Compare to Simpson's

$$\int_{x_i}^{x_{i+2}} f(x)\,dx \approx \frac{h}{3}\left[f(x_i) + 4f(x_{i+1}) + f(x_{i+2}) \right]$$

$$\Rightarrow \mathcal{E}_i = -\frac{1}{90} h^5 f^{(4)}(\xi_i)$$

$$\mathcal{E} = \sum_{i=0,2,4,\ldots}^{n-3} \mathcal{E}_i \Rightarrow \mathcal{E} = -\frac{b-a}{180} h^4 f^{4}(\xi_i)$$

For polynomials up to third degree Simpson gives exact answer (even though we used quadratic)

→ use this method typically

Gaussian Quadrature

No longer use equally spaced x_i

We ignore panels now and consider full integral from $[a,b]$ at the start

For ease we will choose $a=-1$ $b=1$ but we can scale to any other interval $t = \frac{b+a}{2} + \frac{b-a}{2}x$

$x \in [-1,1] \Rightarrow t \in [a,b]$

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{k=0}^{n-1} c_k f(x_k) \quad \left(\begin{array}{c}\text{open method so} \\ x_k \text{ is not } -1 \text{ or } 1\end{array}\right)$$

We can integrate polynomials up to degree $2n-1$ exactly with this method

Take $n=2$ $\quad \int_{-1}^{1} f(x)\,dx = c_0 f(x_0) + c_1 f(x_1)$

x_0 and $x_1 \in (a,b)$

We take $f(x) = \begin{Bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{Bmatrix}$

$\int_{-1}^{1} 1\,dx = 2 = c_0 + c_1$ $\quad \int_{-1}^{1} x^2\,dx = \frac{2}{3} = c_0 x_0^2 + c_1 x_1^2$

$\int_{-1}^{1} x\,dx = 0 = c_0 x_0 + c_1 x_1$ $\quad \int_{-1}^{1} x^3\,dx = 0 = c_0 x_0^3 + c_1 x_1^3$

assuming that all poly up to degree 3 can be integrated exactly to solve for our 4 params

$c_0 x_0 = -c_1 x_1$ $\quad c_0 x_0^3 = -c_1 x_1^3 \Rightarrow x_0^2 = x_1^2$

$x_0 = -x_1$

$c_0 = c_1 \Rightarrow c_0 = 1 = c_1$

$x_0^2 + x_0^2 = 2/3 \Rightarrow x_0 = -1/\sqrt{3} \quad x_1 = 1/\sqrt{3}$

$\int_{-1}^{1} f(x)\,dx \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

[See example]

This generalizes to larger n but not worth the pain [7.5.2 if you prefer]

Monte Carlo

We can do integrals by sampling random #ers

Computer can help us generate pseudorandom #ers
look pretty random but deterministic

Simple method: linear congruential generator

produces integers from 0 to $M-1$ where $M \in$ int

with $u_i = (p\, u_{i-1} + c) \bmod M$ $\quad p, c \in$ int

$u_0 \equiv$ seed let's say $u_0 = 5$ $\quad p = 4 \quad c = 1 \quad M = 15$

$u_0 = 5 \quad u_1 = (4 \times 5 + 1) \bmod 15 = 6 \quad u_2 = (4 \times 6 + 1) \bmod 15 = 10$

$u_3 = (4 \times 10 + 1) \bmod 15 = 11 \quad u_4 = 0 \quad u_5 = 1 \quad u_6 = 5$

5, 6, 10, 11, 0, 1, 5, ... keeps repeating (period of 6)

usually pick large p, M

To get float take $r_i = u_i / M$

integers $[0, M-1]$ $\Rightarrow$ floats $[0, 1)$

floats $[a, b)$ $\Rightarrow$ $x_i = a + (b-a)\, r_i$

- numpy to generate your random #ers

numpy.random.seed(17) ← any integer

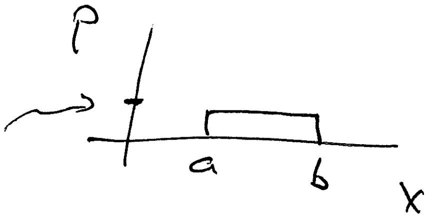
np.random.uniform(a, b, (n, m))

interval output array size

recall: expectation value of function of continuous random variable X: $\mathcal{E}(X) = \int P(x)\, x \, dx$

$$\mathcal{E}[f(X)] = \int_{-\infty}^{\infty} \underbrace{P(x)}_{\text{probability density}} f(x)\, dx$$

For now let $P(x) \equiv \text{uniform} = \dfrac{1}{b-a}$



We call $\mathcal{E}[f(X)] = \mu_f$ population mean

Variance of a function of random var X

$$\sigma_f^2 \equiv \mathcal{E}\{[f(x) - \mathcal{E}[f(x)]]^2\} \equiv \mathbb{V}[f(X)]$$

$$= \mathcal{E}\{[f(x)]^2 - 2 f(x)\mathcal{E}[f(x)] + (\mathcal{E}[f(x)])^2\}$$

$$= \mathcal{E}[f^2(x)] - \mathcal{E}^2[f(x)]$$

$$\sigma_f^2 = \frac{1}{b-a}\int_a^b f^2(x)\, dx - \left[\frac{1}{b-a}\int_a^b f(x)\, dx\right]^2$$

σ_f^2 ≡ population variance

σ_f ≡ pop. std. deviation

We don't know μ_f or σ_f ⇒ will estimate them by monte carlo integration

Sample mean

We will take n samples of the random variables $x_0, x_1, \ldots x_{n-1}$ from $p(x)$ which is uniform

e.g. we roll a 6 sided die n times

each x_i can take the value $1, 2, \ldots 6$ with probability $1/6$ for each

then we calculate a function: $f(x_0), f(x_1), \ldots$

We define $\bar{f} = \frac{1}{n}\sum_{i=0}^{n-1} f(x_i)$ ⇒ sample mean

Not equal to population mean since we only took n samples (they are equal as $n \to \infty$)

We estimate population mean as

$$\varepsilon(\bar{f}) = \varepsilon\left[\frac{1}{n}\sum f(x_i)\right] = \frac{1}{n}\sum \varepsilon[f(x_i)]$$

$$= \frac{1}{n}(\mu_f n) = \mu_f$$

expectation of sample mean = population mean

$\Rightarrow$ $\bar{f}$ is unbiased estimator of μ_f

Variance of sample mean

$$\sigma^2_{\bar{f}} = \mathbb{V}(\bar{f}) = \mathbb{V}\left[\frac{1}{n}\sum f(x_i)\right]$$

For independent (uncorrelated) random variables

x_1 and x_2: $\mathbb{V}(a x_1 + b x_2) = a^2\,\mathbb{V}(x_1) + b^2\,\mathbb{V}(x_2)$

$$\sigma^2_{\bar{f}} = \frac{1}{n^2}\sum \mathbb{V}[f(x_i)] = \frac{1}{n^2}\left[n\sigma_f^2\right] = \sigma_f^2/n$$

uncertainty in mean = standard deviation$/\sqrt{n}$

But we still don't know σ_f^2 (pop variance)

We form estimate $u \equiv \overline{f^2} - (\bar{f})^2$

We calculate $\mathcal{E}(u) = \mathcal{E}(\overline{f^2}) - \mathcal{E}[(\bar{f})^2]$

$$\mathcal{E}(\overline{f^2}) = \mathcal{E}\left[\frac{1}{n}\sum f^2(x_i)\right] = \frac{1}{n}\sum \mathcal{E}[f^2(x_i)]$$

$$\mathbb{V}(f) = \mathcal{E}(f^2) - [\mathcal{E}(f)]^2$$

$$\mathcal{E}(f^2) = \mathbb{V}(f) + [\mathcal{E}(f)]^2 = \sigma_f^2 + \mu_f^2$$

$$\mathcal{E}(\overline{f^2}) = \frac{1}{n}\sum(\sigma_f^2 + \mu^2) = \sigma_f^2 + \mu_f^2$$

And $\mathbb{V}(\bar{f}) = \mathcal{E}(\bar{f}^2) - [\mathcal{E}(\bar{f})]^2$

$$\mathcal{E}[(\bar{f})^2] = V(\bar{f}) + [\mathcal{E}(\bar{f})]^2$$

$$= \sigma^2_{\bar{f}} + \mu_f^2 = \frac{\sigma^2_f}{n} + \mu_f^2$$

$$\mathcal{E}(m) = \sigma_f^2 + \mu_f^2 - \frac{\sigma^2_f}{n} - \mu_f^2 = \sigma_f^2 - \sigma^2_f/n$$

$$\mathcal{E}(m) = \sigma^2_f \frac{n-1}{n} \implies \text{biased estimator of } \sigma^2_f$$

for large n, $\frac{n-1}{n} \approx 1$

$$\sigma^2_{\bar{f}} = \frac{\sigma_f^2}{n} \approx \frac{1}{n}\left[\frac{n}{n-1} m\right] = \frac{1}{n-1}\Big[\underbrace{\overline{f^2} - \bar{f}^2}_{\text{calcuable}}\Big]$$

We estimate population variance with sample variance

$\mu_f \longrightarrow \bar{f}$ from n samples, with uncertainty $\sigma_{\bar{f}}$

Why?!?

$$\int_a^b f(x)\,dx = (b-a)\mu_f$$

$$\int_a^b f(x)\,dx \approx (b-a)\bar{f} \pm (b-a)\sigma_{\bar{f}}$$

$$\approx (b-a)\bar{f} \pm (b-a)\left[\frac{\overline{f^2} - \bar{f}^2}{n-1}\right]^{1/2}$$

$$\int_a^b f(x)\,dx) \approx \frac{b-a}{n}\sum f(x_i) \pm \frac{b-a}{\sqrt{n-1}}\left[\frac{1}{n}\sum f^2(x_i) - \left(\frac{1}{n}\sum f(x_i\right.\right.$$

Choose x_i uniformly from a to b