Differentiable manifolds and the Stokes' Theorem

Jonathan Lau

March 8, 2023

Stokes' Theorem:

$$\int_M d\omega = \int_{\partial M} \omega$$

Special cases:

$$\int_{\partial D} Pdx + Qdy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

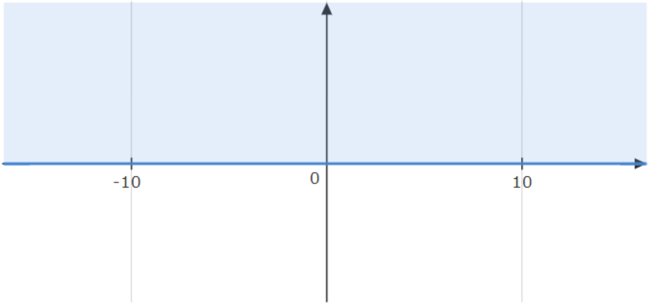
$$\int_{\partial V} F \cdot dS = \int_V \mathrm{div} F$$

Table of Contents

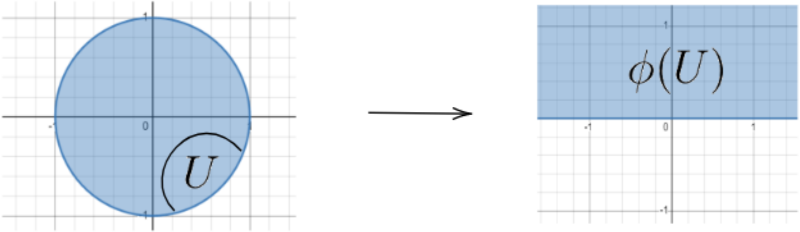
Table of Contents

Upper Half Space

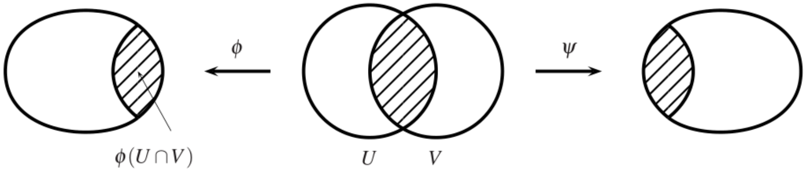
The upper half space is $\mathcal{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$. Its boundary is $\partial\mathcal{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n = 0\}$.



Charts

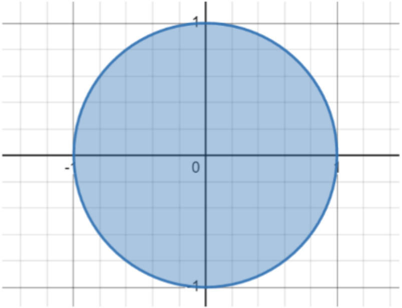


Manifolds

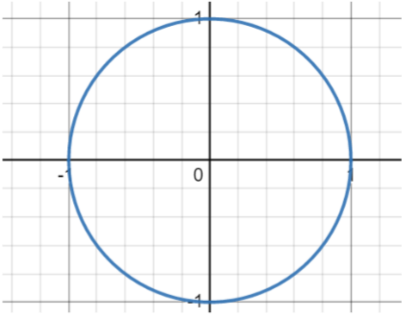


Boundary

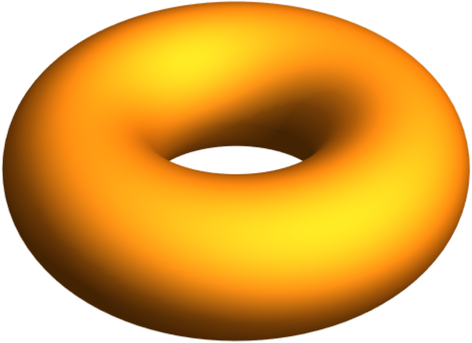
M



∂M



Examples



Examples

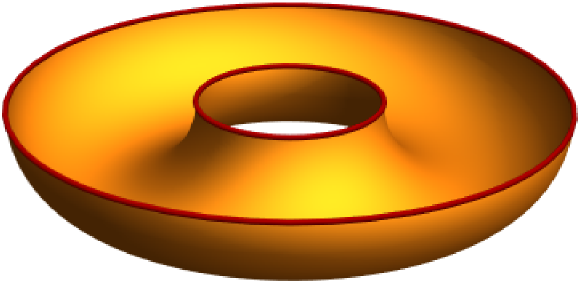
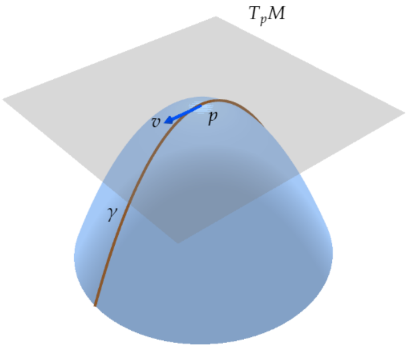
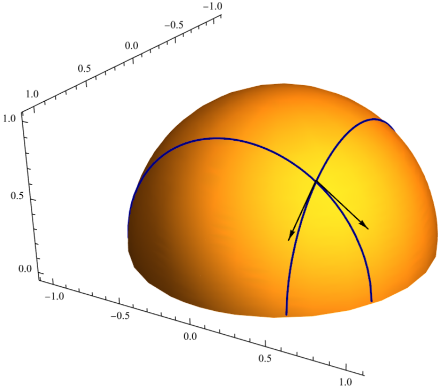


Table of Contents

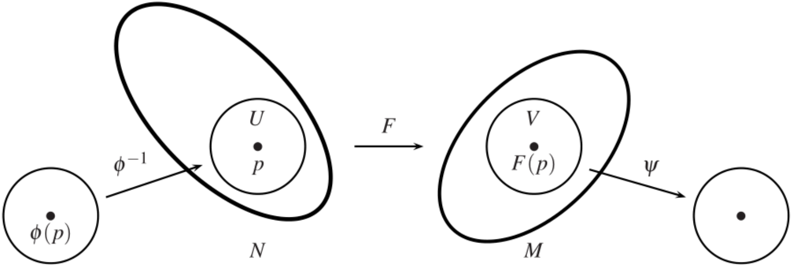
Tangent Space



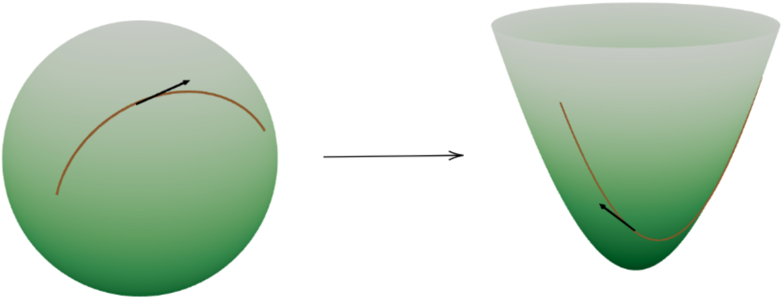
Tangent vectors



Smooth Functions



Differential



Alternating multilinear

Alternating:

$$\omega_p(v_1, \ldots, v_i, v_{i+1}, \ldots, v_k) = -\omega_p(v_1, \ldots, v_{i+1}, v_1, \ldots, v_k)$$

Multilinear:

$$\omega_p(v_1, \ldots, v_i + cw_i, \ldots, v_k) = \omega_p(v_1, \ldots, v_i, \ldots, v_k) + c\omega_p(v_1, \ldots, w_i, \ldots, v_k)$$

Table of Contents

Partition of Unity

Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ be an atlas on M. A partition of unity on a manifold M is a collection of nonnegative smooth functions $\{\rho_\alpha : M \to \mathbb{R}\}_{\alpha \in A}$ such that

1. the collection of supports, $\{\operatorname{supp} \rho_\alpha\}_{\alpha \in A}$, is locally finite,
2. $\sum_{\alpha \in A} \rho_\alpha = 1$.
3. $\operatorname{supp} \rho_\alpha \subset U_\alpha$ for all $\alpha \in A$.

Stokes' theorem

Let M be an oriented n dimensional manifold with non empty boundary, and let ω be a differential $(n-1)$-form on M with compact support. Give ∂M the boundary orientation, and let $\iota : \partial M \to M$ be the inclusion map. Writing $\int_{\partial M} \iota^* \omega$ as $\int_{\partial M} \omega$,

$$\int_{\partial M} \omega = \int_M d\omega$$

$$
\begin{aligned}
\int_0^\infty \frac{\partial f_i}{\partial x_i} dx_i &= \lim_{a\to\infty} \int_0^a \frac{\partial f_i}{\partial x_i} dx_i \\
&= \lim_{a\to\infty} \left(f_i(\dots, a, \dots) - f_i(\dots, 0, \dots)\right) \\
&= - f_i(\dots, 0, \dots) \\
\int_{-\infty}^0 \frac{\partial f_i}{\partial x_i} dx_i &= \lim_{a\to -\infty} \int_a^0 \frac{\partial f_i}{\partial x_i} dx_i \\
&= \lim_{a\to\infty} \left(f_i(\dots, 0, \dots) - f_i(\dots, a, \dots)\right) \\
&= f_i(\dots, 0, \dots) \\
\int_{-\infty}^\infty \frac{\partial f_i}{\partial x_i} dx_i &= 0
\end{aligned}
$$

Stokes' theorem

$$\int_M d\omega = \sum_\alpha \int_{U_\alpha} \rho_\alpha d\omega \tag{1}$$

$$= \sum_\alpha \int_{U_\alpha} d(\rho_\alpha \omega) \tag{2}$$

$$= \sum_\alpha \int_{\phi(U_\alpha)} (\phi^{-1})^* d(\rho_\alpha \omega) \tag{3}$$

$$= \sum_\alpha \int_{\phi(U_\alpha)} d(\phi^{-1})^* (\rho_\alpha \omega) \tag{4}$$

$$= \sum_\alpha \int_{\partial\phi(U_\alpha)} (\phi^{-1})^* \rho_\alpha \omega \tag{5}$$

$$= \sum_\alpha \int_{\phi(\partial U_\alpha)} (\phi^{-1})^* \rho_\alpha \omega \tag{6}$$

$$= \sum_\alpha \int_{\partial U_\alpha} \rho_\alpha \omega \tag{7}$$

$$= \int_{\partial M} \omega \tag{8}$$