Differentiable manifolds and the Hairy Ball Theorem

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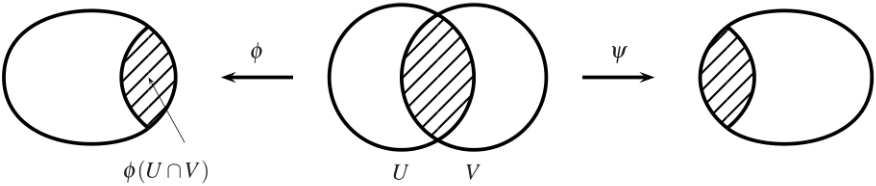
Charts

A chart on a topological space M is a pair (U, ϕ) where $U \subset M$ and $\phi : U \to \mathbb{R}^n$ is a homeomorphism.

Sometimes, we express ϕ as $(x_1, \ldots, x_n)$.

Manifolds

A n dimensional manifold M is a subset of $\mathbb{R}^\ell$ together with a collection of charts $\mathcal{A}$ that cover M such that for all charts (U, ϕ), (V, ψ), the maps $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are smooth on $\phi(U \cap V)$ and $\psi(U \cap V)$ respectively.



Tangent Space

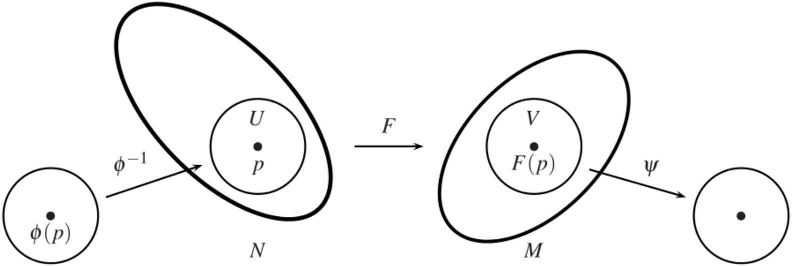
Given a point $p \in M$ and a smooth curve $\gamma : (-1, 1) \to M$ in M such that $\gamma(0) = p$, its velocity vector is $\frac{d\gamma}{dt}|_{t=0}$. The set of all velocity vectors at p is the tangent space at p, denoted T_pM.

With a chart (U, ϕ) such that $\phi(p) = 0$, construct curves $\gamma_i : t \mapsto \phi^{-1} \circ \iota_i(t)$, where ι_i is the inclusion into the ith coordinate. Their velocity vectors form a basis for the tangent space.

Example: sphere

Smooth Functions

A continuous function $F : N \to M$ is smooth if for all charts (U, ϕ) on N and (V, ψ) on M, $\psi \circ F \circ \phi^{-1}$ is smooth.



1-form

A 1-form ω is a linear function from T_pM to $\mathbb{R}$.

The space of 1-forms $\mathrm{Hom}(T_pM, \mathbb{R})$ is a vector space of dimension n. If we fix a basis for T_pM, then we can express vectors in T_pM in terms of the coordinates in that basis. Then, we define $dx_i(a_1, \ldots, a_n) = a_i$.

k-form

A k-form ω is an alternating multilinear function from $(T_pM)^k$ to $\mathbb{R}$.

Theorem

$\{dx_{i_1} \wedge \cdots \wedge dx_{i_k} | 1 \leq i_i < \cdots < i_k \leq n\}$ is a basis for $\Lambda^k(T_pM)$, the set of alternating multilinear functions on $(T_pM)^k$.

This theorem shows that $dx_1, \ldots, dx_n$ is a basis for the vector space of 1-forms. With this basis, we can express all 1-forms ω as $(a_1, \ldots, a_n)$. For a vector $v = (b_1, \ldots, b_n) \in T_pM$,

$$\begin{aligned}
\omega(v) =&(a_1 dx_1 + \cdots + a_n dx_n)(b_1, \ldots, b_n) \\
=&a_1 dx_1(b_1, \ldots, b_n) + \cdots + a_n dx_n(b_1, \ldots, b_n) \\
=&a_1 b_1 + \cdots + a_n b_n
\end{aligned}$$

So, a 1-form ω maps tangent vectors to the length of their projection onto the subspace spanned by ω, up to a multiplicative constant $|\omega|$.

Wedge Product

The wedge product of k 1-forms $\omega_1, \dots, \omega_k$ is the k-form $\omega_1 \wedge \cdots \wedge \omega_k(v_1, \dots, v_k) = \det(\omega_i(v_j))$.

Consider a 3-form $dx_1 \wedge dx_2 \wedge dx_3$ on $T_p\mathbb{R}^5$, and apply it to the vectors $u = (0,0,1,1,1), v = (1,2,0,0,1), w = (2,2,2,1,0)$. The result is $\det\begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 2 \\ 1 & 0 & 2 \end{pmatrix}$.

This is the same as projecting u, v, w onto the subspace $(c_1, c_2, c_3, 0, 0)$, then calculating it's volume.

By the Gram Schmidt process, $\omega_1 \wedge \cdots \wedge \omega_k = c\tau_1 \wedge \cdots \wedge \tau_k$ where $\tau_1, \dots, \tau_k$ are orthonormal. Then,

$$\begin{aligned} \omega_1 \wedge \cdots \wedge \omega_k(v_1, \dots, v_k) &= c\tau_1 \wedge \cdots \wedge \tau_k(v_1, \dots, v_k) \\ &= c\det(\tau_i(v_j)), \end{aligned}$$

which is projecting the $v_i's$ onto the subspace spanned by the $\tau_i's$, then taking the hypervolume and multiplying by a constant c.

Differential Forms

A differential form ω is an object where at each $p \in M$, ω_p is a n-form on T_pM, and for all charts $(U, x_1, \ldots, x_n)$, $\omega = f(p)dx_1 \wedge \cdots \wedge dx_n$ for some smooth f.

Example: dx_1 is a differential 1-form.

The d operator

Let $f : M \to \mathbb{R}$ be a smooth function. We define a differential 1-form df as $\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$. Given a differential form ω on M and a chart (U, ϕ), $\omega = \sum_I f dx_I$ on U. Then, on U, $d\omega$ is defined as $\sum_I df \wedge dx_I$.

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Orientation

stuff ...

Manifolds with Boundary

stuff ...

Integration on a $\mathbb{R}^n$

Let ω be a n-form on (U, ϕ), where $U \subset \mathbb{R}^n$. Then $\omega = f(x)dx_1 \wedge \cdots \wedge dx_n$ for some smooth f. The integral of ω over U, denoted $\int_U \omega$, is defined to be $\int_U f$.

Pullback

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Integration on a chart

Let ω be a n-form on U. The integral of ω over U is defined as

Partition of Unity

A partition of Unity on a manifold M is a collection of nonnegative smooth functions $\{\rho_\alpha : M \to \mathbb{R}\}_{\alpha\in A}$ such that

1. the collection of supports, $\{\operatorname{supp}\rho_\alpha\}_{\alpha\in A}$, is locally finite,
2. $\sum_{\alpha\in A}\rho_\alpha = 1$.

Integration on a Manifold

Let ω be a n-form on M. The integral of ω over M, denoted by $\int_M \omega$, is defined to be $\sum_{\alpha\in A}\int_{U_\alpha}\rho_\alpha\omega$.

Stokes' theorem on $\mathcal{H}^n$

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Stokes' theorem on Manifolds

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