Differentiable manifolds and the Stokes' Theorem

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Stokes' Theorem

Stokes' Theorem:

$$\int_M d\omega = \int_{\partial M} \omega$$

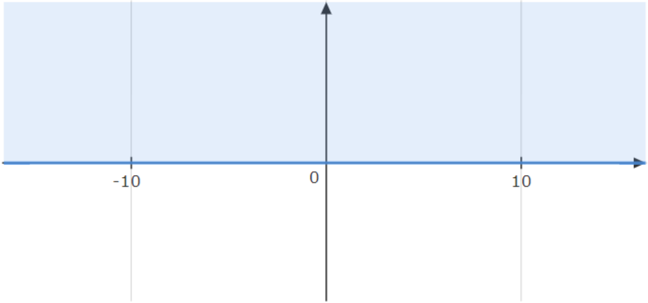
Special cases:

$$\int_{\partial D} Pdx + Qdy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

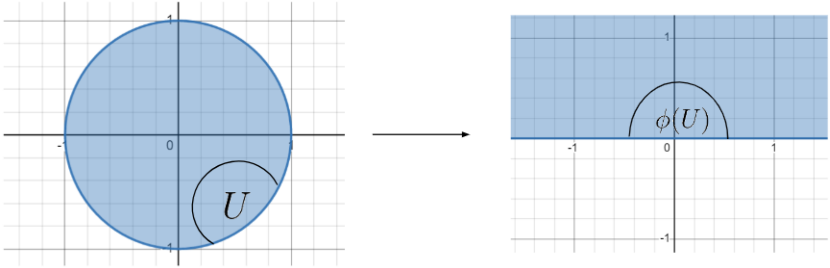
$$\int_{\partial V} F \cdot dS = \int_V \mathrm{div} F$$

Upper Half Space

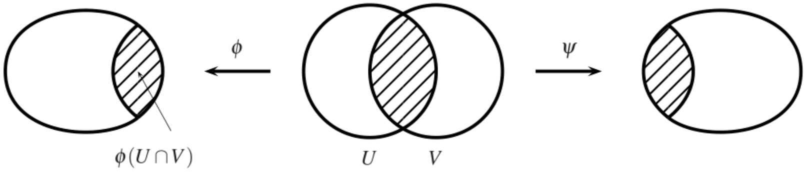
The upper half space is $\mathcal{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$. Its boundary is $\partial\mathcal{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n = 0\}$.



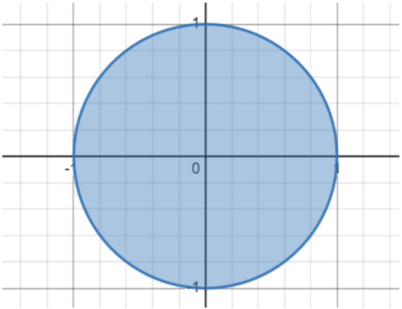
Charts



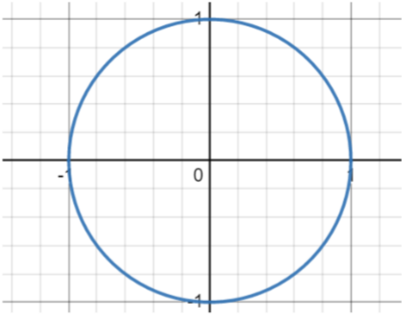
Manifolds



M

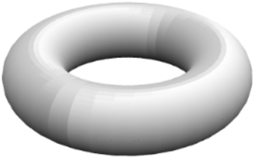


∂M

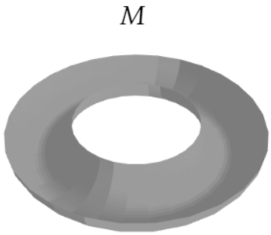


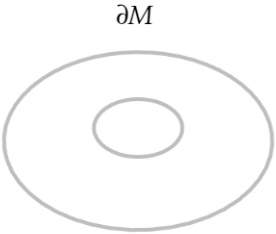
Examples

M



Examples

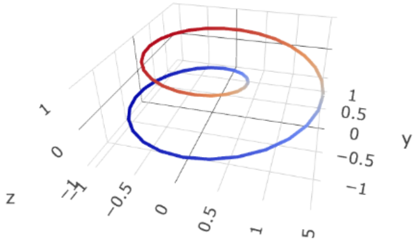




Examples

M

∂M



Examples

$GL(n, \mathbb{R}), SL(n, \mathbb{R})$
e.g.

$$GL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc \neq 0 \right\}$$

$$SL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1 \right\}$$

Proposition: Boundary is manifold

Let M be a manifold with non empty boundary. Then, ∂M is a manifold with empty boundary.

proof: Let $\mathcal{A}$ be an atlas on M. For each $(U, x_1, \dots, x_n) \in \mathcal{A}$, we construct a chart on $(U \cap \partial M, x_1|_{\partial M}, \dots, x_{n-1}|_{\partial M})$ on ∂M.

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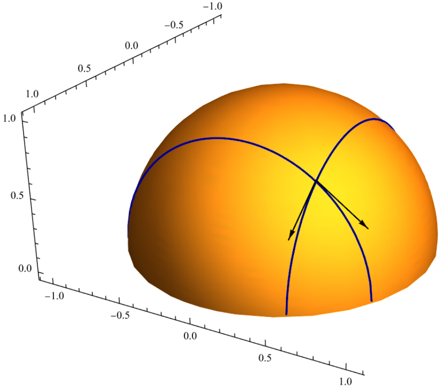
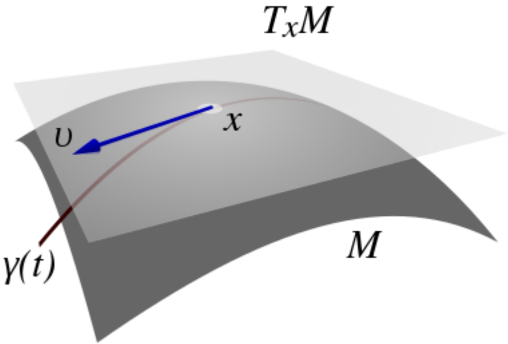
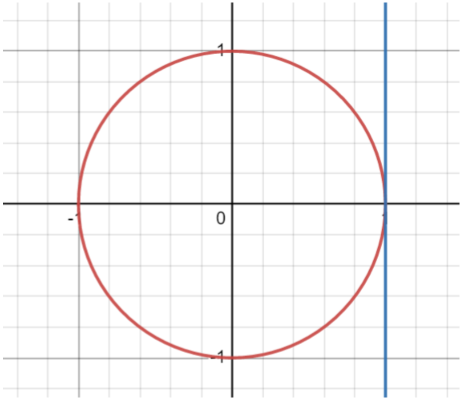
Tangent vectors



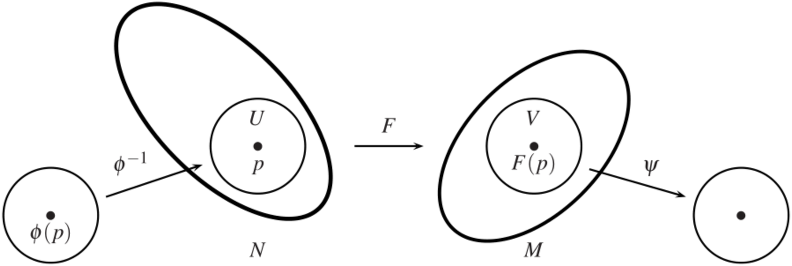
Tangent Space



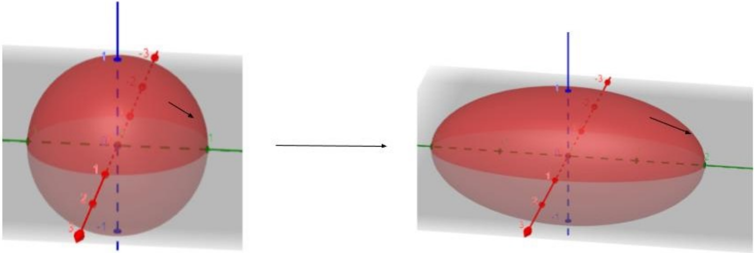
Tangent Space



Smooth Functions

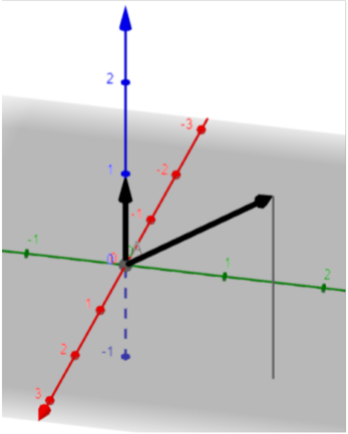


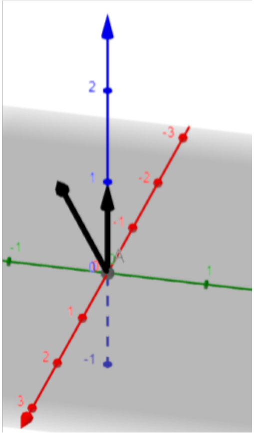
Differential



Wedge product

Example: $dx_1 \wedge dx_3((0,0,1),(2,2,2))$.





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Partition of Unity

A partition of Unity on a manifold M is a collection of nonnegative smooth functions $\{\rho_\alpha : M \to \mathbb{R}\}_{\alpha \in A}$ such that

1. the collection of supports, $\{\operatorname{supp} \rho_\alpha\}_{\alpha \in A}$, is locally finite,
2. $\sum_{\alpha \in A} \rho_\alpha = 1$.

Stokes' theorem

Let M be an oriented n dimensional manifold with non empty boundary, and let ω be a differential $(n-1)$-form on M with compact support. Give ∂M the boundary orientation, and let $\iota : \partial M \to M$ be the inclusion map. Writing $\int_{\partial M} \iota^* \omega$ as $\int_{\partial M} \omega$,

$$\int_{\partial M} \omega = \int_M d\omega$$

Green's theorem

$$\oint_{\partial D} Pdx + Qdy = \int_D (Q_x - P_y) dA$$