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5.2 # 2bd, 3ac, 7, 9, 11

2. For each of the following matrices $A \in M_{n\times n}(\mathbb{R})$, test A for diagonalizability, and if A is diagonalizable, find an invertible matrix Q and a diagonal matrix D s.t. $Q^{-1}AQ = D$.

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b) $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ $\quad \begin{vmatrix} 1-t & 3 \\ 3 & 1-t \end{vmatrix} = (1-t)^2 - 9 = 1 - 2t + t^2 - 9$

$= t^2 - 2t - 8 = (t-4)(t+2) = 0 \Rightarrow \lambda = \{4, -2\}$.

Since 2 distinct e-vals where $n = 2$, A is diagonalizable.

$\lambda = 4$ $\quad \left(\begin{array}{cc|c} -3 & 3 & 0 \\ 3 & -3 & 0 \end{array}\right) = \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow$ soln. set $= t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = -2$ $\quad \left(\begin{array}{cc|c} 3 & 3 & 0 \\ 3 & 3 & 0 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow$ soln. set $= t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$.

d) $A = \begin{pmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{pmatrix}$ $\quad \begin{vmatrix} 7-t & -4 & 0 \\ 8 & -5-t & 0 \\ 6 & -6 & 3-t \end{vmatrix} = (3-t)\begin{vmatrix} 7-t & -4 \\ 8 & -5-t \end{vmatrix}$

$= (3-t)\left((7-t)(-5-t) + 32\right) = (3-t)(-35 + 32 - 2t + t^2)$

$= (3-t)(t^2 - 2t - 3) = -(t-3)(t-3)(t+1) = -(t-3)^2(t+1) = 0$

$\Rightarrow \lambda = \{3, -1\}$.

Need to check if $d_3 = m_3 = 2$.

$\left(\begin{array}{ccc|c} 4 & -4 & 0 & 0 \\ 8 & -8 & 0 & 0 \\ 6 & -6 & 0 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$ soln. set $= t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$d_3 = m_3 = 2$ ✓ so A diagonalizable.

$\left(\begin{array}{ccc|c} 8 & -4 & 0 & 0 \\ 8 & -4 & 0 & 0 \\ 6 & -6 & 4 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ -3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$ soln. set $= t\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$

$Q = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 4 \\ 0 & 1 & 3 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

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3. For each of the following linear operators T on a v.s. V, test T for diagonalizability, and if T is diagonalizable, find a basis β for V s.t. $[T]_\beta$ is a diagonal matrix.

a) $V = P_3(\mathbb{R})$, $T(f(x)) = f'(x) + f''(x)$. Let $\beta = \{1, x, x^2, x^3\}$

$$[T]_\beta = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad f_T(t) = \begin{vmatrix} -t & 1 & 2 & 0 \\ 0 & -t & 2 & 6 \\ 0 & 0 & -t & 3 \\ 0 & 0 & 0 & -t \end{vmatrix} = t^4$$

$m_0 = 4$; if $d_0 = 4$, then diagonalizable.

$$\left(\begin{array}{cccc|c} 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 6 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \text{soln. set} = t\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$d_0 = 1 \Rightarrow$ not diagonalizable

e) $V = \mathbb{C}^2$, $T(z, w) = (z + iw, iz + w)$. Let $F = \mathbb{C}$, $\beta = \{(1, 0), (0, 1)\}$.

$$[T]_\beta = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \quad f_T(t) = (1-t)^2 + 1 = t^2 - 2t + 2 = 0$$

$$\Rightarrow t = \frac{2 \pm \sqrt{4 - 4(2)}}{2} = 1 \pm \sqrt{1-2} = 1 \pm i.$$

T must be diagonalizable, since 2 e-vals

$\lambda = 1+i$ $\quad \left(\begin{array}{cc|c} -i & i & 0 \\ i & -i & 0 \end{array}\right) = \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \text{soln. set} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = 1-i$ $\quad \left(\begin{array}{cc|c} i & i & 0 \\ i & i & 0 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \text{soln. set} = t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\beta = \{(1, 1), (1, -1)\}$

(thus $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $D = \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}$)

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5.2 # 7, 9, 11

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7. For $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$, find an expression for A^n, where n is an arbitrary positive integer.

$D = Q^{-1}AQ \Rightarrow A = QDQ^{-1} \Rightarrow A^n = QD^nQ^{-1}$ (by example 7).

Finding a diagonal matrix representation D and matrix Q:

$$\begin{vmatrix} 1-t & 4 \\ 2 & 3-t \end{vmatrix} = (1-t)(3-t) - 8 = 3 - 4t + t^2 - 8$$
$$= t^2 - 4t - 5 = (t-5)(t+1) \Rightarrow \lambda = \{5, -1\}.$$
$\Rightarrow$ diagonalizable

$$\left(\begin{array}{cc|c} -4 & 4 & 0 \\ 2 & -2 & 0 \end{array}\right) = \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \text{soln. set} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$\left(\begin{array}{cc|c} 2 & 4 & 0 \\ 2 & 4 & 0 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \text{soln. set} = t\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\Rightarrow Q = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Q^{-1} = \frac{1}{1(1) - 2(1)}\begin{pmatrix} -1 & -2 \\ -1 & 1 \end{pmatrix} = \frac{1}{-3}\begin{pmatrix} -1 & -2 \\ -1 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$$

$$A^n = QD^nQ^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 5^n & 0 \\ 0 & (-1)^n \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$$
$$= \frac{1}{3}\begin{pmatrix} 5^n & 2(-1)^n \\ 5^n & -(-1)^n \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 5^n + 2(-1)^n & 2(5^n) - 2(-1)^n \\ 5^n - (-1)^n & 2(5^n) + (-1)^n \end{pmatrix}$$

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9. Let T be the lin. op. on a finite-dim. v.s. V, and suppose there exists an O.B. β for V s.t. $[T]_\beta$ is an upper triangular matrix.

a) Prove that the characteristic polynomial for T splits.

b) State and prove the analogous result for matrices.

PF (a): The characteristic polynomial is $\det(T - tI_V) = \det([T]_\beta - tI)$. $[T]_\beta - tI$ is also upper triangular (b/c sum of upper triangular matrices is upper triangular, and the determinant of an upper triangular matrix is the product of the terms along the diagonal). Thus:

$$f_T(t) = \prod_{i=1}^{n}\left(([T]_\beta)_{ii} - t\right) = \prod_{i=1}^{n}\left((-1)(t - ([T]_\beta)_{ii})\right) = (-1)^n \prod_{i=1}^{n}\left(t - ([T]_\beta)_{ii}\right)$$

b) Let $A \in M_{n\times n}(F)$, and let A be upper triangular. Prove that the characteristic polynomial for A splits.

PF: Let β = S.O.B. F^n. Then $A = [L_A]_\beta$ is upper triangular. By (5.1 exercise 7e), the characteristic polynomial of A equals the characteristic polynomial of L_A, which splits according to (part a).

11. Let A be an $n\times n$ matrix that is similar to an upper triangular matrix (call this B) and has the distinct e-vals $\lambda_1, \lambda_2, \ldots \lambda_k$ w/ corresponding multiplicities $m_1, m_2, \ldots, m_k$. Prove the following.

a) $tr(A) = \sum_{i=1}^{k} m_i \lambda_i$

b) $det(A) = (\lambda_1)^{m_1} (\lambda_2)^{m_2} \cdots (\lambda_k)^{m_k}$

LEM 1: trace of similar matrices are equal

PF: Let A, B be similar $M_{n\times n}(F)$ matrices. Then $\exists$ Q s.t. $A = Q^{-1}BQ$. By (Section 2.3 exercise 13), $tr((Q^{-1}B)Q) = tr(Q(Q^{-1}B)) = tr(IB) = tr(B) = tr(A)$

LEM 2: Using the assumptions from the exercise, all of the entries along the main diagonal of B are e-vals of A, and each value λ_i appears m_i times on B's main diagonal.

PF: Since B, A similar, they have the same characteristic polynomial and e-vals. Since $B - \lambda I$ is upper tri., $f_A(t) = f_B(t) = \prod_{i=1}^{n} (B_{ii} - t)$ (and unique factorization by Thm E.9). Thus $\{B_{ii}\}$, $1 \le i \le n$ is the set of eigenvalues of B (and therefore A), and the multiplicity m_i is the number of times λ_i appears on the main diagonal.

PF (a): By (LEM 2), each e-val λ_i of A appears m_i times on B's diagonal, and by (LEM 1), $tr(A) = tr(B) = \underbrace{\lambda_1 + \lambda_1 + \ldots + \lambda_1}_{m_1} + \underbrace{\lambda_2 + \ldots + \lambda_2}_{m_2} + \ldots + \underbrace{\lambda_k + \lambda_k + \ldots + \lambda_k}_{m_k} = \sum_{i=1}^{k} \lambda_i m_i$.

PF (b): By (LEM 2), each e-val λ_i of A appears m_i times on B's diagonal, and since determinants of similar matrices are equal, $det(A) = det(B) = \underbrace{\lambda_1 \lambda_1 \ldots \lambda_1}_{m_1} \underbrace{\lambda_2 \lambda_2 \ldots \lambda_2}_{m_2} \ldots \underbrace{\lambda_k \lambda_k \ldots \lambda_k}_{m_k} = \prod_{i=1}^{n} \lambda_i^{m_i}$.

E. 3, 4, 5, 6, 7

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THM. E.3. Let $f(x)$ be a polynomial w/ coefficients from a field F, and let T be a linear operator on a v.s. V over F. Then the following statements are true.

a) $f(T)$ is a linear operator on V.

b) If β is a finite OB of V and $A = [T]_\beta$, then $[f(T)]_\beta = f(A)$.

PF (a): Let $f(T) = a_0 I + a_1 T + \dots + a_n T^n$. Clearly, I and T^i $(i \geq 1)$ are linear operators on V, and the sum of linear operators on the same v.s. is a linear operator.

PF (b): Let f be as defined in part (a).

$$[f(T)]_\beta = [a_0 I + a_1 T + \dots + a_n T^n]_\beta = [a_0 I]_\beta + [a_1 T]_\beta + \dots + [a_n T^n]_\beta$$
$$= a_0 [I]_\beta + a_1 [T]_\beta + \dots + a_n [T^n]_\beta = a_0 I + a_1 [T]_\beta + \dots + a_n [T^n]_\beta$$
$$= a_0 I + a_1 [T]_\beta + \dots + a_n [T]_\beta^n = f(A)$$

THM E.4 Let T be a lin. op. over a v.s. V over a field F, and let A be a square matrix with entries from F. Then, for any polynomials $f_1(x)$ and $f_2(x)$ with coefficients from F:

a) $f_1(T) f_2(T) = f_2(T) f_1(T)$

b) $f_1(A) f_2(A) = f_2(A) f_1(A)$

PF (a): Let $f_1(t) = \sum_{i=0}^n a_i t^i$, $f_2(t) = \sum_{j=0}^m b_j t^j$, and $T^0 = I_V$.

Then $f_1(T) f_2(T) = \left(\sum_{i=0}^n a_i T^i\right)\left(\sum_{j=0}^m b_j T^j\right) = \sum_{i=0}^n \sum_{j=0}^m a_i b_j T^i T^j$
$= \left(\sum_{j=0}^m b_j T^j\right)\left(\sum_{i=0}^n a_i T^i\right) = f_2(T) f_1(T)$

PF (b): Let f_1, f_2 as defined in part (a), $A^0 = I$.

Then $f_1(A) f_2(A) = \left(\sum_{i=1}^n a_i A_i\right)\left(\sum_{j=0}^m b_j A^j\right) = \sum_{i=0}^n \sum_{j=0}^m a_i b_j A^i A^j$
$= \left(\sum_{j=0}^m b_j A^j\right)\left(\sum_{i=0}^n a_i A^i\right) = f_2(A) f_1(A)$.

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THM E.5 Let T be a lin. op. on a v.s. V over a field F, and let A be an $n \times n$ matrix with entries from F. If $f_1(x)$ and $f_2(x)$ are prime polynomials with entries from F, then there exist polynomials $q_1(x)$ and $q_2(x)$ with entries from F s.t.

a) $q_1(T) f_1(T) + q_2(T) f_2(T) = I_V$

b) $q_1(A) f_1(A) + q_2(A) f_2(A) = I_n$

(THM. E.2) states that if $f_1(x), f_2(x) \in P(F)$ relatively prime $\exists\, q_1(x), q_2(x) \in P(F)$ s.t. $q_1(x) f_1(x) + q_2(x) f_2(x) = 1$, where $1 \in P(F)$ is the polynomial comprising of the multiplicative identity of the indeterminate variable.

For (part a), the indeterminate is $T \in \mathcal{L}(V)$, and the multiplicative identity is I_V; for (part b), the indeterminate is $A \in M_{n \times n}(F)$, and the multiplicative identity is I_n. Apply (THM E.2) to these indeterminates.

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E.6, 7.

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THM. E.6 Let $\phi(x)$ and $f(x)$ be polynomials. If $\phi(x)$ is irreducible and $\phi(x)$ does not divide $f(x)$, then $\phi(x)$ and $f(x)$ are relatively prime.

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PF. $\phi(x)$ irreducible $\Rightarrow$ it cannot be expressed as a product of polynomials with degree ≥ 1. Thus $\phi(x)$ may only be represented as the product of a scaled multiple of itself and a scalar (polynomial w/ degree 0). Since $\phi(x)$ doesn't divide $f(x)$, the only polynomials that may divide both $\phi(x)$ and $f(x)$ are polynomials of degree 0. Thus relatively prime.

you probably need to flesh this out!

THM E.7. Any two distinct irreducible monic polynomials are relatively prime.

PF: Two polynomials that are both distinct and monic may not be scalar multiples of one another. Thus, neither polynomial divides the other with the quotient polynomial being of degree 0. Since both are irreducible, no polynomial with positive degree divides either polynomial. Thus the only polynomials that can divide both polynomials have degree 0 $\Rightarrow$ the polynomials are relatively prime.