Alg-5

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HW Due 2/8/2021

1 Let G be a group and $a, b \in G$. Proove that $(aba^{-1})^n = ab^n a^{-1}$ for each $n \in \mathbb{N}$.

2. Let $K = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$. For each $a \in K^n$ and $A \in GL(n, K)$, define $f_{A,a} : K^n \to K^n$ by $f_{A,a}(x) = Ax + b$.

(i) Proove that $f_{A,a}$ is a bijective map.

(ii) Let $Aff(n, K) = \{ f_{A,a} : A \in GL(n, K)$ and $a \in K^n \}$

Proove that $Aff(n, K)$ is a group under composition operation.

Group under product or multiplication $(G, \cdot)$ multiplication	Group under sum or addition $(G, +)$
1 — 1 is the identity	$0 \in G$ is the additive identity
2. $a \in G$, $a^{-1} \in G$	$a \in G$, $-a \in G$
3. $a a^{-1} = 1 = a^{-1} a$	$a + (-a) = 0 = (-a) + a$

3 $a a^{-1} = 1 = a^{-1} a$

$a + (-a) = 0 = (-a) + a$

4 $x_1 \cdots x_n, x_{n+1} \in G$

$(x_1 \cdots x_n) x_{n+1} = x_1 \cdots x_n x_{n+1}$

$\prod_{i=1}^{n} x_i = x_1 x_2 \cdots x_n$

$(x_1 + \cdots + x_n) + x_{n+1} = \sum_{i=1}^{n+1} x_i$

5 $x \in G \quad x^2 = x x$

$x^{n+1} = x^n . x = x x^n$

inductively

$x \in G, \quad 2x = x + x$

$(n+1)x = nx + x.$

6 $(x^{-1})^n = (x^n)^{-1}$

$n(-x) = -(nx)$

7 $x^m . x^n = x^{m+n}$

$mx + nx = (m+n)x$

8 $(x^m)^n = x^{mn} = (x^n)^m$

$m(nx) = (mn)x = n(mx)$

X & Y are sets, always taken to be nonempty.

$\varphi : X \to Y$ is

(i) injective or one-to-one iff

$f(x_1) = f(x_2) \implies x_1 = x_2$

$f(x_1) = f(x_2) \Longrightarrow x_1 = x_2$

or $x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2)$

(2) surjective or onto iff for each $y \in Y$, $\exists x \in X$ s.t. $f(x) = y$.

(3) bijective iff both injective and surjective or both one-to-one and onto

Def $B \subseteq Y$, $A \subseteq X$, $\varphi : X \to Y$ is a map

(1) $\varphi(A) = \{ f(a) \in Y : a \in A \}$ called direct image of A under φ

(2) $\varphi^{-1}(B) = \{ x \in X : \varphi(x) \in B \}$, it is called the inverse image of φ.

Def: $f : X \to Y$ is invertible iff $\exists\, g : Y \to X$ s.t.

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$f \circ g = I_Y$ &

$g \circ f = I_X$ where

$I_S : S \to S$ s.t.

$I_S(s) = s \quad \forall s \in S$

I_S is called the identity map on S

If there are $g_i : Y \to X$ s.t.

$g_i \circ f = I_X$ & $f \circ g_i = I_Y, i = 1, 2$

then $g_1 = g_2$ i.e.

$g_1(y) = g_2(y) \quad \forall y \in Y$

The unique $g: Y \to X$ s.t. is called

the ... is called
$f \circ g = I_Y$ & $g \circ f = I_X$ is called
the inverse of f and is
denoted by $g = f^{-1}$, & is
a function

Thm $f: X \to Y$ is invertible
$\iff$ f is a bijection map

Note: $f^{-1}(B) = \{x \in X : f(x) \in B\}$
where f^{-1} does not denote
inverse of f.

$f: \mathbb{R} \to \mathbb{R}$ defined by
$f(x) = x^2$ is neither injective nor
surjective: $f(-1) = f(1)$, but
$-1 \neq 1$, not injective

... there is no $x \in \mathbb{R}$

not surjective : there is no $x \in \mathbb{R}$
s.t. $x^2 = -1$

$$f_{A,a}, f_{B,b}, \quad a, b \in K^n$$
$$A, B \in GL(n, \mathbb{R})$$

$$f_{A,a} \circ f_{B,b} \in Aff(n, K)$$
$$= f_{C,c}, \quad \text{where } C \in GL(n, K)$$
$$c \in K^n$$

where $n \geq 1$ integer

$n = 1$ $\quad Aff(1, K)$ is called $ax+b$ group with $K = \mathbb{R}$

$n \in \mathbb{N}$, $\quad Aff(n, \mathbb{R})$ is used in Image Processing

in Image Processing
(Tekalp)
$\downarrow$
Digital Video Processing

Ex $G_1, \ldots G_n$ are groups

$G = G_1 \times G_2 \times \cdots \times G_n$

$= \prod_{i=1}^{n} G_i \overset{\text{def}}{=} \{(g_1, \ldots, g_n) : g_i \in G_i \;\; i=1 \ldots n\}$

G is a group

for $g, h \in G$,

$g = (g_1 \ldots g_n) \quad g_i \in G_i, \; i=1, \ldots, n$

$h = (h_1 \ldots h_n), \quad h_i \in G_i$

define $gh = (g_1h_1, g_2h_2, \ldots, g_nh_n) \in G$.

as $g_ih_i \in G_i, \; i=1 \ldots n$

$e_G = (e_1 \ldots e_n), \; e_i$ is id. of G_i

$g^{-1} = (g_1^{-1}, \ldots g_n^{-1})$

$g^{-1} = (g_1^{-1}, \ldots, g_n^{-1})$

$O(n) = \{ A \in M_{n \times n}(\mathbb{R}) : A^{-1} = A^t \}$

$AA^t = I_n = A^t A$

= the set of orthogonal matrices

is a group under product

$O(n) \subseteq GL(n, \mathbb{R})$

$U(n) = \{ A \in M_{n \times n}(\mathbb{C}) : A^{-1} = A^* = \overline{A}^t = \overline{A^t} \}$

is a group under product

& is called a unitary group

$U(n) \subseteq GL(n, \mathbb{C})$

$SO(n) = \{ A \in O(n) : \det(A) = 1 \}$

the set of all rotation matrices

is a group under product.

is a group under ...

$SU(n) = \{ A \in U(n) : \det A = 1 \}$ a group, called special unitary gp

$A \in U(n)$

$|\det(A)| = 1$, $\det(A) \in \mathbb{C}$

$S^1 = \mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$ is a group under product

$z, w \in S^1 \implies zw \in S^1$

$z^{-1} = \frac{1}{z}$

$U(1) = S^1$

Ex $G = \{1, -1, i, -i\}$

Then G is a group under product.

$G \subseteq S^1$

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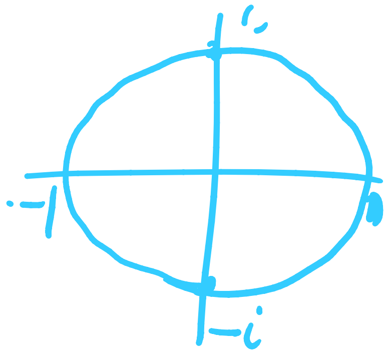

$i^2 = -1$ $\quad 1-i$

Def $H \subseteq G$ is called a subgroup of G iff (1) $e \in H$

(2) $a, b \in H \Rightarrow ab \in H$

(3) $a \in H \Rightarrow a^{-1} \in H$

& denoted by $H \leq G$.

$\subseteq \rightarrow$ subset

$\leq \longrightarrow$ subgroup

Check that $H \leq G$

$\Longleftrightarrow (H, \cdot)$ is a group

Thm $H \subseteq G$

Then $H \leq G \Longleftrightarrow ab^{-1} \in H$

$\forall a, b \in H$.