Learn Bayes, MCMC and Stan 2017!

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New York, NY
August 23 - August 25, 2017

Wednesday

- Foundations of Bayesian Inference
- Bayesian Computation and Markov Chain Monte Carlo
- Bayesian Modeling with Stan

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Thursday

- Regression Modeling with Stan

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Friday

- Hierarchical Modeling with Stan

Foundations of Bayesian Inference

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$\mathcal{D}$

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$$\pi(\mathcal{D})$$

Quick aside: what exactly is a probability distribution?

$\pi(\mathcal{D})$?

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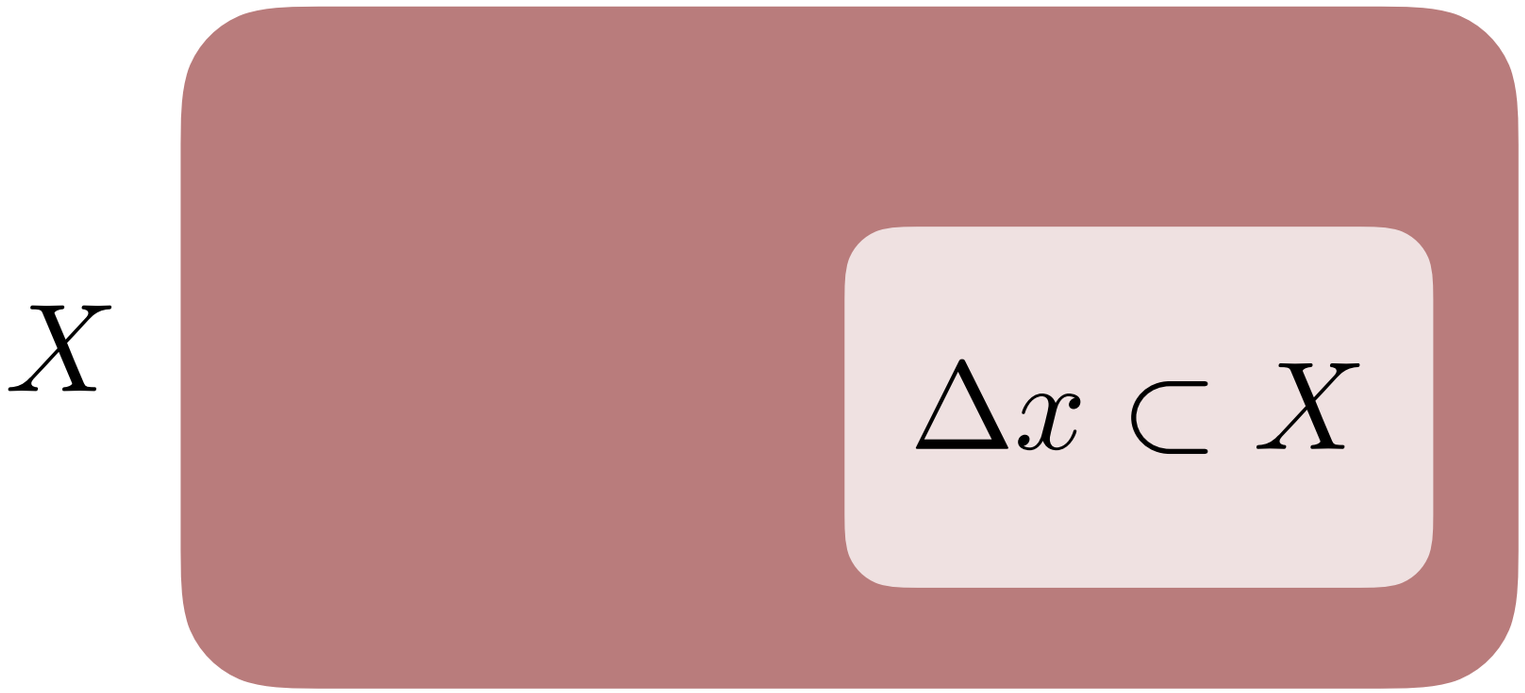

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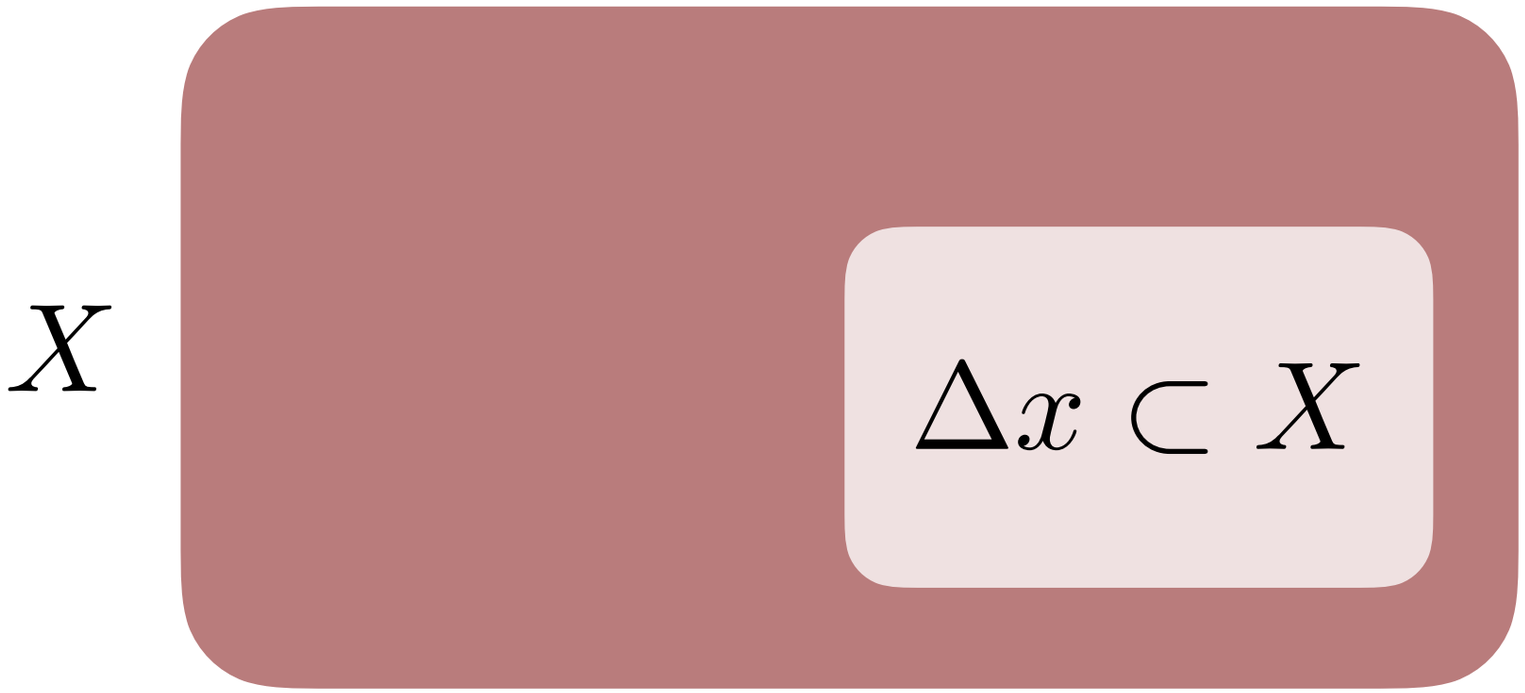
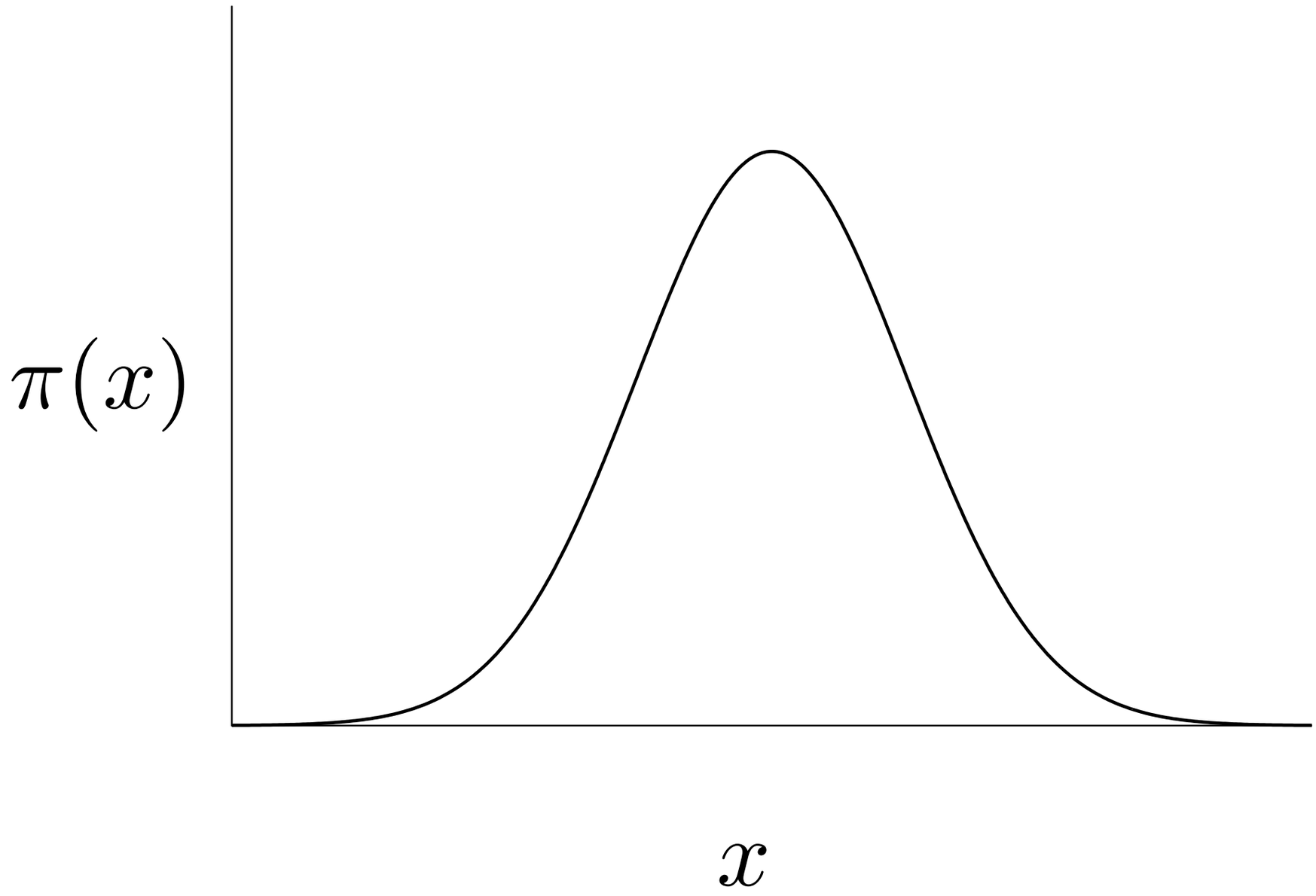


$$\pi[\Delta x] \in [0, 1]$$

$$\pi[X] = 1$$

Typically we represent probability distributions with probability density functions which we can *integrate*.



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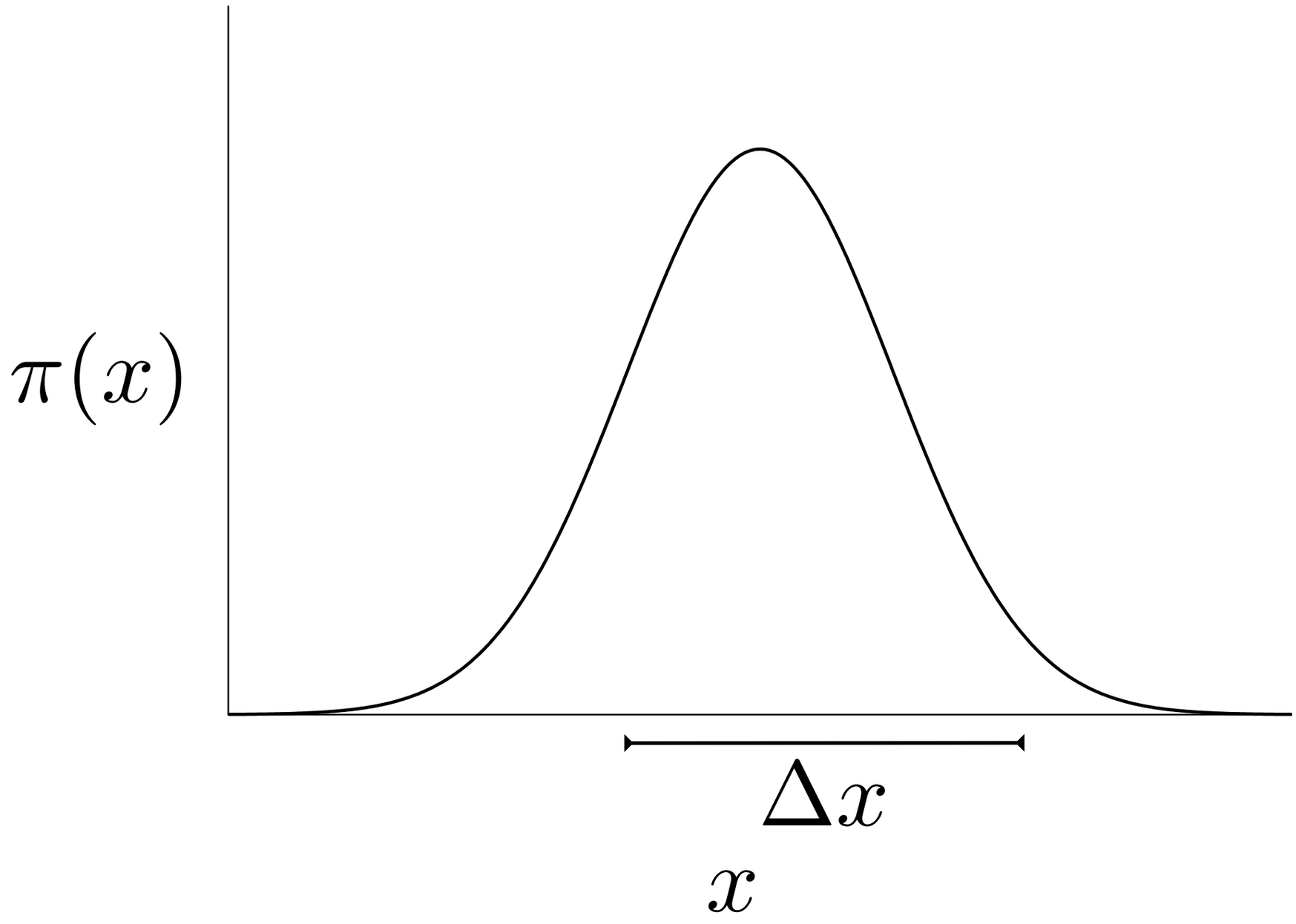

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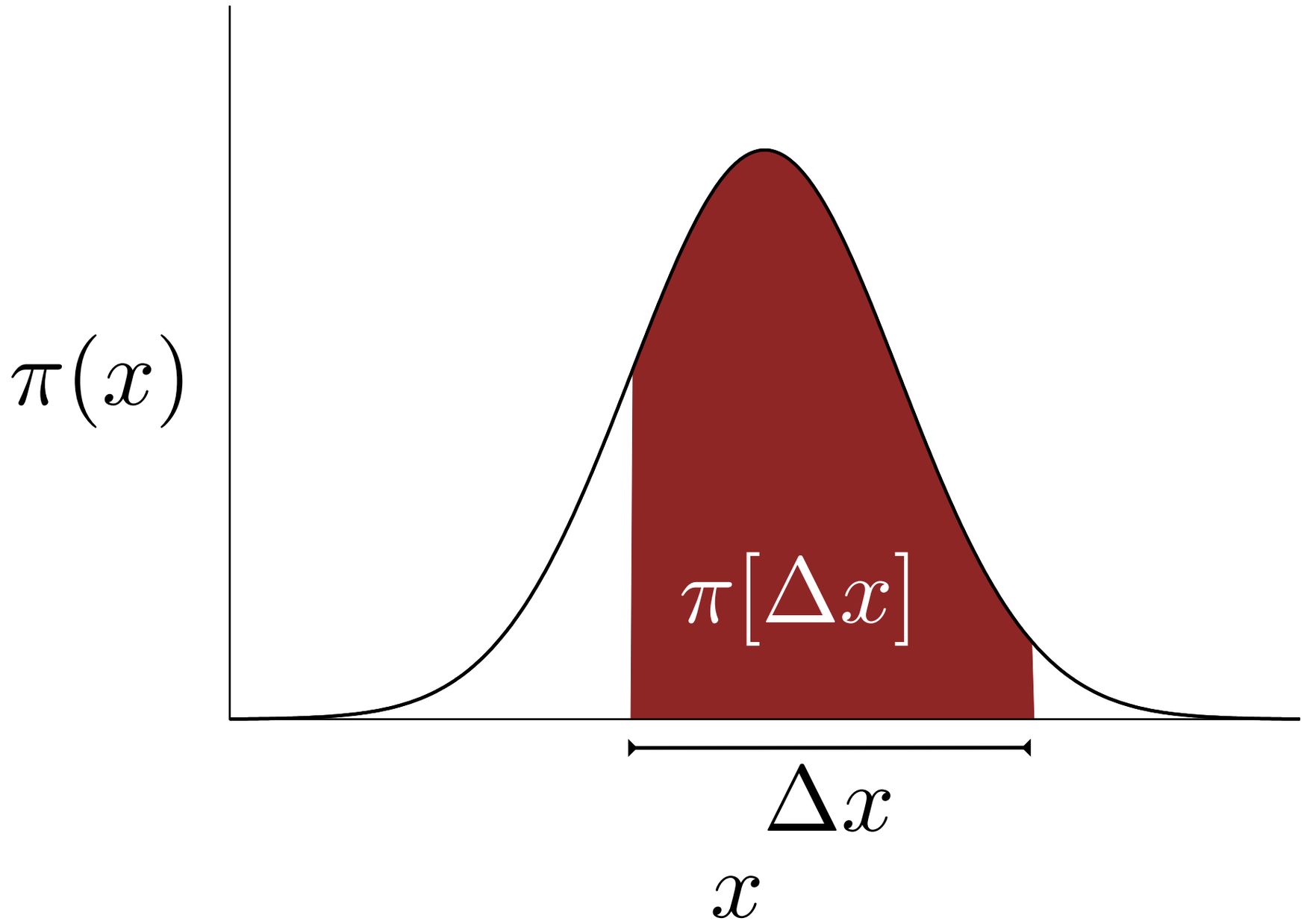

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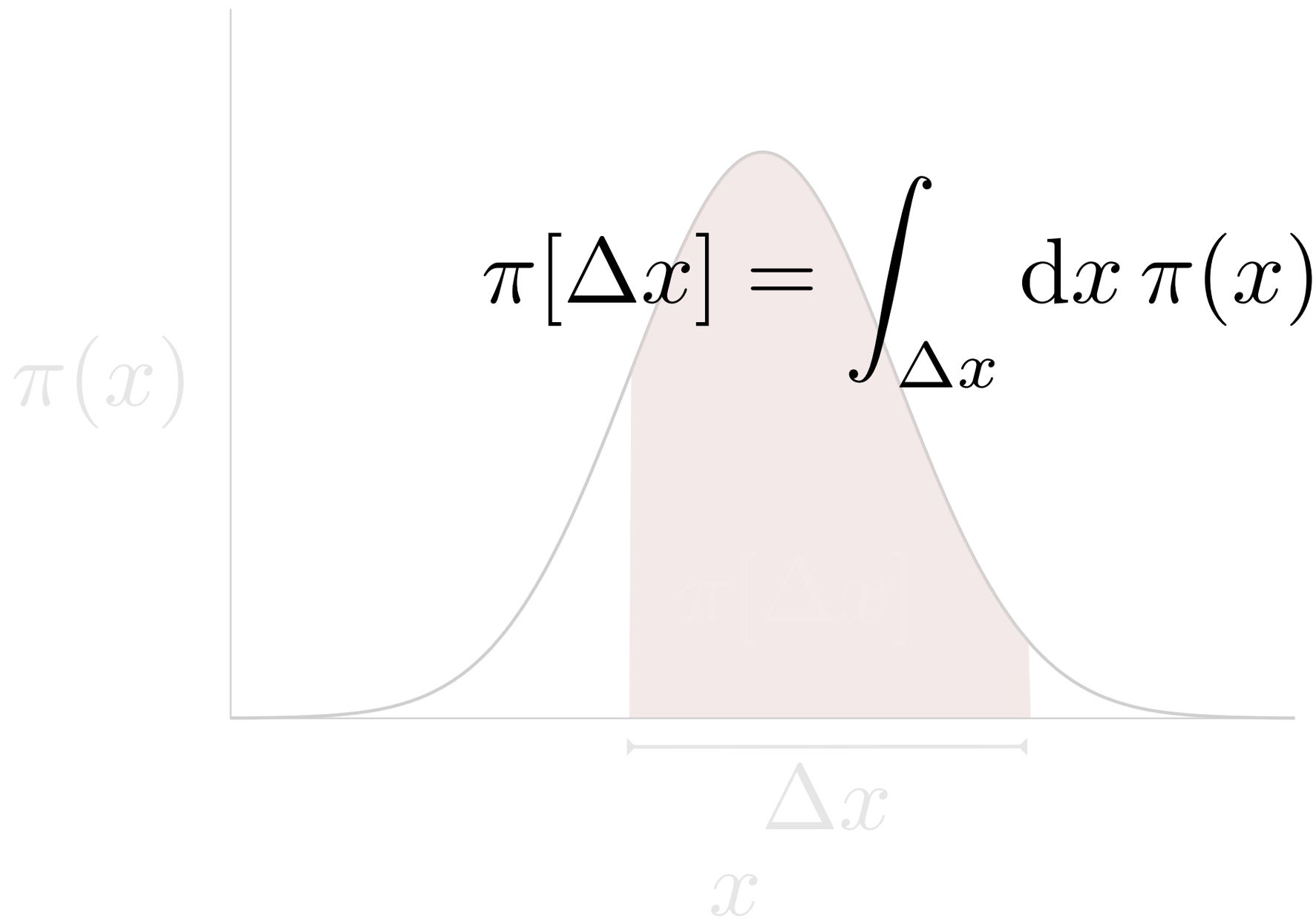


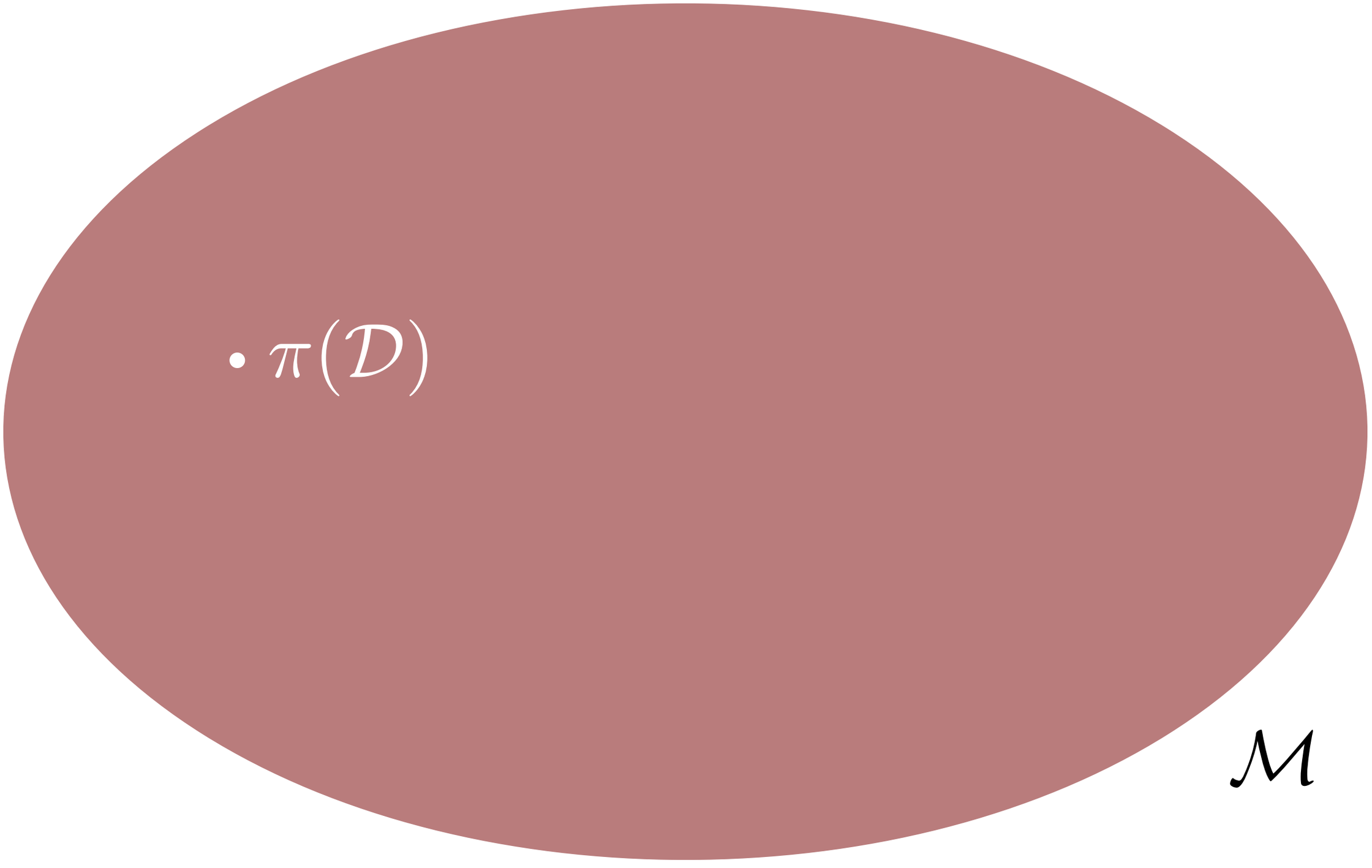
We can also use these probabilities to weight function outputs and define *expectation values*.

$$\mathbb{E}[f] = \int_X \mathrm{d}x\, \pi(x) f(x)$$

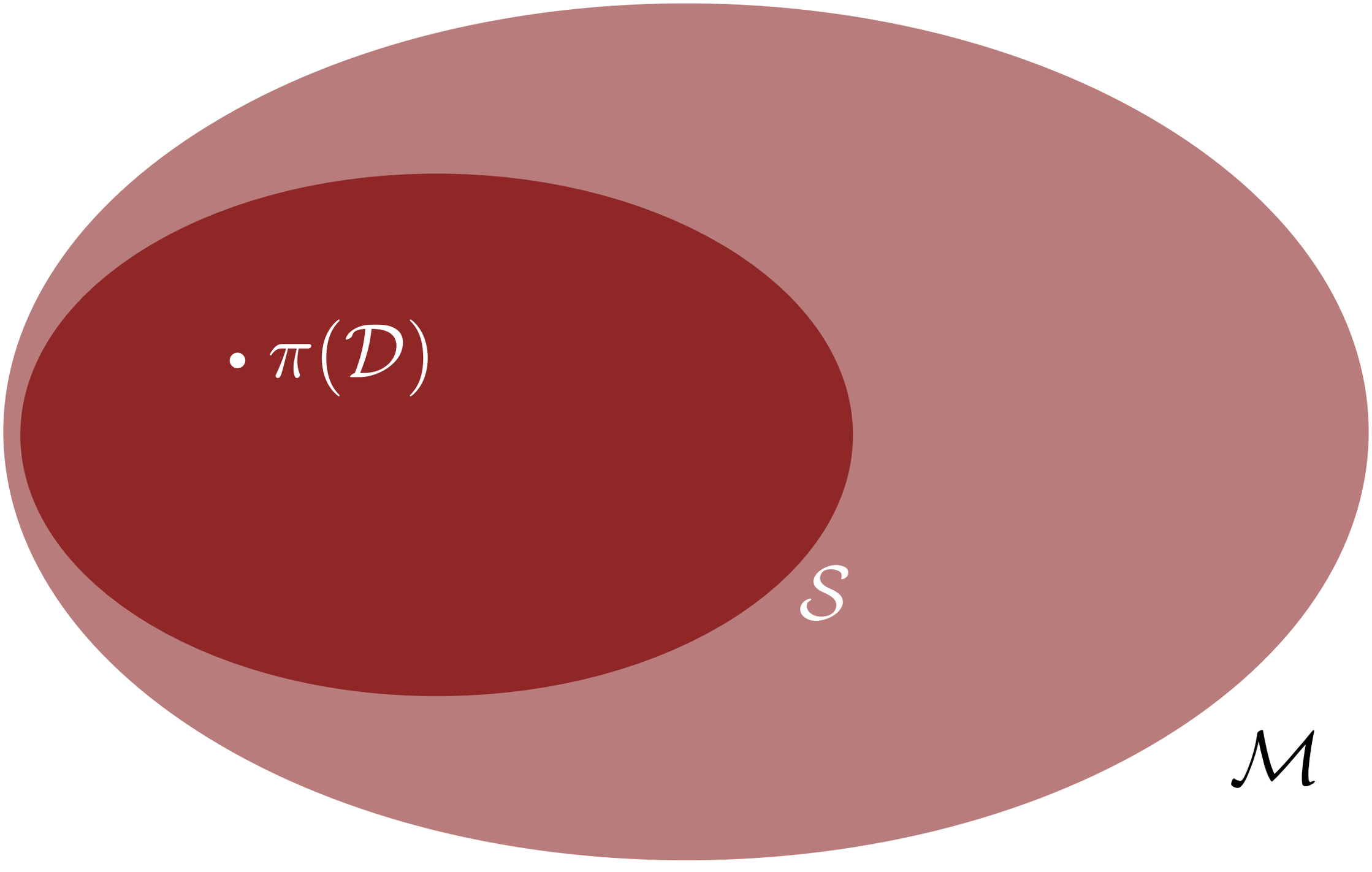
Let's return to our assumed latent data generating process,
a probability distribution over the measurement space.

$$\pi(\mathcal{D})$$

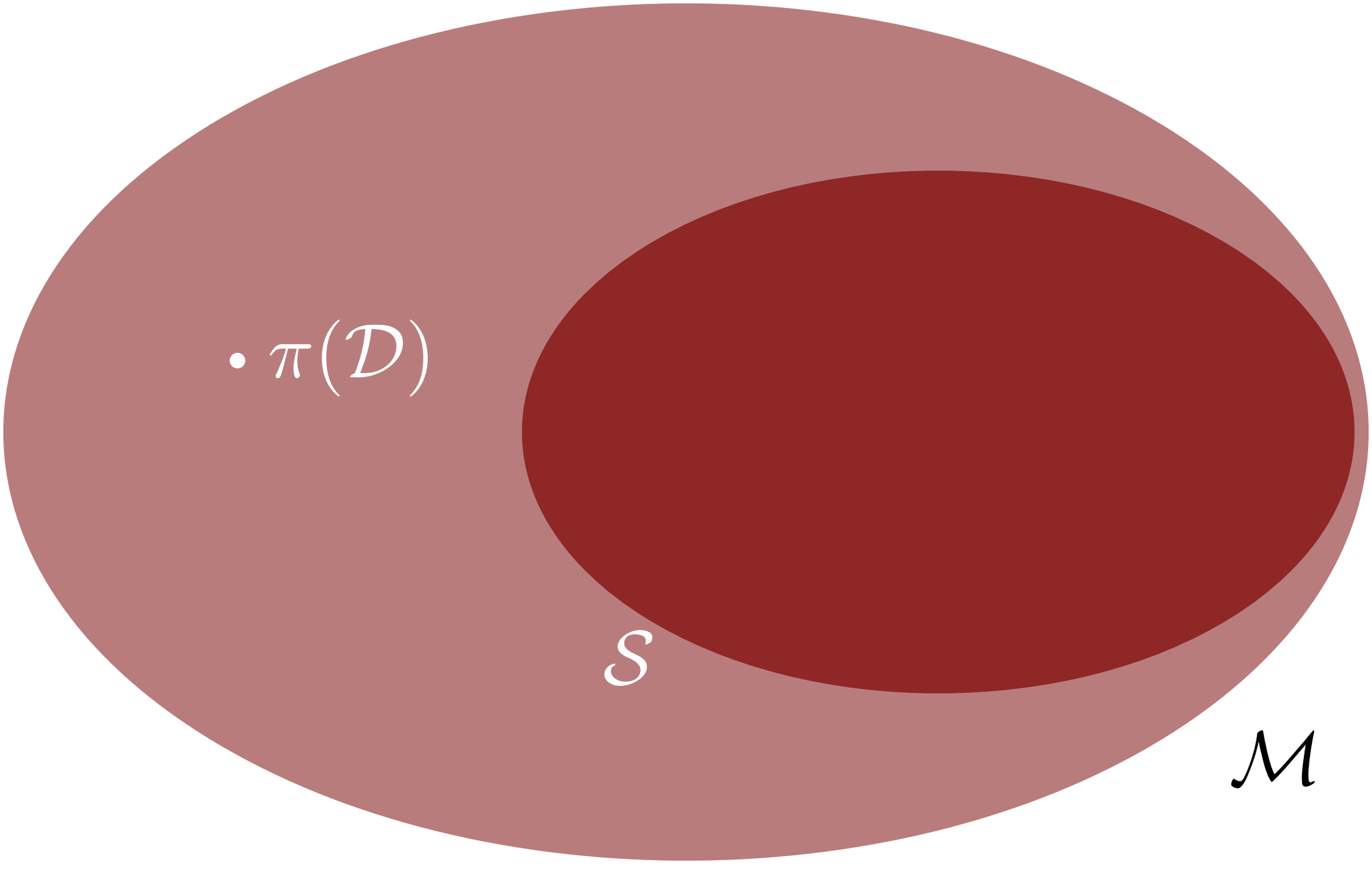
The "true" data generation process must lie in the space of all possible data generating processes, $\mathcal{M}$.



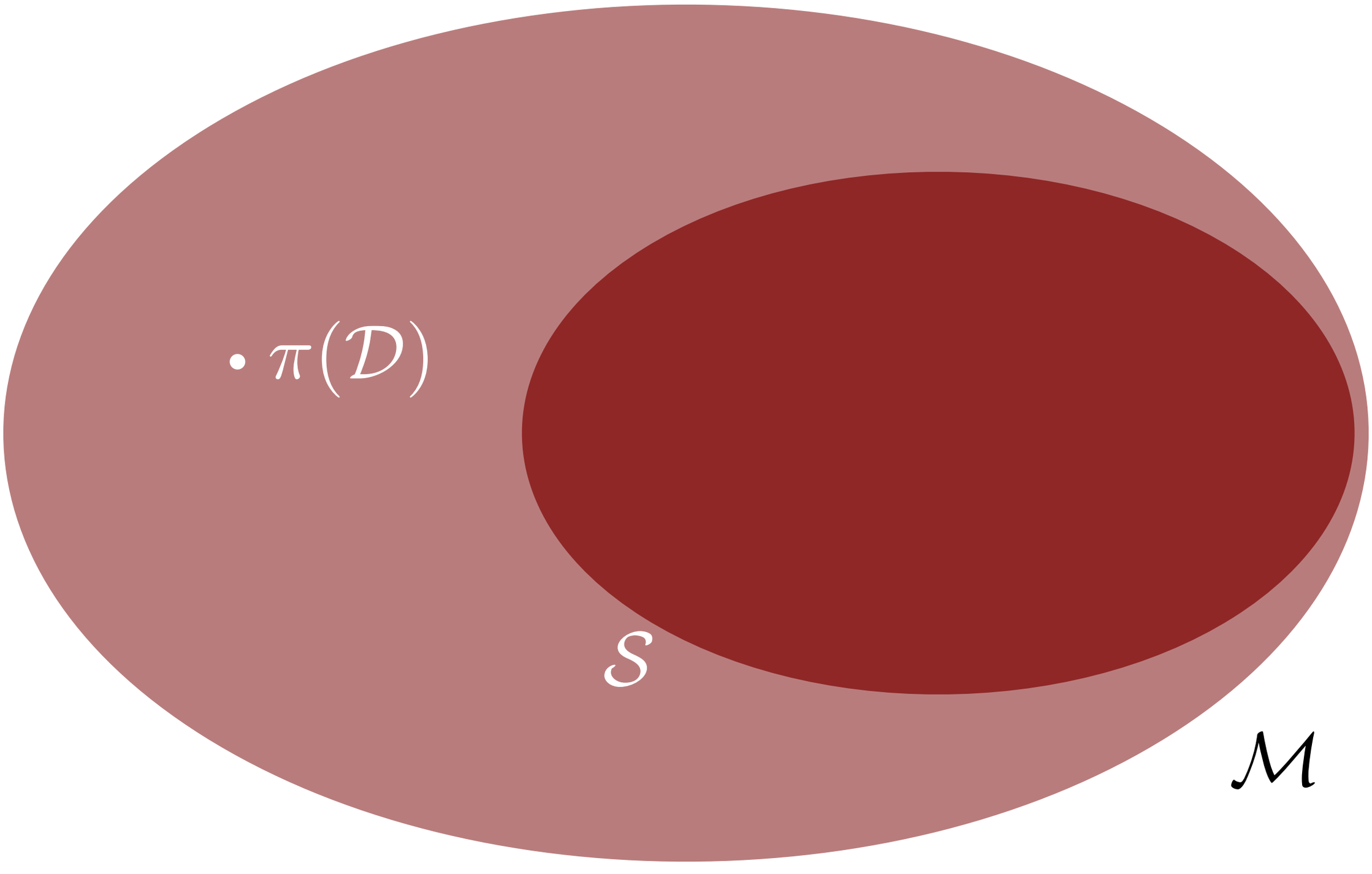
But in practice we have to consider only a small selection of processes $\mathcal{S} \subset \mathcal{M}$, sometimes called the *small world*.



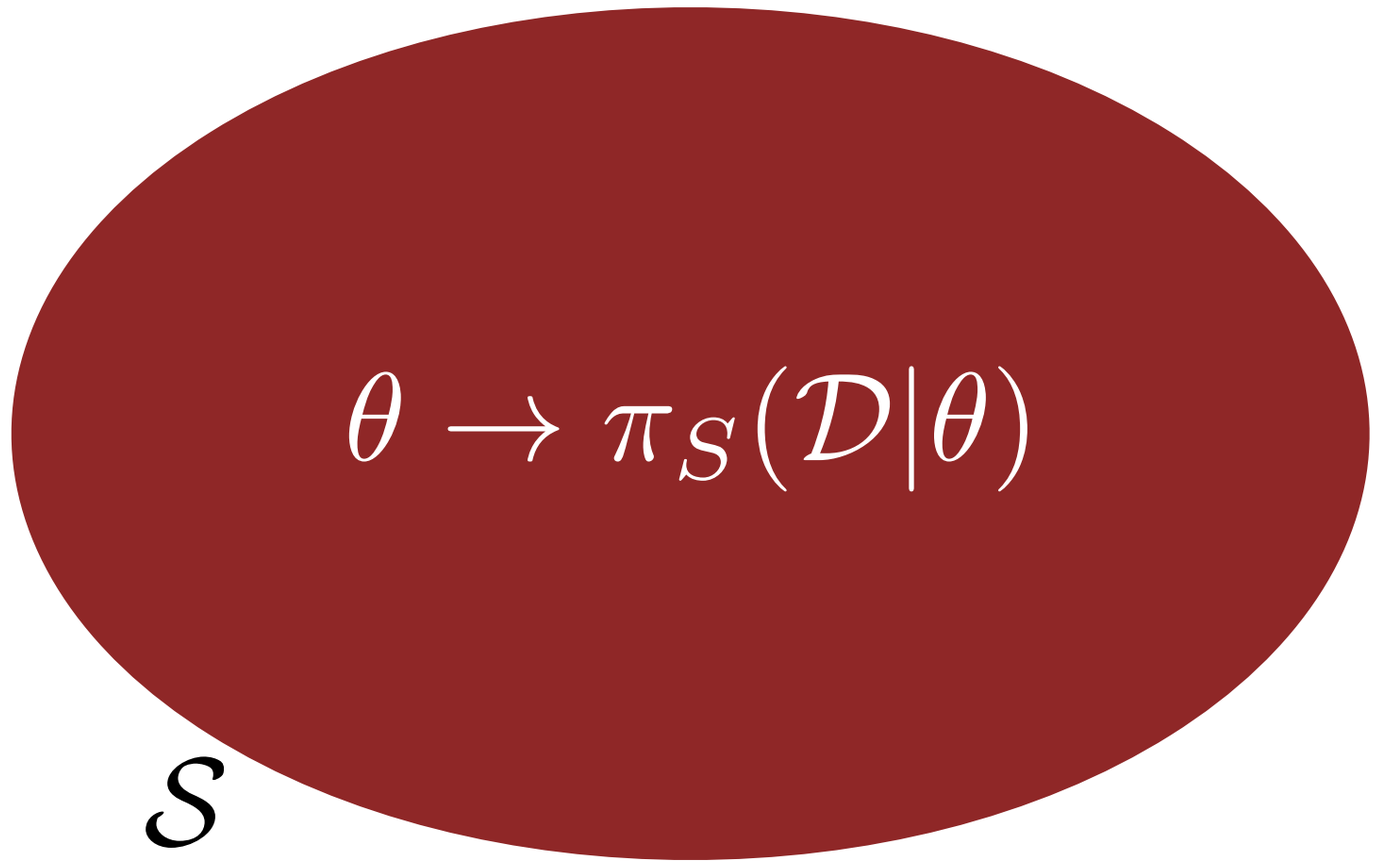
The true process, if it exists, may not be an element of the small world, but our inferences may still be meaningful.



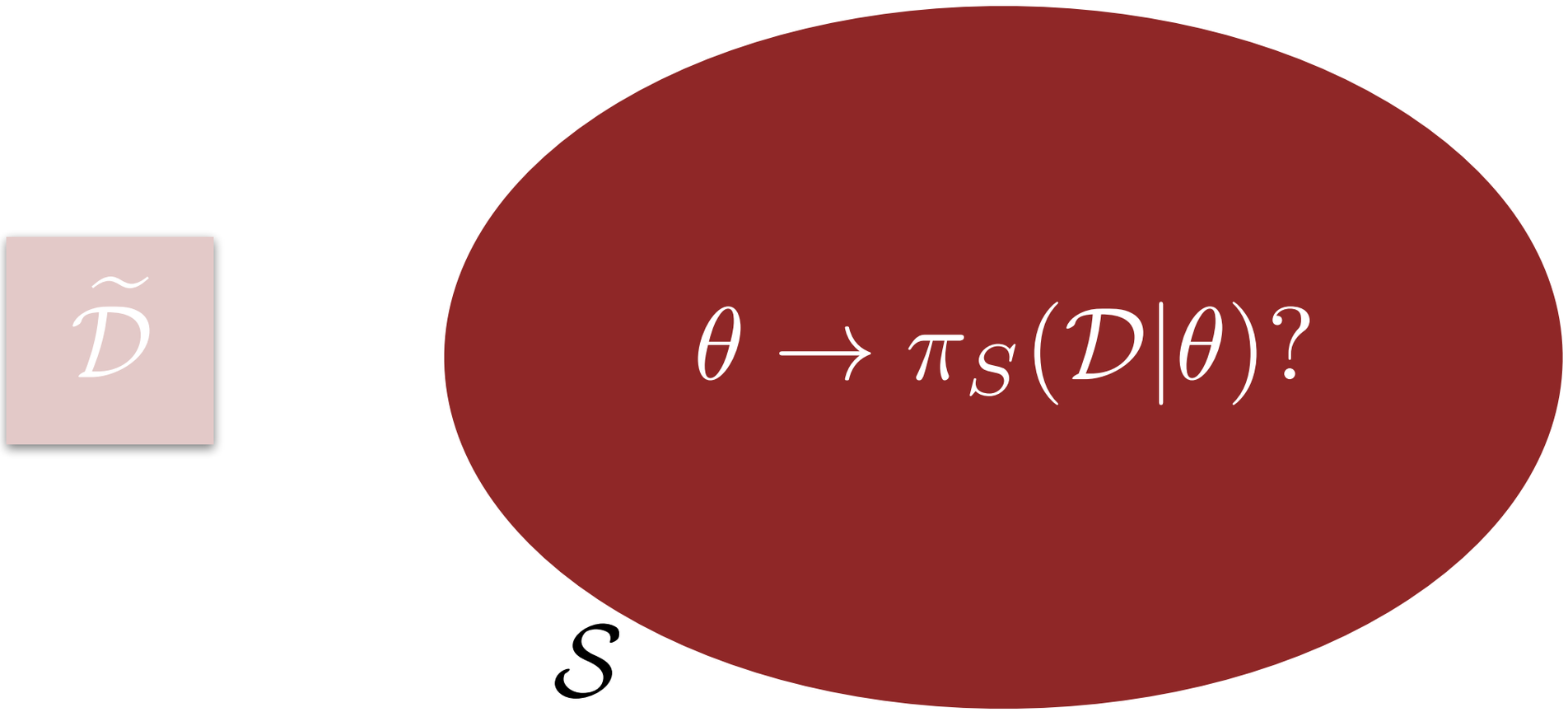
"All models are wrong but some are useful".
-George Box



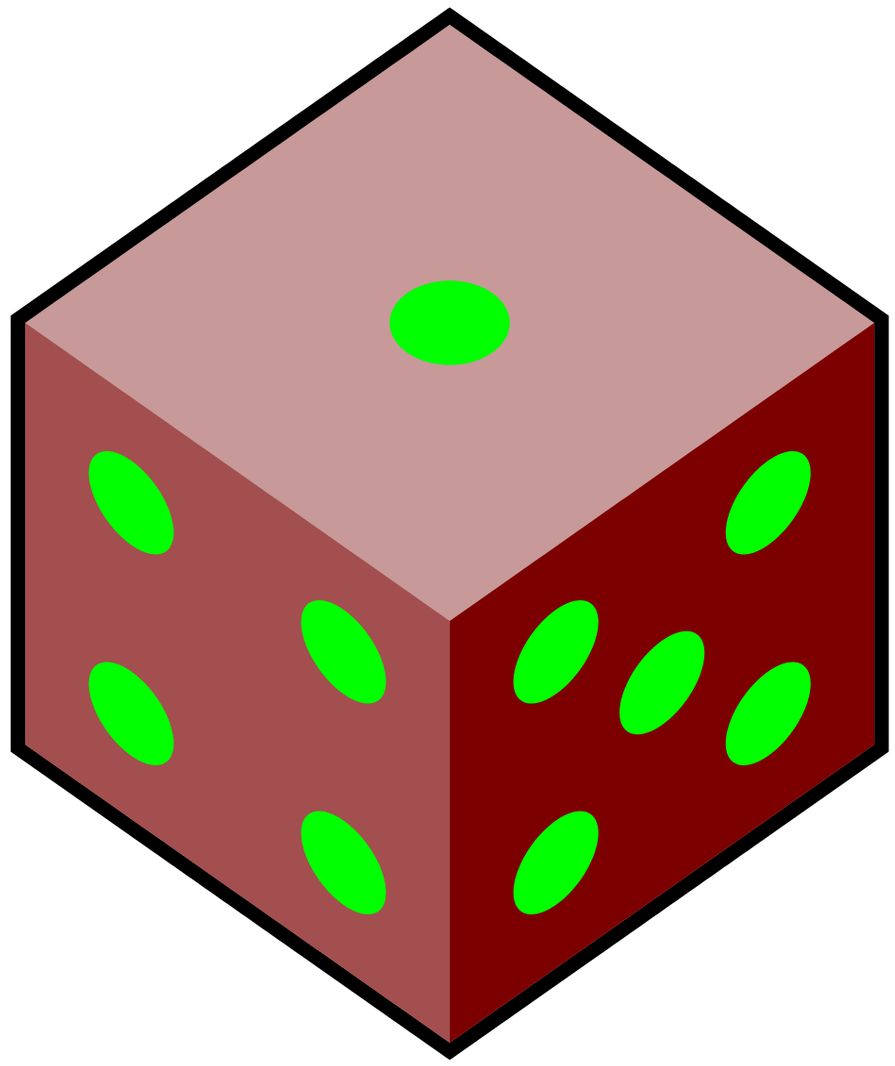
Any inferential model is then a choice of small world,
or a *likelihood* of distributions over measurements.



Inference is the identification of those points in the small world *consistent* with a given measurement.



How we define consistency, however, depends exactly on how we define probability itself.



In *frequentist statistics*, probability is defined in terms of frequencies and so can be applied to only the data.

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Frequentist methods compute expectations with respect to the data to identify estimators that work well *on average*.

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$$\mathcal{L}(\theta) = \int \mathcal{L}(\hat{\theta}(\mathcal{D}), \theta) \pi_S(\mathcal{D} \mid \theta) \mathrm{d}\mathcal{D}$$

Bayesian inference, however, treats probability as a general measure of uncertainty.

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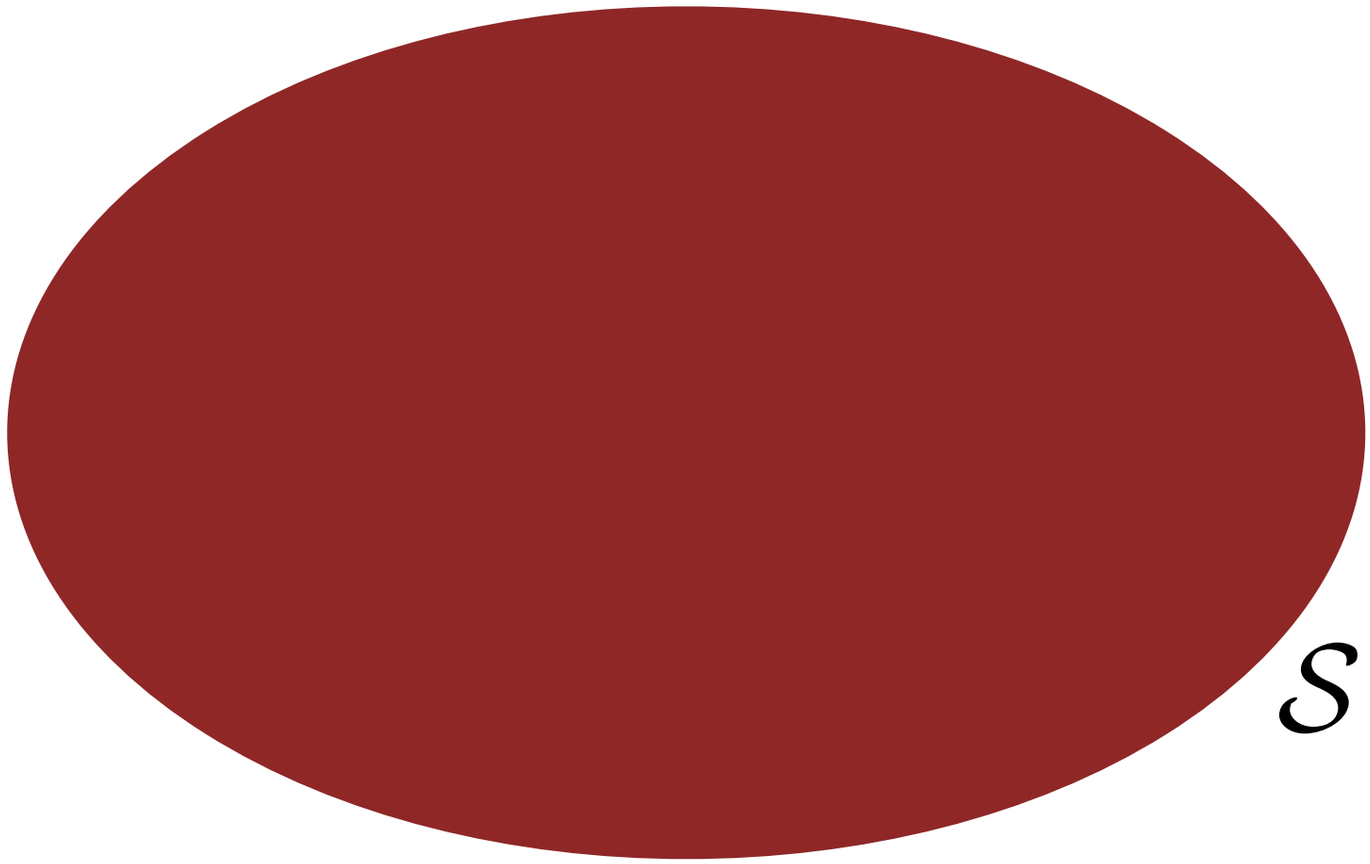
In this more general perspective we quantify consistency using a probability distribution over the small world.



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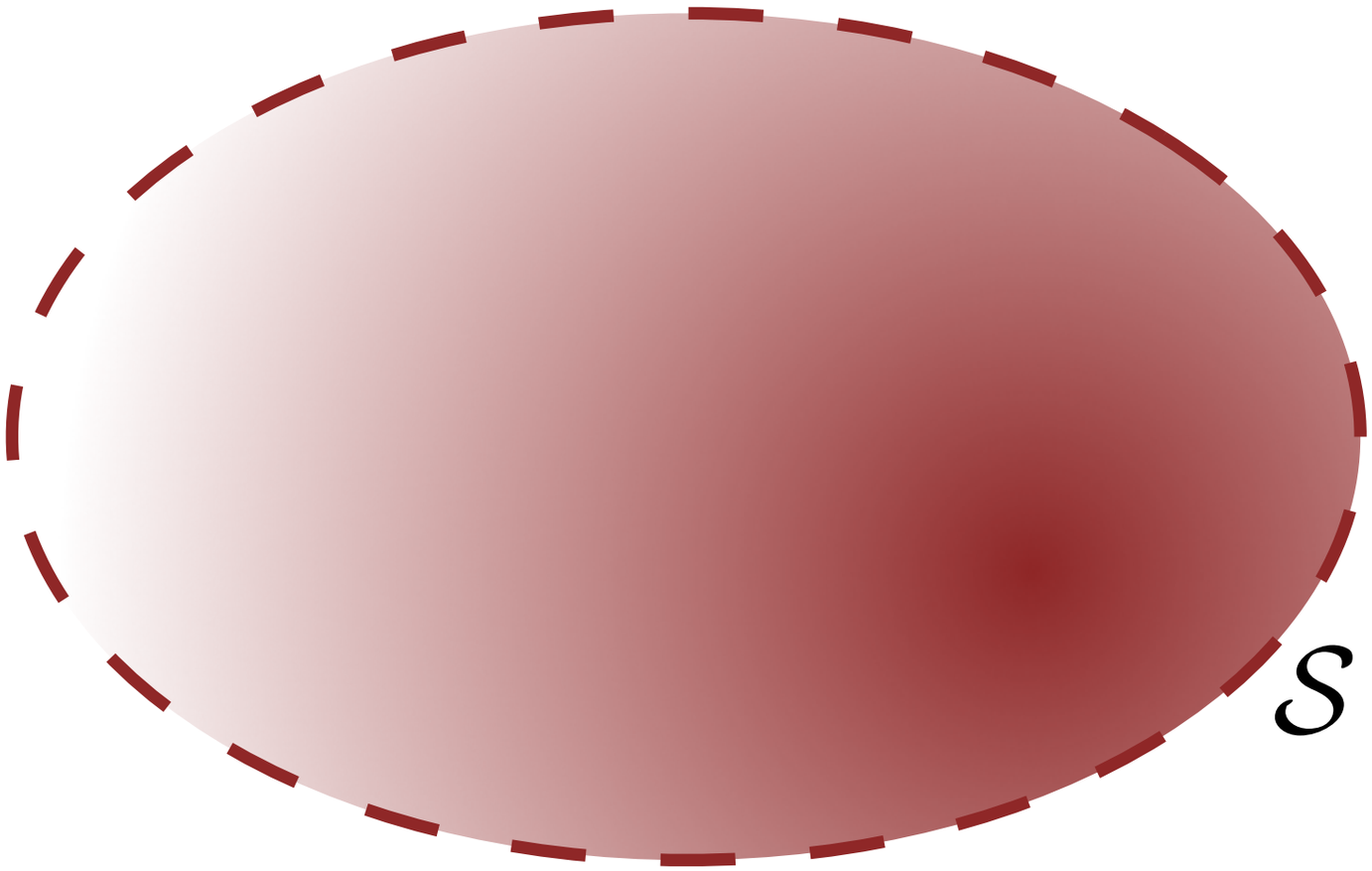

Uncertainty within the model is just
an application of *Bayes' Theorem*.

$$\pi_S(\theta|\widetilde{\mathcal{D}}) = \frac{\pi_S(\widetilde{\mathcal{D}}|\theta)\pi_S(\theta)}{\pi_S(\widetilde{\mathcal{D}})}$$

The *prior* incorporates any prior knowledge about the model space before the data are measured.

$$\pi_S(\theta|\widetilde{\mathcal{D}}) = \frac{\pi_S(\widetilde{\mathcal{D}}|\theta)\pi_S(\theta)}{\pi_S(\widetilde{\mathcal{D}})}$$

The *likelihood* is similar to the frequentist approach:
a generative model of the data.

$$\pi_S(\theta|\widetilde{\mathcal{D}}) = \frac{\pi_S(\widetilde{\mathcal{D}}|\theta)\pi_S(\theta)}{\pi_S(\widetilde{\mathcal{D}})}$$

The *posterior* incorporates both the information into the prior and the data into a final uncertainty in the model.

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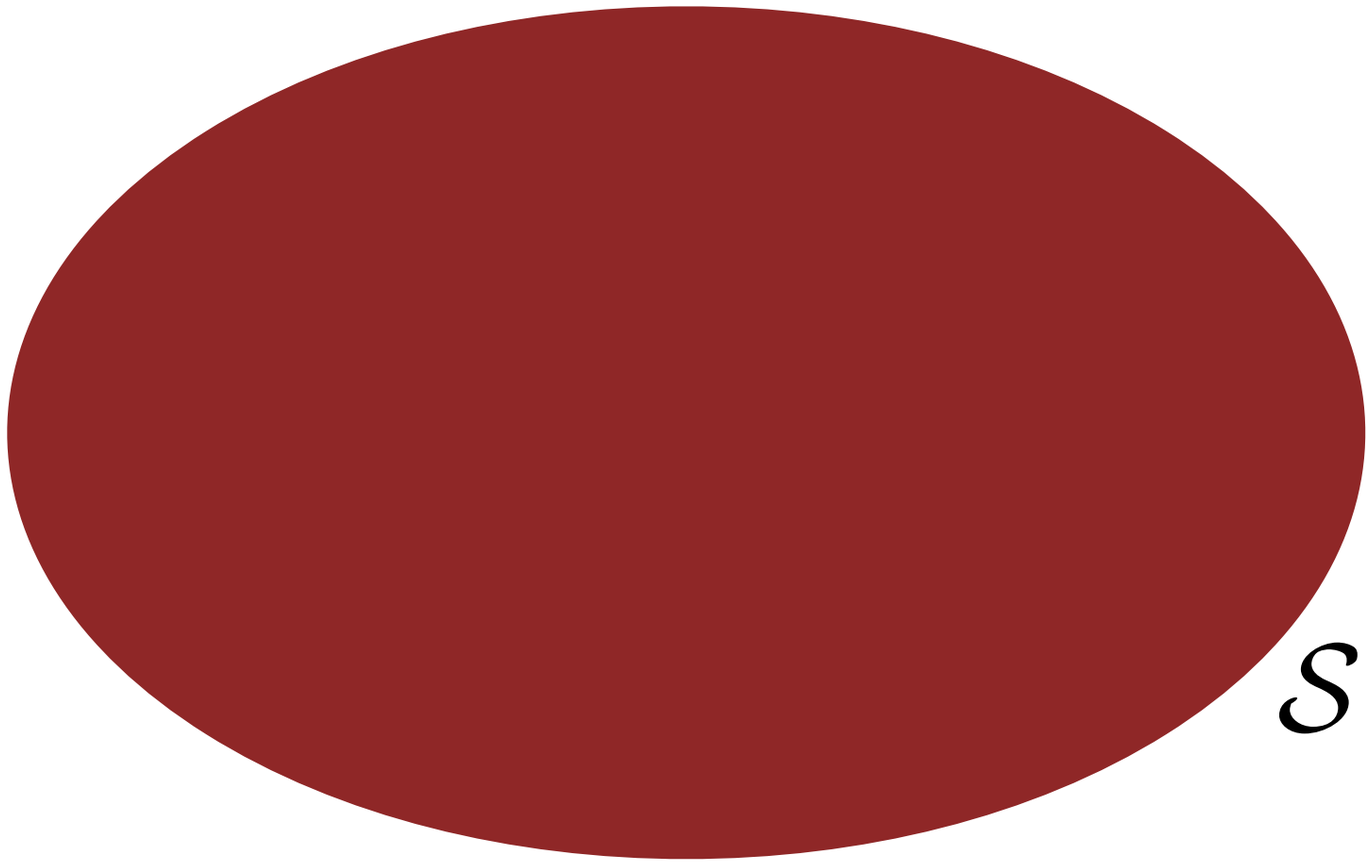
Conditioning on the measurement reduces our uncertainty amongst the data generation processes.



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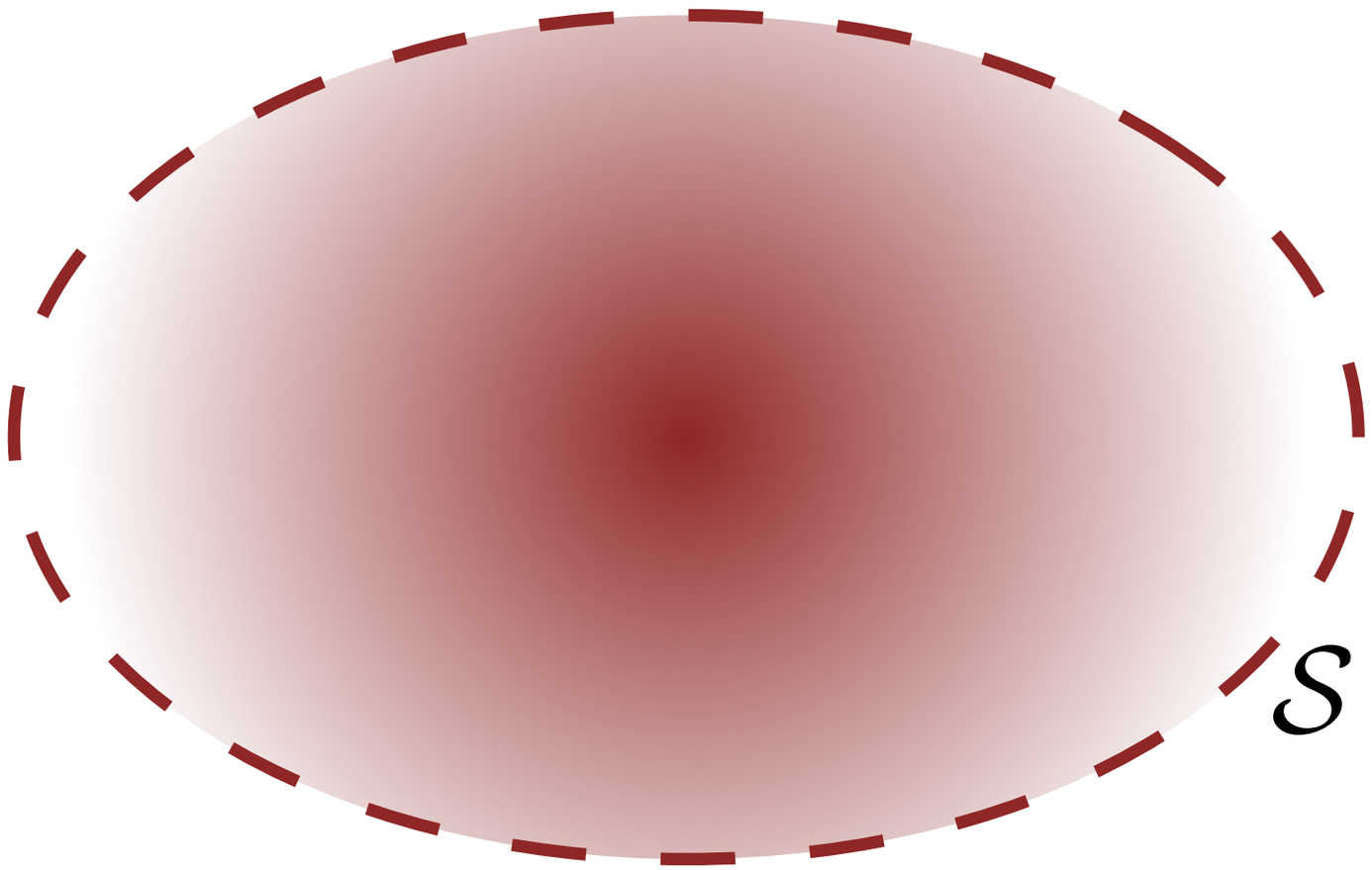

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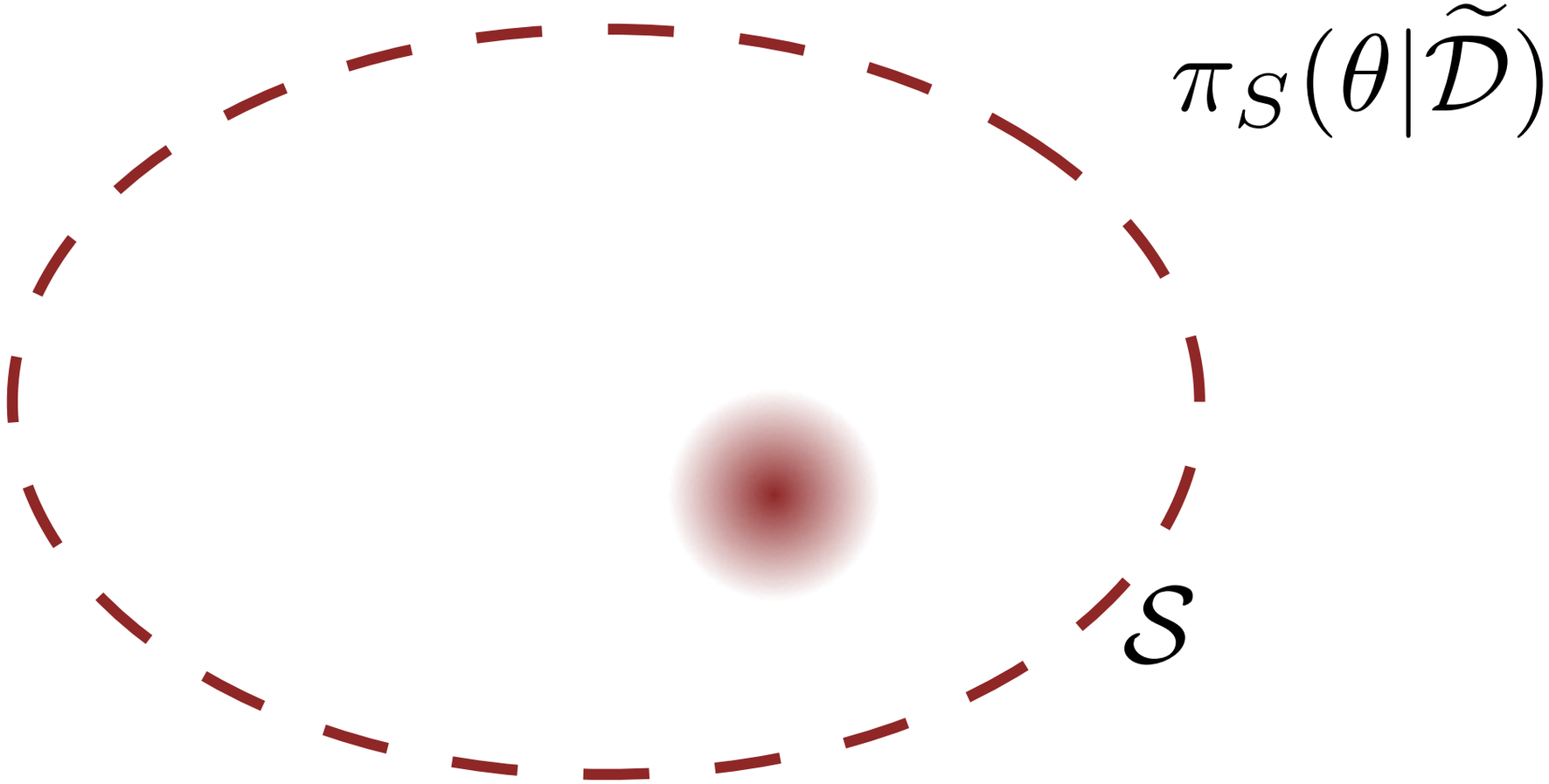

From a Bayesian perspective, all inferential questions are answered by expectations.

$$\mathbb{E}[f] = \int \mathrm{d}\theta \, \pi_S(\theta|\widetilde{\mathcal{D}}) f(\theta)$$

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$$U(A) = \int \mathrm{d}\theta \, \pi_S(\theta|\widetilde{\mathcal{D}}) \, U(A, \theta)$$

Expectations also allow us to incorporate systematic effects via *marginalization*.

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$$\pi_S(\theta_2, \ldots, \theta_n|\widetilde{\mathcal{D}}) = \int \mathrm{d}\theta_1 \, \pi_S(\theta_1, \theta_2, \ldots, \theta_n|\widetilde{\mathcal{D}})$$

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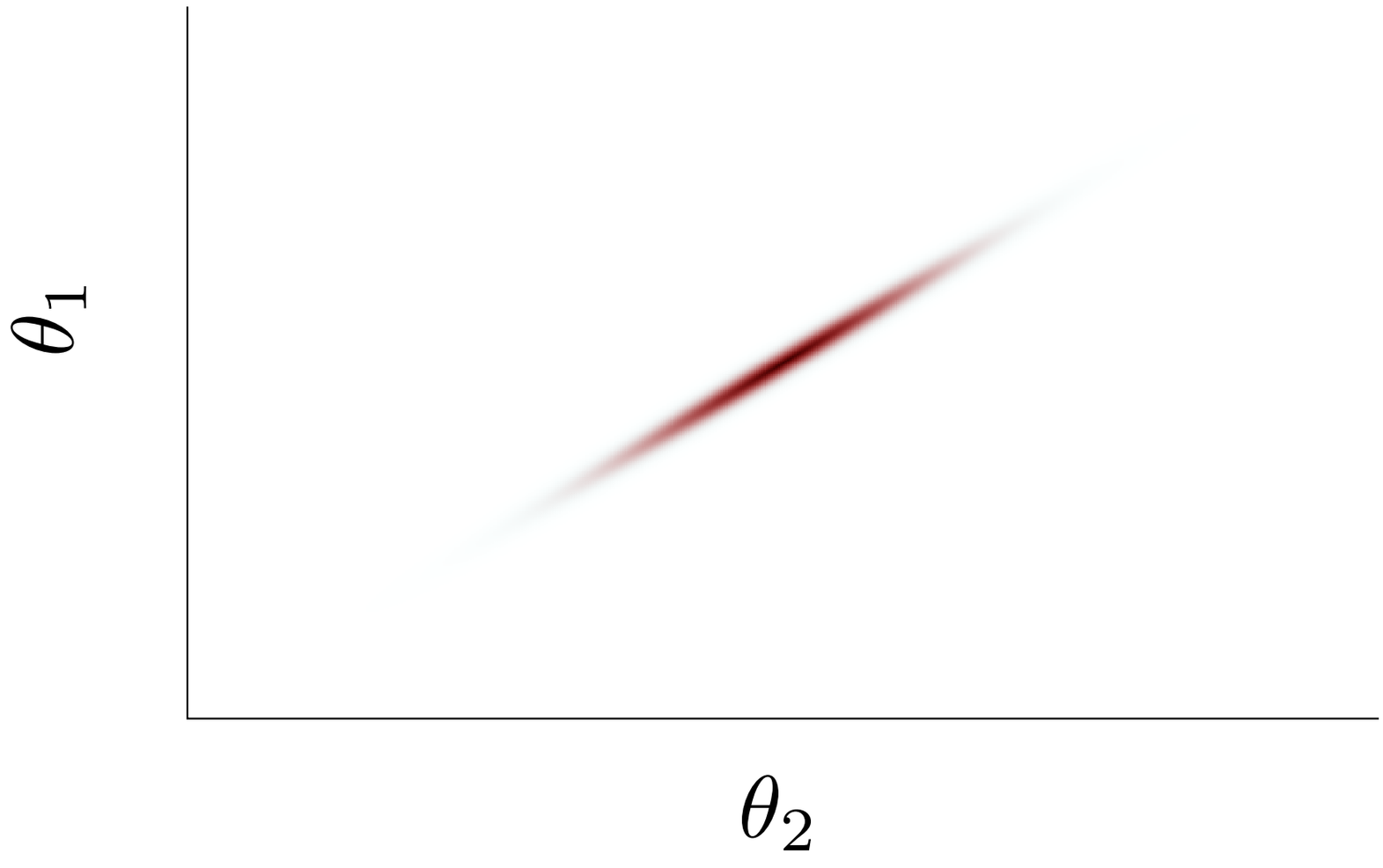

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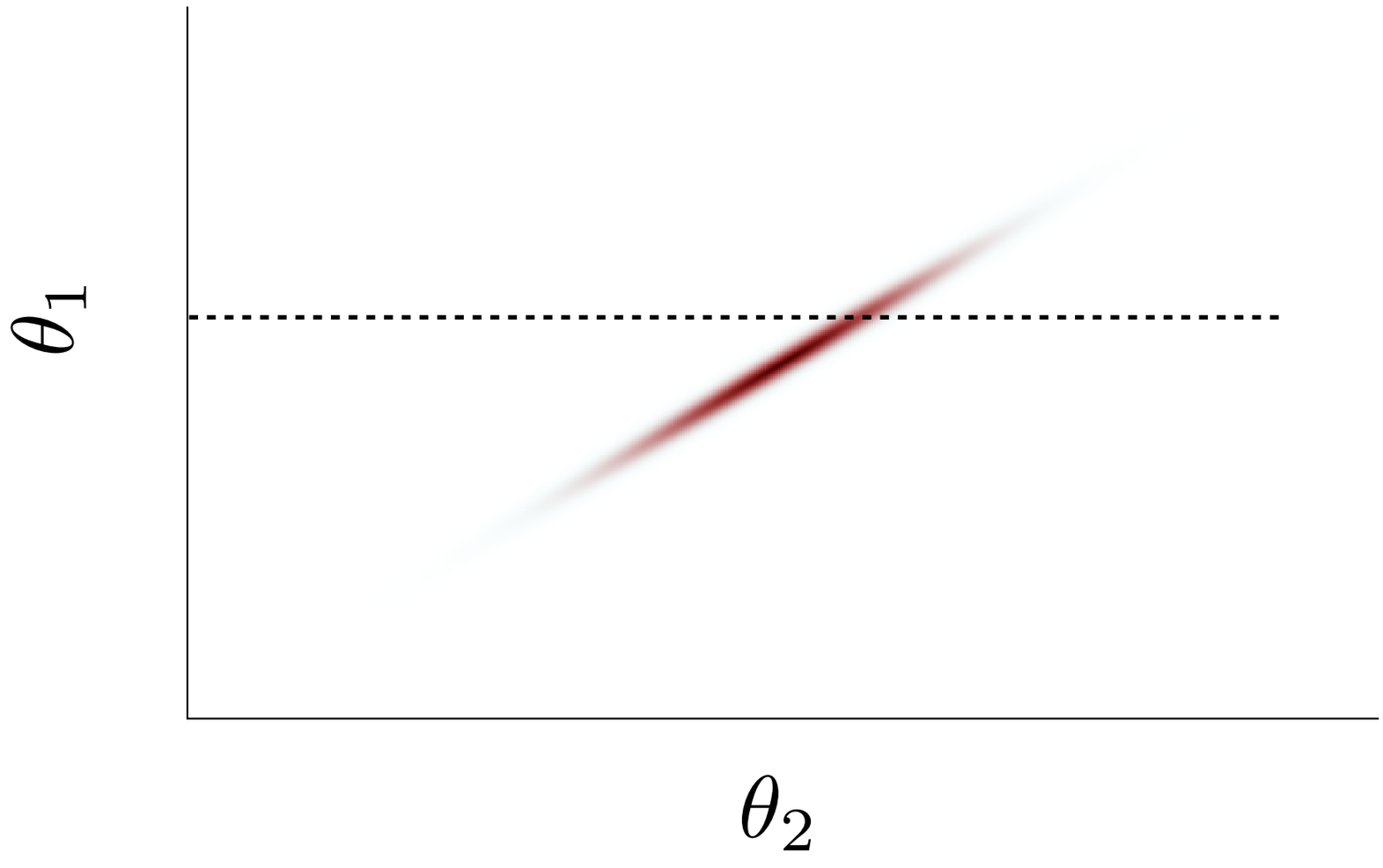

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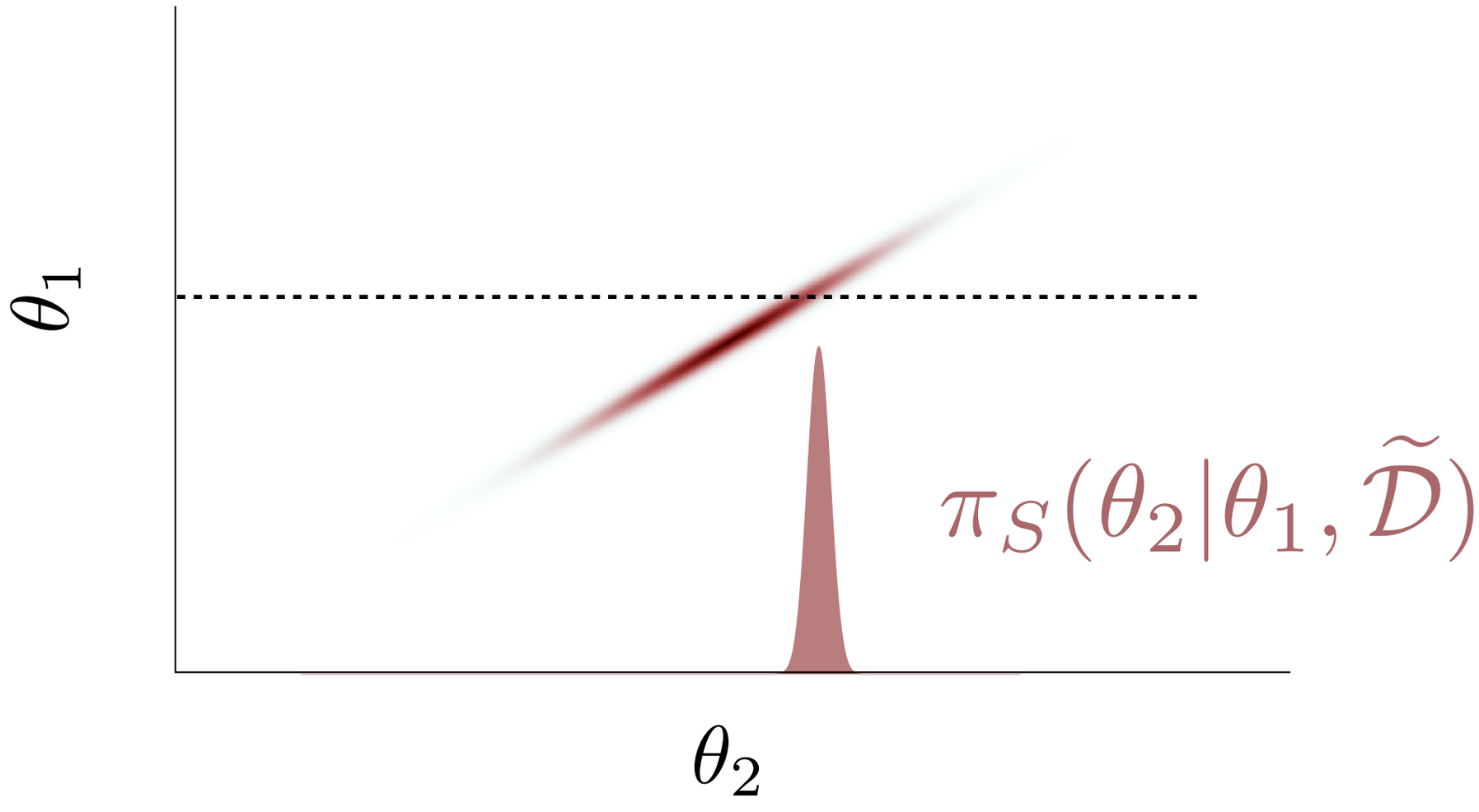

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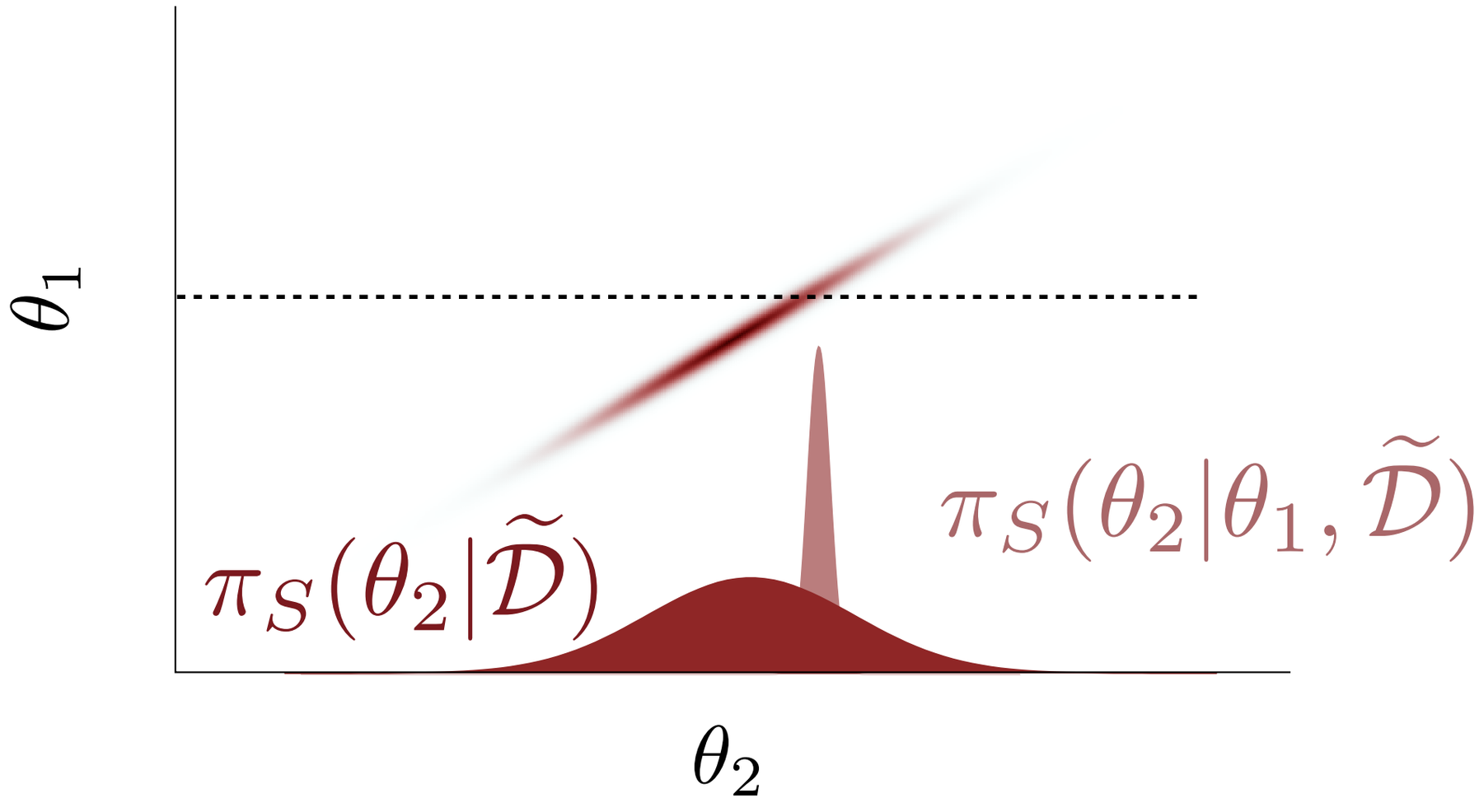
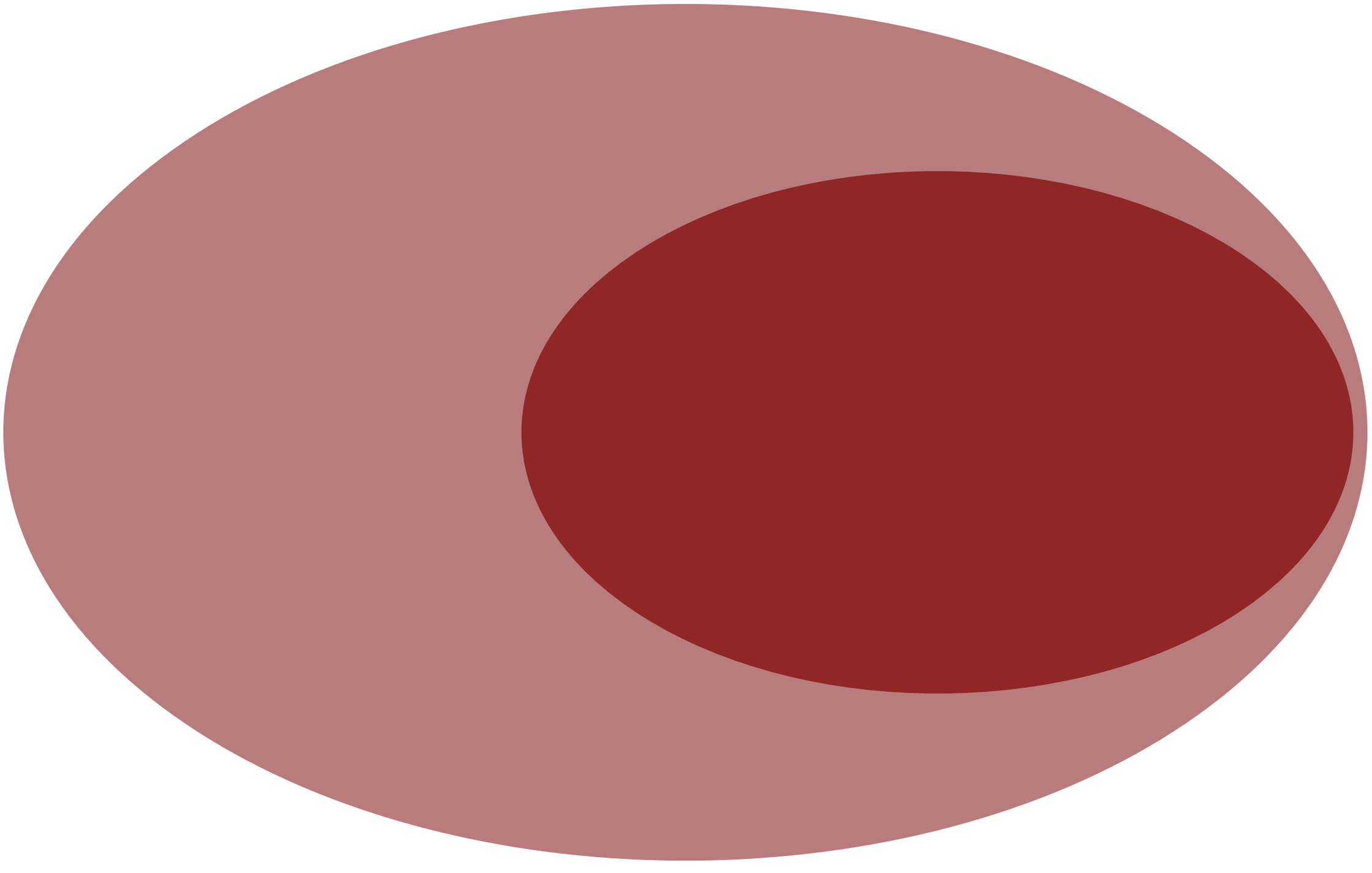
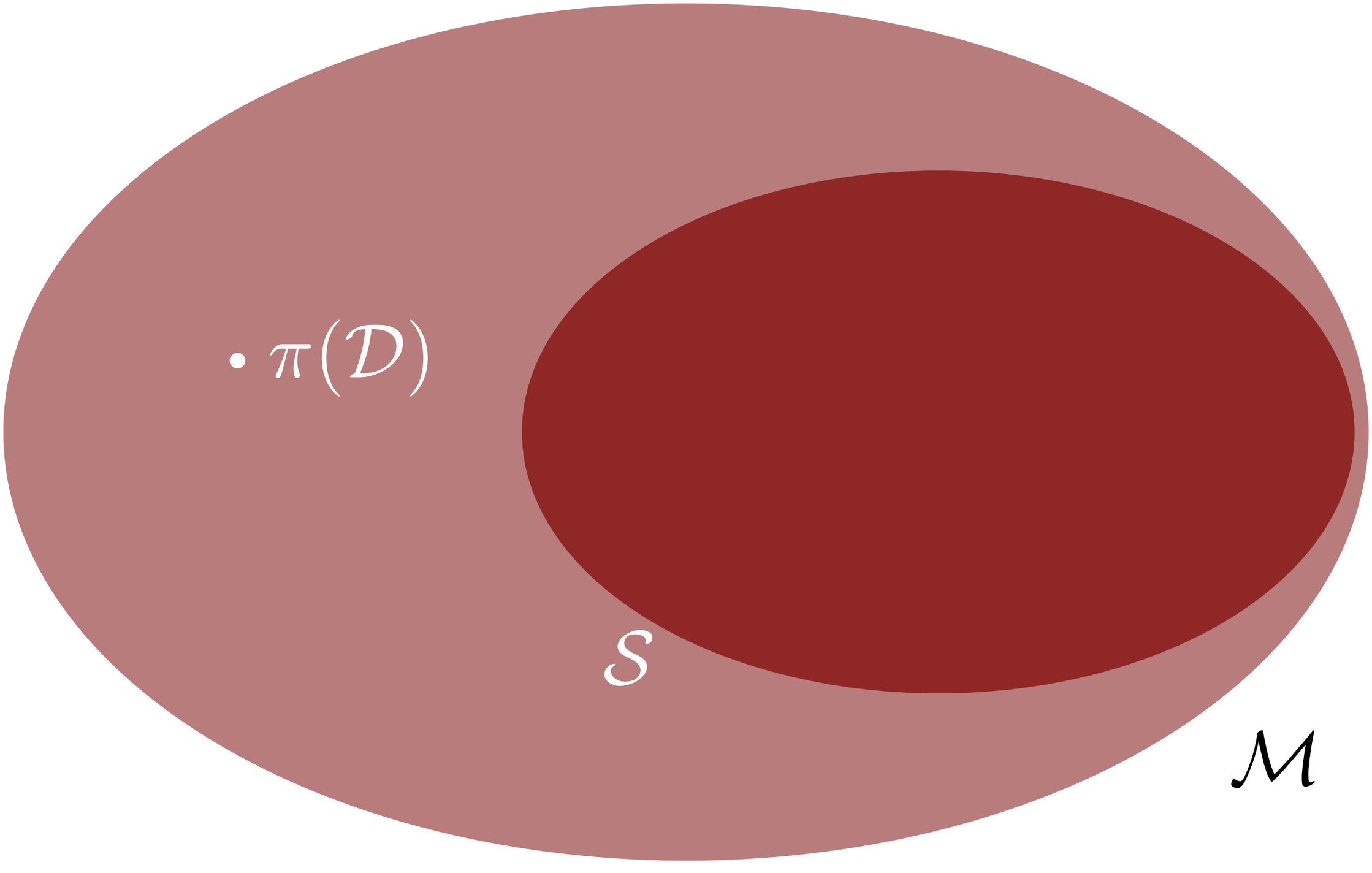


Predictive Model Comparison



The only way to test the assumptions themselves
is by considering *predictive performance*.



The average data generation process over the entire model is given by the *posterior predictive distribution*.

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$$\pi_S(\mathcal{D}|\widetilde{\mathcal{D}}) = \int \mathrm{d}\theta\, \pi_S(\mathcal{D}|\theta)\pi_S(\theta|\widetilde{\mathcal{D}})$$

Comparing the posterior predictive distribution to the true data generation process tests our assumptions.

$$\pi_S(\mathcal{D}|\mathcal{D}) \quad \overset{?}{\approx} \quad \pi(\mathcal{D})$$

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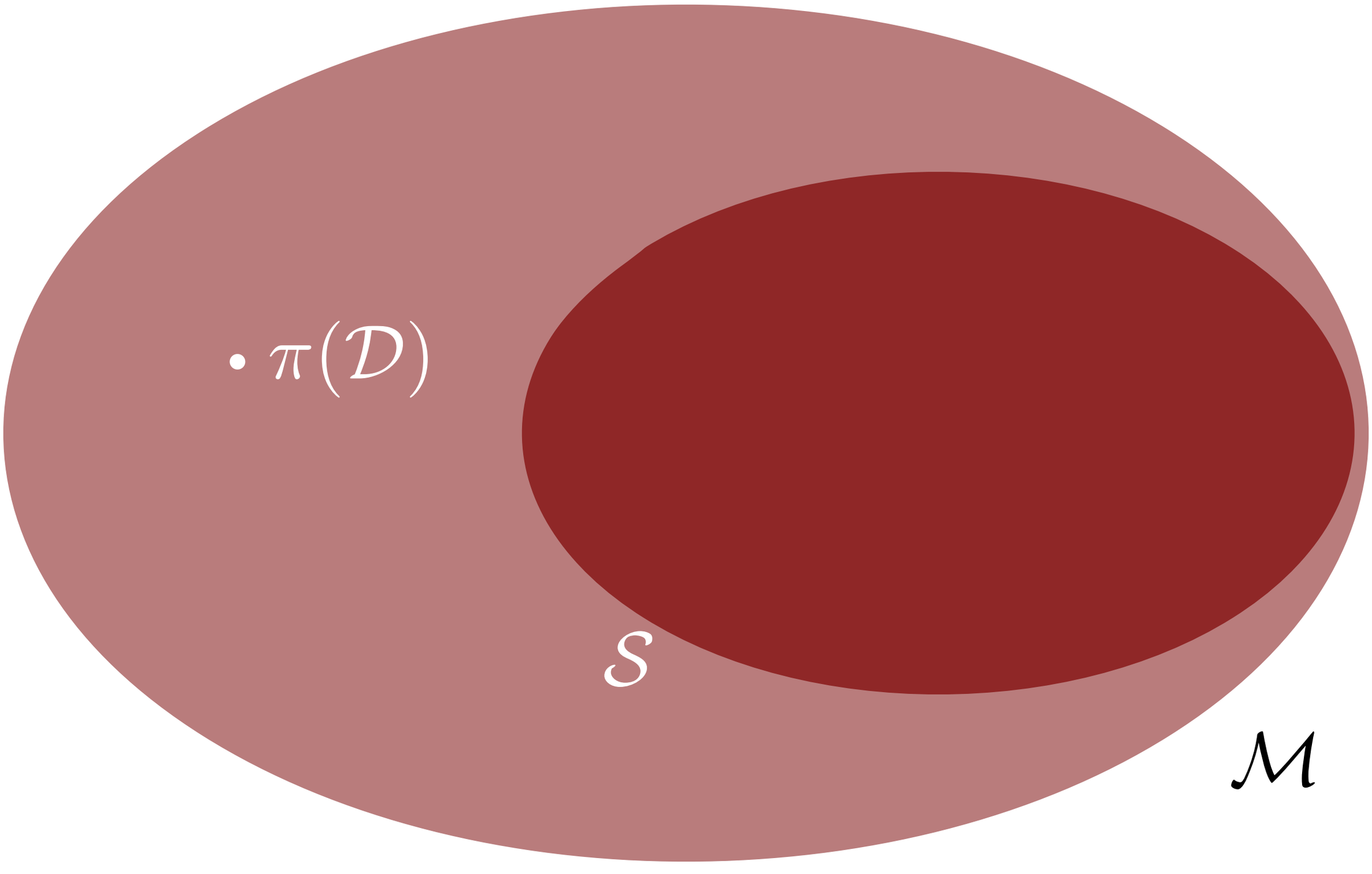
When the small world does not contain the latent data generating process our models will, in general, misfit.



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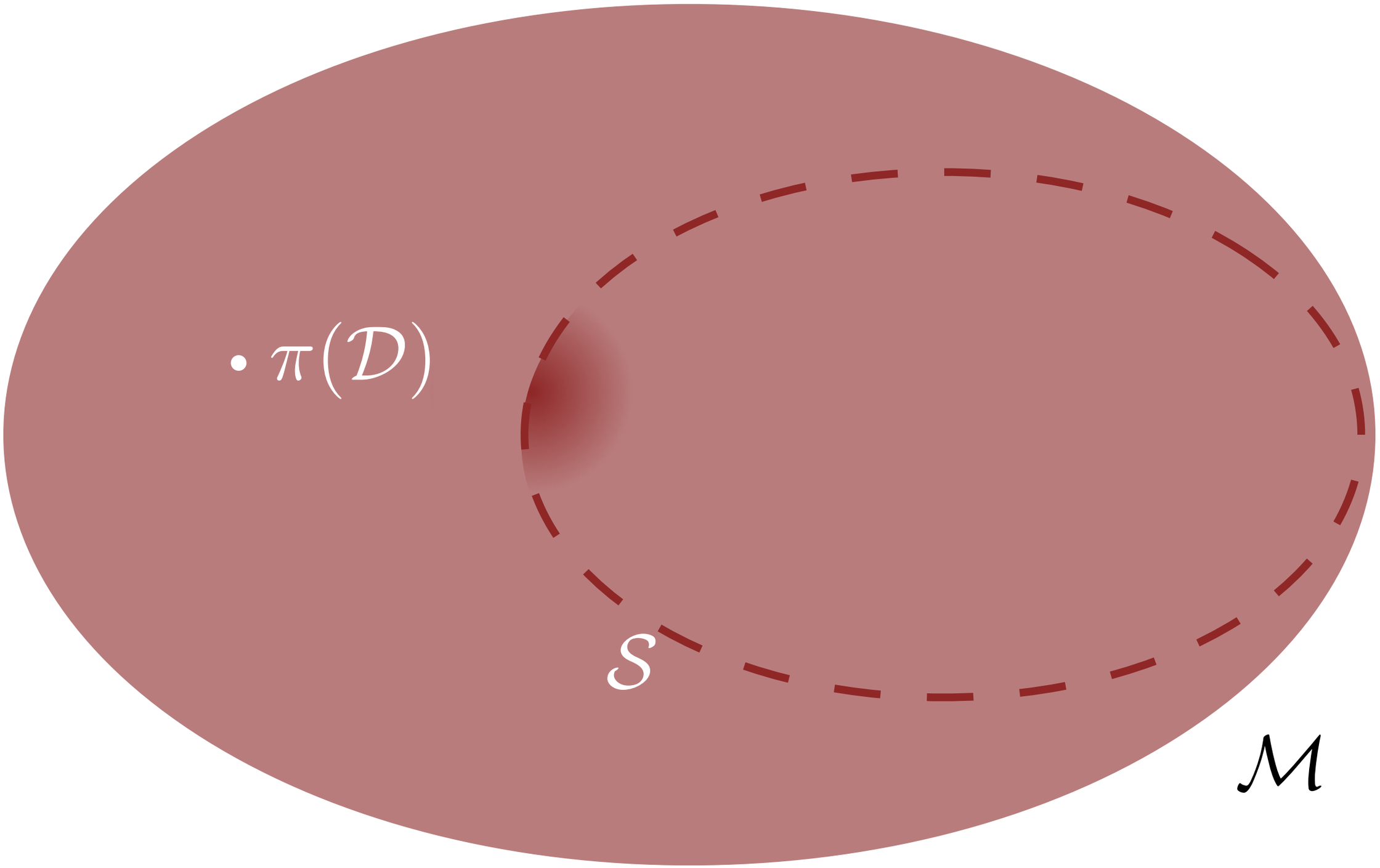
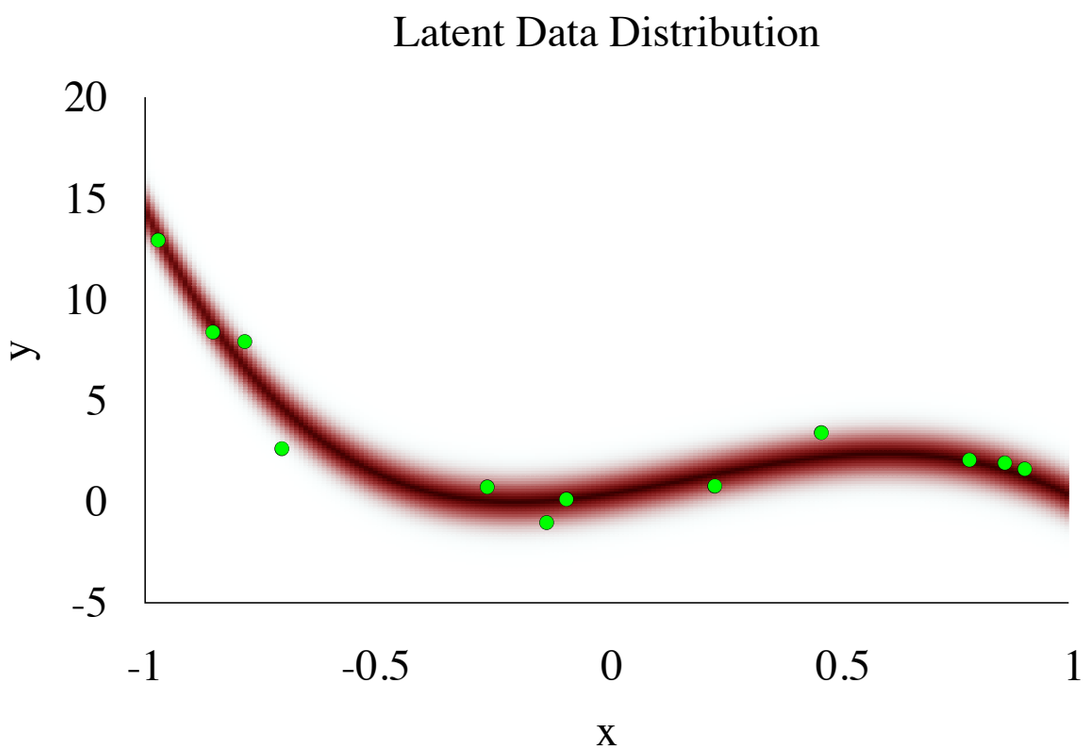
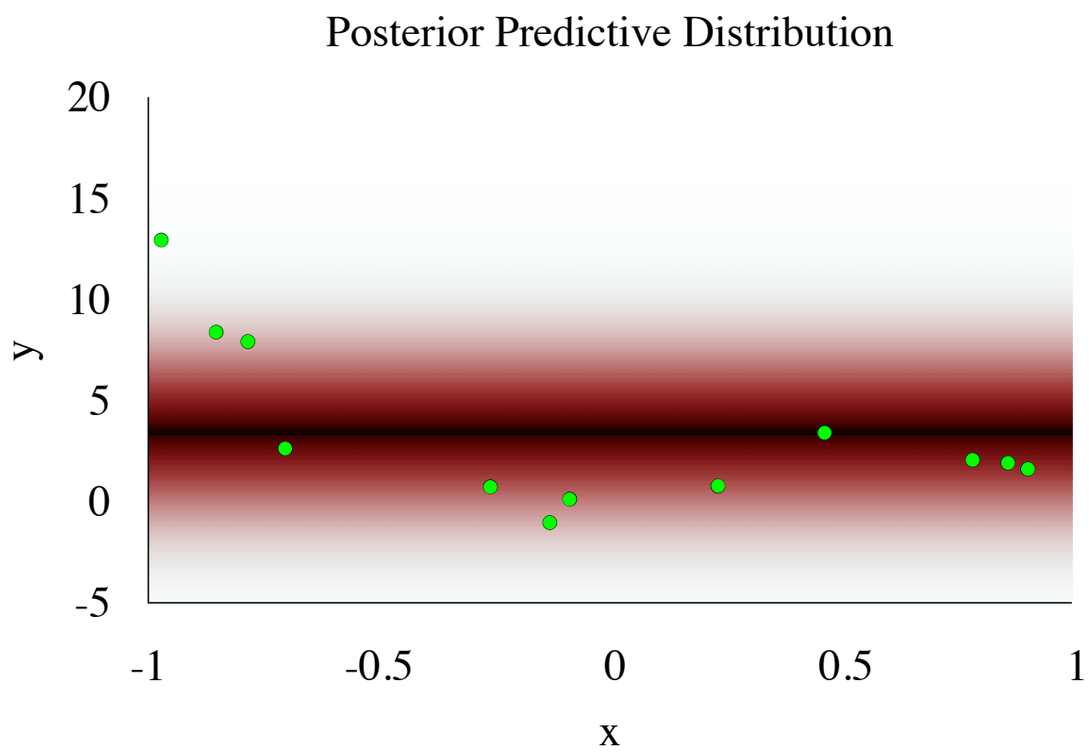
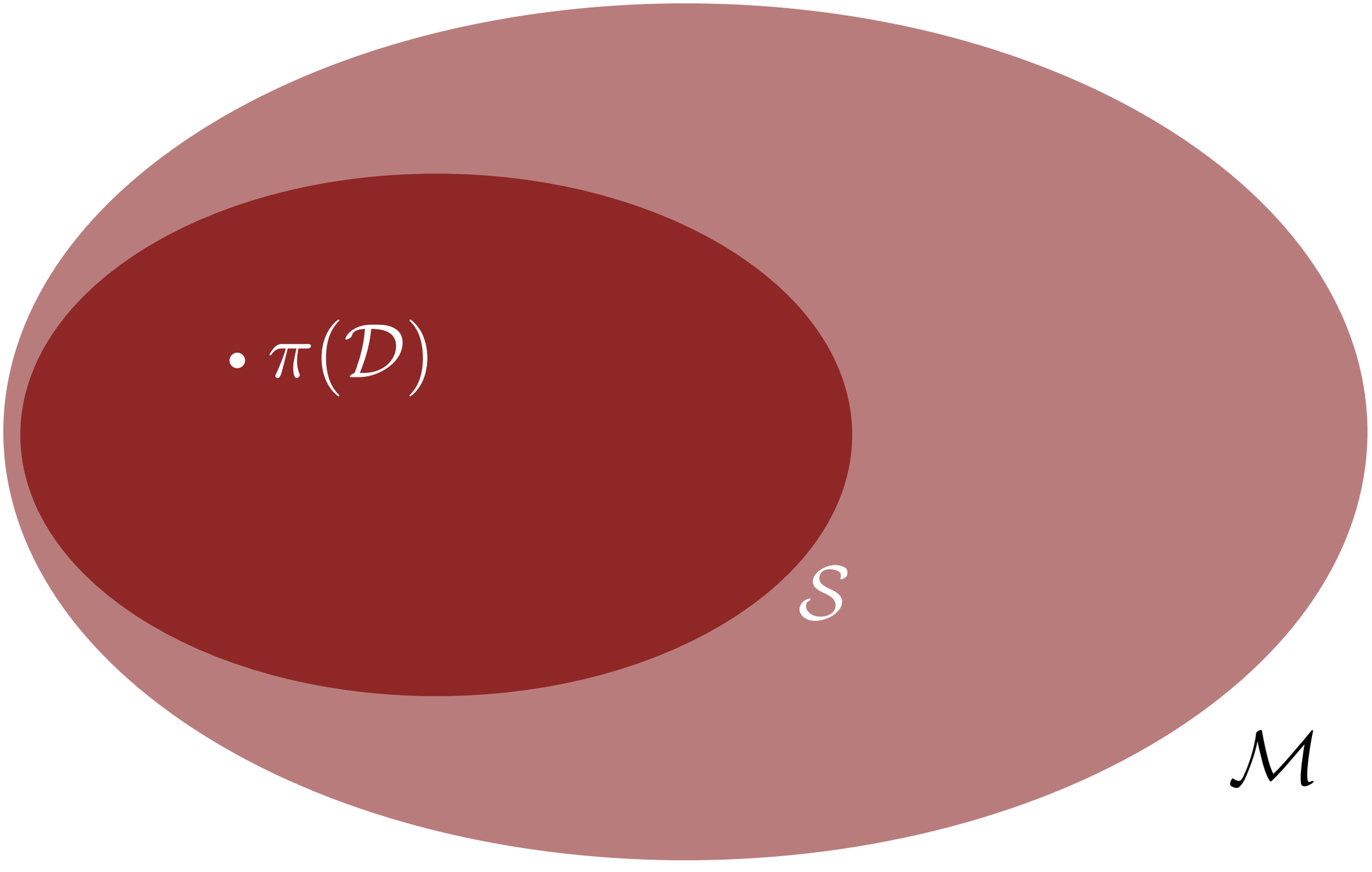


Fortunately, misfit results in tension between predictive distributions and measurements.





Even if the small world does contain the latent data generating process, however, our models can still overfit.



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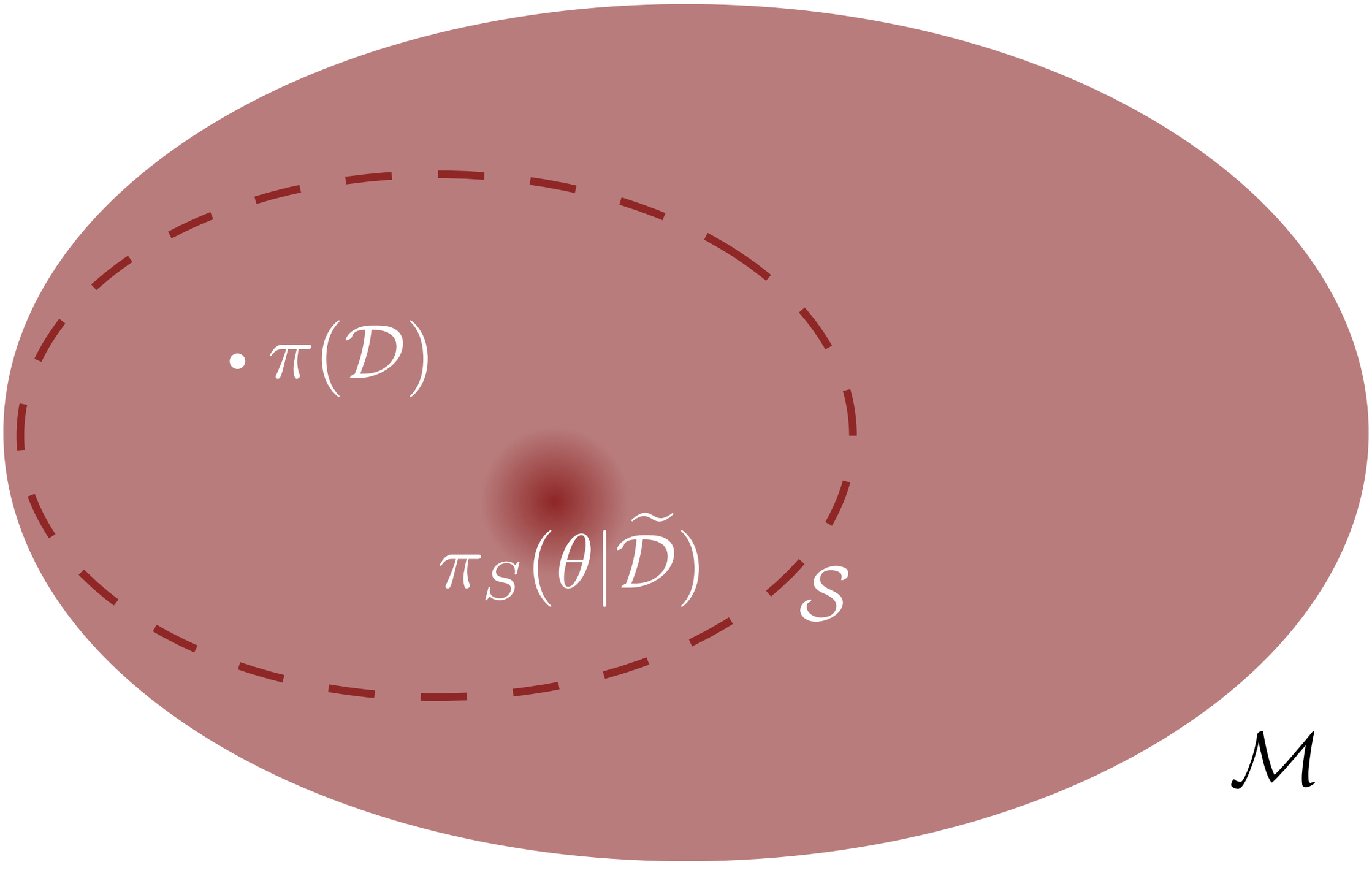
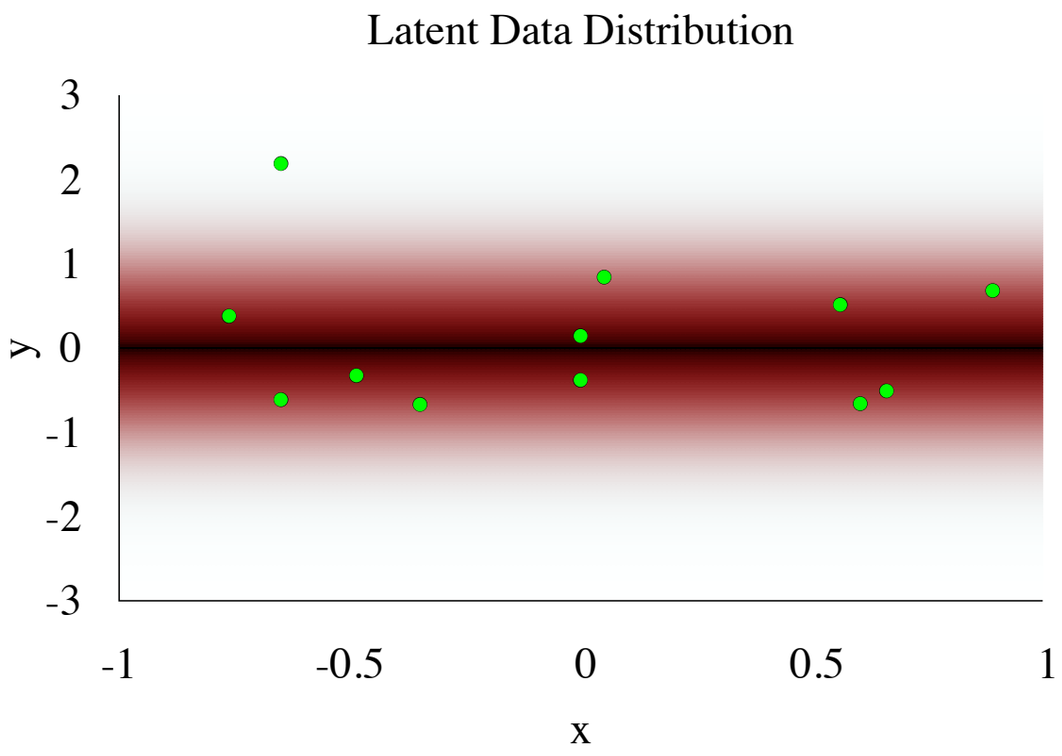
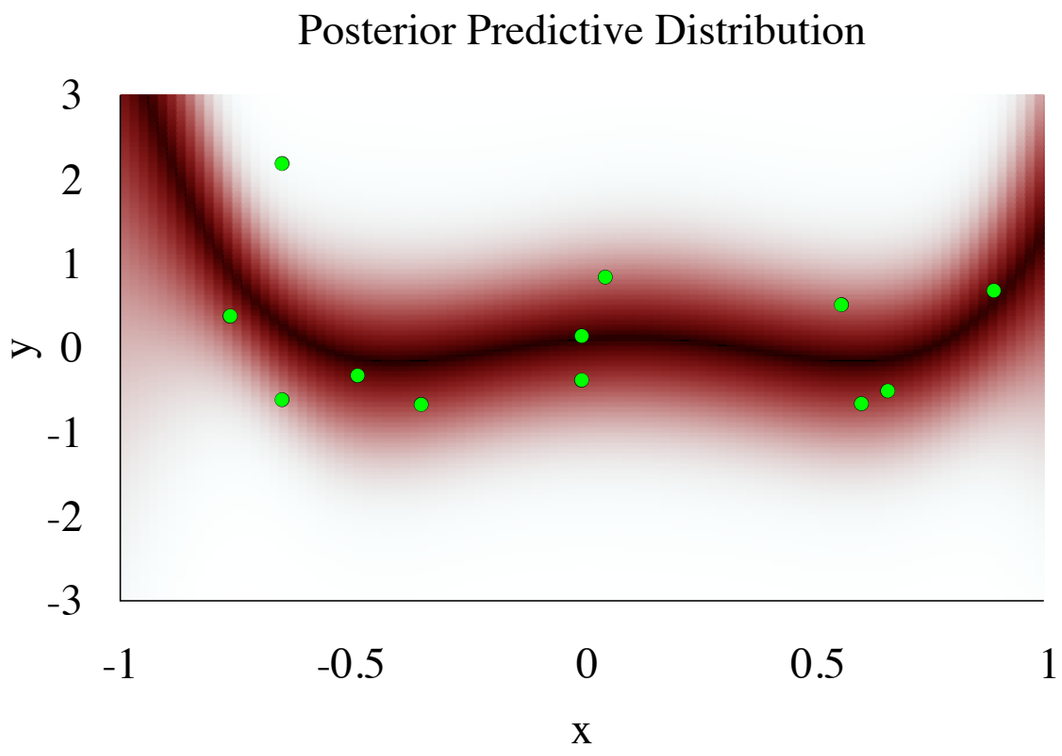


As with misfit, overfitting manifests as tension between predictive distributions and measurements.





Posterior predictive checks visually compare the predictive distribution to the measurement.

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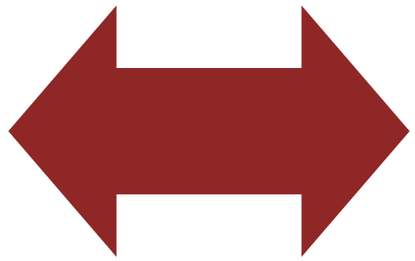

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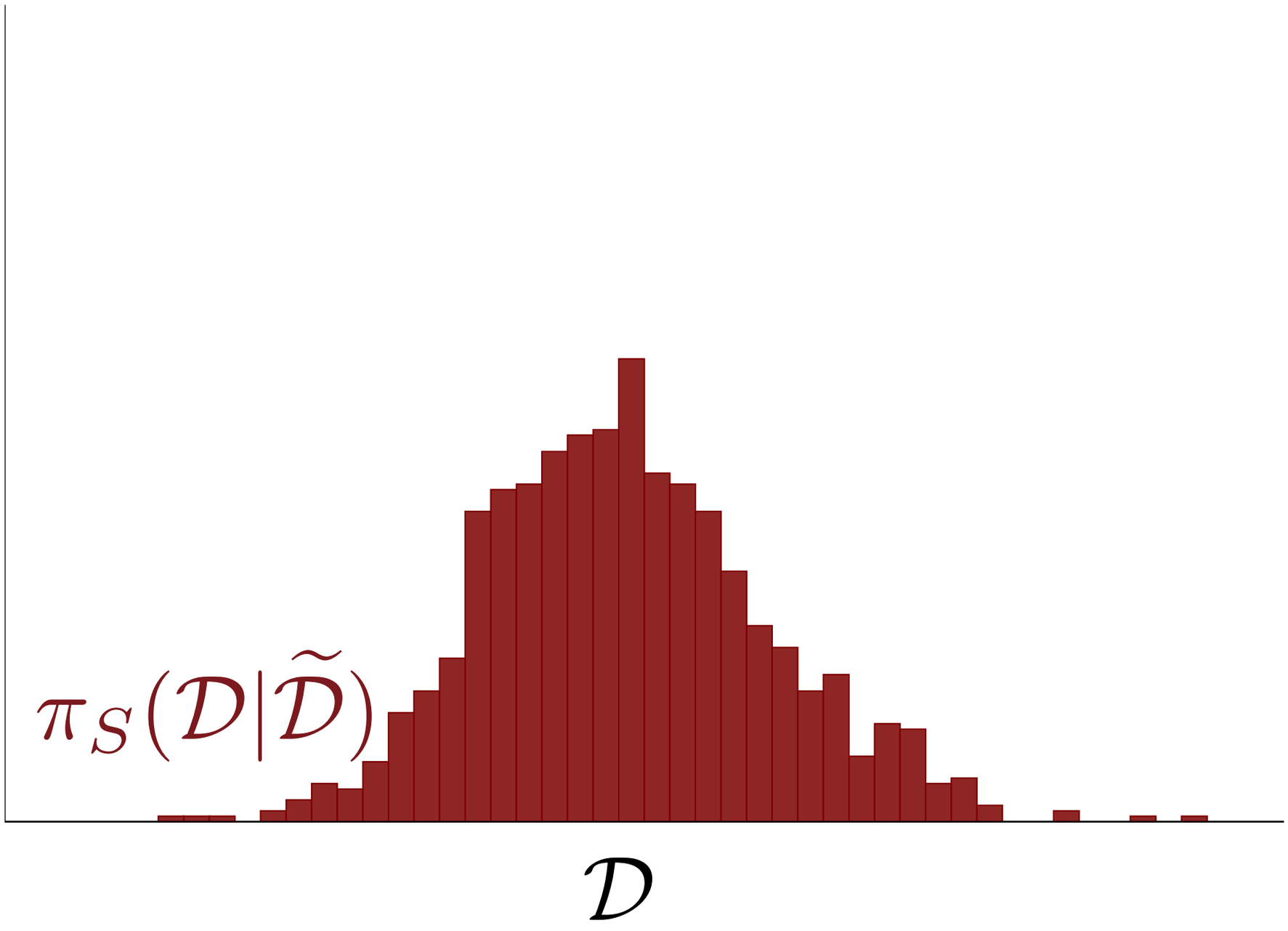
Firstly we can check to see how consistent the measurement is with the inferred predictive distribution.



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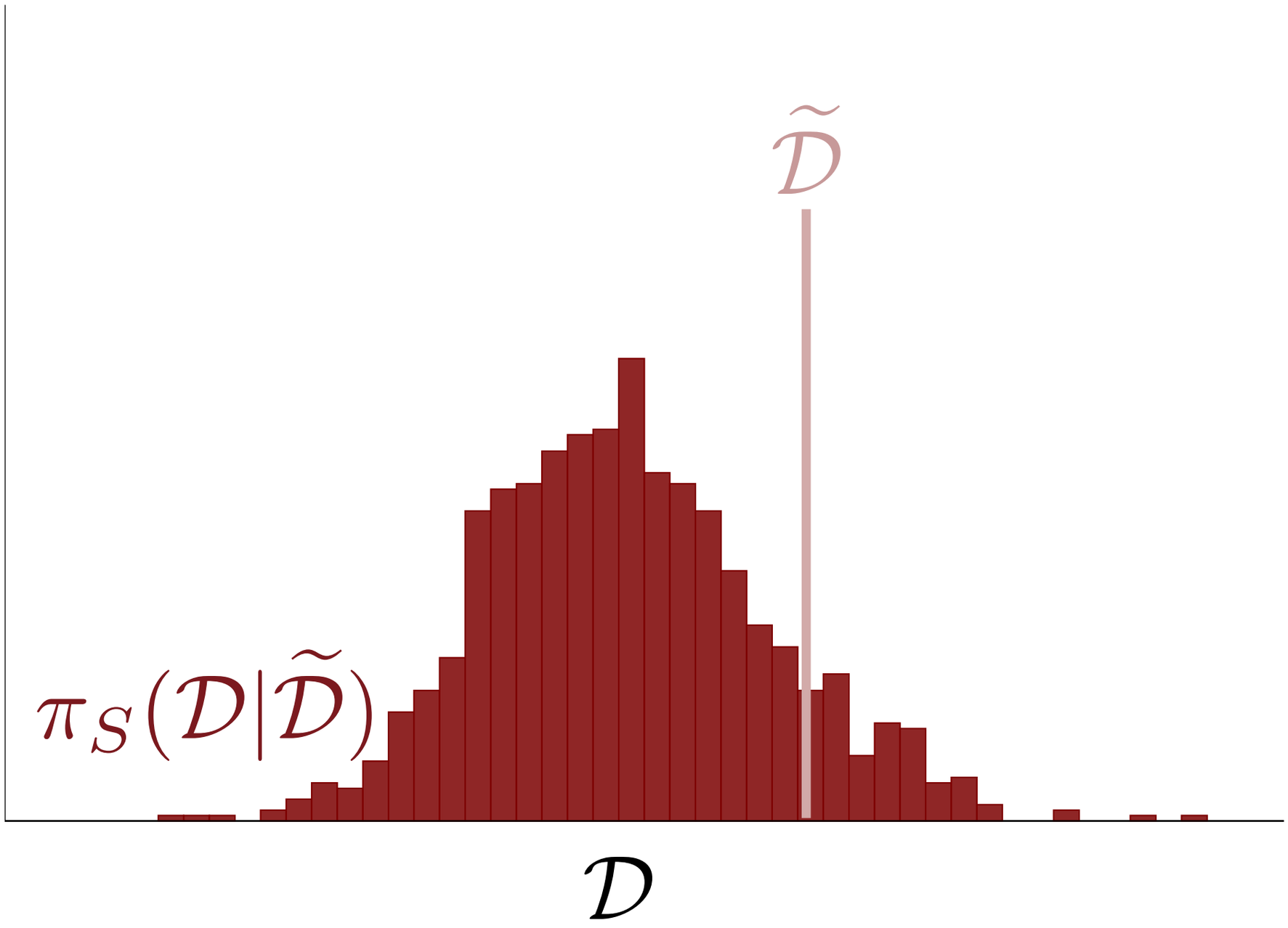

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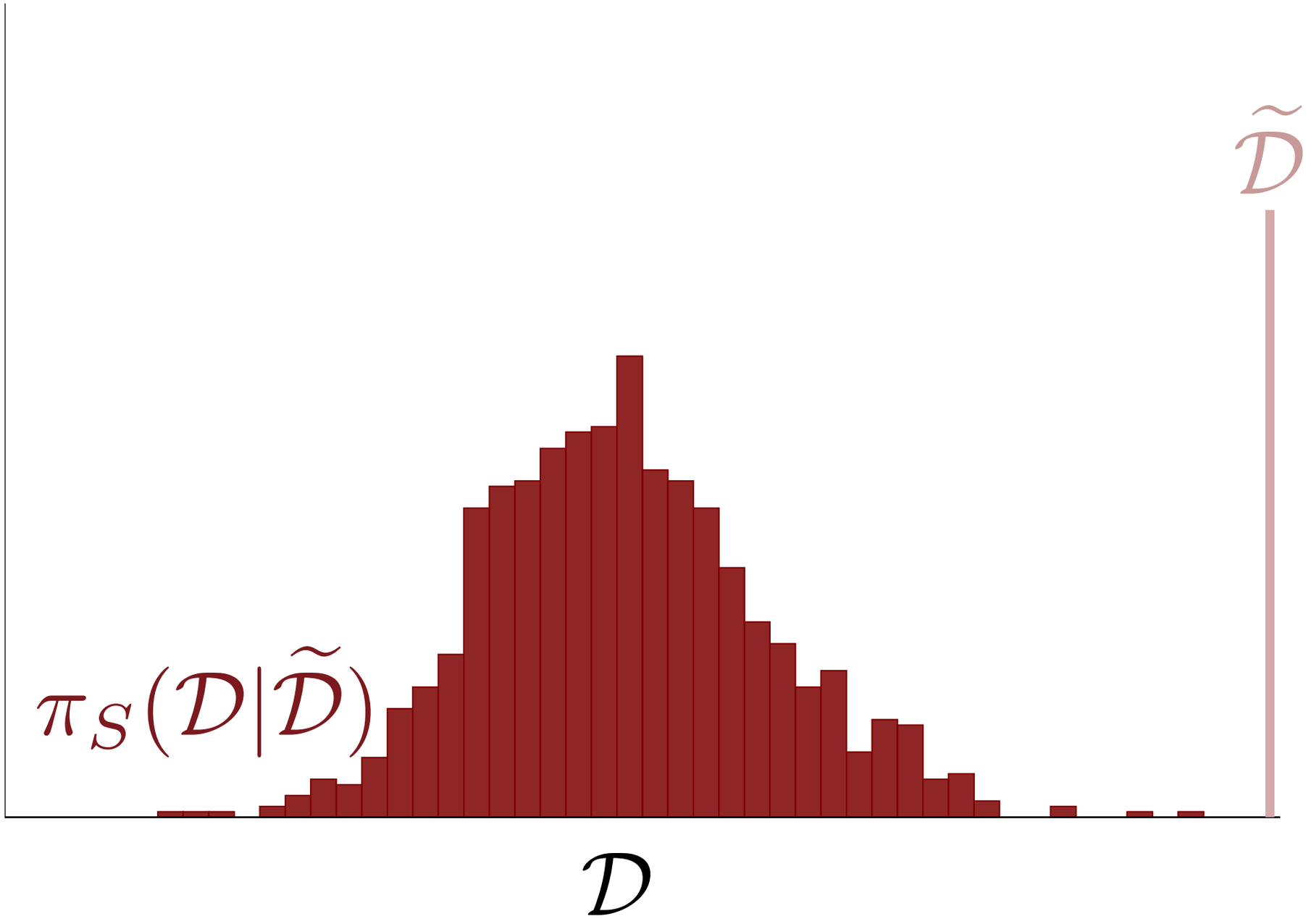
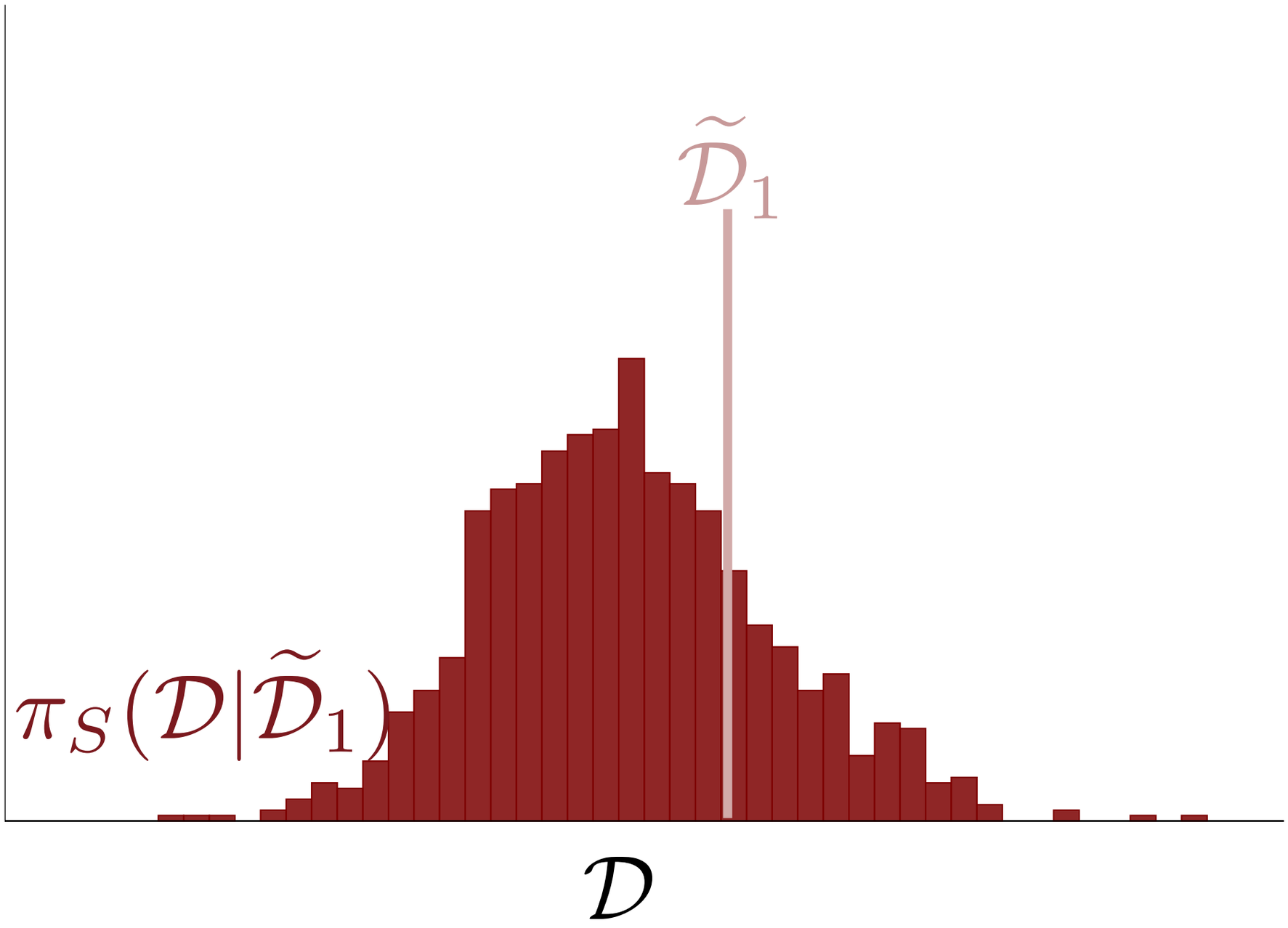


Similarly, we can check for overfitting by comparing held-out or partitioned measurements.



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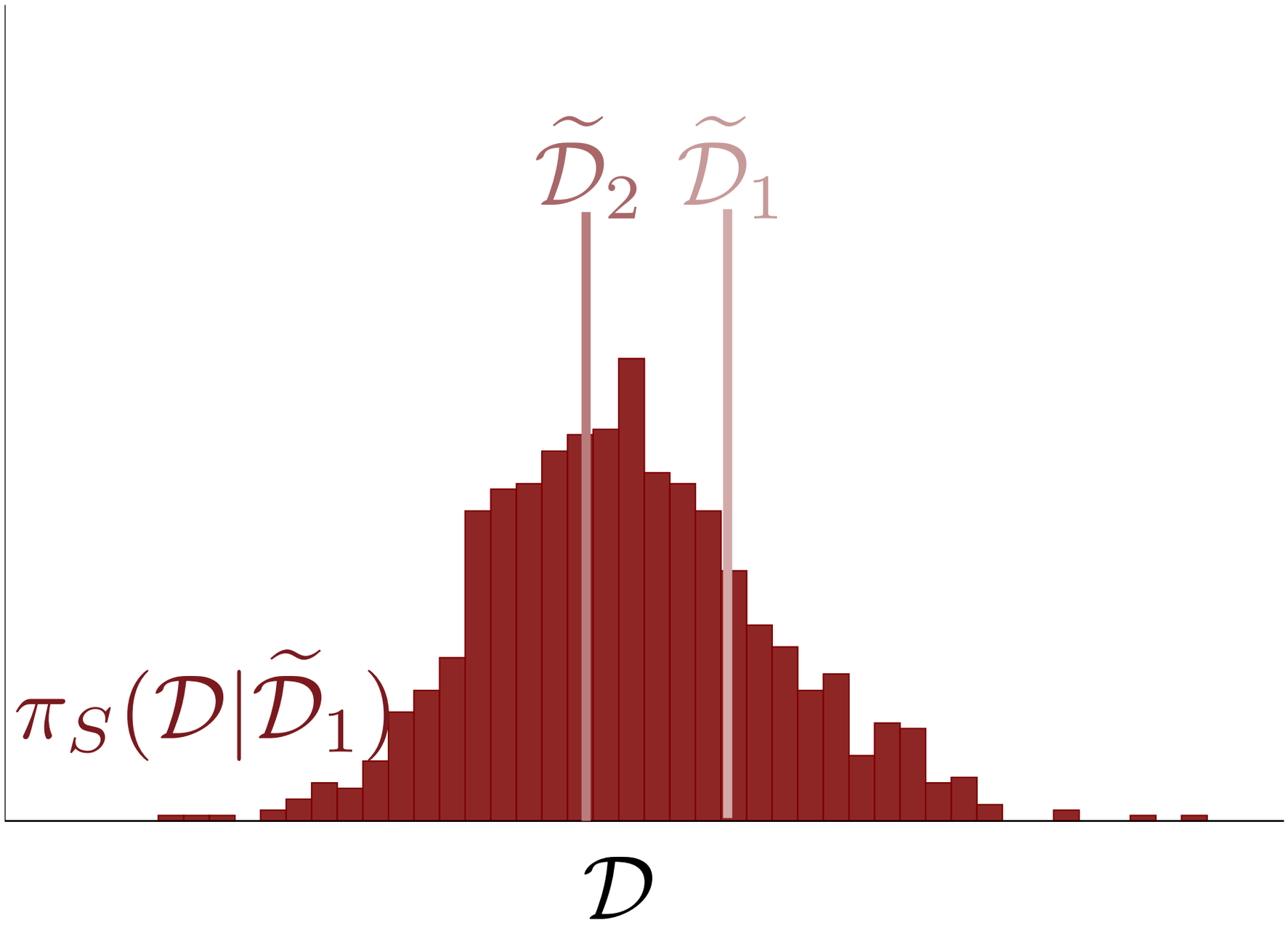

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