n}(F)$ s.t. $A = Q^{-1}BQ$. By (COR. to THM. 4.7), $\det(Q^{-1}) = (\det(Q))^{-1}$. By (THM 4.7), determinants of products of matrices are equivalent to products of the determinants, i.e., $\det(A) = \det(Q^{-1}BQ)$
$= \det(Q^{-1})\det(B)\det(Q) = \frac{1}{\det(Q)}\det(B)\det(Q) = \frac{\cancel{\det(Q)}}{\cancel{\det(Q)}}\det(B)$
$= \det(B)$

4.4 #5, 6

5. Suppose that $M \in M_{(m+n)\times(m+n)}(F)$ can be written in the form, $M = \begin{pmatrix} A & B \\ 0 & I_m \end{pmatrix}$, where A is square. Prove that $\det(M) = \det(A)$.

4

$\sqrt{8}$

induction

PF: Denote the square matrix consisting of the first i rows and i columns of M as M_i. Calculate the determinant of M by the cofactor expansion along row $m+n$ (the last row). The determinant is $\sum_{i=1}^{m+n} (-1)^{i+(m+n)} M_{(m+n)i} \, |\tilde{M}_{(m+n)i}|$. Since all but the last term are zeroes, this sum is equal to $(-1)^{(m+n)+(m+n)} M_{(m+n)(m+n)} |\tilde{M}_{(m+n)(m+n)}| = (1)(1)\,|\tilde{M}_{(m+1)(m+1)}|$. Note that this is the determinant of $M_{(m+n-1)}$, so $\det(M_{m+n}) = \det(M_{m+n-1})$. This identical logic can be applied inductively to $M_{m+n-1}, M_{m+n-2}, \ldots, M_{n+1}$, since the determinant of each of these matrices may be evaluated as a cofactor expansion of their respective bottom rows, which consist of all zeroes with a leading 1 in the last column. Hence,

$\det(M) = \det(M_{m+n}) = \det(M_{m+n-1}) = \ldots = \det(M_{n+1}) = \det(M_n) = \det(A)$

---

6. Prove that if $M \in M_{(m+n)\times(m+n)}(F)$ can be written in the form $M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, where A and C are square matrices, then $\det(M) = \det(A) \cdot \det(C)$.

PF: Let A be an $n\times n$ matrix and C be an $m \times m$ matrix. Let $F = \begin{pmatrix} I_n & 0 \\ 0 & C \end{pmatrix}$. The determinant of F may be obtained by doing the cofactor expansion along the first row, which evaluates to $\sum_{i=1}^{n} (-1)^{i+1} F_{1i} |\tilde{F}_{1i}| = (-1)^{1+1}(1)\,|\tilde{F}_{11}| = |\tilde{F}_{11}|$. This same logic can be recursively applied n times (by similar logic to that stated in (exercise 5, above) to obtain $\det(F) = \det(C)$.

Now, let $G = \begin{pmatrix} A & B \\ 0 & I_m \end{pmatrix}$. Multiplying F and G, we obtain:

$$(FG)_{ij} = \begin{cases} \sum_{k=1}^{n} (e_i)_k A_{kj} = A_{ij}, & i,j \le n \\ \sum_{k=1}^{n} (e_i)_k B_{k(j-n)} = B_{i(j-n)}, & i \le n,\ j > n \\ 0, & i > n,\ j \le n \\ \sum_{k=1}^{m} C_{(i-n)(k+n)} (e_k)_{(k+n)} = C_{(i-n)(j-n)}, & i,j > n \end{cases} \Rightarrow FG = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = M.$$

Then $\det(M) = \det(F)\det(G) = \det(C) \cdot \det(A)$

Jonathan Lam
Prof Matcheu
MA 326
Lin Alg.
11/7/19.

5.1 # 3cd, 4 deg, 7, 12, 14, 15, 19, 22.

3. For each of the following matrices $A \in M_{n\times n}(F)$:
i) Determine all eigenvalues of A.
ii) For each eigenvalue of A, find the set of e-vects corresponding to λ.
iii) If possible, find a basis for F^n consisting of eigenvalues of A.
iv). If successful in finding such a basis, determine an invertible matrix Q and diagonal matrix D st. $D = Q^{-1}AQ$.

c.) $A = \begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix}$, $F = \mathbb{C}$.

i) $\begin{vmatrix} i-t & 1 \\ 2 & -i-t \end{vmatrix} = (i-t)(-i-t) - 2 = 0$

$-i^2 + t^2 - 2 = t^2 - 1 = 0 \Rightarrow t = \pm 1.$ (e-vals)

ii) for $\lambda = 1$: $\left(\begin{array}{cc|c} i-1 & 1 & 0 \\ 2 & -i-1 & 0 \end{array}\right) \Rightarrow (i-1)x_1 = -x_2 \Rightarrow x_2 = (1-i)x_1$

so solution set $= t\begin{pmatrix} 1 \\ 1-i \end{pmatrix}$

for $\lambda = -1$: $\left(\begin{array}{cc|c} i+1 & 1 & 0 \\ 2 & -i+1 & 0 \end{array}\right) \Rightarrow (i+1)x_1 = -x_2 \Rightarrow x_2 = (-i-1)x_1$

So solution set $= t\begin{pmatrix} 1 \\ -i-1 \end{pmatrix}$

iii) $\beta = \left\{ \begin{pmatrix} 1 \\ 1-i \end{pmatrix}, \begin{pmatrix} 1 \\ -i-1 \end{pmatrix} \right\}$

iv) By (THM 5.1), $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

By (COR to THM. 2.23), $Q = \begin{pmatrix} 1 & 1 \\ 1-i & -i-1 \end{pmatrix}$

d) $A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix}$, $F = \mathbb{R}$.

i) $\begin{vmatrix} 2-t & 0 & -1 \\ 4 & 1-t & -4 \\ 2 & 0 & -1-t \end{vmatrix} = (1-t)\begin{vmatrix} 2-t & -1 \\ 2 & -1-t \end{vmatrix} = (1-t)\left((2-t)(-1-t) - (-1)(2)\right)$

$= (1-t)(\cancel{-2} - t + t^2 + \cancel{2}) = t(1-t)(t-1) = -t(t-1)^2$

$\Rightarrow t = 0, 1$ (e-vals).

14

ii) For $\lambda = 0$:

$$\left(\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 4 & 1 & -4 & 0 \\ 2 & 0 & -1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 4 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

then let $x_3 = t$, $x_2 = 2t$, $x_1 = \frac{1}{2}t \Rightarrow$ soln. set $= t\begin{pmatrix} \frac{1}{2} \\ 2 \\ 1 \end{pmatrix}$

For $\lambda = 1$:

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 4 & 0 & -4 & 0 \\ 2 & 0 & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \text{soln. set}$$

$\Rightarrow$ soln. set $= t\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

iii) β: $\left\{\begin{pmatrix} \frac{1}{2} \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$

iv). $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$Q = \begin{pmatrix} \frac{1}{2} & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

Jonathan Lam
Prof. Mintchev
MA326
Lin. Alg.
11/7/19

PSET 7

5.1 # 4 d,e, 7, 12, 14, 15, 19, 22

4. For each linear operator T on V, find the eigenvalues of T and an ordered basis β for V s.t. $[T]_\beta$ is a diagonal matrix.

d). $V = P_1(\mathbb{R})$, $T(ax+b) = (-6a+2b)x + (-6a+b)$

$\beta = \{1, x\}$

$[T]_\beta = \begin{pmatrix} 1 & -6 \\ 2 & -6 \end{pmatrix}$

$\begin{vmatrix} 1-t & -6 \\ 2 & -6-t \end{vmatrix} = (1-t)(-6-t) + 12 = 0$

$= -6 + 5t + t^2 + 12 = t^2 + 5t\ 6 = 0$

$\Rightarrow (t+3)(t+2) = 0 \Rightarrow t = -2, -3$

Finding e-vects:

$\left(\begin{array}{cc|c} 3 & -6 & 0 \\ 2 & -4 & 0 \end{array}\right) = \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow$ solns $= t\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\left(\begin{array}{cc|c} 4 & -6 & 0 \\ 2 & -3 & 0 \end{array}\right) = \left(\begin{array}{cc|c} 2 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow$ solns. $= t\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$\{\lambda\} = \{-2, -3\}$

basis of e-vects of $P_1(\mathbb{R}) = \{2+x,\ 3+2x\}$

---

e) $V = P_2(\mathbb{R})$, $T(f(x)) = xf'(x) + f(2)x + f(3)$

$\beta = \{1, x, x^2\}$

$[T]_\beta = \begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix}$

$\begin{vmatrix} 1-t & 3 & 9 \\ 1 & 3-t & 4 \\ 0 & 0 & 2-t \end{vmatrix} = (2-t)\begin{vmatrix} 1-t & 3 \\ 1 & 3-t \end{vmatrix}$

$= (2-t)((1-t)(3-t) - 3(1))$

$= (2-t)(\cancel{3} - 4t + t^2 - \cancel{3}) = t(2-t)(t-4) = 0$

$\Rightarrow t = 0, 2, 4$ (e-vals).

Finding e-vects:

$$\left(\begin{array}{ccc|c} 1 & 3 & 9 & 0 \\ 1 & 3 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right) = \begin{pmatrix} 1 & 3 & 9 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix} = \left(\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \text{solns.} = t \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -1 & 3 & 9 & 0 \\ 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 1 & 4 & 0 \\ 0 & 4 & 13 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \text{solns} = t \begin{pmatrix} 3 \\ 13 \\ -4 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -3 & 3 & 9 & 0 \\ 1 & -1 & 4 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \text{solns} = t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$\{\lambda\} = \{0, 2, 4\}$

basis of $P_2(\mathbb{R}) = \{3 - x,\ 3 + 13x - 4x^2,\ 1 + x\}$

g) $V = P_3(\mathbb{R})$, $T(f(x)) = xf'(x) + f''(x) - f(2)$

$\beta = \{1, x, x^2, x^3\}$

$$[T]_\beta = \begin{pmatrix} -1 & -2 & -2 & -8 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

finding e-vals: $\begin{vmatrix} -1-t & -2 & -2 & -8 \\ 0 & 1-t & 0 & 6 \\ 0 & 0 & 2-t & 0 \\ 0 & 0 & 0 & 3-t \end{vmatrix} = (-1-t)(1-t)(2-t)(3-t)$

$\Rightarrow$ evals $= -1, 1, 2, 3$

$$\left(\begin{array}{cccc|c} 0 & -2 & -2 & -8 & 0 \\ 0 & 2 & 0 & 6 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{array}\right) = \left(\begin{array}{cccc|c} 0 & 1 & 1 & 4 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) = \left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \text{soln. set} = t \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Jonathan Lam
Prof. Mintchev
MA 326
Lin. Alg.
11/7/19.

PSET 7

5.1 #4g, 7, 12, 14, 15, 19, 22.

(4g, cont'd.)

$$\left(\begin{array}{cccc|c} -2 & -2 & -2 & -8 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right) = \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \text{soln. set} = t\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} -3 & -2 & -2 & -8 & 0 \\ 0 & -1 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) = \left(\begin{array}{cccc|c} 3 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \text{soln. set} = t\begin{pmatrix} -2 \\ 0 \\ 3 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} -4 & -2 & -2 & -8 & 0 \\ 0 & -2 & 0 & 6 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{cccc|c} 2 & 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \text{soln. set} = t\begin{pmatrix} -\frac{7}{2} \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

$\{\lambda\} = \{-1, 1, 2, 3\}$

basis for $P_4(\mathbb{R}) = \{1,\ 1-x,\ -2+3x^2,\ -\frac{7}{2}+3x+x^3\}$

---

7. DEF. (determinant of a linear operator.) Let T be a linear operator on a finite-dim. v.s. V. We define the determinant of T, denoted $\det(T)$ as follows: choose any o.B. β for V, and define $\det(T) = \det([T]_\beta)$.

a) Prove that the preceding definition is independent of the choice of the choice of an o.B. for V. That is, if β and γ are two o.B.s for V, then $\det([T]_\beta) = \det([T]_\gamma)$.

PF: By (THM. 2.23), $[T]_\beta$ and $[T]_\gamma$ are similar (i.e., $[T]_\beta = Q^{-1}[T]_\gamma Q$, where Q is the change-of-basis matrix converting β to γ coordinates). Since Q is invertible, $\det(Q^{-1}) = (\det(Q))^{-1}$, and $\det([T]_\beta) = \det(Q^{-1}[T]_\gamma Q)$

$= \det(Q^{-1})\det([T]_\gamma)\det(Q) = \frac{\cancel{\det(Q)}}{\cancel{\det(Q)}}\det([T]_\gamma) = \det([T]_\gamma)$

b.) Prove that T invertible $\Leftrightarrow$ $\det(T) \neq 0$.

PF: Fix some basis β of V. T invertible $\Leftrightarrow$ $[T]_\beta$ inv. $\Leftrightarrow$ $\det([T]_\beta) = \det(T) \neq 0$.

c). Prove that if T invertible, then $\det(T^{-1}) = (\det(T))^{-1}$.

PF: Fix some basis β of V. $\det(T^{-1}) = \det([T^{-1}]_\beta) = \det([T]_\beta^{-1}) = (\det([T]_\beta))^{-1}$
$= (\det(T))^{-1}$.

d). Prove that if U is also a linear operator on V, then $\det(TU) = \det(T) \cdot \det(U)$.

PF: Fix some basis β of V. $\det(TU) = \det([TU]_\beta) = \det([T]_\beta [U]_\beta)$
$= \det([T]_\beta) \det([U]_\beta) = \det(T) \cdot \det(U)$.

e). Prove that $\det(T - \lambda I_V) = \det([T]_\beta - \lambda I)$ for any scalar λ and any O.B. β for V.

PF: Fix a scalar λ and an O.B. β for V.
$\det(T - \lambda I_V) = \det([T - \lambda I_V]_\beta) = \det([T]_\beta - [\lambda I_V]_\beta)$
$= \det([T]_\beta - \lambda[I_V]_\beta) = \det([T]_\beta - \lambda I)$.

12. a) Prove that similar matrices have the same characteristic polynomial.

PF. Let, A, B, and Q be $n \times n$ matrices, s.t. Q is invertible and $A = Q^{-1}BQ$ (i.e. A similar to B). Then
$A - \lambda I_n = Q^{-1}BQ - \lambda Q^{-1}Q = Q^{-1}(BQ - \lambda Q) = Q^{-1}(B - \lambda I_n)Q$.
Thus $f_A(t) = \det(A - tI_n) = \det(B - tI_n) = f_B(t)$
(exercise 4.3 #15)

b). Show that the definition of the characteristic polynomial of a linear operator over a finite-dim. v.s. V is independent of the choice of basis for V.

PF: Let $T \in \mathcal{L}(V)$, and β, γ be distinct bases for V. Then the characteristic polynomial of T w.r.t. β is $\det([T]_\beta - \lambda I)$, and the characteristic polynomial of T w.r.t. γ is $\det([T]_\gamma - \lambda I)$. Since $[T]_\beta = Q^{-1}[T]_\gamma Q$ (THM 2.23), $[T]_\beta$ is similar to $[T]_\gamma$, so the characteristic polynomials are equal by (part a).

PSET #7

Jonathan Lam
Prof. Mintchev
MA326
Lin. Alg
11/7/19

5.1 # 14, 15, 19, 22.

14. For any square matrix A, prove that A and A^t have the same characteristic polynomial (and hence the same e-vals).

PF: $f_{A^t}(t) = \det(A^t - tI_n) = \det((A - tI_n)^t) = \det(A - tI_n) = f_A(t)$

15. a) Let T be a lin. operator on a v.s. V, and let x be an e-vect of T corresponding to the e-val λ. For any positive integer m, prove that x is an e-vect of T^m corresponding to the e-val λ^m.

PF: (by induction) Base case: $T^1(x) = \lambda^1 x$.

Inductive hyp: assume $T^{n-1}(x) = \lambda^{n-1}x$, $n > 1$. Then $T^n(x) = \lambda^n x$.

Proof of inductive hyp: $T^n(x) = T(T^{n-1}(x)) = T(\lambda^{n-1}x) = \lambda^{n-1}T(x)$
$= \lambda^{n-1}\lambda x = \lambda^n x$. Thus $T^m(x) = \lambda^m x \quad \forall m \in \mathbb{Z}^+$

b) Let $A \in M_{n\times n}(F)$, and let x be an e-vect of A corresponding to the e-val λ. For any positive integer m, prove that x is an e-vect of A^m corresponding to the e-val λ^m.

PF: (by induction) Base case: $A^1 x = \lambda^1 x$.

Inductive hyp: assume $A^{n-1}(x) = \lambda^{n-1}x$, $n > 1$. Then $A^n x = \lambda^n x$.

Proof of inductive hyp: $A^n x = A^1(A^{n-1}x) = A(\lambda^{n-1}x) = \lambda^{n-1}(Ax)$
$= \lambda^{n-1}\lambda x = \lambda^n x$. Thus $A^m(x) = \lambda^m x \quad \forall m \in \mathbb{Z}^+$.

19. Let $A, B \in M_{n\times n}(F)$ and similar. Prove that there exists an n-dim v.s. V, a lin. op. T on V, and o.B.'s β and γ for V s.t. $A = [T]_\beta$, $B = [T]_\gamma$.

5

PF: Let $V = F^n$, β = s.o.B. of V, $T = L_A$, and $B = Q^{-1}AQ$.

Thus $[L_A]_\beta = A$. Also, Q is an inv. $n\times n$ matrix. By (exercise 13 from 2.5), $\exists\, \gamma$ o.B. of V s.t. $Q = [I_V]_\gamma^\beta$. Thus $B = Q^{-1}AQ$
$= [I_V]_\beta^\gamma [L_A]_\beta [I_V]_\gamma^\beta = [L_A]_\gamma$.

22. a) Let T be a lin. op. over a v.s. V over the field F, and let $g(t) = \sum_{n=0}^{\infty} a_n t^n \in P(F)$ be an arbitrary polynomial. Prove that if x is an e-vect of T corresponding to the e-val. λ, then $(g(T))(x) = g(\lambda)x$

<u>PF</u>: $(g(T))(x) = \left(\sum_{n=0}^{\infty} a_n T^n\right)(x) \underbrace{=}_{\text{linearity of } T} \sum_{n=0}^{\infty} a_n T^n(x) \underbrace{=}_{\text{(exercise 15a)}} \sum_{n=0}^{\infty} a_n \lambda^n x$

$= \left(\sum_{n=0}^{\infty} a_n \lambda^n\right) x = g(\lambda)x.$

b) Let $A \in M_{n\times n}(F)$, $g(t)$ as declared in (part a). Prove that if x is an e-vect of A with corresponding e-val λ, then $(g(A))x = g(\lambda)x$.

<u>PF</u>: $(g(A))x = \left(\sum_{n=0}^{\infty} a_n A^n\right)x \underbrace{=}_{\text{linearity of mat. mult.}} \sum_{n=0}^{\infty} a_n A^n x \underbrace{=}_{\text{(exercise 15b)}} \sum_{n=0}^{\infty} a_n \lambda^n x = \left(\sum_{n=0}^{\infty} a_n \lambda^n\right)x = g(\lambda)x$

c) Verify (part b) for $g(t) = 2t^2 - t + 1$, $x = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\lambda = 4$, $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$

$g(A) = 2\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}^2 - \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2\begin{pmatrix} 7 & 6 \\ 9 & 10 \end{pmatrix} + \begin{pmatrix} 0 & -2 \\ -3 & -1 \end{pmatrix} = \begin{pmatrix} 14 & 10 \\ 15 & 19 \end{pmatrix}$

$g(\lambda) = 2(4^2) - 4 + 1 = 29$

$(g(A))x = \begin{pmatrix} 14 & 10 \\ 15 & 19 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 58 \\ 87 \end{pmatrix} = 29\begin{pmatrix} 2 \\ 3 \end{pmatrix} = g(\lambda)x.$ ✓