PMATH 451 - Measure Theory

Fantastic Measures and How to Construct Them

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Preface

These are course notes for PMATH 451 - Measure Theory as offered during the Winter Term of 2019 by Professor Nico Spronk. I hope the reader develops as much character as the writer did while taking the course.

Any errors, lies, or omissions are entirely Alex Rutar's fault, not mine. If you wish to make a complaint, you can click here.

Part I

Measures

Chapter 1

Sigma algebras and measures

1.1 Motivation of the topic

Around the time of Mr Lebesgue, the notion of uniform convergence, and how it differs from pointwise convergence, was developed. As such, he was interested in improving the Riemann integral on $\mathbb{R}^d$, by means of a translation-invatiant measure of $\mathbb{R}^d$. One of his main achievements was the study of L^p spaces, and the rigorous treatment of their functions.

Some time after Lebesgue, Kolmogorov, a great polymath, came around and realised the need to adapt Lebesgue's treatment of measure theory into probability theory.

By way of philosophy, we wish to assign a rigorous of measure. Measure can be thought of volume, area, length, or probability, depending on the geometry or context. Once this is done, we seek to average, or rather integrate, these functions with respect to appropriate measures. The core of the theory is to provide a robust sequence of tools to approximate or calculate the following rigorously:

1. Functional analysis applications
2. L^p spaces and duality
3. Lebesgue's theory of differentiation

Thus, we embark in our quest.

1.2 The stuff you can measure

Definition 1.1 Let X be a non-empty set. A family of subsets $\mathcal{M}$ of X is called a **σ-algebra** on X provided that:

1. $X \in \mathcal{M}$
2. If $A \in \mathcal{M}$ then $A^c = X \setminus A \in \mathcal{M}$ (closed under complementation.
3. If $A_1, A_2, \ldots \in \mathcal{M}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$ (closed under countable unions).

Definition 1.2 We call the pair $(X, \mathcal{M})$ a **measurable space**.

Proposition 1.3 *Given a measurable space* $(X, \mathcal{M})$*, we have:*

1. $\emptyset \in \mathcal{M}$

2. *A σ-algebra is closed under countable intersections.*

3. *If $A, B \in \mathcal{M}$, then $A \setminus B \in \mathcal{M}$ (closed under relative complement).*

Proof. Trivial. ■

Definition 1.4 A **measure** is a function $\mu : \mathcal{M} \to [0, \infty]$ (the non-negative real numbers adjoined with infinity) that satisfies:

1. $\mu(\emptyset) = 0$

2. (σ-additivity). If $A_1, A_2, \ldots \in \mathcal{M}$ such that $A_i \cap A_j = \emptyset$ for $i \neq j$ (that is, they are pairwise disjoint), then
$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

Definition 1.5 We call the triple $(X, \mathcal{M}, \mu)$ a **measure space**.

Remark 1.6 We observe the following properties:

1. If $a \in [0, \infty]$, we declare that $a + \infty = \infty$.

2. We note that $\infty = \sup[0, \infty] = \sup[0, \infty)$.

3. We define for $a_1, a_2, \ldots \in [0, \infty]$:
$$\sum_{i=1}^{\infty} a_i = \sup_{n \in \mathbb{N}} \sum_{i=1}^{n} a_i = \lim_{n \to \infty} \sum_{i=1}^{n} a_i$$

4. If $a_1, a_2, \ldots, b_1, b_2, \ldots \in [0, \infty]$ then we shall say, fairly liberally, that:
$$\sum_{i=1}^{\infty} (a_i + b_i) = \sum_{i=1}^{\infty} a_i + \sum_{i=1}^{\infty} b_i$$
and
$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$$

Theorem 1.7 *Let $(X, \mathcal{M}, \mu)$ be a measure space, then:*

1. *(Monotonicity and difference). If $E, F \in \mathcal{M}$ and $E \subseteq F$ then $\mu(E) \leq \mu(F)$. Furthermore, if $\mu(F) < \infty$ then $\mu(F \setminus E) = \mu(F) - \mu(E)$.*

2. *(σ-subadditivity) If $E_1, E_2, \ldots \in \mathcal{M}$, then*
$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$$

3. *(Continuity from below). If $E_1 \subseteq E_2 \subseteq \ldots \in \mathcal{M}$, then*
$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{n \to \infty} \mu(E_n)$$

4. *(Continuity from above). If $E_1 \supseteq E_2 \supseteq \ldots \in \mathcal{M}$ and $\mu(E_1) < \infty$ then*

$$\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu(E_n)$$

Proof. We attack them one-by-one:

1. This is a simple computation:

$$\mu(F) = \mu(E) + \mu(F \setminus E) \geq \mu(E)$$

Furthermore, if $\mu(F) < \infty$, the above implies that $\mu(F) - \mu(E) = \mu(F \setminus E)$.

2. Let $A_1 = E_1$ and inductively define $A_{n+1} = E_{n+1} \setminus (\bigcup_{i=1}^{\infty} E_i)$ so that $E_i \cap E_j = \emptyset$ if $i \neq j$, and we have each $E_i \in \mathcal{M}$, with

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} A_i$$

Thus,

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) \leq \sum_{i=1}^{\infty} \mu(E_i)$$

where the last inequality follows from (1), since $A_i \subseteq E_i$ for all i.

3. Construct a sequence of sets as above, so that $A_1 = E_1$ and $A_{n+1} = E_{n+1} \setminus (\bigcup_{i=1}^{\infty} E_i)$. Then,

$$\begin{aligned}
\mu\left(\bigcup_{i=1}^{\infty} E_i\right) &= \mu\left(\bigcup_{i=1}^{\infty} A_i\right) \\
&= \sum_{i=1}^{\infty} \mu(A_i) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \mu(A_i) \\
&= \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} A_i\right) \\
&= \lim_{n\to\infty} \mu\left(E_n\right)
\end{aligned}$$

4. We note that

$$\emptyset = E_1 \setminus E_1 \subseteq E_1 \setminus E_2 \subseteq E_1 \setminus E_3 \subseteq \ldots$$

Then, from the difference formula (which is where finiteness shows up), and De Morgan's law, we have,

$$\begin{aligned}
\mu(E_1) - \mu\left(\bigcap_{i=1}^{\infty} E_i\right) &= \mu\left(E_1 \setminus \bigcap_{i=1}^{\infty} E_i\right) \\
&= \mu\left(\bigcup_{i=1}^{\infty} (E_1 \setminus E_i)\right) \\
&= \lim_{n\to\infty} \mu(E_1 \setminus E_n) && \text{(by 3)} \\
&= \lim_{n\to\infty} \left(\mu(E_1) - \mu(E_n)\right) \\
&= \mu(E_1) - \lim_{n\to\infty} \mu(E_n)
\end{aligned}$$

■

1.3 Types of measures

Unfortunately, we will find that our definition is perhaps too general, hopelessly abstract, and might need some tightening up. We do this in this section.

Definition 1.8 A measure space $(X, \mathcal{M}, \mu)$ is called:

1. **finite**, if $\mu(X) < \infty$
2. a **probability space**, if $\mu(X) = 1$
3. **σ-finite**, if there is a countable collection $\{X_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$ such that $(\bigcup_{i=1}^{\infty} X_i = X)$, and $\mu(X_i) < \infty$ for each i.
4. **decomposable**, if there is a set $\Pi \subseteq \mathcal{M}$ such that, for $P, Q \in \Pi$, we have:
   (a) $P \cap Q = \emptyset$ for $P \neq Q$ in Π and X is the disjoint union of Ps (that is Π partitions X).
   (b) if $E \subseteq X$, then $E \in \mathcal{M} \iff E \cap P \in \mathcal{M}$ for each $P \in \Pi$.
   (c) each $\mu(P) < \infty$
   (d) if $E \in \mathcal{M}$ with $\mu(E) < \infty$, then
   $$\mu(E) = \sup_{\substack{\mathcal{F} \subseteq \Pi \\ \mathcal{F} \text{is finite}}} \left(\sum_{P \in \mathcal{F}} \mu(E \cap P) \right) = \sum_{P \in \Pi} \mu(E \cap P)$$
5. **semifinite** if for any E in $\mathcal{M}$, with $\mu(E) > 0$, there is F in $\mathcal{M}$, with $F \subseteq E$ such that $0 < \mu(F) < \infty$ (each set is "finite approximatable from below")
6. **complete**, if whenever $N \subseteq X$ such that $N \subseteq E \in \mathcal{M}$ for which $\mu(E) = 0$, then $N \in \mathcal{M}$.

Example 1.9 If $0 < \mu(X) < \infty$, then $\frac{1}{\mu(X)}\mu$ is a probability measure.

Example 1.10 Let $E_n = \bigcup_{i=1}^{n} X_i$, so that $E \subseteq E_2 \subseteq E_3 \subseteq \ldots$ in $\mathcal{M}$, $X = \bigcup_{i=1}^{\infty} E_i$, and each $\mu(E_i) < \infty$. This is σ-finite. Alternatively, we shall have let $A_1 = X_1$ and $A_{n+1} = X_{n+1} \setminus \bigcup_{i=1}^{n} X_i$, so each $A_i \in \mathcal{M}$, $A_i \cap A_j = \emptyset$ if $i \neq j$, each $\mu(A_i) < \infty$, and X is the disjoint union of the A_is.

The following is a relationship between the types of probability measures:

$$\text{probability} \implies \text{finite} \implies \sigma - \text{finite} \implies \text{decomposable}$$

and σ-finite implies semifinite. None of these implications is generally reversible.

Remark. Completeness has some technical usefulness. Every measure space $(X, \mathcal{M}, \mu)$ has a completion; that is, a complete measure space $(X, \overline{\mathcal{M}}, \overline{\mu})$ such that $\mathcal{M} \subseteq \overline{\mathcal{M}}$, $\mu|_{\mathcal{M}} = \mu$. Most "natural" constructions of measures give us complete measures.

Example 1.11 (Zero measure) Given a measurable space $(X, \mathcal{M})$, let $\mu(E) = 0$ whenever $E \subset X$. This is, perhaps, the most trivial and boring example of a measure.

Example 1.12 (Counting measure) Let X be any non-empty set. Then, the power set $\mathcal{P}(X)$ is a

σ-algebra on X. We let $\gamma : \mathcal{P}(X) \to [0, \infty]$ and

$$\gamma(E) = \begin{cases} |E|, & E \text{ is finite} \\ \infty & \text{otherwise} \end{cases}$$

Then $(X, \mathcal{P}(X), \gamma)$ is a measure space (easy exercise). This space is:

1. finite $\iff$ X is finite
2. σ-finite $\iff$ X is countable
3. always decomposable by choosing $\Pi = \{\{x : x \in X\}\}$
4. always semifinite
5. always complete since $E \in \mathcal{P}(X)$ has $\mu(E) = 0$ if and only if $E = \emptyset$.
6. Since $X \neq \emptyset$, if X is finite, let $\nu = \frac{1}{|X|}\gamma$ is a probability measure (in fact, we call it the uniform probability measure).

Example 1.13 (Point mass or Dirac measure) Let X be a non-empty set and let $a \in X$. Define $\delta_a : \mathcal{P}(X) \to \{0, 1\} \subset [0, \infty]$ by

$$\delta_a(E) = \begin{cases} 1, & \text{if } a \in E \\ 0, & \text{if } a \notin E \end{cases}$$

We show that this is a measure. Notice that:

1. $\delta_a(\emptyset) = 0$ since $a \notin \emptyset$
2. If $E_1, E_2, \ldots \in \mathcal{P}(X)$ which are pairwise disjoint, then exactly one of the following can happen: (i) either a in in the union of the E_is, in which case a is contained in exactly one such set, say E_{i_0}, or (ii) a is not in the union, and $a \notin E_i$ for all i. Thus,

$$\begin{aligned} \delta_a\left(\bigcup_{i=1}^{\infty} E_i\right) &= \begin{cases} 1, & \text{if } a \in E_{i_0} \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} \delta_a(E_{i_0}) = \sum_{i=1}^{\infty} \delta_1(E_i) \\ \sum_{i=1}^{\infty} \delta_a(E_i) \end{cases} \end{aligned}$$

This is a probability measure. It is also complete, by using the null-sets $\mathcal{N}_{\delta_a} = \{E \in \mathcal{P}(X) : a \notin E\}$.

Example 1.14 (Co-countable measure) Let X be an uncountable set and define

$$\mathcal{M} = \{E \subseteq X : \text{either } E \text{ is countable of } X \setminus E \text{ is countable}\}$$

We can check this set is a σ-algebra. Then, define $\mu : \to [0, \infty]$, with:

$$\mu(E) = \begin{cases} 0, & \text{if } E \text{ is countable} \\ \infty, & \text{otherwise} \end{cases}$$

This is a bear. It is not semifinite, decomposable, nor σ-finite. It is, however, complete.

Example 1.15 Let $X = \{x_0\}$. Observe $\mathcal{P}(X) = \{\emptyset, X\}$ and define $\mu(\emptyset) = 0$ and $\mu(\{x_0\}) = \infty$. It is not semifinite, nor decomposable.

1.4 Constructing measures

1.4.1 Outer measures to measures

Definition 1.16 Let X be a non-empty set. An **outer measure** on X is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ such that:

1. $\mu^*(\emptyset) = 0$
2. (Monotonicity) If $A \subseteq B \subseteq X$, then $\mu^*(A) \leq \mu^*(B)$
3. (σ-subadditivity) If $A_1, A_2, \ldots \in \mathcal{P}(X)$ then

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$$

Remark 1.17 Any measure on $\mathcal{P}(X)$ is an outer measure. As an advantage, outer measures are easy to construct and have largest domain. As a disadvantage, we may not have σ-additivity.

Theorem 1.18 *Let $\mathcal{E} \subseteq \mathcal{P}(X)$ be any family such that $\{\emptyset, X\} \subseteq \mathcal{E}$, and there is a function $\rho : \mathcal{E} \to [0, \infty]$ such that $\rho(\emptyset) = 0$. Then, the formula for $A \in \mathcal{P}(X)$ given by*

$$\mu^*(A) = \inf\left\{\sum_{i=1}^{\infty} \rho(E_i) \,:\, E_1, E_2 \ldots \in \mathcal{E} \text{ and } A \subseteq \bigcup_{i=1}^{\infty} E_i\right\}$$

defines an outer measure on X.

Remark 1.19 Unless $(\mathcal{E}, \rho)$ is "nice" we may not be able to recover ρ from μ^*; note that for $E \in \mathcal{E}$, we have $\mu^*(E) \leq \rho(E)$, but we may not get equality.

Proof. First, observe that $0 \leq \mu^*(\emptyset) \leq \rho(\emptyset) = 0$, as $\emptyset \subseteq \emptyset$.

Second, if $A \subseteq B \subseteq B \subseteq X$, then any countable $\mathcal{E}$-cover of B, (say $E_1, E_2, \ldots \in \mathcal{E}$ with $B \subseteq \bigcup_{i=1}^{\infty} E_i$) is evidently a countable $\mathcal{E}$-cover of A. Then $\mu^*(A) \leq \mu^*(B)$ follows from the definition of infimum.

Lastly, suppose $A_1, A_2, \ldots \subseteq X$ and let $\epsilon > 0$. By definition of μ^* to each A_i, we get $E_{i1}, E_{i2}, \ldots \in \mathcal{E}$ such that

$$A_i \subseteq \bigcup_{i=1}^{\infty} E_{ij} \qquad \text{and} \qquad \sum_{j=1}^{\infty} \rho(E_{ij}) \leq \mu^*(A_i) + \frac{\epsilon}{2^i}$$

Then, we have that

$$\bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} E_{ij}$$

so we have

$$\begin{aligned}
\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) &\leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \rho(E_{ij}) \\
&\leq \sum_{i=1}^{\infty} \left(\mu^*(A_i) + \frac{\epsilon}{2^i}\right) \\
&= \sum_{i=1}^{\infty} \mu^*(A_i) + \epsilon
\end{aligned}$$

and since $\epsilon > 0$ is arbitrary, we extract

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$$

and we are done! ■

Definition 1.20 (Caratheodory) Given an outer measure μ^* on X, we say that a set $A \subseteq X$ is μ-measurable provided that for any $E \in \mathcal{P}(X)$ we have

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A)$$

Definition 1.21 Given a non-empty set X, an **algebra** on X is a family $\mathcal{A} \subseteq \mathcal{P}(X)$ such that:

1. $X \in \mathcal{A}$
2. (Complementation) If $A \in \mathcal{A}$ then $X \setminus A \in \mathcal{A}$
3. (Unions) If $A, B \in \mathcal{A}$ then $A \cup B \in \mathcal{A}$

Remark 1.22 Property 3 allows finite unions (by induction) and finite intersection (by De Morgan's law). Properties 1 and 2 then demand that $\emptyset \in \mathcal{A}$.

Remark 1.23 If μ^* is an outer measure, then $\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \setminus A)$, always. Thus, for Definition 1.17, we only have to check that $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \setminus A)$.

Theorem 1.24 *Given an outer measure $\mu^* : \mathcal{P}(X) \to [0, \infty]$, we have that:*

1. *The collection $\mathcal{M} = \{A \in \mathcal{P}(X) :$ for any $E \in \mathcal{P}(X), \mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \setminus A)\}$ is a σ-algebra.*
2. *The function $\mu = \mu^*|_{\mathcal{M}} : \mathcal{M} \to [0, \infty]$ is a complete measure.*

Thus $(X, \mathcal{M}, \mu)$ is a measure space.

Proof. We begin with part 1. Let us verify that $\mathcal{M}$ is an algebra, from the definition. First, if $E \in \mathcal{P}(X)$, then

$$\begin{aligned} \mu^*(E \cap X) + \mu^*(E \setminus X) &= \mu^*(E) + \mu^*(\emptyset) \\ &= \mu^*(E) \\ &\leq \mu^*(E) \end{aligned}$$

so that $X \in \mathcal{M}$.

Now let $A, B \in \mathcal{M}$. We have for $E \in \mathcal{P}(X)$ that:

$$\begin{aligned} \mu^*(E \cap (X \setminus A)) + \mu^*(E \setminus (X \setminus A)) &= \mu^*(E \setminus A) + \mu^*(E \cap A) \\ &\leq \mu^*(E) \end{aligned}$$

so that $X \setminus A \in \mathcal{M}$.

Furthermore, we have, by a sequence of symbolic manipulations,

$$\begin{aligned} \mu^*(E) &\geq \mu^*(E \cap A) + \mu^*(E \setminus A) \\ &\geq \mu^*((E \cap A) \cap B) + \mu^*((E \cap A) \setminus B) + \mu^*((E \setminus A) \cap B) + \mu^*((E \setminus A) \setminus B) \\ &= \mu^*(E \cap (A \cap B)) + \mu^*(E \cap (A \setminus B)) + \mu^*(E \cap (B \setminus A)) + \mu^*(E \setminus (A \cup B)) \\ &\geq \mu^*(E \cap (A \cup B)) + \mu^*(E \setminus (A \cup B)) && \text{(by σ-subadditivity)} \end{aligned}$$

And since we have $A \cup B = (A \cap B) \cup (A \setminus B) \cup (B \setminus A)$, we can conclude that $A \cup B \in \mathcal{M}$, by which we conclude $\mathcal{M}$ is an algebra.

Now we show that $\mathcal{M}$ is a σ-algebra. Since we have shown that it is an algebra, we only need to show that it is closed under countable (disjoint) unions. Let $A_1, A_2, \ldots \in \mathcal{M}$ and $A = \bigcup_{i=1}^{\infty} A_i$. We use our classic trick of letting $B_1 = A_1$ and $B_{n+1} = A_{n+1} \setminus (\bigcup_{i=1}^{n} A_i)$, so that $B_i \cap B_j = \emptyset$ whenever $i \neq j$, and each $B_i \in \mathcal{M}$. Let $E \in \mathcal{P}(X)$. We have,

$$\begin{aligned}
\mu^*\left(E \cap \bigcup_{i=1}^{n} B_i\right) &\geq \mu^*\left(\left(E \cap \bigcup_{i=1}^{n} B_i\right) \cap B_n\right) + \mu^*\left(\left(E \cap \bigcup_{i=1}^{n} B_i\right) \setminus B_n\right) \\
&= \mu^*(E \cap B_n) + \mu^*\left(E \cap \bigcup_{i=1}^{n-1} B_i\right) \\
&= \mu^*(E \cap B_n) + \mu^*(E \cap B_{n-1}) + \mu^*\left(E \cap \bigcup_{i=1}^{n-2} B_i\right) \\
&\vdots \\
&\geq \sum_{i=1}^{n} \mu^*(E \cap B_i)
\end{aligned}$$

Thus we have that,

$$\begin{aligned}
\mu^*(E) &\geq \mu^*\left(E \cap \bigcup_{i=1}^{n} A_i\right) + \mu^*\left(E \setminus \bigcup_{i=1}^{n} A_i\right) \\
&\geq \mu^*\left(E \cap \bigcup_{i=1}^{n} B_i\right) + \mu^*(E \setminus A) && \text{(by re-labelling and monotonicity)} \\
&\geq \sum_{i=1}^{n} \mu^*(E \cap B_i) + \mu^*(E \setminus A) \\
&\to \sum_{i=1}^{\infty} \mu^*(E \cap B_i) + \mu^*(E \setminus A) && (\text{as } n \to \infty)(\dagger) \\
&\geq \mu^*\left(\bigcup_{i=1}^{\infty} (E \cap B_i)\right) + \mu^*(E \setminus A) \\
&= \mu^*(E \cap A) + \mu^*(E \setminus A)
\end{aligned}$$

so that $A \in \mathcal{M}$, proving thus that $\mathcal{M}$ is a σ-algebra.

We wish to see that μ is a measure. We further assume that for $A_1, A_2, \ldots \in \mathcal{M}$ defined above that

$A_i \cap A_j = \emptyset$ for $i \neq j$. But then $B_i = A_i$ for each i. Letting $E = A$ and using †, we show that

$$\begin{aligned} \mu^*(A) &\geq \sum_{i=1}^{\infty} \mu^*(A \cap A_i) + \mu^*(A \setminus A) \\ &= \sum_{i=1}^{\infty} \mu^*(A_i) \\ &\geq \sum_{i=1}^{\infty} \mu^*(A_i) \\ &\geq \mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \\ &= \mu^*(A) \end{aligned}$$

For part 3, we see that if $N \in \mathcal{M}$ with $\mu(N) = 0$, then $E \in \mathcal{M}$ for each $E \subseteq N$, making μ complete. We also have that for any F in $\mathcal{P}(X)$ and E as above, then

$$\mu^*(F \cap E) + \mu^*(F \setminus E) \leq \mu^*(N) + \mu^*(F) = \mu(N) + \mu^*(F) = \mu^*(F) \qquad \blacksquare$$

Victory![1]

1.4.2 Building measures from premeasures, via outer measures

Definition 1.25 Let $\mathcal{A}$ be an algebra on X. A **premeasure** is a function $\mu_0 : \mathcal{A} \to [0, \infty]$ such that

1. $\mu_0(\emptyset) = 0$
2. (Restricted σ-additivity) If $A_1, A_2, \ldots \in \mathcal{A}$ such that $A_i \cap A_j = \emptyset$ whenever $i \neq j$ and $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$, then
$$\mu_0\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu_0(A_i)$$

Remark 1.26 If $A_1, \ldots, A_n \in \mathcal{A}$ are pairwise disjoint, then the sequence $A_1, \ldots, A_n, \emptyset, \emptyset, \ldots \in \mathcal{A}$, then we can obtain finite additivity. Furthermore, as with measures, pre-measures are monotone.

Theorem 1.27 ***(Measures from premeasures)*** *Let $(X, \mathcal{A}, \mu_0)$ be a premeasure space. Let $\mu^* : \mathcal{P}(X) \to [0, \infty]$ be given by*

$$\mu^*(E) = \inf\left\{\sum_{i=1}^{\infty} \mu_0(A_i) \,:\, A_1, A_2, \ldots \in \mathcal{A},\, E \subseteq \bigcup_{i=1}^{\infty} A_i\right\}$$

so that μ^* is an outer measure. Then,

1. *(Recovery of premeasure from outer measure) $\mu^*|_{\mathcal{A}} = \mu_0$*
2. *The set $\mathcal{M}$ of μ^*-measurable sets contains $\mathcal{A}$. Hence, $\mu = \mu^*|_{\mathcal{M}}$ satisfies that $\mu|_{\mathcal{A}} = \mu_0$.*
3. *(Almost uniqueness). If $\nu : \mathcal{M} \to [0, \infty]$ is a measure with $\nu|_{\mathcal{A}} = \mu_0$ then $\nu(E) \leq \mu(E)$ for all $E \in \mathcal{M}$ with $\nu(E) = \nu(E)$ if $\mu(E) < \infty$. In particular, if $(X, \mathcal{M}, \mu)$ is σ-finite, then μ is the unique extension of μ_0 to a measure on $\mathcal{M}$.*

[1] This is quite technical... I wonder what subset of this proof is actually useful for me to become fluent in.

Proof. That μ^* is an outer measure is from a prior proposition.

1. Let $A \in \mathcal{A}$. Since $A \subseteq A$, we have $\mu^*(A) \leq \mu_0(A)$, by definition of μ^*. Conversely, let $A_1, A_2, \ldots \in \mathcal{A}$ be so that $A \subseteq \bigcup_{i=1}^{\infty} A_i$ and construct B_i in the usual way, by making $B_1 = A_1$ and $B_{n+1} = A_{n+1} \setminus (\bigcup_{i=1}^{n} A_i)$. Then,

$$A = A \cap \bigcup_{i=1}^{\infty} A_i = A \cap \bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} (A \cap B_i)$$

where $(A \cap B_i) \cap (A \cap B_j) = \emptyset$ for $i \neq j$. Hence, by the restricted σ-additivity of μ_0, we obtain

$$\begin{aligned} \mu_0(A) &= \mu_0\left(\bigcup_{i=1}^{\infty}(A \cap B_i)\right) \\ &= \sum_{i=1}^{\infty} \mu_0(A \cap B_i) \\ &\leq \sum_{i=1}^{\infty} \mu_0(A_i) && \text{(by monotonicity)} \end{aligned}$$

By definition of μ^*, we see that $\mu_0(A) \leq \mu^*(A)$.

2. Now let $A \in \mathcal{A}$, let $E \in \mathcal{P}(X)$. By definition of $\mu^*(E)$, give $\epsilon > 0$, we may find $A_1, A_2, \ldots \in \mathcal{A}$ such that $E \subseteq \bigcup_{i=1}^{\infty} A_i$ and

$$\sum_{i=1}^{\infty} \mu_0(A_i) \leq \mu^*(E) + \epsilon$$

Then, for each i,

$$\mu_0(A_i) = \mu_0(A_i \cap A) + \mu_0(A_i \setminus A)$$

by finite additivity and

$$E \cap A \subseteq \bigcup_{i=1}^{\infty}(A_i \cap A), \qquad E \setminus A \subseteq \bigcup_{i=1}^{\infty}(A_i \setminus A)$$

Thus,

$$\begin{aligned} \mu^*(E) + \epsilon &\geq \sum_{i=1}^{\infty} \mu_0(A_i) \\ &= \sum_{i=1}^{\infty} \mu_0(A_i \cap A) + \sum_{i=1}^{\infty} \mu_0(A_i \setminus A) \\ &\geq \mu^*(E \cap A) + \mu^*(E \setminus A) \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we see that $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \setminus A)$, so that $A \in \mathcal{M}$, thus $\mathcal{A} \subseteq \mathcal{M}$.

3. We will use continuity from below several times.

If $R \in \mathcal{M}$ and $A_1, A_2, \ldots \in \mathcal{A}$ are such that $E \subseteq \bigcup_{i=1}^{\infty} A_i$, then

$$\begin{aligned}
\nu(E) &\leq \nu\left(\bigcup_{i=1}^{\infty} A_i\right) && \text{(monotonicity)} \\
&\leq \sum_{i=1}^{\infty} \nu(A_i) && \text{(σ-subadditivity)} \\
&= \sum_{i=1}^{\infty} \mu_0\left(A_i\right)
\end{aligned}$$

and it follows from the definition of $\mu = \mu^*|_{\mathcal{M}}$ that $\nu(E) \leq \mu(E)$. Recall, from A1, that

$$\mathcal{A}_\sigma = \left\{\bigcup_{i=1}^{\infty} A_i \;:\; A_1, A_2, \ldots \in \mathcal{A}\right\}$$

Then we have that $\nu|_{\mathcal{A}_\sigma} = \mu|_{\mathcal{A}_\sigma}$. If $A = \bigcup_{i=1}^{\infty} A_i$ with $A_1, A_2, \ldots \in \mathcal{A}$, then

$$\begin{aligned}
\nu(A) &= \lim_{n\to\infty} \nu\left(\bigcup_{i=1}^{\infty} A_i\right) \\
&= \lim_{n\to\infty} \mu_0\left(\bigcup_{i=1}^{\infty} A_i\right) \\
&= \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{\infty} A_i\right) \\
&= \mu\left(A\right)
\end{aligned}$$

Now let $E \in \mathcal{M}$ with $\mu\left(E\right) < \infty$. Given $\epsilon > 0$, let $A_1, A_2, \ldots \in \mathcal{A}$ with $E \subseteq \bigcup_{i=1}^{\infty} A_i = A$, such that

$$\mu\left(E\right) + \epsilon = \mu^*\left(E\right) + \epsilon > \sum_{i=1}^{\infty} \mu_0\left(A_i\right)$$

Hence,

$$\mu\left(E\right) \leq \mu\left(A\right) \leq \sum_{i=1}^{\infty} \mu\left(A_i\right) = \sum_{i=1}^{\infty} \mu_0\left(A_i\right) < \mu\left(E\right) + \epsilon$$

and it follows that $\mu\left(A \setminus E\right) = \mu\left(A\right) - \mu\left(E\right) < \epsilon$. Hence, as $A \in \mathcal{A}_\sigma$, $\mu\left(A\right) = \nu(A)$ and we have

$$\begin{aligned}
\mu\left(E\right) &\leq \mu\left(A\right) \\
&= \nu(A) \\
&= \nu(A \cap E) + \nu(A \setminus E) \\
&\leq \nu(A \cap E) + \mu\left(A \setminus E\right) && \text{(as $\nu \leq \mu$, generally)} \\
&< \nu(E) + \epsilon
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we find that $\mu\left(E\right) \leq \nu(E)$. Hence $\mu\left(E\right) = \nu(E)$.

Now let us focus on the case where $(X, \mathcal{M}, \mu)$ is σ-finite. We can arrange our space as $X = \bigcup_{i=1}^{\infty} X_i$ where each $X_i \in \mathcal{M}$, with $\mu\left(X_i\right) < \infty$ and $X_1 \subseteq X_2 \subseteq \ldots$. If $E \in \mathcal{M}$, we have

$$E = \bigcup_{i=1}^{\infty} (E \cap X_i) \qquad X_1 \cap E \subseteq X_2 \cap E \subseteq \ldots$$

and

$$\begin{aligned}\mu(E) &= \lim_{n\to\infty} \mu(X_n \cap E) \\ &= \lim_{n\to\infty} \nu(X_n \cap E) \\ &= \nu(E)\end{aligned}$$

Victory! ■

Remark 1.28 The uniqueness of the above theorem also holds if we have that $(X, \mathcal{M}, \mu)$ is semifinite. Indeed, by Assignment 1, if $E \in \mathcal{M}$ we get

$$\mu(E) = \sup\{\mu(F) : F \in \mathcal{M}, F \subseteq E, \mu(F) < \infty\} = \sup\{\nu F : F \in \mathcal{M}, F \subseteq E, \nu F < \infty\} \leq \nu(E)$$

implying $\nu(E) = \mu(E)$.

Theorem 1.29 *Given a measure space $(X, \mathcal{M}, \mu)$, there is a complete measure space $(X, \overline{\mathcal{M}}, \overline{\mu})$ such that $\overline{\mu}|_{\mathcal{M}} = \mu$. Furthermore, if $(X, \mathcal{M}, \mu)$ is a σ-finite, then any $E \in \overline{\mathcal{M}}$ admits a representation of the following form:*

$$E = M \cup N \qquad M \in \mathcal{M}, \qquad N \subseteq N' \qquad \text{where } N' \in \mathcal{M} \text{ with } \mu\left(N'\right) = 0$$

Proof. We regard $(X, \mathcal{M}, \mu)$ as a premeasure space. Then, the previous theorem provides an outer measure μ^* with $\mu^*|_{\mathcal{M}} = \mu$ and, furthermore, if

$$\overline{\mathcal{M}} = \{A \in \mathcal{P}(X) : \mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \setminus A)\ , \text{for any } E \in \mathcal{P}(X)\}$$

then $\mathcal{M} \subseteq \overline{\mathcal{M}}$. Let $\overline{\mu} = \mu^*|_{\overline{\mathcal{M}}}$. We appeal to A1Q4 to see the structure of $E \in \overline{\mathcal{M}}$. We have $X \setminus E \in \overline{\mathcal{M}}$ and we have $X \setminus E = A \setminus N$ where $A \in \mathcal{M}_{\sigma\delta} = \mathcal{M}$ and $\mu^*(N) = 0$. For each n, we can find $A_{n1}, A_{n2}, \ldots \in \mathcal{A}$ such that

$$N \subseteq \bigcup_{i=1}^{\infty} A_{ni} = A_n \qquad \text{and} \qquad \sum_{i=1}^{\infty} \mu(A_{ni}) < \frac{1}{n} = \mu^*(N) + \frac{1}{n}$$

so $N \subseteq A_n$ with $A_n \in \mathcal{M}$. Thus $N \subseteq \bigcap_{n=1}^{\infty} A_n = N'$ and $N' \in \mathcal{M}$ and $\mu(N') \leq \mu(A_n) < \frac{1}{n}$ for each n so $\mu(N') = 0$. Now

$$\begin{aligned}E &= X \setminus (X \setminus E) \\ &= X \setminus (A \setminus N) \\ &= (X \setminus A) \cup N \\ &= M\end{aligned}$$

■

1.5 On building σ-algebras and Borel measures

1.5.1 Borel σ-algebra

Theorem 1.30 *Let X be a non-empty set.*

1. *$\{\mathcal{M}_i\}_{i\in I}$ is a family of σ-algebras on X, then $\bigcap_{i\in I} \mathcal{M}_i$ is also a σ-algebra.*
2. *Given $\emptyset \neq \mathcal{E} \subseteq \mathcal{P}(X)$, the family $\sigma\langle\mathcal{E}\rangle = \bigcap\{\mathcal{M} : \mathcal{M} \text{ is a } \sigma\text{-algebra on } X, \mathcal{E} \subseteq \mathcal{M}\}$ is a σ-algebra. It is called the **σ-algebra generated by** $\mathcal{E}$.*

3. *If $\emptyset \neq \mathcal{E} \subseteq \sigma\langle\mathcal{E}\rangle$ in $\mathcal{P}(X)$, then $\sigma\langle\mathcal{F}\rangle \subseteq \sigma\langle\mathcal{E}\rangle$.*

Proof. 1. Here we just check the sigma algebra axioms, which turns out to be easy.

2. This is a straightforward application of the part above.

3. We see that $\sigma\langle\mathcal{E}\rangle$ is a sigma algebra, containing $\mathcal{F}$. Part 2 tells us that $\sigma < \mathcal{F} >$ is the smallest sigma algebra containing $\mathcal{F}$. ∎

Remark 1.31 As with the lemma above, we may define

$$\langle\mathcal{E}\rangle = \bigcap\{\mathcal{A} : \mathcal{A} \text{ is an algebra on } X, \mathcal{E} \subseteq \mathcal{A}\}$$

Example 1.32 Let (X, d) be a metric space. Let $\mathcal{G} = \{G \subseteq X : G \text{ is open}\}$. The **Borel** σ-algebra is given by

$$\mathcal{B}(X, d) = \mathcal{B}(X) = \sigma\langle\mathcal{G}\rangle$$

As a remark, if $\mathcal{F} = \{F \subseteq X : F \text{ is closed}\}$, then $\sigma\langle\mathcal{F}\rangle = \sigma\langle\mathcal{G}\rangle$.

Definition 1.33 We define

$$\mathcal{G}_\delta = \left\{\bigcap_{i=1}^{\infty} G_i : G_1, G_2, \ldots \in \mathcal{G}\right\}$$

and

$$\mathcal{F}_\sigma = \left\{\bigcup_{i=1}^{\infty} F_i : F_1, F_2, \ldots \in \mathcal{F}\right\}$$

Theorem 1.34 *Here we generate $\mathcal{B}(\mathbb{R})$. Let $\mathcal{B}(\mathbb{R})$ denote the Borel σ-algebra on $\mathbb{R}$ with its usual metric. Consider the following families of subsets of $\mathbb{R}$:*

1. *$\mathcal{O} = \{(a, b) : -\infty \leq a \leq b \leq \infty\}$, the set of open intervals,*

2. *$\mathcal{O}_\infty = \{(a, \infty) : -\infty \leq a \in \mathbb{R}\}$, the set of half rays,*

3. *$\mathcal{H} = \{(a, b] : -\infty \leq a \leq b \leq \infty, a, b \in \mathbb{R}\}$, the set of half open and half closed intervals, with the convention that $(a, \infty] = (a, \infty)$ and $(a, a] = \emptyset$,*

4. *$\mathcal{C}_\infty = \{[a, \infty) : -\infty < a \in \mathbb{R}\}$, the family of closed half rays.*

Then $\mathcal{B}(\mathbb{R}) = \sigma\langle\mathcal{O}\rangle = \sigma\langle\mathcal{O}_\infty\rangle = \sigma\langle\mathcal{H}\rangle = \sigma\langle\mathcal{C}_\infty\rangle$.

Proof. This should be fairly intuitive; see Folland for details if necessary. ∎

Definition 1.35 An **elementary family** of sets on X is any $\mathcal{E} \subseteq \mathcal{P}(X)$ such that:

1. $X \in \mathcal{E}$

2. If $E, F \in \mathcal{E}$, then $E \cap F = \bigcup_{i=1}^{n} E_i$, with $E_1, \ldots, E_n \in \mathcal{E}$.

3. If $E \in \mathcal{E}$, then $X \setminus E = \bigcup_{i=1}^{m} E_i$ with $E_i \in \mathcal{E}$.

Remark 1.36 A simple induction argument shows that any finite intersection of elements of $\mathcal{E}$ is a finite union of elements of $\mathcal{E}$.

Example 1.37 In $\mathbb{R}$, the set $\mathcal{H} = \{(a, b] : -\infty \leq a \leq b \leq \infty, a, b \in \mathbb{R}\}$ is an elementary family.

Theorem 1.38 *If $\mathcal{E} \subseteq \mathcal{P}(X)$ is an elementary family, then*

$$\langle \mathcal{E} \rangle = \left\{ \bigcup_{i=1}^{n} E_i \, : \, E_i \in \mathcal{E}, E_i \cap E_j = \emptyset \text{ if } i \neq j, \; n \in \mathbb{N} \right\}$$

Proof. The $\supseteq$ direction is obvious. Hence it suffices to show the $\subseteq$ direction by showing that the RHS is an algebra.

The family is clearly closed under finite intersections. Hence it remains to see that it is closed under complementation. Let $E_1, \ldots, E_n \in \mathcal{E}$, and write each $X \setminus E_i = \bigcup_{j=1}^{m_i} E_{ij}$. Now we consider

$$X \setminus \left(\bigcup_{i=1}^{n} E_i \right) = \bigcap_{i=1}^{n} (X \setminus E_i) = \bigcap_{i=1} \bigcup_{j=1}^{m_i} E_{ij} = \bigcup_{1 \leq j_i \leq n \;\; i=1,\ldots,n} (E_{1j_1} \cap \ldots \cap E_{n_{jn}})$$

where each $E_{1j_1} \cap \ldots \cap E_{n_{jn}}$ is a finite unions of elements of $\mathcal{E}$ by the last remark. ■

Corollary 1.39 *In $\mathbb{R}$ that*

$$\langle \mathcal{H} \rangle = \left\{ \bigcup_{i=1}^{n} (a_i, b_i] \, : \, -\infty \leq a_i \leq b_i \leq \infty \, for i = 1, \ldots, n \in \mathbb{N} \right\}$$

1.5.2 Borel measures on the real line

Definition 1.40 The set of **decreasing, right continuous** functions is:

$$ND_r(\mathbb{R}) = \left\{ F \, : \, \mathbb{R} \to \mathbb{R} \, : \; \text{if } x < y \text{ in } \mathbb{R}, F(x) \leq F(y) \, \text{for a} \in \mathbb{R} \lim_{x \to a^+} F(x) = F(a) \right\}$$

Remark 1.41 For $F \in ND_r(\mathbb{R})$, then $F(\pm\infty) := \lim_{x \to \pm\infty} F(x)$ always exists, allowing $\pm\infty$.

Lemma 1.42 *(Premeasures from $ND_r(\mathbb{R})$) Let $F \in \mathrm{ND}_r(\mathbb{R})$ and $\mathcal{A} = \{\mathcal{H}\} \subset \mathcal{P}(\mathbb{R})$ (algebra generated by half-open, half-closed intervals). Then $\mu_{0,F} : \mathcal{A} \to [0, \infty]$ defined by*

$$\mu_{0,F} \left(\bigcup_{i=1}^{n} (a_i, b_i] \right) = \sum_{i=1}^{n} [F(b_i) - F(a_i)]$$

defines a premeasure on $\mathcal{A}$.

Proof. For simplicity, write $\mu_0 = \mu_{0,F}$. It is evident that μ_0 is well-defined, since $(a, b] = (a, c] \cup (c, b]$ and $\mu_0 ((a, b]) = \mu_0 ((a, c]) + \mu_0 ((c, b])$ and that $\mu_0 (\emptyset) = 0$, where $\emptyset = (a, a]$. It remains to show that μ_0 enjoys restricted σ-additivity.

1. Suppose $(a, b] = \bigcup_{i=1}^{\infty} (c_j, d_j]$, with $-\infty < a < b < \infty$. We wish to see that

   $$\mu_0 ((a, b]) = \sum_{i=1}^{\infty} \mu_0 ((c_i, d_i])$$

   First, given $n \in \mathbb{N}$, there is a bijection $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $c_{\sigma(1)} \leq d_{\sigma(1)} \leq c_{\sigma(2)} \leq d_{\sigma(2)} \leq \ldots \leq c_{\sigma(n} \leq d_{\sigma(n)}$

Then, as F is non-decreasing we have that

$$\begin{aligned}
\sum_{j=1}^{n} \mu_0\left((c_j, d_j]\right) &= \sum_{j=1}^{n} [F(d_j) - F(c_j)] \\
&= \sum_{j=1}^{n} [F(d_{\sigma(j)}) - F(c_{\sigma_j})] \\
&= F(d_{\sigma(n)}) \underbrace{-F(c_{\sigma(n)}) + F(d_{\sigma(n-1)})}_{\leq 0} + F(d_{\sigma(1)}) - F(c_{\sigma(n)}) \\
&\leq F(d_{\sigma(n)}) - F(d_{\sigma(1)}) \\
&\leq F(b) - F(a) \\
&= \mu_0\left((a, b]\right)
\end{aligned}$$

and hence

$$\sum_{j=1}^{\infty} \mu_0\left((c_j, d_j]\right) \leq \mu_0\left((a, b]\right)$$

To see the converse inequality, let $\epsilon > 0$ and since F is right-continuous, we may find:

(a) a $\delta_0 > 0$ such that $a + \delta_0 < b$ and $F(a + \delta) < F(a) + \frac{\epsilon}{2}$

(b) for each j, find $\delta_j > 0$ such that $F(d_j + \delta_j) + \frac{\epsilon}{2^{j+1}}$

Then, $\{(c_j, d_j + \delta_j)\}_{j=1}^{\infty}$ is a cover of $[a+\delta_0, b]$ and hence, by compactness, we have that $[a+\delta_0, b] \subseteq \bigcup_{j=1}^{n}(c_j, d_j + \delta_j)$ for some n. Let $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be as above. Notice that:

(a) If $c_{\sigma(1)} < a + \delta_0$, then $F(c_{\sigma(1)}) \leq F(a + \delta_0) < F(a) + \frac{\epsilon}{2}$

(b) For $j = 1, \ldots n-1$, if $c_{\sigma(j+1)} < d_{\sigma(j)} + \delta_{\sigma(j)}$ then $F(c_{\sigma(j+1)}) \leq F(d_{\sigma(j)} + \delta_{\sigma(j)}) < F(\delta_{\sigma(j)}) + \frac{\epsilon}{2^{\sigma(j)+1}}$

(c) If $b < d_{\sigma(n)} + \delta_{\sigma(n)}$ then $F(b) < F(d_{\sigma(n)}) + \frac{\epsilon}{2^{\sigma(n)+1}}$

Hence, we get,

$$\begin{aligned}
\sum_{j=1}^{\infty} \mu_0\left((c_j, d_j]\right) &\geq \sum_{j=1}^{n} \mu_0\left((c_j, d_j]\right) \\
&= \sum_{j=1}^{n} [F(d_j) - F(c_j)] \\
&= F(d_{\sigma(n)}) + \sum_{j=1}^{n-1} [F(d_{\sigma(j)}) - F(c_{\sigma(j+1)})] - F(c_{\sigma(1)}) \\
&> \left(F(b) - \frac{\epsilon}{2^{\sigma(n)+1}}\right) + \sum_{j=1}^{n-1} \left(-\frac{\epsilon}{2^{\sigma(j)+1}}\right) - [F(a) + \epsilon] \\
&> F(b) - F(a) - \epsilon \\
&= \mu_0\left((a, b]\right) - \epsilon
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we see that $\sum_{j=1}^{\infty} \mu_0\left((c_j, d_j]\right) \geq \mu_0\left((a, b]\right)$ and hence, our desired inequality holds.

2. We argue similarly for intervals of the form $(-\infty, a], (a, \infty] = (a, \infty)$, and $(-\infty, \infty] = \mathbb{R}$. (Exercise).

3. If $A, A_1, A_2, \ldots \in \mathcal{A}$, with $A = \sqcup_{j=1}^{\infty} A_j$. Write $A = \sqcup_{j=1}^{n} (a_i, b_i]$ and for each i, j get

$$(a_i, b_i] \cap A_j = \sqcup_{k=1}^{m_{ij}} (c_{ijk}, d_{ijk}]$$

From parts $1, 2$, we have that $(a_i, b_i] = \sqcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} (c_{ijk}, d_{ijk}]$ so that

$$\mu_0 \left((a_i, b_i] \right) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \mu_0 \left((c_{ijk}, d_{ijk}] \right)$$

so we have

$$\begin{aligned} \mu_0 (A) &= \sum_{i=1}^{n} \mu_0 \left((a_i, b_i] \right) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{\infty} \sum_{k=1}^{m_{ij}} \mu_0 \left((c_{ijk}, d_{ijk}] \right) \\ &= \sum_{j=1}^{n} \sum_{i=1}^{\infty} \sum_{k=1}^{m_{ij}} \mu_0 \left((c_{ijk}, d_{ijk}] \right) \\ &= \sum_{j=1}^{\infty} \mu_0 (A_j) \end{aligned}$$

since each $A_j = \sqcup_{i=1}^{\infty} \sqcup_{k=1}^{m_{ij}} (c_{ijk}, d_{ijk}]$. ■

Definition 1.43 A Borel measure on the real line $\mu : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ is said to be **locally finite** if

$$\mu_0 \left([-a, a] \right) < \infty$$

for $a > 0$ in $\mathbb{R}$.

Remark 1.44 Locally finite is equivalent to having $\mu(K) < \infty$ for each compact $K \subseteq \mathbb{R}$.

Remark 1.45 In $\mathbb{R}$, locally finite implies σ-finite because

$$\mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n]$$

Theorem 1.46 *(Characterisation of locally finite measures)*

1. *For each $F \in \mathrm{ND}_r(\mathbb{R})$, there is a unique locally finite measure $\mu : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ such that*

$$\mu_F \left((a, b] \right) = F(b) - F(a)$$

for each $-\infty < a < b < \infty$

2. *Every locally finite measure appears as in 1.*

3. *If $F, G \in \mathrm{ND}_r(\mathbb{R})$ then $\mu_F = \mu_G$ if and only if $F - G$ is constant.*

Proof. 1. The last lemma provides a premeasure $(\mathbb{R}, \langle\mathcal{H}\rangle, \mu_{0,F})$ where $\mu_{0,F}([a,b]) = F(b) - F(a)$ for $-\infty \le a \le b \le \infty$. This gives rise to an outer measure $\mu_F^* : \mathcal{P}(\mathbb{R}) \to [0,\infty]$, and its σ-algebra $\mathcal{M}_F$ of μ_F^*-measurable sets. Notice that a prior proposition provides that $\mathcal{B}(\mathbb{R}) = \sigma\langle\mathcal{H}\rangle$, so since $\mathcal{H} \subseteq \langle\mathcal{H}\rangle \subseteq \mathcal{M}_F$ (from the pre-measure to measure construction) we have that $\mathcal{B}(\mathbb{R}) = \mathcal{M}\mathcal{H} \subseteq \mathcal{M}_F$.

Then, we let $\mu_F = \mu_F^*|_{\mathcal{B}(\mathbb{R})} : \mathcal{B}(\mathbb{R}) \to [0,\infty]$. Notice, for $a > 0$ in $\mathbb{R}$, that

$$\mu_F([-a,a]) \le \mu_F([-a-1,a]) = F(a) - F(-a-1) < \infty$$

so μ_F is locally finite and hence σ-finite. Hence μ_F is the unique extension of $\mu_{0,F}$ to $\mathcal{B}(\mathbb{R})$ (or even to $\mathcal{M}_F$.

2. Let $\mu : \mathcal{B}(\mathbb{R}) \to [0,\infty]$ be a locally finite measure. Then, for $x \in \mathbb{R}$, we let

$$F(x) = \begin{cases} \mu((0,x]) & x \ge 0 \\ -\mu((x,0]) & x < 0 \end{cases}$$

We will see that $F \in \mathrm{ND}_r(\mathbb{R})$. If $x < y$ in $\mathbb{R}$, we have the following cases:

(a) If $x \ge 0$, then $(0,x) \subseteq (0,y]$ so $F(x) = \mu((0,x]) \le \mu((0,y]) = F(y)$

(b) If $y < 0$ then $(y,0] \subseteq (0,x]$ so that $\mu((y,0]) \le \mu((0,x])$ implying that $F(x) = -\mu((0,x]) \le -\mu((y,0]) = F(y)$

(c) If $x < 0 \le y$ then $F(x) = -\mu((x,0]) \le 0 \le \mu((0,y]) = F(y)$

To see right continuity, it suffices to see for $x \in \mathbb{R}$ we have that $F(x) = \lim_{n\to\infty} F(x_n)$ whenever $x_! \ge x_2 \ge \ldots \ge x$ with $\lim_{n\to\infty} x_n = x$ (exercise). thus, given x and $\{x_n\}_{n=1}^\infty$, as above, we have

$$F(x_n) - F(x) = \mu((x,x_n]) \to \mu(\emptyset) = 0$$

as $n \to \infty$, by continuity from above, since $\mu((x,x_1]) \le \mu([-a,a]) < \infty$ where $a = \max\{|x|, |x_1|\}$. Notice that for $a < b$ in $\mathbb{R}$, $\mu_F((a,b]) = \mu((a,b])$, which by uniqueness in part 1, shows that $\mu = \mu_F$.

3. Observe the following sequence of equivalences:

$$\begin{aligned} \mu_F = \mu_G &\iff \text{for } x \in \mathbb{R} : \begin{cases} F(x) - F(0) = \mu_F((0,x]) = \mu_G((0,x]) = G(x) - G(0) & x \ge 0 \\ F(0) - F(x) = \mu_F((x,0]) = \mu_G((x,0]) = G(0) - G(x) & x < 0 \end{cases} \\ &\iff F(x) - G(x) = F(0) - G(x) \text{ is constant} \end{aligned}$$

■

Remark 1.47 We can witness the following computations:

1. $\mu_F((a,b]) = F(b^-) - F(a)$
2. $\mu_F([a,b]) = F(x) - F(a^-)$
3. $\mu_F(\{a\}) = F(a) - F(a^-)$

Conclusion. $\mu_F(\{a\}) = 0$ if and only if F is continuous at a.

Example 1.48 (Dirac measure) From the above, we construct the point mass or Dirac measure. Fix an $a \in \mathbb{R}$. Let the Heaviside function $H_a \in \mathrm{ND}_r(\mathbb{R})$ be defined as:

$$H_a(x) = 1_{[a,\infty)}(x) = \begin{cases} 1 & x \in [a,\infty) \\ 0 & \text{otherwise} \end{cases}$$

Let $\delta_a : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ be the Dirac measure, so that $\delta_a(A)$ is 1 if $a \in A$ and zero otherwise. Notice that if $c < d$ in $\mathbb{R}$, we get

$$\delta_a((c, d]) = \begin{cases} 1 & c < a \leq d \\ 0 & \text{otherwise} \end{cases}$$

so, by uniqueness, we conclude that $\delta_a = \mu_{H_a}$. In fact, via our canonical construction of measures, we see that $\delta_a = \mu^*_{H_a}$, and it turns out $\mathcal{M}_{H_a} = \mathcal{P}(\mathbb{R})$.

Example 1.49 (Lebesgue measure) Let $I(x) = x$, so that $I \in \mathrm{ND}_r(\mathbb{R})$. We let $\lambda = \mu_I$ and $\mathcal{L} = \mathcal{M}_I$ denote the Lebesgue measure and Lebesgue σ-algebra, respectively. We encode some of its nice properties in the following theorem.

Theorem 1.50 *Let $(\mathbb{R}, \mathcal{L}, \lambda)$ be the Lebesgue measure space. Then,*

1. *$(\mathbb{R}, \mathcal{L}, \lambda)$ is translation invariant; that is, for $x \in \mathbb{R}$ and $E \in \mathcal{L}$, we have $\lambda(E + x) = \lambda(E)$*
2. *(Uniqueness) If $\mu : \mathcal{B}(\mathbb{R})$ is a locally finite measure, which is translation-invariant, then $\mu = c\lambda$ for some $c \geq 0$ in $\mathbb{R}$.*

Proof. 1. If $-\infty \leq a \leq b \leq \infty$, then

$$\lambda((a, b] + x) = \mu_I((a + x, b + x]) = (b + x) - (a + x) = b - a = \lambda((a, b])$$

Hence, if $A \in \langle \mathcal{H} \rangle$, we have $\mu_I(A + x) = \mu_I(A)$. But, if $E \in \mathcal{P}(\mathbb{R})$, then $E \subseteq \bigcup_{i=1}^{\infty} A_i$ where $A_i \in \langle \mathcal{H} \rangle$, which happens if and only if $E + x \subseteq \bigcup_{i=1}^{\infty} (A_i + x)$. Thus, by definition of μ^*_I, we see that $\mu^*_I(E + x) = \mu^*_I(E)$. Now, if $A \in \mathcal{L}$ and $E \in \mathcal{P}(\mathbb{R})$, then

$$\begin{aligned} \mu^*_I(E \cap (A + x)) + \mu^*_I(E \setminus (A + x)) &= \mu^*_I([(E - x) \cap A] + x) + \mu^*_I([(E - x) \setminus A] + x) \\ &= \mu^*_I((E - x) \cap A) + \mu^*_I((E - x) \setminus) \\ &\leq \mu^*_I(E - x) \\ &= \mu^*_I(E) \end{aligned}$$

so that $A + x \in \mathcal{L}$.

2. We let $\mu = \mu_F$ where $F \in \mathrm{ND}_r(\mathbb{R})$. In fact, we may let $F(0) = 0$, so that

$$F(x) = \begin{cases} \mu((0, x]) & x \geq 0 \\ -\mu((x, 0]) & x < 0 \end{cases}$$

Then for $y \geq 0$, we have

$$F(y) = \mu((0, y]) = \mu(x, x + y]) = F(x + y) - F(x)$$

so $F(x + y) = F(x) + F(y)$. Hence if $x \geq 0$, $F(nx) = nF(x)$ for all $n \in \mathbb{N}$, by induction.

We can use these functional equations to argue that $F(x) = F(1)x = cx$. So, by uniqueness, $\mu = \mu_{cI} = c\lambda$. ∎

1.6 Cantor's Sets and Functions

Fix $0 < \alpha \leq 1$.[2] Let $I_{01} = [0,1]$ and J_{01} be the open middle of length $\alpha/3$. Notice that $I_{01} \setminus J_{01} = I_{11} \dot\cup I_{12}$, each a closed interval, with $\lambda(I_{1k}) < 1/2$, $k = 1, 2$. Having constructed closed intervals $I_{m1}, \ldots, I_{m2^m}$, each of length at most $1/2^m$, we let for each $k = 1, \ldots, 2^m$, J_{mk} denote the open middle of length $\alpha/3^{m+1}$. Then each $I_{mk} \setminus J_{mk} = I_{m+1,2k-1} \dot\cup I_{m+1,2k}$.

Let $C_{\alpha,n} = \bigcup_{k=1}^{2^n} I_{nk}$, so $C_{\alpha,n}$ is compact. Notice that $C_{\alpha,1} \supseteq C_{\alpha,2} \supseteq \cdots$, then $C_\alpha := \bigcap_{n=1}^\infty C_{\alpha,n}$ is empty and compact. If $\alpha = 1$, then $C = C_1$ is called the (middle thirds) **Cantor set**.

Remark 1.51 1. C_α is nowhere dense. Indeed, if $x \in C_\alpha$, $\epsilon > 0$, let n be so $1/2^n < 2\epsilon$ and we see that $(x - \epsilon, x + \epsilon) \subsetneq I_{nk}$ for any $k = 1, \ldots, 2^n$. Thus $(x - \epsilon, x + \epsilon) \cap (\mathbb{R} \setminus C_\alpha) \neq \emptyset$.

2. We can compute

$$\begin{aligned} \lambda(C_\alpha) &= \lambda([0,1]) - \lambda([0,1] \setminus C_\alpha) \\ &= 1 - \lambda\left(\bigcup_{n=1}^\infty \bigcup_{k=1}^{2^n} J_{nk}\right) \\ &= 1 - \sum_{n=1}^\infty \sum_{k=1}^{2^n} \lambda(J_{nk}) \\ &= 1 - \sum_{n=1}^\infty \alpha \frac{\alpha}{3}\left(\frac{2}{3}\right)^n \\ &= 1 - \alpha \end{aligned}$$

In particular, $\lambda(C) = 0$.

Write each $I_{nk} = [a_{nk}, b_{nk}]$. Define $\phi_{\alpha,n} : \mathbb{R} \to \mathbb{R}$ by

$$\phi_{\alpha,n} = \begin{cases} 0 & : x \in (-\infty, 0) \\ \frac{2k-1}{2^{m+1}} & : x \in J_{mk} \\ \frac{1}{2^n(b_{mk} - a_{mk})}(x - a_{mk}) + c_{mk} & : x \in I_{mk} \\ 1 & : x \in (1, \infty) \end{cases}$$

Each $\phi_{\alpha,n}$ is continuous and non-decreasing on $\mathbb{R}$, and $\|\phi_{\alpha,n} - \phi_{\alpha,n+1}\| = \frac{1}{2^n}$. Thus $(\phi_{\alpha,n})_{n=1}^\infty$ is uniformly Cauchy, so $\phi_\alpha := \lim_{n\to\infty} \phi_{\alpha,n}$ exists and is continuous. Furthermore, (1) tells us for $x < y$, $\phi_\alpha(x) \leq \phi_\alpha(y)$, so $\phi_\alpha \in \mathrm{ND}_r(\mathbb{R})$ and is, in fact, continuous. We let μ_{ϕ_α} denote the corresponding locally finite measure on $(\mathbb{R}, \mathcal{B}(()\mathbb{R}))$. If $\alpha = 1$, $\mu_\phi = \mu_{\phi_1}$ is called the Cantor singular measure.

Note that $\mu_{\phi_\alpha}(C_\alpha) = 1 = \mu_{\phi_\alpha}(\mathbb{R})$, so $\mu_{\phi_\alpha}(\mathbb{R} \setminus C_\alpha) = 0$. We say that μ_{ϕ_α} is **concentrated** on C_α. $\mathcal{M}_{\phi_\alpha} \supseteq \mathcal{P}(\mathbb{R} \setminus C_\alpha)$ as null sets for $\mathcal{M}_{\phi_\alpha}$.

[2]We thank Alex Rutar for authoring and sharing this section.

Chapter 2

Integration

2.1 Measurable functions

Definition 2.1 Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces and $T : X \to Y$. We say that T is $\mathcal{M} - \mathcal{N}$**-measurable** provided that $T^{-1}(\mathcal{N}) \subseteq \mathcal{M}$.

Lemma 2.2 *(Testing measurability) Suppose* $(X, \mathcal{M})$, $(Y, \mathcal{N})$ *and* $T : X \to Y$ *are as above, and* $\mathcal{N} = \sigma \langle \mathcal{E} \rangle$. *Then* T *is* $\mathcal{M} - \mathcal{N}$*-measurable if and only if* $T^{-1}(E) \in \mathcal{M}$ *for* $E \in \mathcal{E}$.

Proof. ($\Longrightarrow$) This is obvious.

($\Longleftarrow$) As in last Friday's proposition, $\mathcal{N}' = \left\{ A \in \mathcal{P}(Y) : T^{-1}(A) \in \mathcal{M} \right\}$ is a σ-algebra. We have that $\mathcal{E} \subseteq \mathcal{N}'$ so that $\mathcal{M} = \sigma \langle \mathcal{E} \rangle \subseteq \mathcal{M}'$ ■

Proposition 2.3 *Let* $(X, \mathcal{M})$ *be a measurable space, with* $f : X \to \mathbb{R}$. *Then the following are equivalent:*

1. f *is* $\mathcal{M} - \mathcal{B}(\mathbb{R})$*-measurable*
2. $f^{-1}(G) \in \mathcal{M}$ *for open* $G \subseteq \mathbb{R}$
3. $f^{-1}((a, \infty)) = \{x \in X : f(x) > a\} \in \mathcal{M}$ *for* $a \in \mathbb{R}$
4. $f^{-1}([a, \infty) = \{x \in X : f(x) \geq a\} \in \mathcal{M}$ *for* $a \in \mathbb{R}$
5. $f^{-1}((-\infty, a)) \in \mathcal{M}$ *for* $a \in \mathbb{R}$
6. $f^{-1}((-\infty, a]) \in \mathcal{M}$ *for* $a \in \mathbb{R}$

Proof. An earlier proposition showed that

$$\mathcal{B}(\mathbb{R}) = \sigma \langle \mathcal{G} \rangle = \sigma \langle \mathcal{O}_\infty \rangle = \sigma \langle \mathcal{C}_\infty \rangle$$

which establishes the equivalence of $1 - 4$. We establish the equivalence of 4 and 5 by saying

$$f^{-1}((-\infty, a)) = f^{-1}(\mathbb{R} \setminus [a, \infty) = \mathbb{R} \setminus f^{-1}([a, \infty)$$

and likewise to establish the equivalence between 3 and 6. ■

Definition 2.4 A function $f : X \to \mathbb{R}$ satisfying conditions above, will simply be called $(\mathcal{M}-)$**-measurable**.

Proposition 2.5 *If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then it is $\mathcal{B}(\mathbb{R})$-measurable.*

Proof. Indeed, if $G \in \mathcal{G}$, then $f^{-1}(G) \in \mathcal{G}$, and we appeal to the "Testing measurability" proposition. ■

Definition 2.6 Let $f_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$. We let the **pointwise supremum** be defined as

$$\left(\sup_{n\in\mathbb{N}} f_n\right)(x) = \sup_{n\in\mathbb{N}} f_n(x) \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$$

for each $x \in \mathbb{R}$. In $\overline{\mathbb{R}}$, we have $(a,\infty] = \left\{x \in \overline{R} : a < x\right\}$.

Theorem 2.7 *In the extended real line, we have*

$$\mathcal{B}\left(\overline{\mathbb{R}}\right) = \sigma\left\langle\{(a,\infty] \in \{a\} \cup \mathbb{R}\}\right\rangle = \sigma\left\langle\mathcal{G} \cup \{\{\infty\},\{\infty\}\}\right\rangle$$

Proof. Exercise. ■

Definition 2.8 Given a measurable space $(X,\mathcal{M})$ and $f : X \to \overline{\mathbb{R}}$, we say f is $(\mathcal{M}-)$ **measurable** if it is $\mathcal{M} - \mathcal{B}\left(\overline{\mathbb{R}}\right)$-measurable.

Remark 2.9 Notice that if $f_n : X \to \mathbb{R}$, $n \in \mathbb{N}$ then

$$\sup_{n\in\mathbb{N}} f_n\,,\ \inf_{n\in\mathbb{N}} f_n : X \to \overline{R}$$

Theorem 2.10 *Let $(X,\mathcal{M})$ be a measurable space, $f_n : X \to \overline{R}$, $n \in \mathbb{N}$, each be measurable. Then the following are measurable:*

1. $\sup_{n\in\mathbb{N}} f_n$
2. $\inf_{n\in\mathbb{N}} f_n$
3. $\limsup_{n\in\mathbb{N}} f_n$
4. $\liminf_{n\in\mathbb{N}} f_n$

Furthermore, if $\lim_{n\to\infty} f_n$ exists, it too is measurable.

Proof. 1. Fix $a \in \mathbb{R}$. Then,

$$\begin{aligned}\left(\sup_{n\in\mathbb{N}} f_n\right)^{-1}((a,\infty]) &= \left\{x \in X : \sup_{n\in\mathbb{N}} f_n(x) > a\right\} \\ &= \bigcup_{n=1}^{\infty} \{x \in X\, f_n(x) > a\} \in \mathcal{M}\end{aligned}$$

2. For $a \in \mathbb{R}$, we have

$$\left(\inf_{n\in\mathbb{N}} f_n\right)^{-1}([-\infty,a)) = \bigcup_{n=1}^{\infty} \{x \in X : f_n(x) < a\} \in \mathcal{M}$$ ■

3. $\limsup_{n\to\infty} f_n(x) = \inf_{n\in\mathbb{N}} \underbrace{\sup_{k\geq n} f_k(x)}_{\text{measurable functions}}$

4. As above.

5. If the limit exists it is equal to the limsup and liminf.

2.2 Interlude: Product Algebras

Definition 2.11 If $(X, \mathcal{M})$, $(Y, \mathcal{N})$ are measurable spaces, we let the **product σ-algebra** of $\mathcal{M}$ and $\mathcal{N}$ be given by

$$\mathcal{M} \otimes \mathcal{N} = \sigma \langle \{E \times F \,:\, E \in \mathcal{M},\, F \in \mathcal{N}\} \rangle \subseteq \mathcal{P}(X \times Y)$$

Theorem 2.12 *Let $\Pi_X : X \times Y \to X$, $\Pi_Y : X \times Y \to Y$ denote the coordinate projections; that is, $\Pi_X(x, y) = x$. Then:*

1. $\mathcal{M} \otimes \mathcal{N} = \sigma \left\langle \Pi_X^{-1}(\mathcal{M}) \cup \Pi_Y^{-1}(\mathcal{N}) \right\rangle$

2. *If $\mathcal{M} = \sigma \langle \mathcal{E} \rangle$ and $\mathcal{N} = \sigma \langle \mathcal{J} \rangle$ then $\mathcal{M} \otimes \mathcal{N} = \sigma \langle \{E \times F \,:\, E \in \mathcal{E}\,,\, F \in \mathcal{J}\} \rangle$*

Proof. 1. Observe that,

$$E \times F = (E \times Y) \cap (X \times F) = \Pi_X^{-1}(E) \cap \Pi_Y^{-1}(F)$$

We see that

$$\{E \times F\,,\, E \in \mathcal{M},\, F \in \mathcal{N}\} \subseteq \sigma \left\langle \Pi_X^{-1}(\mathcal{M}) \cap \Pi_Y^{-1}(\mathcal{N}) \right\rangle$$

and

$$\Pi_X^{-1}(\mathcal{M}) \cap \Pi_Y^{-1}(\mathcal{N}) \subseteq \sigma \langle \{E \times F\,,\, E \in \mathcal{M},\, F \in \mathcal{N}\} \rangle$$

2. Further observe:

$$\mathcal{M} \otimes \mathcal{N} = \sigma \left\langle \Pi_X^{-1}(\mathcal{M}) \cap \Pi_Y^{-1}(\mathcal{N}) \right\rangle = \sigma \left\langle \Pi_X^{-1}(\mathcal{E}) \cap \Pi_Y^{-1}(\mathcal{J}) \right\rangle$$

since $\sigma \left\langle \Pi_X^{-1}(\mathcal{E}) \right\rangle = \Pi_X^{-1}(\mathcal{M})$, for example. ■

Definition 2.13 Let (X, d) be a metric space and $\mathcal{G}(X)$ be the open sets of X with respect to its metric topology. The **Borel σ-algebra** on (X, d) is

$$\mathcal{B}\left((X, d)\right) = \mathcal{B}\left(X\right) := \sigma \left\langle \mathcal{G}(X) \right\rangle$$

Remark 2.14 If the metrics ρ and d are equivalent on X then these metrics generate the same open sets, and thus $\mathcal{B}\left((X, d)\right) = \mathcal{B}\left((X, \rho)\right)$

Theorem 2.15 *Let (X, d_X) and (Y, d_Y) be separable metric spaces. Let ρ be any metric on $X \times Y$ such that $\rho \sim \rho_\infty$, where*

$$\rho_\infty \left((x, y), (x', y')\right) = \max \left\{d_X(x, x'), d_Y(y, y')\right\}$$

Then $\mathcal{B}\left(X \times Y, \rho\right) = \mathcal{B}\left(X, d_X\right) \otimes \mathcal{B}\left(Y, d_Y\right)$

Proof. For $r > 0$, $(x, y) \in X \times Y$ we have radius r open balls

$$B_r\left((x, y), \rho_\infty\right) = B_r\left(x, d_X\right) \times B_r\left(y, d_Y\right)$$

Let $D_X \subseteq X$ and $D_Y \subseteq Y$ be countable dense subsets. Then for open $G \subseteq X \times Y$, we have

$$G = \bigcup_{(x,y) \in (D_X \times D_Y) \cap G} \; \bigcup_{r \in \mathbb{Q} \cap (0, d_{\rho_\infty}((x,y), X \times Y \setminus G))} B_r(x, d_X) \times B_r(y, d_Y)$$

Thus, $\mathcal{G}(X\times Y)\subseteq\mathcal{B}(X)\otimes\mathcal{B}(Y)$ so that $\mathcal{B}(X\times Y)=\mathcal{B}(X)\otimes\mathcal{B}(Y)$. Conversely,

$$\begin{aligned}\mathcal{B}(X)\otimes\mathcal{B}(Y)&=\sigma\left\langle\{G\times H\,:\,G\subseteq X\text{ open },H\subseteq Y\text{ open }\}\right\rangle\\&\subseteq\sigma\left\langle\mathcal{G}(X\times Y)\right\rangle\\&\subseteq\mathcal{B}(X\times Y)\qquad(\dagger)\end{aligned}$$

■

Remark 2.16 Without separability, (†) always holds. However, the converse inclusion is in doubt.

Example 2.17 (Difficult exercise) Consider the discrete metric space $(\mathbb{R},d)$. The following is food for thought. It is true that

$$\mathcal{P}(\mathbb{R})=\mathcal{B}((\mathbb{R},d))$$

Given this, do we have that $\mathcal{P}(\mathbb{R})\otimes\mathcal{P}(\mathbb{R})\subset\mathcal{P}(\mathbb{R}\times\mathbb{R})$?.

Corollary 2.18 $\mathcal{B}\left(\mathbb{R}^d\right)=\mathcal{B}(\mathbb{R})\otimes\ldots\otimes\mathcal{B}(\mathbb{R})$ *(d-fold product)*

Proof. This is an immediate consequence of the theorem above. ■

Theorem 2.19 *If $(X,\mathcal{M}),(Y,\mathcal{N})$, and $(Z,\mathcal{O})$ are measurable spaces, $S:X\to Y$, $T:Y\to Z$ are $\mathcal{M}-\mathcal{N}$-measurable, $\mathcal{N}-\mathcal{O}$-measurable, then $TS:X\to Z$ is $\mathcal{M}-\mathcal{O}$-measurable.*

Proof. If $E\in\mathcal{O}$, then

$$(TS)^{-1}(E)=S^{-1}(\underbrace{T^{-1}(E)}_{\in\mathcal{N}})\in\mathcal{M}$$

■

Theorem 2.20 *If $(X,\mathcal{M})$ is a measurable space, and $T:X\to\mathbb{R}^d$, then T is $\mathcal{M}-\mathcal{B}\left(\mathbb{R}^d\right)$-measurable if and only if each $\Pi_k\circ T:X\to\mathbb{R}$ is measurable, where the Π_i's are simply the coordinate projections.*

Proof. If $B\in\mathcal{B}(\mathbb{R})$, then

$$(\Pi_k\circ T)^{-1}(B)=T^{-1}(\Pi_k^{-1}(B))\qquad(\dagger)$$

($\Longrightarrow$) We have that $\Pi_k:\mathbb{R}^d\to\mathbb{R}$ is continuous, so $\Pi_k^{-1}(G)$ is open for open G in $\mathbb{R}$ and hence $\Pi^{-1}(B)\in\mathcal{B}\left(\mathbb{R}^d\right)$ for B above. Hence $T^{-1}(\Pi_k^{-1}(B))\in\mathcal{M}$, by (†).

($\Longleftarrow$) We have $(\pi_k\circ T)^{-1}(B)\in\mathcal{M}$ for B above. We have that $\mathcal{B}\left(\mathbb{R}^d\right)=\sigma\left\langle\Pi_1^{-1}(\mathcal{B}(\mathbb{R}))\cup\ldots\cup\Pi_d^{-1}(\mathcal{B}(\mathbb{R}))\right\rangle$. We use (†) to see that T is $\mathcal{M}-\mathcal{B}\left(\mathbb{R}^d\right)$-measurable. ■

As a direct corollary to the above, we have the following version for the complex numbers.

Corollary 2.21 $\mathbb{C}\cong\mathbb{R}^2$ *and if $(X,\mathcal{M})$ is a measurable space, $T:X\to\mathbb{C}$ then T is $\mathcal{M}-\mathcal{B}(\mathbb{C})$-measurable if and only if $Re(T),Im(T):X\to\mathbb{R}$ is $\mathcal{M}$-measurable.*

2.3 End Interlude: Back to measurable functions

Definition 2.22 We call a complex valued function f **measurable** provided that it is $\mathcal{M}-\mathcal{B}(\mathbb{C})$-measurable function. In this case, we simply say that f is a **$\mathcal{M}$-measurable** function.

Theorem 2.23 *(Arithmetic properties of measurable functions). Let $(X,\mathcal{M})$ be a measurable space and $f,g:X\to\mathbb{C}$ each be measurable. Then*

$$f+g,fg:X\to\mathbb{C}$$

defined as pointwise operations, are each $\mathcal{M}$-measurable.

Proof. Consider $\alpha, m : \mathbb{C}^2 \to \mathbb{C}$ given by $\alpha(z,w) = z + w$ and $m(z,w) = zw$. Each of these functions is continuous, and hence $\mathcal{B}\left(\mathbb{C}^2\right) - \mathcal{B}(\mathbb{C})$-measurable. We define $F : X \to \mathbb{C}^2$ by $F(x) = (f(x), g(x))$. By a modification of the last proposition, $\mathbb{C}^2$ playing the role of $\mathbb{R}^d$, we see that F is $\mathcal{M} - \mathcal{B}\left(\mathbb{C}^2\right)$-measurable. Then

$$f + g = \alpha \circ F \qquad fg = m \circ F$$

as desired. ■

Remark 2.24 A constant function is always measurable. Hence, if $f : X \to \mathbb{C}$ is measurable and $c \in \mathbb{C}$, then cf is measurable.

2.4 Constructing the integral

2.4.1 Measurable non-negative simple functions

Definition 2.25 If $(X, \mathcal{M})$ is a measurable space, let the **simple functions** be defined by

$$\mathcal{S}^+(X, \mathcal{M}) = \{\varphi : X \to [0, \infty) : |\varphi(X)| < \infty \text{ and } \varphi \text{ is measurable}\}$$

Lemma 2.26 *The following is a lemma about simple functions.*

1. *If $E \in \mathcal{P}(X)$, then*

$$1_E \in \mathcal{S}^+(X, \mathcal{M}) \iff E \in \mathcal{M}$$

   where 1_E is the indicator function for E.

2. *If $\varphi : X \to [0, \infty)$, then $\varphi \in \mathcal{S}^+(X, \mathcal{M})$ if and only in there are $0 \leq a_1 < a_2 < \ldots < a_n$, and pairwise disjoint $E_1, \ldots, E_n$ in $\mathcal{M}$, such that*

$$\varphi = \sum_{i=1}^{n} a_i 1_{E_i}$$

Proof. With a bit of annoying casework:

1. Clearly $1_E(X) \subseteq \{0,1\} \subseteq [0, \infty$. If $B \in \mathcal{B}(\mathbb{R})$ then,

$$1_E^{-1}(B) = \begin{cases} \emptyset & \{0,1\} \cap B = \emptyset \\ E & \{0,1\} \cap B = \{1\} \\ X \setminus E & \{0,1\} \cap B = \{0\} \\ X & \{0,1\} \subseteq B \end{cases}$$

   Hence 1_E is $\mathcal{M}$-measurable if and only if $E \in \mathcal{M}$.

2. $(\Longleftarrow)$ Use (1) and arithmetic of measurable functions.

   $(\Longrightarrow)$ Let $\{a_1, \ldots, a_n\} = \varphi(X)$, where the a_i's are distinct. Then let $E_i = \varphi^{-1}(\{a_i\})$. ■

Definition 2.27 (Pre-integral) If $(X, \mathcal{M}, \mu)$ is a measure space, define

$$I_\mu : \mathcal{S}^+(X, \mathcal{M}) \to [0, \infty]$$

by

$$I_\mu(\phi) = \sum_{i=1}^{n} a_i \mu(E_i) \qquad \text{if } \phi = \sum_{i=1}^{n} a_i 1_{E_i}$$

where we let

$$a \cdot \infty = \begin{cases} \infty & a > 0 \\ 0 & a = 0 \end{cases}$$

Theorem 2.28 *Let $\phi, \psi \in \mathcal{S}^+(X, \mathcal{M})$. Then,*

1. *If $\phi \le \psi$, then $I_\mu(\phi) \le I_\mu(\psi)$.*
2. *If $c \in [0, \infty)$, then $I_\mu(\phi + c\psi) = I_\mu(\phi) + cI_\mu(\psi)$.*

Proof. Write $\phi = \sum_{i=1}^{n} a_i 1_{E_i}$ and $\psi = \sum_{j=1}^{m} b_j 1_{F_j}$ with $\bigsqcup_{i=1}^{n} E_i = X = \bigsqcup_{i=1}^{n} F_i$. We attack them in parts:

1. We can compute:

$$\begin{aligned} I_\mu(\phi) &= \sum_{i=1}^{n} a_i \mu(E_i) \\ &= \sum_{i=1}^{n} a_i \sum_{j=1}^{m} \mu(E_i \cap F_j) \\ &= \sum_{j=1}^{m} \sum_{i=1}^{n} a_i \mu(E_i \cap F_j) \\ &\le \sum_{j=1}^{m} \sum_{i=1}^{n} \mu(E_i \cap F_j) && a_i \le b_j \text{ if } E_i \cap F_j = \emptyset \\ &= \sum_{j=1}^{m} b_i \mu(F_j) \\ &= I_\mu(\psi) \end{aligned}$$

2. Notice that for $E, F \in \mathcal{P}(X)$, we can write

$$1_E 1_F = 1_{E \cap F} \qquad 1_E + 1_F = 1_{E \cup F} + 1_{E \cap F}$$

We have,

$$\begin{aligned} \phi c\psi &= \underbrace{\sum_{j=1}^{m} 1_{F_j}}_{1_X} \sum_{i=1}^{n} a_i 1_{E_i} + \underbrace{\sum_{i=1}^{n} 1_{E_i}}_{1_X} \sum_{j=1}^{m} cb_j 1_{F_j} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i + cb_j) 1_{E_i \cap F_j} \end{aligned}$$

Let $\{c_1, \dots c_p\} = \{a_i + cb_k : i = 1, \dots, n; j = 1, \dots, m\}$ (distinct enumeration) for $k = 1, \dots, p$ call

$$G_k = \bigsqcup_{i=1,\dots,n; j=1,\dots m \,:\, a_i + cb_j = c_k} E_i \cap F_j$$

so that

$$\phi + c\psi = \sum_{k=1}^{p} c_k 1_{G_k}$$

Then,

$$\begin{aligned}
I_{\phi+c\psi} &= \sum_{k=1}^{p} c_k \mu\left(G_k\right) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i + cb_j)\mu\left(E_i \cap F_j\right) \\
&= \sum_{i=1}^{n} a_i \mu\left(E_i\right) + c \sum_{j=1}^{m} b_j \mu\left(F_j\right) \\
&= I_\mu(\phi) + cI + \mu(\psi)
\end{aligned}$$

■

2.4.2 Non-negative measurable functions

Notation 2.29 Let $(X, \mathcal{M}, \mu)$ be a measure space. Whenever we work with functions into the extended real numbers we shall say:

$$\overline{M}^+(X, \mathcal{M}) = \{f : X \to [0, \infty] \, : \, f \text{ is } \mathcal{M}\text{-measurable}\}$$

Definition 2.30 (Lebesgue) Let $f \in \overline{M}^+(X, \mathcal{M})$ and we let

$$\mathcal{S}^+(X, \mathcal{M}) = S_f^+ = \left\{\phi \in S^+(X, \mathcal{M}) \, : \, \phi \leq f \text{ (pointwise) } \right\}$$

and we let the **Lebesgue integral** of f with respect to μ be given by

$$\int_X f d\mu = \sup_{\phi \in S_f^+} \{I_\mu\} \in [0, \infty]$$

Theorem 2.31 *Here are a few properties of the Lebesgue integral of f.*

1. *If $f \leq g$ (pointwise) in $\overline{M}^+(X, \mathcal{M})$ then $\int_X f d\mu \leq \int_X g d\mu$.*
2. *If $\phi \in \mathcal{S}^+(X, \mathcal{M})$ then $\int_X \phi d\mu = I_\mu(\phi)$.*

Proof. By parts.

1. We have $S_f^+ \subseteq S_g^+$, and we apply the definition of the integral.
2. Since $\phi \in S_\phi^+$ we have $I_\mu(\phi) \leq \int_X \phi d\mu$. Conversely, if $\psi \in S_\phi^+$ then $\psi \leq \phi$ so that $I_\mu(\psi) \leq I_\mu(\phi)$ (by the above) and thus $\int_X d\mu \leq I_\mu(\phi)$ by definition of the integral. ■

Remark 2.32 Hereafter, we write $\int_X \phi d\mu = I_\mu(\phi)$.

Theorem 2.33 ***(Monotone convergence theorem)*** *Let $(X, \mathcal{M}, \mu)$ be a measure space and $f_1 \leq f_2 \leq \ldots$ in $\overline{M}^+(X, \mathcal{M})$. Then*

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X \left(\lim_{n\to\infty} f_n\right) d\mu$$

Proof. First, observe that

$$f = \sup_{n\in\mathbb{N}} f_n = \lim_{n\to\infty} f_n$$

exists and $f \in \overline{M}^+(X, \mathcal{M})$. Also, by the proposition above, we have

$$\int f_1 \leq \int f_2 \leq \ldots \leq \int f$$

so that

$$\lim_{n\to\infty} \int f_n \leq \int f$$

Now if $0 < \eta < 1$ and $\varphi \in S_f^+$ is fixed, we wish to show that

$$\lim_{n\to\infty} \int f_n \geq \eta \int \varphi \qquad (\dagger)$$

To do this, we let

$$A_n = \{x \in X \;:\; f_n(x) \geq \eta\varphi(x)\} = (f_n - \eta\varphi)^{-1}([0, \infty]) \in \mathcal{M}$$

Then:

1. $f_n \leq f_{n+1} \Longrightarrow A_n \subseteq A_{n+1}$
2. $\lim n \to \infty f_n = f \Longrightarrow \bigcup_{n=1}^{\infty} A_n = X$ (that is, if $f(x) > 0$, then $f(x) > \eta\varphi(x)$ so $x \in A_n$ eventually)

Thus, if $E \in \mathcal{M}$, continuity from below provides that

$$\lim_{n\to\infty} \mu\left(A_n \cap E\right) = \mu\left(E\right)$$

Write, in standard form,

$$\varphi = \sum_{i=1}^{m} a_i 1_{E_i}$$

and for each n,

$$f_n \geq 1_{A_n} f_n \geq \eta 1_{A_n}\varphi = \eta \sum_{i=1}^{m} a_i 1_{A_n \cap E_i}$$

and hence

$$\int f_n \geq 1_{A_n} f_n = \eta \sum_{i=1}^{m} a_i \mu\left(A_n \cap E_i\right)$$

and thus, taking the limit,

$$\begin{aligned}\lim_{n\to\infty} \int f_n \geq \lim_{n\to\infty} \left(\eta \sum_{i=1}^{m} a_i \mu\left(A_n \cap E_i\right)\right) &= \eta \sum_{i=1}^{m} a_i \mu\left(E_i\right) \\ &= \eta \int \varphi\end{aligned}$$

so that $(\dagger)$ holds. Now, in $(\dagger)$, we let φ vary in S_f^+ and we see that

$$\lim_{n\to\infty} \int f_n \geq \sup_{\varphi \in S_f^+} \left(\eta \int \varphi\right) = \eta \int f$$

and then allowing η to vary, we get

$$\lim_{n\to\infty}\int f_n \geq \lim_{\eta\to 1^{-1}} \eta \int f = \int f \qquad \blacksquare$$

Lemma 2.34 ***(Fatou's Lemma)*** *Let $(f_n)_{n=1}^\infty \subseteq \overline{M}^+(X,\mathcal{M})$. Then,*

$$\int_X \left(\liminf_{n\to\infty} f_n\right) d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu$$

Proof. For each n, we know that $\int \inf_{k\geq n} f_k \leq \int f_n$, so by the Monotone Convergence Theorem, we get

$$\int \liminf_{n\to\infty} f_n = \int \lim_{n\to\infty}\left(\inf_{k\geq n} f_k\right) = \lim_{n\to\infty}\int \inf_{k\geq n} f_k = \liminf_{n\to\infty}\int \inf_{k\geq n} f_k \leq \liminf \int f_n \qquad \blacksquare$$

Theorem 2.35 *(Approximation by simple functions from below) If $(X,\mathcal{M})$ is a measurable space, and $f \in \overline{M}^+(X,\mathcal{M})$, then there are $\varphi_1 \leq \varphi \leq \dots$ in $S^+(X,\mathcal{M})$ with $\lim_{n\to\infty}\varphi_n = f$.*

Proof. Notice that that

$$[0,n) = \bigsqcup_{k=1}^{n2^n}\left[\frac{k-1}{2^n},\frac{k}{2^n}\right)$$

and we consider

$$F_n = f^{-1}([n,\infty]) \qquad E_{n,k} = f^{-1}\left(\left[\frac{k-1}{2^n},\frac{k}{2^n}\right]\right)$$

for $k = 1,\dots,n2^n$ and let

$$\varphi_n = n1_{F_n} + \sum_{k=1}^{n2^n}\frac{k-1}{2^n}1_{E_{n,k}}$$

Then, $n \leq f$ on F_n and $\varphi_n \leq f$ on $E_{n,k}$, so $\varphi_n \leq f$ on $X = F_n \sqcup \bigsqcup_{k=1}^{n2^n} E_{n,k}$. We also have $E_{n,k} = E_{n+1,2k-1} \sqcup E_{n+1,2k}$ and $\frac{k-1}{2^n} = \frac{2k-2}{2^{n+1}} \leq \frac{2k-1}{2^{n+1}} \leq \frac{2k}{2^{n+1}}$ so $\varphi_n \leq \varphi_{n+1}$ on $E_{n,k}$, likewise on F_n. Finally

$$f - \varphi_n \leq \frac{1}{2^n} \qquad \text{on } \bigsqcup_{k=1}^{n2^n} E_{n,k}$$

and $\varphi_n \geq n$ on F_n and it follows that $\lim_{n\to\infty}\varphi_n = f$. $\blacksquare$

Lemma 2.36 *(An expected result for integration which requires some technology to prove)*

1. *If $f,g \in \overline{M}^+(X,\mathcal{M}), c \geq 0$ then $f + cg \in \overline{M}^+(X,\mathcal{M})$ and*

$$\int_X (f+cg)d\mu = \int_X f d\mu + c\int_X g d\mu$$

2. *If $(f_k)_{k=1}^\infty \subseteq \overline{M}^+(X,\mathcal{M})$ then*

$$\sum_{k=1}^\infty f_k \in \overline{M}^+(X,\mathcal{M}) \qquad \textit{and} \qquad \int_X\left(\sum_{k=1}^\infty (f_k)\right)d\mu = \sum_{k=1}^\infty \int_X f_k d\mu$$

3. *If $f \in \overline{M}^+(X,\mathcal{M})$, then $\mu_f : \mathcal{M} \to [0,\infty]$ with $\mu_f(E) = \int_X 1_E f d\mu$ defines a measure, and we write*

$$\int_E f d\mu = \mu_f(E) = \int_X (1_E f) d\mu$$

Proof. We use our usual tricks in our approximation from below toolbox:

1. Let $(\varphi_n)_{n=1}^\infty \subset S_f^+$, so $\varphi_1 \leq \varphi_2 \leq \ldots, \lim_{n\to\infty} \varphi_n$ and $(\psi_n)_{n=1}^\infty S_g^+$ so that $\psi_1 \leq \psi_2 \leq \ldots \leq \lim_{n\to\infty} \psi_n = g$ by last lemma. Then, $(\varphi_n + c\psi_n)_{n=1}^\infty \subset S_{f+cg}^+$ with $\varphi_1 + c\psi_1 \leq \varphi_2 + c\psi_2 \leq \ldots$ and $\lim_{n\to\infty}(\varphi_n + c\psi_n) = f + cg$. Thus, $\varphi_n + c\psi_n \in \overline{M}^+(X,\mathcal{M})$. Furthermore, the MCT provides

$$\int (f+cg) = \lim_{n\to\infty} \int (\varphi_n + c\psi_n) = \lim_{n\to\infty} \left[\int \varphi_n + c\int \psi_n\right] = \lim_{n\to\infty} \int \varphi_n + c \lim_{n\to\infty} \int \psi_n = \int f + c\int g$$

2. Let $g_n = \sum_{k=1}^n f_k$. Then $g_1 \leq g_2 \leq \ldots$ with $\sum_{k=1}^\infty f_k = \lim_{n\to\infty} g_n$. We note that by part 1,

$$\int g_n = \sum_{k=1}^n \int f_k$$

and an application of the MCT provides

$$\int \sum_{k=1}^\infty f_k = \int \lim_{n\to\infty} g_n = \lim_{n\to\infty} \int g_n = \lim_{n\to\infty} \sum_{k=1}^n \int f_k = \sum_{k=1}^\infty \int f_k$$

3. Notice that $1_\phi = 0$ so $\mu_f(\emptyset) = \int 0 \cdot f = \int 0 = 0$. If $E_1, E_2, \ldots \in \mathcal{M}$ which are pairwise disjoint, then apply part 2 to $f_k = 1_{E_k}$, noting that

$$\sum_{k=1}^\infty 1_{E_k} = 1_{\bigcup_{k=1}^\infty E_k}$$

to get σ-additivity. ■

2.5 $\mathbb{R}, \mathbb{C}$-valued integrable functions

Notation 2.37 Let $(X, \mathcal{M}, \mu)$ be a measure space. We let

$$\begin{aligned} M(X,\mathcal{M}) &= \{f : X \to \mathbb{C} \,:\, f \text{ is } \mathcal{M}\text{-measurable }\} \\ M^{\mathbb{R}}(X,\mathcal{M}) &= \{f : X \to \mathbb{R} \,:\, f \text{is } \mathcal{M} \text{ measurable}\} \\ M^+(X,\mathcal{M}) &= \{f : X \to [0,\infty) \,:\, :\, f\text{is } \mathcal{M} \text{ measurable}\} \end{aligned}$$

Remark 2.38 If $f \in M^{\mathbb{R}}(X,\mathcal{M})$ then

$$f^+ = \max\{f, 0\} \qquad f^- = \max\{-f, 0\}$$

are both $M^+(X,\mathcal{M})$ since $-f \in M^{\mathbb{R}}(X,\mathcal{M})$ as $x \mapsto -x$ is continuous on $\mathbb{R}$ hence Borel measurable. Hence we have $f = f^+ - f^-$ and $|f| = f^+ + f^- \in M^+(X,\mathcal{M})$.

Remark 2.39 If $f \in M(X,\mathcal{M})$, then $|\cdot| : \mathbb{C}[0,\infty)$ is continuous, hence, Borel measurable, so $|f| \in M^+(X,\mathcal{M})$.

Definition 2.40 We let

$$L(X,\mathcal{M},\mu) = L(\mu) = \left\{ f \in M(X,\mathcal{M}) \ : \ \int_X |f| d\mu < \infty \right\}$$

denote the μ **- Lebesgue integrable** functions. Notice that

$$Re(f)^+, Re(f)^-, Im(f)^+, Im(f)^- \leq |f| \leq Re(f)^+ + Re(f)^- + Im(f)^+ + Im(f)^-$$

so we have that

$$f \in L(\mu) \iff Re(f)^+, Re(f)^-, Im(f)^+, Im(f)^- \in L(\mu)$$

We therefore may define for $f \in L(\mu)$ the **Lebesgue integral** with respect to μ by

$$\int_X f d\mu = \int_X Ref^+ d\mu - \int_X Ref^- d\mu + i \left[\int_X Imf^+ d\mu - \int_X Imf^- d\mu \right]$$

Theorem 2.41 *Let $f, g \in L(X,\mathcal{M},\mu)$ and $c \in \mathbb{C}$, then $f+g, cf \in L(\mu)$ with*

$$\int_X (f+g) d\mu = \int_X f d\mu + \int_X g d\mu \qquad \int_X (cf) d\mu = c \int_X f \mu$$

Proof. Assume $f, g \in L^{\mathbb{R}}(\mu)$ and $c \in \mathbb{R}$. Then,

$$(f+g)^+ - (f-g)^- = f + g = f^+ - f^- + g^+ - g^-$$

implying thus that,

$$(f+g)^+ + f^- + g^- = f^+ g^+ + (f+g)^-$$

We then integrate, applying the last corollary and we rearrange,

$$\int (f+g)^+ - \int (f+g)^- = \int f^+ - \int f^- + \int g^+ - \int g^-$$

and additivity follows.

Now, observe that

$$cf = \begin{cases} cf^+ - cf^- & c \geq 0 \\ |c|f^- - |c|f^+ & c < 0 \end{cases}$$

Then, for example, if $c < 0$, we have each of

$$\int |c| f^{\pm} = |c| \int f^{\pm} < \infty$$

and

$$cf = \int |c|f^- - \int |c|f^+ = |c| \int f^- - |c| \int f^+ = c \int f^+ - c \int f^- = c \int f$$

We can then extend this to arithmetic in $\mathbb{C}$ on the real and imaginary parts. ■

Definition 2.42 If $f, g \in M(X,\mathcal{M})$ we say that $f = g$ **μ-almost everywhere** (we write μ-a.e.) if

$$\mu\left(\{x \in X \ : \ f(x) \neq g(x)\}\right) = 0$$

Remark 2.43 Notice that

$$\{x \in X \ : \ f(x) \neq g(x)\} = \begin{cases} (f-g)^{-1}(\mathbb{C} \setminus \{0\}) \\ (f-g)^{-1}((0,\infty)) \cup \left[f^{-1}(\{\infty\}) \cap g^{-1}([0,\infty))\right] \cup \left[f^{-1}([0,\infty)) \cap g^{-1}(\{0\})\right] \end{cases}$$

where in the first case f and g are complex valued and in the second case, f, g are $[0,\infty]$-valued.

Theorem 2.44 *Say $f \sim g$ if and only if $f = g$ μ a.e. Then $\sim$ is an equivalence relation.*

Proof. Exercise. ∎

Notation 2.45 Let $(f_n)_{n=1}^{\infty} \subset M(X, \mathcal{M})$ and $f \in M(X, \mathcal{M})$. We shall write

$$\lim_{n\to\infty} f_n = f \ \mu - a.e.$$

if

$$\mu\left(\{x \in X \ : \ \lim_{n\to\infty} f_n(x) \neq f(x)\}\right) = 0$$

Remark 2.46 Notice that

$$\begin{aligned} E &= \left\{x \in X \ : \ \lim_{n\to\infty} f_n(x) \text{ does not exist}\right\} \\ &= \left\{x \in X \ : \ \liminf_{n\to\infty} Ref_n \neq \limsup_{n\to\infty} Ref_n\right\} \cup \left\{x \in X \ : \ \liminf Imf_n \neq \limsup_{n\to\infty} Imf_n\right\} \in \mathcal{M} \end{aligned}$$

Likewise $\{x \in X \ : \ \lim_{n\to\infty} f_n(x)$ exists, but is not $f(x)\} \in \mathcal{M}$.

Theorem 2.47 *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f \in \overline{M}^{+}(X, \mathcal{M})$. Then*

1. $\int_X f d\mu < \infty \Longrightarrow \mu\left(f^{-1}(\{\infty\})\right) = 0$. *That is f is finite almost everywhere.*
2. $\int_X f d\mu = 0 \iff \mu\left(f^{-1}((0,\infty])\right) = 0$. *That is, f is zero almost everywhere.*

Proof. 1. For each $n \in \mathbb{N}$, we have $n1_{f^{-1}(\{\infty\})} \in S_f^+$ so $0 \leq n\mu\left(f^{-1}(\{\infty\}\right) = \int n1_{f^{-1}(\{\infty\})} \leq \int f < \infty$, so it follows that $\mu\left(f^{-1}(\{\infty\}\right)$

2. $(\Longrightarrow)$ $\frac{1}{n}1_{f^{-1}([\frac{1}{n},\infty])} \in S_f^+$ so that

$$0 \leq \frac{1}{n}\mu\left(f^{-1}([\frac{1}{n},\infty]\right) = \int \frac{1}{n}1_{f^{-1}([\frac{1}{n},\infty]} \leq \int f = 0$$

so that $\mu\left(f^{-1}([\frac{1}{n},\infty]\right) = 0$ Now

$$f^{-1}((0,\infty]) = \bigcup_{n=1}^{\infty} f^{-1}([\frac{1}{n},\infty]$$

and so by σ-subadditivity, the measure of a countable union of μ-null sets is μ-null.

$(\Longleftarrow)$ Let $\varphi = \sum_{i=1}^{n} a_i 1_{E_i} \in S_f^+$ and $a_i > 0$, then $E_i = f^{-1}(\{a_i\}) \subseteq f^{-1}((0,\infty])$ so $\mu(E_i) = 0$. Hence, we see that $\int \varphi = 0$. Then, by definition, $\int f = \sup_{\varphi \in S_f^+} \int \varphi = 0$. ∎

Theorem 2.48 *Let $(X, \mathcal{M}, \mu)$ be a measure space.*

1. *If $f \in \overline{M}^+(X, \mathcal{M})$ then $\int_F f d\mu < \infty \iff$ there is $f_0 \in M^+(X, \mathcal{M})$ such that $f = f_0$ μ-a.e.*

2. *If $f, g \in L(X, \mathcal{M}, \mu)$ then $f = g$ a.e. if and only if $\int_X |f - g| d\mu = 0$.*

Proof. Part 1 is done by letting $f_0 = 1_{f^{-1}([0,\infty))}$ and 2 follows directly from the above. ■

Theorem 2.49 ***(Lebesgue Dominated Convergence Theorem)*** *Let $(f_n)_{n=1}^{\infty} \subseteq L(X, \mathcal{M}, \mu)$, and $f \in M(X, \mathcal{M})$ $(*)$ such that:*

1. *$\lim f_n = f$ μ-a.e.*

2. *There is a $g \in L^+(\mu)$ such that $|f_n| \leq g$ μ-a.e.*

Then $f \in L(\mu)$ and $\lim_{n\to\infty} \int_X f d\mu = \int_X f d\mu$. If, further, $(X, \mathcal{M}, \mu)$ is complete, we may replace $(*)$ with $f : X \to \mathbb{C}$.

Proof. Let $N = \bigcup_{n=1}^{\infty} (|f_n - g|)^{-1}((0, \infty)) \cup \{x \in X : \lim_{n\to\infty} f_n(x) \neq f(x)\}$ so that $\mu(N) = 0$. We may replace f_n by $1_N f_n$ and f by $1_N f$ and assume all limits and inequalities are pointwise. Notice that if $(X, \mathcal{M}, \mu)$ is complete, we do not need the assumption that f is measurable to see that $N \in \mathcal{M}$ (check!).

We then have that $f \in M(X, \mathcal{M})$ with $|f| = \lim_{n\to\infty} |f_n| \leq |g|$ so that $\int |f| \leq \int g < \infty$, that is, f is integrable.

(I) First, let us assume that each f_n, and hence f, is $\mathbb{R}$-valued. Then $(g + f_n)_{n=1}^{\infty}, (g - f_n)_{n=1}^{\infty}, g + f, g - f \subset M^+(X, \mathcal{M})$. Hence, by Fatou's lemma:

$$\begin{aligned}
\int g \pm \int g &= \int (g \pm f) \\
&= \int \liminf_{n\to\infty} (g \pm f_n) \\
&\leq \liminf_{n\to\infty} \int (g \pm f_n) \\
&= \liminf_{n\to\infty} \left[\int g \pm \int f_n \right] \\
&= \begin{cases} \int g + \liminf_{n\to\infty} \int f_n & \text{if } \pm = + \\ \int g - \limsup_{n\to\infty} \int f_n & \text{if } \pm = - \end{cases}
\end{aligned}$$

Then,

1. $\pm = +$ provides $\int g + \int f \leq \int g + \liminf_{n\to\infty} \int f_n$ implying thus that $\int f \leq \liminf_{n\to\infty} \int f_n$.

2. Likewise, if $\pm = -$, we shall get $\int f \geq \limsup_{n\to\infty} \int f_n$.

Putting this all together,

$$\limsup_{n\to\infty} \int f_n \leq \int f_n \leq \liminf_{n\to\infty} \int f_n$$

implying thus that $\lim_{n\to\infty} \int f_n$ exists and equals $\int f$.

(II) Here we use (I) to see that $\lim_{n\to\infty} Re(f_n) = Re(f)$, hence $\lim_{n\to\infty} \int Re(f_n) = \int Re(f)$, and likewise with the imaginary parts. Thus,

$$\begin{aligned}
\lim_{n\to\infty}\int f_n &= \lim_{n\to\infty}\int Re(f_n) + i\lim_{n\to\infty}\int Im(f_n)\\
&= \int Re(f) + i\int Im(f)\\
&= \int f
\end{aligned}$$

■

Remark 2.50 The monotone convergence theorem and Fatou's lemma also work with assumptions of μ-a.e. convergence.

Notation 2.51 We shall denote

$$S(X,\mathcal{M}) = \{\varphi : X \to \mathbb{C} \ : \ \varphi \text{ is } \mathcal{M}\text{-measurable and } |\varphi(X)| < \infty\}$$

Thus, in standard form we shall let $\{c_1,\ldots,c_n\} = \varphi(X)$ (distinct values), $E_i = \varphi^{-1}(\{c_i\}) \in \mathcal{M}$ and we see that $\varphi = \sum_{i=1}^n c_i 1_{E_i}$.

Corollary 2.52 *(Consequences of the LDCT)*

1. *If $(f_n) \subseteq L(\mu)$, $f \in M(X,\mathcal{M})$ with $f = \lim_{n\to\infty} f_n$ μ-a.e. and there is $g \in L^+(\mu)$ with $|f_n| \leq g$ μ-a.e., then $\lim_{n\to\infty}\int_X |f - f_n| d\mu = 0$.*

2. *Given $f \in L(\mu)$, there exists a sequence $(\varphi_n)_{n=1}^\infty \subset S(X,\mathcal{M})$ such that $|\varphi_n| \leq |f|$ and $\lim_{n\to\infty}\varphi_n = f$ (pointwise, hence μ-a.e.) Furthermore, we have that $\int_X f d\mu = \lim_{n\to\infty}\int \varphi_n d\mu$.*

3. *If $f \in L(\mu)$, then $\left|\int_X f d\mu\right| \leq \int_X |f| d\mu$.*

Proof. These are long statements, but not hard to prove.

1. We have $\lim_{n\to\infty}|f - f_n| = 0$ μ-a.e. and $|f - f_n| \leq |f| + |f_n| \leq 2g \in L^+(\mu)$. Apply the LDCT.

2. An earlier lemma tells us that there are sequences $(\varphi_n^+)_{n=1}^\infty, (\varphi_n^-)_{n=1}^\infty, (\psi_n^+)_{n=1}^\infty, (\psi_n^-)_{n=1}^\infty$ so that

   $$0 \leq \varphi_1^+ \leq \varphi_2^+ \leq \ldots \qquad \text{with } \lim_{n\to\infty}\varphi_n = Re(f^+)$$

   and

   $$0 \leq \psi_1^- \leq \psi_2^- \leq \ldots \qquad \text{with } \lim_{n\to\infty}\psi_n = Im(f^-)$$

   and likewise for the other two sequences. Then, let

   $$\varphi_n = \varphi_n^+ - \varphi_n^- + i\left[\psi_n^+ - \psi_n^-\right]$$

   then,

   $$\begin{aligned}
   |\varphi_n| &= \sqrt{|\varphi_n^+ - \varphi_n^-|^2 + |\psi_n^+ - \psi_n^-|^2}\\
   &= \sqrt{|\varphi_n^+ + \varphi_n^-|^2 + |\psi_n^+ + \psi_n^-|^2}\\
   &\leq \sqrt{|Re(f^+) + Re(f^-)|^2 + |Im(f^+) + Im(f^-)|^2}\\
   &= \sqrt{|Re(f)|^2 + |Im(f)|^2}\\
   &= |f|
   \end{aligned}$$

   and also $\lim_{n\to\infty}\varphi_n = f$. We have that since $|\varphi_n| \leq |f|$, we use LDCT to get limit of integrals.

3. If $\varphi \in S(X, \mathcal{M}) \cap L(\mu)$, write $\varphi = \sum_{i=1}^{n} c_i 1_{E_i}$ (in standard form). Then

$$\left|\int \varphi\right| = \left|\sum_{i=1}^{n} c_i \mu\left(E_i\right)\right| \leq \sum_{i=1}^{n} |c_i| \mu\left(E_i\right) = \int |varphi|$$

Now if $f \in L(\mu)$, w obtain a sequence $(\varphi_n)_{n=1}^{\infty} \subset S(X, \mathcal{M})$, as in above, and we have

$$\left|\int f\right| = \lim_{n\to\infty} \left|\int \varphi_n\right| \leq \lim_{n\to\infty} \int |\varphi_n| = \int |f|$$

as $|\varphi_n| \leq |f|$ and $\lim_{n\to\infty} |\varphi_n| = |f|$. ■

Theorem 2.53 *Let $(X, \mathcal{A}, \mu_0)$ be a premeasure space, and $(X, \mathcal{M}, \mu)$ denote the measure space induced by the pre-measure-outer-measure-measure construction. Given $f \in L(\mu)$ and $\epsilon > 0$, there is*

$$\varphi = \sum_{i=1}^{n} a_i 1_{B_i} \quad a_1, \ldots, a_n \in \mathbb{C}, B_1, \ldots, B_n \in \mathcal{A}$$

such that

$$\int_X |\varphi - f| d\mu < \epsilon$$

Proof. (I) Let $E \in \mathcal{M}$, with $\mu\left(E\right) < \infty$. Then, given $\epsilon > 0$, there is $B \in \mathcal{A}$ such that $\mu\left(B \triangle E\right) < \epsilon$. Indeed, let $A_1, A_2, \ldots \in \mathcal{A}$ be so that $E \subseteq \bigcup_{i=1}^{\infty} A_i$ with $\sum_{i=1}^{\infty} \mu_0\left(A_i\right) < \mu^*\left(E\right) + \frac{\epsilon}{2} = \mu\left(E\right) + \frac{\epsilon}{2}$. Let n be so that

$$\sum_{i=n+1}^{\infty} \mu_0\left(A_i\right) < \frac{\epsilon}{2}$$

and let $B = \bigcup_{i=1}^{n} A_i \in \mathcal{A}$. Then

$$B \triangle E = (B \setminus E) \cup (E \setminus B) \subseteq \left(\bigcup_{i=1}^{\infty} A_i \setminus E\right) \cup \left(\bigcup_{i=n+1}^{\infty} A_i\right)$$

and we complete the proof by using σ-subadditivity of μ.

(II) If $\psi \in S(X, \mathcal{M}) \cap L(\mu)$. Then there is φ as above so $\int |\varphi - \psi| < \epsilon$. Indeed, write, $\psi = \sum_{i=1}^{n} a_i 1_{E_i}$ (standard form). By (I), we may find for each i a B_i in $\mathcal{A}$ such that $\mu\left(B_i \triangle E_i\right) < \frac{\epsilon}{a}$ where $a = 1 + \sum_{i=1}^{n}(|a_i|)$. Then

$$\begin{aligned}
\int |\varphi - \psi| &\leq \sum_{i=1}^{n} \int |1_{B_i} - 1_{E_i}| \\
&= \sum_{i=1}^{n} |a_i| \mu\left(B_i \triangle E_i\right) \\
&< \epsilon
\end{aligned}$$

■

(III) If $f \in L(\mu)$, a corollary to LDCT provides ψ in $S(X, \mathcal{M}) \cap L(\mu)$ such that $\int |f - \psi| < \frac{\epsilon}{2}$ (in fact $|\psi| \leq |f|$). We let ψ be as in (II) be so $\int |\psi - \varphi| < \frac{\epsilon}{2}$.

Theorem 2.54 *Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \times (a, b) \to \mathbb{C}$, with $a < b$ in $\overline{\mathbb{R}}$, such that:*

1. $f(\cdot, s) \in L(\mu)$ *for each* s *in* (a, b),

2. $\frac{\partial}{\partial s} f(x, s) = \lim_{h \to 0} \frac{f(x,s+h)-f(x,s)}{h}$ *exists for each* $(x, s) \in X \times (a, b)$

3. *There is* $g \in L^+(\mu)$ *such that* $\left|\frac{\partial}{\partial s} f(\cdot, s)\right| \leq g$ μ*-a.e. for each* s *in* (a, b)

Then, $F(s) = \int_X f(x, s) d\mu(x)$ *then* F *is differentiable on* (a, b) *with*

$$F'(s) = \int_X \frac{\partial}{\partial s} f(x, s) d\mu(x)$$

Proof. We fix $s \in (a, b)$ and an arbitrary sequence $(h_n)_{n=1}^{\infty} \subset \mathbb{R} \setminus \{0\}$, such that $s + h_n \in (a, b)$ for each n and $\lim_{n\to\infty} h_n = 0$. Notive that for each $x \in X$, $f(x, \cdot) : (a, b) \to \mathbb{C}$ is continuous on the intervals $[s, s+h_n], [s+h_n, s]$ (if $h_n < 0$) for $n \in \mathbb{N}$. Hence by the mean value theorem, we find c_n, d_n in $(s, s+h_n)$ and $(s + h_n, s)$, respectively, such that,

$$\begin{aligned} |f(x, s+h) - f(x, s)| &= \left| Re\left(\frac{\partial}{\partial s} f(x, c_n)\right) + iIm\left(\frac{\partial}{\partial s} f(x, d_n)\right)\right| |h_n| \\ &\leq 2|g(x)||hn| \end{aligned}$$

Thus, by LDCT,

$$\begin{aligned} F'(s) &= \lim_{n\to\infty} \frac{f(s+h_n) - f(s)}{h_n} \\ &= \lim_{n\to\infty} \int \left(\frac{f(x, s+h_n) - f(x, s)}{h_n}\right) d\mu(x) && \text{(by the LDCT)} \\ &= \int \frac{\partial}{\partial S} f(x, s) d\mu(x) \end{aligned}$$

■

2.6 Modes of Convergence

Let $(X, \mathcal{M}, \mu)$ be a measure space, $(f_n)_{n=1}^{\infty}$ and f in $M(X, \mathcal{M})$. We want to investigate different modes of the statement $\lim_{n\to\infty} f_n = f$.

Definition 2.55 Let $(X, \mathcal{M}, \mu)$ be a measure space, $(f_n)_{n=1}^{\infty}$ and f in $M(X, \mathcal{M})$. We say that $\lim_{n\to\infty} f_n = f$:

1. **uniformly** if $\lim_{n\to\infty} \sup_{x\in X} |f_n(x) - f(x)| = 0$,

2. **pointwise** if $\lim_{n\to\infty} |f_n(x) - f(x)| = 0$ for each $x \in X$,

3. **pointwise** μ**-a.e.** if $\lim_{n\to\infty} |f_n(x) - f(x)| = 0$ for each $x \in X \setminus N$ where $\mu(N) = 0$,

4. **in** $\boldsymbol{L}^1(\mu)$ if $\lim_{n\to\infty} \int_X |f_n - f| d\mu = 0$,

5. **in** μ**-measure** if for any $\epsilon > 0$ we have that $\lim_{n\to\infty} \mu(\{x \in X \ : \ |f_n(x) - f(x)| \geq \epsilon\}) = 0$

Example 2.56 We look at the following sequences in $M(\mathbb{R}, \mathcal{B}(\mathbb{R}))$:

$$\begin{aligned} f_n &= \frac{1}{n} 1_{[0,n]} \\ g_n &= 1_{[n,n+1]} \\ h_n &= n 1_{[n, \frac{1}{n}]} \\ k_n &= 1_{[\frac{j}{2^k}, \frac{j+1}{2^k}]} \text{ where } n = 2^k + j, j = 0, \ldots, 2^k - 1, k \in \mathbb{N} \end{aligned}$$

These sequences play important roles in analysing modes of convergence. Then

	uniform	pointwise	pointwise $\lambda-$a.e.	in $L^1(\lambda)$	in $\lambda-$measure
f_n	✓	✓	✓	×	✓
g_n	×	✓	✓	×	×
h_n	×	×	✓	×	✓
k_n	×	×	×	✓	✓

Theorem 2.57 *If* $\lim_{n\to\infty} f_n = f$ *in* $L^1(\mu)$ *then* $\lim_{n\to\infty} f_n = f$ *in* μ*-measure.*

Proof. Let $\epsilon > 0$. Denote $E_{n,\epsilon} = \{x \in X : |f_n - f| \geq \epsilon\}$. Then,

$$\int_X |f_n - f| d\mu \geq \int_{E_{n,\epsilon}} |f_n - f| d\lambda \geq \epsilon \mu (E_{n,\epsilon})$$

Since $f_n \to f$ in L^1,

$$\begin{aligned} \mu (E_{n,\epsilon}) &\leq \frac{1}{\epsilon} \int_X |f_n - f| d\lambda \\ &\to 0 \end{aligned}$$

as desired. ■

Theorem 2.58 *Let* $(f_n)_{n=1}^{\infty}$ *and* f *in* $M(X, \mathcal{M})$*.*

1. *If* $\lim_{n\to\infty} f_n = f$ *in* μ*-measure then* $(f_n)_{n=1}^{\infty}$ *is* ***Cauchy in*** μ***-measure****; i.e. given* $\epsilon > 0$*,* $\delta > 0$*, there is* n_0 *in* $\mathbb{N}$ *(dependent on* ϵ, δ*) such that whenever* $n, m \geq n_0$ *we have*

$$\mu (\{x \in X \ : \ |f_n(x) - f_m(x)| \geq \epsilon\})$$

2. *If* $(f_n)_{n=1}^{\infty}$ *is Cauchy in* μ*-measure then there is a subsequence* $\left(f_{n_j}\right)_{j=1}^{\infty}$ *such that* $\lim_{j\to\infty} f_{n_j} = f_0$ *for some* f_0 *in* $M(X, \mathcal{M})$*,* μ*-a.e. Furthermore,* $\lim_{j\to\infty} f_{j_j} = f_0$ *in measure.*

Proof. The first is by definition, the second involves a clever trick.

1. If $m, n \in \mathbb{N}$ then

$$\begin{aligned} \{x \in X : |f_n(x) - f_m(x)| \geq \epsilon\} &\subseteq \{x \in X : |f_n(x) - f(x)| + |f(x) - f_m(x)| \geq \epsilon\} \\ &\subseteq \left\{x \in X : |f_n(x) - f(x)| \geq \frac{\epsilon}{2}\right\} \cup \left\{x \in X : |f(x) - f_m(x)| \geq \frac{\epsilon}{2}\right\} \end{aligned}$$

and now we may apply the definitions.

2. Let $n_1 < n_2 < \ldots$ be chosen such that

$$E_j = \left\{ x \in X \ : \ |f_n(x) - f_m(x)| \geq \frac{1}{2^j} \right\} \text{ for } n, m \geq n_k$$

satisfies $\mu(E_j) < \frac{1}{2^j}$ (that is $\epsilon, \delta = 2^{-j}$). Let $F_k = \bigcup_{j=k}^{\infty} E_j$, so by σ-subadditivity,

$$\mu(F_k) \leq \sum_{j=k}^{\infty} \frac{1}{2^j} = \frac{1}{2^{k-1}}$$

If $x \notin F_k$ then for $i > j >\geq k$ we have

$$|f_{n_j}(x) - f_{n_i}(x)| \leq \sum_{p=j}^{i-1} |f_{n_p}(X) - f_{n_{p+1}}(x)| < \sum_{p=j}^{i-1} \frac{1}{2^p} = \frac{1}{2^{j-1}} \leq \frac{1}{2^{k-1}}$$

Hence $(f_{n_j})_{j=1}^{\infty}$ is pointwise Cauchy on $X \setminus F_k$. Let $F = \bigcap_{k=1}^{\infty} F_k$, so for each k

$$0 \leq \mu(F) \leq \mu \left(F_k \leq \frac{1}{2^{k-1}} \right)$$

so $\mu(F) = 0$. Hence for $x \in X \setminus F = \bigcup_{k=1}^{\infty}(X \setminus F_k)$ we have that $(f_{n_j})_{j=1}^{\infty}$ is pointwise Cauchy. Hence there is $\tilde{f} \in M(X \setminus F, \mathcal{M}|_{X \setminus F})$ where

$$\mathcal{M}|_{X \setminus F} = \{A \cap (X \setminus F) \ \ A \in \mathcal{M}\}$$

Let $f_0 : X \to \mathbb{C}$ be defined by

$$f_0(x) = \begin{cases} \tilde{f}(x) & x \in X \setminus F \\ 0 & x \in F \end{cases}$$

and it is easy to see that $f_0 \in M(X, \mathcal{M})$. Given $\epsilon > 0$, let k be so large that $\frac{1}{2^{k-1}} < \epsilon$. Then, for $x \in X \setminus F_k$,

$$|f_0(x) - f_{n_k}(x)| = \lim_{j \to \infty} |f_{n_j}(x) - f_{n_k}(x)| \leq \frac{1}{2^{k-1}} < \epsilon$$

so

$$E = \{x \in X \ : \ |f_0(x) - f_{n_k}(x)| \geq \epsilon\} \subseteq F_k$$

and hence $\mu(E) \leq \mu(F_k) \leq \frac{1}{2^{k-1}} < \epsilon$, so we get convergence in measure. ■

Theorem 2.59 *If* $\lim_{n\to\infty} f_n = f$ *in* $L^1(\mu)$ *then there is a subsequence* $(f_{n_j})_{j=1}^{\infty}$ *such that* $\lim_{j\to\infty} f_{n_j} = f$ μ*-a.e.*

Proof. By the last proposition, we have $\lim_{n\to\infty} f_n = f$ in μ-measure and by the part 1 of the theorem above, $(f_n)_{n=1}^{\infty}$ is Cauchy in μ-measure. By part 2, there is a subsequence $(f_{n_j})_{j=1}^{\infty}$ so $\lim_{j\to\infty} f_{n_j} = f_0$ μ-a.e. Like in the proof of part 1 of the theorem above,

$$\begin{aligned} E &= \{x \in X : |f_0(x) - f(x)| \geq \epsilon\} \\ &\subseteq \left\{ x \in X : |f_0(x) - f_{n_j}(x)| \geq \frac{\epsilon}{2} \right\} \cup \left\{ x \in X : |f_{n_j}(x) - f(x)| \geq \frac{\epsilon}{2} \right\} \end{aligned}$$

Hence, since $\lim_{n\to\infty} f_n = f$ in measure, and $\lim_{j\to\infty} f_{n_j} = f_0$ in measure, we see that $\mu(E)$ is bounded by arbitrarily small values. ■

Theorem 2.60 *If $a < b$ in $\mathbb{R}$, $f : [a,b] \to \mathbb{R}$ is Riemann integrable, then $f \in L([a,b], \mathcal{B}([a,b]), \lambda)$, with coinciding integrals:*

$$\text{(Riemann)} \quad \int_a^b f = \int_{[a,b]} f \quad \text{(Lebesgue)}$$

Proof. Let $J_{n,i} = \left[a + \frac{i-1}{n}(b-a), a + \frac{i}{n}(b-a)\right)$ for $i = 1, \ldots, n$ and let $I_{n,j} = \overline{J_{n,j}}$, $l_{n,i} = \inf_{x \in I_{n,i}} f(x)$, $u_{n,i} = \sup_{x \in I_{n,i}} f(x)$, $\varphi_n = \sum_{i=1}^n l_{n,i} 1_{J_{n,i}}$, $\psi_n = \sum_{i=1}^n u_{n,i} 1_{J_{n,i}}$, and

$$L_n(f) = \int_{[a,b]} \varphi_n d\lambda \qquad U_n(f) = \int_{[a,b]} \psi_n d\lambda$$

Riemann integrability tells us that

$$\lim_{n\to\infty} (U_n(f) - L_n(f)) = 0$$

Notice that $\varphi_n \le f \le \psi_n$, so

$$\int_{[a,b]} |\psi_n - \varphi_n| d\lambda = U_n(f) - L_n(f) \to 0 \qquad \text{as } n \to \infty$$

so that $\lim_{n\to\infty} |\psi_n - \varphi_n| = 0$. Thus there is a subsequence so that $\lim_{j\to\infty} |\psi_{n_j} - \varphi_{n_j}| = 0$ λ-a.e. Since $\varphi_n \le \varphi_{n+1} \le f \le \psi_{n+1} \le \psi_n$ we conclude that $f = \lim_{j\to\infty} \varphi_{n_j}$ μ-a.e., with integrable majorant $g = |\varphi_1| + |\psi_1|$ so that

$$\begin{aligned} \int_{[a,b]} f d\lambda &= \lim_{j\to\infty} \int_{[a,b]} \varphi_{n_j} d\lambda \\ &= \lim_{j\to\infty} L_{n_j}(f) \\ &= \int_a^b f \end{aligned}$$

■

Remark 2.61 If $a < b$ in $\overline{R}$, $f \ge 0$ is improperly Riemann integrable, then it is Lebesgue integrable on (a,b), by a simple application of the above and the MCT.

Definition 2.62 If $(f_n)_{n=1}^\infty, f$ are in $M(X, \mathcal{M})$, then we say

$$\lim_{n\to\infty} f_n = f$$

μ-almost uniformly if, given $\epsilon > 0$, there is $E \in \mathcal{M}$ with $\mu(E) < \epsilon$ such that

$$\lim_{n\to\infty} \sup_{x \in X\setminus E} |f_n(X) - f(x)| = 0$$

The following is a very neat result that arose from a school of Russian thinkers.

Theorem 2.63 ***(Egoroff's Theorem)*** *Suppose $(X, \mathcal{M}, \mu)$ is a finite measure space. If $(f_n)_{n=1}^\infty, f$ are in $M(X, \mathcal{M})$ such that $\lim_{n\to\infty} f_n = f$ μ-a.e. then $\lim_{n\to\infty} f_n = f$ μ-almost uniformly.*

Remark 2.64 The assumption that $(X, \mathcal{M}, \mu)$ be finite is necessary. In $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ we have $\lim_{n\to\infty} 1_{[n,n+1]} = 0$ μ-a.e. but not μ-almost uniformly. Obviously, this measure space is not finite, violating our hypotheses.

Proof. Let $N = \{x \in X : \lim_{n\to\infty} f_n(x)$ does not exist, or is not equal to $f(x)\}$, so $\mu(N) = 0$. For $k, n \in \mathbb{N}$, let

$$E_{n,k} = \bigcup_{m=n}^\infty \left\{ x \in X \ : \ |f_m(x) - f(x)| \ge \frac{1}{k} \right\}$$

so that $E_{n,k} \in \mathcal{M}$, $E_{n,k} \supseteq E_{n+1,k}$ and $\bigcap_{n=1}^{\infty} E_{n,k} \subseteq N$. Thus, continuity from above (which we can invoke since $\mu(X) < \infty$) we see that $\lim_{n\to\infty} \mu(E_{n,k}) = 0$.

Given an $\epsilon > 0$, choose n_k so $\mu(E_{n_k,k}) < \frac{\epsilon}{2^k}$. Let $E = \bigcup_{k=1}^{\infty} E_{n_k,k}$ so $\mu(E) < \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon$, and for $x \in X \setminus E = \bigcap_{k=1}^{\infty} (X \setminus E_{n_k,k}) \subseteq X \setminus E_{n_k,k}$, for any k, we have $|f_n(x) - f(x)| < \frac{1}{k}$ for $n \geq n_k$. Thus

$$\limsup_{n\to\infty} \sup_{x\in X\setminus E} |f_n(x) - f(x)| \leq \frac{1}{k}$$

for each k which gives

$$\limsup_{n\to\infty} \sup_{x\in X\setminus E} |f_n(x) - f(x)| = 0 \qquad \blacksquare$$

2.7 Product measures

Theorem 2.65 *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be two measure spaces. Let*

$$\mathcal{E} = \{E \times F \ : \ E \in \mathcal{M}, F \in \mathcal{N}\} \subseteq \mathcal{P}(X \times Y)$$

Let $\mathcal{A} = \langle \mathcal{E} \rangle$. Then,

1. *Each element of $\mathcal{A}$ is of the form*

$$A = \bigsqcup_{i=1}^{n} E_i \times F_i$$

   for $E_1, \ldots, E_n \in \mathcal{M}$ and $F_1, \ldots F_n \in \mathcal{N}$, with $(E_i \times F_i) \cap (E_j \times F_j) = \emptyset$ whenever $i \neq j$.

2. *We define $(\mu \times \nu)_0 : \mathcal{A} \to [0, \infty]$ by*

$$(\mu \times \nu)_0(A) = \sum_{i=1}^{n} \mu(E_i)\, \nu(F_i)$$

   if A is as in 1. Then $(\mu \times \nu)_0$ is a pre-measure, hence extends to a measure $\mu \times \nu \mathcal{M} \otimes \mathcal{N} \to [0, \infty]$. If each of μ and ν is σ-finite, then $\mu \times \nu$ is σ-finite and this extension is unique.

Proof. We go in order:

1. We see that $\mathcal{E}$ is an elementary family of sets. That is, if $E, E_1 \in \mathcal{M}$ and $F, F_1 \in \mathcal{N}$; i.e. (proof by picture):

   - $(E \times F) \cap (E_1 \times F_1) = (E \cap E_1) \times (F \cap F_1) \in \mathcal{E}$
   - $(X \times Y) \setminus (E \setminus F) = [(X \setminus E) \times F] \cup [E \times (Y \setminus F)] \cup [(X \setminus E) \cup (Y \setminus F)]$

   Thus, the result follows from an earlier lemma on an elementary family. Note it is easy to arrange pairwise disjointness.

2. We need to establish that the formula for $(\mu \times \nu)_0 (A)$ is well-defined. Suppose

$$A = \bigsqcup_{i=1}^{n} (E_i \times F_i) = \bigsqcup_{j=1}^{m} (M_j \times N_j)$$

Then for each $x \in X$, we see that

$$1_A(x, \cdot) = \sum_{i=1}^{n} 1_{E_i}(x) 1_{F_i} = \sum_{j=1}^{m} 1_{M_j}(x) 1_{F_j}$$

and hence

$$\int_Y 1_A(x, y) d\nu(y) = \sum_{i=1}^{n} \nu(F_i) 1_{E_i}(x) = \sum_{j=1}^{m} \nu(N_j) 1_{M_j}(x)$$

and moreover, an iterated integral computation yields

$$\int_X \left[\int_y 1_A(x, y) d\nu(y) \right] d\mu\left(x\right) = \sum_{i=1}^{n} \mu\left(E_i\right) \nu(F_i) = \sum_{j=1}^{m} \mu\left(M_j\right) \nu(N_j) \qquad (\dagger)$$

which gives an unambiguous value for $\left(\mu \times \nu\right)_0 (A)$, so it is well-defined.

Now we check the pre-measure properties. Evidently, $\emptyset = \emptyset \times \emptyset$ so $\prod m_0(\emptyset) = 0$. Now suppose, $A, (A_n)_{n=1}^{\infty}$ are all in $\mathcal{A}$ with $A = \bigsqcup_{n=1}^{\infty} A_n$. But then

$$A_1 = \sum_{n=1}^{\infty} 1_{A_n}$$

and for $x \in X$, we have

$$1_A(x, \cdot) = \sum_{n=1}^{\infty} 1_{A_n}(x, \cdot)$$

Thus, by two applications of a corollary to the MCT and ($\dagger$), we get,

$$\begin{aligned}
\left(\mu \times \nu\right)_0 (A) &= \int_X \int_Y 1_A(x, y) d\nu(y) d\mu\left(x\right) \\
&= \int_X \int_Y \sum_{n=1}^{\infty} 1_{A_n}(x, y) d\nu(y) d\mu\left(x\right) \\
&= \int_X \left[\sum_{n=1}^{\infty} \int_y 1_{A_n}(x, y) d\nu(y) \right] d\mu\left(x\right) \\
&= \sum_{n=1}^{\infty} \int_X \int_Y 1_{A_n}(x, y) d\nu(y) d\mu\left(x\right) \\
&= \sum_{n=1}^{\infty} \left(\mu \times \nu\right)_0 (A_n)
\end{aligned}$$

We may now appeal to the premeasure-outer measure-measure construction to get $\mu \times \nu$ on $\mathcal{M} \otimes \mathcal{N} = \sigma \langle \mathcal{E} \rangle = \sigma \langle \mathcal{A} \rangle$. If $(X_n)_{n=1}^{\infty} \subseteq \mathcal{M}$ and $(Y_n)_{n=1}^{\infty} \subseteq \mathcal{N}$ display σ-finiteness of μ or ν, respectively, then each

$$(\mu \times \nu)(X_n \times Y_n) = \mu(X_n) \nu(Y_n) < \infty$$

and $X \times Y = \bigcup_{n=1}^{\infty} (X_n \times Y_n)$ showing σ-finiteness if $\mu \times \nu$. ■

Definition 2.66 Let Z be a non-empty set. A **monotone class** on Z is any non-empty family $\mathcal{C}$ such that:

1. If $E_1 \subseteq E_2 \subseteq E_3 \subseteq \dots$ in $\mathcal{C}$ then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{C}$.
2. If $F_1 \supseteq F_2 \supseteq \dots$ in $\mathcal{C}$ then $\bigcap_{i=1}^{\infty} F_i \in \mathcal{C}$.

Remark 2.67 The following are evident facts about monotone classes:

1. If $\{\mathcal{C}_\alpha\}_{\alpha \in A}$ is a family of monotone classes on Z, then $\bigcap_{\alpha \in A} \mathcal{C}_\alpha$ is also a monotone class. Hence if $\mathcal{E} \subseteq \mathcal{P}(\mathbb{Z})$ is non-empty, then

$$\mathcal{C}(\mathcal{E}) = \bigcap \{\mathcal{C} \,:\, \mathcal{E} \subseteq \mathcal{C}, \mathcal{C} \text{ is a monotone class}\}$$

   is a monotone class.

2. Clearly, a σ-algebra is a monotone class. Hence

$$\sigma \langle \mathcal{E} \rangle \supseteq \mathcal{C}(\mathcal{E})$$

3. Not every monotone class is a σ-algebra. For example, the family $\mathcal{C} = \{\{n, n+1, n+2, \dots\} \,:\, n \in \mathbb{N}\} \subseteq \mathcal{P}(\mathbb{N})$ is a monotone class but not a σ-algebra.

Lemma 2.68 ***(Monotone class lemma)*** *Let $\mathcal{A}$ be an algebra of sets on Z. Then $\mathcal{C}(\mathcal{A}) = \sigma \langle \mathcal{A} \rangle$.*

Proof. Since we have $\mathcal{C}(\mathcal{A}) \subseteq \sigma \langle \mathcal{A} \rangle$, as remarked above, it suffices to show that $\mathcal{C}(\mathcal{A})$ is itself a σ-algebra, hence $\sigma \langle \mathcal{A} \rangle \subseteq \mathcal{C}(\mathcal{A})$.

As a device for this, we introduce the following notation. If $E \in \mathcal{C}(\mathcal{A})$, let

$$\mathcal{C}_E = \{F \in \mathcal{C}(\mathcal{A}) \,:\, E \setminus F, F \setminus E, F \cap E \in \mathcal{C}(\mathcal{A})\}$$

Here are some straightforward observations:

1. If $E, F \in \mathcal{C}(\mathcal{A})$, we have that $E \in \mathcal{C}_F \iff F \in \mathcal{C}_E$.
2. If $E \in \mathcal{C}(\mathcal{A})$, then $\mathcal{C}_E$ is a monotone class; if $F_1 \subseteq F_2 \subseteq \dots$ in $\mathcal{C}_E$ we have:
   (a) $E \setminus F_1 \supseteq E \setminus F_2 \supseteq \dots$ are in $\mathcal{C}_E \subseteq \mathcal{C}(\mathcal{A})$, implying that $E \setminus \bigcup_{i=1}^{\infty} F_i = \bigcap_{i=1}^{\infty} (E \setminus F_i) \in \mathcal{C}(\mathcal{A})$,
   (b) $F_1 \setminus E \subseteq F_2 \setminus E \subseteq \dots$ are in $\mathcal{C}_E \subseteq \mathcal{C}(\mathcal{A})$, implying that $\bigcup_{i=1}^{\infty} F_i \setminus E = \bigcup_{i=1}^{\infty} (F_i \setminus E) \in \mathcal{C}(\mathcal{A})$,
   (c) $F_1 \cap E \subseteq F_2 \cap E \subseteq \dots$ are in $\mathcal{C}_E \subseteq \mathcal{C}(\mathcal{A})$ implying that $\bigcup_{i=1}^{\infty} F_i \cap E = \bigcup_{i=1}^{\infty} (F_i \cap E) \in \mathcal{C}(\mathcal{A})$;

   so $\bigcup_{i=1}^{\infty} F_i \in \mathcal{C}_E$; likewise $F_1 \supseteq F_2 \supseteq \dots$ in $\mathcal{C}_E$ we see that $\bigcap_{i=1}^{\infty} F_i \in \mathcal{C}_E$.
3. If $A, B \in \mathcal{A}$, then $A \in \mathcal{C}_B$, as $A \setminus B, B \setminus A, A \cap B \in \mathcal{A} \subseteq \mathcal{C}(\mathcal{A})$.

Now, from (2) and (3) above, we see that $\mathcal{A} \subseteq \mathcal{C}_B$, so $\mathcal{C}(\mathcal{A}) \subseteq \mathcal{C}_B$, while $\mathcal{C}_B \subseteq \mathcal{C}(\mathcal{A})$, so $\mathcal{C}_B = \mathcal{C}(\mathcal{A})$. Meanwhile, by (1), for $B \in \mathcal{A}$ tells us for $E \in \mathcal{C}(\mathcal{A})$, $E \in \mathcal{C}(\mathcal{A}) = \mathcal{C}_B$ so $\mathcal{C}_E = \mathcal{C}_B = \mathcal{C}(\mathcal{A})$. For any $E \in \mathcal{C}(\mathcal{A})$, $\mathcal{C}_E = \mathcal{C}(\mathcal{A})$. Now if $E, F \in \mathcal{C}(\mathcal{A}) = C_E = C_F$ then as $Z \in \mathcal{A} \subseteq \mathcal{C}(\mathcal{A})$, we have

$$\begin{aligned} E \cup F &= Z \setminus (Z \setminus (E \cup F)) \\ &= Z \setminus (\underbrace{(Z \setminus E)}_{\in \mathcal{C}(\mathcal{A})} \cap \underbrace{(Z \setminus F)}_{\in \mathcal{C}(\mathcal{A})}) \\ &\in \mathcal{C}(\mathcal{A}) \end{aligned}$$

So that $\mathcal{C}(\mathcal{A})$ is closed under pairwise, hence finite, unions.

Now suppose $E_1, E_2, \ldots \in \mathcal{C}(\mathcal{A})$, we have that each $F_n = \bigcup_{i=1}^n E_i \in \mathcal{C}(\mathcal{A})$ and $F_1 \subseteq F_2 \subseteq \ldots$ so

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{n=1}^{\infty} F_n \in \mathcal{C}(\mathcal{A})$$

So we have that $\mathcal{C}(\mathcal{A})$ satisfies the σ-algebra axioms. ∎

Definition 2.69 We define **sections**. Let $E \subseteq X \times Y$. Define for (x, y) in $X \times Y$ the x**-section** and y**-section** of E by

$$E_x = \{y \in Y \,:\, (x,y) \in E\} \qquad E^y = \{x \in X \,:\, (x,y) \in E\}$$

If $f : X \times Y \to \mathbb{C}$ (or into $[0, \infty]$) we let

$$\begin{aligned} f_x &: Y \to \mathbb{C} \quad f_x(y) = f(x,y) \\ f^y &: X \to \mathbb{C} \quad f^y(x) = f(x,y) \end{aligned}$$

Theorem 2.70 *The following two are facts:*

1. *If $E \in \mathcal{M} \otimes \mathcal{N}$, then $E_x \in \mathcal{N}$ and $E^Y \in \mathcal{M}$ for $(x,y) \in X \times Y$,*
2. *If $f : X \times Y \to \mathbb{C}$ (or into $[0,\infty]$) is $\mathcal{M} \otimes \mathcal{N}$-measurable, then for any $(x,y) \in X \times Y$, f_x is $\mathcal{N}$-measurable, f^y is $\mathcal{M}$-measurable.*

Proof. This is truly a proposition because the proof is simple:

1. Let $\mathcal{R}_Y = \{E \subseteq X \times Y \,:\, E_x \in \mathcal{N} \text{for each } x \in X\}$. Then,
   (a) $\mathcal{R}_Y$ is a σ-algebra:
      i. $E_1, E_2, \ldots \in \mathcal{R}_Y$, then $(\bigcup_{i=1}^{\infty} E_i)_x = \bigcup_{i=1}^{\infty} E_{ix}$
      ii. If $E \in \mathcal{R}_Y$, then $(X \times Y \setminus E)_x = Y \setminus E_x$
   (b) $\mathcal{E} = \{M \times N \,:\, M \in \mathcal{M}, N \in \mathcal{N}\} \subseteq \mathcal{R}_Y$ as $(M \times N)_x = \begin{cases} N & x \in M \\ \emptyset & x \notin M \end{cases}$. Thus $\mathcal{M} \times \mathcal{N} = \sigma\langle\mathcal{E}\rangle \subseteq \mathcal{R}_Y$. Similarly $E^Y \in \mathcal{M}$ for each $y \in Y$.
2. If $B \in \mathcal{B}(\mathbb{C})$, then

$$\begin{aligned} (f_x)^{-1}(B) &= \{y \in Y \,:\, f_x(y) = f(x,y) \in B\} \\ &= f^{-1}(B)_x \end{aligned}$$

   and similarly, $(f^y)^{-1}(B) = f^{-1}(B)^y$. Then we use part 1 to complete the proof. ∎

Note. Assignment 3 will be given a two-day extension...

Theorem 2.71 ***(Pre-Tonelli theorem)*** *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces. If $E \in \mathcal{M} \otimes \mathcal{N}$, then:*

1. *The function $f : X \to [0, \infty]$ given by $x \mapsto \nu(E_x)$ is $\mathcal{M}$-measurable,*

2. *The function $f : Y \to [0,\infty]$ given by $y \mapsto \mu(E^y)$ is $\mathcal{N}$-measurable,*

3. $\mu \times \nu(E) = \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y)$

Proof. (I) We assume that $\mu(X), \nu(Y) < \infty$. We will let

$$\mathcal{C} = \{E \in \mathcal{M} \otimes \mathcal{N} : \text{(i), (ii), (iii) hold for } E\}$$

We will establish that $\mathcal{A} = \langle \{M \otimes N : M \in \mathcal{M}, N \in \mathcal{N}\} \rangle \subseteq \mathcal{C}$ and that $\mathcal{C}$ is a monotone class. Hence, the monotone class lemma shows that

$$\mathcal{M} \otimes \mathcal{N} = \sigma \langle \mathcal{A} \rangle = \mathcal{C}(\mathcal{A}) \subseteq \mathcal{C} \subseteq \mathcal{M} \otimes \mathcal{N}$$

If $E \in \mathcal{A}$, write $E = \bigsqcup_{i=1}^n A_i \times B_i$, $A_i \in \mathcal{M}$, $B_i \in \mathcal{N}$ for $i = 1, \dots, n$. Then for $x \in X$, we have

$$E_x = \bigcup_{i=1, x \in A_i}^{n} B_i \qquad \text{so } \nu(E_x) = \sum_{i=1}^{n} \nu(B_i) 1_{A_i}(x)$$

But then it is clear that (i) and part of (iii) hold for E. Likewise, (ii) holds, and the other part of (iii), so $E \in \mathcal{C}$. That is $\mathcal{A} \subseteq \mathcal{C}$.

Now, let us see that $\mathcal{C}$ is a monotone class. Let $E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots$ in $\mathcal{C}$. Then, for $x \in X$, $E_{1x} \supseteq E_{2x} \supset \dots$ in $\mathcal{N}$, and $\left(\bigcap_{n=1}^\infty E_n\right)_x = \bigcap_{n=1}^\infty (E_{nx})$. Since $\nu(E_{1x}) \leq \nu(X) < \infty$, by assumption, we may appeal to continuity from above to see that

$$\nu\left(\left(\bigcap_{n=1}^{\infty} E_n\right)_x\right) = \nu\left(\bigcap_{n=1}^{\infty} (E_{nx})\right) = \lim_{n\to\infty} \nu(E_{nx})$$

Hence (i) holds for $\bigcap_{n=1}^\infty E_n$. Furthermore, by LDCT with integrable majorant $\mu(X)\nu(Y)1_{X\times Y}$, and again, continuity from above,

$$\begin{aligned}
(\mu \times \nu)\left(\bigcap_{n=1}^{\infty} E_n\right) &= \lim_{n\to\infty} (\mu \times \nu)(E_n) \\
&= \lim_{n\to\infty} \int_X \nu(E_{nx}) d\mu(x) && \text{(by iii)} \\
&= \int_X \lim_{n\to\infty} \nu(E_{nx}) d\mu(x) \\
&= \int_X \nu\left(\left(\bigcap_{n=1}^{\infty} E_n\right)_x\right) d\mu(x)
\end{aligned}$$

So that $\bigcap_{n=1}^\infty E_n$ satisfies part of (iii). Likewise, if $E_1 \subseteq E_2 \subseteq \dots$ in $\mathcal{C}$ then we apply continuity from below, and MCT, to see that $\bigcup_{n=1}^\infty E_n$ satisfies (i) and part of (iii). Similarly, in each case above, the y-sections of, respectively, intersections of a decreasing sequence, or unions of an increasing sequence, are in $\mathcal{C}$.

(II) Now let each of μ, ν be σ-finite. Hence, there are $X_1 \subseteq X_2 \subseteq \dots$ in $\mathcal{M}$ so $\bigcup_{n=1}^\infty X_n = X$ and $Y_1 \subseteq Y_2 \subseteq \dots$ in $\mathcal{N}$ so that $\bigcup_{n=1}^\infty Y_n = Y$. If $E \in \mathcal{M} \otimes \mathcal{N}$ then $E \cap (X_1 \times Y_1) \subseteq E \cap (X_2 \times Y_2) \subseteq \dots$

and each $E \cap (X_n \times Y_n)$ satisfies $(i), (ii)$, and (iii) in the finite measure space $(\mu \times \nu)|_{X_n \times Y_n}$. Hence, we conclude by continuity from below that

$$y \mapsto \mu(E^y) = \lim_{n\to\infty} \mu(E^y \cap Y_n)$$

since $[E \cap (X_n \times Y_n)]^y = E^y \cap Y_n$, is an increasing sequence, and this function is $\mathcal{M}$-measurable, and, then by MCT, and again continuity from below,

$$\begin{aligned}
\mu \times \nu(E) &= \lim_{n\to\infty} \mu\left(E \cap (X_n \times Y_n)\right) \\
&= \lim_{n\to\infty} \int_Y \nu(E^y \cap Y_n) d\nu(y) \\
&= \int_Y \lim_{n\to\infty} \nu(E^y \cap Y_n) d\nu(y) \\
&= \int_Y \nu(E^y) d\nu(y)
\end{aligned}$$

Thus E satisfies (ii), and part of (iii). Likewise E satisfies (i) and the other part of (iii). ■

The upcoming theorems are usually packaged together and referred to, in singular, as the Tonelli-Fubini theorem. It is a very pleasant, highly desirable, and tremendously useful result.

Theorem 2.72 ***(Tonelli)*** *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces. If $f \in \overline{M}^+(X \times Y, \mathcal{M} \otimes \mathcal{N})$ then:*

$$\begin{aligned}
x &\mapsto \int_Y f_x d\nu \, : X \to [0,\infty] \quad \textit{is } \mathcal{M}\textit{-measurable} \\
y &\mapsto \int_X f^y d\mu \, : Y \to [0,\infty] \quad \textit{is } \mathcal{N}\textit{-measurable}
\end{aligned}$$

and

$$\int_X \left[\int_Y f(x,y) d\nu(y)\right] d\mu(x) = \int_{X\times Y} f d(\mu \times \nu) = \int_Y \left[\int_X f(x,y) d\mu(x)\right] d\nu(y) \qquad (\dagger)$$

Proof. For an indicator function 1_E we have

$$\begin{aligned}
\int_{X\times Y} 1_E d(\mu \times \nu) &= \mu \times \nu(E) \\
&= \int_X \nu(E_x) d\mu(x) \\
&= \int_X \int_Y 1_{E_x} d\nu d\mu(x) \qquad = \int_X \int_Y (1_E)_x d\nu d\mu(x)
\end{aligned}$$

Similarly, this is true for the y-sections and the other iterated integral. Hence Tonelli's theorem holds for indicator functions and thus it holds for $f \in S^+(X \times Y, \mathcal{M} \otimes \mathcal{N})$. If $f \in \overline{M}^+(X \times Y, \mathcal{M} \otimes \mathcal{N})$ we have $(\varphi_n)_{n=1}^\infty \subset S^+(X \times Y, \mathcal{M} \otimes \mathcal{N})$ such that $\varphi_1 \leq \varphi_2 \leq \ldots$ and with $\lim_{n\to\infty} \varphi_n = f$. We may now use the monotone convergence theorem:

$$\int_Y f_x d\nu = \int_Y \lim_{n\to\infty} \varphi_{nx} d\nu = \lim_{n\to\infty} \int_Y \varphi_{nx} d\nu \qquad (*)$$

so $x \mapsto \int_Y f_x$ is $\mathcal{M}$-measurable, and

$$\begin{aligned}
\int_{X\times Y} fd(\mu\times\nu) &= \lim_{n\to\infty}\int X\times Y\varphi_n d(\mu\times\nu) \qquad (**)\\
&= \lim_{n\to\infty}\int_X\int_Y \varphi_{nx}d\nu d\mu(x)\\
&= \int_X \lim_{n\to\infty}\int_Y \varphi nxd\nu d\nu(x)\\
&= \int_X\int_Y \lim_{n\to\infty}\varphi_{nx}d\nu d\mu(x)\\
&= \int_X\int_Y f_x d\nu d\mu(x)
\end{aligned}$$

and similarly for the y-sections. ■

Theorem 2.73 ***(Fubini)*** *Let $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ be σ-finite measure spaces. If $f\in L(\mu\times\nu)$ then*

$$\left(x\mapsto\int_Y f_x d\nu\right)\in L(\mu),\left(y\mapsto\int_X f^y d\mu\right)\in L(\nu)$$

and (†) in Tonelli's theorem holds.

Proof. We proceed with the set-up of Tonelli's theorem. Recall that if $f\in L(\mu\times\nu)$ we can find $(\varphi_n)_{n=1}^\infty\subset S(X\times Y,\mathcal{M}\otimes\mathcal{N})$ such that each $|\varphi_n|\leq|f|$ and $\lim_{n\to\infty}\varphi_n=f$. We use LDCT to see that

$$\int_{X\times Y}|f|d(\mu\times\nu)=\int_X\int_Y\underbrace{|f|_x}_{=|f_x|}d\nu d\mu(x)$$

so that

$$x\mapsto\left|\int_Y f_x d\nu\right|\leq\int_Y|f_x|d\nu$$

which shows that $x\mapsto\int_Y f_x d\nu$ is in $L(\mu)$. Likewise for other sections. ■

Remark 2.74 If $f\in M(X\times Y,\mathcal{M}\otimes\mathcal{N})$ we may wish to see that $f\in L(\mu\times\nu)$. This is equivalent to saying that $|f|\in L(\mu\times\nu)$, and we may be able to compute this, with an iterated integral, using Tonelli's theorem.

2.8 Multidimensional Lebesgue Measure

We shall denote by $\mathcal{B}(\mathbb{R})$ and $\mathcal{L}$ the Borel and Lebesgue σ-algebras, respectively. Recall that in $\mathbb{R}$, the Lebesgue measure is translation invariant. We remark that the maps $T_x(x)=x+y$ and $M_c(y)=cy$ are continuous, hence Borel measurable. Thus, if $E\in\mathcal{B}(\mathbb{R})$ we have that $x+E=T_x(E)=T_{-x}^{-1}(E)\in\mathcal{B}(\mathbb{R})$. Likewise, $cE=M_c(E)=M_{1/c}^{-1}(E)\in\mathcal{B}(\mathbb{R})$.

Theorem 2.75 *Let T be an affine transformation $T:R\to R$ and $f\in L(\lambda)$. Then*

$$\int_R f\circ T=\frac{1}{\det T}\int_R fd\lambda$$

Proof. Follows from the change of variables formula. ■

Recall that $\mathcal{B}\left(\mathbb{R}^d\right) = \mathcal{B}\left(\mathbb{R}\right) \otimes \ldots \otimes \mathcal{B}\left(\mathbb{R}\right)$, so that $\lambda_d = \lambda \times \ldots \times \lambda : \mathcal{B}\left(\mathbb{R}^d\right) \to [0, \infty]$. We remark that we could define λ_d on $\mathcal{L} \otimes \ldots \otimes \mathcal{L}$ or on the completion of $(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right), \lambda_d)$. We ask whether $\mathcal{L} \otimes \mathcal{L} \subset \mathcal{L}_2$ (properly). For notation we shall write

$$\int_{\mathbb{R}^d} d\lambda_d = \int_{\mathbb{R}^d} f(x_1, \ldots, x_n) d(x_1, \ldots, x_n)$$

Tonelli-Fubini tells us that

$$\int_{\mathbb{R}^d} f d\lambda_d = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x_1, \ldots, x_n) dx_{\sigma(1)} \ldots dx_{\sigma(d)}$$

where σ is a permutation on $\{1, \ldots, d\}$.

Theorem 2.76 *Let $f \in L(\lambda_d)$. Then,*

1. *For $x \in \mathbb{R}^d$, let $T_x : \mathbb{R}^d \to \mathbb{R}^d$ be given by $T_x(y) = x + y$. Then $f \circ T_x \in L(\lambda_d)$ with*

$$\int_{\mathbb{R}^d} f \circ T_x d\lambda_d = \int_{\mathbb{R}^d} f d\lambda_d$$

2. *For $A \in GL(d, \mathbb{R})$, $f \circ A \in L(\lambda)$ with*

$$\int_{\mathbb{R}^d} f \circ A d\lambda_d = \frac{1}{|\det A|} \int_{\mathbb{R}^d} f d\lambda_d$$

Proof. We rely on the Tonelli-Fubini theorem.

1. By the Tonelli-Fubini theorem,

$$\begin{aligned}
\int_{\mathbb{R}^d} f \circ T_x d\lambda_d &= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x_1 + y_1, \ldots, x_d + y_d) dy_1 \ldots dy_d \\
&= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} d(y_1, x_2 + y_2, \ldots, x_d + y_d) dy_1 \ldots dy_d \\
&\vdots \qquad\qquad \text{(TF d times)} \\
&= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(y_1, \ldots, y_d) dy_1 \ldots dy_d \\
&= \int_{\mathbb{R}^d} f d\lambda_d
\end{aligned}$$

2. Since A is invertible, from baby linear algebra we have that $A = A_1 \ldots A_n$ where each A_i is an elementary matrix. Recall that the elementary matrices are one of three:

   (a) (Add row to vector) $A_{ij}(x_1, \ldots, x_d) = (x_1, \ldots, x_i + x_j, \ldots, x_d)$. Has determinant 1

   (b) (Swap) $S_{ij}(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_d) = (x_1, \ldots, x_j, \ldots, x_i, \ldots, x_d)$. Has determinant -1

   (c) (Multiply row) $M_{ic}(x_1, \ldots, x_d) = (x_1, \ldots, cx_i, \ldots, x_d)$. Has determinant c.

   We may now exploit the Tonelli-Fubini on each one of these types of matrices to get the required formula. We may again apply the Tonelli-Fubini theorem to then get,

$$\begin{aligned}
\int_{\mathbb{R}^d} f \circ A d\lambda_d &= \int_{\mathbb{R}^d} f \circ A_1 \circ \ldots \circ A_n d\lambda_d \\
&= \frac{1}{|\det(A_n)|} \int_{\mathbb{R}^d} f \circ A_1 \ldots \circ A_{n-1} d\lambda_d \\
&\vdots \qquad\qquad\qquad\qquad \text{(TF repeatedly applied)} \\
&= \frac{1}{|\det(A)|} \int_{\mathbb{R}^d} f d\lambda_d
\end{aligned}$$

■

Chapter 3

Signed Measures

Definition 3.1 Let $(X, \mathcal{M})$ be a measurable space. A (finite) **signed measure** on $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to \mathbb{R}$ such that :

1. $\nu(\emptyset) = 0$
2. For a sequence of pairwise disjoint sets $\{E_n\}_{n=1}^{\infty}$ then

$$\nu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \nu(E_i) \qquad \text{(series converges in } \mathbb{R}\text{)} \quad (\star)$$

Example 3.2 If $\mu_1, \mu_2 L\mathcal{M} \to [0, \infty)$ then $\nu = \mu_1 - \mu_2$ is a signed measure.

Example 3.3 If $\mu : \mathcal{M} \to [0, \infty]$ is a measure and $f \in L(\mu)$, we define $f \cdot \mu : \mathcal{M} \to \mathbb{R}$ by

$$f \cdot \mu(E) = \int_E f d\mu = \int_X 1_E f d\mu$$

This is a signed measure (use LDCT).

Remark 3.4 It is possible to define a signed measure into $(-\infty, \infty]$ or $[-\infty, \infty)$. Only one of $\infty, -\infty$ is allowed. For convenience we shall work only with finite signed measures.

Remark 3.5 Notice that the series in $(\star)$ is always absolutely convergent [we shall delay the proof until later].

Remark 3.6 If $F \subseteq E$ in $\mathcal{M}$, then $\nu(E \setminus F) = \nu(E) - \nu(F)$, and the proof is the same as for measures.

Theorem 3.7 *Here are two familiar continuity theorems:*

1. ***(Continuity from below)*** *If $E_1 \subseteq E_2 \subseteq E_3 \subseteq \dots$ in $\mathcal{M}$ then*

$$\nu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \nu(E_n)$$

2. ***(Continuity from above)*** *If $E_1 \supseteq E_2 \supseteq \dots$ in $\mathcal{M}$ then*

$$\nu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \nu(E_n)$$

Proof. Identical as the one for (non-negative) measures. Notice that $\nu(E_1) \in \mathbb{R}$, so there is no restriction needed for continuity from above. ■

Definition 3.8 Let $(X, \mathcal{M}, \nu)$ be a signed measure space. For a set E in $\mathcal{M}$, then E is **positive** (respectively, **negative** or **null**) if for any $F \subseteq E$, F in $\mathcal{M}$ we have $\nu(F) \geq 0$ (respectively, $\nu(F) \leq 0$, or $\nu(F) = 0$). Notice that null sets are simultaneously positive and negative.

Theorem 3.9 *1. If $P \in \mathcal{M}$ is positive for ν and $\mathbb{Q} \subseteq P$ with $\mathbb{Q} \in \mathcal{M}$, then $\mathbb{Q}$ is positive for ν.*

2. If $P_1, P_2, \ldots$ in $\mathcal{M}$ are each positive for ν, then $P = \bigcup_{i=1}^{\infty} P_i$ is positive for ν.

Proof. 1. If $E \subseteq Q \subseteq P$ with $E \in \mathcal{M}$ then $\nu(E) \geq 0$.

2. If $E \subseteq P$, $E \in \mathcal{M}$, let $Q_1 = Q_{n+1} = P_{n+1} \setminus \bigcup_{i=1}^{n} P_i$ and each Q_n is positive by the part above. Write $E = \bigcup_{i=1}^{\infty}(E \cap Q_i)$ as $E \subseteq \bigcup_{i=1}^{\infty} Q_i = \bigcup_{i=1}^{\infty} P_i$ so $\nu(E) = \sum_{i=1}^{\infty} \nu(E \cap Q_i) \geq 0$. ■

3.1 Hahn Decomposition Theorem

Theorem 3.10 *Let $(X, \mathcal{M}, \nu)$ be a signed measure space. Then there exist P, N in $\mathcal{M}$ such that:*

1. *P is positive for ν*
2. *N is negative for ν*
3. *$P \cup N = X$ and $P \cap N = \emptyset$*

Furthermore, if a pair P', N' satisfies $(i), (ii), (iii)$ then $P \triangle P'$ and $N \triangle N'$ are each null for ν.

Definition 3.11 A pair (P, N), as above, is called a **Hahn decomposition** for ν.

Proof. Every set named in this proof is assumed to be in $\mathcal{M}$.

(I) If $E \in \mathcal{M}$, $\epsilon > 0$, we claim there is $E_\epsilon \subseteq E$ such that:

1. $\nu(E_\epsilon) \geq \nu(E)$
2. For any $B \subseteq E_\epsilon$ we have $\nu(B) > -\epsilon$.

Indeed, if not, then every subset $A \subseteq E$ satisfying the first condition admits a subset $B \subseteq A$ such that $\nu(B) \leq -\epsilon$ (i.e. it violates the second condition). Then, we inductively find:

- $B_1 \subseteq E$ such that $\nu(B_1) \leq -\epsilon$ and hence $\nu(E \setminus B_1) = \nu(E) - \nu(B_1) > \nu(E)$; hence
- $B_2 \subseteq E \setminus B$, such that $\nu(B_2) \leq -\epsilon$ and $\nu(E \setminus (B_1 \cup B_2)) = \nu(E) - \sum_{i=1}^{2} \nu(B_i) > \nu(E)$; hence
- $\vdots$
- $B_{n+1} \subseteq E \setminus \bigcup_{i=1}^{n} B_i$, with $\nu(B_{n+1}) \leq -\epsilon$ and $\nu\left(E \setminus \bigcup_{i=1}^{n+1} B_i\right) > \nu(E)$

But, as $B_i \cap B_j = \emptyset$ for $i \neq j$, we would have

$$\nu\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \underbrace{\nu(B_i)}_{\leq -\epsilon} = -\infty$$

violating that ν is finite.

(II) If $E \in \mathcal{M}$, there is a positive $P \subseteq E$ such that $\nu(P) \geq \nu(E)$. Indeed, let $E_0 = E$ and we use (I) and induction to find $E_n \subseteq E_{n-1}$ for $n \in \mathbb{N}$ such that:

1. $\nu(E_n) \geq \nu(E_{n-1})$
2. If $B \subseteq E_n$, then $\nu(B) > -\frac{1}{n}$ (where $\epsilon = \frac{1}{n}$)

Let $P = \bigcap_{n=1}^{\infty} E_n$. By continuity from above, $\nu(P) = \lim_{n\to\infty} \nu(E_n) \geq \nu(E_0) = \nu(E)$. If $B \subseteq P$, then $B \subseteq E_n$ for each n, so by part 2 above, $\nu(B) > -\frac{1}{n}$, for all n, that is $\nu(B) \geq 0$. Thus, P is positive for ν.

(III) This is the crux of the proof. Let

$$s = \sup\{\nu(E) : E \in \mathcal{M}\} \in [0, \infty]$$

Then, there is a sequence $E_1, E_2, \ldots$ such that $s = \lim_{n\to\infty} \nu(E_n)$. For each n, find $P_n \subseteq E_n$, which is positive for ν, with $\nu(P_n) \geq \nu(E_n)$. Let $P = \bigcup_{i=1}^{\infty} P_i$. We note that P is positive for ν by a theorem above, and we compute:

$$\begin{aligned}
\nu(P) &= \lim_{n\to\infty} \nu\left(\bigcup_{i=1}^{n} P_i\right) \\
&\geq \lim_{n\to\infty} \nu(P_n) \\
&\geq \lim_{n\to\infty} \nu(E_n) \\
&= s
\end{aligned}$$

so $\nu(P) = s$. We let $N = X \setminus P$. If there were $E \subseteq N$ with $\nu(E) > 0$, then $\nu(E \cup P) = \nu(E) + \nu(P) > \nu(P) = s$, violating the definition of s. Hence, $\nu(E) \leq 0$, so N is negative.

(IV) We address the essential uniqueness. If (P', N') is another Hahn decomposition, then $P \triangle P' = (P \setminus P') \cup (P' \setminus P) \subseteq N' \cup N$. Thus $P \triangle P' \subseteq P \cup P'$ is simultaneously positive and negative, and thus is a null set for ν, and likewise for $N \triangle N'$. ■

3.2 Jordan Decomposition Theorem

Definition 3.12 If μ and ν are each either a measure or a signed measure on $\mathcal{M}$. The pair (μ, ν) is called **mutually singular**, denoted $\mu \perp \nu$, provided there is a pair (E, F) in $\mathcal{M} \times \mathcal{M}$ such that:

1. E is null for ν
2. F is null for μ
3. $E \cup F = X$ and $E \cap F = \emptyset$

Notation 3.13 If ν is a measure or signed measure on $\mathcal{M}$, and $E \in \mathcal{M}$, define ν_E on $\mathcal{M}$ by

$$\nu_E(A) = \nu(A \cap E)$$

which defines again a measure, or a signed measure. Then $\mu \perp \nu$ via (E, F) is the statement that:

1. $\mu = \mu_E$, $\nu = \nu_F$

2. $E \cup F = X$ and $E \cap F = \emptyset$

Theorem 3.14 *Let $\nu : \mathcal{M} \to \mathbb{R}$ be a signed measure. Then, there is a unique pair (ν^+, ν^-) of (finite) measures such that:*

1. $\nu = \nu^+ - \nu^-$
2. $\nu^+ \perp \nu^-$

Proof. (Existence) Let (P, N) be a Hahn decomposition of ν. Let $\nu^+ = \nu_P$ and $\nu^- = -\nu_N$. As $P \cup N = X$ and $P \cap N = \emptyset$, we see that $\nu = \nu_P + \nu_N = \nu^+ - \nu^-$ and each of ν^+, ν^- are measures, by assumptions of P, N.

(Uniqueness) Suppose $\mu_1, \mu_2 : \mathcal{M} \to [0, \infty)$ are such that $\nu = \mu_1 - \mu_2$ and $\mu_1 \perp \mu_2$ via (E, F). Then (E, F) is a Hahn decomposition of ν; that is $\nu_E = \mu_1$, $\nu_F = -\mu_2$. Thus $P \triangle E, N \triangle F$ are each null for ν. Then for A in $\mathcal{M}$:

$$P \cap A = [(E \cap A) \sqcup ((P \setminus E) \cap A)] \setminus [(E \setminus P) \cap A]$$

so we have,

$$\begin{aligned}
\nu^+(A) &= \nu_P(A) \\
&= \nu(P \cap A) \\
&= \nu\left((E \cap A) \sqcup [(P \setminus E) \cap A]\right) - \nu\left((E \setminus P) \cap A\right) \\
&= \nu\left(E \cap A\right) + \underbrace{\nu\left((P \setminus E) \cap A\right)}_{=0} - \underbrace{\nu\left((E \setminus P) \cap A\right)}_{=0} \\
&= \nu_E(A) \\
&= \mu_1(A)
\end{aligned}$$

Likewise, $\nu^- = \mu_2$. ■

Definition 3.15 If $\nu : \mathcal{M} \to \mathbb{R}$ is a signed measure, we let its **total variation** be given by:

$$|v| = \nu^+ + \nu^- \qquad \text{i.e.} \qquad |\nu|(E) = \nu^+(E) + \nu^-(E)$$

Remark 3.16 By the triangle inequality, $|\nu(E)| = |\nu^+(E) - \nu^-(E)| \leq \nu^+(E) + \nu^-(E) = |\nu|(E)$.

Remark 3.17 If $E_1, E_2, \ldots$ in $\mathcal{M}$ are pairwise disjoint then:

$$\sum_{i=1}^{\infty} |\nu\left(E_i\right)| \leq \sum_{i=1}^{\infty} |\nu|(E_i) = |\nu| \left(\bigcup_{i=1}^{\infty} E_i\right) < \infty$$

and hence $\sum_{i=1}^{\infty} \nu(E_i)$ is always absolutely converging for pairwise disjoint sets.

Theorem 3.18 *(On null sets) If $\nu : \mathcal{M} \to \mathbb{R}$ is a signed measure and $E \in \mathcal{M}$, then TFAE:*

1. *E is null for ν*
2. *E is simultaneously ν^+-null and ν^- null*
3. *E is $|\nu|$-null.*

Proof. Let (P, N) be a Hahn decomposition for ν.

Suppose E is null for ν. Then $P \cap E$ is both ν^+-null ($\nu^+ = \nu_P$) and ν^--null ($\nu^- = -\nu_N$ and $P \cap N = \emptyset$) and $N \cap E$ is both ν^--null and ν^+-null. This is if and only if E is simultaneously ν^+-null and ν^--null. This then implies that $E = (P \cap E) \cup (N \cap E)$ is $|\nu| = (\nu^+ + \nu^-)$-null.

Conversely, if E is $|\nu|$-null, then $|\nu(B)| \leq |\nu|(B) \leq |\nu|(E) = 0$ for all $B \subseteq E$, $B \in \mathcal{M}$, so E is null for ν. Likewise, $\nu^+, \nu^- \leq |\nu|$ so $|\nu|$-null implies each of ν^+-null and ν^--null. ■

Theorem 3.19 *If $\mu : \mathcal{M} \to [0, \infty]$ is a measure, $f \in L^{\mathbb{R}}(\mu)$:*

1. *$(f \cdot \mu)^+ = f^+ \cdot \mu$, $(f \cdot \mu)^- = f^- \cdot \mu$ and $|f \cdot \mu| = |f| \cdot \mu$*
2. *E in $\mathcal{M}$ satisfies that E is null for $f \cdot \mu$ if and only if $1_E f = 0$ μ-a.e.*

Proof. In parts.

1. Here, we note that the pair $(P = \{x \in X : f(x) \geq 0\}, N = \{x \in X : f(x) < 0\})$ is a Hahn decomposition for $f \cdot \mu$. Furthermore, for E in $\mathcal{M}$, we have:
$$\begin{aligned}(f \cdot \mu)^+(E) &= f \cdot \mu(E \cap P) \\ &= \int_{E \cap P} f d\mu \\ &= \int_E 1_P f d\mu \\ &= \int_E f^+ d\mu \\ &= f^+ \cdot \mu(E)\end{aligned}$$
Likewise, $(f \cdot \mu)^- = f^- \cdot \mu$, $|f \cdot \mu| = (f \cdot \mu)^+ + (f \cdot \mu)^- = (f^+ + f^-) \cdot \mu = |f| \cdot \mu$.
2. We have the following equivalences:
$$\begin{aligned}E \text{ is null for } f \cdot \mu &\iff \text{(by last proposition) } E \text{ is } |f| \cdot \mu\text{-null} \\ &\iff 1_E |f| = 0 \ \mu - a.e \\ &\iff 1_E f = 0 \ \mu - a.e.\end{aligned}$$
■

3.3 Complex measures

Definition 3.20 A **complex measure** on $\mathcal{M}$ is a function $\nu : \mathcal{M} \to \mathbb{C}$ such that:

1. $\nu(\emptyset) = 0$
2. If $E_1, E_2, \ldots$ in $\mathcal{M}$ is a sequence of pairwise disjoint sets, then,
$$\nu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \nu(E_i)$$

Remark 3.21 The functions $Re, Im : \mathbb{C} \to \mathbb{R}$ are each continuous additive functions. Hence if $\nu : \mathcal{M} \to \mathbb{C}$ is a complex measure, then each $\operatorname{Re} \nu, \operatorname{Im} \nu : \mathcal{M} \to \mathbb{R}$ are signed measures. Hence, we obtain Jordan decompositions:
$$\nu = \left[\operatorname{Re} \nu^+ - \operatorname{Re} \nu^-\right] + i \left[\operatorname{Im} \nu^+ - \operatorname{Im} \nu^-\right]$$

with $\mathrm{Re}\,\nu^+ \perp \mathrm{Re}\,\nu^-$ and $\mathrm{Im}\,\nu^+ \perp \mathrm{Im}\,\nu^-$.

Warning. $\mathrm{Re}\,\nu \not\perp \mathrm{Im}\,\nu$ in general.

Definition 3.22 Let $(X, \mathcal{M})$ be a measurable space, $\mu : \mathcal{M} \to [0, \infty]$ is a measure and $\nu : \mathcal{M} \to \mathbb{C}$ is a complex measure. Then ν is said to be **absolutely continuous** with respect to μ, written $\nu \ll \mu$, if for E in $\mathcal{M}$ with $\mu(E) = 0$ implies $\nu(E) = 0$.

Remark 3.23 For $B \subseteq E$, $B \in \mathcal{M}$, $\mu(E) = 0$ implies $\mu(B) = 0$, implies (AC) $\nu(B) = 0$. Hence, absolute continuity is equivalent to: E is μ-null $\implies$ E is null for ν.

Theorem 3.24 *Let μ, ν be as above with μ finite. Then $\nu \ll \mu$ (i.e. AC holds) if and only if (AC') for any $\epsilon > 0$, $\exists \delta > 0$ such that for $E \in \mathcal{M}$, $\mu(E) < \delta$ implies that $|\nu(E)| < \epsilon$.*

Proof. First, since $|\nu(\cdot)| \le \mathrm{Re}\,v^+ + \ldots + \mathrm{Im}\,v^-$ it suffices to show that (AC) $\iff$ (AC') for ν a finite measure.

Suppose (AC') fails. Then there exists an $\epsilon > 0$ such that there is $E_n \in \mathcal{M}$ with $\mu(E_n) < \frac{1}{2^n}$ while $\nu(E_n) \ge \epsilon$ Let $F_n = \bigcup_{i=n}^{\infty} E_i$ so $F_1 \supseteq F_2 \supseteq \ldots$ with $\mu(F_n) \le \frac{1}{2^{n-1}}$ via σ-subadditivity and hence by continuity from above

$$\mu\left(\bigcap_{n=1}^{\infty} F_n\right) = \lim_{n\to\infty} \mu(F_n) = 0$$

while

$$\begin{aligned}\nu\left(\bigcap_{n=1}^{\infty} F_n\right) &= \lim_{n\to\infty} \nu(F_n) \le \liminf_{n\to\infty} \nu(E_n) \\ &\ge \epsilon\end{aligned}$$

■

so (AC) fails. Thus (AC) $\implies$ (AC').

If(AC') holds, there is $\delta_n > 0$ so for $E \in \mathcal{M}$, $\mu(E) < \delta_n$ implies $\nu(E) < \frac{1}{n}$. Hence if $\mu(E) = 0 < \delta_n$ for all n we get $\nu(E) < \frac{1}{n}$ for any n, so that $\nu(E) = 0$.

Lemma 3.25 ***(Dichotomy lemma)*** *Let $\nu, \mu : \mathcal{M} \to [0, \infty)$ be finite measures. Then either:*

1. *$\mu \perp \nu$, or*
2. *there is $\epsilon > 0$ and $E \in \mathcal{M}$ for which $\mu(E) > 0$ and E is positive $\nu - \epsilon\mu$*

Proof. Let (P_n, N_n) be a Hahn decomposition for $\nu - \frac{1}{n}\mu$ and $P = \bigcup_{n=1}^{\infty} P_n$, $N = X \setminus = \bigcap_{n=1}^{\infty} N_n$. Then N is negative for each $\nu - \frac{1}{n}\mu$ so $0 \le \nu(N) \le \frac{1}{n}\mu(N)$ for each n, hence $\nu(N) = 0$. If:

- $\mu(P) = 0$ then $\mu \perp \nu$
- $\mu(P) > 0$ then $\mu(P_n) > 0$ by continuity from below and $E = P_n$ satisfies that $\mu(E) > 0$ and is positive for $\mu - \epsilon\mu$ with $\epsilon = \frac{1}{n}$. ■

3.4 Lebesgue-Radon-Nikodym Theorem

Theorem 3.26 *Let $(X, \mathcal{M})$ be a measurable space $\nu : \mathcal{M} \to \mathbb{C}$ be a complex measure, and $\mu : \mathcal{M} \to [0, \infty]$ be a σ-finite measure. Then:*

1. *there is a unique complex measure $\rho : \mathcal{M} \to \mathbb{C}$ such that $\rho \perp \mu$ and $\nu - \rho \ll \mu$,*

2. *There is f in $L(\mu$ such that $\nu - \rho = f \cdot \mu$. Recall that $f \cdot \mu(E) = \int_E f d\mu$. In particular, if $\nu \ll \mu$ then $\nu = f \cdot \mu$ for some f in $L(\mu)$.*

Notation 3.27 The decomposition

$$\nu = \underbrace{\rho}_{\perp\mu} + \underbrace{(\nu - \rho)}_{\ll\mu}$$

is called the **Lebesgue decomposition** of ν with respect to μ.

Remark 3.28 The element f in $L(\mu)$, above, is called the **Radon-Nikodym derivative** of ν with respect to μ, with the notation $f = \frac{d\nu}{d\mu}$.

Proof. (I) Assume $\mu, \nu : \mathcal{M} \to [0, \infty)$ are finite measures. Let

$$\mathcal{F} = \left\{ f \in \overline{M}^+(X, \mathcal{M}) \ : \ \int_E f d\mu \leq \nu(E) \text{ for all } E \right\}$$

Note that $0 \in \mathcal{F}$, so $\mathcal{F} \neq \emptyset$. If $f, g \in \mathcal{F}$ then $\max\{f, g\} \in \mathcal{F}$. Indeed, let $A = \{x \in X \ : \ f(x) \geq g(x)\}$. Then, for E in $\mathcal{M}$:

$$\begin{aligned}
\int_E \max\{f, g\} d\mu &= \int_{E \cap A} f d\mu + \int_{E \setminus A} f d\mu \\
&\leq \nu(E \cap A) + \nu(E \setminus A) \\
&= \nu(E)
\end{aligned}$$

Thus, if $f_1, \ldots, f_n \in \mathcal{F}$ then $\max\{f_1, \ldots, f_n\} \in \mathcal{F}$. Let

$$s = \sup \left\{ \int_X f d\mu \ : \ f \in \mathcal{F} \right\} \leq \nu(X) < \infty$$

Hence for each nm there is f_n in $\mathcal{F}$ such that

$$s - \frac{1}{n} < \int_X f_n d\mu \leq s$$

We let $g_n = \max\{f_1, \ldots, f_n\} \in \mathcal{F}$ so $g_n \leq g_{n+1}$ and we let $f = \lim_{n\to\infty} g_n$. Then

$$s \geq \lim_{n\to\infty} \int_X g_n d\mu \geq \lim_{n\to\infty} \int_X f_n \geq \lim_{n\to\infty} \left(s - \frac{1}{n} \right) = s$$

so that

$$\infty > s = \lim_{n\to\infty} \int_X g_n d\mu = \int_X f d\mu$$

by the MCT. In particular, $f \in \overline{L}^+(\mu)$, so we can and will assume that $f \in L^+(\mu)$ (i.e. $\mathbb{R}$-valued). If $E \in \mathcal{M}$ we again use MCT:

$$\begin{aligned}
\int_E f d\mu &= \lim_{n\to\infty} \int_E g_n d\mu \\
&\leq \lim_{n\to\infty} \nu(E) \\
&= \nu(E)
\end{aligned}$$

so $f \in \mathcal{F}$.

Now let $\rho = \nu - f \cdot \mu$ which is non-negative as $f \in \mathcal{F}$. Arguing by contradiction, if $\rho \not\perp \mu$ then the dichotomy lemma provides $\epsilon > 0$ and $E \in \mathcal{M}$ which is satisfies $\mu(E) > 0$ and is positive for

$$\rho - \epsilon\mu = (\nu - f \cdot \mu) - \epsilon\mu = \nu - (f + \epsilon 1)\mu$$

That is, for $B \subseteq E$, $B \in \mathcal{M}$,

$$\int_B (f + \epsilon 1) d\mu = (f + \epsilon 1) \cdot \mu(B) \leq \nu(B) \qquad (\dagger)$$

Hence if $A \in \mathcal{M}$ we have

$$\begin{aligned} \int_A (f + \epsilon \underbrace{1_E}_{\neq 1}) d\mu &= \int_{A \setminus E} f d\mu + \int_A (f + \epsilon 1) d\mu \\ &\leq \underbrace{\nu(A \setminus E)}_{f \in \mathcal{F}} + \underbrace{\nu(A \cap E)}_{\text{by } (\dagger)} \end{aligned}$$

So $f + \epsilon 1_E \in \mathcal{F}$. However,

$$\begin{aligned} \int_X (f + \epsilon 1_E) d\mu &= \int_X f d\mu + \epsilon\mu(E) \\ &= s + \epsilon > s \end{aligned}$$

But these last two statements contradict definitions of $\mathcal{F}$ and s. Thus $\rho \perp \mu$.

(II) (Glueing together) Assume $\nu : \mathcal{M} \to [0, \infty)$ and $\mu : \mathcal{M} \to [0, \infty]$ is σ-finite. We get $(X_n)_{n=1}^\infty \subseteq \mathcal{M}$ such that $X = \bigsqcup_{n=1}^\infty X_n$ and each $X_n \in \mathcal{M}$ with $\mu(X_n) < \infty$. Let $\nu_i = \nu_{X_i}$, $\mu_i = \mu_{X_i}$. We apply (I) to pairs (ν_i, μ_i) on measurable spaces $(X_i, \mathcal{M}_{X_i} = \{E \cap X_i \ : \ E \in X_i\})$ to obtain measures $\rho_i : \mathcal{M}_{X_i} \to [0, \infty)$ (by construction from (I))

$$\rho_i \perp \mu_i \qquad \nu_i - \rho_i = f \cdot \mu_i \ll \mu_i$$

where $f_i \in L^+(\mu_i)$. Define:

1. $\rho : \mathcal{M} \to [0, \infty]$ by $\rho(E) = \sum_{i=1}^\infty \rho_i(E \cap X_i)$
2. $f : X \to [0, \infty)$ by $f(x) = f_i(x)$ if $x \in X_i$

It is easily checked that ρ defines a measure and that $f \in M^+(X, \mathcal{M})$. If (E_i, F_i) realises $\rho_i \perp \mu_i$, then $(\bigcup_{i=1}^\infty E_i, \bigcup_{i=1}^\infty F_i)$ realises $\rho \perp \mu$. Furthermore, for $E \in \mathcal{M}$ we have

$$\begin{aligned} \nu(E) &= \sum_{i=1}^\infty \nu(E \cap X_i) \\ &= \sum_{i=1}^\infty \left(\rho_i(E \cap X_i) + \underbrace{\int_{E \cap X_i} f_i d\mu_i}_{\int_E 1_{X_i} f d\mu} \right) \\ &= \rho(E) + \int_E f d\mu \qquad \text{(by MCT)} \end{aligned}$$

In particular, as $\nu(X) < \infty$, we see that ρ is a finite measure and $f \in L^+(\mu)$.

(III) Now suppose $\nu : \mathcal{M} \to \mathbb{C}$ and $\mu : \mathcal{M} \to [0, \infty]$ is σ-finite.

We apply Jordan decomposition to get

$$\nu = (\operatorname{Re}\nu^+ - \operatorname{Re}\nu^-) + i(\operatorname{Im}\nu^+ - \operatorname{Im}\nu^-)$$

Apply (II) to each of $\nu_1 = \operatorname{Re}\nu^+, \ldots, \nu_4 = \operatorname{Im}\nu^-$ to get for $k = 1, 2, 3, 4$ measures $\rho_k : \mathcal{M} \to [0, \infty)$, $\rho_k \perp \mu$, $\nu - \rho_k = f_k \cdot \mu \ll \mu$, $f_k \in L^+(\mu)$ and we let $\rho = \rho_1 - \rho_2 + i(\rho_3 - \rho_4)$ and $f = f_1 - f_2 + i(f_3 - f_4)$. Clearly, $\rho \perp \mu$ and $\nu - \rho = f \cdot \mu = (f_1 - f_2 + i(f_3 - f_4)) \cdot \mu \ll \mu$.

(IV) (Uniqueness) Suppose that $\rho, \rho' : \mathcal{M} \to \mathbb{C}$ such that $\rho, \rho' \perp \mu$ and $\nu - \rho, \nu - \rho' \ll \mu$. Since

$$\rho + (\nu - \rho) = \nu = \rho' + (\nu - \rho')$$

we have

$$\rho - \rho' = (\nu - \rho') - (\nu - \rho)$$

simultaneously singular with respect to μ and absolutely continuous with respect to μ. Hence $\rho - \rho' = 0$.■

3.5 The Radon-Nikodym Derivative

Let us assume, above, that $\nu \ll \mu$, so the Lebesgue-Radon-Nikodym tells us that $\nu = f \cdot \mu$ for some $f \in L(\mu)$, where $f \cdot \mu(E) = \int_E f d\mu$. Let us remind ourselves of some facts:

1. If $f \in L(\mu)$, then $f \cdot \mu = 0 \iff 1_E f = 0$ μ-a.e. for each $E \in \mathcal{M}$, $\iff f = 0$ μ-a.e. Hence if $f, g \in L(\mu)$, then $f \cdot \mu = g \cdot \mu \iff (f - g) \cdot \mu = 0 \iff f = g$ μ-a.e.

2. We let $L^1(\mu) = L(\mu) \big/ \sim_\mu$ where $f \sim_\mu g$ if and only if $f = g$ μ-a.e. We may think og elements of $L^1(\mu)$ as "functions in the large", i.e. the expression

$$\int_X f(x) d\mu(x)$$

makes sense interpreting $f(x)$ as some representative that agrees with the class μ-a.e. Most arguments will involve treating f in $L^1(\mu)$ as representative elements of its equivalence class. Pointwise μ-a.e. operations are legal.

Notation 3.29 If $\nu = f \cdot \mu$ as above, we write $f = \frac{d\nu}{d\mu}$ in $L^1(\mu)$. Hence $\nu = \frac{d\nu}{d\mu}\mu$.

Definition 3.30 Let $\nu : \mathcal{M} \to \mathbb{C}$ be a complex measure. We let

$$L(\nu) = L(\operatorname{Re}\nu^+) \cap \ldots \cap L(\operatorname{Im}\nu^-)$$

and define for f in $L(\nu)$ the **Lebesgue integral** by

$$\int_X f d\mu = \int_X f d(\operatorname{Re}\nu^+) - \int_X f d(\operatorname{Re}\nu^-) + i\left[\int_X f d(\operatorname{Im}\nu^+) - \int_X f d(\operatorname{Im}\nu^-)\right]$$

We let $L^1(\nu) = L(\nu) \big/ \sim_\nu$ where $f \sim_\nu g$ if and only if $f = g$ ν-a.e (simultaneously for ν's decomposition into real and imaginary positive and negative parts) if and only if $f = g$ $|\nu|$-a.e.

Theorem 3.31 *Let ν be a complex measure, μ a finite measure, and λ a σ-finite measure, on a measurable space X. Then:*

1. *If $\nu \ll \lambda$ then for $g \in L(\nu)$, $g\frac{d\nu}{d\lambda} \in L^1(\lambda)$ with*

$$\underbrace{\int_X g\frac{d\nu}{d\lambda} = \int_X g d\nu}_{\star}$$

2. *if $\nu \ll \mu$, $\mu \ll \lambda$ then $\nu \ll \lambda$ and*

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}$$

Proof. By parts.

1. If $E \in \mathcal{M}$ then,

$$\int 1_E d\nu = \nu\left(E\right) =_{L-R-N} \frac{d\nu}{d\lambda}\lambda(E) = \int_E \frac{d\nu}{d\lambda}d\lambda = \int 1_E \frac{d\nu}{d\lambda}d\lambda$$

   Hence ($\star$) holds for $g \in S(X, \mathcal{M})$ and by the usual LDCT argument for $g \in L(\nu)$.

2. If $E \in \mathcal{M}$ we have $\lambda(E) = 0$ implying $\mu(E) = 0$ implying $\nu(E) = 0$, so that $\nu \ll \lambda$ by condition (AC). Then, for $E \in \mathcal{M}$, we may apply part 1 to see

$$\begin{aligned}
\int 1_E \frac{d\nu}{d\lambda}d\lambda &= \nu\left(E\right) \\
&= \int 1_E \frac{d\nu}{d\mu}d\mu \\
&= \int 1_E \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}d\lambda && \text{(by 1)}
\end{aligned}$$

   and from the discussion above,

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}$$

   λ-a.e. ■

Part II

Applications of measures to functional analysis

Chapter 4

L^p-spaces

Let $(X, \mathcal{M}, \mu)$ be a measure space. Recall that $L^1(\mu) = L(\mu) \Big/ \sim_\mu$ by the remarks in the chapter above. Likewise, if $1 < p < \infty$ then we let

$$L^p(\mu) = \left\{ f \in M(X, \mathcal{M}) \ : \ \int_X |f|^p d\mu < \mu \right\} \Big/ \sim_\mu$$

Remark 4.1 The functional $\|\cdot\|_1$ on $L^1(\mu)$ given by $\|f\|_1 = \int_X |f| d\mu$ is a norm on $L^1(\mu)$. Indeed, check that:

1. $\|f\|_1 = \|g\|_1$ if $f = g$ μ-a.e.
2. $\|f\|_1 \geq 0$, and is equal to zero if $f = 0$ μ-a.e.
3. $\|f + g\|_1 \leq \|f\|_1 + \|g\|_1$
4. $\|cf\|_1 = |c| \, \|f\|_1$ for $c \in \mathbb{C}$.

Fact. If $\varphi : \mathbb{R} \to \mathbb{R}$ is twice differentiable and for which $\varphi'' > 0$. Then, φ is **strictly convex**. If $x < y$ in $\mathbb{R}$, $0 < t < 1$,

$$\varphi\left((1-t)x + ty\right) < (1-t)\varphi(x) + t\varphi(y)$$

Example 4.2 The function $\varphi(x) = e^x$ is strictly convex.

Theorem 4.3 ***(Young's inequality)*** *If $a, b \geq 0$, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

where equality happens if $a^p = b^q$.

Proof. We compute:

$$\begin{aligned} ab &= e^{\log(ab)} \\ &= \exp\left(\frac{1}{p}\log a^p + \frac{1}{q}\log b^q\right) \\ &\leq \frac{1}{p}e^{\log(a^p)} + \frac{1}{q}e^{\log(a^q)} \\ &= \frac{a^p}{p} + \frac{b^q}{q} \end{aligned}$$

where we have used the fact that e^x is convex. ■

Remark 4.4 If $f, g \in L^{\mathbb{R}}(\mu)$, $f \geq g$ μ-a.e. and $f \neq g$ μ-a.e. , then

$$\int_X f d\mu > \int_X g d\mu$$

Indeed, $(f-g)\cdot\mu$ is a non-zero positive measure.

Theorem 4.5 ***(Hölder's inequality)*** *Let $p, q > 1$ and $\frac{1}{p}+\frac{1}{q}=1$ ("conjugate indices"), $f \in L^p(\mu)$, $g \in L^q(\mu)$. Then, $fg \in L^1(\mu)$ with*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

with equality holding only if there are $\alpha, \beta \geq 0$ such that $\alpha|f|^p = \beta|g|^p$ μ-a.e. (in fact, we can let $\alpha = \|g\|_q$ and $\beta = \|f\|_p$).

Proof. By Young's inequality, for μ-a.e. x (we assume that $\|f\|_p^p, \|g\|_q^q > 0$, to avoid trivialities), get

$$\frac{|f(x)g(x)|}{\|f\|_p \|g\|_q} \leq \frac{1}{p}\frac{|f(x)|^p}{\|f\|_p^p} + \frac{1}{q}\frac{|g(x)|^q}{\|g\|_q^q}$$

with equality holding only when $\|g\|_q^q |f|^p = \|f\|_p^p |g|^q$. We now simply integrate over Xm and multiply by $\|f\|_p^p \|g\|_q^q$ to see that

$$\begin{aligned} \|fg\|_q &\leq \frac{1}{p}\frac{\|f\|_p^p}{\|f\|_p^{p-1}}\|g\|_q + \frac{1}{q}\frac{\|g\|_q^q}{\|g\|_q^{q-1}}\|f\|_p \\ &\leq \left(\frac{1}{p}+\frac{1}{q}\right)\|f\|_p \|g\|_q \end{aligned}$$

with equality holding only if $\|g\|_q^q |f|^p = \|f\|_p^p |g|^q$ μ-a.e. ■

Theorem 4.6 ***(Minkowski's Inequality)*** *If $p > 1$ and $f, g \in L^p(\mu)$ then $f+g \in L^p(\mu)$ with*

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

with equality only if $\mathrm{sgn} f = \mathrm{sgn} g$ μ-a.e. and there are $\alpha, \beta \geq 0$ such that $\alpha|f| = \beta|g|$ μ-a.e.

Notation 4.7 ("Signum") The function $\mathrm{sgn} : \mathbb{C} \to \mathbb{C}$ is given by

$$\mathrm{sgn}(z) = \begin{cases} \frac{z}{|z|} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

Proof. We have, by Holder's inequality used twice,

$$\begin{aligned} |f+g|^p &= |f+g|\,|f+g|^{p-1} \\ &\leq (|f|+|g|)\,|f+g|^{p-1} && (\dagger) \\ &\leq \|f\|_p \left\||f+g|^{p-1}\right\|_q + \|g\|_q \left\||f+g|^{p-1}\right\|_q && (\dagger\dagger) \\ &= \left(\|f\|_p + \|g\|_q\right)\left\||f+g|^{p-1}\right\|_q && (\star\star) \end{aligned}$$

where equality holds at (†) iff $\text{sgn} f = \text{sgn} g$ μ-a.e., and at (††) iff $\alpha|f|^p = \|f\|_q \left\||f+g|^{p-1}\right\|_q$ and $\alpha|g|^p = \|g\|_p \left\||f+g|^{p-1}\right\|_q$ where $\alpha = \left\||f+g|^{p-1}\right\|_q$. Notice that

$$\frac{1}{1} = 1 - \frac{1}{p} \qquad q(p-1) = p$$

so

$$\begin{aligned}\left\||f+g|^{p-1}\right\|_q &= \left(\int |f+g|^{(p-1)q}\right)^{1/q} \\ &= \left(\int |f+g|^p\right)^{1/q} \\ &= \|f+g\|_p^{p/q}\end{aligned}$$

Further (exercise),

$$|f+g|^p \leq (|f|+|g|)^p \leq 2^p \max(|f|,|g|)^p \in L^1(\mu)$$

Then ($\star\star$) tells us that, if $\|f+g\|_0 \neq 0$ we have

$$\|f+g\|_p = \frac{\|f+g\|_p}{\|f+g\|_p^{p/q}} \leq \|f\|_p + \|g\|_q \qquad \blacksquare$$

Where equality holds in the situation described above.

Conclusion. $(L^p(\mu), \|\cdot\|_p)$ is a normed space.

Lemma 4.8 *Let $(L, \|\cdot\|)$ be a normed space. Then $(L, \|\cdot\|)$ is a Banach space if and only if $\sum_{k=1}^{\infty} f_k$ converges in L, whenever $\sum_{k=1}^{\infty} \|f_k\| < \infty$ in $\mathbb{R}$.*

Proof. ($\Longleftarrow$) Let $(f_n)_{n=1}^{\infty}$ be Cauchy in $(L, \|\cdot\|)$. Then we can find a subsequence $(f_{n_k})_{k=1}^{\infty}$ such that $\left\|f_{n_{k+1}} - f_{n_k}\right\| < \frac{1}{2^k}$ for each k. We may then use our assumption to let

$$f = f_{n_1} + \sum_{k=1}^{\infty} \left(f_{n_{k+1}} - f_{n_k}\right) \in L$$

Check that $f = \lim_{k\to\infty} f_{n_k}$m hence $f = \lim_{n\to\infty} f_n$ $\blacksquare$

Theorem 4.9 *Let $(X, \mathcal{M}, \mu)$ be a measure space, and $1 \leq p < \infty$, then $(L^p(\mu), \|\cdot\|_p)$ is a Banach space.*

Proof. We use the lemma. Let $(f_k)_{k=1}^{\infty} \subset L^p(\mu)$ such that

$$s = \sum_{k=1}^{\infty} \|f_k\|_p < \infty$$

We hammer this with Minkowski, MCT, and LDCT. We shall think of each f_k as an element of $M(X, \mathcal{M})$ (abusing notation, but performing operation on them which are defensible almost everywhere). Let, for n in $\mathbb{N}$,

$$g_n = \sum_{k=1}^{n} |f_k| \qquad g = \sum_{k=1}^{\infty} |f_k| \in \overline{M}^+(X, \mathcal{M})$$

Now, by Minkowski's inequality, each

$$\|g_n\|_p \le \sum_{k=1}^{n} \|f_k\|_p \le s$$

Thus,

$$\int |g_n|^p = \|g_n\|_p^p \le s^p$$

and hence, by MCT,

$$\int |g|^p = \lim_{n\to\infty} \int |g_n|^p \le s^p < \infty$$

so $|g|^p \in \overline{L}^p(\mu)$, and hence, replacing by values on a null set, me may assume $|g|^p \in L^+(\mu)$ (i.e. $\mathbb{R}$-valued). Now we let

$$f(x) = \sum_{k=1}^{\infty} f_k(x) \qquad (\star)$$

for μ-a.e. x in X. Then

$$|f| \le \sum_{k=1}^{\infty} |f_k| \le |g|$$

which shows that $(\star)$ is finite, μ-a.e., and thus μa.e. equivalent to an element of $M(X, \mathcal{M})$, which we again call f. Since $|f|^p \le |g|^p$ we see that $f \in L^p(\mu)$. Now, for each n,

$$\begin{aligned} \left| f - \sum_{k=1}^{n} f_k \right|^p &\le \left(|f| + \sum_{k=1}^{n} f_k \right)^p \\ &\le |g|^p \in L(\mu) \end{aligned}$$

and $\lim_{n\to\infty} |f - \sum_{k=1}^{n} f_k| = 0$ μ-a.e. Thus, by the LDCT, we get

$$\left\| f - \sum_{k=1}^{n} f_k \right\|_p^p = \int \left| f - \sum_{k=1}^{n} f_k \right|^p \to 0$$

as $n \to \infty$, so $f = \sum_{k=1}^{\infty} f_k \in L^p(\mu)$. ■

4.1 Dual spaces

Definition 4.10 Let $(L, \|\cdot\|)$ be a $\mathbb{C}$-normed Banach space. We let its **dual space** be

$$L^* = \left\{ \Phi : L \to \mathbb{C} \ \ {}^{\Phi \text{is linear}}_{\|\Phi\|_*=\sup\{|\Phi(f)|: f\in L, \|f\|\le 1\}<\infty} \right\}$$

Remark 4.11 The following are beautiful facts of life:

1. L^* is itself a $\mathbb{C}$-vector space with norm $\|\cdot\|_*$. To check this is a norm, observe that:

(a)

$$\begin{aligned}\|\Phi\|_* = 0 &\iff |\Phi(f)| = 0 \text{ for all } f \text{ in } L \text{ with } \|f\| \leq 1 \\ &\iff \Phi(f) = \|f\| \Phi\left(\frac{1}{\|f\|} f\right) = 0 \text{ for all } f \in L \setminus \{0\} \\ &\iff \Phi = 0\end{aligned}$$

(b)

$$\begin{aligned}\|\Phi + \Psi\| &= \sup\{|\Phi(f) + \Psi(f)| : f \in L, \|f\| \leq 1\} \\ &= \sup\left\{|\Phi(f)| + |\Psi(f')| \ : \ f, f' \in L, \|f\|, \left\|f'\right\| \leq 1\right\} \\ &= \|\Phi\|_* + \|\Psi\|_*\end{aligned}$$

(c) $\|c\Phi\|_* = |c| \|\Phi\|_*$

2. If $\Phi \in L^*$, Φ is Lipschitz, continuous. Indeed, if $f \in L \setminus \{0\}$, $|\Phi(f)| = \|f\| \left|\Phi\left(\frac{1}{\|f\|} f\right)\right| \leq \|\Phi\|_* \|f\|$ and hence if $f, g \in L$ we get

$$|\Phi(f) - \Phi(g)| = |\Phi(f - g)| \leq \|\Phi\|_* \|f - g\|$$

Now, onto the fun stuff that makes us all happy.

Theorem 4.12 *Let $(X, \mathcal{M}, \mu)$ be a measure space, $p, q > 1$ conjugate indices. Then:*

1. *For $g \in L^q(\mu)$ we have $\Phi_g \in L^p(\mu)^*$ given by*

$$\Phi_g(f) = \int_X fg d\mu$$

   satisfies $\|\Phi_g\|_* = \|g\|_q$

2. *If $\Phi \in L^p(\mu)^*$, then $\Phi = \Phi_g$ for some g in $L^q(\mu)$*

Hence, $g \mapsto \Phi_g : L^q(\mu) \to L^p(\mu)^*$ is an isometric surjection.

Proof. In parts.

1. First notice for f in $L^p(\mu)$, we have

$$\int |fg| = \|fg\|_1 \leq \|f\|_p \|g\|_q$$

   by Hölder's inequality, so $fg \in L^1(\mu)$, so that $\Phi_g(f) = \int fg$ is well-defined.

   Again, we use Hölder's inequality to see for $f \in L^p(\mu)$ with $\|f\|_p \leq 1$ we have

$$|\Phi_g(f)| = \left|\int fg\right| \leq \int |fg| = \|fg\|_1 \leq \|f\|_p \|g\|_q \leq \|g\|_q$$

   so that $\|\Phi_g\|_* \leq \|g\|_q$.

To see the converse inequality, for $g \neq 0$ let

$$f = \frac{1}{\|g\|_q^{q-1}} |g|^{q-1} \overline{\operatorname{sgn}(g)}$$

Then $\frac{1}{p} = 1 - \frac{1}{q} = \frac{q-1}{1}$, and $q = (q-1)p$ and we have

$$\begin{aligned} \int |f|^p &\leq \frac{1}{\|g\|_q^{(q-1)p}} \int |g|^{(q-1)p} \\ &= \frac{1}{\|g\|_q^q} \int |g|^q \\ &= 1 \end{aligned}$$

so $\|f\|_p \leq 1$. Thus,

$$\begin{aligned} \|\Phi_g\|_* &\geq |\Phi_g(f)| \\ &= \left| \frac{1}{\|g\|_q^{q-1}} \int |g|^{q-1} \underbrace{\overline{\operatorname{sgn}(g)} g}_{=|g|} \right| \\ &= \frac{1}{\|g\|_q^{q-1}} \int |g|^q \\ &= \frac{\|g\|_q^q}{\|g\|_q^{q-1}} \\ &= \|g\|_q \end{aligned}$$

as desired.

2. (I) Let us restrict us to a finite setting first. Let $\Phi \in L^p(\mu)^*$. Suppose that $\mu(X) < \infty$. Let $\nu : \mathcal{M} \to \mathbb{C}$, by

$$\nu(E) = \Phi(1_E)$$

Then

$$\nu(\emptyset) = \Phi(1_\emptyset) = \Phi(0) = 0$$

If $E_1, E_2, \ldots \in \mathcal{M}$ are pairwise disjoint, then $E = \bigsqcup_{i=1}^\infty E_i$, we have,

$$\begin{aligned} \left\| 1_E - \sum_{i=1}^n 1_{E_i} \right\|_p^p &= \int \left| 1_{\bigsqcup_{i=n+1}^\infty E_i} \right|^p d\mu \\ &= \mu\left(\bigsqcup_{i=n+1}^\infty E_i \right) \\ &= \sum_{i=n+1}^\infty \mu(E_i) \\ &\to 0 \end{aligned}$$

as $n \to \infty$, as this is the tail of a converging series. Thus, $1_E = \lim_{n\to\infty} \sum_{i=1}^{n} 1_{E_i}$ in $L^p(\mu)$. Thus, as Φ is linear and continuous, we have,

$$\begin{aligned}
\nu(E) &= \Phi(1_E) \\
&= \Phi\left(\lim_{n\to\infty} \sum_{i=1}^{n} 1_{E_i}\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \Phi(1_{E_i}) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \nu(E_i) \\
&= \sum_{i=1}^{\infty} \nu(E_i)
\end{aligned}$$

and thus ν is a complex measure. Furthermore, if $E \in \mathcal{M}$ satisfies $\mu(E) = 0$, then $1_E = 0$ μ-a.e. so $\nu(E) = \Phi(1_E) = \Phi(0) = 0$, so that $\nu \ll \mu$. Thus, the Lebesgue-Radon-Nikodym theorem provides $g = \frac{d\nu}{d\mu}$ in $L^1(\mu)$ such that $\nu(E) = \int_E g d\mu$.

(Ia) We show that $g \in L^q(\mu)$. First, if $f \in M(X, \mathcal{M})/\sim_\mu$ is **essentially bounded**; namely, there exists an $M > 0$ in $\mathbb{R}$ such that $|f| \leq M1_X$ μ-a.e., then

$$\begin{aligned}
\int |fg| d\mu &\leq \int M|g|d\mu \\
&= M \|g\|_1 \\
&< \infty
\end{aligned}$$

so that $fg \in L^1(\mu)$. We then note that,

$$M(g) \geq \sup\left\{\left|\int |fg|\right| : f \in M(X, \mathcal{M})/\sim_\mu \text{ is essentially bounded and } \|f\|_p \leq 1\right\} \qquad (\star)$$

For f as in $(*)$, we find $(\varphi_n)_{n=1}^{\infty} \subset S(X, \mathcal{M})/\sim_\mu$ such that $f = \lim_{n\to\infty} \varphi_n$ μ-a.e. and such that $|\varphi_n| \leq |f|$. Notice for $\phi \in S(X, \mathcal{M})/\sim_\mu$ with $\phi = \sum_{j=1}^{n} c_i 1_{E_i}$ (in standard form), that

$$\begin{aligned}
\Phi(\varphi) &= \sum_{j=1}^{m} c_i \Phi(1_{E_i}) \\
&= \sum_{j=1}^{n} c_i \nu(E_j) \\
&= \sum_{j=1}^{m} c_i \int 1_{E_i} g d\mu \\
&= \int \varphi g d\mu
\end{aligned}$$

Then,

$$|\varphi_n - f|^p \leq (|\varphi_n| + |f|)^p \leq 2^p |f|^p \in L^1(\mu)$$

so by the LDCT, get

$$\lim_{n\to\infty} \|\varphi_n - f\|_p^p = \lim_{n\to\infty} \int |\varphi_n - f|^p d\mu = 0$$

and

$$|\varphi_n g| = |\varphi_n|\,|g| \le |f|\,|g| = |fg| \in L^1(\mu)$$

Thus for such f, using continuity of Φ, and then LDCT,

$$\begin{aligned}\Phi(f) &= \lim_{n\to\infty} \Phi(\varphi_n)\\ &= \lim_{n\to\infty} \int \varphi_n g d\mu\\ &= \int fgd\mu && (\dagger)\end{aligned}$$

Thus we see that $M(g) \le \|\Phi\|_* < \infty$. Now we let $(\phi_n)_{n=1}^\infty \subset S(X, \mathcal{M})/\sim_\mu$ such that

$$\lim_{n\to\infty} \phi_n = g \quad \mu\text{-a.e.}$$

with $|\phi_n| \le |\phi_{n+1}| \le |g|$. We define

$$f_n = \frac{1}{\|\phi_n\|_q^{q-1}} \overline{\mathrm{sgn}(g)}$$

which is essentially bounded, and with

$$\int |f_n^p| \le 1$$

Furthermore, by MCT,

$$\int |g|^p d\mu = \lim_{n\to\infty} \int |\phi)n|^q d\mu$$

and we compute that

$$\begin{aligned}\|g\|_q &= \lim_{n\to\infty} \|\phi_n\|_q\\ &= \lim_{n\to\infty} \frac{1}{\|\phi\|_q^{q-1}} \int |\phi_n|^q\\ &= \lim_{n\to\infty} \int |f_n|\,|\phi_n| && \text{(via the formula for } f_n)\\ &\le \liminf_{n\to\infty} \int |f_n|\,|g| d\mu\\ &= \liminf_{n\to\infty} \int f_n g d\mu\\ &\le \|\Phi\|_*\\ &< \infty\end{aligned}$$

so that $g \in L^q(\mu)$. We then see that $\Phi = \Phi_g$ by mimicking the calculation in $(\dagger)$, but for f not necessarily essentially bounded.

(II) Assume that μ is a general measure. If $E \in \mathcal{M}$, identify $L^p(\mu_E) \cong 1_E L^p(\mu) \subseteq L^p(\mu)$, and likewise for q.

If $F \in \mathcal{M}$ has $\mu(F) < \infty$, then (I) provides g_F in $1_F L^p(\mu)$ such that

$$\Phi(1_F f) = \int_F f g_F d\mu = \int_X f g_F d\mu$$

as $g_F = 1_F g_F$. Notice that if $F \subseteq F'$, where $F' \in \mathcal{M}$, $\mu(F') < \infty$, then $g_F = g_{F'}$, μ-a.e. Hence, if $F_1, F_2, \ldots \in \mathcal{M}$, each with $\mu(F_i) < \infty$ then on $E = \bigcup_{i=1}^{\infty} F_i$ we may uniquely define g_E so $g_E = g_{F_n}$, μ_{F_n}-a.e. and $1_E g_e = g_E$. Let $E_n = \bigcup_{i=1}^{n} F_i$, and MCT and (I) and (Ia) provide:

$$\begin{aligned}\int |g_E|^q d\mu &= \lim_{n\to\infty} |g_{E_n}|^q \\ &= \lim_{n\to\infty} \left\| \Phi|_{1_{E_n} L^p(\mu)} \right\| \\ &\leq \|\Phi\|_*\end{aligned}$$

so $g_E \in L^q(\mu)$. In fact, $g_E = 1_E L^q(\mu)$. We then let:

$$s = \sup\left\{ \int |g_E|^q d\mu \ : \ E \in \mathcal{M} \text{ is } \sigma\text{-finite for } \mu \right\} \underbrace{\leq}_{\text{as above}} \leq \|\Phi\|_* < \infty$$

Then let $E_1, E_2, \ldots \in \mathcal{M}$ each be σ-finite for μ such that

$$\lim_{n\to\infty} \int |g_{E_n}|^q d\mu = s$$

Then, $E = \bigcup_{i=1}^{\infty} E_i$ is σ-finite, and, again using MCT,

$$\begin{aligned}s &\geq \int |g_E|^q d\mu \\ &= \lim_{n\to\infty} \int \left| g_{\bigcup_{i=1}^{n} E_i} \right|^q d\mu \\ &\geq \lim_{n\to\infty} \int |g_{E_n}|^q \, d\mu \\ &= s\end{aligned}$$

so that $\int |g_E|^q = s$. Now if $E' \in \mathcal{M}$ is σ-finite, for μ, such that:

$$\begin{aligned}\int \left|g_{E'\setminus E}\right|^q d\mu = \int |g_E|^q d\mu + \int \left|g_{E'\setminus E}\right|^q d\mu \\ \int |g_{E'}|^q \, d\mu \\ \leq s\end{aligned}$$

implying thus that $\int \left|g_{E'\setminus E}\right|^q d\mu = 0$ and we conclude that $g_{E'\setminus E} = 0$, μ-a.e. If $f \in L^p(\mu)$, we think of f as a function and let

$$E_f = \bigcup_{n=1}^{\infty} \underbrace{\left\{ x \in X \ : \ |f(x)|^p > \frac{1}{n} \right\}}_{\text{each is finite for } \mu}$$

so that E_f is σ-finite. Decompose:

$$E_f \cap F = \bigcup_{i=1}^{\infty} E_i$$

each $E_i \in \mathcal{M}$ with $\mu(E_i) < \infty$ and $E_1 \subseteq E_2 \subseteq \dots$ and we have:

1. $\lim_{n\to\infty} \|f - 1_{E_n} f\|_p = 0$ (by LDCT)

2. $|fg_{E_n}| \leq |fg_E| \in L^1(\mu)$ (By Hölder's inequality)

Thus, by continuity of Φ, LDCT, and (I), we have:

$$\begin{aligned}\Phi(f) &= \lim_{n\to\infty} \Phi(1_{E_n} f) \\ &= \lim_{n\to\infty} \int 1_{E_n} f g_E d\mu \\ &= \int f g_E d\mu\end{aligned}$$

Hence $\Phi = \Phi_E$. ■

Chapter 5

Radon measures

5.1 Smaller fish to fry

Definition 5.1 Let (X,d) be a metric space. We say that (X,d) is **locally compact** if for each $x \in X$, there is an $\epsilon_x > 0$ such that $\overline{B(x,\epsilon_x)}$ (closure of ϵ_x-ball, centred at x) is compact.

Example 5.2 $\mathbb{R}^d$ is locally compact, since, by the Heine-Borel theorem, every closed ball is compact.

Example 5.3 Let (X,d) be a discrete metric space. If $x \in X$, then $B(x,\epsilon) = \overline{B(x,\epsilon)}$ is compact only for $0 < \epsilon \leq 1$. In this case, we distinguish $\overline{B(x,\epsilon)}$ from $\overline{B}(x,\epsilon)$, where the former is simply the closure of the open ball of radius ϵ. This metric space is locally compact.

Example 5.4 If C is a closed subset of a locally compact space, and U is an open subset of a locally compact space, then $C, U, C \cap U, C \cup U$ are locally compact. For instance, $[0,\infty), [a,b) \subseteq \mathbb{R}$ are locally compact.

Definition 5.5 Let (X,d) be a locally compact metric space. A measure $\mu : \mathcal{B}(X) \to [0,\infty]$ is called a **Radon measure** if it satisfies the following:

1. (Outer regularity) for $E \in \mathcal{B}(X)$, $\mu(E) = \inf\{\mu(U) \ : \ E \subseteq U, U \text{ open}\}$
2. (Locally finiteness) for $K \subseteq X$ compact $\mu(K) < \infty$, and
3. (Inner regularity on open sets) if $U \subseteq X$ is open, then $\mu(U) = \sup\{\mu(K) : K \subseteq U, K \text{ compact}\}$

Theorem 5.6 *Let μ be a Radon measure, as above. Then if $E \in \mathcal{B}(X)$ such that $\mu(E) < \infty$, then*

$$\mu(E) = \sup\{\mu(K) \ : \ K \subseteq E\,, K \ \textit{compact}\} \qquad (\star)$$

Hence, if X is σ-finite for μ, then μ is inner regular $(\star)$ for each E in $\mathcal{B}(X)$.

Proof. Let us assume that $\mu(E) < \infty$. Let $\epsilon > 0$. Let:

1. $E \subseteq U$, U, open, $\mu(U) < \mu(E) + \epsilon$
2. $F \subseteq E$, F compact, $\mu(U) < \mu(F) + \epsilon$, and
3. $U \setminus E \subseteq V$, so $\mu(V) < \epsilon$

The first property gives us that $\mu(U \setminus E) < \epsilon$. Let

$$\begin{aligned} K &= F \setminus V \\ &= F \cap (X \setminus V) \\ &\subseteq F \setminus (U \setminus E) \\ &\subseteq F \cap E \\ &\subseteq E \end{aligned}$$

and is compact. Then,

$$\begin{aligned} \mu(K) &= \mu(F) - \mu(F \cap V) \\ &> \underbrace{\mu(U)}_{\geq \mu(E)} - \epsilon - \underbrace{\mu(V)}_{< \epsilon} \\ &> \mu(E) - 2\epsilon \end{aligned}$$

Now, if E is σ-finite for μ write $E = \bigcup_{i=1}^{\infty} E_i$, each $E_i \in \mathcal{B}(X)$, $\mu(E_i) < \infty$, $E_1 \subseteq E_2 \subseteq \ldots$. For each n, let $K_n \subseteq E_n$, such that $\mu(K_n) \leq \mu(E_n) < \mu(K+n) + \frac{1}{n}$. Then, by continuity from below,

$$\mu(E) = \lim_{n \to \infty} \mu(E_n) = \lim_{n \to \infty} \mu(K_n)$$

Thus,

$$\mu(E) = \sup_{n \in \mathbb{N}} \{\mu(K_n)\} \qquad \blacksquare$$

Remark 5.7 We say that (X, d) is **σ-compact** if $X = \bigcup_{n=1}^{\infty} K_n$, each K_n compact. If μ is a Radon measure, then σ-compact implies σ-finite.

Definition 5.8 Let (X, d) be a locally compact metric space. If $f : X \to \mathbb{C}$ is continuous, we define its **support** as

$$\operatorname{supp}(f) = \overline{\{x \in X \ : \ f(x) \neq 0\}}$$

We let

$$C_c(X) = \{f : X \to \mathbb{C} \ : \ f \text{ is continuous and } \operatorname{supp}(f) \text{ is compact}\}$$

Remark 5.9 If X were not to be locally compact, then $C_c(X) = \{0\}$ is possible.

Notation 5.10 If $U \subseteq X$ is open and $f \in C_c(X)$ we write $f \prec U$ (read, U dominates f) if and only if $\operatorname{supp}(f) \subseteq U$ and $0 \leq f \leq 1$.

Lemma 5.11

1. ***(Metric Uryzohn's Lemma)*** *If $K \subseteq U \subseteq X$, K compact, and U open, then there exists $f \prec U$ such that $f|_K = 1$. We write $K \prec f$ in this situation.*

2. ***(Partition of unity)*** *If K is compact, $\bigcup_{i=1}^{n} U_i$, each U_i open, then for each i there is $g_i \prec U_i$ such that*

$$K \prec g_1 + \ldots + g_n \prec \bigcup_{i=1}^{n} U_i$$

*Then $\{g_1, \ldots, g_n\}$ is a **partition of unity** for K, subordinate to $\{U_1, \ldots, U_n\}$*

Proof. 1. For each x in K, get $\epsilon_x > 0$, so that $B(x, 2\epsilon_x) \subseteq U$ and $\overline{B(x, 2\epsilon_x)}$ is compact. Then

$$K \subseteq \bigcup_{x \in K} B(x, \epsilon_x)$$

so that by compactness, we can let

$$K \subseteq \bigcup_{i=1}^{n} B(x_i, \epsilon_{x_i}) = V$$

for some $x_1, \ldots, x_n$. Furthermore,

$$\overline{V} \subseteq \bigcup_{i=1}^{n} \overline{B(x_i, \epsilon_{x_i})} \subseteq \bigcup_{i=1}^{n} B(x_i, 2\epsilon_{x_i}) \subseteq U$$

so that $\overline{V}$ is compact with $\overline{V} \subseteq U$. Then let

$$f(x) = \min \left\{ \frac{d(x, X \setminus V)}{d(K, X \setminus V)}, 1 \right\}$$

for $x \in X$. For $x \in X \setminus V$, we have $f(x) = 0$, so $\operatorname{supp}(f) \subseteq \overline{V}$ and is compact. Also, $f|_K = 1$.

2. Let $K_i = K \setminus \left(\bigcup_{i=1, j \neq i}^{n} U_j \right)$ for each i, so K_i is compact and $K_i \subseteq U_i$. Then let f_i be so $K_i \prec f_i \prec U_i$. Then $f = f_1 + \ldots + f_n > 0$ on K, so that $K \subseteq W = f^{-1}((0, \infty))$. Let h be so that $K \prec h \prec W$. Let
$$g_i = \frac{f_i h}{f}$$
for $i = 1, \ldots, n$. ■

Remark 5.12 $C_c(X)$ is a complex vector space with pointwise operations.

5.2 Riesz Representation Theorem

Mr Riesz, as a great functional analyst, produced many representation theorems. Here we explore the positive Radon measure version of his theorems.

Definition 5.13 Let X be a locally compact metric space. A linear functional$I : C_c(X) \to \mathbb{C}$ is said to be a **positive linear functional** if $I(f) \geq 0$ whenever $f \geq 0$.

Theorem 5.14 ***(Riesz representation theorem)*** *Let X be a locally compact metric space and let $I : C_c(X) \to \mathbb{C}$ be a positive linear functional, then there exists a unique Radon measure $\mu : \mathcal{B}(X) \to [0, \infty]$ such that*

$$I(f) = \int_X f d\mu$$

for each f in $C_c(X)$.

Proof. Given an open set $U \subseteq X$, we let

$$\mu^0(U) = \sup \{ I(f) \ : \ f \prec U \} \in [0, \infty]$$

Notice that $\mu^0(U) = 0$. We define for $E \in \mathcal{P}(X)$

$$\mu^*(E) = \inf \left\{ \sum_{i=1}^{\infty} \mu^0(U_i) \ : \ E \subseteq \bigcup_{i=1}^{\infty} U_i, \text{ each } U_i \text{ open} \right\}$$

Then μ^* is an outer measure. We let:

- $\mathcal{M} = \{A \in \mathcal{P}(X) \ : \ \mu^*(E) \geq \mu^*(A \cap A) + \mu^*(E \setminus A), \forall E \in \mathcal{P}(X)\}$
- $\mu = \mu^*|_{\mathcal{M}}$ (We will show that $\mathcal{B}(X) \subseteq \mathcal{M}$ and we will further let $\mu = \mu^*|_{\mathcal{B}(X)}$)

(I) If $E \in \mathcal{P}(X)$, then $\mu^*(E) = \inf \{\mu^0(U) \ : \ E \subseteq U, U \text{ open}\}$. In particular, we see that $\mu^*(U) = \mu^0(U)$ for U open. Let U be open and $U \subseteq \bigcup_{i=1}^{\infty} U_i$, each U_i open, and $f \prec U$. Then, $\text{supp}(f) \subseteq U \subseteq \bigcup_{i=1}^{\infty}$ so, since the support of f is compact, $\text{supp}(f) \subseteq \bigcup_{i=1}^{n} U_i$ (compactness). Let $\{g_1, \ldots, g_n\}$ be a partition of unity for $\text{supp}(f)$, subordinate to $\{U_1, \ldots, U_n\}$. Then each $fg_i \prec U_i$ and $f = f(g_1 + \ldots + g_n)$. Hence:

$$I(f) = \sum_{i=1}^{n} I(fg_i) \leq \sum_{i=1}^{n} \mu^0(U_i) \leq \sum_{i=1}^{\infty} \mu^0(U_i)$$

Now, take supremum on LHS, over all $f \prec U$ and take infimum on RHS, over all countable open covers, to get

$$\mu^0(U) \leq \mu^*(U)$$

Conversely, since $U \subseteq U$, we have that $\mu^*(U) \leq \mu^0(U)$, hence $\mu^*(U) = \mu^0(U)$, on open U. Now, if $E \in \mathcal{P}(X)$, with $E \subseteq \bigcup_{i=1}^{\infty} U_i = u$ each U_i open, so that U is open, then by the monotonicity and σ-subadditivity of the outer measure μ^*, we have

$$\mu^*(E) \leq \mu^*(U) \leq \sum_{i=1}^{\infty} \mu^*(U_i) = \sum_{i=1}^{\infty} \mu^0(U_i)$$

and hence, taking infimum over all open covers on RHS and using the squeeze theorem, we see that

$$\mu^*(E) = \inf \{\mu^0(U) \ : \ E \subseteq U, U \text{ open}\}$$

(II) If $K \subseteq X$ is compact and $K \prec f$, then $\mu^*(K) \leq I(f)$ (we shall see that $\mathcal{B}(X) \subseteq \mathcal{M}$, so we will conclude $\mu(K) \leq I(f)$). In particular, μ will be locally finite, so that $\mu(K) < \infty$.

Let $0 < \epsilon < 1$, and let $V = f^{-1}((1-\epsilon, \infty)) \supseteq K$. Hence if $g \prec V$, then $(1-\epsilon)g \leq f$ so by positivity of I, $I(g) \leq \frac{1}{1-\epsilon} I(f)$. Hence

$$\mu^*(K) \leq \mu^0(V) = \sup \{I(g) \ : \ g \prec V\} \leq \frac{1}{1-\epsilon} I(f)$$

Taking $\epsilon \to 0^+$, we get $\mu^*(K) \leq I(f)$.

(III) We have that $\mathcal{B}(X) \subseteq \mathcal{M}$ (μ^*-measurable sets). In particular, $\mu = \mu^*|_{\mathcal{B}(X)}$ satisfies $\mu(U) = \mu^0(U)$ for U open and μ is outer regular by (I), and locally finite by (II).

It suffices to show that $U \in \mathcal{M}$ whenever U is open. Suppose $V \subseteq V$ is open with $\mu^*(V) < \infty$ (say $\overline{V}$ is compact), and let $\epsilon > 0$. We let:

- $f \prec U \cap V$ be so $\mu^*(U \cap V) < I(f) + \epsilon$
- $g \prec V \setminus \operatorname{supp}(f)$ be so $\mu^*(V \setminus \operatorname{supp}(f)) < I(g) + \epsilon$

Then $f + g \prec V$ as $\operatorname{supp}(f) \cap \operatorname{supp}(g) = \emptyset$, and we have

$$\begin{aligned}
\mu^*(V \cap U) + \mu^*(V \setminus U) &< I(f) + \epsilon + \mu^*(V \setminus \operatorname{supp}(f)) \\
&< I(f) + I(g) + 2\epsilon \\
&= I(f+g) + 2\epsilon \\
&\leq \mu^0 + 2\epsilon \\
&= \mu^*(V) + 2\epsilon
\end{aligned}$$

so, since $\epsilon > 0$ is arbitrary,

$$\mu^*(V \cap U) + \mu^*(V \setminus U) \leq \mu^*(V)$$

Now if $E \subseteq X$, $\mu^*(E) < \infty$, we find, for $\epsilon > 0$, open V such that $E \subseteq V$ and $\mu^*(V) = \mu^0(V) < \mu^*(E) + \epsilon$. Then,

$$\begin{aligned}
\mu^*(E) + \epsilon > \mu^*(V) \\
\mu^*(V \cap U) + \mu^*(V \setminus U) \\
&\geq \mu^*(E \cap U) + \mu^*(E \setminus U)
\end{aligned}$$

and hence, as $\epsilon > 0$ was arbitrary,

$$\mu^*(E) \geq \mu^*(E \cap U) + \mu^*(E \setminus U) \qquad (\star)$$

Notice that $(\star)$ is trivial if $\mu^*(E) = \infty$.

(IV) Now the best part of the proof:

$$I(f) = \int_X f d\mu \qquad \text{for } f \text{ in } C_c(X)$$

First, if $f \in C_c(X)$, we may write $f = f_1 - f_2 + i(f_3 - f_4)$ where each $f_i \geq 0$. Let $M_i = \sup\{f_i(x) \,:\, x \in X\}$ and we see that each $f_i = (M_i + 1)\frac{1}{M_i+1} f_i$, where $0 \leq \frac{1}{M_i+1} f_i \leq 1$. Hence, it suffices to establish this for $0 \leq f \leq 1$. Now let

$$K_0 = \operatorname{supp}(f) \qquad K_j = f^{-1}(\left[\frac{j}{n}, 1\right]) \quad j = 1, 2, \ldots, n$$

so each $K_0, \ldots, K_n$ is compact and $K_0 \supseteq K_1 \supseteq \ldots \supseteq K_n$. Then let

$$f_j = \min\left\{\max\left\{f - \frac{j-1}{n} \cdot 1\right\}, \frac{1}{n}\right\}$$

Then

$$f = \sum_{j=1}^{n} f_j$$

and $1_{k_j} \leq n f_j \leq 1_{K_{j-1}}$, $j = 1, \ldots, n$. Hence, taking integrals we see

$$\mu(K_j) \leq n \int_X f_j d\mu \leq \mu(K_{j-1})$$

Therefore,

$$\frac{1}{n}\sum_{j=1}^{m}\mu\left(K_j\right) \le \int_X f d\mu \le \frac{1}{n}\sum_{j=1}^{n}\mu\left(K_{j-1}\right) \qquad (\star)$$

On the other hand we have $K_j \prec nf_j \prec K_{j-1}^{\circ}$ (interior) so, using (II),

$$\mu\left(K_j\right) \le nI(f_j) \le \mu\left(K_{j-1}^{\circ}\right) \le \mu\left(K_{j-1}\right)$$

Averaging out over all j's,

$$\frac{1}{n}\sum_{j=1}^{n}\mu\left(K_j\right) \le I(f) \le \frac{1}{n}\sum_{j=1}^{n}\mu\left(K_{j-1}\right) \qquad (\dagger)$$

Hence, by $(\star)$ and $(\dagger)$ we obtain:

$$\begin{aligned}\left|I(f) - \int_X f d\mu\right| &\le \frac{1}{n}\left(\mu\left(K_0\right) - \mu\left(K_n\right)\right) && \text{(Telescoping)}\\ &\le \frac{1}{n}\mu\left(K_0\right)\end{aligned}$$

for any $n \in \mathbb{N}$. Hence we are done!

(V) Inner regularity on open sets. Let $U \subseteq X$ be open. Find $(f_n)_{n=1}^{\infty} \subseteq C_c(X)$ each $f_n \prec U$ so $\lim_{n\to\infty} I(f_n) = \mu^0(U) = \mu\left(U\right)$. Let $K_n = \operatorname{supp}\left(f_n\right) \subseteq U$. Then by (IV),

$$I(f_n) = \int_X f_n d\mu \le 1_{K_n} d\mu = \mu\left(K_n\right) \le \mu\left(U\right)$$

Hence, by a squeeze argument, taking limits $\lim_{n\to\infty}\mu\left(K_n\right) = \mu\left(U\right)$. Namely.

$$\mu\left(U\right) \le \sup\left\{\mu\left(K\right) \;:\; K \subseteq U, K \text{ compact}\right\}$$

where the converse inequality is obvious.

(VI) Uniqueness. Let μ' be a Radon measure for which $\int f d\mu' = I(f)$ for f in $C_c(X)$. Then, if U is open, and $K \prec f \prec U$ (K compact in U), then

$$\begin{aligned}\mu'(K) &= \int 1_K d\mu'\\ &\le \int f d\mu'\\ &= I(f)\\ &\le \int 1_U d\mu'\\ &= \mu'(U)\end{aligned}$$

So that

$$\sup\left\{\mu'(K) \;:\; K \subseteq U, K \text{ compact}\right\} \le \sup\left\{I(f) \;\; f \prec U\right\} \le \mu'(U)$$

But, by inner regularity of μ' on open sets, and definition of $\mu\left(U\right) = \mu'(U)$, we see

$$\mu'(U) \le \mu\left(U\right) \le \mu'(U)$$

■

So $\mu' = \mu$ on open sets. But each is outer regular, hence $\mu' = \mu$ on $\mathcal{B}\left(X\right)$.

Victory!

Theorem 5.15 *Let (X, d) be a locally compact metric space, and $\mu : \mathcal{B}(X) \to [0, \infty]$ be a Radon measure. Then, for $1 \le p < \infty$, we have that $C_c(X)/\sim_\mu$ is dense in $L^p(\mu)$.*

Proof. Notice that $C_c(X)/\sim_\mu \subseteq L^p(\mu)$ as μ is locally finite. If $E \in \mathcal{B}(X)$ with $\mu(E) < \infty$, then inner and outer regularity, we can find for $\epsilon > 0$, compact K and open U such that $K \subseteq E \subseteq U$ and $\mu(E) < \mu(K) + \frac{\epsilon}{2}$, and $\mu(U) < \mu(E) + \frac{\epsilon}{2}$. Thus,

$$\mu(U \setminus K) = \mu(U \setminus E) + \mu(E \setminus K) < \epsilon$$

Then for $K \prec f \prec U$, we have

$$\|f - 1_E\|_p^p = \int |f - 1_E|^p d\mu \le \int |1_U - 1_K|' d\mu = \int 1_{U \setminus K} d\mu = \mu(U \setminus K) < \epsilon$$

Hence, simple elements of $L^p(\mu)$ are approximated from $C_c(X)/\sim_\mu$, and hence arbitrary elements. ■

Theorem 5.16 *Let (X, d) be a σ-compact locally compact metric space. Then every locally finite measure $\nu : \mathcal{B}(X) \to [0, \infty]$ (i.e. $\nu(K) < \infty$, for K compact) is a Radon measure. In particular, ν is automatically outer regular and inner regular.*

Proof. Since ν is locally finite, each f in $C_c(X)$ is Borel measurable, and $|f| \le 1_{\mathrm{supp}(f)}$ so $f \in L(\nu)$. Since ν is non-negative, $I(f) = \int_X f d\nu$ defines a positive linear functional on $C_c(X)$. Hence, the Riesz Representation Theorem provides us with a Radon measure μ such that

$$\int_X f d\mu = I(f) = \int_X f d\mu$$

We wish to see that $\nu = \mu$ on Borel sets.

(I) Let $U \subseteq X$ be open. Since X is σ-compact, we may write $X = \bigcup_{n=1}^\infty L_n$, each $L_n \subseteq X$ compact and $L_1 \subseteq L_2 \subseteq \ldots$. For each n let

$$F_n = \left\{ x \in U \ : \ d(x, X \setminus U) \ge \frac{1}{n} \right\}$$

and let $K_n = L_n \cap F_n \subseteq U$. Since $F_1 \subseteq F_2 \subseteq \ldots$, so $K_1 \subseteq K_2 \subseteq \ldots$. Furthermore, if $x \in U$, there is n_1 so that $d(x, X \setminus U) \ge \frac{1}{n_1}$ and n_2 such that $x \in L_{n_2}$. Hence for $n \ge \max(n_1, n_2)$ we have $x \in K_n \cap L_n$. Thus,

$$U = \bigcup_{n=1}^\infty K_n$$

We shall choose $(f_n)_{n=1}^\infty \subset C_c(X)$ inductively as follows:

- $K_1 \prec f_1 \prec U$
- $K_2 \cup \mathrm{supp}(f_2) \prec f_2 \prec U$
- $\vdots$
- $K_{n+1} \cup \mathrm{supp}(f_n) \prec f_{n+1} \prec U$

Hence $f_1 \leq f_2 \leq \ldots$ and $\lim_{n\to\infty} f_n = 1_U$. Thus, by MCT, we have

$$\begin{aligned}
\nu(U) &= \int 1_U d\nu \\
&= \lim_{n\to\infty} \int f_n d\nu \\
&= \lim_{n\to\infty} \int f_n d\mu \\
&= \int 1_U d\mu \\
&= \mu(U)
\end{aligned}$$

(II) Now let $E \in \mathcal{B}(X)$ be an arbitrary Borel set with $\mu(E) < \infty$. Given $\epsilon > 0$ find $K \subseteq E \subseteq V$, K compact, V open, so that

$$\mu(E) < \mu(K) + \frac{\epsilon}{2} \qquad \mu(V) < \mu(E) + \frac{\epsilon}{2}$$

(using inner and outer regularity of μ). Hence, by (I),

$$\begin{aligned}
\nu(V) - \nu(K) &= \nu(V \setminus K) \\
&= \mu(V \setminus K) && \text{(since } V \setminus K \text{ is open)} \\
&= \mu((V \setminus E) \cup (E \setminus K)) \\
&= \mu(V \setminus E) + \mu(E \setminus K) \\
&= \epsilon
\end{aligned}$$

So, by the above, $\nu(E) \leq \nu(V) \leq \nu(K) + \epsilon \leq \nu(E) + \epsilon$ Thus,

$$\nu(E) = \inf\{\nu(V) \ : \ E \subseteq V, V \text{ open}\} = \inf\{\mu(V) \ : \ E \subseteq V, V \text{ open}\} = \mu(E)$$

(III) If $E \in \mathcal{B}(X)$ with $\mu(E) = \infty$, write $E = \bigcup_{i=1}^{\infty} E_i$, each $\mu(E_i) < \infty$ arranged such that $E_1 \subseteq E_2 \subseteq \ldots$ in $\mathcal{B}(X)$ (e.g. $E_i = E \cap L_i$, with L_i as in (I)). Then, by (II) and continuity from below, we have

$$\begin{aligned}
\mu(E) &= \lim_{n\to\infty} \mu(E_n) \\
&= \lim_{n\to\infty} \nu(E_n) \\
&= \nu(E)
\end{aligned}$$

■

Thus $\mu = \nu$ on $\mathcal{B}(X)$.

Theorem 5.17 *If (X, d) is a σ-compact locally compact metric space and $\mu : \mathcal{B}(X) \to \mathbb{C}$, then μ is a linear combination of up to four finite Radon measures.*

Proof. We consider, for example, the Jordan decomposition

$$\mu = \mu_1 - \mu_2 + i[\mu_3 - \mu_4]$$

Each μ_k is a finite measure, hence a Radon measure. ■

Remark 5.18 We may call μ, above, a "complex Radon measure".

Exam note. Nico will ask, on the final exam, to prove that the dual of $L_1(\lambda)$ is $L_\infty(\lambda)$, where λ is a finite measure.

Corollary 5.19 *The d-dimensional Lebesgue measure $\lambda_d : \mathcal{B}\left(\mathbb{R}^d\right) \to [0,\infty]$ is inner and outer regular.*

Proof. We note that $\mathbb{R}^d = \bigcup_{n=1}^\infty \overline{B(0,n)}$ is σ-compact, by the Heine-Borel theorem. If $K \subseteq \mathbb{R}^d = \bigcup_{n=1}^\infty (-n,n)^d$ is compact, then $K \subsetneq (-n_0,n_0)^d$ for some n_0. Hence,

$$\lambda_d(K) \leq \lambda_d\left((-n_0,n_0)^d\right) = (2n_0)^d < \infty$$

Thus, λ_d is a locally finite Borel measure on a σ-compact space, hence Radon. ■

Remark 5.20 If $\emptyset \neq U \subseteq \mathbb{R}^d$ is open, then $\lambda_d(U) > 0$. Indeed, if $x \in U$, we may find an $\epsilon > 0$ such that $\prod_{j=1}^d (x_j - \epsilon, x_j + \epsilon) = B(x, d_\infty) \subseteq U$ and we have $\lambda_d(U) \geq (2\epsilon)^d > 0$.

Chapter 6

Differentiation

Recall. (Fundamental Theorem of Calculus) If $f : (a,b) \to \mathbb{C}$ is continuous and bounded (with $\lim_{t\to a^+} f(t) = f(a)$) then for $x \in (a,b)$

$$f(x) = \frac{d}{dt}\left[\int_a^t f(s)ds\right]_{t=x} = \lim_{r\to 0^+} \frac{1}{2r}\int_{x-r}^{x+r} f(s)ds$$

We shall generalise this to integrable f and to $d > 1$.

Notation 6.1 (Euclidean balls) If $x \in \mathbb{R}^d$, $r > 0$, we let

$$B(x,r) = \left\{y \in \mathbb{R}^d \ : \ \|x-y\|_2 < r\right\}$$

Remark 6.2 In what follows we could replace $\|\cdot\|_2$ with any norm on $\mathbb{R}^d$ and results will remain true as stated. We shall retain $\|\cdot\|_2$, however, for concreteness. For this chapter, our goal is to consider statements of the form

$$\lim_{r\to 0^+} \frac{1}{\lambda_d(B(x,r))}\int_{B(x,r)} f(y) \underbrace{dy}_{d\lambda_d(y)} = f(x) \quad \lambda_d - a.ex$$

under mild assumptions for f.

Lemma 6.3 ***(Sufficiently many disjoint balls lemma)*** *Let $\mathcal{C}$ be a collection of Euclidean balls in $\mathbb{R}^d$. Let $U = \bigcup_{B\in\mathcal{C}} B$. Then for any $0 < c < \lambda_d(U)$ there exist $B_1, \ldots, B_n$ in $\mathcal{C}$ such that $B_i \cap B_j = \emptyset$ for $i \neq j$ and $3^d \sum_{i=1}^n \lambda_d(B_j) > c$.*

Proof. Since $U \neq \emptyset$, there is c as above. By inner regularity, there is $K \subseteq U$ compact such that $\lambda_d(K) > c$. Since $K \subseteq U = \bigcup_{B\in\mathcal{C}} B$, by compactness there is $B_1', \ldots, B_m'$ in $\mathcal{C}$ such that $K \subseteq \bigcup_{j=1}^m B_j'$. Write each $B_j' = B(x_j', r_j')$, we may relabel $r_1' \geq \ldots \geq r_m'$. Then,

- $B_1 = B_1'$
- $B_2 = B_{j_2}'$ where $j_2 = \min\left\{j \in \{1,\ldots,m\} \ : \ B_j' \cap B_1 = \emptyset\right\}$
- $\vdots$

- $B_n = B'_{j_n}$ where $j_n = \min\left\{j \in \{j_{n-1}+1, \ldots, m\} \ : \ B'_j \cap \bigcup_{i=1}^{n=1} B_i\right\}$

where n is determined by where this algorithm ends. If $B'_j \notin \{B_1, \ldots, B_n\}$, then $B'_j \cap B_i = B'_{j_i}$ for some $j_i < j$, so $r_i := r'_{j_i} \geq r'_j$. If we write $B_i = B(x_i, r_i)$ then $B'_j \subseteq B(x_i, 3r_i)$. Notice that

$$\lambda_d\left(B(x_i, 3r_i)\right) = \lambda_d\left(3I(B(0, r_i)) + x_i\right) = 3^d \lambda_d\left(B(0, r_i)\right) = 3^d \lambda_d\left(B_i\right)$$

Thus,

$$\begin{aligned} c < \lambda_d\left(K\right) & \\ & \leq \lambda_d\left(\bigcup_{j=1}^n B'_j\right) \\ & \leq \lambda_d\left(\bigcup_{i=1}^n B(x_i, 3r_i)\right) \\ & \leq \sum_{i=1}^n \lambda_d\left(B(x_i, 3r_i)\right) \\ & = 3^d \sum_{i=1}^n \lambda_d\left(B_i\right) \end{aligned}$$

■

Definition 6.4 If $f \in L(\lambda_d)$, we let

$$A_r f(x) = \frac{1}{\lambda_d\left(B(x,r)\right)} \int_{B(x,r)} f(y) dy$$

be the **"average value"**, for $r > 0, x \in \mathbb{R}^d$. We let the **Hardy-Littlewood maximal function** be given as follows:

$$Hf(x) = \sup_{r>0} \left\{A_r|f|(x)\right\}$$

Remark 6.5 The mapping $(r, x) \mapsto A_r f(x) \ : \ (0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ is continuous. First, as above,

$$\lambda_d\left(B_(x, r)\right) = \lambda_d\left(rI(B(0,1)) + x\right) = r^d \lambda_d\left(B(0, r)\right)$$

Second, if $((r_n, x_n))_{n=1}^\infty$ with $\lim_{n\to\infty}(r_n, x_n) = (r, x)$, then

$$1_{B(x_n, r_n)}|f| \leq |f|$$

and

$$\lim_{n\to\infty} 1_{B(x_n, r_n)} f = f$$

pointwise. Hence, by the LDCT,

$$A_{r_n} f(x_n) = \frac{1}{r_n^d \lambda_d\left(B(0,1)\right)} \int 1_{B(x_n, r_n)} f \to \frac{\int 1_{B(x,r)} f}{r^d \lambda_d\left(B(0,1)\right)} = A_r f(x)$$

Remark 6.6 The Hardy-Littlewood maximal function can be realised as:

$$Hf(x) = \sup_{r>0} A_r|f|(x) = \sup_{r \in (0,\infty)\cap\mathbb{Q}} A_r|f|(x)$$

So the Hardy-Littlewood is the supremum of a countable family of continuous functions hence Borel measurable.

Remark 6.7 We may define $A_r f$ and hence Hf for f in

$$L_{loc}(\lambda_d) = \left\{f \in M(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)) \ : \ 1_K f \in L(\lambda_d) \text{ for any compact } K \subset \mathbb{R}^d\right\}$$

Lemma 6.8 ***(Hardy-Littlewood maximal inequality)*** *If $f \in L(\lambda_d)$ and $\alpha > 0$ then*

$$\lambda_d\left(Hf^{-1}(\alpha, \infty]\right) \leq \frac{3^d}{\alpha} \int \mathbb{R}^d |f| d\lambda_d$$

Proof. Let $E_\alpha = Hf^{-1}\left((\alpha, \infty]\right)$. Then, for each $x \in E_\alpha$, $Hf(x) > \alpha$ so there is $r_x > 0$ such that $A_{r_x}|f|(x) > \alpha$. Now $E_\alpha \subseteq \bigcup_{x \in E_\alpha} B(x, r_x) = U$, so if $o < \lambda_d\left(E_\alpha\right)$ and $0 < c < \lambda_d\left(E_\alpha\right) \leq \lambda_d\left(U\right)$ the sufficiently many disjoint balls lemma provides $x_1, \ldots, x_n \in E_\alpha$, $B_i = B(x_i, r_{x_i})$ for $i = 1, \ldots, n$ such that $B_i \cap B_j = \emptyset$ and $c < 3^d \sum_{i=1}^n \lambda_d\left(B_i\right)$. Then, for each i,

$$\frac{1}{\lambda_d\left(B_i\right)} \int_{B_i} |f| = A_{r_{x_i}}(x_i) > \alpha \qquad \Longrightarrow \qquad \frac{1}{\alpha} \int_{B_i} |f| > \lambda_d\left(B_i\right)$$

and hence,

$$\begin{aligned} c &< 3^d \sum_{i=1}^n \lambda_d\left(B_i\right) \\ &< \frac{3^d}{\alpha} \sum_{i=1}^n \int_{B_i} |f| \\ &= \frac{3^d}{\alpha} \int_{\bigcup_{i=1}^n B_i} |f| \\ &\leq \frac{3^d}{\alpha} \int |f| \end{aligned}$$

■

Lemma 6.9 ***(Chebyshev's Inequality)*** *If $f \in \overline{M}^+(X, \mathcal{M})$, $\mu : \mathcal{M} \to [0, \infty]$ is a measure, and $\alpha > 0$, then*

$$\frac{1}{\alpha} \int_{f^{-1}((\alpha, \infty])} f d\mu \geq \mu\left(f^{-1}\left((\alpha, \infty]\right)\right)$$

Proof. Very economically:

$$\int_{f^{-1}((\alpha, \infty])} f d\mu \geq \int_{f^{-1}((\alpha, \infty])} \alpha 1 d\mu = \alpha \mu\left(f^{-1}\left((\alpha, \infty]\right)\right)$$

■

6.1 The unoriginally named: Differentiation Theorems

Theorem 6.10 ***(Differentiation Theorem I)*** *If $f \in L_{loc}(\lambda_d)$, then*

$$\lim_{r \to 0^+} A_r f(x) = f(x)$$

for λ_d-a.e. x in $\mathbb{R}^d$.

Proof. Since $\mathbb{R}^d = \bigcup_{N=1}^{\infty} B(0,N)$, it suffices to prove this result for $1_{B(0,N)}f$. Hence $f \in L(\lambda_d)$. Given $\epsilon > 0$, since λ_d is a Radon measure, there is $h \in C_c(\mathbb{R}^d)$ such that $\int |h - f| < \epsilon$. Notice that

$$\begin{aligned}
|A_r h(x) - h(x)| &= \left| \frac{1}{\lambda_d\left(B(x,r)\right)} \int_{B(x,r)} (h(y) - h(x))\, dy \right| \\
&\leq \frac{1}{\lambda_d\left(B(x,r)\right)} \int_{B(x,r)} |h(y) - h(x)|\, dy \\
&\leq \sup_{y \in B(x,r)} |h(y) - h(x)| \\
&\to 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{as } r \to 0^+
\end{aligned}$$

Hence,

$$\begin{aligned}
\limsup_{r\to 0^+} |A_r f(x) - f(x)| &\leq \limsup_{r \to 0^+} \left[|A_r f(x) - A_r h(x)| + |A_r h(x) - h(x)| + |h(x) - f(x)| \right] \\
&\leq \lim_{r \to 0} \sup_{r' \in (0,r)} \left[A_r |f - h|(x) + |h(x) - f(x)| \right] \\
&\leq H(f-h)(x) + |f(x) - h(x)|
\end{aligned}$$

Given $\delta > 0$, let

$$E_\delta = \left\{ x \in \mathbb{R}^d \ : \ \limsup_{r \to 0^+} |A_r f(x) - f(x)| > \delta \right\}$$

Then,

$$E_\delta \subseteq \left\{ x \in \mathbb{R}^d \ : \ H(f-h)(x) > \frac{\delta}{2} \right\} \cup \left\{ x \in \mathbb{R}^d \ : \ |h(x) - f(x)| > \frac{\delta}{2} \right\}$$

Therefore, by the Hardy-Littlewood maximal theorem and Chebyshev's inequality,

$$\begin{aligned}
\lambda_d\left(E_\delta\right) &\leq \lambda_d \left(H(f-h)^{-1} \left((\frac{\delta}{2}, \infty] \right) \right) + \lambda_d \left(|h - f|^{-1} \left((\frac{\delta}{2}, \infty] \right) \right) \\
&\leq \frac{2 \cdot 3^d}{\delta} \int |f - h| + \frac{2}{\delta} \int_{|f-h|^{-1}\left((\frac{\delta}{2},\infty]\right)} \\
&< \frac{2 \cdot 3^d + 2}{\delta} \epsilon
\end{aligned}$$

Then, since $\epsilon > 0$ is arbitrary, $\lambda_d\left(E_\delta\right) = 0$. Then for $x \in \mathbb{R}^d \backslash \bigcup_{n=1}^{\infty} E_{1/n}$ we have $\lim_{r \to 0^+} |A_r f(x) - f(x)| = 0$. ■

Definition 6.11 Let for f in $L_{loc}(\lambda_d)$ its **Lebesgue set** to be

$$L_f = \left\{ x \in \mathbb{R}^d : \lim_{r \to 0^+} \frac{1}{\lambda_d\left(B(x,r)\right)} \int_{B(x,r)} |f(y) - f(x)| dy = 0 \right\}$$

Theorem 6.12 *Let $f \in L_{loc}(\lambda_d)$. Then $\lambda_d^* \left(\mathbb{R}^d \setminus L_f\right) = 0$ where λ_d^* is the outer measure associated with λ_d.*

Proof. Let $\{c_n\}_{n=1}^{\infty}$ we a countable dense subset of $\mathbb{C}$. Then let

$$E_n = \left\{ x \in \mathbb{R}^d \ : \ \limsup_{r \to 0^+} |A_r |f - c_n 1|(x) - |f(x) - c_n|| > 0 \right\}$$

so E_n is λ_d-null set, and $E = \bigcup_{n=1}^{\infty} E_n$ is a null set as well. If $x \in \mathbb{R}^d \setminus E$, and $\epsilon > 0$, then $|f(x) - c_n| < \epsilon$ for some n. Hence, for any y in $\mathbb{R}^d$,

$$|f(y) - f(x)| \leq |f(y) - c_n| + |c_n - f(x)| < |f(y) - c_n| + \epsilon$$

Thus, as $x \notin E_n$,

$$\begin{aligned}
\frac{1}{\lambda_d\left(B(x,r)\right)} \int_{B(x,r)} |f(y) - f(x)| dy &\leq \frac{1}{\lambda_d\left(B(x,r)\right)} \int_{B(x,r)} \left[|f(y) - c_n| + \epsilon\right] dy \\
&= \frac{1}{\lambda_d\left(B(x,r)\right)} \int_{B(x,r)} |f(y) - c_n 1(y)| dy + \epsilon \\
&\to |f(x) - c_n| + \epsilon \qquad\qquad (r \to 0^+) \\
&< 2\epsilon
\end{aligned}$$

Thus, as $\epsilon > 0$ is arbitrary, the limit

$$\lim_{r \to 0^+} \frac{1}{\lambda_d\left(B(x,r)\right)} \int_{B(x,r)} |f(y) - f(x)| dy = 0$$

for $x \in E$. We have $\mathbb{R}^d \setminus E \subseteq L_f$ so that $\mathbb{R}^d \setminus L_f \subseteq E$. ∎

Theorem 6.13 ***(Differentiation Theorem II)*** *Let $\mu : \mathcal{B}\left(\mathbb{R}^d\right) \to [0, \infty]$ be a locally finite measure such that $\mu \perp \lambda_d$. Then*

$$\lim_{r \to 0^+} \frac{\mu\left(B(x,r)\right)}{\lambda_d\left(B(x,r)\right)} = 0$$

for λ_d-a.e. x.

Proof. Let (E, F) be a Borel partition of $\mathbb{R}^d$ such that $\mu(F) = 0 = \lambda_d(E)$. Let for $\delta > 0$,

$$F_\delta = \left\{ x \in F : \limsup_{r \to 0^+} \frac{\mu\left(B(x,r)\right)}{\lambda_d\left(B(x,r)\right)} > \delta \right\}$$

Since μ is a Radon measure, given an $\epsilon > 0$, there is an open $U \supseteq F$ such that $\mu(U) < \epsilon$. If $x \in F_\delta \subseteq F \subseteq U$, there is $r_x > 0$ be so that $B_x = B(x, r_x) \subseteq U$ and $\frac{\mu(B_x)}{\lambda_d(B_x)} \geq \delta$ (via definition of F_δ). Thus,

$$F_\delta \subseteq \bigcup_{x \in F_\delta} B_x := V \subseteq U$$

and given $0 < c < \lambda_d(V)$ we may find $B_{x_1}, \ldots, B_{x_n}, x_1, \ldots, x_n \in F_\delta$ such that $B_{x_i} \cap B_{x_j} = \emptyset$ and $c < 3^d \sum_{i=1}^{n} \lambda_d\left(B_{x_i}\right)$, by the sufficiently many disjoint balls lemma.

Hence,

$$\begin{aligned}
c &< 3^d \sum_{i=1}^{n} \lambda_d\left(B_{x_i}\right) \\
&< \frac{3^d}{\delta} \sum_{i=1}^{n} \mu\left(B_{x_i}\right) \\
&= \frac{3^d}{\delta} \sum_{i=1}^{n} \mu\left(\bigcup_{i=1}^{n} B_{x_i}\right) \\
&\leq \frac{3^d}{\delta} \mu(V) \\
&\leq \frac{3^d}{\delta} \mu(U) \\
&< \frac{3^d}{\delta} \epsilon
\end{aligned}$$

But then we have,

$$\begin{aligned}
\lambda_d^*\left(F_\delta\right) \leq \lambda_d(V) \\
&= \lim_{c \to \lambda_d(V)^-} c \\
&\leq \frac{3^d}{\delta} \epsilon
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we see that $\lambda_d^*\left(F_\delta\right) = 0$. Hence, if $x \in \mathbb{R}^d \setminus \bigcup_{k=1}^{\infty} F_{1/k}$ then

$$\lim_{r \to 0^+} \frac{\mu\left(B(x,r)\right)}{B(x,r)} = 0 \qquad \blacksquare$$

Definition 6.14 A collection of sets $\left\{E_r(x) \ : \ x \in \mathbb{R}^d, r > 0\right\} \subseteq \mathcal{B}\left(\mathbb{R}^d\right)$ is called **nicely shrinking** if:

1. $E_r(x) \subseteq B_r(x)$
2. $\lambda_d\left(E_r(x)\right) \geq \alpha \lambda_d\left(B(x,r)\right)$ where $\alpha > 0$ is constant.

Example 6.15 The balls are nicely shrinking. Furthermore, the sets

$$\left\{\left[x_1, x_1 + \frac{r}{\sqrt{d}}\right) \times \ldots \times \left[x_d, x_d + \frac{r}{\sqrt{d}}\right) \ : \ r > 0, x = (x_1, \ldots, x_d) \in \mathbb{R}^d\right\}$$

is nicely shrinking.

Theorem 6.16 *Let $\nu : \mathcal{B}\left(\mathbb{R}^d\right) \to \mathbb{C}$ be a complex measure with Lebesgue-Radon-Nikodym decomposition*

$$\nu = \rho + f \cdot \lambda_d \qquad \rho \perp \lambda_D \quad f \in L(\lambda_d)$$

[In effect, $f = \frac{d\nu}{d\lambda_d}$ λ_d-a.e.]. Then, for any nicely shrinking family $\left\{E_r : x \in \mathbb{R}^d, r > 0\right\}$ we have

$$\lim_{r \to 0^+} \frac{\nu\left(E_r(x)\right)}{\lambda_d\left(E_r(x)\right)} = f(x)$$

for λ_d-a.e. x in $\mathbb{R}^d$.

Proof. Write $\rho = \operatorname{Re}\rho^+ - \operatorname{Re}\rho^- + i[\operatorname{Im}\rho^+ - \operatorname{Im}\rho^-]$ (Jordan decomposition), with $\operatorname{Re}\rho^+, \ldots, \operatorname{Im}\rho^- \leq |\rho| \leq \operatorname{Re}\rho^+ + \ldots + \operatorname{Im}\rho^-$ so each $\operatorname{Re}\rho^+, \ldots, \operatorname{Im}\rho^- \perp \lambda_d$. By the Second Differentiation Theorem, we see that for each $\mu = \operatorname{Re}\rho^+, \ldots, \operatorname{Im}\rho^-$ we have

$$\lim_{r\to 0^+} \frac{\mu(E_r(x))}{\lambda_d(E_r(x))} \leq \lim_{r\to 0^+} \frac{\mu(B(x,r))}{\alpha\lambda_d(B(x,r))} = 0 \qquad \lambda_d - a.e.$$

Hence we conclude the same for ρ, and

$$\begin{aligned}\left|\frac{1}{\lambda_d(E_r(x))}\int_{E_r(x)} f(y)dy - f(x)\right| &\leq \frac{1}{\lambda_d(E_r(x))}\int_{E_r(x)} |f(y)-f(x)|dy \\ &\leq \frac{1}{\alpha\lambda_d(B(x,r))}\int_{B(x,r)} |f(y)-f(x)|dy \\ &\to 0 \qquad\qquad (\text{as } r \to 0^+)\end{aligned}$$

provided that $x \in L_f$. ■

6.2 Functions and measures on $\mathbb{R}$

Theorem 6.17 *If $F \in ND_r(\mathbb{R})$ (non-decreasing, right continuous) then $F'(x)$ exists for λ-a.e. x in $\mathbb{R}$.*

Proof. If $h \neq 0$, then

$$\frac{F(x+h)-F(x)}{h} = \begin{cases} \frac{\mu_F((x,x+h])}{\lambda_d((x,x+h])} & h > 0 \\ \frac{\mu_F((x+h,x])}{\lambda_d((x+h,x])} & h < 0 \end{cases}$$

Since each family $\{(x, x+h] : x \in \mathbb{R}, h > 0\}$, $\{(x-h, x] \ : \ x \in \mathbb{R}, h > 0\}$ is nicely shrinking we see that

$$\lim_{h\to 0^+} \frac{F(x+h)-F(x)}{h} \qquad \lim_{h\to 0^-} \frac{F(x+h)-F(x)}{h}$$

converge for λ-a.e. to right and left derivatives exist for such x. However, each is λ-a.e. equal to $\frac{d\mu_F}{d\lambda}$, thanks to the last corollary. Hence F' exists λ-a.e. ■

Example 6.18 Consider the Cantor ternary function $\varphi \in ND(\mathbb{R})$. Notice that $\varphi'(x) = 0$ whenever $x \in \mathbb{R} \setminus C$.

6.2.1 Interlude: Variation of functions

Definition 6.19 Let $F : \mathbb{R} \to \mathbb{C}$. If $a < b$ in $\mathbb{R}$ we define the **variation** of F on $[a, b]$ by

$$V_F[a,b] = \sup\left\{\sum_{i=1}^n |F(a_i) - F(a_{i-1})| \ : \ a = a_0 < a_1 < \ldots < a_n = b,\ n \in \mathbb{N}\right\}$$

Example 6.20 Let

$$F(x) = \begin{cases} x \sin\frac{1}{x} & x > 0 \\ 0 & x = 0 \end{cases}$$

Then $V_F[0, \epsilon] = \infty$ for all $\epsilon > 0$.

Theorem 6.21 ***(Properties of variation)***

1. *If $a < b < c$, then $V_F[a,c] = V_F[a,b] + V_F[b,c]$.*

2. *If $a' \leq a < b \leq b'$ then $V_F[a,b] \leq V_F[a',b']$.*

Proof. Exercise. ■

Definition 6.22 The **variation** of other intervals is given by:

$$V_F(a,b] = \lim_{x \to a^+} V_F[x,b] \qquad V_F(-\infty,b] = \lim_{x \to -\infty} V_F[x,b]$$

Theorem 6.23 ***(Effect of right continuity on variation)***

1. *If F is right continuous at a and $V_F[a,b] < \infty$, then $V_F(a,b] = V_F[a,b]$*

2. *If $V_F(-\infty,b] < \infty$, then $\lim_{x \to -\infty} V_F(-\infty,x] = 0$.*

Proof. We prove these properties one at a time.

1. By property 2 in the theorem above and definition of $V_F(a,b]$, we have $V_F(a,b] \leq V_F[a,b]$.

   To see the converse inequality, given $\epsilon > 0$, let $\delta > 0$ be so that $a < x < a + \delta$ and we have that $|F(x) - F(a)| < \epsilon$. Now we let $a < a_0 < \ldots < a_n = b$ be so

   $$\sum_{i=1}^{n} |F(a_i) - F(a_{i-1})| > V_F[a,b] - \epsilon \qquad a < a_1 < a + \delta$$

   Then,

   $$\begin{aligned} V_F[a,b] &< |F(a_1) - F(a_0)| + \sum_{i=2}^{n} |F(a_i) - F(a_{i-1})| + \epsilon \\ &< \epsilon + V_F[a_1,b] + \epsilon \\ &\leq V_F(a,b] + 2\epsilon \end{aligned}$$

   Since $\epsilon > 0$ was arbitrary, $V_F[a,b] \leq V_F(a,b]$.

2. For fixed $x < b$, then by property 1 above,

   $$\begin{aligned} V_F(-\infty,b] &= \lim_{y \to -\infty} V_F[y,b] \\ &= \lim_{\substack{y \to -\infty \\ y < x}} (V_F[y,x] + V_F[x,b]) \qquad (\star) \\ &= V_F(-\infty,x] + V_F[x,b] \end{aligned}$$

   Then, taking $x \to -\infty$, the result follows. ■

Definition 6.24 If $V_F(-\infty,x] < \infty$ for each x in $\mathbb{R}$, we define the **total variation** function of F by

$$T_F(x) = V_F(-\infty,x] \in [0,\infty)$$

If $\sup_{x \in \mathbb{R}} T_F(x) < \infty$, we say that F is of **bounded variation** and we write $F \in BV(\mathbb{R})$.

We further let

$$BV_r(\mathbb{R}) = \{F \in BV(\mathbb{R}) \ : \ F \text{ is right continuous}\}$$

Remark 6.25 Here are a few facts about the total variation function:

1. It follows from (2) above that $T_F(-\infty) = \lim_{x\to-\infty} T_F(x) = 0$.
2. If $F \in BV_r(\mathbb{R})$, then T_F is right continuous. To see this, let $a < x < b$, and we use $(\star)$ and part (1) of the previous two propositions to see that

$$T_F(x) - T_F(a) = V_F[a,x] = V_F[a,b] - V_F[x,b] = V_F(a,b] - V_F[x,b] \to 0 \text{ (as } x \to a^+)$$

   Namely, $\lim_{x\to a^+} T_F(x) = T_F(a)$.

Theorem 6.26 *1. $F \in BV(\mathbb{R})$ if and only if $\operatorname{Re} F, \operatorname{Im} F \in BV(\mathbb{R})$.*

2. If $F \in BV^{\mathbb{R}}(\mathbb{R})$ (real-valued), then each of $T_F \pm F$ is non-decreasing and bounded.

3. (Jordan decomposition) If $F \in BV(\mathbb{R})$, we let $F_1, \frac{1}{2}(T_{\operatorname{Re} F} + \operatorname{Re} F), F_2 \frac{1}{2}(T_{\operatorname{Re} F} - \operatorname{Re} F), F_3 = \frac{1}{2}(T_{\operatorname{Im} F} + \operatorname{Im} F), F_4 = \frac{1}{2}(T_{\operatorname{Im} F} - \operatorname{Im} F)$, then $F = F_1 - F_2 + i[F_3 - F_4]$, hence F is bounded and $F(\pm\infty) = \lim_{x\to\pm\infty} F(x)$ exists.

Proof. 1. If $x < y$ in $\mathbb{R}$, then by using definitions of $V_H, H = F, \operatorname{Re} F, \operatorname{Im} F$, we see

$$V_{\operatorname{Re} F}[x,y], V_{\operatorname{Im} F}[x,y] \le V_F[x,y] \le V_{\operatorname{Re} F}[x,y] + V_{\operatorname{Im} F}[x,y]$$

Taking $x \to -\infty$, we see that

$$T_{\operatorname{Re} F}(y), T_{\operatorname{Im} F}(y) \le T_F(y) \le T_{\operatorname{Re} F}(y) + T_{\operatorname{Im} F}(y)$$

and then taking $y \to \infty$ does the job.

2. If $x < y$ in $\mathbb{R}$ then

$$\begin{aligned}(T_G \pm G)(y) - (T_G \pm G)(x) &= T_G(y) - T_G(x) \pm [G(y) - G(x)] \\ &= V_G[x,y] + [G(y) - G(x)] \\ &\ge |G(y) - G(x)| \pm [G(y) - G(x)] \\ &\ge 0\end{aligned}$$

Furthermore $T_G(\pm\infty)$ always exist. We shall check later that they are bounded (use remark 2 below).

3. This is obvious. ■

Remark 6.27 1. If F above is right continuous, so too are $\operatorname{Re} F, \operatorname{Im} F$ and hence $F_1, \dots, F_4$.

2. If $F : \mathbb{R} \to \mathbb{R}$ and is bounded then $F \in BV^{\mathbb{R}}(\mathbb{R})$.

Corollary 6.28 *If $F \in BV_r(\mathbb{R})$, then $F'(x)$ exists for λ-a.e. x in $\mathbb{R}$.*

Proof. Corollary to theorem above. ■

Theorem 6.29 ***(Complex Borel measures on $\mathbb{R}$)*** *Let $F \in BV_r(\mathbb{R})$.*

1. There is a complex measure $\mu_F : \mathcal{B}(\mathbb{R}) \to \mathbb{C}$ such that

$$\mu_F((a,b]) = F(b) - F(a) \qquad \text{for } a < b \text{ in } \mathbb{R} \quad (\dagger)$$

2. *If $G \in BV_r^{\mathbb{R}}(\mathbb{R})$ (real-valued), then $|\mu_G| = \mu_{T_G}$. Hence $\mu_G^{\pm} = \mu_{\frac{1}{2}(T_G \pm G)}$, and the notions of Jordan decomposition coincide.*

3. *If $\nu : \mathcal{B}(\mathbb{R}) \to \mathbb{C}$ is any measure such that*

$$\nu((a,b]) = F(b) - F(a) \qquad \text{for } a < b \text{ in } \mathbb{R} \qquad (\dagger\dagger)$$

Proof. 1. Let $F = F_1 - F_2 + i[F_3 - F_4]$ (Jordan). Then each $F_k \in ND_r(\mathbb{R})$ so corresponds to a measure μ_{F_k}, satisfying the analogue of (†). Let $\mu_F = \mu_{F_1} - \mu_{F_2} - i[\mu_{F_3} - \mu_{F_4}]$.

2. Let $a < b$ in $\mathbb{R}$. We recall that:

$$\begin{aligned}
|\mu_G|((a,b]) &= \sup\left\{\sum_{i=1}^{n} |\mu_G(E_i)| \;:\; \{E_1, \ldots, E_n\} \text{ is a Borel partition of } (a,b], n \in \mathbb{N}\right\} \\
|\mu_G| &= \mu_G^+ + \mu_G^- \qquad \text{[true, but less useful]} \\
\mu_{T_G}((a,b]) &= T_G(b) - T_g(a) = V_G[a,b] \\
&= \sup\left\{\sum_{i=1}^{n} |G(a_i) - G(a_{i-1})| \;:\; (a,b] = \bigsqcup_{i=1}^{n}(a_{i-1}, a_i]\right\}
\end{aligned}$$

Hence, it is immediate that $\mu_{T_G}((a,b]) \leq |\mu_G|((a,b])$. Now,

$$|\mu_G(a,b]| = |G(b) - G(a)| \leq V_G[a,b] = T_G(b) - T_G(a) = \mu_{T_G}((a,b])$$

We let $\mathcal{H} = \{(c,d] \;:\; a \leq c \leq d \leq b\}$ and for any $A \in \langle H \rangle \subseteq \mathcal{P}((a,b])$. We have $A = \bigsqcup_{i=1}^{n}(c_i, d_i]$ (elementary family) and hence we have,

$$\begin{aligned}
|\mu_G(A)| &= \left|\sum_{i=1}^{n} \mu_G((c_i, d_i])\right| \\
&\leq \sum_{i=1}^{n} |\mu_G((c_i, d_i])| \\
&\leq \sum_{i=1}^{n} \mu_{T_G}((c_i, d_i]) \\
&= \mu_{T_G}(A)
\end{aligned}$$

We let

$$\mathcal{C} = \{E \in \mathcal{B}((a,b]) \;:\; |\mu_G(E)| \leq \mu_{T_G}(E)\}$$

Then,

(a) $\langle H \rangle \subseteq \mathcal{C}$

(b) If $E_1 \supseteq E_2 \supseteq \ldots$ in $\mathcal{C}$ then, by continuity from above,

$$\begin{aligned}
\left|\mu_G\left(\bigcap_{n=1}^{\infty} E_n\right)\right| &= \lim_{n\to\infty} |\mu_G(E_n)| \\
&\leq \lim_{n\to\infty} \mu_{T_G}(E_n) \\
&= \mu_{T_G}\left(\bigcap_{n=1}^{\infty} E_n\right)
\end{aligned}$$

(c) If $E_1 \subseteq E_2 \subseteq \ldots$ in $\mathcal{C}$, then continuity from below gives:

$$\left|\mu_G\left(\bigcup_{n=1}^{\infty} E_n\right)\right| \leq \mu_{T_G}\left(\bigcup_{n=1}^{\infty} E_n\right)$$

Hence, by the Monotone Class Lemma, $\mathcal{C} \supseteq \sigma\langle\mathcal{H}\rangle = \mathcal{B}((a,b])$, so $\mathcal{B}((a,b])$. Thus, for any Borel partition $\{E_1,\ldots,E_n\}$ of $(a,b]$ we have

$$\sum_{i=1}^{n} |\mu_G(E_i)| \leq \sum_{i=1}^{n} \mu_{T_G}(E_i) = \mu_{T_G}\left(\bigsqcup_{i=1}^{n} E_i\right) = \mu_{T_G}((a,b])$$

Thus, $|\mu_G|((a,b]) \leq \mu_{T_G}((a,b])$. In conclusion, $|\mu_G|((a,b]) = \mu_{T_G}((a,b])$ and hence, by characterisation of (locally) finite Borel measures on $\mathbb{R}$, $|\mu_G| = \mu_{T_G}$.

We have

$$\mu_G^{\pm} = \frac{1}{2}(|\mu_G| \pm \mu_G) = \frac{1}{2}(\mu_{T_G} \pm \mu_G) = \mu_{\frac{1}{2}(T_G \pm G)}$$

3. If ν satisfies (††) then we see for $a < b$ in $\mathbb{R}$ that

$$\operatorname{Re}\nu\,(a,b]) = \operatorname{Re}F(b) - \operatorname{Re}F(a) = \mu_{\operatorname{Re}F}((a,b])$$

Thus $\operatorname{Re}\nu$, $\mu_{\operatorname{Re}F}$ admit the same Jordan decomposition at least on intervals of the form $(a,b]$. Hence, by uniqueness for measures, $\operatorname{Re}\nu = \mu_{\operatorname{Re}F}$. Likewise $\operatorname{Im}\nu = \mu_{\operatorname{Im}F}$. ■

Remark 6.30 If $F, G \in BV_r(\mathbb{R})$, then $\mu_F = \mu_G$ if and only if $F - G = c1$ (constant).

Remark 6.31 If ν is as in part (3) above, then $F(x) = \nu((-\infty,x])$ defines an element of $BV_r(\mathbb{R})$.

6.3 Fundamental Theorem of Calculus

Definition 6.32 If $F : \mathbb{R} \to \mathbb{C}$ is **absolutely continuous**, write $F \in AC(\mathbb{R})$ provided that ($AC_\dagger$) given $\epsilon > 0$, there is $\delta > 0$ such that $a_1 \leq b_1 \leq a_2 \leq b_2 \leq \ldots \leq a_n \leq b_n$ such that $\sum_{i=1}^{n}(b_i - a_i) < \delta$ we have $\sum_{i=1}^{n} |F(b_i) - F(a_i)| < \epsilon$.

Scope. Lipschitz $\implies$ absolutely continuous $\implies$ uniformly continuous $\implies$ continuous. None of the backward implications are true in general.

Theorem 6.33 *$F \in BV \cap AC(\mathbb{R})$ implies $T_F \in AC(\mathbb{R})$.*

Proof. Given $\epsilon > 0$, find $\delta > 0$ as in ($AC_\dagger$) with $a_i < b_i$. Then, as $F \in BV(\mathbb{R})$, for each $i = 1,\ldots,n$, we find $a_i = T_{i,0} < \ldots < t_{i,m_i} = b_i$ be so that

$$\sum_{j=1}^{m_i} |F(t_{i,j}) - F(t_{i,j-1})| > V_F[a_i,b_i] - \frac{\epsilon}{2^i}$$

Then,

$$\begin{aligned}
\sum_{i=1}^{n}|T_F(b_i)-T_F(a_i)| &= \sum_{i=1}^{n} V_F[a_i,b_i] \\
&< \sum_{i=1}^{n}\left(\sum_{j=1}^{m_i}|F(t_{i,j})-F(t_{i,j-1})|+\frac{\epsilon}{2^i}\right) \\
&< 2\epsilon
\end{aligned}$$

Since $\sum_{i=1}^{n}\sum_{j=1}^{m_i}(t_{i,j}-t_{i,j-1})=\sum_{i=1}^{n}(b_i-a_i)<\delta$. ■

For the last theorem of the course, we prove the most general form of the loftily named theorem that gives the name to this chapter.

Theorem 6.34 ***(Fundamental Theorem of Calculus)***

1. *If $F \in BV \cap AC(\mathbb{R}) \subseteq BV_r(\mathbb{R})$, then $\mu_F \ll \lambda$.*
2. *If $f \in L(\lambda)$, then $F(x)=\int_{-\infty}^{x} f(t)d\lambda(t)$ satisfies $F \in BV \cap AC(\mathbb{R})$.*

Proof. 1. By Jordan decomposition of F, it suffices to thiw this for $F \in AC \cap ND(\mathbb{R})$.

Let $E \in \mathcal{B}(\mathbb{R})$ be so that $\mu(E)=0$. Given $\epsilon>0$, let $\delta>0$ be as in $(AC_\dagger)$ and let $\{(a_i,b_i]\}_{i=1}^{\infty}$ be so $E \subseteq \bigcup_{i=1}^{\infty}(a_i,b_i]$ and $\sum_{i=1}^{\infty}(b_i-a_i)=\sum_{i=1}^{\infty}\mu\left((a_i,b_i]\right)<\delta$. Find a sequence $\{(a_i',b_i']\}_{i=1}^{\infty}$ be such that there are $m_1<m_2<\dots$ such that

$$\bigcup_{i=1}^{n}(a_i,b_i]=\bigsqcup_{i=1}^{m_n}(a_i',b_i']$$

and $(a_i',b_i']\cap(a_j',b_j']=\emptyset$ if $i \neq j$. Then for each n, $\sum i=1^{m_n}(b_i'-a_i') \le \sum_{i=1}^{n}(b_i-a_i)<\delta$ so,

$$\begin{aligned}
\mu_F(E) &\le \mu_F\left(\bigcup_{i=1}^{\infty}(a_i,b_i]\right) \\
&= \lim_{n\to\infty}\mu_F\left(\bigcup_{i=1}^{n}(a_i,b_i]\right) \\
&= \lim_{n\to\infty}\mu_F\left(\bigsqcup_{i=1}^{m_n}(a_i',b_i']\right) \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}[F(b_i')-F(a_i')] \\
&\le \epsilon
\end{aligned}$$

As $\epsilon>0$ was arbitrary, we conclude that $\mu_F(E)=0$.

2. Write $F=\operatorname{Re} f^{+}-\operatorname{Re} f^{-}+i[\operatorname{Im} f^{+}-\operatorname{Im} f^{-}]$, so that

$F(x)=f\cdot\mu\left((-\infty,x]\right)=\operatorname{Re} f^{+}\cdot\mu\left((-\infty,x]\right)-\operatorname{Re} f^{-}\cdot\mu\left((-\infty,x]\right)+i[\operatorname{Im} f^{+}\cdot\mu\left((-\infty,x]\right)-\operatorname{Im} f^{-}\cdot\mu\left((-\infty,x]\right)]$

is a linear combination of 4 non-decreasing bounded measures. Thus $F \in BV(\mathbb{R})$.

We recall a proposition proved prior to L-R-N theorem, since $|f| \cdot \lambda \ll \lambda$ (property AC) implies (AC') given $\epsilon > 0$, there is $\delta > 0$ such that $\lambda(E) < \delta$ implies $|f| \cdot \lambda(E) = \int -E|f|d\lambda < \epsilon$.

Hence, if $a \leq b_1 \leq a_2 \leq b_2 \leq \ldots \leq a_n \leq b_n$ in $\mathbb{R}$ with

$$\lambda\left(\bigcup_{i=1}^{n}(a_i, b_i]\right) = \sum_{i=1}^{n}(b_i - a_i) < \delta$$

then

$$\begin{aligned}\sum_{i=1}^{n}|F(b_i) - F(a_i)| &= \sum_{i=1}^{n}\left|\int_{(a_i,b_i]} f d\lambda\right| \\ &\leq \sum_{i=1}^{n}\int_{(a_i,b_i]}|f|d\lambda \\ &= |f| \cdot \lambda\left(\bigsqcup_{i=1}^{n}(a_i, b_i]\right) \\ &< \epsilon\end{aligned}$$

Hence, $F \in AC(\mathbb{R})$. ■

Remark 6.35 $F \in BV \cap AC(\mathbb{R})$ if and only if there is f in $L(\lambda)$ such that $F' = f$ λ-a.e. and $F(x) = \int_{-\infty}^{x} f d\lambda$. Indeed, we saw earlier that $F \in BV_r(\mathbb{R})$ is λ-a.e. differentiable. Since $F \in BV \cap AC(\mathbb{R})$ $\mu_F \ll \lambda$ implies, by the Radon-Nikodym theorem, that $\mu_F = f \cdot \lambda$ and hence $F' = f$ λ-a.e. by Differentiation Theorem. Converse is just given.