Functional Analysis Preliminaries

Recall from last time the following problems:

Exercise 2.25 Prove that matrix inversion in $GL(n,\mathbb{C})$ with the topology of $\mathbb{C}^{n^2}$ is continuous.

Proof. Last time we wrote a heinous expression using adjugate matrices. Yuck. Here is a more elegant proof that is less computationally expensive. Let $A \in GL(n,\mathbb{C})$. Then, $\det A \neq 0$. That implies that the characteristic polynomial

$$p_A(t) = a_n t^n + \ldots + a_1 t + a_0$$

satisfies $a_0 \neq 0$. Consider the polynomial

$$q_A(t) = -\frac{1}{a_0}\left(a_n t^{n-1} + \ldots + a_1\right)$$

We claim that $A^{-1} = q_A(A)$. Indeed, arm ourselves with the Cayley-Hamilton theorem and compute:

$$\begin{aligned} Aq_A(A) &= A \cdot \frac{-1}{a_0}\left(a_n A^{n-1} + \ldots + a_1 I\right) \\ &= -\frac{1}{a_0}\left(a_n A^n + \ldots a_1 A\right) \\ &= -\frac{1}{a_0}\left(p_A(A) - a_0 I\right) \\ &= I \end{aligned}$$

Which is what any educated person would have done. ■

Exercise 2.26 Exhibit a group which is not unimodular.

We first prove a lemma.

Lemma 2.27 *Let G be a locally compact group that is homeomorphic to an open subset of $\mathbb{R}^d$, and let φ be a homeomorphism of G onto U. Then:*

1. *If for each $a \in G$ the function $u \mapsto \varphi(a\varphi^{-1}(u))$ is the restriction of U of an affine map $L_a : \mathbb{R}^d \to \mathbb{R}^d$, then the formula*
$$\mu(A) = \int_{\varphi(A)} |\det L_{\varphi^{-1}(u)}|^{-1} d\lambda(u)$$
defines a left-Haar measure on G.

2. *If for each $a \in G$ the function $u \mapsto \varphi(\varphi^{-1}(u)a)$ is the restriction of U of an affine map $R_a : \mathbb{R}^d \to \mathbb{R}^d$, then the formula*
$$\mu(A) = \int_{\varphi(A)} |\det R_{\varphi^{-1}(u)}|^{-1} d\lambda(u)$$
defines a right-Haar measure on G.

Proof. We only prove the first problem and claim the second one is identical. Note first that μ is Borel. Since G is homeomorphic to an open subset of $\mathbb{R}^d$, which is separable, every open set in G is σ-compact. Since taking a determinant is continuous, and the Lebesgue measure is finite on compact sets, μ is locally finite. These two conditions imply that μ is Radon.

To show it is left-invariant, use the change of variables formula to compute:

$$\begin{aligned}
\mu(xA) &= \int_{\varphi(xA)} |\det L_{\varphi^{-1}(u)}| d\lambda(u) && y = \varphi(x^{-1}\varphi^{-1}(u)) \\
&= \int_{\varphi(A)} |\det L_{x\varphi^{-1}(y)}|^{-1} |\det L_x| d\lambda(y) \\
&= \int_{\varphi(A)} |\det L_{\varphi^{-1}(y)}| d\lambda(y) \\
&= \mu(A)
\end{aligned}$$

■

We may now exhibit a group which is not unimodular.

Proof. We claim that the group

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in GL(n, \mathbb{R}) : a > 0, b \in \mathbb{R} \right\}$$

■

is not unimodular. Indeed, if we identify G with the open right-half plane in $\mathbb{R}^2$, we may use the lemma above to get that $d\mu = x^{-2}dxdy$ and $d\mu = x^{-1}dxdy$ are left- and right-Haar measures, respectively.

Functional Analysis Preliminaries

PMATH 451 Fact. For $1 < p < \infty$, the dual of $L^p(G)$ is $L^q(G)$ where p, q are Hölder conjugates. If the Haar measure on G is σ-finite then the dual of $L^1(G)$ is $L^\infty(G)$. So what does $L^\infty(G)^*$ look like?

PMATH 451 Fact. For now, we restrict ourselves to cases where μ is σ-finite. Then the dual of $L^\infty(\mu)$ is the set of finitely additive set functions which are absolutely continuous with respect to μ; denote it $A(\mathcal{B}(G), \mu)$. There is an obvious embedding $L^1(G) \hookrightarrow A(\mathcal{B}(G), \mu)$.

Definition 3.1 Let G be a locally compact group and let E be a subspace of $L^\infty(G)$ containing the constant functions. A **mean** on E is a functional $M \in E^*$ such that $\langle 1, M \rangle = \|M\| = 1$. It is said to be **left-invariant** if

$$\langle L_x\phi, M \rangle = \langle \phi, M \rangle \qquad (\phi \in E, x \in G)$$

where $L_x\phi(t) = \phi(xt)$.

Definition 3.2 A locally compact group G is **amenable** if there is a left invariant mean on $L^\infty(G)$.

Notice how we say that G is amenable and not *left*-amenable. We advertise the following theorem before digressing into the functional analysis background necessary to start hacking at it.

Theorem 3.3 *For a locally compact group G, the following are equivalent:*

1. *G is amenable*
2. *There is a right invariant mean on $L^\infty(G)$*
3. *There is an invariant mean on $L^\infty(G)$*

Proof. (1) $\implies$ (2) Let M be a left-invariant mean on $L^\infty(G)$. For $\phi \in L^\infty(G)$ define $\tilde{\phi} = \phi(x^{-1})$ for $x \in G$. Then, the functional

$$\tilde{M} : L^\infty(G) \to \mathbb{C} \qquad \phi \mapsto \langle \tilde{\phi}, M \rangle$$

is exactly what we require. (2) $\implies$ (1) is exactly the same.

(3) $\implies$ (1) is obvious.

(2) $\implies$ (3) requires a bit more work, and that work comes in the form of functional analysis background.■

Definition 3.4 A **directed set** is a set Λ with a relation $\lesssim$ such that :

1. $\lambda \lesssim \lambda, \forall \lambda \in \Lambda$
2. If $\lambda_1 \lesssim \lambda_2$ and $\lambda_2 \lesssim \lambda_3$ then $\lambda_1 \lesssim \lambda_3$
3. For any $\lambda_1, \lambda_2 \in \Lambda$ there exists λ_3 such that $\lambda_1, \lambda_2 \lesssim \lambda_3$

Example 3.5 A **net** in a set X is a mapping from Λ to X via $\lambda \mapsto x_\lambda$. We write $(x_\lambda)_{\lambda \in \Lambda}$.

Example 3.6 $\mathbb{N}$ and $\mathbb{R}$ with $\leq$. The neighbourhoods around a point $x \in X$ ordered by reverse inclusion. Tagged partitions ordered by fineness of mesh.

Definition 3.7 We say that a net (x_λ) converges to x if for every neighbourhood U of x, (x_λ) is eventually in U (there exists a λ_0 such that for all $\lambda_0 \lesssim \lambda$ we have $x_\lambda \in U$).

Definition 3.8 Let (X, τ) be a topological space. A **sub-basis** for the topology is a collection $\sigma \subset \mathcal{P}(X)$ such that τ is the union of finite intersections of sets in σ.

Observation. Let X be a set and for each $\gamma \in \Gamma$ let f_γ be a map from X to the topological space (X_γ, τ_γ). There is a unique weakest topology τ that makes the maps $\{f_\gamma : \gamma \in \Gamma\}$ continuous. A sub-basis for this topology is given by:

$$\sigma = \left\{ f_\gamma^{-1}(U_\gamma) : U_\gamma \subset X_\gamma \text{ is open in } \tau_\gamma \right\}$$

Definition 3.9 Let $\mathcal{F} = \{f_\gamma : \gamma \in \Gamma\}$ be as above. Then we denote $\sigma(X, \mathcal{F})$ to be **the weak topology generated by $\mathcal{F}$.**

Remark 3.10 A set $U \subset X$ is open if and only if for every $x \in U$ there are indices $\gamma_1, \ldots, \gamma_n \in \Gamma$ and $U_1 \in \tau_1, \ldots, U_n \in \tau_n$ such that

$$x \in \bigcap_{i=1}^{n} f_{\gamma_i}^{-1}(U_{\gamma_i}) \subset U$$

Definition 3.11 Given a normed vector space X with dual X^*, the **weak topology** on X is $\sigma(X, X^*)$.

Observation. With the above observation, we say that $U \subset U$ is open in the weak topology iff for every $x \in U$ there are bounded functionals $f_1, \ldots, f_n$ and positive reals $\epsilon_1, \ldots, , \epsilon_n$ such that

$$\{y \in U : |f_i(x) - f_i(y)| < \epsilon_i\} \subset U$$

Definition 3.12 Given a normed space X with dual X^*, the **weak $*$-topology** on X^* is the weak topology generated by the elements $\hat{x} \in X^{**}$, for $x \in X$.

Observation. A set $G \subset X^*$ is open in the weak-star topology iff for every $g \in G$ there are points $x_1, \ldots, x_n \in X$ and positive reals $\epsilon_1, \ldots, \epsilon_n$ such that

$$\{f \in X^* : |f(x_i) - g(x_i) < \epsilon_i|\} \subset G$$

Definition 3.13 Let X be a normed space, X^* its dual, and X^{**} its double dual. A net (x_λ) in X is said to **converge weakly** to x if $f(x_\lambda) \to f(x)$ for all $f \in X^*$. A net (f_λ) in X^* **converges weak$-*$** iff $f_\alpha(x) \to f(x)$ (pointwise convergence).

We coalesce all of these facts to prove a happy result.

Theorem 3.14 ***(Banach-Alaoglu's Theorem)*** *If X is a normed vector space, the closed unit ball $B^* = \{f \in X^* : \|f\| \leq 1\}$ in X^* is compact in the weak$-*$ topology.*

Proof. For each $x \in X$, let $D_x = \{z \in \mathbb{C} : |z| \leq \|x\|\}$. Being closed and bounded in $\mathbb{C}$, D_x is compact and, via Tychonoff, so is $D = \prod_{x \in X} D_x$. We ask: what is the relationship between B^* and D? Well, D, being endowed with the product topology, can be identified as the complex-valued functions ϕ on X with the property that $|\phi(x)| \leq \|x\|$. The set B^* is precisely those above which are linear, so B^* sits in D. But B^* is closed: indeed, given a net (f_λ) in B^* converging to f in D, we have

$$f(ax+by) = \lim_\lambda f_\lambda(ax+by) = \lim_\lambda (af_\lambda(x) + bf_\lambda(y)) = af(x) + bf(y)$$

so f is linear and thus $f \in B^*$. ∎

Back to Theorem 3.3. For now we shall accept the following lemma as a fact.

Lemma 3.15 *For a locally compact group, TFAE:*

1. *G is amenable*
2. *There is a net $(m_\lambda)_\lambda$ of non-negative functions of norm one in $L^1(G)$ such that*

$$\|\delta_x * m_\lambda - m_\lambda\|_1 \to 0 \qquad (x \in G)$$

Equipped with this we may tackle (2) $\implies$ (3) from Theorem 3.3.

Theorem 3.3 *For a locally compact group G, the following are equivalent:*

1. *G is amenable*
2. *There is a right invariant mean on $L^\infty(G)$*
3. *There is an invariant mean on $L^\infty(G)$*

Proof. $(1), (2) \implies (3)$. Suppose (1) and, equivalently, (2) hold. Let (m_α) be a net of norm one functions with the above property and let (m'_β) be a net of the similar form for the right version:

$$\|m_\beta * \delta_x - m_\beta\|_1 \to 0 \qquad (x \in G)$$ ∎

Then any weak$-*$ accumulation point of $(m_\alpha * m'_\beta)_{\alpha,\beta}$ in $L^\infty(G)^*$ is an invariant mean on $L^\infty(G)$.