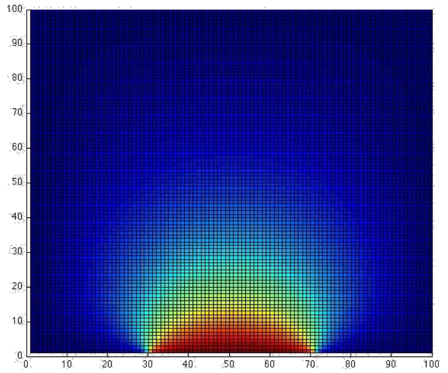

# Lecture 02:
# Linear Algebra Refresher

CS 111: Intro to Computational Science

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# Recall: Gaussian Elimination…

- In the form of **Ax = b**

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \xleftrightarrow{\textit{equivalent to}} \begin{aligned} 2x - y &= 0 \\ -x + 2y &= 3 \end{aligned}$$

***Multiplying*** *an equation in a system with a scalar gives you a new equation that is also "true" about the system.*

***Adding 2*** *equations together in a system does the same also!*

- Using **Gaussian Elimination**, we take advantage of certain Algebra rules:

$$\begin{array}{c} row_0 \\ row_1 \end{array} \left[\begin{array}{cc|c} 2 & -1 & 0 \\ -1 & 2 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} -1 & 2 & 3 \\ 2 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} -1 & 2 & 3 \\ 0 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

*Augmented Form* — *Switched rows* — *$row_1 = 1/3 x row_0 + 2/3 x row_1$* — *$row_0 = -row_0 + 2x row_1$*

- So, this means that our system can be re-written as: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

*SO:*
**x = 1, y = 2**

# Another Example

Example: 3 eq., 3 unknowns

$$\begin{cases} x_1 + 2x_3 = 1 \\ 2x_1 + 2x_2 = 1 \\ 3x_1 + 2x_2 + x_3 = 1 \end{cases}$$

Solve for vector **x** in **Ax** = **b** using Gaussian Elimination.

<u>ANS</u>: $\mathbf{x} = \begin{bmatrix} -1 \\ 1.5 \\ 1 \end{bmatrix}$

# Matrix Multiplication

Given:

$$\mathbf{U} = \begin{bmatrix} 2 & 7 & 1 & 8 \\ 0 & 2 & 8 & 1 \\ 0 & 0 & 8 & 2 \\ 0 & 0 & 0 & 8 \end{bmatrix} \quad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 1 \end{bmatrix}$$

**Find A = L.U** *(i.e. matrix multiplication)*

***Do you expect it to be the same as A = U.L?***

# Matrix Multiplication using L and U

A = L. U

*Lower triangle matrix* *Upper triangle matrix*

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 0 & 0.5 & 1 & 0 \\ -0.5 & -0.5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 7 & 1 & 8 \\ 0 & 2 & 8 & 1 \\ 0 & 0 & 8 & 2 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 7 & 1 & 8 \\ 1 & 5.5 & 8.5 & 5 \\ 0 & 1 & 12 & 2.5 \\ -1 & -4.5 & -4.5 & 3.5 \end{bmatrix}$$

*This is 1x row1 of U*

*This is ½ row 1 + 1 x row 2 of U*

*This is ½ row 2 + 1 x row 3 of U*

*This is -½ row 1 + -½ row 2 + 1 x row 4 of U*

# A = L.U *aka* LU Decomposition

- It is often useful to be able to factor *any* matrix **A** into an **L.U**
  - So we would be doing the *reverse* of that previous example…
  - Why do you *think* it is “useful”??!??! *(think like an engineer!!)*

- More generally, we can factor any matrix: **P.A = L.U**
  - **P** is called a “permutation matrix” (more on this later)

# Identity Matrix

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

What's **A**.**I**?

*They're both equal to **A***

What's **I**.**A**?

# Diagonal Matrix

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

What's **D.A**? *It's **A**, but the **rows** have each been multiplied by the same-position diagonal values of D*

What's **A.D**? *It's **A**, but the **columns** have each been multiplied by the same-position diagonal values of D*

# Permutation Matrices

P is like an identity matrix, but with ***rearranged rows/columns***

Example: a 3x3 **P** could be: $\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- Think of what happens if **A** = $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and we multiplied it with **P**

- If I multiply **P.A**: I rearrange the **rows** of A
- If I multiply **A.P**: I rearrange the **columns** of A

# Transpose Operation on Matrices

- $\mathbf{A}^T$: Transpose the rows in A into columns in A

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \Longrightarrow A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

2x3

3x2

- So, if A is a size $\mathbf{m}\times n$ matrix, then $A^T$ is $n\times\mathbf{m}$
- If $\mathbf{A} = \mathbf{A}^T$, then we say that A is a **<u>symmetrical matrix</u>**.
  - Symmetrical matrices have important properties (more on this later)…

# Determinant of a Square Matrix

- A scalar value that has properties of the linear transformation
- Use: **det(A)** or **||A||**
- In a 2x2 matrix:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies \det(\mathbf{A}) = ad - bc$$

- In a 3x3 matrix and larger, it becomes a more complicated formula (see board)

***<u>The 2 VERY important rules to remember:</u>***

1. If det(**A**) != 0 $\implies$ **A** is ***invertible*** $\implies$ $\mathbf{A}^{-1}$ <u>exists</u>, where $\mathbf{A}.\mathbf{A}^{-1} = \mathbf{A}^{-1}.\mathbf{A} = \mathbf{I}$ (the identity matrix)
2. If det(**A**) = 0 $\implies$ **A** is ***singular*** $\implies$ $\mathbf{A}^{-1}$ <u>does not exist</u> (i.e. **A** is not invertible!)

# Consider these Matrices…

$$M1 = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 3 & -3 & 6 \end{bmatrix}$$

*$3^{rd}$ row = 3 x $1^{st}$ row!*
*det(M1) = 1(6-0) –(-1)(12-0) + 2(-6-3) = 6+12-18 = 0*
*M1 is singular!*

*What is "it" about them that makes them "special"?*

$$M2 = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 4 & 1 \\ 4 & 10 & 3 \end{bmatrix}$$

*$2^{nd}$ column = $1^{st}$ column + 2 x $3^{rd}$ column!*
*det(M2) = 1(12-10) – (1)(6-4) + 0 = 2 – 2 = 0*
*M2 is singular!*

# Invertible Matrices

- If and only if (*iff*) **det(A)** **!=** **0**, can we say that a matrix $\mathbf{A}^{-1}$ exists (i.e. that matrix **A** is "invertible"), such that: $A.A^{-1} = A^{-1}.A = I$

- Inverse matrix properties:

$$(A^{-1})^{-1} = A \qquad \det(A^{-1}) = (\det(A))^{-1}$$

$$(A^{T})^{-1} = (A^{-1})^{T} \qquad (AB)^{-1} = B^{-1}A^{-1}$$

# How to Calculate Invertible Matrices

- Consider **A** = $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} 0.667 & 0.333 \\ 0.333 & 0.667 \end{bmatrix}$$

- **det(A)** = 4 – 1 = 3 != 0 ➔ **A** is invertible!
- So what's $\mathbf{A}^{-1}$?

$$AA^{-1} = I \Rightarrow \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So, $2a - c = 1$ ; $-a + 2c = 0$ ➔ $3c = 1$ ➔ $c = 0.333$ ➔ $a = 0.667$

Also, $2b - d = 0$ ; $-b + 2d = 1$ ➔ $3d = 2$ ➔ $d = 0.667$ ➔ $b = 0.333$

# Your TO DOs!

- Are you all on Canvas, Piazza, and Gradescope?
- Go to lab/section tomorrow!
- New readings for you – See Canvas (under "Modules" → "**Week 2**")
- Start on your 1st assignment that's due by MONDAY at 11:59 PM

</LECTURE>