$$\mathcal{E}[(\bar{f})^2] = V(\bar{f}) + [\mathcal{E}(\bar{f})]^2$$

$$= \sigma^2_{\bar{f}} + \mu_f^2 = \frac{\sigma^2 f}{n} + \mu_f^2$$

$$\mathcal{E}(\eta) = \sigma_f^2 + \mu_f^2 - \frac{\sigma^2 f}{n} - \mu_f^2 = \sigma_f^2 - \sigma^2_f/n$$

$$\mathcal{E}(\eta) = \sigma^2_f \frac{n-1}{n} \implies \text{biased estimator of } \sigma^2_f$$

for large n, $\frac{n-1}{n} \approx 1$

$$\sigma^2_{\bar{f}} = \frac{\sigma_f^2}{n} \approx \frac{1}{n}\left[\frac{n}{n-1}\eta\right] = \frac{1}{n-1}\left[\underbrace{\overline{f^2} - \bar{f}^2}_{\text{calcuable}}\right]$$

We estimate population variance with sample variance

$\mu_f \longrightarrow \bar{f}$ from n samples, with uncertainty $\sigma_{\bar{f}}$

<u>Why?!?</u>

$$\int_a^b f(x)\,dx = (b-a)\mu_f$$

$$\int_a^b f(x)\,dx \approx (b-a)\bar{f} \pm (b-a)\sigma_{\bar{f}}$$

$$\approx (b-a)\bar{f} \pm (b-a)\left[\frac{\overline{f^2} - \bar{f}^2}{n-1}\right]^{1/2}$$

$$\int_a^b f(x)\,dx) \approx \frac{b-a}{n}\sum f(x_i) \pm \frac{b-a}{\sqrt{n-1}}\left[\frac{1}{n}\sum f^2(x_i) - (\frac{1}{n}\sum f(x_i$$

Choose x_i uniformly from a to b

first term looks like $\frac{b-a}{n} f(x_i)$

like $h f(x_i)$ from before

Weighting

We have been using flat probability $P(x) = \frac{1}{b-a}$

We can generalize more easily to probability

$\frac{w(x)}{b-a}$ from a to b and $\emptyset$ elsewhere

$$\Rightarrow \mu_f = \mathcal{E}[f(x)] = \frac{1}{b-a} \int_a^b \underbrace{w(x)}_{\text{weight function}} f(x)\, dx$$

sample mean $\bar{f} = \frac{1}{n} \sum f(x_i)$

but x_i are randomly drawn from $\frac{w(x)}{b-a}$

everything else remains unchanged

$$\int_a^b w(x) f(x)\, dx \approx \frac{b-a}{n} \sum f(x_i) \pm \text{same uncertainty}$$

and x_i chosen from $w(x)/(b-a)$

We either need to draw randomly from non uniform distribution or convert $\int w(x) f(x)$ to $\int f(x)$ by substitution and use our original formula

Side note

Transformation method to generate random variable from some non-uniform probability distribution

We draw a uniform random # $u \in [0,1]$ and want to generate a random # from any probability distribution $P(x)$

$$p(u) = \begin{cases} 1 & \text{for } 0 \le u < 1 \\ 0 & \text{otherwise} \end{cases} \qquad \int_{-\infty}^{\infty} p(u)\,du = 1$$

Conservation of probability requires that

$$|p(u)\,du| = |P(x)\,dx|$$

$$\Rightarrow \int_{-\infty}^{u} p(u')\,du' = \int_{-\infty}^{x} P(x')\,dx'$$

$$\int_{0}^{u} 1\,du' = \int_{-\infty}^{x} P(x')\,dx' \Rightarrow u = \int_{-\infty}^{x} P(x')\,dx$$

need to solve for x given randomly generated u

Let $P(x) = A(1+ax^2)$ $\quad -1 \le x < 1$

Need $\int_{-1}^{1} P(x)\,dx = 1 \Rightarrow$ solve for A

$$u = \int_{-\infty}^{x} P(x')\,dx' = \int_{-1}^{x} A(1+ax'^2)\,dx'$$

$$u = A(x + ax^3/3 + 1 + a/3) \Rightarrow \text{solve for } x$$

Easier: rejection method

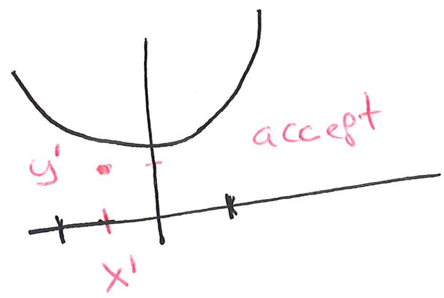
Let $P(x) = 1 + ax^2$ (don't need to normalize)

$-1 \leq x < 1$

Generate uniform x' in range $[-1, 1)$

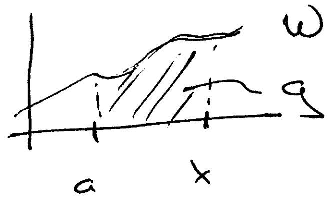
Generate uniform y' in range $[0, 1+a]$

If $y' < P(x')$ accept the drawn value x'



Let $g(x)$ be cumulative distribution function

$$g(x) = \int_a^x w(x')\,dx'$$



$g(a) = 0 \qquad g(b) = b - a$

fund theorem of calc $\quad F(x) = \int_a^x f(x)\,dx \Rightarrow F'(x) = f(x)$

$$\int_a^b w(x)\,f(x)\,dx = \int_a^b f(x)\,\frac{dg}{dx}\,dx$$

Let $\quad u = g(x) \quad$ so $\quad x = g^{-1}(u) \qquad du = g'(x)\,dx$

$$= \int_a^b f\left(g^{-1}(g(x))\right)\frac{dg}{dx}\,dx$$

$$= \int_0^{b-a} f\left(g^{-1}(u)\right)du$$

$$\Rightarrow \int_a^b w(x)\,f(x)\,dx = \int_0^{b-a} f\left(g^{-1}(u)\right)du \approx \frac{b-a}{n}\sum_{i=0}^{n-1} f(g^{-1}(u_i))$$

u_i are uniformly distributed from 0 to $b-a$

<u>ex:</u> $\quad I = \int_1^3 e^{-x}\sin x\,dx$

Let $\quad w(x) = ce^{-x} \qquad f(x) = \sin x / c$

$$\int_1^3 ce^{-x}\,dx = (3-1) \Rightarrow c = \frac{2e^3}{e^2-1}$$

$$g(x) = c\int_1^x e^{-x'}\,dx' = c\left(e^{-1} - e^{-x}\right)$$

$u(x) = c(e^{-1} - e^{-x})$

$e^{-x} = e^{-1} - U/c \Rightarrow x = -\ln(e^{-1} - u/c) = g^{-1}(u)$

$$\int_1^3 c e^{-x} \frac{\sin x}{c} dx = \int_0^2 \frac{1}{c} \sin\left[-\ln\left(e^{-1} - \frac{u}{c}\right)\right] du$$

$$\approx \frac{2}{n} \sum \frac{1}{c} \sin\left(-\ln\left[e^{-1} - u_i/c\right]\right)$$

u_i randomly drawn from 0 to 2

[code]

Kind of artificial since e^{-x} was obviously weight fcn

We can generalize to <u>importance sampling</u>

Let's start with our old friend $f(x)$

$$\int_a^b f(x)\,dx = \int_a^b w(x) \frac{f(x)}{w(x)} dx$$

We just saw $\int_a^b w(x) f(x)\,dx = \int_0^{b-a} f(g^{-1}(u))\,du$

where $g(x) = \int_a^x w(x')\,dx'$ and $u = g(x) \Leftrightarrow x = g^{-1}(u)$

So $$\int_a^b w(x) \frac{f(x)}{w(x)} dx = \int_0^{b-a} \frac{f(g^{-1}(u))}{w(g^{-1}(u))} du$$

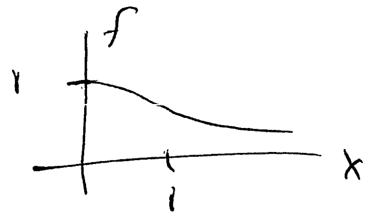
$$\approx \frac{b-a}{n} \sum_i \frac{f(g^{-1}(u_i))}{w(g^{-1}(u_i))}$$ u_i are uniform in 0 to $b-a$

Treat f/w as the unweighted integrand

Idea is to choose $w(x)$ that behaves similarly to $f(x)$ $\Rightarrow$ then random #ers are picked from the important regions instead of uniformly

f/w will vary less than f, reducing the error

Ex: $f(x) = [1+x^2]^{-1/2}$ on $[0,1]$



try $w(x) = C_0 + C_1 x$

normalize $\int_0^1 w(x)\,dx = 1 = C_0 + \frac{1}{2} C_1$

Set $\frac{f(1)}{w(1)} = \frac{f(0)}{w(0)}$ $\quad C_0 = (C_0 + C_1)\sqrt{2}$

Solve $\quad C_0 = 4 - 2\sqrt{2} \quad C_1 = -6 + 4\sqrt{2}$

$g(x) = \int_0^x (C_0 + C_1 x')\,dx' = C_0 x + \frac{1}{2} C_1 x^2 = u$

$x^2 + 2\frac{C_0}{C_1} x - 2u/C_1 = 0 \Rightarrow x = g^{-1}(u) = \frac{-C_0 + \sqrt{2C_1 u + C\ldots}}{C_1}$

Choose positive root so $u \in [0,1] \Rightarrow x$ from 0 to 1

$$\int_0^1 \frac{1}{[x^2+1]^{1/2}}\,dx \approx \frac{1}{n} \sum \frac{1}{[(g^{-1}(u_i))^2+1]^{1/2}[C_0 + C_1 g^{-1}(u_i)]}$$

u_i are uniform on $[0,1]$

Monte Carlo extends well to multidimensional integration

Let $\vec{x} = (x_0, x_1, x_2, \ldots x_{d-1})$ d-component vector

$f(\vec{x}) \rightarrow$ # with d variables as input

$$\mu_f \equiv \mathcal{E}[f(\vec{x})] = \frac{1}{V} \int f(\vec{x})\, d^d x$$

like $\frac{1}{b-a}$

$$\int f(\vec{x})\, d^d x \approx \frac{V}{n} \sum_{i=0}^{n-1} f(\vec{x}_i) \pm \text{uncertainty}$$

$\propto \frac{1}{\sqrt{n}}$