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PSET 3

1.5 # 2adf, 3, 5, 16

29/30

1.5.5 | 5
2.1.14 | 4
| 9

2. Determine whether the sets of vectors are linearly independent or dependent.

a) $\left\{\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix}\right\}$ in $M_{2\times 2}(\mathbb{R})$

linearly independent $\Leftrightarrow$ no nontrivial lin. combs. of vectors in the set, i.e., determine if system has nontrivial solutions.

$$a\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix} + b\begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$a(1) + b(-2) = 0$

$(a(-3) + b(6) = 0) \rightarrow$ divide by $(-3) \rightarrow a(1) + b(-2) = 0$

$(a(-2) + b(4) = 0) \rightarrow$ divide by $(-2) \rightarrow a(1) + b(-2) = 0$

$(a(4) + b(-8) = 0) \rightarrow$ divide by $4 \rightarrow a(1) + b(-2) = 0$

Now only 1 equation remaining with two variables. Any combination $(a, b) = (2t, t)$ $\forall t \in \mathbb{R}$ is a solution, therefore there are nontrivial sol'ns to this lin comb $\Rightarrow$ dependent set.

d) $\{x^3 - x, 2x^3 + 4, -2x^3 + 3x^2 + 2x + 6\}$ in $P_3(\mathbb{R})$

same method as in (2a):

$a(x^3 - x) + b(2x^2 + 4) + c(-2x^3 + 3x^2 + 2x + 6) = 0 = 0x^3 + 0x^2 + 0x + 0$

polynomials equal $\Leftrightarrow$ coefficients match:

$$\begin{cases} a - 2c = 0 \rightarrow \text{multiply by } -1 \rightarrow -a + 2c = 0, \text{ same as 3rd eq.} \\ 2b + 3c = 0 \rightarrow \text{multiply by } 2 \rightarrow 4b + 6c = 0, \text{ same as 4th eq.} \\ -a + 2c = 0 \\ 4b + 6c = 0 \end{cases}$$

$$\begin{cases} -a + 2c = 0 \\ 2b + 3c = 0 \end{cases}$$

Let $b = t$, then $2t + 3c = 0 \Rightarrow c = -\frac{2}{3}t$

$-a + 2c = 0 \Rightarrow -a + 2\left(-\frac{2}{3}t\right) = 0 \rightarrow a = -\frac{4}{3}t$

This means $\left(-\frac{4}{3}t, t, -\frac{2}{3}t\right)$, $\forall t \in \mathbb{R}$ is a sol'n. Since there are nontrivial lin. combs. that sum to 0, this set is linearly dependent.

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2f.) using same method as in (2a), (2d):

$\{(1,-1,2), (2,0,1), (-1,2,-1)\}$ in $\mathbb{R}^3$.

$a(1,-1,2) + b(2,0,1) + c(-1,2,-1) = (0,0,0)$;

$a + 2b - c = 0$
$-a + 2c = 0$
$2a + b - c = 0$

$a + 2b - c = 0$
$2b + c = 0$
$-3b + c = 0$

$a + 2b - c = 0$
$2b + c = 0$
$\frac{5}{2}c = 0 \Rightarrow c = 0$

$2b + (0) = 0 \Rightarrow 2b = 0 \Rightarrow b = 0$

$a + 2(0) - (0) \Rightarrow a = 0$

The only solution to this system of lin. eqs. is $(a,b,c) = (0,0,0)$, thus there are only trivial soln's to the lin. comb $\Rightarrow$ the set is linearly independent.

3. <u>CLAIM</u>: the set

$$\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \right\} \quad (\text{in } M_{2\times3}(F))$$

is linearly dependent.

<u>PROOF</u>:

Since
$$1\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} + (-1)\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} + (-1)\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

then there exists a nontrivial lin. comb. which sums to the zero vector in $M_{2\times3}(F)$, thus the set is linearly dependent.

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1.5 # 5, 16

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probably better to index a single variable...

5. CLAIM: the set $S = \{1, x, x^2, \ldots, x^n\}$ is linearly independent in $P_n(F)$.

PROOF: The linear combination over elements in S:
$a(1) + b(x) + c(x^2) + \ldots + zx^n$ is only equal to the zero vector of $P_n = 0 + 0x + 0x^2 + \ldots + 0x^n$ if all the coefficients are equal, i.e., $a=0, b=0, c=0, \ldots, z=0$. Thus, the only linear combination over the elements in S that sum to 0 is the trivial lin. comb., so S is linearly independent.

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16. Prove: A set S of vectors is linearly independent $\Leftrightarrow$ each finite subset of S is linearly independent.

CLAIM: ($\Rightarrow$) If S is lin. ind., then each finite subset of S is also linearly ind.

PROOF: Let T be a subset of S. By (THM 1.6, COR. 1), since $T \subseteq S$ and S is lin. ind., T is also lin. ind. T includes all finite subsets of S, so all finite subsets of S are lin. ind.

CLAIM: ($\Leftarrow$) If each finite subset of S is linearly independent, then S is also linearly independent.

PROOF: Proof by contrapositive: This claim is logically equivalent to proving S is linearly dependent $\Rightarrow$ not all finite subsets of S are linearly independent. Thus, assume S is lin dep. By (DEF lin. dep.), $\exists$ (finite) $T = \{u_1, u_2, u_3, \ldots, u_n\} \subseteq S$, $\{a_1, a_2, \ldots, a_n\} \in F$ s.t. $a_1u_1 + a_2u_2 + \ldots + a_nu_n$ yields the zero vector in the vector space. Since $T \subseteq T$, T is also linearly dep. by definition. Since T is a finite subset of S, $\exists$ linearly dep. subset of $S \Rightarrow$ not all subsets of S are linearly ind.

1.6 # 20, 24, 29, 31.

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20. Let V be a v.s. with dimension n, and let S be subset of V, $\text{span}(S) = V$.

a) CLAIM: There exists a subset of S that is a basis for V.
PROOF: Let $\beta = \{b_1, b_2, \dots, b_n\}$ be a basis for V. Then $\beta_1 \subseteq V = \text{span}(V)$, so each vector b_i, $1 \leq i \leq n$ can be represented as the lin. comb. $b_i = \sum_{j=1}^{k_i} a_{ij} s_{ij}$,

for some $a_{ij} \in F$, $s_{ij} \in S$, k_i finite (since lin. combos involve a finite # of vectors).

Let $T = \{ s_{ij} \in V : \forall i, 1 \leq i \leq n, 1 \leq j \leq k_i \}$, i.e., T is the set of all vectors from S in the linear combinations forming the basis vectors, and T is finite since n is finite, k_i is finite $\forall 1 \leq i \leq n$.

Since β generates V, and β can be represented as a lin. comb. of vectors over S, then $\forall v \in V$,

$$v = \sum_{i=1}^{n} c_i b_i = \sum_{i=1}^{n} c_i \sum_{j=1}^{k_i} a_{ij} s_{ij} = c_1 a_{11} s_{11} + c_1 a_{12} s_{12} + \dots + c_1 a_{1k_1} s_{1k_1} + c_2 a_{21} s_{21} + \dots + c_n a_{nk_n} s_{nk_n}$$

which is a linear combination of vectors over T. Thus V is generated by T, a finite subset of S. By (THM 1.9), there exists a subset of T that is a basis for V. Since $T \subseteq V$, that basis must also be in S.

b.) CLAIM: S contains at least n vectors
PROOF: By (THM 1.10, COR 1), all bases of a v.s. have the same cardinality, which is the dimension of the v.s., n. Since a basis is the subset of S, then S must contain at least as many vectors as the basis, i.e., n vectors.

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1.6 #24

24. Let $f(x)$ be a polynomial of degree n in $P_n(\mathbb{R})$. Prove that $\forall g(x) \in P_n(\mathbb{R})$, $\exists a_0, a_1, \ldots, a_n$ s.t.
$g(x) = a_0 f(x) + a_1 f'(x) + \ldots + a_n f^{(n)}(x)$

CLAIM: $\{f, f', \ldots, f^{(n)}\}$ is lin. ind.

PROOF: Let z be the zero function in $P(\mathbb{R})$, i.e., $z = 0 + 0x + 0x^2 + \ldots$. Assume a lin. comb. over $\{f, f', \ldots, f^{(n)}\}$: $c_0 f + c_1 f' + \ldots + c_n f^{(n)} = z$. By calculus, $\forall h \in P_i(\mathbb{R})$, $h' \in P_{i-1}(\mathbb{R})$, and degree$(h') <$ degree(h). Since f is the only function with degree n, it is the only polynomial with a nonzero x^n term; thus $c_0 = 0$ or else the result of the lin. comb. would have a nonzero x^n coefficient.

Now the lin. comb. over $\{f, f', \ldots, f^{(n)}\}$ is equivalent to $c_1 f' + c_2 f'' + \ldots + c_n f^{(n)} = z$. By the same reasoning as above, c_1 must be 0, or else the result of the lin. comb. would have a nonzero x^{n-1} term. By induction, $c_0 = c_1 = \ldots = c_n = 0$. Thus, $\{f, f', \ldots, f^{(n)}\}$ is lin. ind. //

CLAIM: Any $g \in P_n(\mathbb{R})$ can be expressed as a lin. comb over $\{f, f', \ldots, f^{(n)}\}$, $f \in P_n(\mathbb{R})$ w/ degree n.

PROOF: $\{f, f', \ldots, f^{(n)}\}$ was shown above to be lin. ind., and has cardinality $n+1$ (by inspection). $\dim(P_n(\mathbb{R})) = n+1$. By (THM 1.10 COR. 2), $\{f, f', \ldots, f^{(n)}\}$ is a basis for $P_n(\mathbb{R})$, and thus $g \in \text{span}(\{f, f', \ldots, f^{(n)}\})$.

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1.6 # 29

29 a) Given that W_1, W_2 finite-dimensional subspaces of v.s. V, then $W_1 + W_2$ finite-dimensional, and $\dim(W_1+W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

Let $\dim(W_1) = n$, $\dim(W_2) = m$, $\dim(W_1 \cap W_2) = l$
Note that $W_1 \cap W_2$ subsp. W_1, W_2 (by THM. 1.4), so $\dim(W_1 \cap W_2) \le \dim(W_1), \dim(W_2)$ (by THM. 1.11).
Fix a basis β_l of $W_1 \cap W_2$, $\beta_l = \{u_1, u_2, \ldots, u_l\}$.
By (THM 1.10 COR 2), we can extend β_l (which is a linearly ind. subset of W_1) to a basis β_n for W_1, $\beta_n = \beta_l \cup \{v_1, v_2, \ldots, v_{n-l}\}$. The same argument applies to extend β_l to a basis β_m for W_2; $\beta_m = \beta_l \cup \{w_1, w_2, \ldots, w_{m-l}\}$.
Finally, let $\beta = \beta_n \cup \beta_m = \{u_1, u_2, \ldots, u_l, v_1, v_2, \ldots, v_{n-l}, w_1, w_2, \ldots, w_{m-l}\}$.

CLAIM: β generates V.
PROOF: $\forall v \in V$, $v = v_1 + v_2$, $v_1 \in W_1$, $v_2 \in W_2$. Then:
$v_1 = \sum_{i=1}^{l} a_i u_i + \sum_{i=1}^{n-l} b_i v_i$,
$v_2 = \sum_{i=1}^{l} c_i u_i + \sum_{i=1}^{m-l} d_i w_i$
$v = \sum_{i=1}^{l} (a_i + c_i) u_i + \sum_{i=1}^{n-l} b_i v_i + \sum_{i=1}^{m-l} d_i v_i$
$\therefore v \in \text{span}(\beta)$, so β generates V.

CLAIM: β is lin. ind.
PROOF: Assume $\sum_{i=1}^{l} a_i u_i + \sum_{i=1}^{n-l} b_i v_i + \sum_{i=1}^{m-l} c_i w_i = 0$.
Then, let $x = -\sum_{i=1}^{m-l} c_i w_i = \sum_{i=1}^{n} a_i u_i + \sum_{i=1}^{n-l} b_i v_i$.
Since x can be expressed as a lin. comb over W_i, then $x \in W_2$. Since x can be expressed as a lin. comb. over $\beta_n = \{u_1, u_2, \ldots, u_l, v_1, v_2, \ldots, v_{n-l}\}$, $x \in W_2$ as well, so $x \in W_1 \cap W_2$ and can be expressed as a lin. comb over β_l: $x = \sum_{i=1}^{l} d_i u_i$. Substituting this back into the original equation: $\sum_{i=1}^{l} (a_i + d_i) u_i + \sum_{i=1}^{n-l} b_i v_i = 0$. Since this is a lin. comb over the lin. ind. set β_n, $b_1 = b_2 = \ldots = b_{n-l} = 0$. Thus the original lin. comb. is equivalent to $\sum_{i=1}^{l} a_i u_i + \sum_{i=1}^{m-l} c_i w_i = 0$. Since this is a lin. comb. over the lin. ind. set β_m, $a_1 = a_2 = \ldots = a_l = c_1 = c_2 = \ldots = c_{m-l} = 0$. Thus all the coefficients must be 0, so β is lin. ind.

CLAIM: $\dim(W_1+W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

PROOF: By its construction, $\text{card}(\beta) = l + (n-l) + (m-l)$
$= n+m-l$, where $n = \dim(W_1)$, $m = \dim(W_2)$, and
$l = \dim(W_1 \cap W_2)$. Since β is a basis for $W_1 + W_2 = V$,
$\dim(W_1+W_2) = \text{card}(\beta) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

b) Let W_1, W_2 be finite-dimensional subsp. of V, and let
$W_1 + W_2 = V$. Deduce that $V = W_1 \oplus W_2 \iff \dim(W_1) + \dim(W_2) =$
$\dim(V)$.

PF ($\Rightarrow$): By (DEF $\oplus$), $W_1 \cap W_2 = \{0\}$. By (part a),
$\dim(V) = \dim(W_1) + \dim(W_2) - \dim(\{0\})$
$= \dim(W_1) + \dim(W_2) - 0 = \dim(W_1) + \dim(W_2)$.

PF ($\Leftarrow$): $W_1 + W_2 = V$, and W_1, W_2 subsp. V by hyp. Also,

$$\underbrace{\dim(W_1) + \dim(W_2) = \dim(V)}_{\text{by hyp}} \quad \underbrace{\dim(V) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)}_{\text{by (part a)}}$$

Thus $\dim(W_1 \cap W_2) = 0 \Rightarrow W_1 \cap W_2 = \{0\}$. Thus
$W_1 \oplus W_2 = V$ by def.

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1.6 #31

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31) Let W_1, W_2 subsp. V, $\dim(W_1) = m$, $\dim(W_2) = n$, $m \geq n$.

a) CLAIM: $\dim(W_1 \cap W_2) \leq n$.
PROOF: $W_1 \cap W_2$ subsp. V (by THM 1.4), and since $W_1 \cap W_2 \subseteq W_1$, it is also a subsp. of W_1.
By (THM 1.11), $\dim(W_1 \cap W_2) \leq \dim(W_1) = n$.

b) CLAIM: $\dim(W_1 + W_2) \leq m + n$.
PROOF: By the result of (1.6 exercise #29),
$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$
$= m + n - \dim(W_1 \cap W_2)$. Since dimension of any v.s. is nonnegative, $\dim(W_1 + W_2) \leq m + n$.

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2.1 # 14, 26

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or a vector space?

14. Let V, W v.s., $T: V \to W$ linear.

a) CLAIM: T 1-1 $\Leftrightarrow$ T carries linearly independent subsets of V onto linearly independent subsets of W.

abuse of notation leads to utter confusion. is V a set or a vector

4

very poor choices of notation

Please conform to standards from textbook or lecture!

PROOF ($\Rightarrow$): Let T be 1-1, $v \subseteq V$ ind., $w = T(v)$.

Assume some lin. comb. over $w = 0$, i.e., $\sum_{i=0}^{n} a_i w_i = 0$, $0 \le n \le card(w)$

Then: $\sum_{i=0}^{n} a_i T(v_i) = T(\sum_{i=0}^{n} a_i v_i) = 0 = T(0)$ (by linearity of T).

Since T is 1-1, $\sum_{i=0}^{n} a_i v_i = 0$, and since this is a lin. comb over a lin. ind. set v, $a_1 = a_2 = \dots = a_n = 0$.

Since these are also the coefficients to the lin. comb. over w yielding 0, w is also linearly independent.

PROOF ($\Leftarrow$): Proof by contrapositive statement: if T not 1-1, then $\exists$ a lin. ind. subset $\{x\}$ of V such that $\{T(x)\}$ is linearly dependent. Since T not 1-1, by (THM 2.4) $N(T) \neq \{0\}$, so $\exists\, v_0 \in V \neq 0$ s.t. $T(v_0) = 0$.

Since the singleton set of any nonzero vector is lin. ind., and any set containing the zero vector is lin. dep, then $\{v_0\}$ is ind., but $\{T(v_0)\} = \{0\}$ is dep.

b) Suppose T 1-1, $S \subseteq V$.

CLAIM: S lin. ind. $\Leftrightarrow$ $T(S)$ lin. ind.

PROOF ($\Rightarrow$): This was proved in (part a).

PROOF ($\Leftarrow$): This proof similar to (part a proof $\Rightarrow$): Assume lin. comb. over S, i.e., $\sum_{i=1}^{n} a_i s_i = 0$, $1 \le n \le card(S)$. Take the transform, i.e., $T(\sum_{i=1}^{n} a_i s_i) = \sum_{i=1}^{n} a_i T(s_i) = T(0) = 0$. Since $T(S)$ lin. ind., all coefficients $a_1 = a_2 = \dots = a_n = 0$, thus S must also be lin. ind.

c) suppose $\beta = \{v_1, v_2, \dots, v_n\}$ is a basis for V and T is 1-1 and onto. Prove that $T(\beta) = \{T(v_1), T(v_2), \dots, T(v_n)\}$ is a basis for W.

PROOF: By (part b), since β lin. ind. and T 1-1, then $T(\beta)$ is lin. ind. By (THM 2.5), since T 1-1, rank(T) $=$ dim(T) $=$ card(β) $=$ card$(T(\beta))$. Since T is onto, $R(T) = W$; since card$(T(\beta))$ = rank(T), by (THM 1.10 COR 2) $T(\beta)$ is a basis for $R(T) = W$.

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PSET 3
2.1 #26

26. $T: V \to V$ is the projection on W_1 along W_2.
i.e., $W_1 \oplus W_2 = V$, $\forall v \in V$, $v = v_1 + v_2$, $v_1 \in W_1$, $v_2 \in W_2$,
$T(v) = T(v_1 + v_2) = v_1$

a) Prove that T is linear, and $W_1 = \{x \in V : T(x) = x\}$.

<u>CLAIM</u>: T linear.
<u>PROOF</u>: $\forall$ $x = x_1 + x_2$, $y = y_1 + y_2$, $x, y \in V$, $x_1, y_1 \in W_1$, $x_2, y_2 \in W_2$, $\forall a \in F$, then:

$$\begin{aligned} T(ax+y) &= T(a(x_1+x_2) + (y_1+y_2)) = T(ax_1 + ax_2 + y_1 + y_2) \\ &= T((ax_1+y_1) + (ax_2+y_2)) \\ &= ax_1 + y_1 = aT(x) + T(y) \end{aligned}$$

<u>CLAIM</u>: $W_1 = \{x \in V : T(x) = x\}$.
<u>PROOF</u>: (Proof by containment both ways).
$\forall x \in W_1$, then x also in V (since $W_1 \subseteq V$).
Since $0 \in W_2$, $x = x + 0 \in V$, and $T(x) = T(x+0) = x$.
Thus $x \in \{x \in V : T(x) = x\}$, so $W_1 \subseteq \{x \in V : T(x) = x\}$.
$\forall x \in \{x \in V : T(x) = x\}$, x lies in the codomain W_1 of T (since $T(x) = x$). Thus $x \in W_1$, so $\{x \in V : T(x) = x\} \subseteq W_1$.
By containment both ways, $W_1 = \{x \in V : T(x) = x\}$.

b) <u>CLAIM</u>: $W_1 = R(T)$
<u>PROOF</u>: Since the codomain of T is W_1, $R(T) \subseteq W_1$.
By (part a), $W_1 = \{x \in V : T(x) = x\} \subseteq R(T)$ (since W_1 is a subset of values of $T(x)$). By containment both ways, $W_1 = R(T)$

<u>CLAIM</u>: $W_2 = N(T)$
<u>PROOF</u>: Let $v \in V$. If $v \in W_2$, then since $0 \in W_1$, $T(v) = T(0+v) = 0$, so $v \in N(T)$. If $v \notin W_2$, then $v = v_1 + v_2$, $v_1 \in W_1 \neq 0$, $v_2 \in W_2$, and $T(v) = T(v_1 + v_2) = v_1 \neq 0$, so $v \notin N(T)$. Thus $v \in N(T) \Leftrightarrow v \in W_2$, so $W_2 = N(T)$.

c) Describe T if $W_1 = V$.
By (Part a), then $\forall x \in V$, $T(x) = x$, so $T = 1$ (identity transform).
Some properties of T: $\dim(W_1) = \text{rank}(T) = \dim(V)$ (by THM 1.11), and by (THM 2.5) and (THM 2.4), T is onto, 1-1, and $N(T) = W_2 = \{0\}$.

d) Describe T if W_1 is the zero subspace.
Then $\text{rank}(T) = 0$, and by the dimension theorem (THM 2.3), $\text{nullity}(T) = \dim(V)$. Since $W_2 = N(T)$ (by part b), by (THM 1.11), $N(T) = V$. This means that T is the zero transformation, i.e., $\forall x \in V$, $T(x) = 0$.

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2.2 # 2(af), 5(bf), 11

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2. Let β, γ be standard ordered bases for $\mathbb{R}^n$, $\mathbb{R}^m$, respectively. For each lin. transformation $T: \mathbb{R}^n \to \mathbb{R}^m$, compute $[T]_\beta^\gamma$.

a.) $T: \mathbb{R}^2 \to \mathbb{R}^3$, $T(a_1, a_2) = (2a_1 - a_2, 3a_1 + 4a_2, a_1)$.

$T(1,0) = (2,3,1) = 2(1,0,0) + 3(0,1,0) + 1(0,0,1)$

$T(0,1) = (-1,4,0) = -1(1,0,0) + 4(0,1,0) + 0(0,0,1)$

$\Rightarrow [T]_\beta^\gamma = \begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{pmatrix}$

f.) $T: \mathbb{R}^n \to \mathbb{R}^n$, $T(a_1, a_2, \ldots, a_n) = (a_n, a_{n-1}, \ldots, a_1)$

$T(1,0,0,\ldots,0,0) = (0,0,0,\ldots,0,1)$

$T(0,1,0,\ldots,0,0) = (0,0,0,\ldots,1,0)$

$T(0,0,1,\ldots,0,0) = (0,0,0,\ldots,1,0,0)$

$\vdots$

$T(0,0,\ldots,1,0) = (0,1,\ldots,0,0)$

$T(0,0,\ldots,0,1) = (1,0,\ldots,0,0)$

$\Rightarrow [T]_\beta = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ \vdots & & \cdots & 1 & 0 \\ & & \iddots & & \vdots \\ 0 & 1 & & & \vdots \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$

(i.e., ones on counterdiagonal, 0's elsewhere).

5 b) $T: P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$, $T(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix}$

Compute $[T]_\beta^\alpha$. $\left(\beta = \{1, x, x^2\}, \alpha = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}\right)$

$T(1) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$T(x) = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = 1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$T(x^2) = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} = 0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow [T]_\beta^\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

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f) $f(x) = 3 - 6x + x^2$. Compute $[f(x)]_\beta$. $(\beta = \{1, x, x^2\})$

$f(x) = 3(1) + (-6)(x) + (1)x^2$,

so $[f(x)]_\beta = \begin{pmatrix} 3 \\ -6 \\ 1 \end{pmatrix}$

11. Let V be v.s., $\dim(V) = n$, $T: V \to V$ linear.
Suppose W is T-invariant subsp. of V, $\dim(W) = k$.
Show that there is a basis β for V s.t. $[T]_\beta$ has the form:
$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, where A is $k \times k$ matrix, 0 is the $(n-k)\times k$ zero matrix

PROOF: Let $\beta' = \{u_1, u_2, \ldots, u_k\}$ be a basis for W.
Since W subsp. V, $\dim(W) \leq \dim(V)$, so $k \leq n$.
By Thm 1.10 Cor 2), β' can be grown into a basis
$\beta = \{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_{n-k}\}$ for V.

Construct the matrix representation of T in the ordered basis β by finding the coordinate vector of each basis vector. Since W is T-invariant, $T(u_i) \in W$, $1 \leq i \leq k$, so
$T(u_i) = \sum_{i=1}^{k} a_i u_i + \sum_{i=1}^{k-n} (0) v_i$, and the coordinate vector is $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ a \\ a \\ a \end{matrix}}\right\} k \\ \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} k-n \end{matrix}$ for all u_i (first k basis vectors), where a_i are arbitrary scalars

This will be the form of the first k column vectors (corresponding to β') in the matrix representation of T in the ordered basis β:

$$\begin{array}{c} k\left\{\vphantom{\begin{matrix} a \\ a \\ a \\ a \end{matrix}}\right. \\ n-k\left\{\vphantom{\begin{matrix} a \\ a \\ a \\ a \end{matrix}}\right. \end{array} \left(\begin{array}{cccc|ccc} a_{11} & a_{12} & \cdots & a_{1k} & b_{11} & \cdots & b_{1(n-k)} \\ a_{21} & a_{22} & & a_{2k} & b_{21} & & \\ \vdots & \vdots & & & & & \\ a_{k1} & a_{k2} & & a_{kk} & b_{k1} & \cdots & b_{k(n-k)} \\ \hline 0 & 0 & & 0 & c_{11} & \cdots & c_{1(n-k)} \\ \vdots & \vdots & \cdots & \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 & c_{(n-k)1} & \cdots & c_{(n-k)(n-k)} \\ \multicolumn{4}{c|}{\underbrace{\hphantom{a_{11}\ a_{12}\ \cdots\ a_{1k}}}_{k}} & \multicolumn{3}{c}{\underbrace{\hphantom{b_{11}\ \cdots\ b_{1(n-k)}}}_{n-k}} \end{array}\right) = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where $A \in M_{k\times k}(F)$, $B \in M_{k\times(n-k)}(F)$, $C \in M_{(n-k)\times(n-k)}(F)$, and 0 is the $(n-k)\times k$ zero matrix. Elements in A, B, C are arbitrary.