Examples of Amenable Groups

3. In our last meeting...

Recall from last time the following problems:

Exercise 3.16 Let X be a Banach space which is known to have a pre-dual. Is the pre-dual unique?

No. We claim that $\ell^1(\mathbb{N})$ does not admit a unique pre-dual. The Riesz Representation Theorem (a nuclear bomb) gives us

$$(c_0)^* = (C_0(\mathbb{N}))^* \cong M(\mathbb{N}) \cong L^1(\mathbb{N}) = \ell^1(\mathbb{N})$$

Now, let us define another space:

$$c = \left\{(x_n)_{n=1}^\infty \in \ell^\infty \,:\, \lim_{n\to\infty} x_n \in \mathbb{R}\right\}$$

It is pretty evident that this is a closed subspace of $\ell^\infty(\mathbb{N})$ when it inherits its norm. It is fairly evident that the mapping $L : c \to \mathbb{R}$ is a bounded linear functional. Furthermore, given $y = (y_n)_{n=1}^\infty \in \ell^1$, the map $f_y : c \to \mathbb{R}$ given by

$$f_y(x) = y_1 L(x) + \sum_{n=1}^\infty x_n y_{n+1}$$

is a bounded linear functional with $\|f_y\| = \|y\|$. The triangle inequality gives us that $\|f_y\| \le \|y\|$. Given $\epsilon > 0$, setting $x_n = \operatorname{sgn}(y_n)$ for $1 \le n \le N$ for N large and $x_n = \operatorname{sgn}(y_1)$ for $n > N$ achieves:

$$|f_y(x)| \ge \|y\|_1 - \epsilon$$

To show the mapping $y \mapsto f_y$ is surjective observe that $c = \overline{\operatorname{span}}(\{1\} \cup \{e_n : n \in \mathbb{N}\})$. Since a continuous function is determined on a dense set, given a functional $f \in c$ we can let $y_n = f(e_n)$ and $y_\infty = f(1)$. Since f is bounded, letting $y = (y_\infty, y_1, y_2, \ldots)$ gives $y \in \ell^1$ (surjectivity).

Lastly, c and c_0 are not isometrically isomorphic. This is because the unit ball of c_0 has no extreme points, while the unit ball of c has many. To show the first statement, let $x \in c$, so that $x_n \to 0$. Pick N so large that $|x_n| < 0.5$ for all $n > N$. Let $y, z \in c$ such that $y_n = z_n = x_n$ for $1 \le n \le N$ and $y_n = x_n + 2^{-n}$ and $z_n = x_n - 2^{-n}$ for $n > N$. Both of these are in the unit ball with their average equal to x. The fact that c has many extreme points follows since $1 \in c$ is an extreme point.

Bessaga-Pelczynski, Mazurkiewicz-Sierpinski. More is true: in fact $\ell^1(\mathbb{N})$ admits a ton of pre-duals. It is known that X is a countable compact Hausdorff space, then X is homeomorphic to a closed ordinal interval $[0, a]$, with its natural order topology. Sixty years ago, Bessaga and Pelczynski shows that if a and b are infinite countable ordinals and $a < b$ then $C([0, a])$ is isomorphic to $C([0, b])$ if and only if $b < a^{*\omega*}$, where ω is the first infinite ordinal. If we combine these two results, we get that for a countable compact space X, $C(X)$ is isomorphic to $C([0, \omega^{\omega^{*a*}}])$. But the domains of all these spaces are compact, so that the Riesz representation theorem gives us:

$$C(X)^* \cong M(X) \cong \ell^1(\mathbb{N})$$

There are more preduals of ℓ^1, not of the form $C(X)$, but I shall say no more about them. I will remark, however, that each of the weak-$*$ topologies is distinct, so it makes no sense to speak of **the** weak-$*$ topology; however, when we say that it is because the dual pair is understood. To exemplify this we remark the following:

Let $(x_n) \subset \ell^1$. Then $\langle x_n, y\rangle \to 0$ for all $y \in c_0$ if and only if $(x_n)_{n=1}^{\infty}$ is bounded and $\lim_{n\to\infty} x_i^{(n)} = 0$ for every i.

The above result was known to me (see Bollobas Ch. 8), and from this I built the following example show show that the weak$-*$ topologies generated by c and c_0 disagree. Let $x_n = e_n$; then by the conditions given above, $x_n \to 0$ in $\sigma(\ell^1, c_0)$. However, the evaluation $\langle x_n, 1\rangle = 1$ for all n where $1 \in c$ is the constant unit sequence, showing that $x_n \not\to 0$ in $\sigma(\ell^1, c)$.

I am unaware as to how to construct the disagreements in topologies (if any) with respect to the $C(K)$ spaces introduced above.

4. Examples of Amenable Groups

We first recall a few examples of amenable and non-amenable groups.

Example 4.1 $\mathbb{F}_2$, the free group on two generators, is not amenable. This is a consequence of the Banach-Tarski paradox and Tarski's theorem.

Example 4.2 Any compact group G is amenable. If G is compact, $L^\infty(G) \subset L^1(G)$. Indeed, integration against a normalised left Haar measure provides a left-invariant mean on $L^\infty(G)$, so that G is amenable.

Example 4.3 Locally compact abelian groups are amenable.

Proof. To see this, let K be the space of all means on $L^\infty(G)$. We claim that K is weak$-*$ compact and convex. Indeed, K is a subset of the unit ball; furthermore, it is closed. Take any converging net (m_α) in K, then evaluating this net at 1 yields the constant net 1 and certainly the limit point must have norm 1. Since K is closed within a compact set (Banach-Alaouglu), it is itself compact.

For $x \in G$ let $T_x : L^\infty(G)^* \to L^\infty(G)^*$ be the adjoint of L_x[1]; then $T_x(K) \subseteq K$ and T_x is weak$-*$ continuous. Furthermore, $T_{xy} = T_x T_y$, for $x, y \in G$. By the Markov-Kakutani Fixed Point Theorem[2], there is a $M \in K$ such that $T_x M = M$ for all $x \in G$, so that M is an invariant mean on $L^\infty(G)$. ■

More is true: abelian groups are amenable, but this is a bit harder.

Now, we specialise to countable groups for the remainder of this talk. In particular, we shall use Folner's characterisation to exhibit more examples of amenable groups. Let us equip ourselves with one more definition:

Definition 4.4 A **finite mean** is a non-negative, finitely supported function $\mu : G \to \mathbb{R}^+$ such that $\|\mu\|_{\ell^1(G)} = 1$.

Remark 4.5 Every finite mean can be viewed as a mean M_μ via the formula $M_\mu(f) = \langle f, \mu\rangle$

Theorem 4.6 *Let G be a countable discrete group. Then the following are equivalent:*

1. *G is amenable.*
2. *For every finite set $S \subset G$ and every $\epsilon > 0$, there is a finite mean ν such that $\|\nu - L_x\nu\|_{\ell^1(G)} \leq \epsilon$ for all $x \in S$.*
3. *For every finite set $S \subset G$ and every $\epsilon > 0$, there is a non-empty finite set $A \subset G$ such that*

[1] The unique map $T^* : Y^* \to X^*$ such that for $T \in B(X,Y)$, $\langle x, T^* g\rangle = \langle Tx, g\rangle$.

[2] Let K be a non-empty compact convex subset of a normed space X, and let $\mathcal{F}$ be a commuting family of continuous affine maps on X such that $T(K) \subset K$ for all $T \in K$. Then some $x_0 \in K$ is a fixed point of all $T \in \mathcal{F}$.

$\frac{|(x \cdot A)\Delta A|}{|A|} \leq \epsilon$ *for all* $x \in S$.

4. *(Følner sequence) There exists a sequence* A_n *of non-empty finite sets such that*

$$\frac{|(x \cdot A_n)\Delta A_n|}{|A_n|} \to 0 \qquad \text{as } n \to \infty$$

Proof. We exhibit an argument presented in a paper by Namioka[3], and borrow liberally from Tao's brilliant exposition of it.

(1) $\implies$ (2) Let us argue by contradiction. The negation of (2) is: there exist S finite and $\epsilon > 0$ such that $\sup_{x \in S} \|\nu - L_x \nu\|_{\ell^1(G)}$ for all means ν. The set

$$\left\{(\nu - L_x\nu)_{x \in S} : \nu \in \ell^1(G)\right\}$$

is now convex and bounded subset of $\ell^1(G)^S$ which lives away from zero. The Hahn-Banach separation theorem then gives us a linear functional $\rho \in (\ell^1(G)^S)^*$ such that $\rho((\nu - L_x\nu)_{x \in S}) \geq 1$ for all means ν. Since $(\ell^1(G)^S)^* \cong \ell^\infty(G)^S$, for $x \in S$ there exist $m_x \in \ell^\infty(G)$ such that $\sum_{x \in S} \langle \nu - \delta_x * \nu, m_x \rangle \geq 1$, and so $\langle \nu, \sum_{x \in S} m_x - L_{x^{-1}} m_x \rangle \geq 1$. Letting $\nu = \delta_y$, we get that $\sum_{x \in S} m_x - L_{x^{-1}} m_x \geq 1$, pointwise. Since G is assumed to be amenable, we may apply a left-invariant mean M to get

$$\sum_{x \in S} M(m_x) - M(L_{x^{-1}} m_x) \geq 1$$

a contradiction to the left invariance of M.

(2) $\implies$ (3) Fix $S \neq \emptyset$ and let $\epsilon > 0$ be small (we shall say how small in a bit). By assumption, get a finite mean ν with

$$\|\nu - L_x\nu\|_{\ell^1(G)} < \frac{\epsilon}{|S|}$$

for all $x \in S$. Write, via a layer cake decomposition, $\nu = \sum_{i=1}^k c_i \chi_{E_i}$ for nested, non-empty sets $E_1 \supseteq E_2 \supseteq \ldots \supseteq E_k$ and $c_1, \ldots, c_k \in \mathbb{R}^+$. Since ν is a mean, $\sum_{i=1}^k c_i |E_i| = 1$. On the other hand, observe that $|\nu - \text{Ł}_x \nu|$ is at least c_i on $(x \cdot E_i)\Delta E_i$, allowing us to conclude that

$$\sum_{i=1}^k c_i |(x \cdot E_i)\Delta E_i| \leq \frac{\epsilon}{|S|} \sum_{i=1}^k c_i |E_i| \qquad \forall x \in S$$

$$\sum_{i=1}^k c_i \sum_{x \in S} |(x \cdot E_i)\Delta E_i| \leq \epsilon \sum_{i=1}^k c_i |E_i| \qquad \text{(counting)}$$

The pidgeonhole principle implies that there exists an index i such that

$$\sum_{x \in S} |(x \cdot E_i)\Delta E_i| \leq \epsilon |E_i|$$

which proves the claim.

(3) $\implies$ (4) Write G as the increasing union of finite sets S_n and apply (3) with $\epsilon = \frac{1}{n}$ and $S = S_n$ to create A_n.

(4) $\implies$ (1) By the Hahn-Banach Theorem, select an infinite mean $\rho \in \ell^\infty(\mathbb{N})^* \setminus \ell^1(\mathbb{N})$ and define

$$M(m) = \rho\left(\left(\langle m, \frac{1}{|A_n|}\chi_{A_n}\rangle\right)_{n \in \mathbb{N}}\right)$$

This mean is left-invariant. ■

[3] Namioka, I. Følner's conditions for amenable semi-groups. Math. Scand. 15 1964 18–28.

Equipped with Følner's condition, we can exhibit a few examples.

Example 4.7 Every finite group is amenable. The normalised counting measure achieves this.

Example 4.8 The integers $\mathbb{Z} = (\mathbb{Z}, +)$ are amenable. We can let the sets $A_N = \{1, 2, \ldots, N\}$ determine a Følner sequence.

Example 4.9 $\mathbb{F}_2$ is not amenable. Of course, we already know this from Tarski's theorem. Tarski's theorem, however, is hard. Følner's characterisation provides an easier proof.

Proof. Arguing by contradiction, suppose $\mathbb{F}_2$ were amenable. For any $\epsilon > 0$, we may find a non-empty finite set K such that $x \cdot K$ differs from K by at most $\epsilon|K|$ points for $x \in \{a, b, a^{-1}, b^{-1}\}$. Observe that

$$a \cdot \big(K \cap (W(b) \cup W(a^{-1}) \cup W(b^{-1}))\big) \subseteq a \cdot K(\text{and } W(a))$$

So we may count:

$$\begin{aligned} |a \cdot (K \setminus W(a))| &\leq |K \cap W(a)| + \epsilon|K| \\ |K| - |K \cap W(a)| &\leq |K \cap W(a)| + \epsilon|K| \end{aligned} \tag{$\dagger$}$$

We may do this for all four permutations, sum them up and obtain:

$$4|K| - |K| \leq |K| + 4\epsilon|K|$$

which is a contradiction if we pick $\epsilon < 0.5$. ∎

Let us construct amenable groups from amenable groups.

Example 4.10 Let $G_1 \subset G_2 \subset G_3 \subset \ldots$ be a sequence of countable amenable groups. Then $G = \bigcup_n G_n$ is amenable.

Proof. There is an invariant means argument, which requires ultralimits. We argue via Følner sequences. Given any finite set $S \subset G$ and $\epsilon > 0$, we have that $S \subset G_n$ for some n. Since G_n is amenable, it admits a set $A \subset G_n$ such that $|(x \cdot A)\Delta A| \leq \epsilon|A|$ for all $x \in S$. Victory! ∎