Differentiable manifolds and the Hairy Ball Theorem

Jonathan Lau

February 24, 2023

Table of Contents

1. Manifolds
2. Integration of Differential n-Forms
3. The Hairy Ball Theorem

Contents

1. Manifolds
2. Integration of Differential n-Forms
3. The Hairy Ball Theorem

Stokes' Theorem:

$$\int_M d\omega = \int_{\partial M} \omega$$

Special cases:

$$\int_{\partial D} Pdx + Qdy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\int_{\partial V} F \cdot dS = \int_V \mathrm{div} F$$

Upper Half Space

The upper half space is $\mathcal{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n >= 0\}$. Its boundary is $\partial\mathcal{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n = 0\}$.

Charts

A chart on a topological space M is a pair (U, ϕ) consisting of an open $U \subset M$ and a homeomorphism $\phi : U \to V \subset \mathbb{H}^n$.

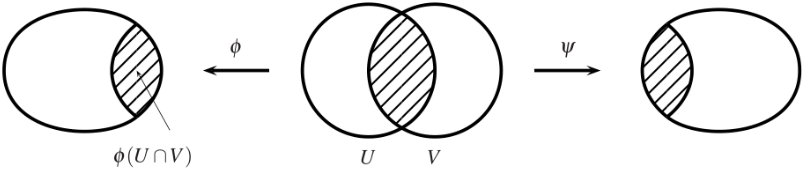
We can express ϕ as $(x_1, \ldots, x_n)$.
Example: sphere

Smooth maps

Let S be a subset of $\mathbb{R}^n$. A map $f : S \to \mathbb{R}^m$ is smooth at p if there exists a neighborhood U of p and a smooth function $f' : U \to \mathbb{R}^m$ such that $f' = f$ on $U \cap S$. If f is smooth at every $p \in S$, then f is smooth on S.
In addition, if F is bijective and F^{-1} is smooth, then F is called a diffeomorphism.

Manifolds with boundary

A n dimensional manifold with boundary M is a subset of $\mathbb{R}^\ell$ together with a collection of charts $\mathcal{A}$ that cover M such that for all charts (U, ϕ), (V, ψ), the maps $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are smooth on $\phi(U \cap V)$ and $\psi(U \cap V)$ respectively.
Such a collection is called an atlas.



Boundary

Let M be a manifold with boundary. A point p is a boundary point if for some chart (U, ϕ), $\phi(p) \in \partial\mathcal{H}^n$. The set of all boundary points is the boundary ∂M.

Manifold

A manifold with boundary M with empty boundary is called a manifold.

Proposition: Boundary is manifold

Let M be a manifold with boundary. Then, ∂M is a manifold.

proof: Let $\mathcal{A}$ be an atlas on M. For each $(U, x_1, \ldots, x_n) \in \mathcal{A}$, we construct a chart on $(U \cap \partial M, x_1|_{\partial M}, \ldots, x_{n-1}|_{\partial M})$ on ∂M.

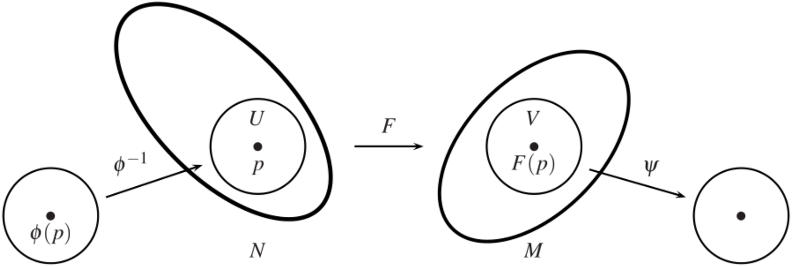
Tangent Space

Let M be a manifold. Given a point $p \in M$ and a smooth curve $\gamma : (-1, 1) \to M$ in M such that $\gamma(0) = p$, its velocity vector is $\frac{d\gamma}{dt}|_{t=0}$. The set of all velocity vectors at p is the tangent space at p, denoted T_pM.

Let (U, ϕ) with $\phi(p) = 0$, and let $\gamma_i : t \mapsto \phi^{-1} \circ \iota_i(t)$. Then, $e_i = \frac{\partial \gamma_i}{\partial t}$ form a basis for T_pM. We call this the basis induced by ϕ.
Example: sphere

Smooth Functions

Let N, M be manifolds. A continuous function $F : N \to M$ is smooth if for all charts (U, ϕ) on N and (V, ψ) on M, $\psi \circ F \circ \phi^{-1}$ is smooth.



Differential

Let $F : N \to M$ be a smooth function, $p \in N$, and $(U, x_1, \ldots, x_n)$, $(V, y_1, \ldots, y_m)$ are charts on N and M. Let $v \in T_pM$. Then, there exists a curve γ such that $\frac{d\gamma}{dt} = v$. We define the differential $F_*(v) = \frac{dF \circ \gamma}{dt}$.

In the bases induced by ϕ, ψ, the differential $F_* : T_pM \to T_{F(p)}N$ at p is a linear transformation represented by the matrix $J(\psi \circ F \circ \phi^{-1})$, where $J(f) = \left(\frac{\partial f_i}{\partial x_j}\right)$ is the m by n Jacobian matrix.
Example: $F : S^2 \to \mathbb{R}^3, (x, y, z) \mapsto (2x, y, z)$, $p = 1/3(1, 2, 2)$, $\phi : (x, y, z) \mapsto (x, y)$, $\psi : (x, y, z) \mapsto (x, y)$.

1-form

Let (U, ϕ) be a chart on M. A 1-form ω is a linear function from T_pM to $\mathbb{R}$.

$\mathrm{Hom}(T_pM, \mathbb{R}) \equiv \mathbb{R}^n$. Fixing a basis for T_pM, we define $(dx_i)_p(a_1, \ldots, a_n) = a_i$.

k-form

Let (U, ϕ) be a chart on M. A k-form ω is an alternating multilinear function from $(T_pM)^k$ to $\mathbb{R}$.

Wedge Product

The wedge product of k 1-forms $\omega_1, \dots, \omega_k$ is the k-form $\omega_1 \wedge \cdots \wedge \omega_k(v_1, \dots, v_k) = \det(\omega_i(v_j))$.

On $T_p\mathbb{R}^5$, $dx_1 \wedge dx_3((0,0,1),(2,2,2)) = \det \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}$. This is the same as projecting $(0,0,1),(2,2,2)$ onto the $x-z$ plane, then calculating it's area.

Theorem

$\{(dx_{i_1})_p \wedge \cdots \wedge (dx_{i_k})_p | 1 \leq i_i < \cdots < i_k \leq n\}$ is a basis for $\Lambda^k(T_pM)$, the set of alternating multilinear functions on $(T_pM)^k$.

So, $(dx_1)_p, \dots, (dx_n)_p$ is a basis for $\mathrm{Hom}(T_pM, \mathbb{R})$, and $(dx_1)_p \wedge \cdots \wedge (dx_n)_p$ is a basis for $\Lambda^n(T_pM)$, i.e. every $\omega \in \Lambda^n(T_pM)$ is a multiple of $(dx_1)_p \wedge \cdots \wedge (dx_n)_p$.

Differential Forms

A differential k-form ω is a function $\omega : p \in M \mapsto \omega_p \in \Lambda^k T_p M$, and for all charts $(U, x_1, \ldots, x_n)$, $\omega_p = \sum_I f_I(p) dx_I$ on U for some smooth f_i's.

Example: dx_I is a differential k-form.

The d operator

Let $f : M \to \mathbb{R}$ be a smooth function. We define a differential 1-form df as $\sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$. Given a differential form ω on M and a chart (U, ϕ), $\omega = \sum_I f dx_I$ on U. Then, on U, $d\omega$ is defined as $\sum_I df \wedge dx_I$.

Table of Contents

1 Manifolds

2 Integration of Differential n-Forms

3 The Hairy Ball Theorem

Orientation

An orientation on a manifold with boundary is a non-vanishing differential n-form.

Oriented Atlas

An oriented atlas is an atlas $\mathcal{A}$ such that for all $(U,\phi),(V,\psi) \in \mathcal{A}$, $\det(J(\psi \circ \phi^{-1})) > 0$.

Theorem

$$Orientations \iff Orientedatlas$$

$$\omega \iff \omega_p(e_1,\ldots,e_n) > 0,$$

where $e_1,\ldots,e_n$ is the basis induced by (U,ϕ).

Integration on a $\mathbb{R}^n$

Let ω be a n-form on (U, ϕ), where $U \subset \mathbb{R}^n$. Then $\omega = f(x)dx_1 \wedge \cdots \wedge dx_n$ for some smooth f. The integral of ω over U is $\int_U \omega = \int_U f$.

Pullback

Let $\omega_p \in \Lambda^k T_p M$ and $F : N \to M$. The pullback of ω_p by F is $F^*(\omega_p) \in \Lambda^k T_p N$, defined as

$$F^*(\omega_p)(v_1, \ldots, v_k) = \omega_p(F_*(v_1), \ldots, F_*(v_k)).$$

The pullback of a differential form ω is defined as $(F^*\omega)_p =^* (\omega_p)$.

Integration on a chart

Let ω be a n-form on (U, ϕ). The integral of ω over U is $\int_U \omega = \int_{\phi^{-1}(U)} (\phi^{-1})^* \omega$.

Partition of Unity

A partition of Unity on a manifold M is a collection of nonnegative smooth functions $\{\rho_\alpha : M \to \mathbb{R}\}_{\alpha \in A}$ such that

1. the collection of supports, $\{\operatorname{supp} \rho_\alpha\}_{\alpha \in A}$, is locally finite,
2. $\sum_{\alpha \in A} \rho_\alpha = 1$.

Integration on a Manifold

Let ω be a n-form on M. The integral of ω over M, denoted by $\int_M \omega$, is defined to be $\sum_{\alpha \in A} \int_{U_\alpha} \rho_\alpha \omega$.

Stokes' theorem on $\mathcal{H}^n$

stuff ...

Stokes' theorem on Manifolds

stuff ...

Table of Contents

1 Manifolds

2 Integration of Differential n-Forms

3 The Hairy Ball Theorem

stuff

stuff ...