Looks pretty similar but big diff actually

$$\underline{\underline{y_{j+1}}} = y_j + hf(x_{j+1}, \underline{\underline{y_{j+1}}})$$ Need y_{j+1} to find y_{j+1}

$$z = y_j + hf(x_{j+1}, z)$$ implicit

Look at $y'(x) = \mu y(x)$ $\quad [\ y(x) = e^{\mu x}\ ]$

$$y_{j+1} = y_j + h\mu y_{j+1} \Rightarrow y_{j+1} = \frac{y_j}{1-\mu h}$$

when $\mu < 0$ we always get $|y_{j+1}| < |y_j|$ ✓

unconditionally stable solution [see code]

Runge-Kutta

Start with higher order Taylor expansion

$$y(x_{j+1}) = y(x_j) + hy'(x_j) + \frac{h^2}{2}y''(x_j) + \frac{h^3}{6}y'''(x_j) + O(h^4)$$

$$= y(x_j) + hf(x_j, y(x_j)) + \frac{h^2}{2}f' + \frac{h^3}{6}f''' + O(h^4)$$

[since $y'(x) = f(x, y(x))$ is exact]

Note: $$f' = \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f$$

$$f'' = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right)\frac{dy}{dx}$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + f\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y \partial x} f + \frac{\partial^2 f}{\partial y^2} ff + \left(\frac{\partial f}{\partial y}\right)^2 f$$

$$y(x_{j+1}) = y(x_j) + h f(x_j, y(x_j)) + \frac{h^2}{2}\left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right)$$

$$+ \frac{h^3}{6}\left[\frac{\partial^2 f}{\partial x^2} + 2f\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + f^2\frac{\partial^2 f}{\partial y^2} + f\left(\frac{\partial f}{\partial y}\right)^2\right] + O(h^4)$$

Match to <u>2nd order R-K</u> up to $O(h^2)$

$$y_{j+1} = y_j + c_0 h f(x_j, y_j) + c_1 h \underbrace{f[x_j + c_2 h, y_j + c_2 h f(x_j, y_j)]}$$

Need to determine c_0, c_1, c_2

Note: let $k_0 = h f(x_j, y_j)$

$$\Rightarrow \quad y_{j+1} = y_j + c_0 k_0 + c_1 h f[x_j + c_2 h, y_j + c_2 k_0]$$

If $c_0, c_1, c_2, x_j, y_j, f$ are known, we can evaluate RHS $\Rightarrow$ explicit

Taylor expand last term

$$f[x_j + c_2 h, y_j + c_2 h f(x_j, y_j)] = f(x_j, y_j) + c_2 h \frac{\partial f}{\partial x}$$

$$+ c_2 h f(x_j, y_j)\frac{\partial f}{\partial y} + \frac{c_2^2 h^2}{2}\frac{\partial^2 f}{\partial x^2} + \frac{1}{2}(c_2 h)(c_2 h)\, 2f \frac{\partial^2 f}{\partial x \partial y}$$

$$+ \frac{c_2^2 h^2}{2} f^2(x_j, y_j)\frac{\partial^2 f}{\partial y^2} + O(h^3)$$

$$y_{j+1} = y_j + c_0 h f + c_1 h f + c_1 c_2 h^2\left[\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right]$$

$$+ \frac{1}{2} c_1 c_2^2 h^3\left[\frac{\partial^2 f}{\partial x^2} + 2f\frac{\partial^2 f}{\partial x \partial y} + f^2\frac{\partial^2 f}{\partial y^2}\right]$$

all $f \rightarrow f(x_j, x_j)$

$\Rightarrow \quad c_0 + c_1 = 1 \qquad c_1 c_2 = 1/2$ matches to order h and h^2

No way to match h^3 since our last expression is missing $\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}$ and $f\left(\frac{\partial f}{\partial y}\right)^2$ terms

Explicit midpoint: Choose $c_0 = 0 \quad c_1 = 1 \quad c_2 = 1/2$

$$y_{j+1} = y_j + h f[\, x_j + \frac{h}{2},\ y_j + \frac{h}{2} f(x_j, y_j))$$

-or-

$$K_0 = h f(x_j, y_j)$$

$$y_{j+1} = y_j + h f\left[\, x_j + \frac{h}{2},\ y_j + \frac{K_0}{2}\right]$$

Use the slope at the midpoint and move it to x_j, y_j to get to y_{j+1}

Explicit trapezoid: Choose $c_0 = 1/2 \quad c_1 = 1/2 \quad c_1 = 1$

$$K_0 = h f(x_j, y_j)$$

$$y_{j+1} = y_j + \frac{h}{2}\left[f(x_j, y_j) + f(x_{j+1}, y_j + K_0)\right]$$

Note: similarity in these methods and integration

$$y'(x) = f(x, y(x)) \qquad \int y'(x) = \int f(x, y(x))\, dx$$

$$y(x_{j+1}) - y(x_j) = \int_{x_j}^{x_{j+1}} \underbrace{f(x, y(x))\, dx}_{\text{but integral depends on } x,\ y(x)}$$

Could approx integral with

LH rectangle $y_{j+1} = y_j + h f(x_j, y_j)$ $\longleftrightarrow$ Forward euler

RH rectangle $y_{j+1} = y_j + h f(x_{j+1}, y_{j+1})$ $\longleftrightarrow$ Backward euler, implicit

midpoint $y_{j+1} = y_j + h f[x_j + \frac{h}{2}, \underbrace{y(x_j + \frac{h}{2})}_{\text{unknown}}]$ ~~accurate~~

estimate $y(x_j + \frac{h}{2}) \approx \frac{1}{2}(y_j + \underbrace{y_{j+1}}_{\text{use forward euler}})$ implicit midpoint

$\hookrightarrow$ explicit midpoint

Same for trapezoid

<u>Best method for our purposes</u> : 4th order Runge Kutta

$$k_0 = h f(x_j, y_j)$$
$$k_1 = h f\left(x_j + \frac{h}{2}, y_j + \frac{k_0}{2}\right)$$
$$k_2 = h f\left(x_j + \frac{h}{2}, y_j + \frac{k_1}{2}\right)$$
$$k_3 = h f(x_j + h, y_j + k_2)$$
$$y_{j+1} = y_j + \frac{1}{6}(k_0 + 2k_1 + 2k_2 + k_3)$$

4 function evaluations

(no derivatives)

$O(h^5)$

too much algebra in taylor expansion

k's give approx to slope at endpoints and midpoint