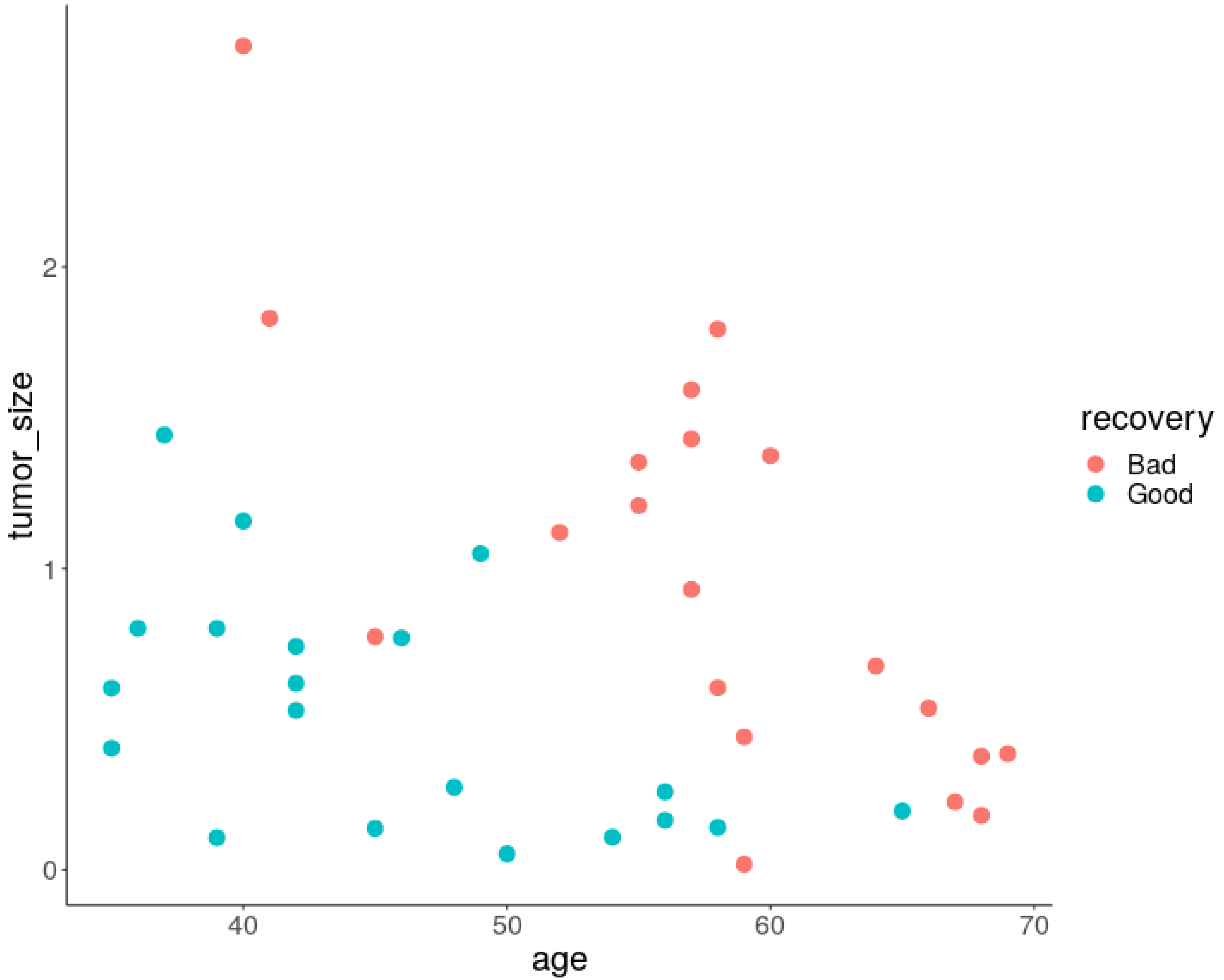
Statistical Learning

Juan C. Laria

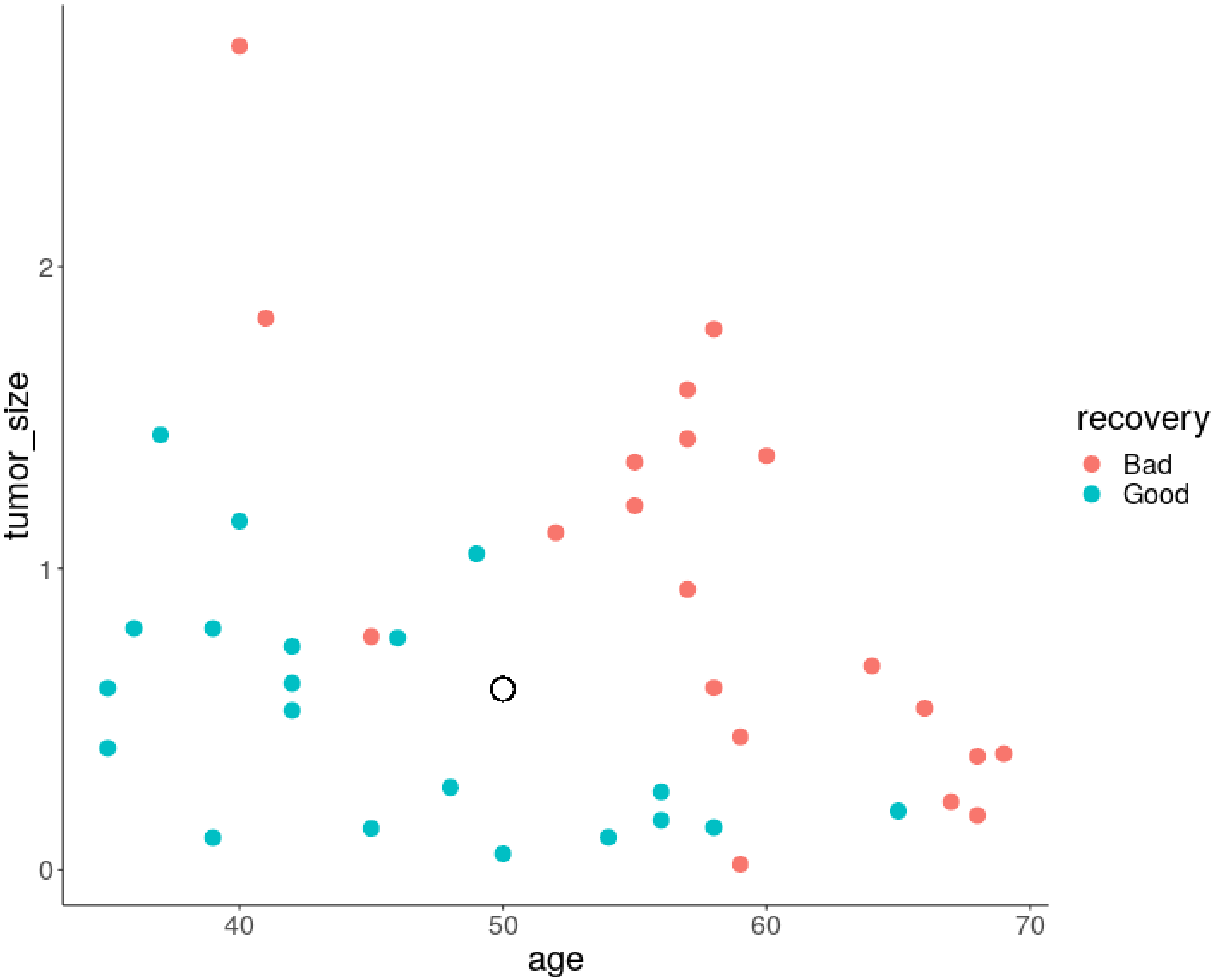
2018/11/14

... with R

Getting Started with k Nearest Neighbors



tumor_size
2
1
0
40
50
60
70
age
recovery
Bad
Good



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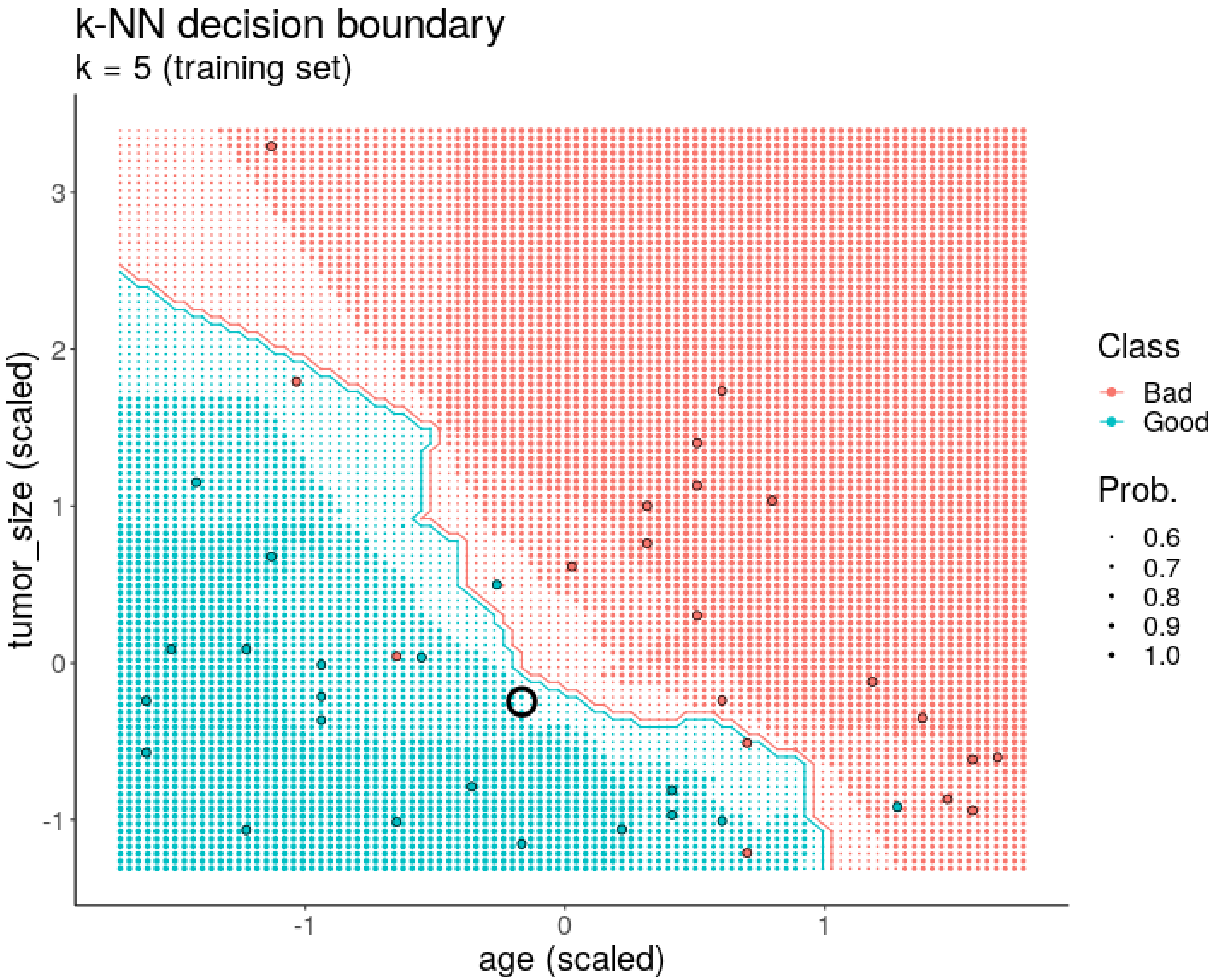

k-NN decision boundary
k = 5 (training set)
tumor_size (scaled)
age (scaled)
3
2
1
0
-1
-1
0
1
Class
Bad
Good
Prob.
0.6
0.7
0.8
0.9
1.0

How does it work?

Euclidean distance

$$d(\mathbf{x}_1, \mathbf{x}_2) = ((\mathbf{x}_1 - \mathbf{x}_2)'(\mathbf{x}_1 - \mathbf{x}_2))^{1/2} = \sqrt{\sum_{j=1}^{p} (x_{1j} - x_{2j})^2}.$$

What does it do?

$$k = 5$$

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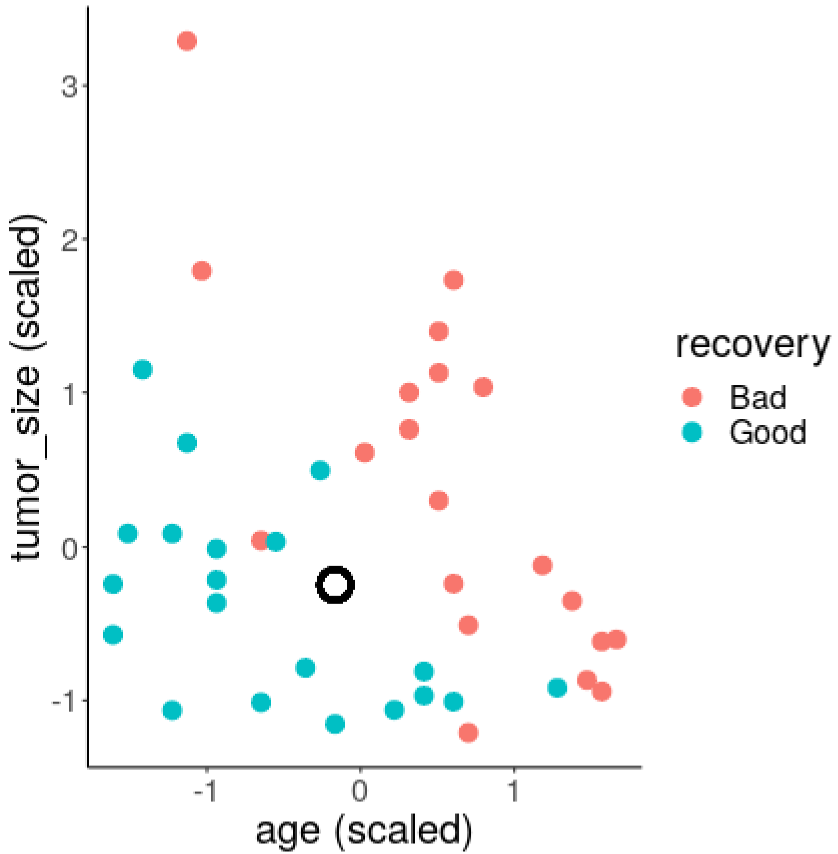

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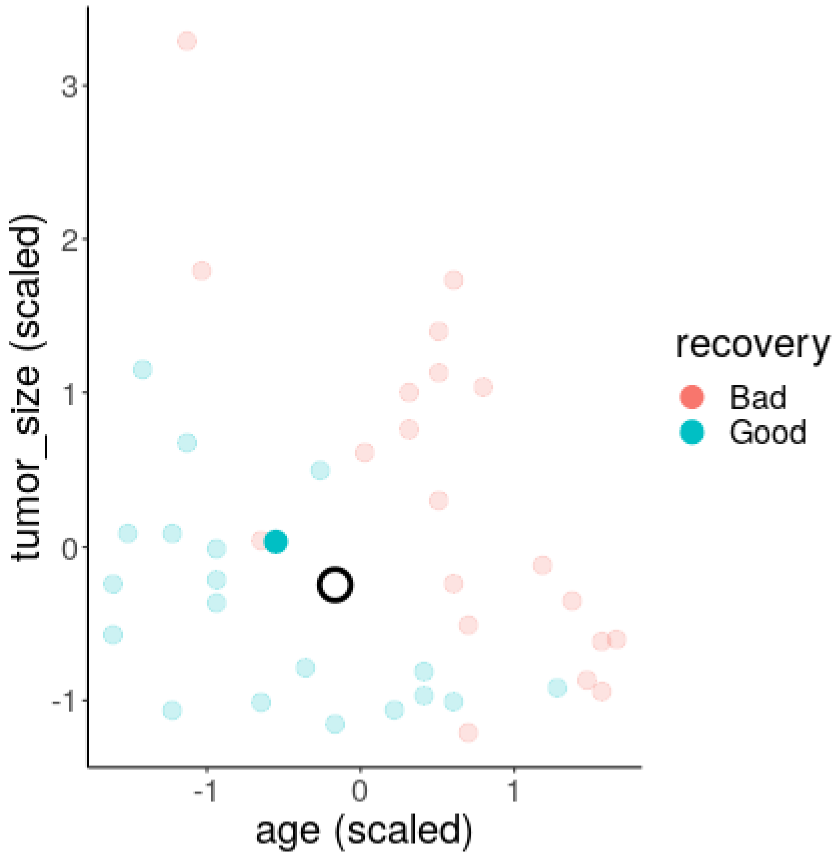

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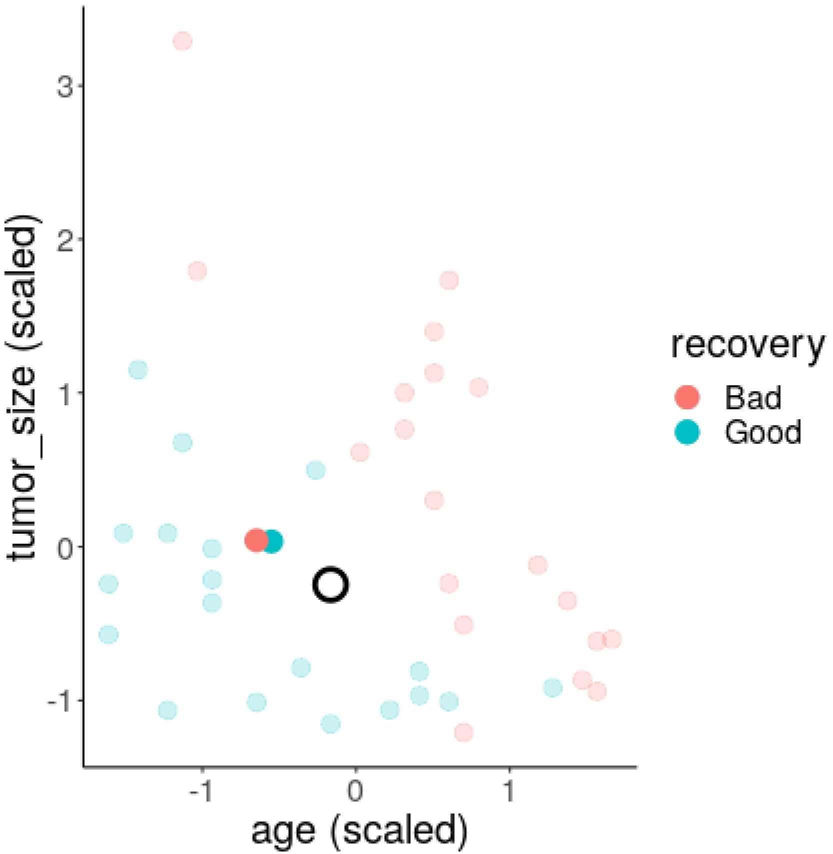

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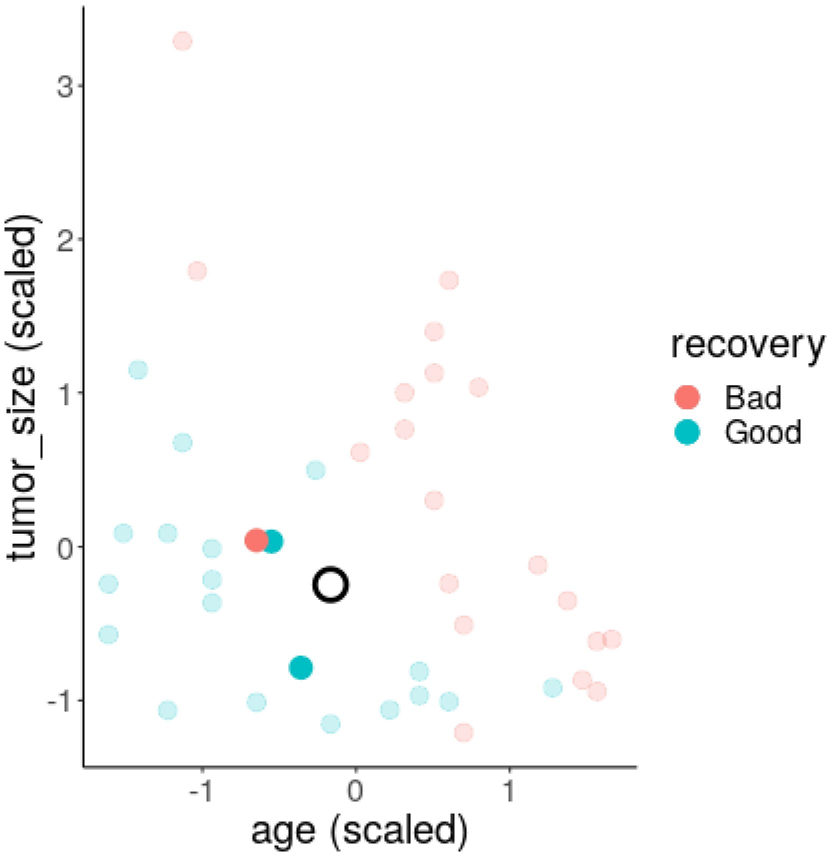

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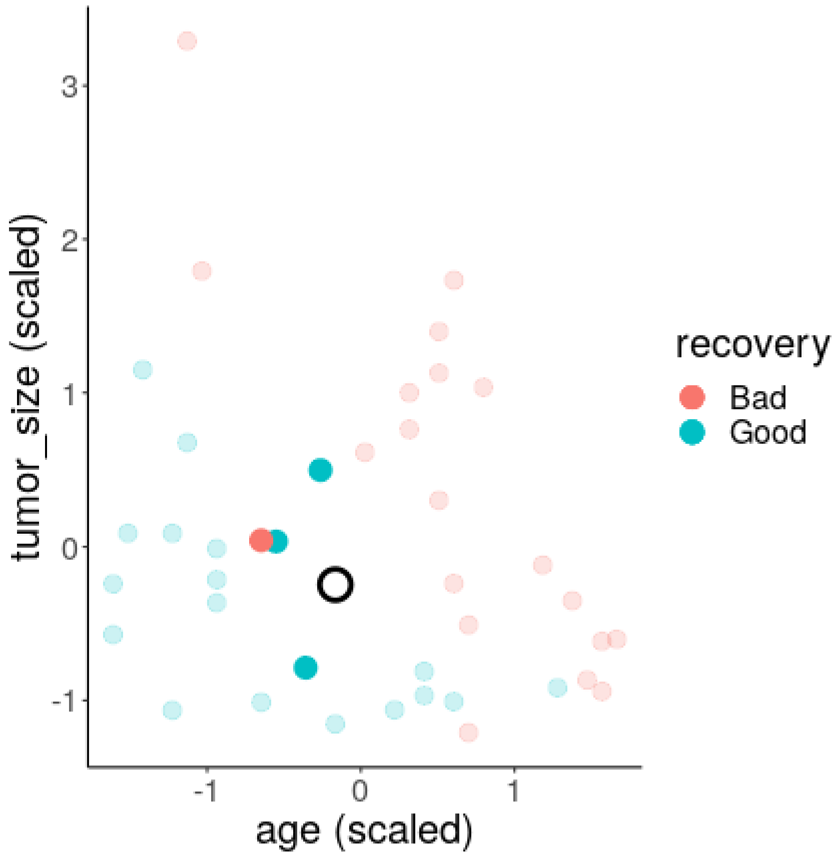

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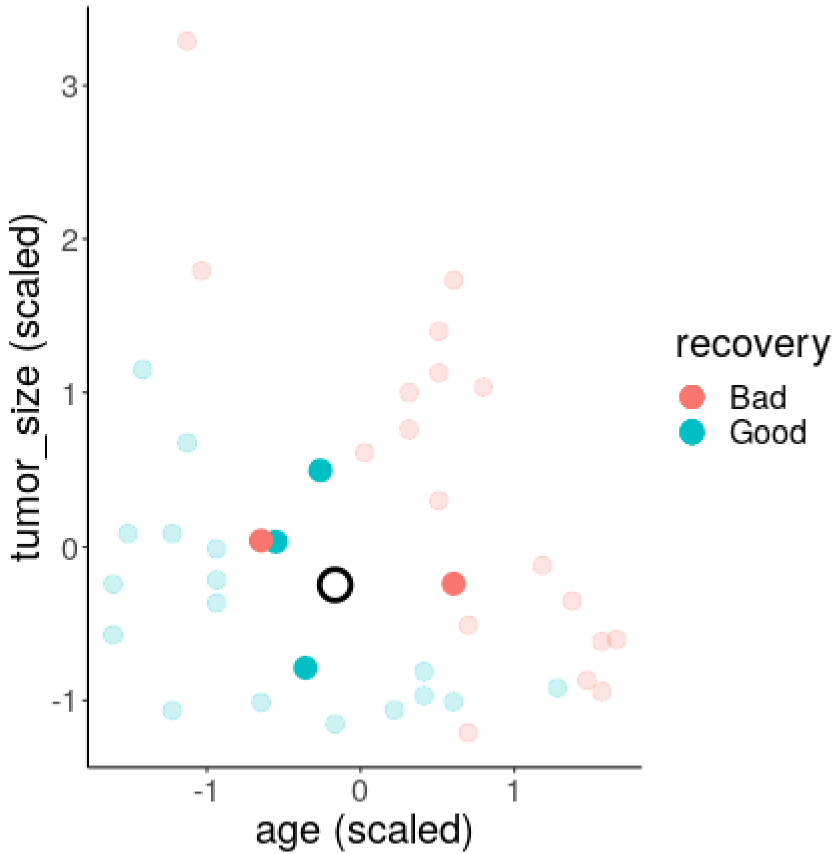

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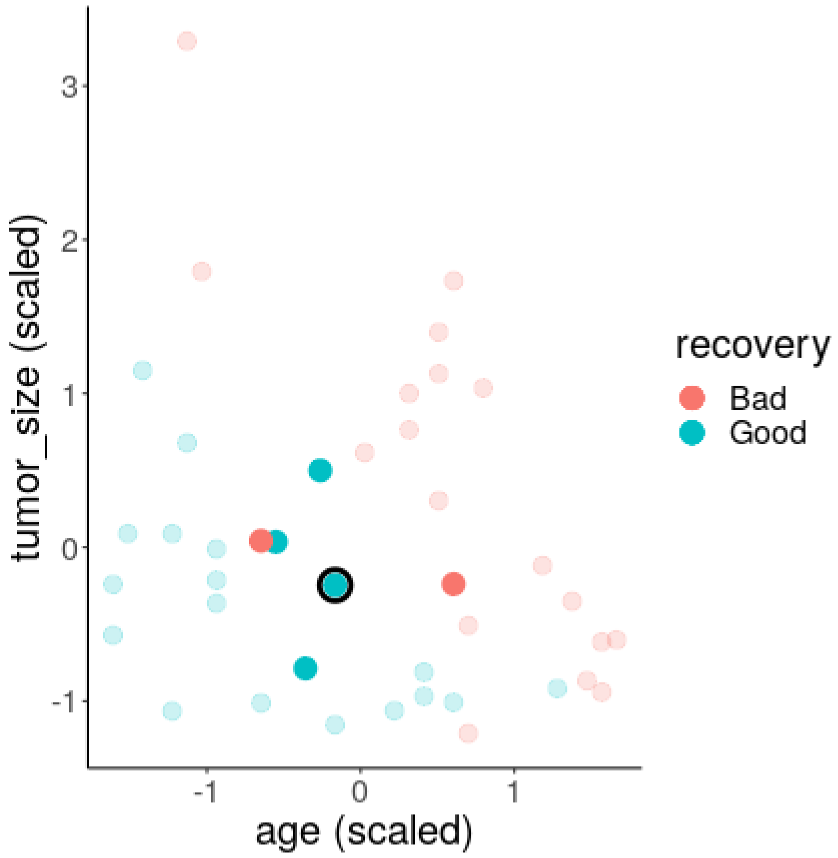


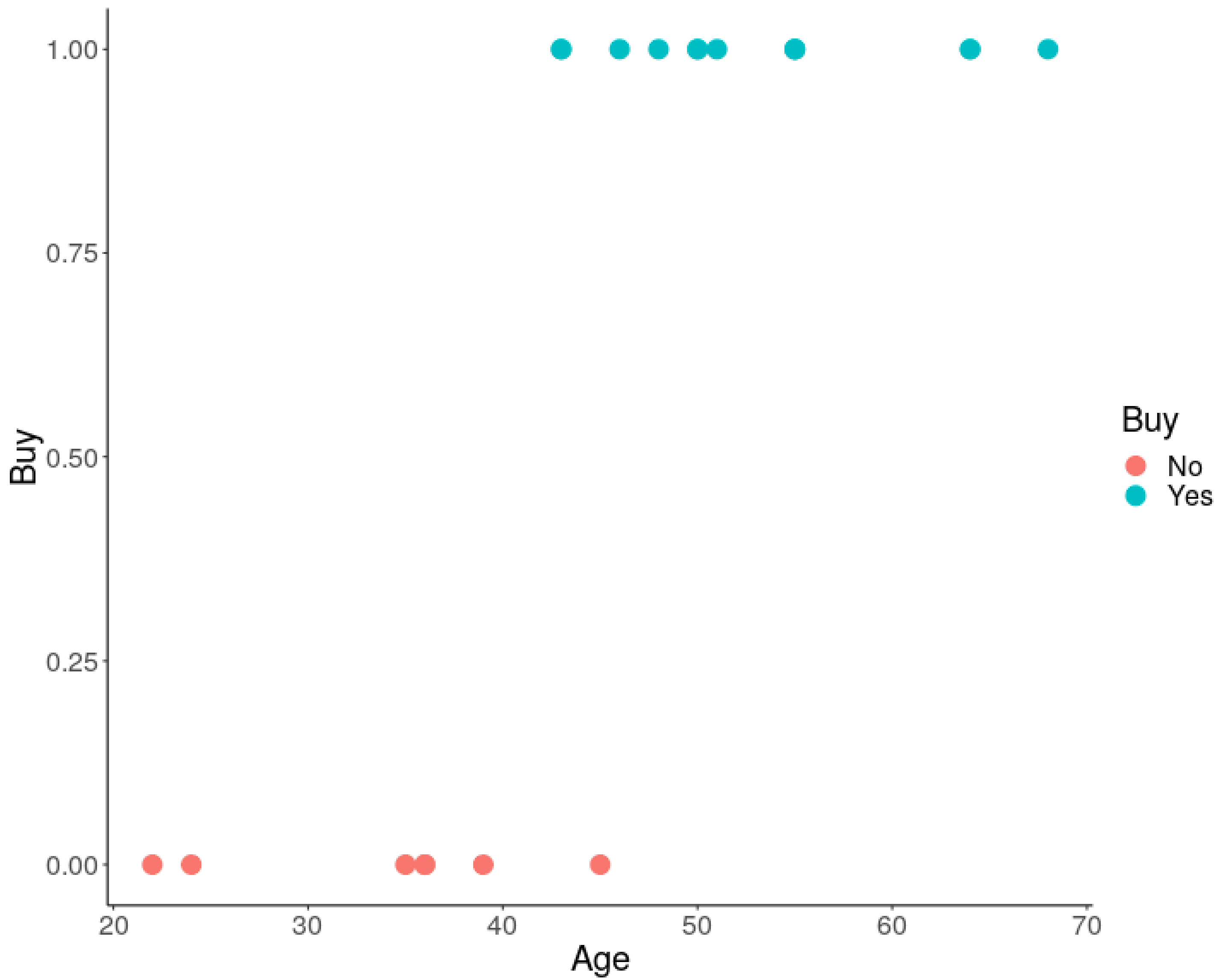
Practice time!

https://jlaria.github.io/SUsl/knn

https://raw.githubusercontent.com/jlaria/SUsl/master/source/knn_script.R

Lunch time!

Logistic Regression


1.00
0.75
0.50
0.25
0.00
Buy
20
30
40
50
60
70
Age
Buy
No
Yes

The intuition

We can think of the action of buying the product as a random variable defined as,

$$Y = \begin{cases} 0, & \text{Not buying} \\ 1, & \text{buying} \end{cases}$$

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Conditioning Y on the value of X, we can model the following probabilities.

$$P(Y=1|X=x) = p(x),$$

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$$P(Y=1|X=x) = p(x),$$

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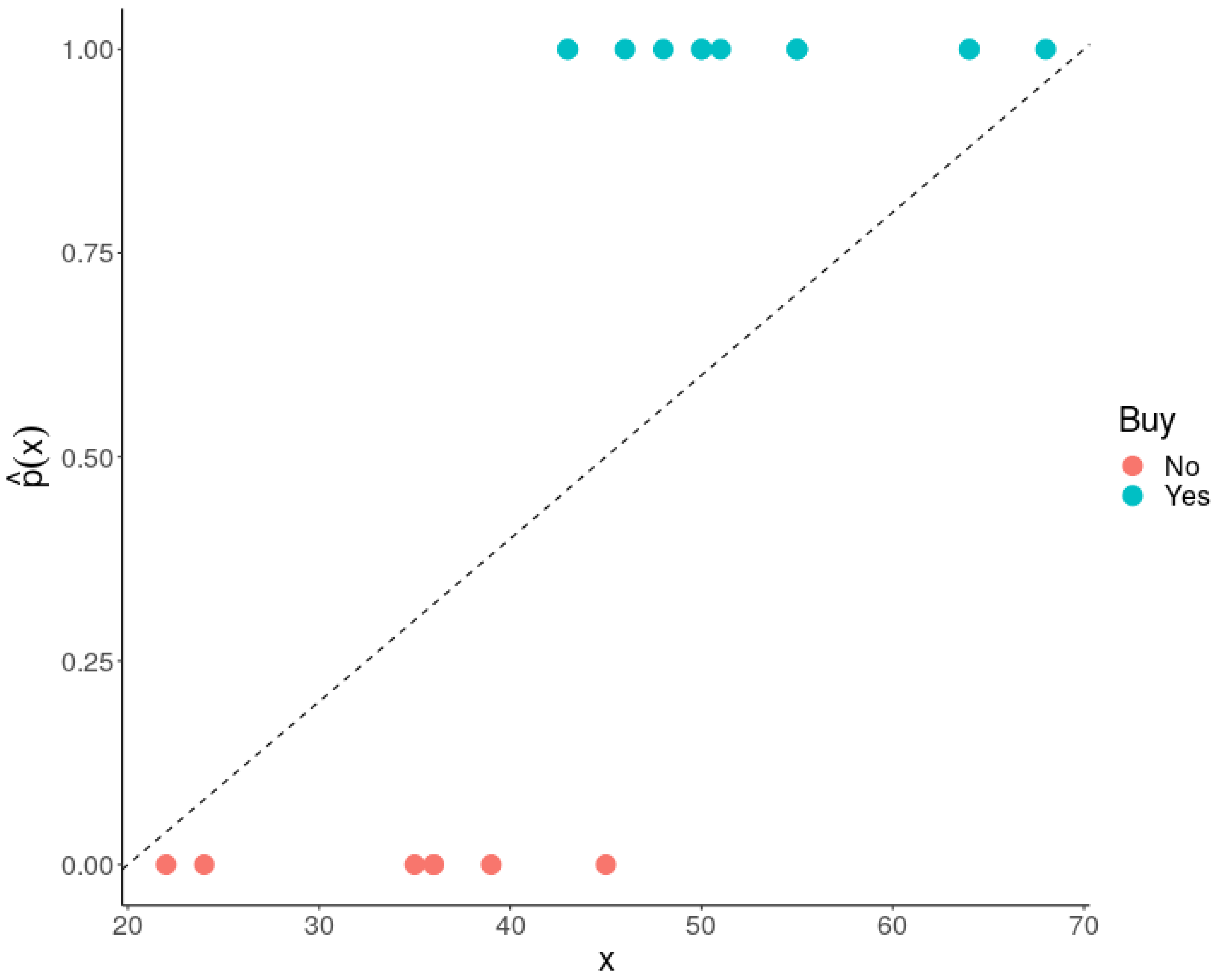
This means that $Y|X=x$ follows a Bernoulli distribution with probability of success $p(x)$. Instead of modelling the response Y, we model $p(x)$.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- To tackle this unbalance, we can take logarithms.
    - $-\infty \leq \log(p/(1-p)) \leq +\infty$

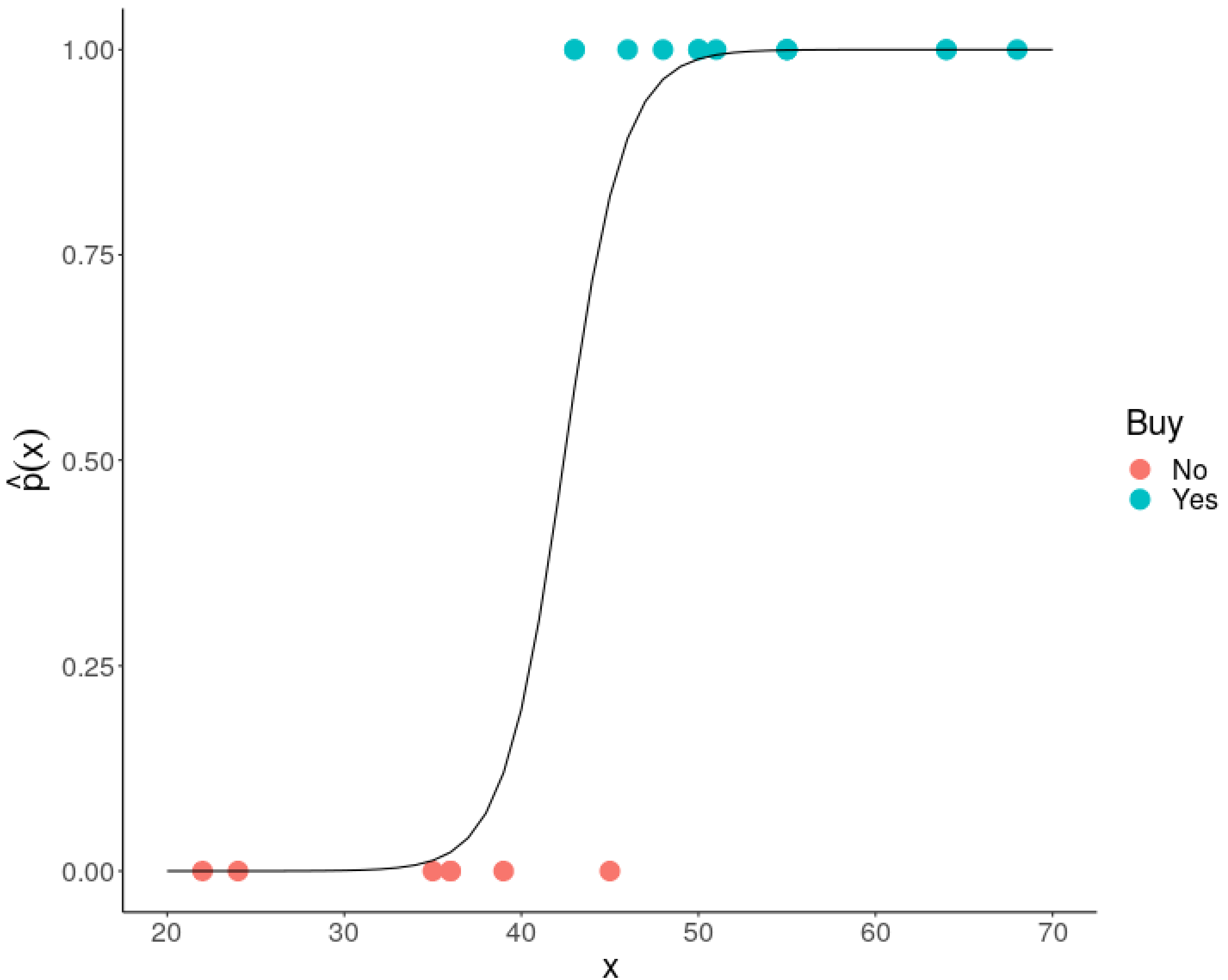
The intuition

- The *log-odds* may be the perfect candidates for a linear approximation.

$$\log\left(\frac{p}{1-p}\right) = \beta_0 + \beta X$$

- Taking p out,

$$p = \left[1 + \exp(-\beta_0 - \beta X)\right]^{-1}$$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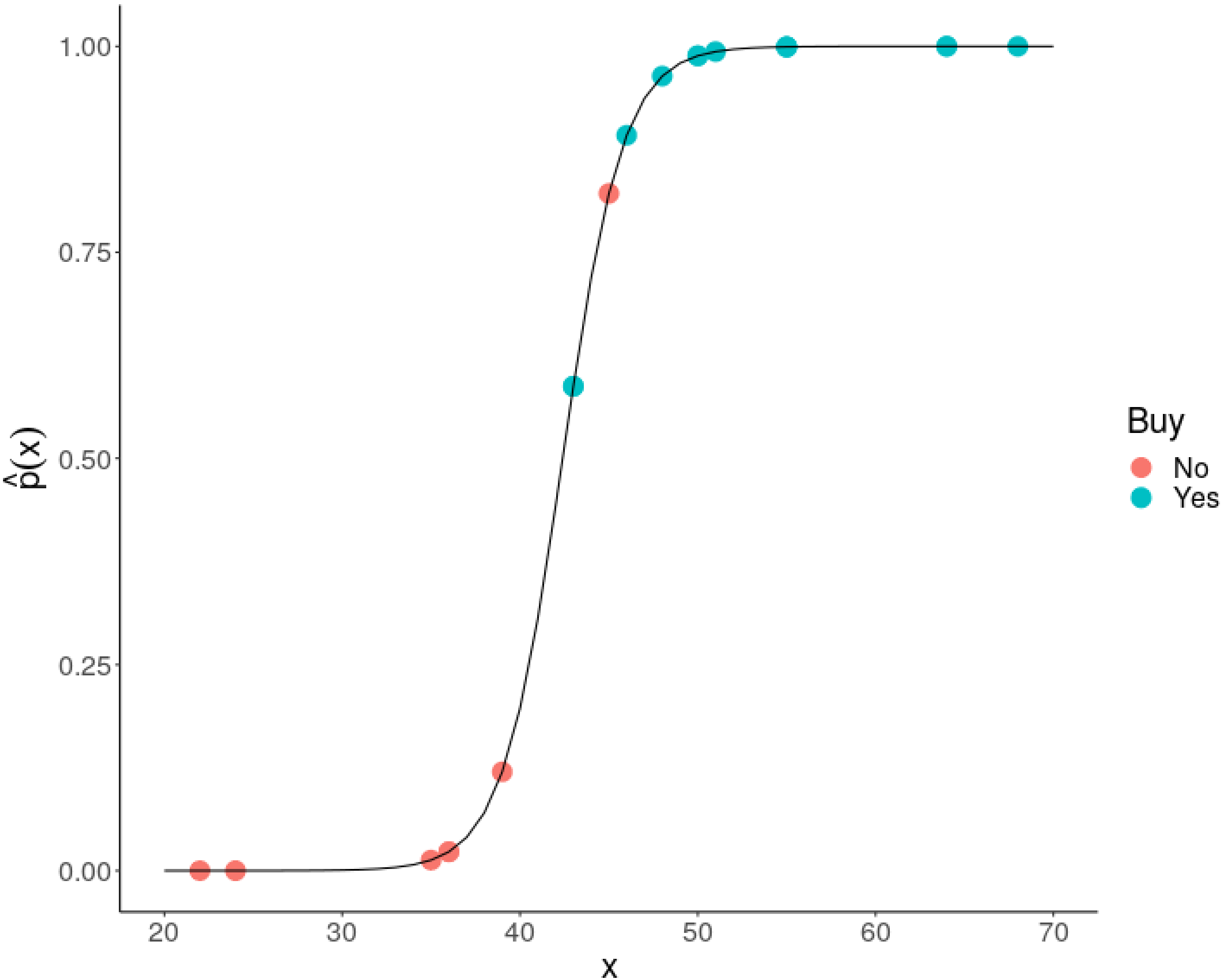
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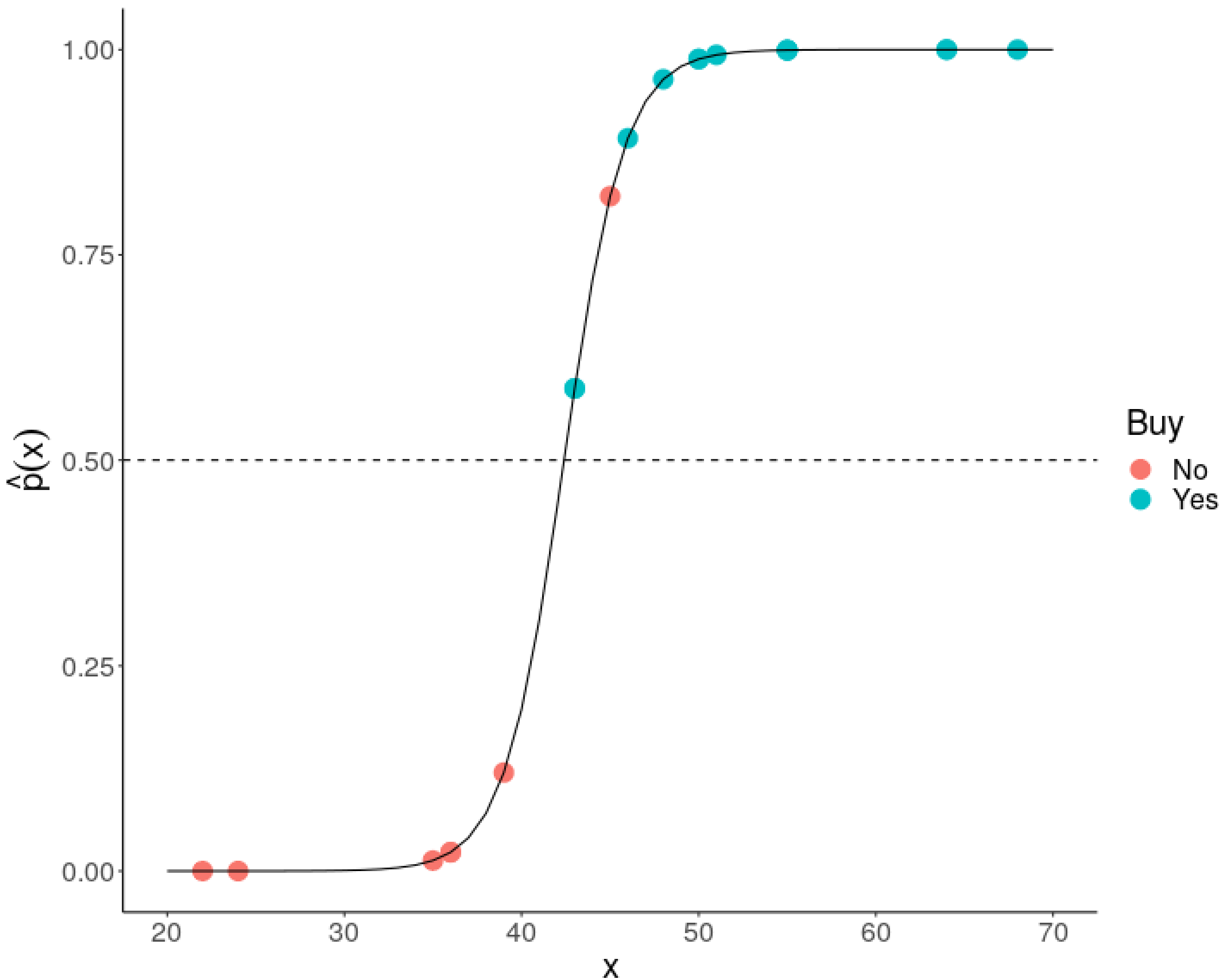
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x
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How does it work?

Let $y_1, y_2, \ldots y_N$ be the responses, and $p_1, p_2, \ldots p_N$ the underlying probabilities that generated them from the Bernoulli model. Then, the likelihood is,

$$L(y_1 \ldots y_N | p_1 \ldots p_N) = \prod_{i=1}^{N} p_i^{y_i} (1 - p_i)^{(1-y_i)} = \prod_{i=1}^{N} \left(\frac{p_i}{1 - p_i} \right)^{y_i} (1 - p_i)$$

Then, the log-likelihood becomes,

$$l(y_1 \ldots y_N | p_1 \ldots p_N) = \sum_{i=1}^{N} \left\{ y_i \log\left(\frac{p_i}{1 - p_i} \right) + \log(1 - p_i) \right\}$$

Let $\eta_i = \mathbf{x}_i^T \beta$, and notice that the odds can be written in terms of η_i as

$$\frac{p_i}{1 - p_i} = \exp(\eta_i)$$

How does it work?

Then, the log-likelihood can be written

$$l(y_1 \dots y_N | p_1 \dots p_N) = \sum_{i=1}^{N} \left\{ y_i \eta_i + \log([1 + \exp(\eta_i)]^{-1}) \right\}$$

$$l(y_1 \dots y_N | p_1 \dots p_N) = \sum_{i=1}^{N} \{ y_i \eta_i - \log(1 + e^{\eta_i}) \}$$

Therefore, the objective of logistic regression is to minimize with respect to β the function

$$R(\beta) = \sum_{i=1}^{N} \log\bigl(1 + \exp(\mathbf{x}_i^T \beta)\bigr) - \sum_{i=1}^{N} y_i \mathbf{x}_i^T \beta$$

Practice time!

https://jlaria.github.io/SUsl/logit

https://raw.githubusercontent.com/jlaria/SUsl/master/source/logit_script.R

Unsupervised statistical learning

k-means clustering

Hierarchical clustering

The intuition

https://www.naftaliharris.com/blog/visualizing-k-means-clustering/

Practice time!

https://jlaria.github.io/SUsl/kmeans

https://jlaria.github.io/SUsl/hclust

Thanks!

Slides created via the R package **xaringan**.

remark.js, **knitr**, and R Markdown.