Alg-15

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S_7 $\quad (a_i\, a_j\, a_k)(a_l\, a_m\, a_n)$ has order 3

$(3)(3)(1)$ $\quad 600$

$$\frac{1}{2}\left(\frac{7.\cancel{6}^{2}.5}{\cancel{3}}\right)\left(\frac{4.3.2}{3}\right) = \frac{70.8}{2} = \frac{560}{2} = 280$$

$(3)(1)(1)(1)(1)$

$$\frac{7.6.5}{3} = 70$$

$70+280 = 350$ elements of S_7 has order 3

How many elements in S_{10} has order 3?

$(3)(3)(3)(1)$ $\quad$ 120 ~~240~~ $\quad$ 70

$(3)(3)(1)(1)(1)(1)$ $\quad \frac{1}{2}\left(\frac{10.9.8}{3}\right)\left(\frac{7.\cancel{6}^{2}.5}{\cancel{3}}\right) = 8400$

$(3)\underbrace{(1)\ldots(1)}$ $\quad$ —— $\quad \frac{10.9.8}{3} = 240$

$(3)\underbrace{(1)\cdots(1)}_{7}$ ———— $3 = 240$

$+\frac{1}{3!}\left(\frac{10.9.8}{3}\right)\left(\frac{7.6.5}{3}\right)\left(\frac{4.3.2}{3}\right)$

$= \frac{1}{6}\; 240 \;\; 70 \;\; 8$

$= 40.70.8 = 2800.8 = 22400$

$22400 + 8400 + 240 = 31040$

So S_{10} has 31040 of order 3

Sign of a permutation.

$\mathcal{F}(\mathbb{R}^n, \mathbb{R}) = \{f : \mathbb{R}^n \to \mathbb{R} \text{ functions}\}$

for $f, g \in \mathcal{F}(\mathbb{R}^n, \mathbb{R})$, define

$f+g, \alpha f$ and $fg : \mathbb{R}^n \to \mathbb{R}$ by

$(f+g)(x_1, \ldots x_n) = f(x_1, \ldots x_n) + g(x_1 \ldots x_n)$

$(\alpha f)(x_1, \ldots x_n) = \alpha f(x_1 - x_n), \; \alpha \in \mathbb{R}$

$(fg)(x_1, \ldots x_n) = f(x_1, \ldots x_n) . g(x_1 \ldots x_n)$

Then $(\mathcal{F}(\mathbb{R}^n, \mathbb{R}), +)$ is an

abelian group &

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- $f(gh) = (fg)h$
- $fg = gf$
- $f(g+h) = fg + fh$
- $(\alpha f)g = \alpha(fg) = f(\alpha g)$

For $\sigma \in S_n$, we define

$\sigma f \equiv f_\sigma \in \mathcal{F}(\mathbb{R}^n, \mathbb{R})$ by

$$(\sigma f)(x_1 \ldots x_n) \equiv (f_\sigma)(x_1 \ldots x_n)$$
$$= f(x_{\sigma(1)} \ldots x_{\sigma(n)})$$

We have the following

(1) $\sigma(\tau f) = (\sigma\tau) f$ or $f_{\sigma\tau} = f_\sigma f_\tau$

(2) $\varepsilon f = f$ where $\varepsilon = id_{S_n}$ i.e. $\varepsilon(k) = k$, $\forall k$.

(3) $\sigma(f+g) = \sigma f + \sigma g$

(4) $\sigma(cf) = c\,\sigma(f)$ $c \in \mathbb{R}$

(5) $\sigma(fg) = \sigma(f)\sigma(g)$.

Proof (2) $(\varepsilon f)(x_1 \ldots x_n) = f(x_{\varepsilon(1)} \ldots x_{\varepsilon(n)})$

Proof (2) $(\varepsilon f)(x_1 \ldots x_n) = f(x_{\varepsilon(1)} \ldots x_{\varepsilon(n)})$
$= f(x_1 \ldots x_n)$
$\forall (x_1 \ldots x_n) \in \mathbb{R}^n$

$\Rightarrow \quad \varepsilon f = f$

(3) $(\sigma(f+g))(x_1 \ldots x_n) = (f+g)(x_{\sigma(1)} \ldots x_{\sigma(n)})$

$= f(x_{\sigma(1)} \ldots x_{\sigma(n)}) + g(x_{\sigma(1)} \ldots x_{\sigma(n)})$

$= (\sigma f)(x_1 \ldots x_n) + (\sigma g)(x_1 \ldots x_n)$

def of sum

$= (\sigma f + \sigma g)(x_1 \ldots x_n), \ \forall (x_1 \ldots x_n) \in \mathbb{R}^n$

$\Rightarrow \sigma(f+g) = \sigma f + \sigma g$

Try (4) & (5) similarly

To proove (1) $((\sigma\tau)f)(x_1 \ldots x_n)$

$= f(x_{\sigma(\tau(1))} \ldots x_{\sigma(\tau(n))})$

$\sigma(\tau f) = \sigma(g) \quad (\sigma(\tau f))(x_1 \ldots x_n)$

$= \sigma(g)(x_1 \ldots x_n)$

$= g(x_{\sigma(1)} \ldots x_{\sigma(n)})$

$$= (\tau f)(y_1 \ldots y_n), \quad y_j = x_{\sigma(j)}$$

$$= f(y_{\tau(1)} \ldots y_{\tau(n)})$$

$$y_{\tau(j)} = x_{\sigma(\tau(j))} \quad j = 1 \ldots n$$

$$= f(x_{\sigma(\tau(1))} \ldots x_{\sigma(\tau(n))})$$

$$= ((\sigma\tau)f)(x_1 \ldots x_n)$$

$$\forall (x_1 \ldots x_n) \in \mathbb{R}^n$$

$$\Rightarrow \sigma(\tau f) = (\sigma\tau) f.$$

$$S_n \xrightarrow{\pi} \mathrm{Perm}(\mathcal{F}(\mathbb{R}^n, \mathbb{R}))$$

$$\sigma \longmapsto \pi(\sigma)(f) = f_\sigma = \sigma f$$

Then π is a homo

$$\pi(\sigma\tau) = f_{\sigma\tau} = f_\sigma f_\tau = \pi(\sigma)\pi(\tau)$$

and $\pi(\varepsilon) = f_\varepsilon = f.$

$\boxed{\begin{array}{l} S_n \text{ acts on } \mathcal{F}(\mathbb{R}^n, \mathbb{R}) \\ \text{Group acting on a set} \end{array}}$

Parity group $G = \{1, -1\}$ a group under product
$= \{\text{even, odd}\}$

Thm. There is a map $\text{Sign} : S_n \to G$
s.t. (1) τ is a transposition (a 2-cycle)
then $\text{sign}(\tau) = -1$

(2) $\text{sign}(\sigma\tau) = \text{sign}(\sigma)\,\text{sign}(\tau)$

(2) says that sign is a homo.

$\ker \text{sign} = A_n$.

$\sigma \in \ker \text{sign} \iff \text{sign}(\sigma) = 1$

$\sigma = \tau_1 \cdots \tau_m \quad \text{sin}(\sigma) = (-1)^m = 1$
m is even

$\Rightarrow$ ~~σ~~ is even, $\sigma \in A_n$.

$\Rightarrow \sigma$ is even, $\sigma \in A_n$.

$\sigma \in A_n \Rightarrow \sigma = \tau_1 \cdots \tau_m$

where m is even

$$\operatorname{sign}(\sigma) = \operatorname{sign}(\tau_1 \cdots \tau_m)$$

as sign homo $\overset{\downarrow}{=} \operatorname{sign}(\tau_1) \cdots \operatorname{sign}(\tau_m)$

$$= (-1) \cdots (-1)$$

$$= (-1)^m = 1.$$

By 1st iso thm $S_n \xrightarrow{\text{sign}} G$ homo

& surjective $\operatorname{sign}(S_n) = G$

$$S_n / \ker(\operatorname{sign}) \simeq \operatorname{sign}(S_n) = G$$

$$S_n / A_n \simeq G$$

$$2 = |G| = |S_n / A_n|$$

$$= \frac{|S_n|}{|A_n|} = \frac{n!}{|A_n|}$$

$$\Longrightarrow |A_n| = \frac{n!}{2}.$$

Proof. Define $\Delta : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

Proof. Define $\Delta : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$\Delta(x_1 \dots x_n) = \prod_{i<j} (x_j - x_i)$$

Then $\Delta \in \mathcal{F}(\mathbb{R}^n, \mathbb{R})$

$n=2$ $\quad \Delta(x_1, x_2) = x_2 - x_1$

$n=3$ $\quad \Delta(x_1, x_2, x_3) = (x_3 - x_1)(x_3 - x_2)(x_2 - x_1)$

$n=4$

$$\underbrace{\{(i,j)\}}_{i<j} \subseteq \{1 \dots n\} \qquad \binom{n}{2}$$

For