PSET 4

2.3 # 13, 15

19/22

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13. Let A, B be $n \times n$ matrices. Prove that $tr(AB) = tr(BA)$, $tr(A^t) = tr(A)$.

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2.3.13 | 4.5
2.4.4 | 5
| 9.5

CLAIM: $tr(A^t) = tr(A)$.

PF: $tr(A^t) = \sum_{i=1}^{n} (A^t)_{ii} = \sum_{i=1}^{n} (A)_{ii} = tr(A)$

since $(A^t)_{ij} = (A)_{ji}$, $i = j$

what is this saying?

CLAIM: $tr(AB) = tr(BA)$.

PF: $tr(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} (A)_{ij}(B)_{ji}\right) = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} (B)_{ji}(A)_{ij}\right)$

$= \sum_{j=1}^{n} (BA)_{jj} = tr(BA)$

you are doing two things at once; you need to explain!

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15. Let M, A be matrices s.t. MA is defined. If the jth column of A is a lin. comb. of a set of columns of A, prove that the jth column of MA is a lin. comb. of the corresponding columns of MA with the same corresponding coefficients.

PF: Let A_j indicate the jth column vector of A. Then $A_j = \sum_{i=1}^{n} c_i A_i$, $c_i \in F$ (by hyp.). By (THM 2.13 a),

$(MA)_j = M(A_j) = M\left(\sum_{i=1}^{n} c_i A_i\right) = \sum_{i=1}^{n} c_i (MA_i) = \sum_{i=1}^{n} c_i (MA)_i$.

(By THM 2.13 a)

Thus $(MA)_j$ is a lin. comb. over the column vectors of MA with the same coefficients as A_j over the column vectors of A.

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2.3 #16.

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16. Let V be finite-dimensional v.s., $T \in \mathcal{L}(V)$.

a) If $\text{rank}(T) = \text{rank}(T^2)$, prove that $R(T) \cap N(T) = \{0\}$. Deduce that $V = R(T) \oplus N(T)$.

LEM 1 If $\text{rank}(T^i) = \text{rank}(T^{2i})$, $i \in \mathbb{Z}^+$, $R(T) \cap N(T) = \{0\}$.

PF: By (DIM THM), $\text{rank}(T^i) + \text{nullity}(T^i) = \dim(V) = \text{rank}(T^{2i}) + \text{nullity}(T^{2i})$. Since $\text{rank}(T^i) = \text{rank}(T^{2i})$, $\text{nullity}(T^i) = \text{nullity}(T^{2i})$. Since $N(T^i)$ subsp. $N(T^{2i})$ (since $\forall x \in N(T^i)$, $T^{2i}(x) = T^i(T^i(x)) = T^i(0) = 0$), by (THM 1.11) $N(T^i) = N(T^{2i})$.

Let $x \in R(T^i) \cap N(T^i)$. Then, $T^i(x) = 0$, and $\exists y \in V$ s.t. $T^i(y) = x$. Since $T^2(y) = T^i(T^i(x)) = 0$, $y \in N(T^{2i}) = N(T^i)$. Thus $T(y) = 0 = x$ $\therefore$ $R(T^i) \cap N(T^i)$ consists only of the 0 vector. //

CLAIM: If $\text{rank}(T) = \text{rank}(T^2)$, $R(T) \cap N(T) = \{0\}$.

PF: Use (LEM. 1) with $i = 1$.

CLAIM: $V = R(T) \oplus N(T)$.

PF:
1) $R(T)$, $N(T)$ subsp. V by (THM 2.1)
2) $R(T) \cap N(T) = \{0\}$ by (LEMMA 1)
3) $R(T) + N(T)$ subsp. V, and $\dim(R(T) + N(T))$
$= \dim(R(T)) + \dim(N(T)) - \dim(R(T) \cap N(T))$
$= \text{rank}(T) + \text{nullity}(T) - 0 = \dim(V)$
(by (1.6 exercise #29 a), (DIM. THM)). Since $R(T) + N(T)$ subsp. V and $\dim(R(T) + N(T)) = \dim(V)$, $R(T) + N(T) = V$ by (THM 1.11).

By (DEF $\oplus$), $V = R(T) \oplus N(T)$.

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b) Prove that $V = R(T^k) \oplus N(T^k)$ for some positive integer k.

Since $R(T^{i+1}) \subseteq R(T^i)$ $\forall\, i \in \mathbb{Z}^+$, $\operatorname{rank}(T^{i+1}) \leq \operatorname{rank}(T^i)$.
Thus rank is non-increasing with increased iterations of applying T.
Since $\operatorname{rank}(T^i)$ is lower bounded by 0, there is some (finite) integer k s.t. $\operatorname{rank}(T^k) = \operatorname{rank}(T^{k+1})$. $R(T^k)$ is thus T-invariant, so $\forall\, j > k$, $\operatorname{rank}(T^k) = \operatorname{rank}(T^j)$. Since $R(T^k) = R(T^{2k})$, by (LEM 1), $R(T^k) \cap N(T^k) = \{0\}$.

$V = R(T^k) \oplus N(T^k)$ is proved by definition analogously to in (part a). The identical conditions 1, 2, and 3 may be used, replacing T with T^k and using the new result above.

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2.3 #17.

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17. Let V be a v.s. Determine all linear transformations $T \in \mathcal{L}(V)$ s.t. $T = T^2$.

LEM 1 $R(T) = \{y : T(y) = y\}$.

PF: Let $x \in V$, and $R' = \{y : T(y) = y\}$.
If $T(x) = x$, then $x \in R'$.
If $T(x) = z$, $z \neq x \in V$, then $T(z) = T^2(x) = T(x) = z$
thus $z \in R'$.
Since all possible images of x lie in R', $R(T) \subseteq R'$.
Also, $R' \subseteq R(T)$ $\therefore$ $R(T) = R'(T) = \{y : T(y) = y\}$.

LEM 2 $V = R(T) \oplus N(T)$

PF: Let $x \in N(T) \cap R(T)$. Then $T(x) = 0$, and since $T(x) = x$, x must be 0. Thus $N(T) \cap R(T) = \{0\}$.
$T = T^2 \Rightarrow R(T) = R(T^2)$. By the previous exercise (2.3 #16 a),
$V = R(T) \oplus N(T)$.

CLAIM: $\{T\} \subseteq \mathcal{L}(V)$ s.t. $T = T^2$ is the set of projections of V onto one its subspaces

PF: $\forall x \in V$, $x = T(x) + (x - T(x))$, where $T(x) \in R(T)$, $(x - T(x)) \in N(T)$ (since $T(x - T(x)) = T(x) - T^2(x) = T(x) - T(x) = 0$), and $T(x) = T(T(x) + T(x - T(x))) = T^2(x) + 0 = T(x) + 0$.
By (DEF projection), T is a projection onto $R(T)$ along $N(T)$.

Remarks / Characterizations of V:

- T exists for every projection of V onto one of its subspaces; i.e., whenever $\exists\, W, Z$ subsp. V s.t. $W \oplus Z = V$, $\exists\, T = T^2$ projecting V onto W
- By (LEM 1), $\forall x \in R(T)$, $y \notin R(T)$, $T(x) = x$, $T(y) = 0$. (In other words, T maps V to $R(T)$, and is the identity map within $R(T)$).

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2.4 # 2bef, 4

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2. Determine whether T is invertible. Justify your answer.

b) $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) = (3a_1 - a_2, a_2, 4a_1)$.

Not invertible, since $\dim(\mathbb{R}^2) \neq \dim(\mathbb{R}^3)$, so $\mathbb{R}^2 \not\cong \mathbb{R}^3$ (THM 2.19) so no invertible lin. transforms from $\mathbb{R}^2$ to $\mathbb{R}^3$.

e) $T: M_{2\times2}(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + 2bx + (c+d)x^2$.

Not invertible. For some $f \neq 0 \in F$, $T\begin{pmatrix} 0 & 0 \\ f & -f \end{pmatrix} = 0 + 0x + (f-f)x^2$

$= 0 + 0x + 0x^2 = 0$, thus $N(T) \neq \{0\}$, so T not 1-1 (THM 2.4) and thus not invertible (since invertible transforms must be 1-1).

f). $T: M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix}$.

Invertible: let $U\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} b & a-b \\ c & d-c \end{pmatrix}$

Let $x = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in M_{2\times2}(\mathbb{R})$. Then:

$(UT)(x) = U\begin{pmatrix} e+f & e \\ g & g+h \end{pmatrix} = \begin{pmatrix} e & (e+f)-e \\ g & (g+h)-g \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} = x$

$(TU)(x) = T\begin{pmatrix} f & e-f \\ g & h-g \end{pmatrix} = \begin{pmatrix} f+(e-f) & f \\ g & g+h-g \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} = x$

Since $T \in \mathcal{L}(M_{2\times2}(\mathbb{R}))$, and $UT = TU = I_2$, T invertible.

4. Let A, B be $n\times n$ invertible matrices. Prove that (AB) invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

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You should probably still have parentheses, since you are using associativity!

$(AB)(B^{-1}A^{-1}) = ABB^{-1}A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$

$(B^{-1}A^{-1})(AB) = B^{-1}A^{-1}AB = B^{-1}I_nB = B^{-1}B = I_n$.

AB is invertible, and its inverse is $(B^{-1}A^{-1})$, since $(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I_n$.

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2.4 # 17, 20.

17. Let V, W be finite-dim. v.s., $T \in \mathcal{L}(V, W)$ isomorphism. Let V_0 subsp. V.

a) Prove $T(V_0)$ subsp. W.

Since V_0 v.s., we can define $U: V_0 \to W$, $U(x) = T(x)$ $\forall x \in V_0$.
$T(V_0) = R(U)$, and $R(U)$ is a subsp. of W (THM 2.1)

b) Prove that $\dim(V_0) = \dim(T(V_0))$.

Let β_0 be a basis for V_0. Since T 1-1, $T(\beta_0)$ is also lin. ind, and it spans $T(V_0)$, so $T(\beta_0)$ is a basis for $T(V_0)$. $\dim(V_0) = \text{card}(\beta_0) = \text{card}(T(\beta_0)) = \dim(T(V_0))$.

20. Let $T \in \mathcal{L}(V, W)$ be a lin. transf. from n-dim. v.s. V to m-dim. v.s. W. Let β, γ be o.b.'s for V, W, respectively. Prove $\text{rank}(T) = \text{rank}(L_A)$, $\text{nullity}(T) = \text{nullity}(L_A)$, where $A = [T]_\beta^\gamma$.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\varphi_\beta"'] & W \arrow[d, "\varphi_\gamma"] \\ F^n \arrow[r, "L_A"'] & F^m \end{tikzcd}$$

CLAIM: $\text{rank}(T) = \text{rank}(L_A)$.

PF: $\varphi_\beta, \varphi_\gamma$ are isomorphisms (THM 2.21), and $L_A \varphi_\beta = \varphi_\gamma T$. Since φ_β isomorphism, it is onto, so $R(\varphi_\beta) = \varphi_\beta(V) = F^n$. Then $R(L_A) = L_A(F^n) = L_A(\varphi_\beta(V))$
$= \varphi_\gamma(T(V)) = R(\varphi_\gamma T)$.

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By exercise 17, $R(T) = T(V)$ subsp. W, and
$\dim(R(T)) = \dim(T(V)) = \dim(\varphi_\gamma(T(V))) = \dim(R(\varphi_\gamma T))$
$= \dim(R(L_A))$
$\Rightarrow \operatorname{rank}(T) = \operatorname{rank}(L_A)$.

<u>CLAIM:</u> $\operatorname{nullity}(T) = \operatorname{nullity}(L_A)$

<u>PF</u> Since φ_β, φ_γ isomorphisms, $\dim(V) = \dim(F^n) = n$.
By (dim THM), $\operatorname{rank}(L_A) + \operatorname{nullity}(L_A) = \dim(F^n)$
$= \dim(V) = \operatorname{rank}(T) + \operatorname{nullity}(T)$, and since
$\operatorname{rank}(T) = \operatorname{rank}(L_A)$, then $\operatorname{nullity}(L_A) = \operatorname{nullity}(T)$
(by cancellation law of addition).