Matrices - Linear Algebra

We will focus mostly on square matrices

$$A = \begin{bmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & A_{12} \\ A_{20} & A_{21} & A_{22} \end{bmatrix} \qquad A_{ij} \equiv i^{th} \text{ row}, \; j^{th} \text{ column}$$

$$\vec{x} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \quad \text{vector}$$

matrix-vector multiplication $\quad \vec{y} = A\vec{x}$

$$y_i = \sum_{j=0}^{n-1} A_{ij} x_j$$

Ex $\quad y_1 = \sum_{0}^{2} A_{1j} x_j = A_{10} x_0 + A_{11} x_1 + A_{12} x_2$

matrix multiplication

$$C = AB \qquad C_{ij} = \sum_{k=0}^{n-1} A_{ik} B_{kj}$$

transpose $\quad B = A^T \quad B_{ij} = A_{ji}$

Ex $\quad A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \qquad AB = ?$

$$AB = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$$

Determinant is a number : det(A)

2x2 $|A| = \begin{vmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{vmatrix} = A_{00}A_{11} - A_{01}A_{10}$

3x3 $|A| = \begin{vmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & A_{12} \\ A_{20} & A_{21} & A_{22} \end{vmatrix} = \cancel{A_{00}\,\mathrm{det}}$

$$= A_{00} \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} - A_{01} \begin{vmatrix} A_{10} & A_{12} \\ A_{20} & A_{22} \end{vmatrix} + A_{02} \begin{vmatrix} A_{10} & A_{11} \\ A_{20} & A_{21} \end{vmatrix}$$

Triangular matrix (upper and lower)

$$\begin{bmatrix} A_{00} & A_{01} & A_{02} \\ 0 & A_{11} & A_{12} \\ 0 & 0 & A_{22} \end{bmatrix} \qquad \begin{bmatrix} A_{00} & 0 & 0 \\ A_{10} & A_{11} & 0 \\ A_{20} & A_{21} & A_{22} \end{bmatrix}$$

Note det(triangular matrix) = product main diagonal

$$|A_{\text{triangular}}| = \prod_{0}^{n-1} A_{ii}$$

Inverse $A^{-1} \Rightarrow A^{-1}A = AA^{-1} = \mathbb{I}$ ~ identity matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

If det(A) = 0 A is singular and A^{-1} does not exist

(rows, columns of A are not linearly independent)

Solving n linear equations w/ n unknowns

$$A\vec{x} = \vec{b} \qquad n \text{ unkowns} \quad \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$n \times n$ coefficients in A, and n constants in $\vec{b} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{pmatrix}$

we need $\det(A) \neq 0$ for solution.

We create augmented coeff matrix of A and b

$$(A|b) = \left[\begin{array}{cccc|c} A_{00} & A_{01} & \cdots & A_{0,n-1} & b_0 \\ \vdots & & & \vdots & \vdots \\ A_{n-1,0} & A_{n-1,1} & \cdots & A_{n-1,n-1} & b_{n-1} \end{array}\right]$$

In solving linear eqns we can

scale row by a constant

pivot: two rows can be interchanged.

eliminate: a row can be replaced by the sum of that row w/ a multiple of any other row.

We will also solve eigenvalue problem

$$A\vec{v} = \lambda\vec{v} \qquad \lambda \in \text{a scalar (number)}$$

$$A\vec{v} - \lambda\vec{v} = \vec{0}$$

$$(A - \lambda \mathbb{I})\vec{v} = \vec{0}$$

We have n linear equations and

n unkowns in $\vec{v}$ plus 1 unknown in λ

→ We won't find unique solutions

For a non trivial solution ($\vec{v} \neq \vec{0}$) we need

$A - \lambda \mathbb{I}$ not to have an inverse so

$$|A - \lambda \mathbb{I}| = 0$$

Error Analysis (worst case scenario)

Consider special 2x2 example

$$(A|b) = \left(\begin{array}{cc|c} 0.2161 & 0.1441 & 0.1440 \\ 1.2969 & 0.8648 & 0.8642 \end{array}\right)$$

You are given approx solution $\tilde{x} = \begin{pmatrix} .9911 \\ -.4870 \end{pmatrix}$

Create residual vector $\vec{r}$

$$\vec{r} = \vec{b} - A\vec{\tilde{x}} \Rightarrow \vec{r} = \begin{pmatrix} -1e-8 \\ 1e-8 \end{pmatrix}$$ not bad

exact solution $\vec{x} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$!?!

Weird. Very small changes to A will yield very big changes to our solution.

Maybe A is 'close to' having $\det(A) = 0$
(close to being singular)
(in our case $\det(A) = -9.999e-9$)

Define

Frobenius norm $\|A\|_F = \left[\sum_i^{n-1} \sum_j^{n-1} |A_{ij}|^2 \right]^{1/2}$

Infinity norm $\|A\|_\infty = \max_{0 \le i \le n-1} \sum_{j=0}^{n-1} |A_{ij}|$

(maximum absolute row-sum norm)

We can show that condition number $K(A)$

$K(A) = \|A\| \, \|A^{-1}\|$ determines sensitivity to small perturbation (Eqn 4.31)

Solving $A\vec{x} = \vec{b}$

First solve $L\vec{x} = \vec{b}$ $\quad L \equiv$ lower triangular matrix

For $n = 3$

$$\begin{pmatrix} L_{00} & 0 & 0 \\ L_{10} & L_{11} & 0 \\ L_{20} & L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}$$

$L_{00} x_0 = b_0$ $\qquad x_0 = b_0 / L_{00}$

$L_{10} x_0 + L_{11} x_1 = b_1$ $\qquad x_1 = \dfrac{b_1 - L_{10} x_0}{L_{11}}$

$L_{20} x_0 + L_{21} x_1 + L_{22} x_2 = b_2$ $\qquad x_2 = \dfrac{b_2 - L_{20} x_0 - L_{21} x_1}{}$