$A U = U \Lambda$ and $U^{-1} A U = \Lambda$ ▨

Full solution is too advanced until we have taken linear algebra

We will use Power Method to find largest λ

$|\lambda_0| > |\lambda_1| > |\lambda_2| > \dots |\lambda_{n-1}|$ since all λ_i distinct

Start with a guess $\vec{z}^{(0)}$ and increment

$$\vec{z}^{(k)} = A \vec{z}^{(k-1)} \qquad k = 1, 2, \dots$$

We note $\vec{z}^{(k-1)} = A \vec{z}^{(k-2)} = A A \vec{z}^{(k-3)}$ so

$$\vec{z}^{(k)} = A^k \vec{z}^{(0)}$$

Let us assume $\vec{z}^{(0)}$ has a component along $\vec{v}^{(0)}$

We write $\vec{z}^{(0)} = \sum_{i=0}^{n-1} c_i \vec{V}_i$ linear combination of eigenvectors

$$\vec{z}^k = A^k \vec{z}^{(0)} = \sum_i c_i A^k \vec{V}_i = \sum_{i=0}^{n-1} c_i \lambda_i^k \vec{V}_i$$

$$\vec{z}^k = c_0 \lambda_0^k \vec{V}_0 + \lambda_0^k \sum_{i=1}^{n-1} c_i \underbrace{\left(\frac{\lambda_i}{\lambda_0}\right)^k}_{} \vec{V}_i \qquad k = 1, 2, \dots$$

$$\left(\frac{\lambda_i}{\lambda_0}\right)^k \to 0 \text{ as } k \to \infty$$

So $\vec{z}^{(k)}$ will tend to $c_0 \lambda_0^k \vec{v}_0$ as k gets large

To find eigenvalue we will use Rayleigh quotient of a vector $\vec{x}$

$$\mu(\vec{x}) = \text{Rayleigh quotient} = \frac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}}$$

if $\vec{x}$ is an eignevector of $\vec{A}$

$$\mu(\vec{x}) = \lambda_x \frac{\vec{x}^T \vec{x}}{\vec{x}^T \vec{x}} = \lambda_x$$ eigenvalue corresponding to eigenvector $\vec{x}$

Since $\vec{z}^{(k)}$ approaches $\vec{v}_0$ $\mu(\vec{z}^{(k)})$ will be a good estimate for λ_0

One caveat: λ_0^k may become very large if $|\lambda_0| > 1$ leading to very large $\vec{z}^{(k)}$ $\Rightarrow$ we normalize or scale

Define $$\vec{q}^{(k)} = \frac{\vec{z}^{(k)}}{\|\vec{z}^{(k)}\|}$$ like a unit vector $\|\vec{q}^{(k)}\| = 1$

So $\mu(\vec{q}^{(k)}) = [\vec{q}^{(k)}]^T A \vec{q}^{(k)}$ since $[\vec{q}^{(k)}]^T \vec{q}^{(k)} = 1$

We start by making a guess $\vec{q}^{(0)}$
(it must have a component along $\vec{v}_0$)

algorithm: $\vec{z}^{(k)} = A\,\vec{q}^{(k-1)}$

$$\vec{q}^{(k)} = \vec{z}^{k} / \|\vec{z}^{k}\|$$

$$\mu(\vec{q}^{(k)}) = [\vec{q}^{(k)}]^T A\,\vec{q}^{(k)}$$

After enough iterations $\mu(\vec{q}^{(k)}) \to \lambda_0$ $\vec{q}^{(k)} \to \vec{v}_0$

We use relative change in $\vec{q}^{(k)}$ to set a tolerance to stop the iteration

[See power.py code]