Differentiable Manifolds and the Hairy Ball Theorem

Jonathan Lau

March 12, 2023

1 Manifolds and Tangent Spaces

Definition 1.1. The upper half space is

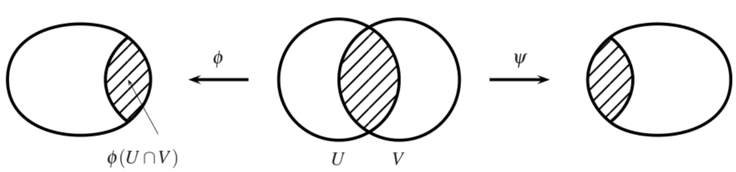
$$\mathcal{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}.$$

Its boundary is $\partial\mathcal{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n = 0\}$, and its interior is $\mathcal{H}^n \backslash \partial\mathcal{H}^n$.

Definition 1.2 (Smooth maps)**.** Let S be a subset of $\mathbb{R}^n$. A map $f : S \to \mathbb{R}^m$ is smooth at p if there exists a neighborhood U of p and a smooth function $f' : U \to \mathbb{R}^m$ such that $f' = f$ on $U \cap S$. If f is smooth at every $p \in S$, then f is smooth on S. In addition, if F is bijective and F^{-1} is smooth, then F is called a diffeomorphism.

Many properties of smooth maps on open sets also hold for smooth maps on arbitrary subsets. We will not prove them.

Definition 1.3. Let M be a Hausdorff, second countable topological space. A chart on M is a pair (U, ϕ) where U is open in M and $\phi : U \to \mathcal{H}^n$ is a homeomorphism onto its image. Two charts $(U, \phi), (V, \psi) \in \mathcal{A}$ are compatible if the functions $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are smooth on $\phi(U \cap V)$ and $\psi(U \cap V)$ respectively. An atlas $\mathcal{A}$ on M is a collection of pairwise compatible charts that cover M.



Theorem 1.4. Each atlas is contained in a unique maximal atlas. That is, for each atlas $\mathcal{A}$ on M, there exists a unique atlas $\mathcal{U}$ on M such that if $\mathcal{U} \subset \mathcal{U}'$, then $\mathcal{U}' = \mathcal{U}$.

Proof. Let $\mathcal{A}$ be an atlas on M. Let $\mathcal{U}$ be the set of charts that are compatible with every chart in $\mathcal{A}$. Let $(U, \phi), (V, \psi) \in \mathcal{U}$, and let $p \in U \cap V$. There exists $(W, \sigma) \in \mathcal{A}$ such that $p \in W$. So, $\psi \circ \sigma^{-1}$ and $\sigma \circ \phi^{-1}$ are smooth at $\sigma(p)$ and $\phi(p)$ respectively. Therefore,

$$\psi \circ \phi^{-1} = (\psi \circ \sigma^{-1}) \circ (\sigma \circ \phi^{-1})$$

is smooth at $\phi(p)$. As p was arbitrary, $\psi \circ \phi^{-1}$ is smooth on $\phi(U \cap V)$. Similarly, $\phi \circ \psi^{-1}$ is smooth on $\psi(U \cap V)$, so $(U, \phi), (V, \psi)$ are compatible, and $\mathcal{U}$ is indeed an atlas.

If $\mathcal{U} \subset \mathcal{U}'$, then every chart in $\mathcal{U}'$ is compatible with every chart in $\mathcal{U}$, in particular, with every chart in $\mathcal{A}$. By construction of $\mathcal{U}$, these charts are in $\mathcal{U}$, so $\mathcal{U}' \subset \mathcal{U}$, and $\mathcal{U}$ is maximal.

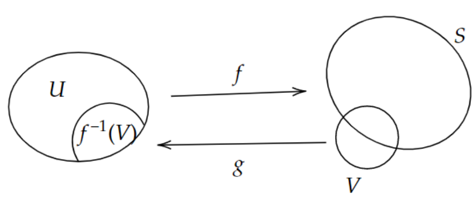
Suppose $\mathcal{V}$ is a maximal atlas containing $\mathcal{A}$. Then, every chart in $\mathcal{V}$ is compatible with $\mathcal{A}$, so $\mathcal{V} \subset \mathcal{U}$. Similarly, $\mathcal{U} \subset \mathcal{V}$. This shows uniqueness, and concludes the proof. $\square$

Definition 1.5. A n dimensional manifold M is a Hausdorff, second countable topological space together with a maximal atlas.

By Theorem 1.4, in order to construct a manifold, we only need to specify a topological space and an atlas. We write M instead of $(M, \mathcal{A})$ for manifolds. From now on, when we say a chart on M, we mean a chart in $\mathcal{A}$.

Theorem 1.6 (Invariance of domain)**.** Let $f : U \to S$ be a diffeomorphism, where U is open in $\mathbb{R}^n$ and $S \subset \mathbb{R}^n$ is an arbitrary subset. Then S is open in $\mathbb{R}^n$.

Proof. Let $p \in U$. Since f^{-1} is smooth, there exists a neighborhood V of $f(p)$ and a smooth function $g : V \to \mathbb{R}^n$ such that $g|_{V \cap S} = f^{-1}$.



Then, $g \circ f$ is the identity on $f^{-1}(V)$, which is open. So,

$$(Jg(f(p)))(Jf(p)) = I,$$

and $\det(Jf(p)) \neq 0$. By the inverse function theorem, there are neighborhoods $U_p \subset U, V_{f(p)} \subset V$ such that $f : U_p \to V_{f(p)}$ is a diffeomorphism. We also have

$$V_{f(p)} = f(U_p) \subset f(U) = S.$$

For each $p \in U$, we can find an open set $V_{f(p)}\mathbb{R}^n$ such that $V_{f(p)} \subset S$. So, S is open. $\square$

Corollary 1.7. Let U and V be open subsets of $\mathcal{H}^n$, and let $f : U \to V$ be a diffeomorphism. Then, f maps interior points to interior points and boundary points to boundary points.

Proof. Suppose $p \subset U$ is an interior point. Then, it has an neighborhood U_p open in $\mathbb{R}^n$. By the above theorem, $f(p)$ lies in the open set $f(U_p)$, so $f(p)$ is an interior point. Similarly, if $f(p)$ is an interior point, $f^{-1}(f(p)) = p$ is an interior point. $\square$

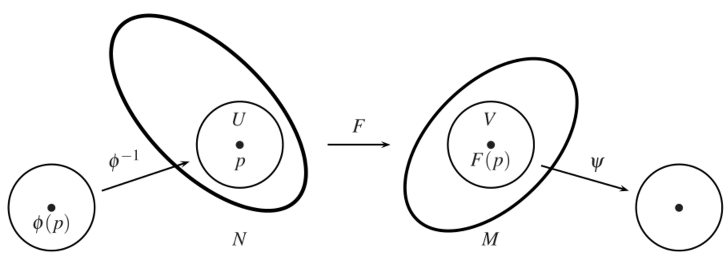
Definition 1.8. Let M be a manifold. A point $p \in M$ is a boundary point if for some chart (U, ϕ), $\phi(p) \in \partial\mathcal{H}^n$. The set of all boundary points is the boundary ∂M.

Suppose (U, ϕ) and (V, ψ) are charts containing p, with $\phi(p) \in \partial\mathcal{H}^n$. By Corollary 1.7, $\psi(p) = \psi \circ \phi^{-1}(\phi(p))$ is also a boundary point. Thus, $\psi(p)$ is a boundary point for all charts (V, ψ). If the boundary of a manifold is empty, then for any chart (U, ϕ), $\phi(U)$ is open in $\mathbb{R}^n$.

Proposition 1.9. Let M be a manifold with non empty boundary. Then, ∂M is a manifold with empty boundary.

Proof. Let $\mathcal{A}$ be an atlas on M. For each $(U, \phi) \in \mathcal{A}$, we construct a chart $(U \cap \partial M, \phi|_{\partial M})$ on ∂M. These charts are compatible, and map into $\mathbb{R}^{n-1}$, so ∂M is a manifold of dimension $n - 1$. $\square$

Definition 1.10 (Smooth Functions)**.** Let N, M be manifolds. A continuous function $F : N \to M$ is smooth if for all charts (U, ϕ) on N and (V, ψ) on M, $\psi \circ F \circ \phi^{-1}$ is smooth. If F is bijective and F^{-1} is also smooth, then F is a diffeomorphism.



Unless stated otherwise, all functions are smooth.

Definition 1.11 (Partial derivatives)**.** Let $f : M \to \mathbb{R}$ and $(U, x_1, \ldots, x_n)$ be a chart. The partial derivative of f with respect to x_i at p is the derivative with respect to standard coordinates

$$\left.\frac{\partial}{\partial x_i}\right|_p f = \left.\frac{\partial}{\partial r_i}\right|_{\phi(p)} (f \circ \phi^{-1}).$$

Definition 1.12 (Germs of functions)**.** Let M be a manifold, and $p \in M$. Let S be the set of all smooth functions defined on a neighborhood of p. For $f, g \in S$, we define an equivalence relation by $f \sim g$ if there is a neighborhood of p on which $f = g$. These equivalence classes are the germs of M at p, denoted $C_p^\infty(M)$.

Definition 1.13. Let M be a manifold, and $p \in M$. A derivation at p is a linear map $D : C_p^\infty(M) \to \mathbb{R}$ such that $D(fg) = D(f)g(p) + f(p)D(g)$.

Definition 1.14. A tangent vector at p is a derivation at p. The tangent space of M at p is the set of all derivations, denoted T_pM.

$\frac{\partial}{\partial x_1}|_p, \ldots, \frac{\partial}{\partial x_n}|_p$ are examples of tangent vectors.

Definition 1.15. Let $F : N \to M$ be a smooth function. We define the differential of F to be a function

$$F_* : T_pN \to T_{F(p)}M$$

defined by

$$F_*(X_p)(f) = X_p(f \circ F) \quad \text{for } f \in C^\infty_{F(p)}(M).$$

It is straightforward to check that T_pM is a vector space, and that F_* is a linear map.

Theorem 1.16 (Chain rule)**.** Let $F : N \to M$, $G : M \to P$, and $p \in N$. Then, $(G \circ F)_* = G_* \circ F_*$.

Proof. Let $X_p \in T_pN$ and f be a smooth function in a neighborhood of $G(F(p))$. Then

$$(G \circ F)_*(X_p)f = X_p(f \circ G \circ F) = (F_*X_p)(f \circ G) = (G_* \circ F_*(X_p))f.$$

□

Theorem 1.17. Let $(U, x_1, \ldots, x_n)$ be a chart. Then,

$$\left.\frac{\partial}{\partial x_1}\right|_p, \ldots, \left.\frac{\partial}{\partial x_n}\right|_p$$

is a basis for T_pM.

Proof. By the chain rule, $\mathrm{id} = (\phi^{-1} \circ \phi)_* = \phi^{-1}_* \circ \phi_*$, so ϕ_* is an isomorphism. We see that

$$\phi_* \left(\left.\frac{\partial}{\partial x_i}\right|_p \right) f = \left.\frac{\partial}{\partial x_i}\right|_p (f \circ \phi) = \left.\frac{\partial}{\partial r_i}\right|_{\phi(p)} (f \circ \phi \circ \phi^{-1}) = \frac{\partial}{\partial r_i}|_{\phi(p)} f.$$

It remains to show that $\frac{\partial}{\partial r_i}|_{\phi(p)}$ is a basis of $T_{\phi(p)}\mathbb{R}^n$. For simplification, we replace $\phi(p)$ by p.

Suppose $\sum_{i=1}^n a_i \frac{\partial}{\partial r_i}|_p = 0$. Applying to the coordinate functions r_i, we see that $a_i = 0$.

Next, we show that they span the space. Let D be a derivation, and f a smooth function. We technically require f to be a germ, but the argument still holds. By Taylor's theorem,

$$f(x) = f(p) + \sum_{i=1}^n (x_i - p_i) g_i(x) \quad g(p) = \frac{\partial f}{\partial x_i}(p).$$

Using the Leibniz rule and linearity,

$$Df(x) = \sum_{i=1}^n Dx_i g_i(p) = \sum_{i=1}^n Dx_i \frac{\partial}{\partial r_i}\bigg|_p f.$$

We have cancelled the terms $Df(p)$ and Dp_i since

$$D(1) = D(1 \cdot 1) = 1D(1) + D(1)1 = 2D(1),$$

so $D(1) = 0$ and $D(c) = 0$ by linearity. Thus,

$$Df = \sum_{i=1}^n Dx_i \frac{\partial}{\partial r_i}\bigg|_p.$$

□

Definition 1.18. Let $p \in M$. The set of all alternating multilinear functions from $(T_pM)^k$ to $\mathbb{R}$ is denoted by $\Lambda^k T_pM$. That is, for $\omega_p \in \Lambda^k T_pM$ and $v_1, \dots, v_n, w_i \in T_pM$,

$$\begin{aligned} \omega_p(\dots, v_i, v_{i+1}, \dots) &= -\,\omega_p(\dots, v_{i+1}, v_i, \dots) && 1 \le i < n \\ \omega_p(\dots, v_i + cw_i, \dots) &= \omega_p(\dots, v_i, \dots) + c\omega_p(\dots, w_i, \dots) && 1 \le i \le n \end{aligned}$$

Definition 1.19. A k-form is a map that sends $p \in M$ to $\omega_p \in \Lambda^k T_pM$.

We define $(dx_1)_p, \dots, (dx_n)_p$ to be the dual basis of $\frac{\partial}{\partial x_1}|_p, \dots, \frac{\partial}{\partial x_n}|_p$. By a result in multilinear algebra, $\{(dx_{i_1})_p \wedge \cdots \wedge (dx_{i_k})_p \mid 1 \le i_1 < \cdots < i_n \le n\}$ is a basis for $\Lambda^k T_pM$. If we fix a chart $(U, x_1, \dots, x_n)$, then we can write $\omega = \sum a_I dx_I$, where a_I are real valued functions on U.

Definition 1.20. Let ω be a k-form. We say that ω is smooth if for every chart $(U, x_1, \dots, x_n)$, the coefficients a_I in $\omega = \sum a_I dx_I$ are smooth. The set of smooth k-forms is denoted by $\Omega^k(M)$. The graded algebra $\bigoplus \Omega^k(M)$ is denoted $\Omega^*(M)$.

Definition 1.21. A vector field X on M is a function that assigns to each $p \in M$ a tangent vector $X_p \in T_pM$. A vector field is smooth (continuous) if for every chart $(U, x_1, \dots, x_n)$, the coefficients a_i in $X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}|_p$ are smooth (continuous). Given a 1-form ω, we define $\omega(X) : M \to \mathbb{R}, p \mapsto \omega_p(X_p)$.

Definition 1.22. An exterior derivative on M is an $\mathbb{R}$-linear map $\Omega^*(M) \to \Omega^*(M)$ such that

1. $D(\omega \cdot \tau) = (D\omega) \cdot \tau + (-1)^k \omega \cdot D\tau$ for $\omega \in \Omega^k(M)$, $\tau \in \Omega^\ell(M)$,
2. $D\omega \in \Omega^{k+1}(M)$ for $\omega \in \Omega^k(M)$,
3. $D \circ D = 0$,
4. if $f : M \to \mathbb{R}$ is smooth and X is a smooth vector field, then $(Df)X = Xf$.

Theorem 1.23. Let $(U, x_1, \dots, x_n)$ be a chart containing p. We define an operator $d_U : \Omega^*(U) \to \Omega^*(U)$ by

$$d_U f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i, \; d_U \omega = \sum_I d_U f \wedge dx_I, \text{ where } \omega = \sum_I f_I dx_I.$$

Next, we define an operator $d : \Omega^*(M) \to \Omega^*(M)$ given by $(d\omega)_p = (d_U \omega)_p$ for some chart (U, ϕ) containing p. Then, d is well defined, and it is the unique exterior derivative on M.

Proof. See [Tu, Theorem 19.4, p.214]. □

Definition 1.24 (Pullback)**.** Let $\omega_p \in \Lambda^k T_p M$ and $F : N \to M$. The pullback of ω_p by F is $F^*(\omega_p) \in \Lambda^k T_p N$, defined as

$$F^*(\omega_p)(v_1, \dots, v_k) = \omega_p(F_*(v_1), \dots, F_*(v_k)).$$

The pullback of a differential form ω is defined as $(F^*\omega)_p = F^*(\omega_p)$.

Some properties of the pullback include

Proposition 1.25. $F^*(\omega \wedge \tau) = F^*\omega \wedge F^*\tau$.

Proposition 1.26. $F^* d\omega = dF^*\omega$.

Proposition 1.27. If $\omega \in \Omega^k(M)$, then $F^*\omega \in \Omega^k(N)$.

2 Integration of Differential n-Forms

From now on, we assume that for manifolds M and charts (U, ϕ), M and U are path connected.

Let ω, τ be non-vanishing differential n-forms, and (U, ϕ) be a chart. That is, ω_p and τ_p are not the zero maps for all $p \in M$. Then, $\omega = \sum_{i=1}^n f dx_1 \wedge \cdots \wedge dx_n$ and $\tau = \sum_{i=1}^n g dx_1 \wedge \cdots \wedge dx_n$ for smooth functions f, g. Since ω, τ are non vanishing f, g are non vanishing. By path connectedness and the intermediate value theorem, we see that f and g are either positive or negative. Furthermore, $\omega = f/g\tau$ on U. This holds for all charts, so $\omega = f\tau$ for some positive or negative f.

Definition 2.1. Let $\omega \sim \tau$ if $\omega = f\tau$ for some positive f, where ω, τ are non vanishing differential n-forms. This is an equivalence relation, and the equivalence classes are called orientations. If a manifold has an orientation, it is orientable. A manifold together with an orientation is an oriented manifold.

Definition 2.2. An oriented atlas is an atlas $\mathcal{A}$ such that for all $(U, \phi), (V, \psi) \in \mathcal{A}$, $\det(J(\psi \circ \phi^{-1})) > 0$.

There is actually an equivalence between orientations and equivalence classes of oriented atlases. Let $\{(U_\alpha, \phi_\alpha)\} \sim \{(V_\beta, \psi_\beta)\}$ if $\det(J\psi_\beta \circ \phi_\alpha^{-1}) > 0$ for all α, β.

Theorem 2.3. Given an orientation $[\omega]$, there exists an oriented atlas $\mathcal{A}_\omega$ such that $\omega_p(\partial_1|_p, \ldots, \partial_n|_p) > 0$ for all $(U, x_1, \ldots, x_n)$. Given an oriented atlas, it is possible to construct a non vanishing n-form. Then, the identification $[\omega] \Leftrightarrow [\mathcal{A}_\omega]$ is well defined.

Proof. See [Tu, Theorem 21.5, Theorem 21.10]. □

Definition 2.4. A diffeomorphism $F : N \to M$ between oriented manifolds is orientation preserving if $[F^*\omega]$ is the orientation for N, where $[\omega]$ is the orientation for M.

The above definition is well defined, and is in fact equivalent to the condition that $\det(J(\psi \circ F^\circ \phi^{-1})) > 0$ for all charts (U, ϕ) on N and (V, ψ) on M [Tu, Proposition 21.8, p.244].

Next, we define the orientation for the boundary.

Definition 2.5. Let $X_p \in T_pM$, $\omega_p \in \Lambda^k T_pM$. We define a $\iota_{X_p}\omega_p \in \Lambda^{k-1}T_pM$ called the interior multiplication of ω_p with X_p, defined by

$$(\iota_{X_p}\omega_p)(v_1, \ldots, v_{k-1}) = \omega_p(X_p, v_1, \ldots, v_{k-1}).$$

We also define the interior multiplication of a differential form ω with a vector field X by

$$(\iota_X\omega)_p = \iota_{X_p}\omega_p.$$

Proposition 2.6. Let $\omega_1, \ldots, \omega_k \in \Lambda^1 T_pM$, and $v \in T_pM$. Then

$$\iota_v(\omega_1 \wedge \cdots \wedge \omega_k) = \sum_{i=1}^{k} (-1)^{k-1} \omega_i(v) \omega_1 \wedge \cdots \wedge \widehat{\omega_i} \wedge \cdots \wedge \omega_k,$$

where the $\widehat{}$ on ω_i means that ω_i is omitted from the wedge product.

Proof.

$$
\begin{aligned}
&\iota_{v_1}(\omega_1 \wedge \cdots \wedge \omega_k)(v_2, \ldots, v_k)\\
=&\omega_1 \wedge \cdots \wedge \omega_k(v_1 \ldots, v_k)\\
=&\det(\omega_i(v_j))\\
=&\sum_{i=1}^{k}(-1)^{k-1}\omega_i(v_1)\det(\omega_\ell(v_j))_{\ell\neq i, j>1}\\
=&\sum_{i=1}^{k}(-1)^{k-1}\omega_i(v_1)\omega_1 \wedge \cdots \wedge \widehat{\omega_i} \wedge \cdots \wedge \omega_k(v_2, \ldots, v_k)
\end{aligned}
$$

□

Definition 2.7. A vector $v \in T_pM$ is outward pointing if for some chart (U, ϕ), $v = \sum a_i \partial_i|_p$, where $a_n < 0$. An outward pointing vector field is a map that assigns to each $p \in \partial M$ an outward pointing vector $v_p \in T_pM$.

Proposition 2.8. Every manifold has an outward pointing vector field.

Proof. See [Tu, Proposition 22.10, p.254]. □

Proposition 2.9. Let M be an oriented manifold with orientation $[\omega]$, and X be an outward pointing vector field. Then $\iota_X\omega$ is a non vanishing $(n-1)$-form on ∂M. The orientation $[\iota_X\omega]$ for ∂M is well defined.

Proof. See [Tu, Proposition 22.11, p.255]. □

For example, $-\partial_n$ is an outward pointing vector field for $\mathcal{H}^n$. Then, the boundary of $\mathcal{H}^n$ is

$$\iota_{-\partial_n}(dx_1 \wedge \cdots \wedge dx_n) = -\iota_{\partial_n}(dx_1 \wedge \cdots \wedge dx_n) = (-1)^n dx_1 \wedge \cdots \wedge dx_{n-1}.$$

Finally, we define integration of differential forms.

Definition 2.10. Let $f : S \to \mathbb{R}$ for some topological space S. The support of f, $\operatorname{supp}(f)$, is the closure of the set $\{p \in S \mid f(p) \neq 0\}$. Let $\omega \in \Omega^k(M)$. The support of ω is the closure of the set $\{p \in M \mid \omega_p \neq 0\}$. The set of compactly supported differential k-forms on M is denoted by $\Omega^k_c(M)$.

Definition 2.11 (Integration on a $\mathbb{R}^n$)**.** Let $\omega \in \Omega^n_c(U)$, where $U \subset \mathbb{R}^n$. Then $\omega = f(x)dx_1 \wedge \cdots \wedge dx_n$ for some smooth f. The integral of ω over U is $\int_U \omega = \int_U f$.

Definition 2.12 (Integration on a chart)**.** Let $\omega \in \Omega^n_c(U)$ for some (U, ϕ). The integral of ω over U is $\int_U \omega = \int_{\phi(U)} (\phi^{-1})^*\omega$.

To show that the integral does not depend on ϕ, we first need a lemma.

Lemma 2.13. Let U, V be open in $\mathbb{R}^n$, $T : V \to U$ be a diffeomorphism, and $\omega \in \Omega^n_c(U)$. If $\det(JT) > 0$ on V, then $\int_V T^*\omega = \int_U \omega$.

Proof.

$$\begin{aligned}
\int_V T^*\omega &= \int_V T^*(f dx_1 \wedge \cdots \wedge dx_n) \\
&= \int_V (f \circ T) T^* dx_1 \wedge \cdots \wedge T^* dx_n \\
&= \int_V (f \circ T) dT^* x_1 \wedge \cdots \wedge dT^* x_n \\
&= \int_V (f \circ T) dx_1 \circ T \wedge \cdots \wedge dx_n \circ T
\end{aligned}$$

Expressing each $dx_i \circ T$ as $\sum_j \frac{\partial x_i \circ T}{\partial y_j} dy_j$, we see that

$$dx_1 \circ T \wedge \cdots \wedge dx_n \circ T = \det(JT) dy_1 \wedge \cdots \wedge dy_n.$$

So, this becomes

$$\begin{aligned}
&\int_V (f \circ T) \det(JT) dy_1 \wedge \cdots \wedge dy_n \\
= &\int_V (f \circ T) |\det(JT)| dy_1 \wedge \cdots \wedge dy_n \\
= &\int_V (f \circ T) |\det(JT)| dy_1 \dots dy_n \\
= &\int_U f dx_1 \dots dx_n \\
= &\int_U \omega
\end{aligned}$$

□

Let ϕ, ψ be two coordinate functions on U such that $\det(J(\phi \circ \psi^{-1})) > 0$. By the lemma, $\int_{\phi(U)} (\phi^{-1})^* \omega = \int_{\psi(U)} (\phi \circ \psi^{-1})^* (\phi^{-1})^* \omega = \int_{\psi(U)} (\psi^{-1})^* \omega$, so $\int_U \omega$ is well defined.

To define integration on manifolds, we need a partition of unity.

Definition 2.14. A collection $\{A_\alpha\}$ of subsets of a topological space S is said to be locally finite if every point q in S has a neighborhood that meets only finitely many of the sets $\{A_\alpha\}$.

Definition 2.15 (Partition of Unity)**.** A partition of unity on a manifold M is a collection of nonnegative smooth functions $\{\rho_\alpha : M \to \mathbb{R}\}_{\alpha \in A}$ such that

(i) the collection of supports, $\{\operatorname{supp} \rho_\alpha\}_{\alpha \in A}$, is locally finite,

(ii) $\sum_{\alpha \in A} \rho_\alpha = 1$.

Definition 2.16. Let $\{U_\alpha\}_{\alpha \in A}$ be a collection of subsets on M. A partition of unity $\{\rho_\alpha\}$ is subordinate to $\{U_\alpha\}$ if $\operatorname{supp} \rho_\alpha \in U_\alpha$.

Theorem 2.17. Let $\{U_\alpha\}$ be an open cover of M. Then, there exists a partition of unity subordinate to $\{U_\alpha\}$.

Proof. See [Tu, Theorem 13.7, p.347]. $\square$

Proposition 2.18. Let $\{\rho_\alpha\}_{\alpha\in A}$ be a collection of functions on M, and ω be a differential k-form with compact support. If $\{\operatorname{supp}\rho_\alpha\}_{\alpha\in A}$ is locally finite, then $\rho_\alpha\omega \equiv 0$ for all but finitely many α.

Proof. For each $p \in \operatorname{supp}\omega$, let U_p be a neighborhood of p that intersects finitely many $\operatorname{supp}\rho_\alpha$. Then the U_p is an open cover of $\operatorname{supp}\omega$, so there is a finite subcover $U_{p_1}, \dots, U_{p_k}$. Each U_{p_i} only intersects finitely many $\operatorname{supp}\rho_\alpha$, so $\operatorname{supp}\omega$ only intersects finitely many $\operatorname{supp}\rho_\alpha$. Finally, since

$$\{p \in M \mid \rho_\alpha(p)\omega_p\} \subset \{p \in M \mid \rho_\alpha(p) \neq 0\} \cap \{p \in M \mid \omega_p \neq 0\},$$

$\operatorname{supp}(\rho_\alpha\omega) \subset \operatorname{supp}\rho_\alpha \cap \operatorname{supp}\omega$. So, $\operatorname{supp}(\rho_\alpha\omega)$ is nonempty for finitely many α. $\square$

Definition 2.19 (Integration on a Manifold)**.** Let ω be a differential n-form on an oriented manifold M, and let $\{\rho_\alpha\}$ be a partition of unity subordinate to the oriented atlas. The integral of ω over M, denoted by $\int_M \omega$, is defined to be $\sum_{\alpha\in A}\int_{U_\alpha}\rho_\alpha\omega$.

By Lemma 2.18, the sum is finite. For the definition to make sense, we need to show that the integral does not depend on the atlas and the partition of unity. Let $\{(U_\alpha, \phi_\alpha)\}, \{(V_\beta, \psi_\beta)\}$ be oriented atlases and $\{\rho_\alpha\}, \{\chi_\beta\}$ be subordinate partitions of unity.

$$\begin{aligned}
\sum_\alpha \int_{U_\alpha} \rho_\alpha\omega &= \sum_\alpha \int_{U_\alpha} \rho_\alpha\Big(\sum_\beta \chi_\beta\Big)\omega \\
&= \sum_\alpha \sum_\beta \int_{U_\alpha} \rho_\alpha\chi_\beta\omega \\
&= \sum_\alpha \sum_\beta \int_{U_\alpha\cap V_\beta} \rho_\alpha\chi_\beta\omega
\end{aligned}$$

where the last equality holds since the support of $\rho_\alpha\chi_\beta$ is contained in $U_\alpha \cap V_\beta$.

To prove Stokes' theorem, we first prove it for $\mathcal{H}^n$.

Theorem 2.20. Let ω be a smooth $(n-1)$-form on $\mathcal{H}^n$ with compact support, and $\iota : \partial M \to M$ be the inclusion map. Using $\int_{\partial\mathcal{H}^n}\omega$ as shorthand for $\int_{\partial\mathcal{H}^n}\iota^*\omega$, $\int_{\mathcal{H}^n} d\omega = \int_{\partial\mathcal{H}^n}\omega$.

Proof. We know that $\omega = \sum f_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$. So,

$$\begin{aligned} d\omega &= \sum_i \left(\sum_j \frac{\partial f_i}{\partial x_j} dx_j \right) \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\ &= \sum_i \frac{\partial f_i}{\partial x_i} dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\ &= \sum_i (-1)^{i-1} \frac{\partial f_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_n. \end{aligned}$$

Then,

$$\int_{\mathcal{H}^n} d\omega = \sum_i (-1)^{i-1} \int_{\mathcal{H}^n} \frac{\partial f_i}{\partial x_i}.$$

For $i < n$,

$$\int_{\mathcal{H}^n} \frac{\partial f_i}{\partial x_i} = \int_0^\infty \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \frac{\partial f_i}{\partial x_i} dx_i dx_1 \ldots dx_n + (-1)^{n-1}.$$

Since ω is compactly supported, the support is contained in $[-a, a]^{n-1} \times [0, a]$ for some a. So, the innermost integral is

$$\begin{aligned} \int_{-\infty}^\infty \frac{\partial f_i}{\partial x_i} dx_i = &\int_{-a}^a \frac{\partial f_i}{\partial x_i} dx_i \\ =&f_i(\ldots, a, \ldots) - f_i(\ldots, -a, \ldots) \\ =&0 \end{aligned}$$

Thus,

$$\begin{aligned} \int_{\mathcal{H}^n} d\omega =&(-1)^{n-1} \int_{\mathcal{H}^n} \frac{\partial f_n}{\partial x_n} \\ =&(-1)^{n-1} \int_{\mathbb{R}^{n-1}} \int_0^a \frac{\partial f_n}{\partial x_n} dx_n dx_1 \ldots dx_{n-1} \\ =&(-1)^n \int_{\mathbb{R}^{n-1}} f_n(x_1, \ldots, x_{n-1}, 0) \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_{\partial\mathcal{H}^n} \iota^* \omega &= \int_{\partial\mathcal{H}^n} \iota^* \sum f_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\ &= \int_{\partial\mathcal{H}^n} \sum \iota^* f_i d\iota^* x_1 \wedge \cdots \wedge \widehat{d\iota^* x_i} \wedge \cdots \wedge d\iota^* x_n. \end{aligned}$$

But $\iota^* x_n = 0$ on $\partial\mathcal{H}^n$. Writing $\iota^* x_i$ as just x_i, this becomes

$$\begin{aligned} \int_{\partial\mathcal{H}^n} \iota^* f_n dx_1 \wedge \cdots \wedge dx_{n-1} =&(-1)^n \int_{\partial\mathcal{H}^n} \iota^* f_n d(-1)^n x_1 \wedge \cdots \wedge dx_{n-1} \\ =&(-1)^n \int_{\mathbb{R}^{n-1}} f_n(x_1, \ldots, x_{n-1}, 0), \end{aligned}$$

as $(\partial\mathcal{H}^n, (-1)^n x_1, \ldots, x_{n-1})$ is a chart in the oriented atlas for $\partial\mathcal{H}^n$ with the boundary orientation. $\square$

Now, we prove Stokes' theorem for manifolds.

Theorem 2.21. Let ω be a smooth $(n-1)$-form on an oriented manifold with boundary M. Then $\int_M d\omega = \int_{\partial M} \omega$.

Proof.

$$\begin{aligned}
\int_M d\omega &= \sum_\alpha \int_{U_\alpha} d\rho_\alpha \omega \\
&= \sum_\alpha \int_{\phi(U_\alpha)} (\phi^{-1})^* d\rho_\alpha \omega \\
&= \sum_\alpha \int_{\phi(U_\alpha)} d(\phi^{-1})^* \rho_\alpha \omega \\
&= \sum_\alpha \int_{\partial\phi(U_\alpha)} \iota_{\mathcal{H}^n}^* (\phi^{-1})^* \rho_\alpha \omega \\
&= \sum_\alpha \int_{\phi(\partial U_\alpha)} (\phi^{-1} \circ \iota_{\mathcal{H}^n})^* \rho_\alpha \omega
\end{aligned}$$

Let $\phi' = (x_1|_{\partial M}, \ldots, x_{n-1}|_{\partial M})$, and $\iota_M : \partial M \to M$ be the inclusion map. Since $\phi^{-1} \circ \iota_{\mathcal{H}^n} = \iota_M \circ \phi'^{-1}$, this becomes

$$\begin{aligned}
&\sum_\alpha \int_{\phi'(\partial U_\alpha)} (\iota_M \circ \phi'^{-1})^* \rho_\alpha \omega \\
=&\sum_\alpha \int_{\phi'(\partial U_\alpha)} (\phi'^{-1})^* \iota_M^* \rho_\alpha \omega \\
=&\sum_\alpha \int_{\partial U_\alpha} \iota_M^* \rho_\alpha \omega \\
=&\int_{\partial M} \omega
\end{aligned}$$

$\square$

3 Hairy Ball Theorem

Theorem 3.1. Hairy Ball Theorem There exists a nowhere vanishing vector field on S^n if and only if n is odd.

To prove this, we first need a lemma on homotopies between manifolds.

Definition 3.2. If X and Y are topological spaces and $F_0, F_1 : X \to Y$ are continuous maps, a homotopy from F_0 to F_1 is a continuous map $H : X \times I \to Y$

satisfying

$$H(x,0) = F_0(x)$$
$$H(x,1) = F_1(x)$$

for all $x \in X$, where I is the unit interval $[0,1]$. If there exists a homotopy from F_0 to F_1, we say that F_0 and F_1 are homotopic. If there exists a smooth homotopy $H : X \times I \to Y$ from F_0 to F_1, then we say that F_0 and F_1 are smoothly homotopic.

Lemma 3.3. Suppose N is a manifold, M is a manifold with empty boundary, and $F, G : N \to M$ are smooth maps. If F and G are homotopic, then they are smoothly homotopic.

Proof. See [Lee, Theorem 6.29, p.142]. □

We now prove the Hairy Ball Theorem.

Theorem 3.4. The following are equivalent:

1. There exists a nowhere-vanishing continuous vector field on S^n.
2. There exists a continuous map $V : S^n \to S^n$ satisfying $V(x) \perp x$ (with respect to the Euclidean dot product on $\mathbb{R}^{n+1}$) for all $x \in S^n$.
3. The antipodal map $\alpha : S^n \to S^n$, $x \mapsto -x$ is homotopic to id_{S^n}
4. The antipodal map α is orientation-preserving.
5. n is odd.

Proof. $1 \implies 2$: Suppose there is a nowhere vanishing vector field on S^n. Let $\iota : S^n \to \mathbb{R}^{n+1}$ be the inclusion map. Define $V : S^n \to \mathbb{R}^{n+1}$ by $V(p) = \iota_*(X_p)$, where we use the standard identification of $T_p\mathbb{R}^{n+1}$ with $\mathbb{R}^{n+1}$. As X is non-vanishing, we can normalize V to get a continuous map $V' : S^n \to S^n$ satisfying $V(x) \perp x$.

$2 \implies 3$: Let $V : S^n \to S^n$ be a continuous map satisfying $V(x) \perp x$ for all $x \in S^n$. Let $H : S^n \times I \to S^n$ given by $H(x,t) = x\cos(t\pi) + V(x)\sin(t\pi)$. Then, $H(x,0) = x = \mathrm{id}_{S^n}(x)$ and $H(x,1) = -x = \alpha(x)$. Since V is continuous, H is continuous. Also, since $V(x) \perp x$, $|H(x,t)|^2 = 1$, so $H(x,t) \in S^n$. Thus, H is a homotopy from id_{S^n} to α.

$3 \implies 4$: We prove a more general version for this implication: suppose M and N are oriented, compact, connected, smooth manifolds, and $F, G : M \to N$ are homotopic diffeomorphisms. Then F and G are either both orientation-preserving or both orientation-reversing. The implication $(3 \Rightarrow 4)$ is a direct consequence of the lemma.

To prove this, first note that by Lemma 3.3, there is a smooth homotopy $H : M \times I \to N$ from F to G. Let $[\omega]$ be the orientation for N. Then, since

$d\omega = 0$, so

$$\begin{aligned} 0 &= \int_{M\times I} H^* d\omega \\ &= \int_{M\times I} dH^*\omega \\ &= \int_{\partial M\times I} H^*\omega \\ &= \int_{M\times\{0\}} H^*\omega + \int_{M\times\{1\}} H^*\omega \\ &= \pm\left(\int_M G^*\omega - \int_M F^*\omega\right) \end{aligned}$$

where the $\pm$ comes from which orientation we choose for $M\times I$. The application of Stokes' theorem requires that the forms have compact support, which they do as they are non-vanishing on compact manifolds. So, $\int_M G^*\omega = \int_M F^*\omega$. They cannot be equal to 0 as M is connected. So, they are either both positive or both negative, which means that $[F^*\omega] = [G^*\omega]$ and concludes the proof for the lemma.

$4 \implies 5$: Let B^{n+1} be the closed unit ball in $\mathbb{R}^{n+1}$, $A : B^{n+1} \to B^{n+1}$ be the map $x \mapsto -x$, $\Omega = dx_1 \wedge \cdots \wedge dx_{n+1}$, and $X = \sum x_i \frac{\partial}{\partial x_i}$. The orientation of S^n is given by $\omega = \iota^*(\iota_X(\Omega))$, where ι is the inclusion map. What we want to show is that $\alpha^*\omega = \omega$. First, note that $A\circ\iota = \iota\circ\alpha$, so

$$\alpha^*\omega = \alpha^*\iota^*(\iota_X\Omega) = \iota^*A^*(\iota_X\Omega).$$

Also, $A_*X_p = X_{A(p)}$ by direct computation, and $A^*\Omega = (-1)^{n+1}\Omega$, using the distributive property of the pullback and that it commutes with d. Therefore,

$$\begin{aligned} A^*(\iota_X\Omega)_p(v_1,\ldots,v_n) &= (\iota_X\Omega)_{A(p)}(A_*v_1,\ldots,A_*v_n) \\ &= \Omega_{A(p)}(X_{A(p)}, A_*v_1,\ldots,A_*v_n) \\ &= \Omega_{A(p)}(A_*X_p, A_*v_1,\ldots,A_*v_n) \\ &= (A^*\Omega)_p(X_p, v_1,\ldots,v_n) \\ &= (-1)^{n+1}\Omega_p(X_p, v_1,\ldots,v_n) \\ &= (-1)^{n+1}(\iota_X\Omega)_p(v_1,\ldots,v_n). \end{aligned}$$

Thus,

$$\alpha^*\omega = \iota^*A^*(\iota_X\Omega) = (-1)^{n+1}\iota^*(\iota_X\Omega) = (-1)^{n+1}\omega,$$

so α is orientation preserving if and only if n is odd.

$5 \implies 1$: If n is odd, we obtain a continuous map $V : p \mapsto ip$ by considering points in S^n as points in $\mathbb{C}^{(n+1)/2}$. Then, we get a non-vanishing vector field $X_p = \iota^{-1}(V(p)) = \gamma_*(\frac{d}{dt}|_{t=0})$, where $\gamma(t) = e^{it}p$.

$\square$

References

[1] Loring W. Tu (2010) *An Introduction to Manifolds*, Springer.

[2] John M. Lee (2013) *Introduction to Smooth Manifolds*, Springer.