Differentiable manifolds and the Hairy Ball Theorem

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Manifolds

A n dimensional manifold is a subset of $\mathbb{R}^\ell$ such that for each point p, there is a neighborhood U and a homeomorphism $\phi : U \to V \subset \mathbb{R}^n$ such that ϕ and ϕ^{-1} are smooth.

Tangent Space

Given a point $p \in M$ and a smooth curve $\gamma : (-1, 1) \to M$ in M such that $\gamma(0) = p$, its velocity vector is $\frac{d\gamma}{dt}|_{t=0}$. The set of all velocity vectors at p is the tangent space at p, denoted T_pM.

If we fix a neighborhood U of p and a $\phi : U \to V \subset \mathbb{R}^n$ centered at p, construct curves $\gamma_i : t \mapsto \phi^{-1} \circ \iota_i(t)$, where ι_i is the inclusion into the ith coordinate. Their velocity vectors form a basis for the tangent space.
Example: sphere

1-form

A 1-form ω is a linear function from T_pM to $\mathbb{R}$. $\hom(T_pM, \mathbb{R})$ is a vector space of dimension n, with basis dx^i, $dx^i(a_1, \ldots, a_n) = a_i$.

k-form

A k-form ω is an alternating multilinear function from $(T_pM)^k$ to $\mathbb{R}$.

Wedge Product

The wedge product of k 1-forms $\omega_1, \ldots, \omega_k$ is the k-form $\omega^1 \wedge \cdots \wedge \omega^k(v_1, \ldots, v_k) = det(\omega_i(v_j))$.

Theorem

$\{dx^{i_1} \wedge \cdots \wedge dx^{i_k} | 1 \leq i_i < \cdots < i_k \leq n\}$ is a basis for $\Lambda^k(T_pM)$, the set of alternating multilinear functions on $(T_pM)^k$.

Example: On $\mathbb{R}^3$, the form $dx \wedge dy$ applied to $(1, 2, 3), (0, 1, 2)$ is $det((1, 2)^T(0, 1)^T) = 1$. We see that

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