Topological groups and Haar measures

Recall from last lecture definition 1.1:

Let G be a group acting on a set S. Then $E \subset S$ is G**-paradoxical** if there exist pairwise disjoint sets $A_1, \ldots, A_n, B_1, \ldots, B_m \subset E$ and elements $x_1, \ldots, x_n, y_1, \ldots, y_m \in G$ such that

$$E = \bigcup_{j=1}^{n} x_j \cdot A_j = \bigcup_{j=1}^{m} y_j \cdot B_j$$

We have not yet defined what an amenable group is, but we want it to be some form of "not paradoxical" group. The following turns out to be true:

Theorem 2.1 ***(Tarski's theorem)*** *Let G be a group acting on a set S and let $E \subseteq S$. The following are equivalent:*

1. *There is a finitely additive, G-invariant set function $\mu : \mathcal{P}(S) \to [0, \infty]$ with $0 < \mu(E) < \infty$;*
2. *E is not G-paradoxical*

It is the third characteristic in Corollary 2.2 that shall result in the "natural" way to study amenability. If you are an attentive reader, you will notice that I have overloaded notation. My objective for the remainder of this seminar is to demystify what the above means.

2.1 Topological groups

Definition 2.2 A group G is a **topological group** if it is endowed with a topology τ with which the maps

$$\begin{aligned} G \times G &\to G \\ (x, y) &\mapsto xy \end{aligned} \qquad\qquad \begin{aligned} G &\to G \\ x &\mapsto x^{-1} \end{aligned}$$

are continuous.

Example 2.3 The groups $(\mathbb{R}, +)$ and $(\mathbb{R}, \times)$ are topological groups with the usual topology.

Remark 2.4 The following are immediate observations from the definition of a topological group:

1. Given $a, b \in G$, if $ab \in U$ for an open set U, then there are open sets V, W such that $a \in V, b \in W$ such that $V \cdot W = \{xy : x \in V, y \in W\} \subseteq U$.
2. Each of the mappings $l_a(x) = ax$, $r_a(x) = xa$, and $inv(x) = x^{-1}$ is a homeomorphism of G onto G.
3. If F is a closed subset of G, then so are aF, Fa, F^{-1} for any $a \in G$
4. If U is an open subset of G and S is a non-empty subset of G then the sets $S \cdot U, U \cdot S, U^{-1}$ are open subsets of G.

Remark 2.5 From now on we shall assume that all our topological groups are Hausdorff.

We wish to demonstrate that the algebraic properties of groups play nicely with the analysis behind them. This proposition showcases it nicely:

Proposition 2.6 *If H is an open subgroup of G, then H is closed.*

Proof. Want to show that $\overline{H} = H$. Let $g \in \overline{H}$. Then, every open set containing g meets H; in particular gH meets H, so that $gH \cap H \neq \emptyset$. But gH being a left-coset of H is either disjoint from H or is H itself; since it is not disjoint, $gH = H$. Then $g = ge \in gH = H$, so that $\overline{H} \subseteq H$, as desired. ■

We add further structure to our topology by the following natural definition. A compact group is exactly what you expect, and so ...

Example 2.7 A group is said to be **locally compact** if there is a compact neighbourhood U around every point $x \in G$.

As many sensible people do, we may associate $M_n(\mathbb{C})$ with the topology of $\mathbb{C}^{n^2}$ to make it into a topological group. We have more:

Proposition 2.8 *$GL(n, \mathbb{C})$ is a locally compact metrisable group.*

Proof. By identifying $M_n(\mathbb{C})$ with the topology of $\mathbb{C}^{n^2}$ and observing that the determinant map is a polynomial in the entries of a matrix, we have that the determinant map is continuous. Then

$$GL(n, \mathbb{C}) = \overset{-1}{\det}\{\mathbb{C}^*\}$$

so that $GL(n, \mathbb{C})$ is a an open subset of $M_n(\mathbb{C})$. Since it is homeomorphic to an open subset of $\mathbb{C}^{n^2}$, it is metrisable. ■

Proposition 2.9 *The groups $U(n), O(n), SU(n), SO(n)$ are compact metric topological groups.*

Proof. Since $O(n), SU(n), SO(n)$ are closed subgroups of $U(n)$, it suffices to show that the latter is compact. We may achieve this by showing it is closed and bounded, and apply the Heine-Borel property. ■

2.2 Haar measures

Definition 2.10 Let G be a locally compact group. A Borel measure μ on G is said to be **left-invariant** if $\mu(xA) = \mu(A)$ for all $x \in G$ and $A \in \mathcal{B}(G)$.

Definition 2.11 Let G be a locally compact group. A **left Haar measure** is a non-zero left-invariant Radon measure μ on G.

Example 2.12 In $(\mathbb{R}^n, +)$, the n-dimensional Lebesgue measure is a left Haar measure.

Example 2.13 In $(\mathbb{R}^*, \times)$, the measure $\mu(A) = \int_A \frac{1}{|t|} d\lambda(t)$ is a left Haar measure.

Example 2.14 If μ is a left-Haar measure, then so is $c\mu$ for $c > 0$.

Theorem 2.15 *Every locally compact topological group has a left Haar measure.*

Theorem 2.16 *Left Haar measures are unique up to a multiplicative constant.*

Definition 2.17 Let G be a locally compact group and let $p \in [1, \infty]$. We write $L^p(G)$ for the L^p spaces with respect to the left Haar measure

Definition 2.18 Let G be a locally compact group. Then, for $f, g \in L^1(G)$ the integral

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)dy$$

exists for almost every x. It is called the **convolution**.

The multiplication operation above converts $(L^1(G), *)$ into a Banach algebra. We call this the **group algebra of** G.

2.3 Invariant means

Definition 2.19 Let G be a locally compact group and let E be a subspace of $L^\infty(G)$ containing the constant functions. A **mean** on E is a functional $M \in E^*$ such that $\langle 1, M \rangle = \|M\| = 1$.

Definition 2.20 Let G be a locally compact group, and let E be a subspace of $L^\infty(G)$ that contains the constants. A mean M on E is **left-invariant** if

$$\langle L_x \phi, M \rangle = \langle \phi, M \rangle$$

where $L_x \phi(t) = \phi(xt)$.

Definition 2.21 A locally compact group G is **amenable** if there is a left invariant mean on $L^\infty(G)$.

Example 2.22 From our previous talk we have that $\mathbb{F}_2$ is not amenable.

Example 2.23 Let G be a compact group, so that $L^\infty(G) \subset L^1(G)$. Then, integration with respect to a normalised Haar measure defines a left-invariant mean on $L^\infty(G)$, so that G is amenable.