Differentiable Manifolds and the Hairy Ball Theorem

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1 Manifolds and Tangent Spaces

Definition 1.1. The upper half space is

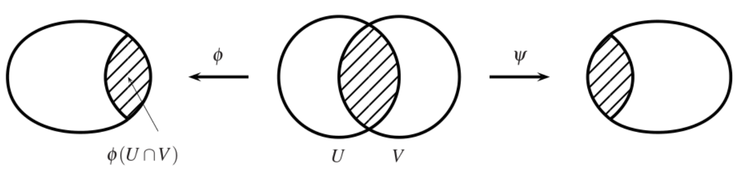
$$\mathcal{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}.$$

Its boundary is $\partial\mathcal{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n = 0\}$, and its interior is $\mathcal{H}^n \backslash \partial\mathcal{H}^n$.

Definition 1.2 (Smooth maps)**.** Let S be a subset of $\mathbb{R}^n$. A map $f : S \to \mathbb{R}^m$ is smooth at p if there exists a neighborhood U of p and a smooth function $f' : U \to \mathbb{R}^m$ such that $f' = f$ on $U \cap S$. If f is smooth at every $p \in S$, then f is smooth on S. In addition, if F is bijective and F^{-1} is smooth, then F is called a diffeomorphism.

Many properties of smooth maps on open sets also hold for smooth maps on arbitrary subsets. We will not prove them.

Definition 1.3. Let M be a Hausdorff, second countable topological space. A chart on M is a pair (U, ϕ) where U is open in M and $\phi : U \to \mathcal{H}^n$ is a homeomorphism onto its image. Two charts $(U, \phi), (V, \psi) \in \mathcal{A}$ are compatible if the functions $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are smooth on $\phi(U \cap V)$ and $\psi(U \cap V)$ respectively. An atlas $\mathcal{A}$ on M is a collection of pairwise compatible charts that cover M.



Theorem 1.4. Each atlas is contained in a unique maximal atlas. That is, for each atlas $\mathcal{A}$ on M, there exists a unique atlas $\mathcal{U}$ on M such that if $\mathcal{U} \subset \mathcal{U}'$, then $\mathcal{U}' = \mathcal{U}$.

Proof. Let $\mathcal{A}$ be an atlas on M. Let $\mathcal{U}$ be the set of charts that are compatible with every chart in $\mathcal{A}$. Let $(U, \phi), (V, \psi) \in \mathcal{U}$, and let $p \in U \cap V$. There exists $(W, \sigma) \in \mathcal{A}$ such that $p \in W$. So, $\psi \circ \sigma^{-1}$ and $\sigma \circ \phi^{-1}$ are smooth at $\sigma(p)$ and $\phi(p)$ respectively. Therefore,

$$\psi \circ \phi^{-1} = (\psi \circ \sigma^{-1}) \circ (\sigma \circ \phi^{-1})$$

is smooth at $\phi(p)$. As p was arbitrary, $\psi \circ \phi^{-1}$ is smooth on $\phi(U \cap V)$. Similarly, $\phi \circ \psi^{-1}$ is smooth on $\psi(U \cap V)$, so $(U, \phi), (V, \psi)$ are compatible, and $\mathcal{U}$ is indeed an atlas.

If $\mathcal{U} \subset \mathcal{U}'$, then every chart in $\mathcal{U}'$ is compatible with every chart in $\mathcal{U}$, in particular, with every chart in $\mathcal{A}$. By construction of $\mathcal{U}$, these charts are in $\mathcal{U}$, so $\mathcal{U}' \subset \mathcal{U}$, and $\mathcal{U}$ is maximal.

Suppose $\mathcal{V}$ is a maximal atlas containing $\mathcal{A}$. Then, every chart in $\mathcal{V}$ is compatible with $\mathcal{A}$, so $\mathcal{V} \subset \mathcal{U}$. Similarly, $\mathcal{U} \subset \mathcal{V}$. This shows uniqueness, and concludes the proof. $\square$

Definition 1.5. A n dimensional manifold with boundary M is a Hausdorff, second countable topological space together with a maximal atlas.

By Theorem 1.4, to construct a manifold, we only need to specify a topological space and an atlas. We write M instead of $(M, \mathcal{A})$ for manifolds. From now on, when we say a chart on M, we mean a chart in $\mathcal{A}$.

Theorem 1.6 (Smooth invariance of domain)**.** Let $f : U \to S$ be a diffeomorphism, where U is open in $\mathbb{R}^n$ and $S \subset \mathbb{R}^n$ is an arbitrary subset. Then S is open in $\mathbb{R}^n$.

Proof. Let $p \in U$. Since f^{-1} is smooth, there exists a neighborhoog V of $f(p)$ and a smooth function $g : V \to \mathbb{R}^n$ such that $g|_{V \cap S} = f^{-1}$. Then, $g \circ f$ is the identity on $f^{-1}(V)$, which is open. So,

$$(Jg(f(p)))(Jf(p)) = I$$

, and $\det(Jf(p)) \neq 0$. By the inverse function theorem, there are neighborhoods $U_p \subset U, V_{f(p)} \subset V$ such that $f : U_p \to V_{f(p)}$ is a diffeomorphism. We also have

$$V_{f(p)} = f(U_p) \subset f(U) = S.$$

For each $p \in U$, we can find an open set $V_{f(p)}\mathbb{R}^n$ such that $V_{f(p)} \subset S$. So, S is open. $\square$

Corollary 1.7. Let U and V be open subsets of $\mathcal{H}^n$, and let $f : U \to V$ be a diffeomorphism. Then, f maps interior points to interior points and boundary points to boundary points.

Proof. Suppose $p \subset U$ is an interior point. Then, it has an neighborhood U_p open in $\mathbb{R}^n$. By the above theorem, $f(p)$ lies in the open set $f(U_p)$, so $f(p)$ is an interior point. Similarly, if $f(p)$ is an interior point, $f^{-1}(f(p)) = p$ is an interior point. $\square$

Definition 1.8. Let M be a manifold with boundary. A point p is a boundary point if for some chart (U, ϕ), $\phi(p) \in \partial\mathcal{H}^n$. The set of all boundary points is the boundary ∂M.

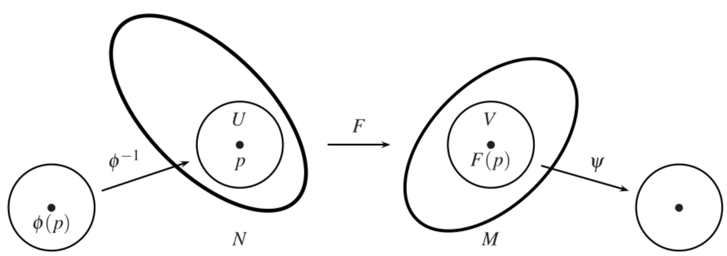
Suppose $(U, \phi), (V, \psi)$ are charts containing p, with $\phi(p) \in \partial\mathcal{H}^n$. By Corollary 1.7, $\psi(p) = \psi \circ \phi^{-1}(\phi(p))$ is also a boundary point, so boundary points are well defined.

Definition 1.9. A manifold with boundary M with empty boundary is called a manifold. For all charts (U, ϕ), the image of ϕ will be an open set in $\mathbb{R}^n$.

Proposition 1.10. Let M be a manifold with non empty boundary. Then, ∂M is a manifold.

Proof. Let $\mathcal{A}$ be an atlas on M. For each $(U, \phi) \in \mathcal{A}$, we construct a chart $(U \cap \partial M, \phi|_{\partial M})$ on ∂M. These charts are compatible, and map into $\mathbb{R}^{n-1}$, so ∂M is a manifold of dimension $n - 1$. $\square$

Definition 1.11 (Smooth Functions)**.** Let N, M be manifolds. A continuous function $F : N \to M$ is smooth if for all charts (U, ϕ) on N and (V, ψ) on M, $\psi \circ F \circ \phi^{-1}$ is smooth. If F is bijective and F^{-1} is also smooth, then F is a diffeomorphism.



Unless stated otherwise, all functions are smooth.

Definition 1.12 (Partial derivatives)**.** Let $f : M \to \mathbb{R}$ and $(U, x_1, \ldots, x_n)$ be a chart. The partial derivative of f with respect to x_i at p is the derivative with respect to standard coordinates

$$\left.\frac{\partial}{\partial x_i}\right|_p f = \left.\frac{\partial}{\partial r_i}\right|_{\phi(p)} (f \circ \phi^{-1}).$$

Definition 1.13 (Germs of functions)**.** Let M be a manifold, and $p \in M$. Let S be the set of all smooth functions defined on a neighborhood of p. For $f, g \in S$, we define an equivalence relation by $f \sim g$ if there is a neighborhood of p on which $f = g$. These equivalence classes are the germs of M at p, denoted $C_p^\infty(M)$.

Definition 1.14. Let M be a manifold, and $p \in M$. A derivation at p is a linear map $D : C_p^\infty(M) \to \mathbb{R}$ such that $D(fg) = D(f)g(p) + f(p)D(g)$.

Definition 1.15. A tangent vector at p is a derivation at p. The tangent space of M at p is the set of all derivations, denoted T_pM.

$\frac{\partial}{\partial x_1}|_p, \ldots, \frac{\partial}{\partial x_n}|_p$ are examples of tangent vectors.

Definition 1.16. Let $F : N \to M$ be a smooth function. We define the differential of F to be a function

$$F_* : T_pN \to T_{F(p)}M$$

defined by

$$F_*(X_p)(f) = X_p(f \circ F) \quad \text{for } f \in C^\infty_{F(p)}(M).$$

It is straightforward to check that T_pM is a vector space, and that F_* is a linear map.

Theorem 1.17 (Chain rule)**.** Let $F : N \to M$, $G : M \to P$, and $p \in N$. Then, $(G \circ F)_* = G_* \circ F_*$.

Proof. Let $X_p \in T_pN$ and f be a smooth function in a neighborhood of $G(F(p))$. Then

$$(G \circ F)_*(X_p)f = X_p(f \circ G \circ F) = (F_*X_p)(f \circ G) = (G_* \circ F_*(X_p))f.$$

□

Theorem 1.18. Let $(U, x_1, \ldots, x_n)$ be a chart. Then,

$$\left.\frac{\partial}{\partial x_1}\right|_p, \ldots, \left.\frac{\partial}{\partial x_n}\right|_p$$

is a basis for T_pM.

Proof. By the chain rule, $\mathrm{id} = (\phi^{-1} \circ \phi)_* = \phi_*^{-1} \circ \phi_*$, so ϕ_* is an isomorphism. We see that

$$\phi_* \left(\left.\frac{\partial}{\partial x_i}\right|_p \right) f = \left.\frac{\partial}{\partial x_i}\right|_p (f \circ \phi) = \left.\frac{\partial}{\partial r_i}\right|_{\phi(p)} (f \circ \phi \circ \phi^{-1}) = \frac{\partial}{\partial r_i}|_{\phi(p)} f.$$

It remains to show that $\frac{\partial}{\partial r_i}|_{\phi(p)}$ is a basis of $T_{\phi(p)}\mathbb{R}^n$. For simplification, we replace $\phi(p)$ by p.

Suppose $\sum_{i=1}^n a_i \frac{\partial}{\partial r_i}|_p = 0$. Applying to the coordinate functions r_i, we see that $a_i = 0$.

Next, we show that they span the space. Let D be a derivation, and f a smooth function. We technically require f to be a germ, but the argument still holds. By Taylor's theorem,

$$f(x) = f(p) + \sum_{i=1}^n (x_i - p_i) g_i(x) \quad g(p) = \frac{\partial f}{\partial x_i}(p).$$

Using the Leibniz rule and linearity,

$$Df(x) = \sum_{i=1}^{n} Dx_i g_i(p) = \sum_{i=1}^{n} Dx_i \frac{\partial}{\partial r_i}\bigg|_p f.$$

We have cancelled the terms $Df(p)$ and Dp_i since

$$D(1) = D(1 \cdot 1) = 1D(1) + D(1)1 = 2D(1),$$

so $D(1) = 0$ and $D(c) = 0$ by linearity. Thus,

$$Df = \sum_{i=1}^{n} Dx_i \frac{\partial}{\partial r_i}\bigg|_p.$$

$\square$

Definition 1.19. A k-form is a map that sends $p \in M$ to $\omega_p \in \Lambda^k T_p M$.

We define $(dx_1)_p, \dots, (dx_n)_p$ to be the dual basis of $\frac{\partial}{\partial x_1}|_p, \dots, \frac{\partial}{\partial x_n}|_p$. By a result in multilinear algebra, $\{(dx_{i_1})_p \wedge \cdots \wedge (dx_{i_k})_p \mid 1 \le i_1 < \cdots < i_n \le n\}$ is a basis for $\Lambda^k T_p M$. If we fix a chart $(U, x_1, \dots, x_n)$, then we can write $\omega = \sum a_I dx_I$, where a_I are real valued functions on U.

Definition 1.20. Let ω be a k-form. If for every chart $(U, x_1, \dots, x_n)$, the coefficients a_I in $\omega = \sum a_I dx_I$ are smooth, then ω is smooth. The set of smooth k-forms is denoted $\Omega^k(M)$. The graded algebra $\bigoplus \Omega^k(M)$ is denoted $\Omega^*(M)$.

Definition 1.21. A vector field X on M is a function that assigns to each $p \in M$ a tangent vector $X_p \in T_p M$. A vector field is smooth if for every chart $(U, x_1, \dots, x_n)$, the coefficients a_i in $X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}|_p$ are smooth. Given a 1-form ω, we define $\omega(X) : M \to \mathbb{R}, p \mapsto \omega_p(X_p)$.

Definition 1.22. An exterior derivative on M is an $\mathbb{R}$-linear map $\Omega^*(M) \to \Omega^*(M)$ such that

1. $D(\omega \cdot \tau) = (D\omega) \cdot \tau + (-1)^k \omega \cdot D\tau$ for $\omega \in \Omega^k(M)$, $\tau \in \Omega^\ell(M)$,
2. $D\omega \in \Omega^{k+1}(M)$ for $\omega \in \Omega^k(M)$,
3. $D \circ D = 0$,
4. if $f : M \to \mathbb{R}$ is smooth and X is a smooth vector field, then $(Df)X = Xf$.

Theorem 1.23. Let $(U, x_1, \dots, x_n)$ be a chart containing p. We define an operator $d_U : \Omega^*(U) \to \Omega^*(U)$ by

$$d_U f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i, \; d_U \omega = \sum_I d_U f \wedge dx_I, \text{ where } \omega = \sum_I f_I dx_I.$$

Next, we define an operator $d : \Omega^*(M) \to \Omega^*(M)$ given by $(d\omega)_p = (d_U \omega)_p$ for some chart (U, ϕ) containing p. Then, d is well defined, and it is the unique exterior derivative on M.

Proof. See [Tu, Theorem 19.4, p.214]. □

Definition 1.24 (Pullback)**.** Let $\omega_p \in \Lambda^k T_pM$ and $F : N \to M$. The pullback of ω_p by F is $F^*(\omega_p) \in \Lambda^k T_pN$, defined as

$$F^*(\omega_p)(v_1, \ldots, v_k) = \omega_p(F_*(v_1), \ldots, F_*(v_k)).$$

The pullback of a differential form ω is defined as $(F^*\omega)_p = F^*(\omega_p)$.

Proposition 1.25. $F^*d\omega = dF^*\omega$

Proposition 1.26. If $\omega \in \Omega^k(M)$, then $F^*\omega \in \Omega^k(N)$.

2 Integration of Differential n-Forms

Definition 2.1. An orientation on a manifold with boundary is a non-vanishing differential n-form.

Definition 2.2. An oriented atlas is an atlas $\mathcal{A}$ such that for all $(U, \phi), (V, \psi) \in \mathcal{A}$, $\det(J(\psi \circ \phi^{-1})) > 0$.

Theorem 2.3.

$$\text{Orientations} \iff \text{Oriented atlas}$$

$$\omega \iff \omega_p(e_1, \ldots, e_n) > 0,$$

where $e_1, \ldots, e_n$ is the basis induced by (U, ϕ).

Definition 2.4 (Integration on a $\mathbb{R}^n$)**.** Let ω be a n-form on (U, ϕ), where $U \subset \mathbb{R}^n$. Then $\omega = f(x)dx_1 \wedge \cdots \wedge dx_n$ for some smooth f. The integral of ω over U is $\int_U \omega = \int_U f$.

Definition 2.5 (Integration on a chart)**.** Let ω be a n-form on (U, ϕ). The integral of ω over U is $\int_U \omega = \int_{\phi^{-1}(U)} (\phi^{-1})^*\omega$.

Definition 2.6 (Partition of Unity)**.** A partition of Unity on a manifold M is a collection of nonnegative smooth functions $\{\rho_\alpha : M \to \mathbb{R}\}_{\alpha \in A}$ such that

(i) the collection of supports, $\{\operatorname{supp} \rho_\alpha\}_{\alpha \in A}$, is locally finite,

(ii) $\sum_{\alpha \in A} \rho_\alpha = 1$.

Definition 2.7 (Integration on a Manifold)**.** Let ω be a n-form on M. The integral of ω over M, denoted by $\int_M \omega$, is defined to be $\sum_{\alpha \in A} \int_{U_\alpha} \rho_\alpha \omega$.

Theorem 2.8. Let ω be a smooth $(n-1)$-form on $\mathcal{H}^n$ with compact support. Then $\int_{\mathcal{H}^n} d\omega = \int_{\partial \mathcal{H}^n} \omega$.

Theorem 2.9. Let ω be a smooth $(n-1)$-form on an oriented manifold with boundary M. Then $\int_M d\omega = \int_{\partial M} \omega$.