Alg-30

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Recall that R ring with 1 is called a division ring or a skew field if $(R^*, \cdot)$ is a group

A commutative skew field or division ring is a field

Ex Real quaternions (W.R. Hamilton 1843)

(complex — . . . (— — —)

$$H \equiv \mathbb{H}(\mathbb{R}) = \{a_0 + a_1 i + a_2 j + a_3 k : a_0 \ldots a_3 \in \mathbb{R}\}$$

space of real quaternions

i, j, k are symbols (not in $\mathbb{R}$)

$$\boxed{\begin{aligned} i^2 = ii = j^2 = k^2 = -1, \quad ij &= k = -ji \\ jk &= i = -kj \\ ki &= j = -ik. \end{aligned}}$$

&

$ijk = -1$

$$G = \{\pm 1, \pm i, \pm j, \pm k\}$$

$(G, \cdot)$ is a noncommutative group of order 8.

$\alpha, \beta \in \mathbb{H}(\mathbb{R})$

$\alpha = a_0 + a_1 i + a_2 j + a_3 k.$

$$\alpha = a_0 + a_1 i + a_2 j + a_3 k.$$
$$\beta = b_0 + b_1 i + b_2 j + b_3 k$$

define $\alpha + \beta = (a_0 + b_0) + (a_1 + b_1) i + (a_2 + b_2) j + (a_3 + b_3) k$

$$\begin{aligned} \alpha \cdot \beta = \alpha\beta = & (a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3) \\ & + (a_0 b_1 + a_1 b_0 + a_2 b_3 - a_3 b_2) i \\ & + (a_0 b_2 + a_2 b_0 + a_3 b_1 - a_1 b_3) j \\ & + (a_0 b_3 + a_3 b_0 + a_1 b_2 - a_2 b_1) k. \end{aligned}$$

$$0_{\mathbb{H}} = 0 + 0i + 0j + 0k.$$
$$1_{\mathbb{H}} = 1 + 0i + 0j + 0k.$$

$(\mathbb{H}(\mathbb{R}), +, \cdot)$ is a division ring or skew-field

not commutative ring
$ij \neq ji$

$\mathbb{H}$ is a a $\mathbb{R}$-v.s

$$\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C} = -(\vec{C} \times (\vec{A} \times \vec{B}))$$

$$\vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

$$= -(\vec{A}(\vec{C} \cdot \vec{B}) - (\ ))$$

$$\dim_{\mathbb{R}} \mathbb{H} = 4 \quad ,$$

$(a - \alpha b^{-1}$

dim

$\mathbb{R}$

$\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}(\mathbb{R})$ $\left(\frac{a}{b} = ab^{-1} \text{ or } b^{-1}a \text{ if } ab = ba \right)$

1-dim 2-dim Real 4-dim

Cayley #s: 8-dim

$\alpha \in \mathbb{H}(\mathbb{R}), \alpha = a_0 + a_1 i + a_2 j + a_3 k$

$\bar{\alpha} = a_0 - a_1 i - a_2 j - a_3 k.$

$\alpha \bar{\alpha} = a_0^2 + a_1^2 + a_2^2 + a_3^2 = b \neq 0$ i.e. $\alpha \neq 0$

$\alpha^{-1} = \frac{1}{b}\bar{\alpha} = \frac{a_0}{b} - \frac{a_1}{b} i - \frac{a_2}{b} j - \frac{a_3}{b} k.$

How do define complex quaternions?
(complete).

Jacobson: Lectures in Abstract Alg Vol 2 (1953)

Linear Algebra
Vector space over a field

uses ~~vector~~ vector space over a skew field

modules over

A modules over rings

Jordan & Von Neumann (1936)

$T: V \to V$, $V/\mathbb{C}$

$Tv = \lambda v$ for some $v \neq 0$

λ is an eigenvalue of T with corresponding eigenvector v

V is $\mathbb{H}(\mathbb{C})$ - Hilbert space

$\mathbb{H}(\mathbb{C})^n$, $\mathbb{R}^n$, or $\mathbb{C}^n$

Hypercomplex Analyses

(Analyses in $\mathbb{H}(\mathbb{C})$

$f * g$.

Euclidean Domains.

$\mathbb{F}$ is a field

$\mathbb{F}[t]$ = the set of all poly with coeff in $\mathbb{F}$, is an integral.

PID (principal ideal

is a PID (principal ideal domain)

Every ideal I of $\mathbb{F}[t]$

$I = (f(t))$ for some poly $f(t) \in \mathbb{F}[t]$ of smallest degree.

Ex $\mathbb{Z}$, $I = (n) = n\mathbb{Z}$.

PID

$\mathbb{F}[t]^{*} \to \mathbb{Z}_{+} = \{0, 1, 2, \dots\}$

$d(f(t)) = \deg f(t)$. called the degree map

(1) $d(f(t)) \geq 0 \quad \forall f(t) \in \underbrace{\mathbb{F}[t]^{*}}_{\mathbb{F}[t] - \{0\}}$

(2) $d(f(t)g(t)) = d(f(t)) + d(g(t))$

(3) $d(f(t)) \leq \deg(f(t)g(t))$

(4) $f(t), g(t) \in \mathbb{F}[t]^{*}$, $\exists\, q(t), r(t) \in \mathbb{F}[t]$

s.t. $f(t) = q(t)g(t) + r(t)$ s.t

quotient

$r(t) \equiv 0$, zero poly

or $0 \le d(r(t)) < \deg(g(t))$

remainder

Def. An integral domain R is called an Euclidean domain iff there is a map $d: R^* \to \mathbb{Z}$

s.t. (1) $d(a) \ge 0 \quad \forall a \in R^*$

(2) $d(a) \le d(ab) \quad \forall a, b \in R^*$

(3) $\forall a, b \in R^*$, there are unique $q, r \in R$ s.t.

$a = qb + r$ with

$0 = r$ or $r \ne 0$ & $d(r) < d(b)$

① $\mathbb{F}[t]$ is an Euclidean domain

$d(f(t)) = $ degree of $f(t)$, $f(t) \in \mathbb{F}[t]^*$

② m, n are nonzero integers.

$\in \mathbb{Z}$ s.t.

(2) m, n are integers

$\exists!\, q, r \in \mathbb{Z}$ s.t.

$m = qn + r$,

or $0 \leq r < |n|$

$d : \mathbb{Z}^* \to \mathbb{Z}$ by

$d(a) = |a|$. . Then (1) $d(a) \geq 0$

(2) $d(a) \leq d(ab)$

(3) $\forall\, a, b \in \mathbb{Z}^*$

$\exists\, q, r \in \mathbb{Z}$ s.t

$a = qb + r$ where

$0 \leq r < |b| = d(b)$

$d(r)$.

Thm Every Euclidean domain is a PID

Euclidean Domains $\subseteq$ PIDs $\subseteq$ UFD

$\neq$

unique factorization domain