Matrix approach

Use finite-diff approx for second and first deriv.

$$w'' = f(x, w, w') \qquad w(a) = c \qquad w(b) = d$$

We found
$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}$$
$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h}$$

$$\frac{w_{j+1} - 2w_j + w_{j-1}}{h^2} = f\left(x_j, w_j, \frac{w_{j+1} - w_{j-1}}{2h}\right) \qquad j = 1, 2, .. n-2$$

$$w_0 = c \qquad w_{n-1} = d$$

Need to solve equations simultaneously for all $w_j$

If $f$ is linear function we have linear system of equations with single solution

otherwise, it could be difficult w/out good starting guess to nonlinear equations

<u>Ex</u>: $$f(x, w, w') = \frac{-30}{1-x^2} w(x) + \frac{2x}{1-x^2} w'(x)$$

$$w_{j-1} - 2w_j + w_{j+1} = -h^2 \frac{30}{1-x_j^2} w_j + \frac{h x_j}{1-x_j^2}(w_{j+1} - w_{j-1})$$

$$w_0 = c = 0.09265\ldots \qquad w_{n-1} = d = 0.111770\ldots$$

Regroup equations by $w$ index

$$w_0 = c$$

$$\alpha_j w_{j-1} + \beta_j w_j + \gamma_j w_{j+1} = 0$$

$$w_{n-1} = d$$

$$\alpha_j = 1 + \frac{h x_j}{1-x_j^2} \qquad \beta_j = -2 + \frac{3ch^2}{1-x_j^2} \qquad \gamma_j = 1 - \frac{h x_j}{1-x_j^2}$$

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \alpha_1 & \beta_1 & \gamma_1 & \cdots & 0 & 0 & 0 \\ 0 & \alpha_2 & \beta_2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & \beta_{n-3} & \gamma_{n-3} & 0 \\ 0 & 0 & 0 & \cdots & \alpha_{n-2} & \beta_{n-2} & \gamma_{n-2} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_{n-1} \end{pmatrix} = \begin{pmatrix} c \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ d \end{pmatrix}$$

$$\Rightarrow A\vec{x} = \vec{b}$$

only the $n$ $w_j$ are unknown

use gaussian elimination to solve (or simpler method in this particular case case $A$ is tridiagonal)

# Eigenvalue Problems

2nd order ODE

$$w'' = f(x, w, w' ; s) \quad w(a) = c \quad w(b) = d$$

$\uparrow$ family of ODEs

only solutions for interesting values of $s$

focus on linear 2nd order

$$w''(x) = \underline{\xi(x)}\, w'(x) + \underline{M(x)}\, w(x) + \underline{\Theta(x)}\, s\, w(x)$$

all known

and we choose an example called Mathieu eqn

$$w''(x) = (2q \cos 2x - s)\, w(x) \qquad w(0) = w(2\pi) = 0$$

periodic BC

$M(x) = 2q \cos 2x$ $\quad \xi(x) = 0$

$\Theta(x) = 1$

Note if $q = 0 \Rightarrow w''(x) = -s\, w(x)$ Mass on a spring

Also note if $w(x)$ is a solution so is $\alpha w(x)$

$\alpha \in$ constant

With our shooting method we tried to find a $w'(0)$ to get us to $w(\text{end pt})$

But if $\alpha w(x)$ is a solution, $w'(0)$ could be anything

Old shooting method doesn't work here (see Fig 8.8 left)

Instead : Pick any $w'(0)$ value

$$w'(0) = \sigma$$

Use $w_0, \sigma$ in IVP $\to$ $w_{n-1}$ will only work for particular $s$

need $g(s) = w_{n-1} (= 0)$ in this case

$g$ is rk4 like solution from $0 \to 2\pi$

use root finder to solve for $s$ so $w(2\pi) = 0$

Little more complex since rk4 needs $s$ value to proceed

[see code]

each value of $s$ has corresponding $w(x)$
eigenvalue eigenfunction

all satisfy $w(0) = w(2\pi) = 0$

We can also modify Matrix approach to same problem

$$w''(x) = (2q\cos 2x - s)\,w(x) \qquad w(0) = w(2\pi) \;(\neq 0)$$

$$x_j = \frac{2\pi j}{n} \qquad j = 0, 1, \ldots n-1$$ (skip $x_j = 2\pi$ since it is same as $x_0$)

need estimate for $w''(x)$ but not $w'(x)$

Central diff approx to 2nd deriv was

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2} + \mathcal{O}(h^2)$$

$$w_{j+1} - 2w_j + w_{j-1} = h^2(2q\cos 2x_j - s)w_j \qquad j = 0, 1, \ldots n-1$$

$$w_{j+1} + \alpha_j w_j + w_{j-1} = -h^2 s w_j \qquad \alpha_j \equiv -2 - 2h^2 q \cos 2x_j$$

Note: we define $w_{-1} \equiv w_{n-1}$ (when $j=0$) $\quad w_n \equiv w_0$ (when $j = n-1$)

$$\begin{bmatrix} \alpha_0 & 1 & 0 & \cdots & 0 & 0 & 1 \\ 1 & \alpha_1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \alpha_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{n-3} & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & \alpha_{n-2} & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 & \alpha_{n-1} \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ \vdots \\ \\ \\ w_{n-1} \end{bmatrix} = -h^2 s \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ \\ \\ \\ w_{n-1} \end{bmatrix}$$

$$A\vec{v} = \lambda \vec{v}$$

$\Rightarrow$ gives estimates for $n$ eigenvalues