Differential Equations

Projectile motion in 2-D with drag

$$\vec{F} = m\vec{a} \qquad \vec{a} = \frac{d^2\vec{r}}{dt^2} \qquad \vec{r} = x\hat{x} + y\hat{y}$$

Drag force (large $|\vec{v}|$) $\quad \vec{F}_d = -kmv^2\hat{v}$

$$\frac{d^2x}{dt^2} = -k\frac{dx}{dt}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{1/2} \quad \text{and for } y$$

two simultaneous ordinary differential equations
for $x(t)$ and $y(t)$

if given $x(0)$ $y(0)$ $v_x(0)$ $v_y(0)$ $\Rightarrow$ initial value

if given $x(0)$ $y(0)$ $v_x(0)$ $y(T)$ $\Rightarrow$ boundary value

QM in 1-D

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = E\,\Psi(x,t) \qquad E = KE + V(x,t)$$

if $V \Rightarrow V(x)$ only $\quad \Psi(x,t) = \psi(x)\,e^{-iEt/\hbar}$

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\psi(x) = E\,\psi(x) \qquad \underline{\text{ODE}}$$

<u>ODE</u> We start with single diff eq for $y(x)$

x is independent, y is dependent variable

We start with initial value problem (IVP)

$$y'(x) = f(x, y(x)) \qquad y(a) = c$$

where $y' = dy/dx$ and $f(x,y)$ is known

We want to solve for y

$$\int \frac{dy}{dx} = \int f(x', y(x'))\, dx'$$

$$y(x) = c + \int_a^x f(z, y(z))\, dz$$

Note that if $f = f(z)$ and not y then

we need to find $\int f(z)\, dz$ which we

did in Chapter 7 (trapezoid, simpsons, etc)

Might have higher order ODE

2nd order $\quad y'' = f(x, y, y') \quad \underbrace{y(a) = c \quad y'(a) = d}_{IVP}$

- or -

$$y'' = f(x, y, y') \quad \underbrace{y(a) = c \quad y(b) = d}_{BVP}$$

- or -

$$y'' = f(x, y, y'; s) \quad y(a) = c \quad y(b) = d$$

eigenvalue problem (EVP)

EVP only has non trivial solutions for some values of s

-or- dependent variable ϕ depends on more than one indep variables (x,y) (not same y as previously)

E&M Poisson Eqn $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = f(x,y)$

<u>IVP</u> $y'(x) = \underbrace{f(x, y(x))}_{\text{Known}} \quad y(a) = c$

solve for $y(x)$ with $x \in [a,b]$

We will typically solve for $y(x)$ at n grid points x_j

$x_j = a + jh \qquad j = 0, 1, .. n-1$ and

$$h = \frac{b-a}{n-1}$$

notation: $y(x_j)$ is exact ODE solution at x_j

y_j is approx solution

<u>Euler's Method</u>

Recall forward diff $f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{h}{2} f''(x) + \ldots$

$$\Rightarrow y'(x_j) = \frac{y(x_{j+1}) - y(x_j)}{h} - \frac{h}{2} y''(\xi_j)$$

ξ_j is point between x_j and x_{j+1}

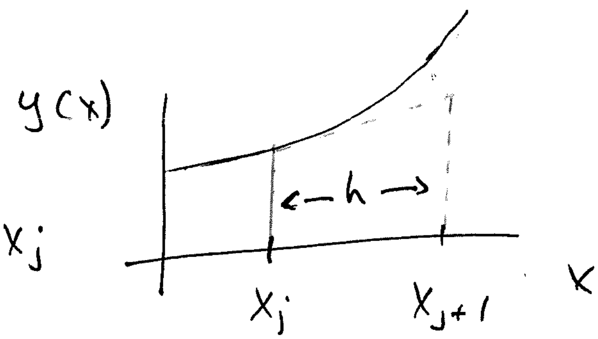
$y(x_j)$ is exact solution to ODE

$$y'(x_j) = f(x_j, y(x_j))$$

$$y(x_{j+1}) = y(x_j) + h f(x_j, y(x_j)) + \frac{h^2}{2} y''(\xi_j) \quad \left[\text{still exact}\right]$$

Now we make approx.

$$y_{j+1} = y_j + h f(x_j, y_j) \quad j = 0, 1, \ldots n-2 \quad y_0 = C$$



$f(x_j, y_j)$ is our approx to slope of tangent to exact soln at x_j

"It can be shown" that the total error (global) that accumulates from $x_0 = a$ to $x_{n-1} = b$

$$|\mathcal{E}| = |y(b) - y_{n-1}| \leq h \times \text{const}$$

$\Rightarrow$ method converges and error decreases linearly, $\mathcal{O}(h)$
$\mathcal{E} \to 0$ as $h \to 0$

Stability examine 'test equation'

$$y'(x) = \mu\, y(x) \quad y(0) = 1 \qquad \mu \in \text{Real, constant}$$

exact solution is $y(x) = e^{\mu x}$

Euler $\Rightarrow$ $y_{j+1} = y_j + h \mu y_j = y_j (1 + h\mu)$

$$y_1 = y_0(1+h\mu)$$
$$y_2 = y_1(1+h\mu) = y_0(1+h\mu)^2$$
$$y_3 = y_2(1+h\mu) = y_0(1+h\mu)^3$$

$$y_{n-1} = y_0(1+h\mu)^{n-1} = (1+h\mu)^{n-1} \quad \text{since } y_0 = 1$$

Consider when $\mu < 0$ (expo decay)

In order for $|y_{j+1}| < |y_j|$ we need

$|1+h\mu| < 1 \implies h < 2/|\mu|$

[See code] <u>unstable</u> method if $h > 0.1$ for this problem

stability is separate from accuracy
$\implies$ needs to be checked also

<u>Backward Euler</u>

analagous to backward diff

$$\underbrace{y'(x_{j+1})}_{f(x_{j+1},\, y(x_{j+1}))} = \frac{y(x_{j+1}) - y(x_j)}{h} + \frac{h}{2} y''(\xi_j)$$

$$y(x_{j+1}) = y(x_j) + h f(x_{j+1}, y(x_{j+1})) - \frac{h^2}{2} y''(\xi_j)$$

approx: $y_{j+1} = y_j + h f(x_{j+1}, y_{j+1})$ $\quad j = 0, 1, \ldots n-2$
$y_0 = c$