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5/11/20

§94 #5, 7a.

5) Suppose that a function f is analytic inside and on a POPCC C and has no zeros on C. Show that if f has n zeros z_k $(k=1,2,\dots,n)$ inside C, where each z_k is of multiplicity m_k, then:

$$\int_C \frac{z f'(z)}{f(z)}\,dz = 2\pi i \sum_{k=1}^{n} m_k z_k$$

By the winding # thm, $\frac{1}{2\pi}\Delta_C \arg f(z) = Z - P$,

where Z is the number of enclosed zeros and P the number of enclosed poles (counted by multiplicity). Since f is AOIC, $P = 0$, and in the context of this problem, $Z = \sum_{k=1}^{n} m_k$.

(by Thm 1, §82)

At $z = z_k$ $(k=1,2,\dots,n)$, $f(z) = (z-z_k)^{m_k} g(z)$,
where g is a function that is AANZ @ z_k.

Thus $f'(z) = m_k (z-z_k)^{m_k-1} g(z) + (z-z_k)^{m_k} g'(z)$.

Let h = integrand of contour integral in question. Then

$$h(z) = \frac{z\left(m_k(z-z_k)^{m_k-1} g(z) + (z-z_k)^{m_k} g'(z)\right)}{(z-z_k)^{m_k} g'(z)}$$

$$= \underbrace{z_k \Big(\overbrace{m_k (z-z_k)^{-1}}^{\text{Residue}}}_{\substack{\text{pseudo power series}\\ \text{centered at } z=z_k}} + \underbrace{\frac{g'(z_k)}{g(z_k)}\Big)}_{\substack{\text{power series}\\ \text{centered @ } z=z_k}}$$

This is clearly a Laurent series centered @ $z=z_k$, where the first term is the pseudo-power series and the second term is analytic @ $z=z_k$ (since g is AANZ @ z_k) and thus forms an ordinary power series @ z_k.

Thus $\underset{z=z_k}{\text{Res}}\ h(z) = b_1 = m_k z_k$.

Residue thm.

$$\int_C \underbrace{h(z)}\, dz = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{\text{Res}}\ h(z)$$

POSCC

AOIC except for (finitely many) poles inside C when $f(z)=0$ (i.e., not analytic @ $z=z_k$, $k=1,2,\ldots n$)

$$= 2\pi i \sum_{k=1}^{n} m_k z_k$$

7a). Determine the number of zeros, counting multiplicities, of $z^4 - 2z^3 + 9z^2 + z - 1$ inside the circle $|z|=2$.

Let $f(z) = 9z^2$,
$g(z) = z^4 - 2z^3 + z - 1$.

On C: ($|z|=2$):

$|f(z)| = |9z^2| = 9|z|^2 = 36$.

$|g(z)| = |z^4 - 2z^3 + z - 1| \underset{\substack{\uparrow \\ \text{extended} \\ \Delta_1}}{\leq} |z^4| + |-2z^3| + |z| + |-1|$

$= |z|^4 + 2|z|^3 + |z| + 1$

$= 2^4 + 2(2)^3 + 2 + 1 = 16 + 16 + 2 + 1 = 35$

Since $|g(z)| \leq 35 < 36 = |f(z)|$ on C, and f, g are entire (and thus AOIC). Thus we can apply Rouche's Thm: f and $f+g$ have the same number of zeros inside C (counting multiplicity). Clearly f only has a zero of multiplicity 2 inside C, so $f+g$ (the function of interest) must also have 2 zeros inside C (counted by multiplicity).

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§98 #3.

3) Show that the image of the half-plane $y > c_2$ under the transformation $w = \frac{1}{z}$ is the interior of a circle when $c_2 > 0$. Find the image when $c_2 < 0$ and when $c_2 = 0$.

Use the result from §98.

For a line or circle in the xy plane,

$$A(x^2+y^2) + Bx + Cy + D = 0 \quad (B^2+C^2 > 4AD)$$

the transformation $w = \frac{1}{z}$ maps this line or circle into the line or circle $D(u^2+v^2) + Bu - Cv + A = 0$ in the uv plane.

Consider the images of horizontal lines in the xy plane, $y = c_0$.

Case I: $c_0 \neq 0$.

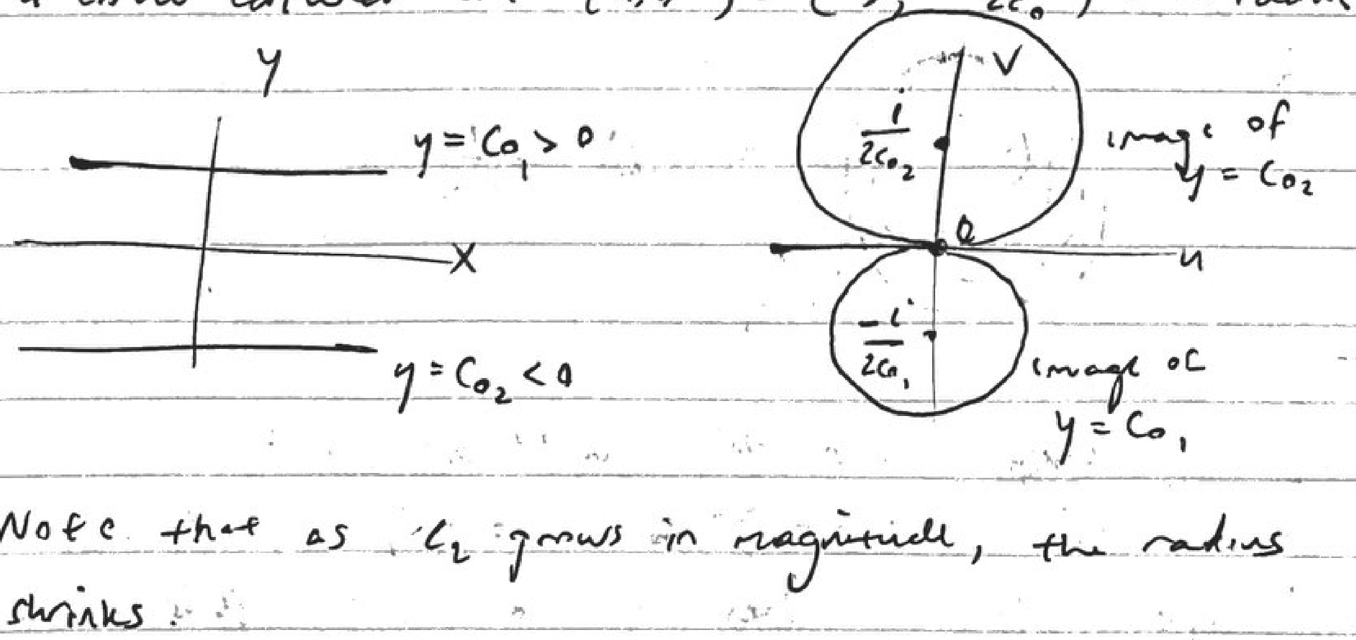
$$\underbrace{A = 0,\ B = 0,\ C = 1,\ D = -c_0}_{\text{mapped to}},\ (B^2+C^2 = 1 > 0 = 4AD)$$

$$-c_0(u^2+v^2) - v = 0$$

$$u^2 + v^2 + \frac{v}{c} + \left(\frac{1}{2c_0}\right)^2 = \left(\frac{1}{2c_0}\right)^2$$

$$u^2 + \left(v + \frac{1}{2c_0}\right)^2 = \left(\frac{1}{2c_0}\right)^2.$$

(a circle centered at $(u,v) = (0, -\frac{1}{2c_0})$ with radius $\frac{1}{2c_0}$)



Note that as c_2 grows in magnitude, the radius shrinks.

Case 2: $c_0 = 0$.

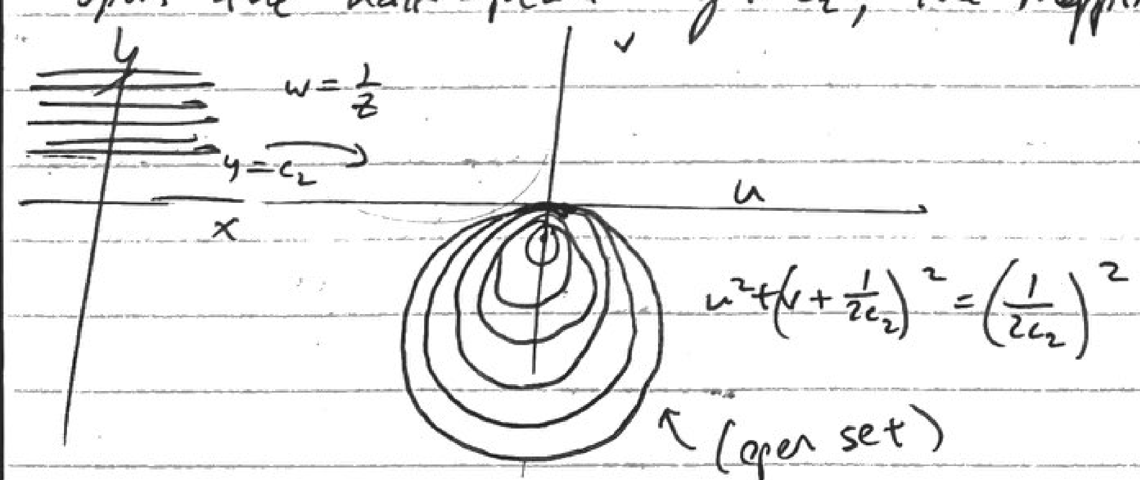
$\underbrace{A = 0,\ B = 0,\ C = 1,\ D = 0}_{\text{gets mapped to}}$ $\qquad (B^2 + C^2 = 1 > 0 = 4AD)$

$-v = 0$

Thus $y = 0$ gets mapped to $v = 0$ under this transformation.

$y = c_0 > c_2 > 0$ is the set of circles with maximum radius $\frac{1}{2c_2}$. As $c_2 \to \infty$, their centers move toward the origin but their radii get smaller so that each circle $u^2 + (v + \frac{1}{2c_0})^2 = (\frac{1}{2c_0})^2$ is contained within the outermost circle, $u^2 + (v + \frac{1}{2c_2})^2 = (\frac{1}{2c_2})^2$.

Also, this set of circles spans the outermost circle; for any point (u_0, v_0) in the outer circle $u^2 + (v + \frac{1}{2c_2})^2 = (\frac{1}{2c_2})^2$, you can find a circle $u^2 + (v + \frac{1}{2c_0})^2 = (\frac{1}{2c_0})^2$, $c_0 > c_2$ that passes through this point, so this set of circles spans the disk $u^2 + (v + \frac{1}{2c_2})^2 = (\frac{1}{2c_2})^2$. Since the preimage of this set is the set $y = c_0 > c_2$, which spans the half-plane $y > c_2$, the mapping is established.



If $c_2 = 0$, then this would be the set of all circles $u^2 + (v + \frac{1}{2c_0})^2 = (\frac{1}{2c_0})^2$, $c_0 > 0$. (the images of all lines $y = c_0 > 0$). This would be all circles whose top point (where "top" is the maximum v-coordinate point) is the origin, extending out to

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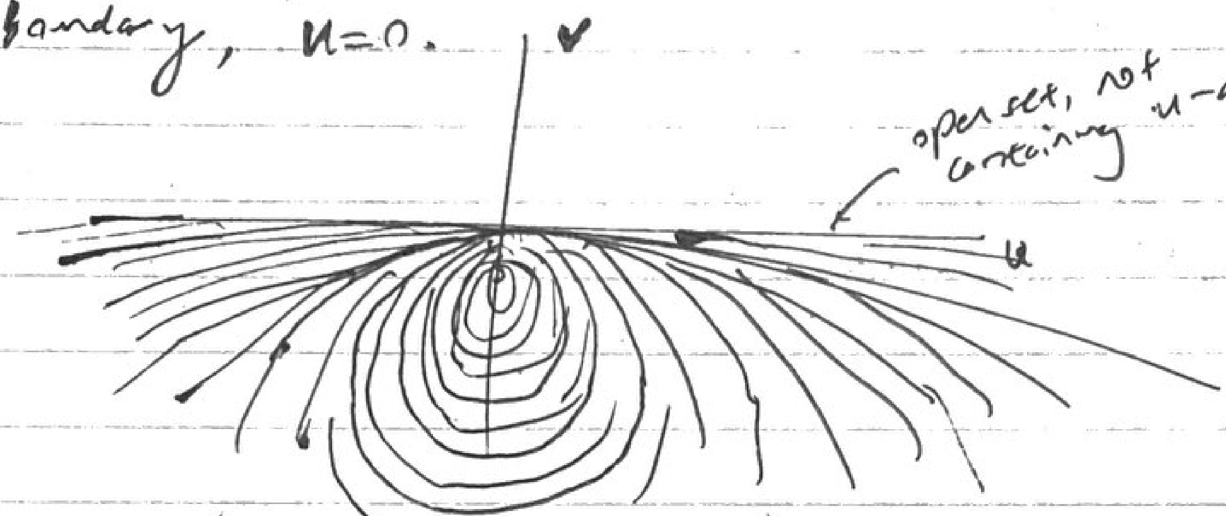
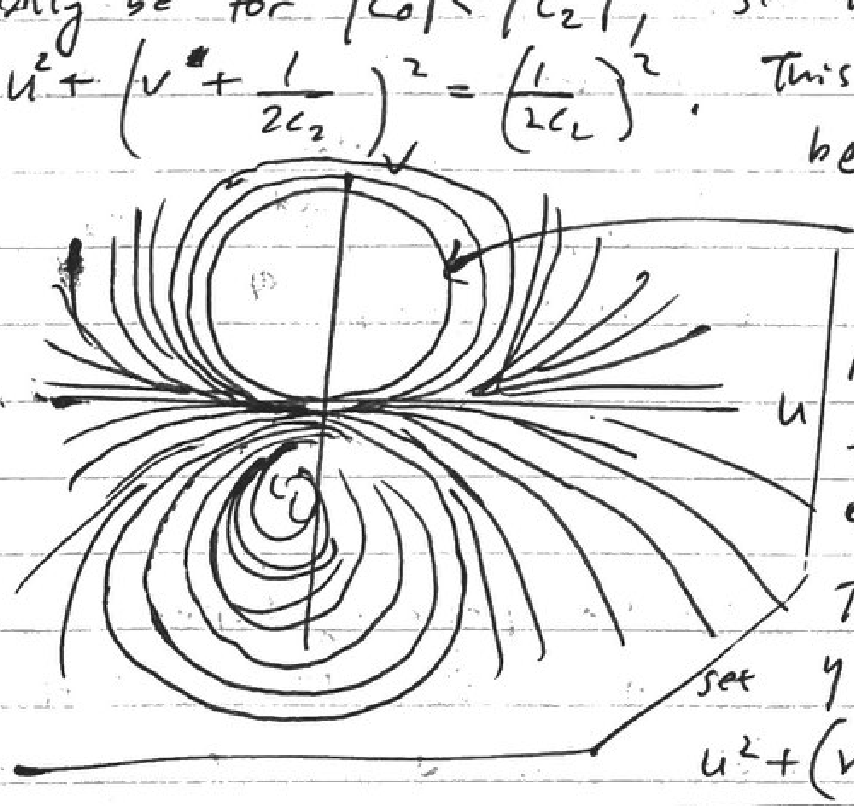
§98 #3, cont'd.

3, cont'd) arbitrarily large radii. This mapping spans the entire LHP of the uv plane, as any point (u_0, v_0), $v_0 < 0$ has some circle of the form $u + \left(v + \frac{1}{2c_0}\right)^2 = \left(\frac{1}{2c_0}\right)^2$ through it. Again, this is an open set, because it doesn't include its boundary, $u=0$.



(entire LHP of uv plane spanned by $y > 0$ mapped by $w = \frac{1}{z}$).

In the case that $c_2 < 0$, then we ~~include~~ add the v-axis (mapped to by $y=0$) and some circles in the UHP of the uv plane. These circles would only be for $|c_0| < |c_2|$, so these are circles <u>outside of</u> $u^2 + \left(v + \frac{1}{2c_2}\right)^2 = \left(\frac{1}{2c_2}\right)^2$. This is the open set illustrated below:



$u^2 + \left(v + \frac{1}{2c_2}\right)^2 = \left(\frac{1}{2c_2}\right)^2$

By the same arguments as before, this spans the set outside of the circle $u^2 + \left(v + \frac{1}{2c_2}\right)^2 = \left(\frac{1}{2c_2}\right)^2$. Thus the whole image of the set $y > (c_2 < 0)$ is the set $u^2 + \left(v + \frac{1}{2c_2}\right)^2 > \left(\frac{1}{2c_2}\right)^2$.

§100 #2.

2). Find the linear fractional transformation that maps the points $z_1 = -i$, $z_2 = 0$, $z_3 = i$ onto the points $w_1 = -1$, $w_2 = i$, $w_3 = 1$. Into what curve is the imaginary axis $x = 0$ transformed?

$W = \dfrac{az+b}{cz+d}$ $(ad - bc \neq 0)$

$\Rightarrow w(cz+d) = az+b.$

$-1\,(c(-i)+d) = a(-i)+b \Rightarrow ci - d = b - ai$ ①

$i\,(c\cancel{(0)}+d) = \cancel{a(0)}+b \rightarrow id = b.$ ②

$1\,(ci+d) = ai+b \Rightarrow ci+d = b+ai$ ③

①+③ : $2ci = 2b \Rightarrow b = ci \underset{②}{=} id$

③−① : $2d = 2ai \Rightarrow d = ai$

$\Rightarrow$ let $M := c = d = ai = -b. \Rightarrow c = d = M,\ a = -iM,\ b = iM$

$w = \dfrac{az+b}{cz+d} = \dfrac{-Miz+Mi}{Mz+M} = \dfrac{M(-iz+i)}{M(z+1)} = \dfrac{-iz+i}{z+1}.$

Reiterating the argument in §99, we show that the linear fractional transformation of a line or circle in the xy plane must be a line or circle in the uv plane:

when $c \neq 0$ and $w = \dfrac{az+b}{cz+d} = \dfrac{a}{c} + \dfrac{bc-ad}{c} \cdot \dfrac{1}{cz+d}$,

then w is the composition $w_3 \circ w_2 \circ w_1$:

$w_1 = cz+d$

$w_2 = \dfrac{1}{z}$

$w_3 = \dfrac{a}{c} + \dfrac{bc-ad}{c} z$

w_1 and w_3 are linear and thus preserve general shape. w_2 is the inverse map, which was shown in §98 to map lines and circles only to lines and circles. Thus w must map lines and

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§100 #2, cont'd

circles only to lines and circles.

Let $z = iy$ (the line $x = 0$).

Then $w = \dfrac{-i(iy)+i}{iy+1} = \dfrac{y+i}{iy+1}$

Thus $|w| = \dfrac{|y+i|}{|iy+1|} = \dfrac{\sqrt{y^2+1}}{\sqrt{y^2+1}} = 1$.

Since mappings of a line in the xy plane must map to a line or a circle in w, and ~~[illegible]~~ always on the circle $|w|=1$, then the line $x=0$ must map onto the (entire) circle $|w|=1$.

§102 #1, 2

1). Recall from Ex 1 in §102 that the transformation

$w = \dfrac{i-z}{i+z}$ maps the half-plane $\text{Im}\, z > 0$ onto the

disk $|w|<1$ and the boundary of the half-plane onto the boundary of the disk. Show that a point $z = x$ is mapped onto the point

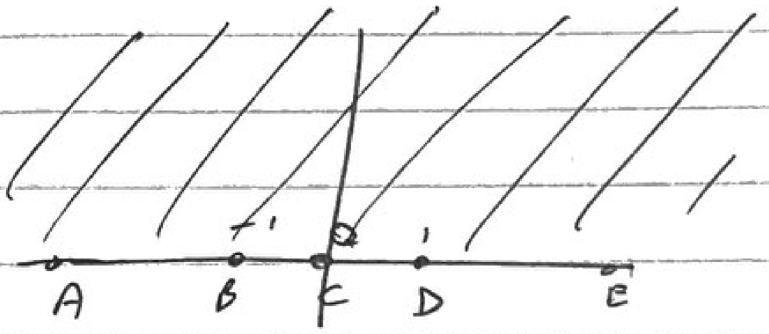
$$w = \frac{1-x^2}{1+x^2} + i\frac{2x}{1+x^2}$$

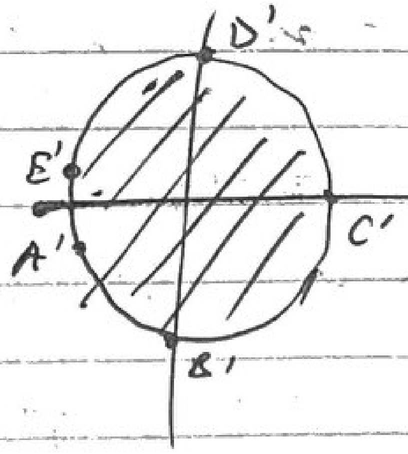
and then complete the verification of the mapping illustrated in Fig. 13, Appendix 2.

$$w = \frac{i-x}{i+x} = \frac{(i-x)(-i+x)}{(i+x)(-i+x)} = \frac{1+ix+ix-x^2}{x^2+1}$$

$$= \frac{1-x^2}{x^2+1} + i\frac{2x}{x^2+1}.$$

from fig 13, appendix 2:





$A = -a_1 \ , \quad a_1 > 1$

$B = -1$

$C = 0$

$D = 1$

$E = a_2 \ , \quad a_2 > 1$

$$A' = \frac{1-(-a_1)^2}{1+(-a_1)^2} + i\frac{2(-a_1)}{1+(-a_1)^2} = \frac{1-a_1^2}{1+a_1^2} - \frac{i a_1}{1+a_1^2}.$$

$$B' = \frac{1-(-1)^2}{1+(-1)^2} + \frac{i(-2)}{1+(-1)^2} = -i.$$

$$C' = \frac{1-0^2}{1+0^2} + \frac{i(2(0))}{1+0^2} = 1.$$

$$D' = \frac{1-1^2}{1+1^2} + \frac{i(2(1))}{1+1^2} = i.$$

$$E' = \frac{1-a_2^2}{1+a_2^2} + \frac{i2a_2}{1+a_2^2}.$$

B', C', and D' clearly match the transformed points from the figure.

A' is in the 3rd quadrant, approaching -1 as $a_1 \to \infty$.
C' is in the 2nd quadrant, approaching -1 as $a_2 \to \infty$.
These values also match the figure.

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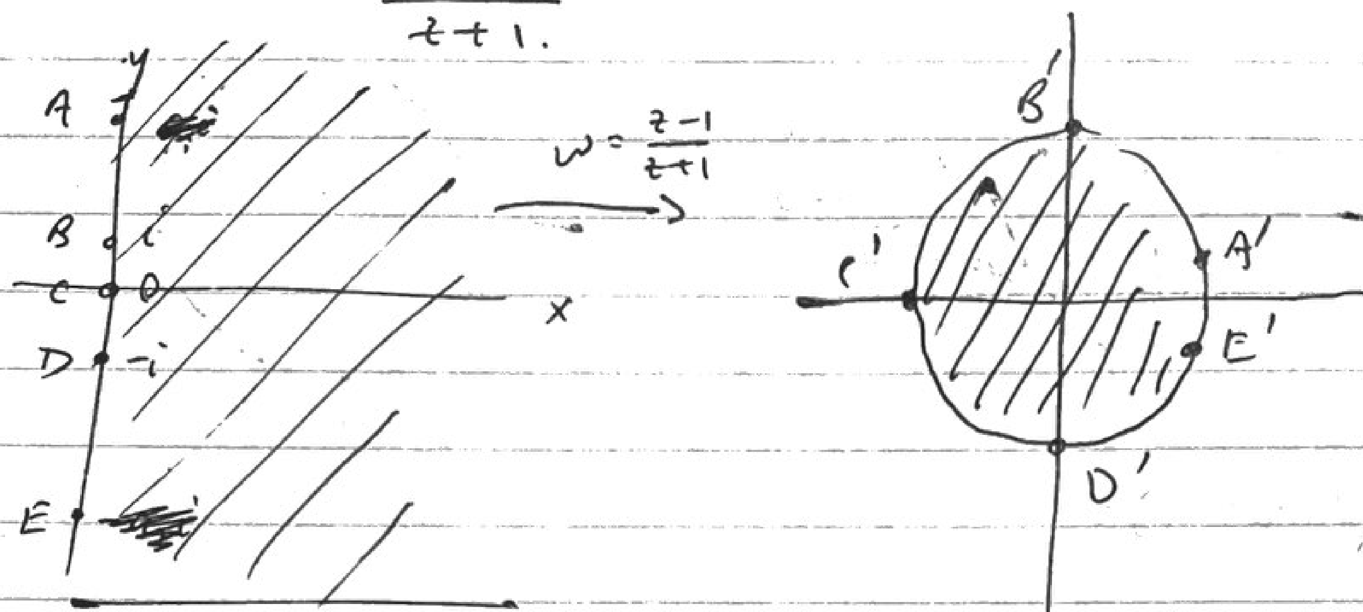
§102 #2.

2.). Verify the mapping shown in Appendix 2, Fig 13.,
where $w = \dfrac{z-1}{z+1}$.



$$w = \frac{z-1}{z+1} = -\left(\frac{1-z}{z+1}\right) = -\left(\frac{i-iz}{i+iz}\right) = -\left(\frac{i-(iz)}{i+(iz)}\right)$$

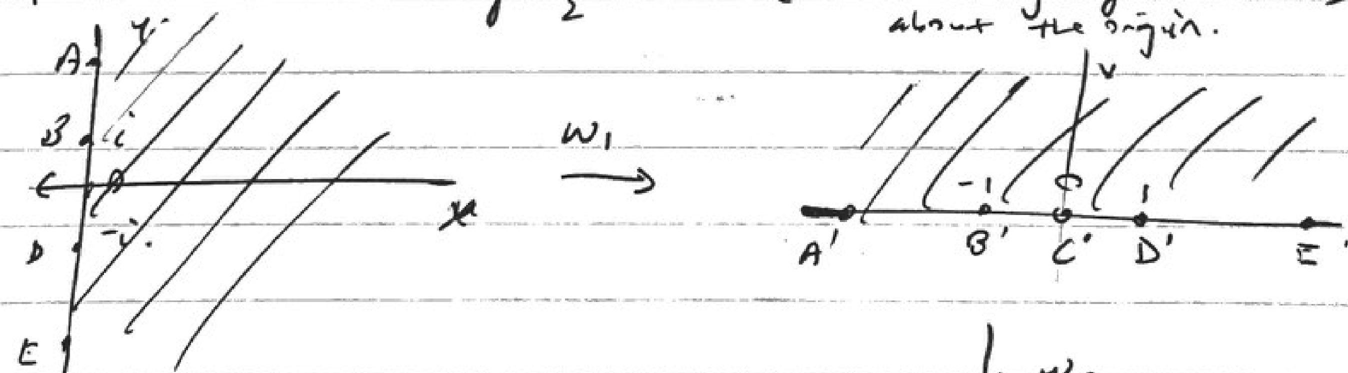
i.e., $w = w_3 \circ w_2 \circ w_1$, $\quad w_1 = iz,$

$$w_2 = \frac{i-z}{i+z},$$

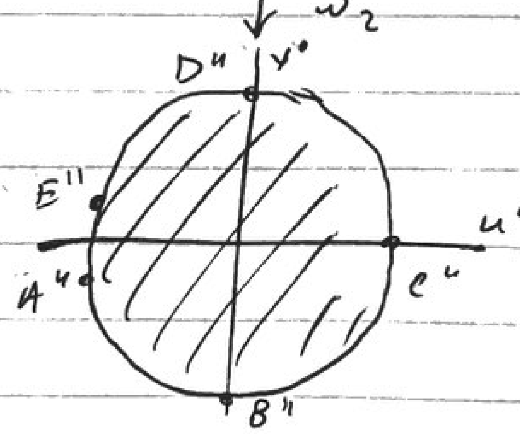
$$w_3 = -z.$$

---

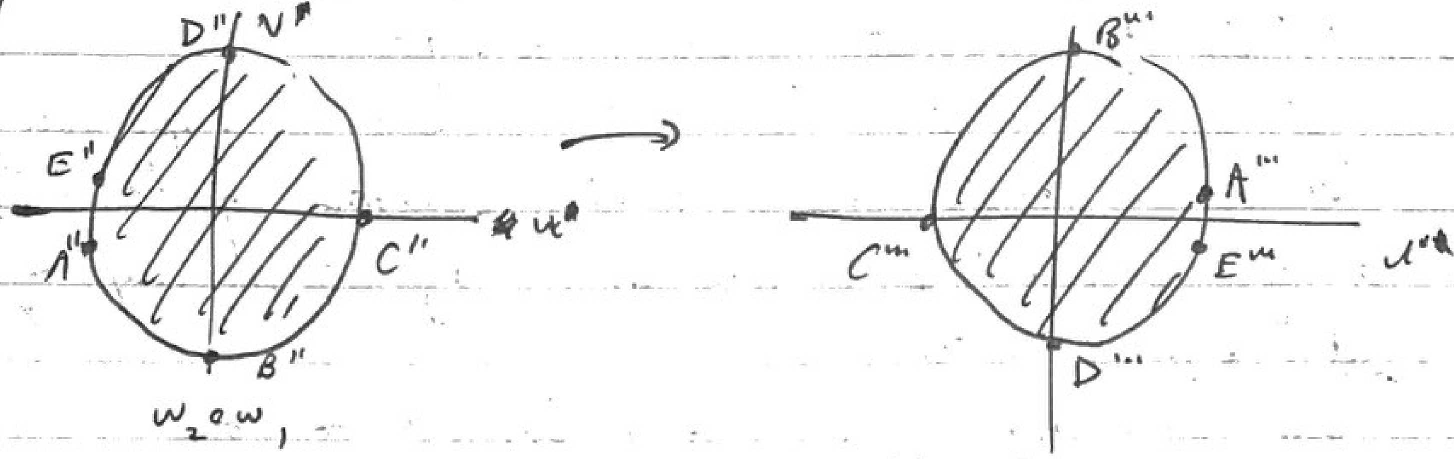
w_1 is a rotation by $\frac{\pi}{2}$ rad (and scaling by $|i| = 1$):
about the origin.



w_2 is the transformation from the previous exercise:



Lastly, applying $w_3 = -z$ flips the signs of the real and imaginary parts of $w_2 \circ w_1$.



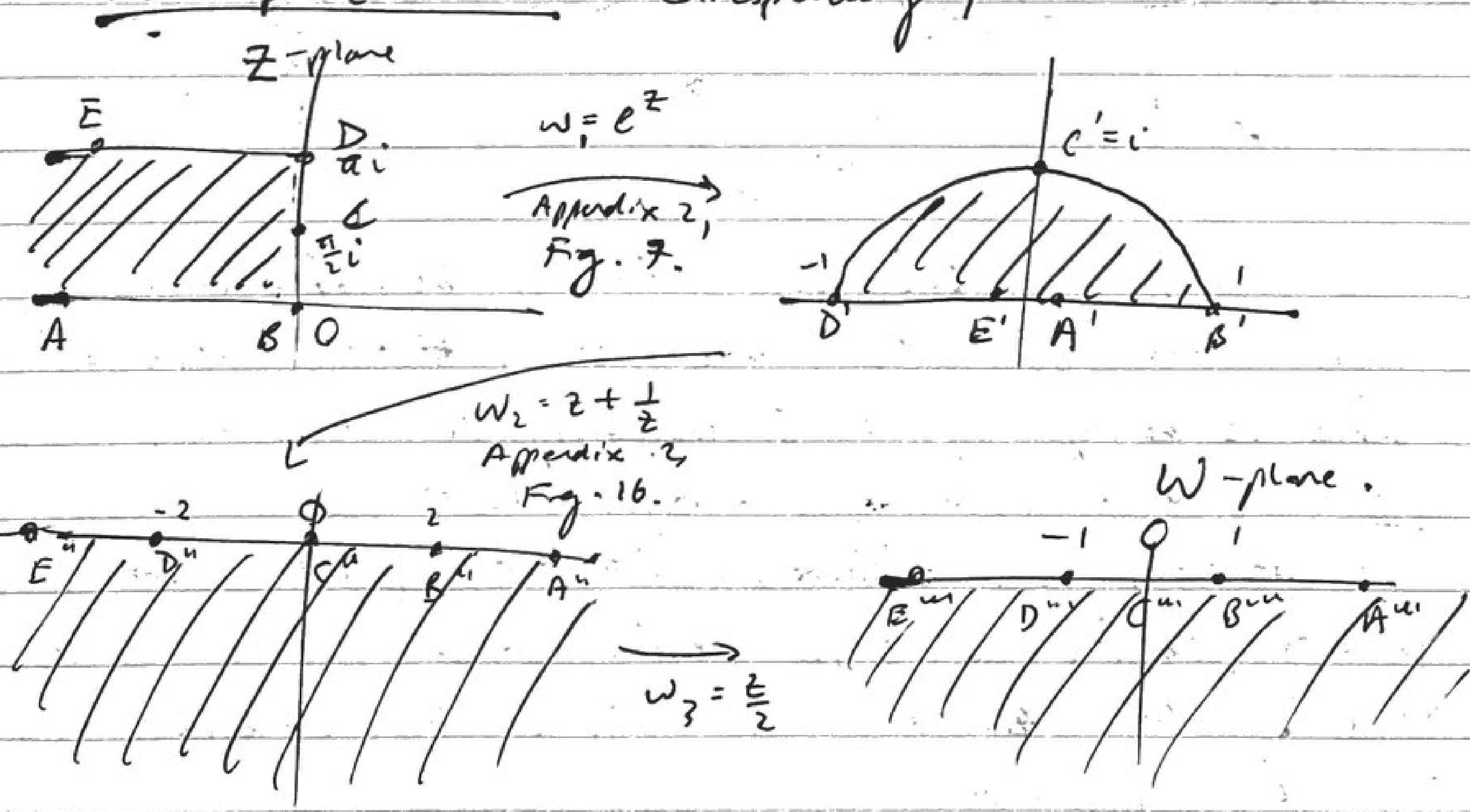
$W = w_3 \circ w_2 \circ w_1$.

§106 #10.

10). Observe that the transformation $w = \cosh z$ can be expressed as a composition of the mappings:

$$W = w_3 \circ w_2 \circ w_1, \quad w_1 = e^z, \quad w_2 = z + \frac{1}{z}, \quad w_3 = \frac{z}{2}.$$

By figures 7 and 16 in Appendix 2, show that when $w = \cosh z$, the semi-infinite strip $x \leq 0$, $0 \leq y \leq \pi$ in the z-plane is mapped to the lower half $v \leq 0$ of the w-plane. Indicate corresponding parts of boundaries.



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§114 # 3, 10.

3.) Show that under the transformation $w = \frac{1}{z}$, the images of the lines $y = x - 1$ and $y = 0$ are the circle $u^2 + v^2 - u - v = 0$ and the line $v = 0$, respectively. Sketch all four curves, determine corresponding directions along them, and verify the conformality of the mapping at the point $z_0 = 1$.

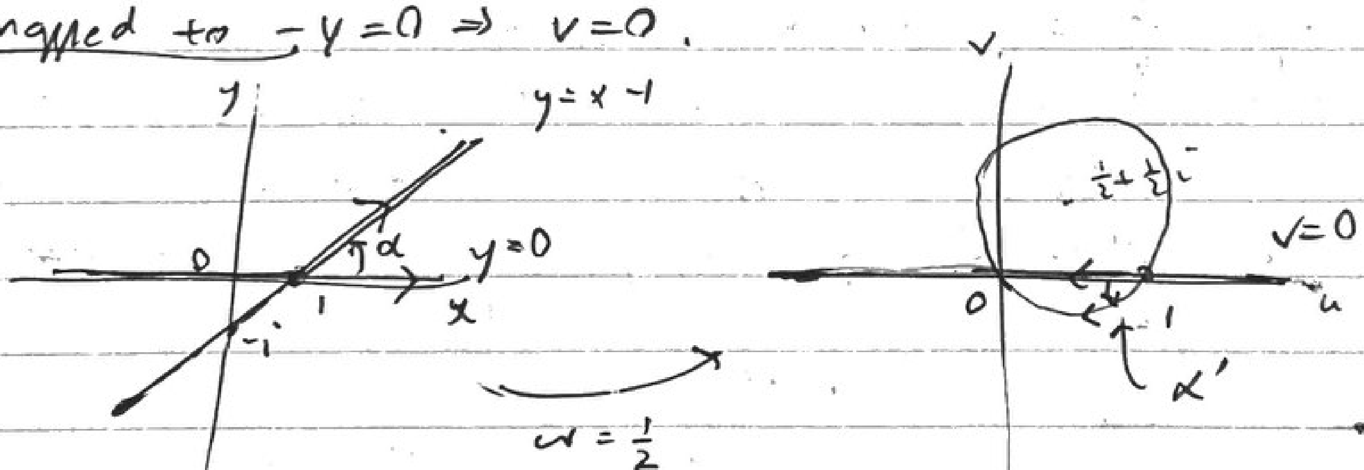
Recall from §98, that a line or circle of the form
$A(x^2 + y^2) + Bx + Cy + D = 0$ gets mapped by $w = \frac{1}{z}$
to $D(u^2 + v^2) + Bu - Cv + A = 0$.

$y = x - 1 \Rightarrow A = 0,\ B = -1,\ C = 1,\ D = 1.$

mapped to $u^2 + v^2 - u - v = 0 \Rightarrow (u - \frac{1}{2})^2 + (v - \frac{1}{2})^2 = \frac{1}{2}$

$y = 0 \Rightarrow A = 0,\ B = 0,\ C = 1,\ D = 0$

mapped to $-v = 0 \Rightarrow v = 0.$

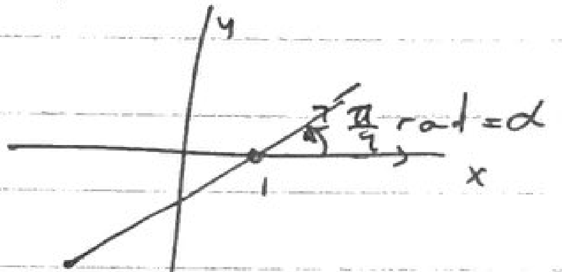


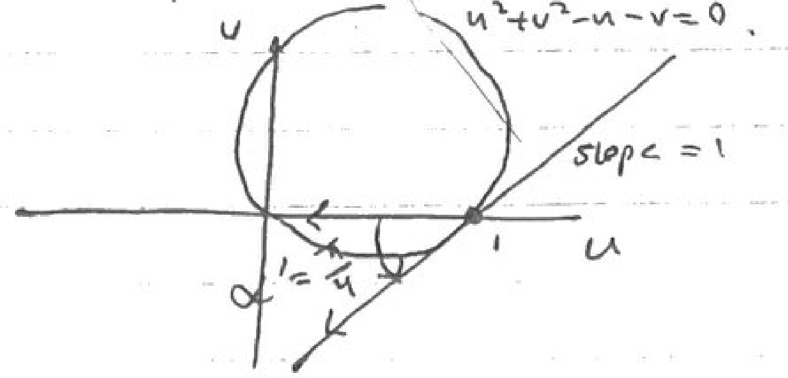
For $y = 0$, let the positive direction be moving to the right. Then $z(x) = x \quad (-\infty \le x \le \infty)$, $w(x) = \frac{1}{x}$, $w'(x) = -\frac{1}{x^2}dx$ which is always negative. So the corresponding direction for $v = 0$ in the uv plane is from right to left.

For $y = x - 1$, let the positive direction be up and to the right. In this direction, $\arg z$ is increasing, so $\arg w = -\arg z$ is decreasing (since $w = \frac{1}{z} = \frac{1}{r}e^{-i\theta}$), so this corresponds to CCW rotation around the circle.

Let α be the angle from $y = 0$ to $y = x - 1$ in the CCW direction, and α' be the angle between their respective images in the same direction, @ $z=1$ and $w=1$ ($z=1$ maps to $w=1$).

By inspection, the angle between $y=x-1$ and $y=0$ is $\frac{\pi}{4}$ rad.





The slope of the circle $u^2+v^2-u-v=0$ at $w=1$ is (by a Calc II thm for slope of implicit fn):

$$\frac{dv}{du} = -\frac{F_v}{F_u} = -\frac{\frac{\partial}{\partial v}(u^2+v^2-u-v)}{\frac{\partial}{\partial u}(u^2+v^2-u-v)} = -\frac{2v-1}{2u-1}$$

at $(u,v)=(1,0)$, $\frac{dv}{du} = \frac{-2(0)-1}{2(1)-1} = 1$.

Using the orientations previously established, it is clear that $\alpha' = \frac{\pi}{4}$ rad $= \alpha$, thus confirming conformality.

10.) Suppose that a function f is analytic @ z_0 and has a zero of order m there ($m \geq 1$). Also write $w_0 = f(z_0)$.

a) Use the Taylor series for f about z_0 to show that there is a nbd of z_0 in which the difference $f(z)-w_0$ can be written. $f(z)-w_0 = (z-z_0)^m \frac{f^{(m)}(z_0)}{m!}[1+g(z)]$

where $g(z)$ is continuous and zero valued @ z_0.

Taylor series centered at $z=z_0$:

$$f(z) = \underbrace{\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n}_{}$$ → zero coefs because of zero of order m.

$$= \underset{\text{nonzero}}{w_0} + \cancel{\frac{f'(z_0)}{1!}}(z-z_0) + \cancel{\frac{f''(z_0)}{2!}}(z-z_0)^2 + \dots +$$

$$\underbrace{\cancel{\frac{f^{(m-1)}(z_0)}{(m-1)!}}(z-z_0)^{m-1}}_{\text{zero coef}} + \underbrace{\frac{f^{(m)}(z_0)}{m!}(z-z_0)^m + \dots}_{\text{nonzero coef}}$$

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§114 #10, cont'd.

$$f(z) = w_0 + \frac{f^{(m)}(z_0)}{m!}(z-z_0)^m + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0)^{m+1} + \dots$$

$$f(z) - w_0 = \frac{f^{(m)}(z_0)}{m!}(z-z_0)^m \Bigg(1 + \underbrace{\frac{f^{(m+1)}(z_0)}{f^{(m)}(z_0)}\frac{m!}{(m+1)!}(z-z_0) + \frac{f^{(m+2)}(z_0)}{f^{(m)}(z_0)}\frac{m!}{(m+2)!}(z-z_0)^2 + \dots}_{\text{let this } = g(z).} \Bigg)$$

$$= \frac{f^{(m)}(z_0)}{m!}(z-z_0)^m \left[1 + g(z)\right]$$

$g(z)$ is clearly an ordinary power series and thus analytic (and continuous) @ z_0, and $g(z_0) = 0$ since every term has a (positive) factor of $(z-z_0)$ in it.

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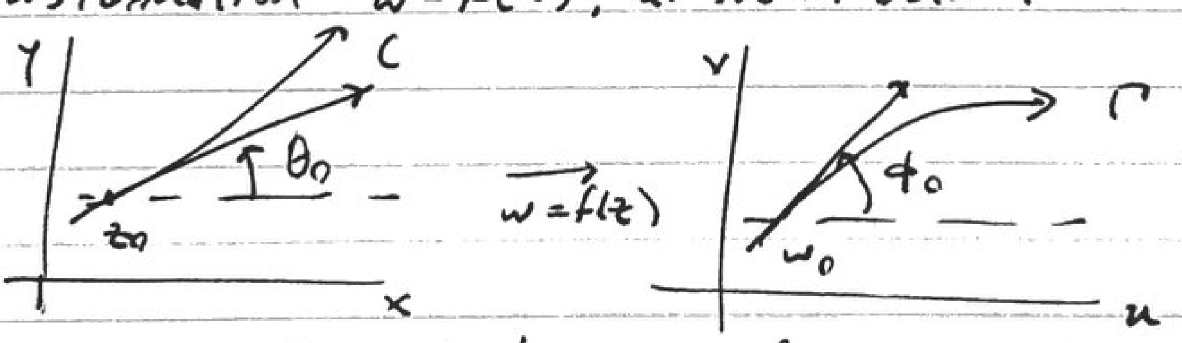
b). Let Γ be the image of a smooth arc C under the transformation $w = f(z)$, as shown below.



Note that θ_0 and ϕ_0 are $\lim_{z\to z_0} \arg(z-z_0)$ and $\lim_{z\to z_0} \arg(f(z)-w_0)$, respectively. Use the result from the previous section to show that θ_0 and ϕ_0 are related by: $\phi_0 = m\theta_0 + \arg(f^{(m)}(z_0))$.

Recall that arg of a product is the sum of the args of the factors.

$$\arg(f(z)-w_0) = \arg\left((z-z_0)^m \frac{f^{(m)}(z_0)}{m!}(1+g(z))\right)$$

$$= \underbrace{\arg\left((z-z_0)^m\right)}_{} + \underbrace{\arg\left(\frac{f^{(m)}(z_0)}{m!}\right)}_{} + \underbrace{\arg(1+g(z))}_{}$$

$= m\arg(z-z_0)$ by splitting product into sum of factors by above remark

$\Downarrow$ $= \arg f^{(m)}(z_0)$ (scalar multiplication doesn't change arg)

$\searrow$ $g(z)=0$ @ $z=z_0$

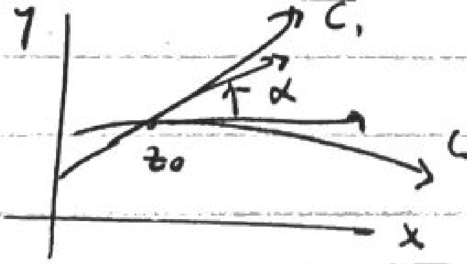
$\phi_0 =$

$$\lim_{z\to z_0}\arg(f(z)-w_0) = \lim_{z\to z_0}\left[m\arg(z-z_0)\right] + \lim_{z\to z_0}\left[\arg\left(f^{(m)}(z_0)\right)\right] + \lim_{z\to z_0}\left[\arg\left(1+g(z)\right)\right]$$

$$= m\underbrace{\lim_{z\to z_0}\arg(z-z_0)}_{=\theta_0} + \arg\left(f^{(m)}(z_0)\right) + \underbrace{\cancel{\arg(1+\underbrace{g(z_0)}_{0})}}_{\arg(1)=0}$$

$$= m\theta_0 + \arg\left(f^{(m)}(z_0)\right).$$

c) Let α denote the angle between two smooth arcs C_1 and C_2 passing through z_0. Show how it follows from the relation in part (b) that the corresponding angle between the image curves Γ_1 and Γ at the point $w_0 = f(z_0)$ is $m\alpha$.



angle between $C_1, C_2 = \theta_1 - \theta_2 = \alpha$.

angle between $\Gamma_1, \Gamma_2 = \phi_1 - \phi_2$

$$= \left(m\theta_1 + \cancel{\arg(f^{(m)}(z_0))}\right) - \left(m\theta_2 + \cancel{\arg f^{(m)}(z_0)}\right)$$

$$= m(\theta_1 - \theta_2) = m\alpha.$$

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§115 #2b.

2). Show that the function $u(x,y)$ is harmonic throughout the xy plane. Then, find its harmonic conjugate. Also, write the corresponding fn. $f(z) = u(x,y) + iv(x,y)$ in terms of z.

b) $u(x,y) = y^3 - 3x^2y$.

Show harmonicity:

$u_x = -6xy, \quad u_{xx} = -6y, \quad u_y = 3y^2 - 3x^2, \quad u_{yy} = 6y$

$u_{xx} + u_{yy} = -6y + 6y = 0$ ✓

Finding harmonic conjugate:

$$v(x,y) = \int_{(x_0,y_0)}^{(x,y)} -u_t(s,t)\,ds + u_s(s,t)\,dt.$$

$$= \int_{(0,0)}^{(x,y)} -\frac{\partial}{\partial t}(t^3 - 3s^2t)\,ds + \frac{\partial}{\partial s}(t^3 - 3s^2t)\,dt$$

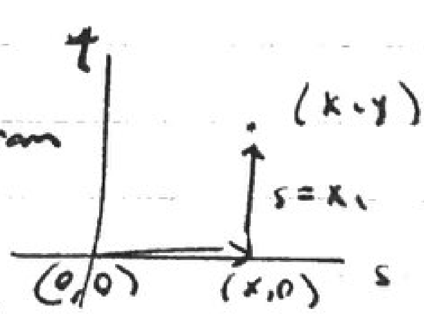
$$= \int_{(0,0)}^{(x,y)} -(3t^2 - 3s^2)\,ds - (6st)\,dt$$

$$= 3\int_{(0,0)}^{(x,y)} s^2 - t^2\,ds - 6\int_{(0,0)}^{(x,y)} st\,dt$$

$$= 3\int_0^x s^2 - \cancel{0^2}\,ds$$

$$-6\int_0^y xt\,dt$$

use this diagram and conservativity of this line integral



$$= 3\left.\frac{s^3}{3}\right|_0^x - 6x\left.\frac{t^2}{2}\right|_0^y$$

$$= x^3 - 3xy^2$$

$f(z) = \cancel{u(x,y) + iv(x,y)} = y^3 - 3x^2y + ix^3 - 3ixy^2$

$= i(-iy^3 - 3xy^2 + 3x^2yi + x^3) = i((iy)^3 + 3x(iy)^2 + 3x^2(iy) + x^3)$

$= i(x+iy)^3 = iz^3.$

§117 #4.

Under the transformation $w = \exp z$, the image of the segment $0 \le y \le \pi$ of the y-axis is the semicircle $u^2+v^2=1$, $v \ge 0$. Also, the function $h(u,v) = \mathrm{Re}\left(2 - w + \frac{1}{w}\right) = 2 - u + \frac{u}{u^2+v^2}$ is harmonic everywhere in the w plane except the origin, and it assumes the value $h=2$ on the semicircle. Write an explicit expression for the function $H(x,y)$ in the form of §117. Then illustrate the thm. by directly showing that $H=2$ along the segment $0 \le y \le \pi$ of the y axis.

$$H = h(u(x,y),\ v(x,y))$$

$$= h(e^x \cos y,\ e^x \sin y)$$

$$= 2 - e^x \cos y + \frac{e^x \cos y}{(e^x \cos y)^2 + (e^x \sin y)^2}$$

$$= 2 - e^x \cos y + \frac{e^x \cos y}{e^{2x}(\cos^2 y + \sin^2 y)}$$

$$= 2 - e^x \cos y + e^{-x} \cos y = 2 - 2\cos y \sinh x$$

On the line segment $0 \le y \le \pi$ of the y-axis, $x=0$, so $\sinh x = 0$, and

$$H(0,y) = 2 - 2\cos y \sinh(0) = 2 - \cancel{2\cos y(0)} = 2 \checkmark.$$

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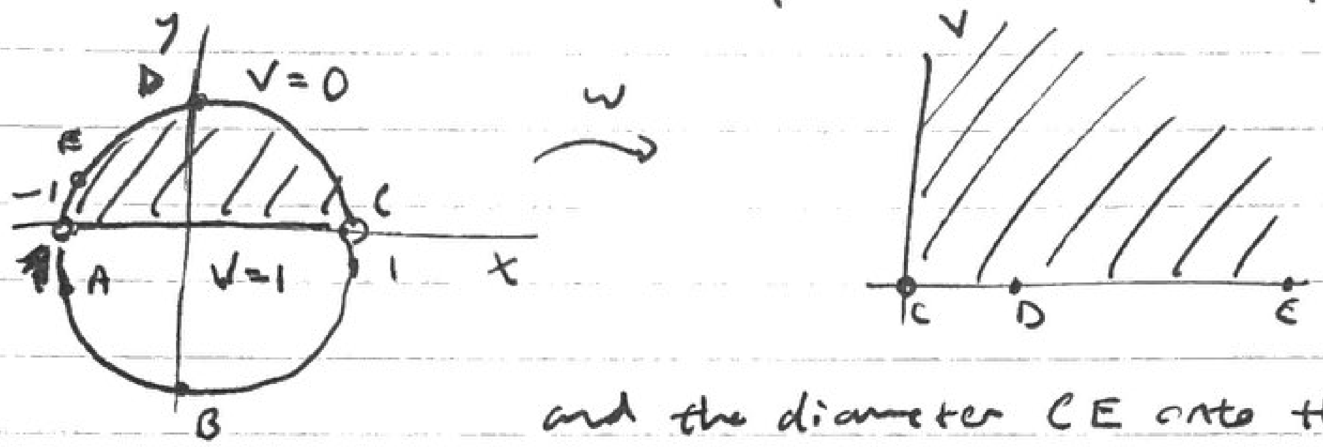
§123 #2, 10.

2). Show that the transformation $w = i\left(\frac{1-z}{1+z}\right)$ of §123 maps the upper half of the circular region shown below onto the first quadrant of the w plane



and the diameter CE onto the positive v-axis. Then find the electrostatic potential in the space enclosed by the half-cylinder $x^2+y^2=1$, $y \geq 0$ and the plane $y=0$ when $V=0$ on the cylindrical surface and $V=1$ on the planar surface.

we already know that the arc CDE gets mapped to the positive real axis by $w = i\left(\frac{1-z}{1+z}\right)$ from Fig. 13 in Appendix 2 and the example in §123. Thus, we only need to show that CE maps to the positive v-axis.

EC: $z = x$, $-1 \leq x \leq 1$

$w = i\left(\frac{1-x}{1+x}\right)$ is clearly pure imaginary, continuous, and its imaginary component is monotonic on $x \in (-1,1)$.

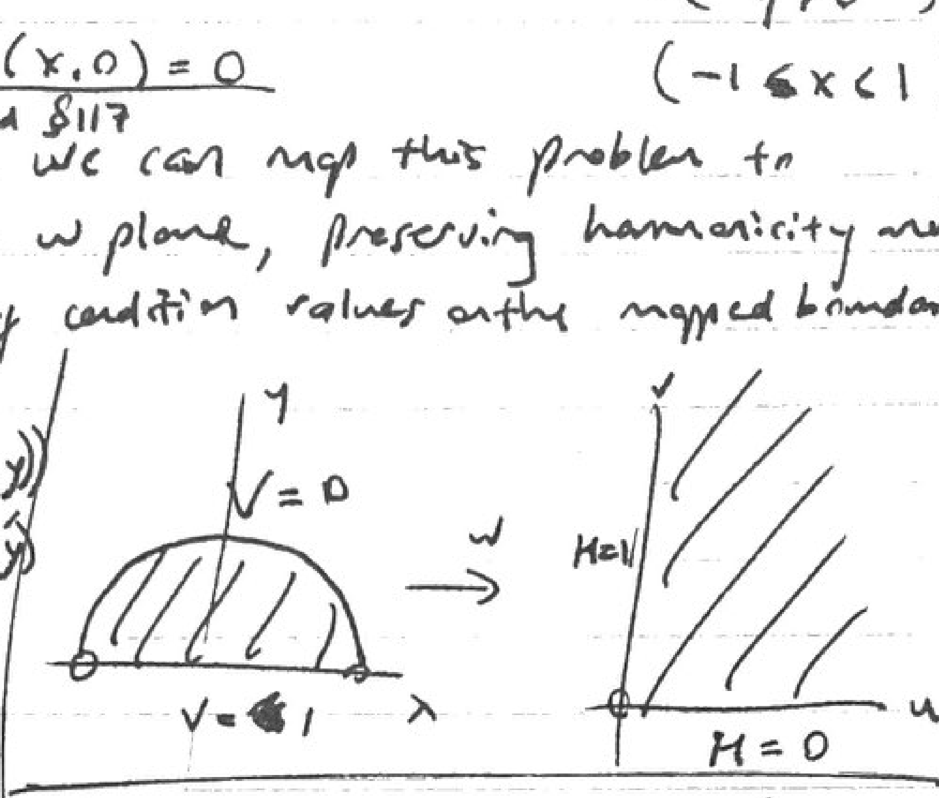
$\lim_{x \to -1} w = +\infty$, $w = 0$ @ $x=1$.

Thus CE must map to the positive v-axis, and the semicircular region must map to the first quadrant in the w-plane.

BVP setup:
$$\begin{cases} V_{xx}(x,y) + V_{yy}(x,y) = 0 & \left(\begin{matrix} x^2+y^2<1, \\ y>0 \end{matrix}\right) \\ V(x,y) = 1 & \left(\begin{matrix} x^2+y^2=1, \\ y>0 \end{matrix}\right) \\ V(x,0) = 0 & (-1 \leq x < 1) \end{cases}$$

By the Thm in §116 and §117 we can map this problem to a similar one in the w plane, preserving harmonicity and maintaining the boundary condition values on the mapped boundaries:

Let $H = V(u(x,y), v(x,y))$
where $w = u(x,y) + iv(x,y)$



This translates to the problem:
$$\begin{cases} H_{uu}(u,v) + H_{vv}(u,v) = 0 & (u, v > 0) \\ H(0,v) = 1 & (v>0) \\ H(u,0) = 0 & (u>0) \end{cases}$$

This can be solved by inspection. The imaginary part of the analytic function
$$\frac{2}{\pi} \operatorname{Log} w = \frac{2}{\pi} \ln|w| + i \underbrace{\frac{2}{\pi} \arg w}_{H(u,v)}$$
(analytic on all but negative real axis and origin, which includes desired domain)

is harmonic and matches these boundary conditions.
Here, (in the first quadrant) the arctangent function matches the value of arg,

so $H = \frac{2}{\pi} \arctan\left(\frac{v}{u}\right) = V(u,v) \quad \left(0 < \arctan\theta < \frac{\pi}{2}\right)$

(Need to find v, u in terms of x, y)
$$w = i\left(\frac{1-z}{1+z}\right) = i\left(\frac{(1-z)(1+z^*)}{|1+z|^2}\right) = \frac{i}{(1+x)^2+y^2}(1 + z^* - z - zz^*)$$
$$= \frac{i}{(1+x)^2+y^2}(1 - x^2 - y^2 - 2yi) = \underbrace{\frac{2y}{(1+x)^2+y^2}}_{u(x,y)} + i \underbrace{\frac{1-x^2-y^2}{(1+x)^2+y^2}}_{v(x,y)}$$

So $\underline{V(x,y) = \frac{2}{\pi} \arctan\left(\frac{1-x^2-y^2}{2y}\right)}$

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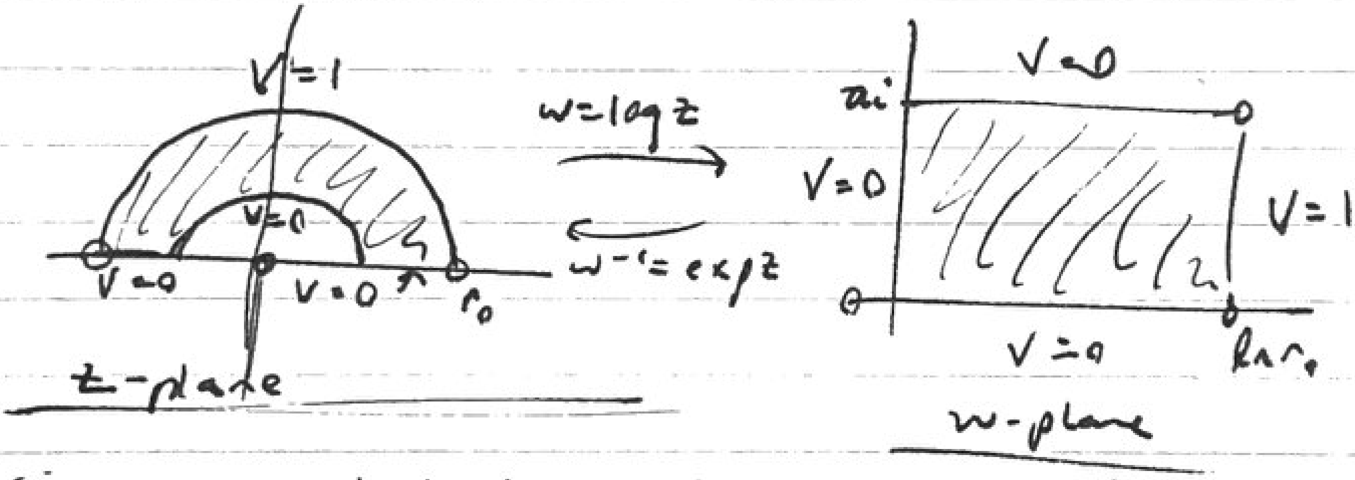
§123 # 10.

10) The soln. to the Dirichlet problem on the right of the below figure is
$$V = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sinh mu}{m \sinh(m \ln r_0)} \sin mv$$
where $m = 2n-1$. By using the branch
$$\log = \ln r + i\theta \quad \left(r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}\right)$$
of the logarithmic function, derive the following soln of the Dirichlet problem on the bottom left:
$$V(r,\theta) = \frac{4}{\pi}\sum_{n=1}^{\infty}\left(\frac{r^m - r^{-m}}{r_0^m - r_0^{-m}}\right)\frac{\sin m\theta}{m}.$$
where $m = 2n-1$.



First, to establish the mapping. We see that this mapping is the inverse of an exponential mapping similar to that shown in Fig 8. of Appendix 2. As expected, the inverse map maps horizontal line segments to segments of radially-oriented lines, and vertical line segments are mapped to circular arcs centered at the origin. In particular:

$$\left.\begin{array}{l}\text{top edge of rectangle}\\ \text{left " " "}\\ \text{bottom " " "}\\ \text{right " " "}\end{array}\right\} \xrightarrow[z = \exp w]{w^{-1}} \left\{\begin{array}{l}\text{bottom left segment on } x\text{-axis}\\ \text{bottom arc, } r=1\\ \text{bottom right segment on } x\text{-axis}\\ \text{top arc, } r = r_0\end{array}\right.$$

Since $\exp z = w^{-1}$, we may use the branch of the log:

$$w = \log z = \ln r + i\theta \quad (r > 0,\ -\tfrac{\pi}{2} < \theta < \tfrac{3\pi}{2})$$

to map the z-plane to the w plane. This branch of log is singular only on the negative imaginary axis, so it is analytic on the desired domain. Thus a conformal mapping is established.

Since we have a conformal mapping, and a harmonic function, by the thm in §116, $V(u,v)$ is harmonic in the mapped region, and by the thm in §117 this harmonic function maintains the same values for the boundary conditions in the z-plane as the mapped conditions in the w-plane.

Note that the mapped boundary conditions already match, i.e.:

$$\left.\begin{matrix}\text{lower edges in } z\text{-plane,} \\ V = 0\end{matrix}\right\} \xrightarrow{w} \left\{\begin{matrix}\text{left, top, bottom edges} \\ \text{in } w\text{-plane, } V = 0\end{matrix}\right.$$

$$\left.\begin{matrix}\text{top arc in } z \text{ plane,} \\ V = 1\end{matrix}\right\} \xrightarrow{w} \left\{\begin{matrix}\text{right edge in } w\text{-plane,} \\ V = 1\end{matrix}\right.$$

and we are already provided with a solution $V(u,v)$. Thus, to obtain the solution for the preimage, all we must do is express (u,v) in terms of z-plane coordinates (r,θ).

$$u = \ln r, \quad v = \theta, \quad (-\tfrac{\pi}{2} < \theta < \tfrac{3\pi}{2})$$

$$V(\ln r, \theta) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sinh(m \ln r)}{m \sinh(m \ln r_0)} \sin(m\theta)$$

$$= \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cancel{\tfrac{1}{2}}(e^{m\ln r} - e^{-m\ln r}) \sin(m\theta)}{m \cdot \cancel{\tfrac{1}{2}}(e^{m\ln r_0} - e^{-m\ln r_0})}$$

$$= \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{r^m - r^{-m}}{r_0^{\,m} - r_0^{\,-m}} \frac{\sin(m\theta)}{m}$$

(where $m = 2n-1$)