Alg-17

22 March 2021 12:16

Alg-17

Action of a group G on a set S,

or G-set S. This means

there is a homo $G \xrightarrow{\pi} Perm(S)$,

π is called an action of G on S. For $x \in G$

$\pi(x) \in Perm(S)$, $\pi(x) \equiv \pi_x$, $s \in S$

$\pi_x(s) \equiv xs$

π is homo $\Rightarrow$ $\pi_x \circ \pi_y = \pi_{xy}$

i.e. $\pi(x) \circ \pi(y) = \pi(xy)$.

Conjugation action of G

S = the set of all subsets of G

= the power set $\mathcal{P}(S)$

$x \in G$, $A, B \in S$ i.e. A & B are subsets of G. B is conjugate to A iff $B = xAx^{-1}$

Define $\pi_x : S \longrightarrow S$ by

$\pi_x(A) = xAx^{-1}$, which is again a subset of G.

$$\pi_{xy}(A) = (xy)A(xy)^{-1} = (xy)A(y^{-1}x^{-1})$$
$$= x(yAy^{-1})x^{-1}$$
$$= \pi_x(\pi_y(A))$$
$$= (\pi_x \pi_y)(A) \quad \forall A \in S$$

$$\Rightarrow \pi_{xy} = \pi_x \pi_y \quad \text{--- (1)}$$

$\Rightarrow \quad \pi_e = I_S$

So the map $\pi : G \to Perm(S)$ given by $\pi(x) = \pi_x$ is a homo

$A \subseteq G$ i.e. $A \in S$

$GA = \{x.A : x \in A\}$ — orbit of A in S

$= \{xAx^{-1} : x \in G\}$ — conjugacy class of $A \in S$

$= \mathcal{O}(A)$ — orbit of A in S

$A \in S$. $\quad G_A = \{x \in G : \underset{\sim}{x.A} = A\}$

$= \{x \in G : xAx^{-1} = A\}$

is a subgroup of G

isotropy or stabilizer group of A in G

G_s is an isotropy of s in G

$G = S$, and each $A \subseteq G$ is to be $A = \{a\}$

the previous example

$\pi_x(a) = x a x^{-1}$, $\quad \pi : G \to Perm(S) = Perm(G)$

$x.a = \pi_x(a) := xax^{-1}$.

is a homo

$Ga = \mathcal{O}(A) = \{x.a : x \in G\}$

$= \{xax^{-1} : x \in G\}$

— conjugacy class of a

$= C(a).$

$$G_a = \{x \in G : x.a = a\}$$

$$= \{x \in G : xax^{-1} = a\}$$

called the centralizer of a & write

$$\equiv C(a)$$

S = the set of all subgroups of G

Define for each $x \in G$, a permutation $\pi_x : S \longrightarrow S$ by

$\pi_x(H) = xHx^{-1}$, $H \leq G$ i.e. $H \in S$

& $xHx^{-1} \leq G$ i.e. $xHx^{-1} \in S$

π is a homo from $G \to \mathrm{Perm}(S)$

For $H \in S$, $GH = \{x.H : x \in G\}$

$= \{xHx^{-1} : x \in G\}$ – conjugacy class of $H = c(H)$

$$G_H = \{x \in G : x.H = H\}$$

$$= \{x \in G : xHx^{-1} = H\}$$

$= N_G(H)$ – normalizer of H in G

$= N_H$ ⟵ Lang's notation

$= N_G(H)$

G_s isotropy group of $s \in S$

is a subgroup of G

G_s isotropy group ...
is a subgroup of G

$Gs = \mathcal{O}(s)$, orbit of s in S
is a subset of S

G acts on S, $G \xrightarrow{\Pi} Perm(S)$

Define for s, t S $\quad s \underset{\pi}{\sim} t$ iff
$t = \pi_x(s)$ for some $x \in G$
$t = xs$ for some $x \in G$

Then (1) $s \underset{\pi}{\sim} s$
(2) $s \underset{\pi}{\sim} t \Rightarrow t \underset{\pi}{\sim} s$
(3) $s \underset{\pi}{\sim} t$, $t \underset{\pi}{\sim} u$
$\Rightarrow s \underset{\pi}{\sim} u$

$$[s]_\pi = \{t \in S : t \underset{\pi}{\sim} s\}$$
$$= \{t \in S : t = \pi_x(s), x \in G\}$$
$$= \{xs : x \in G\} = \mathcal{O}(s)$$
$$= Gs$$

How Gs is related to the set of left cosets of G_s i.e. G/G_s

Thm $|Gs| = \underbrace{|G/G_s|}$ $\leftarrow$ # of left cosets of G_s

G is acting on S

$G \xrightarrow{\Pi} Perm(S)$ homo. For $s \in S$ fixed.
... S by

$G \longrightarrow \mathrm{Perm}(S)$ which ... fixed.

define $\bar{f} : G/G_s \longrightarrow S$ by

$$\bar{f}(xG_s) = xs = \pi_x(s) \in S$$

$\boxed{\text{where } f : G \longrightarrow S \text{ defined by } f(x) = x.s}$

$\bar{f}$ is called the induced map

$\bar{f}$ is (1) well defined

(2) injective

(3) $\bar{f}(G/G_s) = \mathrm{Im}\,\bar{f}$

$= Gs$

$\bar{f} : G/G_s \longrightarrow Gs$ is a bijection

Well defined:

$xG_s = yG_s \iff y^{-1}x \in G_s$

$\iff (y^{-1}x)s = s$ defn of G_s

$\iff y[(y^{-1}x)s] = ys$

$\iff (yy^{-1})(xs) = ys$

$\iff xs = ys$

$\bar{f}$ is well defined

$\bar{f}$ is injective: $\bar{f}(xG_s) = \bar{f}(yG_s)$

$\Rightarrow xG_s = yG_s$

Now $\bar{f}(xG_s) = \bar{f}(yG_s)$

$\Rightarrow xs = ys$

$\Rightarrow (y^{-1}x)s = s$

$y^{-1}x \in G_s$

$\Rightarrow (y^{-1}x) \cdot s = \ldots$

$\Rightarrow y^{-1}x \in G_s$

$\Rightarrow xG_s = yG_s$

Thus $f : G/G_s \longrightarrow Gs$ is a bijection $\Rightarrow |G/G_s| = |Gs|$

$\Rightarrow |Gs| = (G : G_s)$ index of G_s in G.

S is a G-set, $\pi : G \longrightarrow \mathrm{Perm}(S)$ is a homo. Then $\underset{\pi}{\sim}$ is an equivalence rel where $s \underset{\pi}{\sim} t \iff t = \pi_x(s)$ for some $x \in G$.

$[s]_\pi = Gs$ — orbit

$S/\underset{\pi}{\sim} = \{ [s]_\pi : s \in S \}$

$S = \bigcup_{s \in S} [s] = \bigcup_{s \in S} Gs$

$\{Gs\}_{s \in S}$ is a partition of S

i.e. (1) $Gs \cap Gt = \phi$ $s \neq t$

(2) $S = \bigcup_s Gs$

S is finite. Then

$$|S| = \sum_{s \in S} |Gs|$$

, $s_1, \ldots s_r$ are distinct.

$= \sum^{r} |\ldots|$

$$= \sum_{i=1}^{r} |G s_i|$$ distinct

$$= \sum_{i=1}^{r} \left| G/G_{s_i} \right|$$

If G is also finite

$$|G| = \sum_{i=1}^{r} \frac{|G|}{|G_{s_i}|}$$