PMATH 450 - Lebesgue Integration and Fourier Analysis

Fantastic Measures and How to Construct Them

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Part I

Introduction

Chapter 1

Motivation

We motivate the subject by asking for a way to measure the "size" of a set $A \subseteq \mathbb{R}$. We may intuitively say that for any interval $A = (a, b)$, or $(a, b]$, or $[a, b)$, or $[a, b]$ is $\lambda(A) = b - a$.

Interestingly enough, we can say that for $\mathbb{Q}$ or any countable set, we have $\lambda(\mathbb{Q}) = 0$ by centering balls of radius $\frac{\epsilon}{2^n}$ for the n-th enumerated element for arbitrary ϵ.

Using these concepts, we can then develop a notion of integration more general than the Riemann integral. This type of integral is better behaved with respect to limits.

On later chapters, we discover that certain equivalence classes, called the L_p spaces are complete with respect to their usual norm. These are interesting as Banach spaces, and, when $p = 2$, Hilbert spaces.

The course acquires a linear algebraic flavour when we talk more about Hilbert spaces, linear functionals, and Fourier series. These tend to have tons of applications, which is why those of us in Pure Mathematics may have taken more time to arrive at them than physicists.

Part II

Lebesgue Measure

Chapter 2

Constructing the Lebesgue Measure

Definition 2.1. When $a, b \in \mathbb{R}$ with $a \leq b$ and $I = (a, b)$, or $(a, b]$, or $[a, b)$, or $[a, b]$, we define $|I| = b - a$. When I is an unbounded interval, we say $|I| = \infty$.

Definition 2.2. When $A \subseteq \mathbb{R}$ is bounded, we define the **outer Jordan content** of A to be

$$c^*(A) = \inf \left\{ \sum_{i=1}^{n} |I_k| \,:\, \text{each } I_k \text{ is a bounded open interval and their union covers } A \right\}$$

Theorem 2.3. *These are properties of the outer Jordan content. Let $A, B, A_1, \ldots, A_n \subseteq \mathbb{R}$ be bounded. Then*

1. *(Translation invariance) For $b \in \mathbb{R}$, we have $c^*(b + A) = c^*(A)$ where $b + A = \{b + x \,:\, x \in A\}$.*
2. *(Scaling) For $r \in \mathbb{R}$, $c^*(rA) = rc^*(A)$ where $rA = \{rx \,:\, x \in A\}$*
3. *(Monotonicity) If $A \subseteq B$, then $c^*(A) \leq c^*(B)$.*
4. *If A is finite, $c^*(A) = 0$.*
5. *When I is a bounded interval $c^*(I) = |I|$.*
6. *(Finite subadditivity) $c^*\left(\bigcup_{k=1}^{n} A_k\right) \leq \sum_{i=1}^{n} c^*(A_k)$*
7. *$c^*(\overline{A}) = c^*(A)$*

Definition 2.4. For a bounded set $A \subseteq \mathbb{R}$, we say that A has **well-defined Jordan content** when $c^*(A) = |I| - c^*(I \setminus A)$ for some bounded interval I with $A \subseteq I$; or, equivalently, when $|I| = c^*(I \cap A) + c^*(I \setminus A)$.

In the case A has a well defined Jordan content, we drop the start and define the **Jordan content** to be $c(A) = c^*(A)$.

Theorem 2.5. *(Properties of Jordan content) Let $A, B, A_k \subseteq \mathbb{R}$ be bounded. Then,*

1. *(Translation invariance) For all $b \in \mathbb{R}$, A has well-defined Jordan content if and only if $b + A$ does. In this case $c(b + A) = c(A)$*
2. *(Scaling property) For all $0 \neq r \in \mathbb{R}$, A has well-defined Jordan content if and only if rA does. In this case, $c(rA) = rc(A)$*
3. *(Finiteness) If A is finite, then A has well-defined Jordan content (and is 0).*
4. *If $c^*(A) = 0$ then A has well-defined Jordan content and $c(A) = 0$*
5. *If A is a bounded interval, then A has a well-defined Jordan content (and the Jordan content is the length of the interval).*
6. *If A and B have well defined Jordan content, then so do $A \cup B$, $A \cap B$, and $A \setminus B$*

7. *(Finite additivity) If $A_1, A_2, \ldots, A_n$ are disjoint and all have well-defined Jordan content, then*

$$c\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} c(A_k)$$

8. *A has well defined Jordan content if and only if $c^*(\partial A) = 0$.*

Definition 2.6. Let $A \subseteq \mathbb{R}$. We define the **outer Lebesgue measure** of A to be

$$\lambda^*(A) = \inf\left\{\sum_{i=1}^{\infty} |I_k| \, : \, \text{each } I_k \text{ is a bounded open interval and their union covers } A\right\} \in [0, \infty]$$

Theorem 2.7. *(Properties of Outer Measure) Let $A, B, A_k \subseteq \mathbb{R}$. Then,*

1. *The outer Lebesgue measure is translation invariant.*
2. *For all $r \in \mathbb{R}$, $\lambda^*(rA) = |r|\lambda^*(A)$.*
3. *(Monotonicity) If $A \subseteq B$, then $\lambda^*(A) \leq \lambda^*(B)$*
4. *If A is finite or countable, then $\lambda^*(A) = 0$.*
5. *If A is an interval, its outer Lebesgue measure is its length.*
6. *(Countable subadditivity) For any sequence $\{A_k\}$ we have*

$$\lambda^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \lambda^*(A_k)$$

Proof. We prove assertion 4. Let $\epsilon > 0$. Denumerate $A = \{a_1, a_2, a_3, \ldots\}$. For each index k, pick the interval I_k centred at k of positive radius strictly less than $\frac{\epsilon}{2^k}$. Then, $A \subseteq \bigcup_{k=1}^{\infty} I_k$, so by monotonicity we have

$$\lambda^*(A) \leq \lambda^*\left(\bigcup_{k=1}^{\infty} I_k\right) \leq \sum_{k=1}^{\infty} |I_k| < \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon$$

Now we prove assertion 5.

Let $\epsilon > 0$ and let A be an interval with end points $a < b$. Pick $I_1 = \left(a - \frac{\epsilon}{2}, b + \epsilon b2\right)$ so that $A \subseteq I_1$ and $|I_1| = |A| + \epsilon$ and for $k \geq 2$ let $I_k = \emptyset$. Then A is contained in the union of the I_k (which is just I_1) and $\lambda^*(A) \leq |A| + \epsilon$, by monotonicity, which implies that $\lambda^*(A) \leq |A|$.

Conversely, let $I_1, I_2, \ldots$ be any sequence of bounded open intervals that covers A and call the collection $\mathcal{U}$. Let $0 < \epsilon\frac{b-a}{2}$ and note that $K = [a + \epsilon, b - \epsilon] \subseteq A$ is compact. Since $\mathcal{U}$ is an open cover for A, it is also an open cover for K, which is compact, so that $\mathcal{U}$ admits a finite subcover for K, say $\mathcal{V}$. We can pick $J_1 \in \mathcal{V}$ so that $a + \epsilon \in J_1$, say $J_1 = (a_1, b_1)$ (with $a_1 < a + \epsilon < b_1$).If $b_1 > b + \epsilon$, we are done; otherwise, pick $J_2 \in \mathcal{V}$ with $J_2 = (a_2, b_2)$ so that $a_2 < b_1 < b_2$. Continue inductively with $J_k = (a_k, b_k)$. This procedure stops because $\mathcal{V}$ is finite. Then $\{J_1, \ldots, J_m\}$ covers K. We have the following sum:

$$\begin{aligned}
\lambda^*(A) \geq \sum_{k=1}^{m} |J_k| &= (b_1 - a_1) + \ldots + (b_m - a_m) \\
&> (b_1 - (a + \epsilon)) + (b_2 - b_1) + \ldots + (b - \epsilon - b_m) \\
&= (b - \epsilon) - (a + \epsilon) \\
&= (b - a) - 2\epsilon \\
&= |A| - 2\epsilon
\end{aligned}$$

and taking infimums (over all possible choices of covers) we can conclude that $\lambda^*(A) \geq |A|$[1], from which it follows that $\lambda^*(A) = |A|$.

When A is an unbounded interval, we can choose bounded intervals $I_n \subseteq A$ with $|I_n| \geq n$ for all $n \in \mathbb{Z}^+$. Since $I_n \subseteq A$, we have

$$\lambda^*(A) \geq \lambda^*(I_n) = |I_n| \geq n$$

We prove countable subadditivity.

Let$\epsilon > 0$ and $A_1, A_2, \ldots \subseteq \mathbb{R}$. For each A_n, choose a family $\{I_n^k\}_{k=1}^{\infty}$ of bounded open intervals so that

$$A_n \subseteq \bigcup_{k=1}^{\infty} I_n^k \qquad \text{and} \sum_{k=1}^{\infty} |I_n^k| \leq \lambda^*(A_n) + \frac{\epsilon}{2^n}$$

Then,

$$\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} \left(\bigcup_{k=1}^{\infty} I_n^k \right)$$

so that

$$\begin{aligned} \lambda^* \left(\bigcup_{n=1}^{\infty} A_n \right) &\leq \sum_{n,k} |I_n^k| \\ &\leq \sum_{n=1}^{\infty} \left(\lambda^*(A_n) + \frac{\epsilon}{2^n} \right) \\ &= \left(\sum_{n=1}^{\infty} \lambda^*(A_n) \right) + \epsilon \end{aligned}$$

and since $\epsilon > 0$ was arbitrary, we are done. ■

Definition 2.8. Let $A \subseteq \mathbb{R}$. We say that A is **Lebesgue measurable** when

$$\lambda^*(X) = \lambda^*(X \cap A) + \lambda^*(X \setminus A)$$

for all $X \subseteq \mathbb{R}$. When A is measurable, we denote the **Lebesgue measure** of A as $\lambda(A) = \lambda^*(A)$.

Remark. Notice that by subadditivity, since $X = (X \cap A) \cup (X \setminus A)$, we have

$$\lambda^*(X) \leq \lambda^*(X \cap A) + \lambda^*(X \setminus A)$$

for all subsets of the real line. Thus we only have to prove one side of the inequality in the definition above.

Theorem 2.9. *(Properties of the Lebesgue measure). Let $A, B, A_k \subseteq \mathbb{R}$. Then:*

1. *For all $a, b \in \mathbb{R}$, $b + A$ is measurable if and only if A is measurable. In this case $\lambda(b + A) = \lambda(A)$.*
2. *For all $0 \neq r \in \mathbb{R}$, rA is measurable if and only if A is measurable. In this case, $\lambda(rA) = |r|\lambda(A)$.*
3. *If $\lambda^*(A) = 0$, (in particular, if A is finite our countable), then A is measurable. In this case $\lambda(A) = 0$.*
4. *Both $\emptyset$ and $\mathbb{R}$ are measurable, with $\lambda(\emptyset)$ and $\lambda(\mathbb{R}) = \infty$.*
5. *If A is measurable, then so is $A^c = \mathbb{R} \setminus A$.*

[1] Maybe my typesetting got sloppy here, but a full proof can be found in: `https://planetmath.org/proofthattheouterlebesguemeasureofanintervalisitslength`.

6. *If A and B are measurable, then so are $A \cup B$, $A \cap B$, and $A \setminus B$. (By induction, so are the finite unions)*

7. *If $A_1, A_2, \ldots$ are all measurable, then so is $\bigcup_{k=1}^{\infty} A_k$*

8. *(Countable additivity) If $A_1, A_2, \ldots$ are all measurable and disjoint, then*

$$\lambda\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \lambda(A_k)$$

9. *If A is an interval, then A is measurable (with $\lambda(A) = |A|$).*

Proof. Parts 1 and 2 are left as exercise. Part 3 goes as follows. Suppose $\lambda^*(A) = 0$ and $X \subseteq \mathbb{R}$. Then,

$$\lambda^*(X \cap A) + \lambda^*(X \setminus A) \leq \lambda^*(A) + \lambda^*(X) = 0 + \lambda^*(X) = \lambda^*(X)$$

so, by definition, the proof is done.

Part 4 follows directly from the definition.

For part 5, note that we prove the measurability of A^c by computing

$$\lambda^*(X \cap A^c) + \lambda^*(X \setminus A^c) = \lambda^*(X \setminus A) + \lambda^*(X \cap A) = \lambda^*(X)$$

where the last equality follows since A was measurable.

We prove parts $6, 7$, and 8 together. Suppose $A, B \subseteq \mathbb{R}$ are disjoint and measurable and let $X \subseteq \mathbb{R}$. Then,

$$\begin{aligned}
\lambda^*(X \cap (A \cup B)) &= \lambda^*((X \cap (A \cup B)) \cap A) + \lambda^*((X \cap (A \cup B)) \setminus A) && \text{since } A \text{ is measurable} \\
&= \lambda^*(X \cap A) + \lambda^*(X \cap B) && \text{since } A \text{ and } B \text{ are disjoint}
\end{aligned}$$

since A is measurable. By induction, we can extend this to the finite case. For the countable case, suppose $A_1, A_2, \ldots \subseteq \mathbb{R}$ are disjoint and measurable, and let $X \subseteq \mathbb{R}$ be arbitrary. For all $n \in \mathbb{Z}^+$ we have

$$\begin{aligned}
\sum_{k=1}^{n} \lambda^*(X \cap A_k) &= \lambda^*\left(X \cap \bigcup_{k=1}^{n} A_k\right) \\
&= \lambda^*\left(\bigcup_{k=1}^{n} (X \cap A_k)\right) \\
&\leq \lambda^*\left(\bigcup_{k=1}^{\infty} (X \cap A_k)\right)
\end{aligned}$$

by subadditivity of λ^* we obtain the opposite inequality, thus obtaining equality. Letting $X = \mathbb{R}$, we get the formula

$$\lambda^*\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \lambda^*(A_k)$$

Still assuming that the A_k are disjoint and measurable, we get

$$\begin{aligned}
\lambda^*(X) &= \lambda^*\left(X \cap \bigcup_{k=1}^{n} A_k\right) + \lambda^*\left(X \setminus \bigcup_{k=1}^{n} A_k\right) \\
&\geq \lambda^*\left(X \cap \bigcup_{k=1}^{n} A_k\right) + \lambda^*\left(X \setminus \bigcup_{k=1}^{\infty} A_k\right) && \text{since } X \setminus \bigcup_{k=1}^{n} A_k \subseteq X \setminus \bigcup_{k=1}^{\infty} A_k \\
&= \sum_{k=1}^{n} \lambda^*(X \cap A_k) + \lambda^*\left(X \setminus \bigcup_{k=1}^{\infty} A_k\right)
\end{aligned}$$

We can take the limit as $n \to \infty$ to get

$$\begin{aligned}\lambda^*(X) &\geq \sum_{k=1}^{\infty} \lambda^*(X \cap A_k) + \lambda^*\left(X \setminus \bigcup_{k=1}^{\infty} A_k\right) \\ &= \lambda^*\left(X \cap \bigcup_{k=1}^{\infty} A_k\right) + \lambda^*\left(X \setminus \bigcup_{k=1}^{\infty} A_k\right)\end{aligned}$$

thus proving that the countable union of disjoint measurable sets is measurable. If the A_k were not assumed to be disjoint, then we can write $B_1 = A_1$ and $B_{n+1} = A_{n+1} \setminus \bigcup_{k=1}^{n} A_k$ and obtain the desired result.

For the last part, let $A = (a, b)$ with $a < b$ be a bounded open set. Let $X \subseteq \mathbb{R}$ be arbitrary and $\epsilon > 0$, and let us cover A by open intervals $I_1, I_2, \ldots$ such that

$$A \subseteq \bigcup_{k=1}^{\infty} I_k \qquad \text{and} \qquad \sum_{k=1}^{\infty} |I_k| < \lambda^*(A) + \epsilon$$

For each $n \in \mathbb{Z}^+$, let $J_k = I_k \cap A = I_k \cap (a, b)$. Likewise, let $L_k = I_k \cap (-\infty, a)$ and $M_k = I_k \cap (b, \infty)$, all of which are bounded open intervals. We

$$X \cap A \subseteq \bigcup_{k=1}^{\infty} \qquad \text{and} \qquad X \setminus A \subseteq \bigcup_{k=1}^{\infty} L_k \cup \bigcup_{k=1}^{\infty} M_k \cup (a - \epsilon, a + \epsilon) \cup (b - \epsilon, b + \epsilon)$$

and so,

$$\lambda^*(X \cap A) \leq \sum_{k=1}^{\infty} |J_k|$$

and,

$$\lambda^*(X \setminus A) \leq \sum_{k=1}^{\infty} |L_k| + \sum_{k=1}^{\infty} |M_k| + 4\epsilon$$

Thus,

$$\lambda^*(X \cap A) + \lambda^*(X \setminus A) \leq \sum_{k=1}^{\infty} (|J_k| + |L_k| + |M_k|) + 4\epsilon \leq \sum_{i=1}^{\infty} |I_k| + 4\epsilon < \lambda^*(X) + 5\epsilon$$

because the J_k, L_k, M_k cover the I_k, perhaps with the exception of a and b, which do not matter as they have measure zero. We remark that this is sufficient because all the other intervals can be formed by either adding the necessary endpoints, or simply using countable set operations. ■

Theorem 2.10. *Let $A_1, A_2, \ldots \subseteq \mathbb{R}$ be measurable. Then,*

1. *If $A_1 \subseteq A_2 \subseteq \ldots$ then $\lambda\left(\bigcup_{n=1}^{\infty}\right) = \lim_{n\to\infty} \lambda(A_n)$.*
2. *If $A_1 \supseteq A_2 \supseteq \ldots$ and $\lambda(A_n) < \infty$ for some $m \in \mathbb{Z}^+$, then $\lambda\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \lambda(A_n)$*

Proof. We use the classic trick of converting unions to disjoint unions for the first part. For more details, see my PMATH 451 notes. ■

Theorem 2.11. *Open and closed sets in $\mathbb{R}$ are measurable.*

Proof. We can generate these via countable set operations from the open bounded intervals. In $\mathbb{R}$ we may argue via the open connected components, using the fact that $\mathbb{R}$ is separable. ■

Definition 2.12. This definition is actually a notational remark. Let $C \subseteq S$. Then,

$$\begin{aligned}C_\sigma &= \left\{\bigcup_{n=1}^{\infty} A_n \,:\, A_n \in C\right\} \\ C_\delta &= \left\{\bigcap_{n=1}^{\infty} A_n \,:\, A_n \in C\right\}\end{aligned}$$

Remark. Note that $C \subseteq C_\sigma$, $C \subseteq C_\delta$, $C_{\sigma\sigma} = C_\sigma$, and $C_{\delta\delta} = C_\delta$.

Definition 2.13. A **σ-algebra** in a set S is a collection $\mathcal{M}$ such that:

1. $\emptyset \in \mathcal{M}$.
2. If $A \in \mathcal{M}$, then $A^c = S \setminus A \in \mathcal{M}$.
3. If $A_1, A_2, \ldots \in \mathcal{M}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$.

Remark. Note that if $\mathcal{M}$ is a σ-algebra, then $\mathcal{M}_\sigma = \mathcal{M}_\delta = \mathcal{M}$.

Definition 2.14. This is a notational remark. Let X be a metric space (or topological space). Then we denote the set of all open sets by $\mathcal{G}$ and the set of all closed sets by $\mathcal{F}$.

Definition 2.15. When X is a metric space (or topological space), the **Borel σ-algebra** in X, denoted by $\mathcal{B}$, is the smallest σ-algebra in X which contains $\mathcal{G}$ (and hence also $\mathcal{F}$). This σ-algebra is smallest in the sense that it is the intersection of all σ-algebras containing $\mathcal{G}$ and $\mathcal{F}$.

Theorem 2.16. *All Borel sets in $\mathbb{R}$ are (Lebesgue) measurable.*

Example 2.17. Let $A = [0,1] \cap \mathbb{Q} = \{a_1, a_2, \ldots\}$. Construct

$$I_{n,k} = \left(a_k - \frac{1}{2^{n+k+1}}, a_k + \frac{1}{2^{n+k+1}}\right)$$

and let $U_n = \bigcup_{k=1}^{\infty} I_{n,k}$. Let $B = \bigcap_{n=1}^{\infty} U_n$. Observe what happens and explain it!

Chapter 3

Applications and computations of the Lebesgue measure

3.1 Cantor sets

Example 3.1. (The standard Cantor set). The Cantor set $C \subseteq [0,1]$ can be constructed in the usual way:

$$\begin{aligned} C_0 &= [0,1] & & \\ C_1 &= C_0 \setminus I_1 & I_1 &= \left(\frac{1}{3}, \frac{2}{3}\right) \\ C_2 &= C_1 \setminus (I_2 \cup I_3) & I_2 = \left(\frac{1}{9}, \frac{2}{9}\right) \text{ and } I_3 &= \left(\frac{7}{9}, \frac{8}{9}\right) \\ &\vdots & &\vdots \end{aligned}$$

Having constructed C_n as the disjoint union of 2^n closed intervals, each of size $\frac{1}{3^n}$, and forming C_{n+1} by taking C_n and subtracting the middle thirds of the interval components of it. Finally, let $C = \bigcap_{n=1}^{\infty} C_n$.

Since $C_0 \supseteq C_1 \supseteq \ldots$, we have by continuity from above

$$\lambda(C) = \lim_{n\to\infty} \lambda(C_n) = \lim_{n\to\infty} \frac{2^n}{3^n} = 0$$

Observe that

$$C = \{(x)_3 = 0.x_1x_2x_3\ldots \in [0,1] \,:\, x_i \in \{0,2\}\}$$

that is: where x is the ternary representation of a number in the interval which does not contain the digit 1. Observe that with this definition, there is a natural bijection with $2^{\mathbb{N}}$ so that $|C| = |2^{\mathbb{N}}| = |R|$.

Example 3.2. (The generalised Cantor set) We can construct $C \subseteq [0,1]$ with $\lambda(C) = m$ for any $0 \leq m < 1$. Choose a sequence of positive real numbers $\{a_n\}_{n\geq}$ with $sum_{n=1}^{\infty} a_n = 1 - m$. Let I_1 be an open interval with $|I_1| = a_1$, so that $C_1 = C_0 \setminus I_1$ is the disjoint union of two closed non-degenerate intervals, each of size less that $\frac{1}{2}$. Thus we have $|C_2| = 1 - a_1 - a_2$.

Repeat the procedure so that C_k is the disjoint union of 2^k non-degenerate closed intervals, each of size $\leq \frac{1}{2^k}$ with $|C_k| = 1 - a_1 - a_2 - \ldots - a_k$. Then choose 2^k open intervals I_j with $2^k \leq 2^{k+1}$ with $\sum_{j=2^k}^{2^{k+1}-1}$ in each of the components of C_k, so that C_{k+1} is a disjoint union of 2^{k+1} disjoint closed intervals, each of size $\leq \frac{1}{2^{k+1}}$.

Finally, let $C = \bigcap_{n=1}^{\infty} C_n$, and then

$$\lambda(C) = \lim_{n\to\infty} \lambda(C_n) = 1 - \sum_{k=1}^{\infty} a_k = m$$

3.2 Baire Category Theorem and Measures

Definition 3.3. Let X be a metric space and $A \subseteq X$. We say that A is **dense** in X if $\overline{A} = X$.

Definition 3.4. Let X be a metric space and $A \subseteq X$. Then A is said to be **nowhere dense** if $\operatorname{int}(\overline{A}) = \emptyset$. Equivalently, A is nowhere dense if for exery open ball $B \subseteq X$, there exists an open ball $C \subseteq B \subseteq X$ such that $C \cap A = \emptyset$.

Remark. If A is nowhere dense, then $\operatorname{int}(A^c)$ is open and dense.

Definition 3.5. A set A is said to be of **first category** (or meagre) when A is equal to a countable union of nowhere dense sets.

Definition 3.6. A set A is said to be of **second category** when A is not first category.

Definition 3.7. A set A is said to be **residual** or (co-meagre) when $A^c = X \setminus A$ is first category.

Theorem 3.8. *(Baire Category Theorem) Let X be a complete metric space.*

1. *If A is first category, then A has empty interior.*
2. *If A is residual then A is dense.*
3. *If A is a countable union of closed sets with empty interior, then A has empty interior.*
4. *If A is a countable intersection of dense open sets then A is dense.*

Proof. Refer to PMATH 351 notes. ■

Remark. In some sense we can define a family of sets $\mathcal{C}$ which contains small sets which satisfies:

1. For all $A, B \subseteq \mathbb{R}$, if $A \subseteq B$ and $B \in \mathcal{C}$ then $A \in \mathcal{C}$,
2. $\mathcal{C}$ is closed under countable unions
3. For all $A \in \mathcal{C}$, then $\operatorname{int}(A) = \emptyset$.

It turns out $\mathcal{C}$ can be any of the: (i) at most countable sets, (ii) sets of Lebesgue measure zero, or (iii) sets of first category.

We formalise this thought into a theorem.

Theorem 3.9. *Every subset of $\mathbb{R}$ is the union of a set of measure zero and a set of first category.*

Proof. Denumerate $\mathbb{Q} = \{a_1, a_2, \ldots\}$. For $n, k \in \mathbb{Z}^+$, let

$$I_{n,k} = \left(a_k - \frac{1}{2^{n+k+1}}, a_k + \frac{1}{2^{n+k+1}}\right)$$

so that $a_k \in I_{n,k}$ and $|I_{n,k}| = \frac{1}{2^{n+k}}$. For $n \in \mathbb{Z}^+$ let

$$U_n = \bigcup_{k=1}^{\infty} I_{n,k}$$

so that each U_n is open with $\mathbb{Q} \subseteq U_n$ and

$$\lambda(U_n) \le \sum_{k=1}^{\infty} |I_{n,k}| = \frac{1}{2^n}$$

Let

$$B = \bigcap_{n=1}^{\infty} U_n$$

so that $\mathbb{Q} \subseteq B$ and

$$\lambda(B) = \lim_{n\to\infty} \lambda(U_n) = 0$$

so that B is a dense $\mathcal{G}_\delta$ set. Since $B^c = \bigcup_{n=1}^{\infty} U_n^c$, B^c is first category, so for any $A \subseteq \mathbb{R}$ we have $A = (A \cap B) \cup (A \cap B^c)$. Since $A \cap B \subseteq B$ we have $\lambda(A \cap B) = 0$. Since $A \cap B^c \subseteq B^c$, $A \cap B^c$ is first category. Both B and B^c cannot be first category, because that would contradict the Baire Category theorem. ■

Theorem 3.10. *There exists a non-measurable set $A \subseteq \mathbb{R}$.*

Proof. Define an equivalence relation $\sim$ on $[0,1]$ by declaring $x \sim y$ if and only if $x - y \in \mathbb{Q}$. For $x \in [0,1]$ write $[x] = \{y \in [0,1] : y \sim x\}$. Let $K = [0,1]/\sim$. For each $c \in K$, by the Axiom of Choice, choose $x_c \in c$. Let $A = \{x_c : c \in K\} \subseteq [0,1]$. Let $[0,2] \cap \mathbb{Q} = \{a_1, a_2, \ldots\}$ with the element a_k distinct. For each $k \in \mathbb{Z}^+$, let

$$A_k = a_k + A$$

We claim the sets A_k are disjoint. Suppose otherwise, and pick $y \in A_k \cap A_l \in \emptyset$. Since $y \in A_k = a_k + A$, we have $y = a_k + x_c$ for some $c \in K$. Since $y \in A_l$, we have $y = a_l + x_d$ for some $d \in K$. Then $a_k + x_c = a_l + x_d$ so that $x_c - x_d = a_l - a_k \in \mathbb{Q}$ so that $x_c \sim x_d$, so that they generate the same equivalence class, a contradiction, showing that the A_k's are disjoint.

We claim that $[1,2] \subseteq \bigcup_{k=1}^{\infty} A_k$. Let $y \in [1,2]$. Then $y - 1 \in [0,1]$ so that $y - 1 \in c$ for some $c \in K$, so that $y-1 \sim x_c$, implying that $y-1-x_c \in \mathbb{Q}$ and thus $y - x_c \in \mathbb{Q}$. Since $1 \leq y \leq 2$ and $0 \leq x_c \leq 1$ we get $0 \leq y - x_c \leq 2$. Since $y - x_c \in [0,2]$ and $y - x_c \in \mathbb{Q}$, we have that $y - x_c = a_k$ for some $k \in \mathbb{Z}^+$, hence $y = a_k + x_c \in a_k + A = A_k$.

Arguing by contradiction, suppose A is Lebesgue measurable. Since the Lebesgue measure is translation invariant, all A_k would be measurable, so that $\lambda(A_k) = \lambda(A)$. Furthermore, since $[1,2] \subseteq \bigcup_{k=1}^{\infty} A_k \subseteq [0,3]$ and we have

$$1 \leq \lambda\left(\bigcup_{k=1}^{\infty}\right) A_k \leq 3$$

but, we also have

$$\begin{aligned}
\lambda\left(\bigcup_{k=1}^{\infty} A_k\right) &= \sum_{k=1}^{\infty} \lambda(A_k) \\
&= \sum_{k=1}^{\infty} \lambda(A) \\
&= \begin{cases} 0 & \lambda(A) = 0 \\ \infty & \lambda(A) > 0 \end{cases}
\end{aligned}$$

which gets us our desired contradiction, so that A is not measurable. ■

Part III

Lebesgue Integration

Chapter 4

The very unfortunate Riemann integral

Definition 4.1. Fix $A \subseteq \mathbb{R}$. For each $E \subseteq A$, the **characteristic function** of E in A is the function $\chi_E : A \to \{0,1\}$ given by

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

Definition 4.2. A **step function** on the set $A = [a,b]$ where $a < b$ is a function $s : [a,b] \to \mathbb{R}$ of the form

$$S = \sum_{k=1}^{n} c_k \chi_{I_k}$$

where $n \in \mathbb{Z}^+$, each $c_k \in \mathbb{R}$, the I_k are disjoint non-empty intervals and $\bigcup_{k=1}^{n} I_k = [a,b]/$

Remark. The c_k and I_k are uniquely determined from s if we require that for each index k, I_{k-1} is to the left of I_k and $c_{k-1} \neq c_k$.

Definition 4.3. When $s : [a,b] \to \mathbb{R}$ is the step function

$$s = \sum_{k=1}^{n} c_k \chi_{I_k}$$

the **Riemann integral of** s is defined to be

$$\int_a^b s = \int_a^b s(x)dx = \sum_{k=1}^{n} c_k |I_k|$$

Definition 4.4. Let $f : [a,b] \to \mathbb{R}$ be a bounded function. We define the **upper Riemann integral** of f to be

$$U(f) = \inf \left\{ \int_a^b s \,:\, s : [a,b] \to \mathbb{R} \text{ is a step function with } f \leq s \right\}$$

and the **lower Riemann integral** by

$$L(f) = \sup \left\{ \int_a^b s \,:\, s : [a,b] \to \mathbb{R} \text{ is a step function with } s \leq f \right\}$$

We say that f is **Riemann integrable** if $U(f) = L(f)$.

Theorem 4.5. *(Properties of the Riemann integral) Let $a < b$, and let $f, g : A = [a,b] \to \mathbb{R}$. Then:*

1. *If f is Riemann integrable and $c \in \mathbb{R}$, then cf is Riemann integrable and*

$$\int_A cf = c \int_A f$$

2. *If f and g are both Riemann integrable, then so is $f+g$ and*

$$\int_A (f+g) = \int_A f + \int_A g$$

3. *If $a < c < b$ then f is Riemann integrable if and only if f is Riemann integrable on both $[a,c]$ and $[c,b]$ and, in this case,*

$$\int_a^b f = \int_a^c f + \int_c^b f$$

4. *Let $f(x) = g(x)$ for all but finitely many $x \in A$. Then f is Riemann integrable if and only if g is and, in this case,*

$$\int_A f = \int_A g$$

Proof. Follows from MATH 148. ■

Theorem 4.6. *(Fundamental Theorem of Calculus) Let $f, g : [a,b] \to \mathbb{R}$. Suppose that g is differentiable on $[a,b]$ with $g'(x) = f(x)$ for all $x \in [a,b]$ and suppose f is Riemann integrable. Then*

$$\int_a^b f(x)dx = g(b) - g(a)$$

Proof. Follows from MATH 148. ■

Theorem 4.7. *(Lebesgue's characterisation of Riemann integrability) Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable if and only if f is discontinuous on a set measure zero.*

Proof. Exercise. ■

Theorem 4.8. *Let $f : [a,b] \to \mathbb{R}$ be continuous. Then it is Riemann integrable.*

Proof. From MATH 148. ■

Example 4.9. The indicator function for the rational numbers is not Riemann integrable.

Example 4.10. Define $f : [0,1] \to [0,1]$ by

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{k}{n} \text{with } k \text{ and } n \text{ co-prime} \\ 0 & \text{otherwise} \end{cases}$$

This function is Riemann integrable, because its set of discontinuities is $\mathbb{Q} \cap [0,1]$.

Example 4.11. Define $s : \mathbb{R} \to [0,1]$ by

$$s(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

Define

$$f(x) = \sum_{k=1}^{\infty} \frac{s(x-a_k)}{2^k}$$

and observe that f is non-decreasing, continuous everywhere, except at the rational points, and it is Riemann integrable.

Example 4.12. Let $[0,1] \cap \mathbb{Q} = \{a_1, a_2, \ldots\}$ and define

$$f(x) = \sum_{k=1}^{\infty} \frac{(x-a_k)^{1/3})}{2^k}$$

Then f is strictly increasing, f is differentiable except at the points a_k where $f'(a_k) = \infty$. Interestingly , it is also a homeomorphism, and the inverse of f is differentiable and, by the inverse function theorem, $0 \leq g'(x) \leq 3$ for all x. However, $(f^{-1})'$ is not Riemann integrable.

Example 4.13. As an exercise, we can modify our construction of a non-measurable set to show that every set A, such that $\lambda^*(A) > 0$, then it contains a non-measurable set.

Chapter 5

Constructing the Lebesgue Integral

Definition 5.1. We shall use the **extended real numbers**, $[-\infty, \infty] = \mathbb{R} \cup \{-\infty, \infty\}$, with the usual ordering, its usual topology, and its usual operations.

Definition 5.2. A function $f : A \subseteq \mathbb{R} \to B \subseteq [-\infty, \infty]$ is said to be **Lebesgue measurable** whenever $f^{-1}(U)$ is measurable for every open set $U \subseteq B$.

Remark. Note that for $f : A \to [-\infty, \infty]$ to be measurable, then A itself has to be measurable.

Remark. Note that if f is measurable and ϕ is continuous, then ϕf is measurable.c

Definition 5.3. A Lebesgue measurable **simple function** on A is a function of the form

$$s = \sum_{k=1}^{n} c_k \chi_{A_k}$$

for some $n \in \mathbb{Z}^+$, some constants $c_k \in R$ and some disjoint measurable sets A_k with $\bigcup_{k=1}^{n} A_k = A$.

Definition 5.4. We define the **Lebesgue integral** of the simple function $s : A \to \mathbb{R}$ given by

$$s = \sum_{k=1}^{n} c_k \chi_{A_k}$$

to be

$$\int_A s = \int_A s(x)dx = \sum_{k=1}^{n} c_k \lambda(A_k)$$

Theorem 5.5. *(Properties of Lebesgue Integration of Simple Functions) Let $A \subseteq \mathbb{R}$ be measurable. Let $r, s : A \to \mathbb{R}$ be simple functions. Then:*

1. *(Linearity) For $c \in \mathbb{R}$, $\int_A cs = c \int_A s$ and $\int_A (r + s) = \int_A r + \int_A s$*
2. *If $r \leq s$, then $\int_A r \leq \int_A s$.*
3. *If $A = B \cup C$ where B and C are disjoint and measurable, then*
$$\int_A s = \int_B s|_B + \int_C s|_C$$
4. *If $E \subseteq A$ is measurable then*
$$\int_E s|_E = \int_A s\chi_E$$
5. *If $\lambda(A) = 0$, then*
$$\int_A s = 0$$

6. *If $r(x) = s(x)$ for all $x \in A \setminus E$ for some set E with $\lambda(E) = 0$, then*

$$\int_A r = \int_A s$$

7. *If $s \geq 0$ and $\int_A s = 0$ then $s(x) = 0$ almost everywhere (for all $x \in A \setminus E$ for some $\lambda(E) = 0$).*

Proof. We prove 1 and 2; the remainder are left as an exercise.

(1) Let

$$r = \sum_{k=1}^{n} a_k \chi_{A_k} \qquad s = \sum_{l=1}^{m} b_l \chi_{B_l}$$

where the A_k are disjoint and measurable with $\bigcup_{k=1}^{n} A_k$ and the B_l are disjoint and measurable with $\bigcup_{l=1}^{m} B_l = A$ For all k, l, let $C_{k,l} = A_k \cap B_l$. Note that the sets $C_{k,l}$ are all disjoint and

$$\bigcup_{l=1}^{m} C_{k,l} = A_k \qquad \bigcup_{k=1}^{m} C_{k,l} = B_l$$

We have

$$r = sum_{k=1}^{n} a_k \chi_{A_k} = \sum_{k,l} a_{k,l} \chi_{C_{k,l}}$$

where $a_{k,l} = a_k$ for all l and

$$s = \sum_{l=1}^{m} b_l \chi_{B_l} = \sum_{k,l} b_{kl} \chi_{C_{k,l}}$$

where $b_{k,l} = b_l$ for all k. Thus we have,

$$\begin{aligned}
\int_A (r+s) &= \sum_{k,l} (a_{k,l} + b_{k,l}) \lambda(C_{k,l}) \\
&= \sum_{k,l} a_{k,l} \lambda(C_{k,l}) + \sum_{k,l} b_{kl} \lambda(C_{k,l}) \\
&= \sum_{k} a_k \sum_{l} \lambda(C_{k,l}) + \sum_{l} b_l \sum_{k} \lambda(C_{k,l}) \\
&= \sum_{k} a_k \lambda(A_k) + \sum_{l} b_l \lambda(B_l) \\
&= \int_A r + \int_A s
\end{aligned}$$

(2) From the above construction, it follows that

$$\int_A r = \sum_{\substack{k,l, \\ C_{k,l} \neq \emptyset}} a_{k,l} \lambda(C_{k,l}) \leq \sum_{\substack{k,l, \\ C_{k,l} \neq \emptyset}} b_{k,l} \lambda(C_{k,l}) = \int_A s$$

■

Note. Given any non-negative measurable function $f : A \to [0, \infty]$, we can construct a sequence of measurable nonnegative simple functions $\{s_k\}$ such that $s_k(x) \leq s_{k+1}(x)$ for all k and x, such that the sequence converges pointwise everywhere. We may write a neat formula. For $s_n(x)$ we partition the interval $[0, n]$ (in the co-domain) into sub-intervals of size $\frac{1}{2^n}$ using $y_k = \frac{k}{2^n}$ then we define:

$$s_n(x) = \begin{cases} \frac{k-1}{2^n} & \text{with } \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n} \quad \text{for } 1 \leq k \leq n2^n \\ n & f(x) \geq n \end{cases}$$

Definition 5.6. For a non-negative measurable function $f : A \subseteq \mathbb{R} \to [0, \infty]$ we define the **Lebesgue integral** of f to be

$$\int_A f = \sup \left\{ \int_A s \ : \ s : A \to \mathbb{R} \text{ is a non-negative step function with } s \leq f \right\}$$

We say that f is **Lebesgue integrable** if it has a finite Lebesgue integral.

Remark. For simple functions, the definitions of integrals for simple functions and integrals for non-negative functions agree.

Definition 5.7. We say that a property P is true **almost everywhere** in A (we write a.e. in A), or holds for **almost every** $x \in A$ whenever P holds for $x \in A \setminus E$ where $E \subseteq A$ satisfies $\lambda(E) = 0$.

Theorem 5.8. *(Properties of the Lebesgue integral of non-negative functions). Let $f, g : A \subseteq \mathbb{R} \to [0, \infty]$ be non-negative measurable functions. Then:*

1. *(Linearity)*
$$\int_A cf = c \int_A f \qquad \int_A (f + g) = \int_A f + \int_A g$$
2. *If $f \leq g$ then $\int_A f \leq \int_A g$*
3. *(Decomposition) If $A = B \sqcup C$ where B and C are disjoin measurable sets, then $\int_A f = \int_B f + \int_C f$*
4. *If $E \subseteq A$ is measurable, then $\int_E f = \int_A f \cdot \chi_E$.*
5. *If $\lambda(A) = 0$ then $\int_A f = 0$*
6. *If $f = g$ almost everywhere in A, then $\int_A f = \int_A g$*
7. *If $\int_A f = 0$ then $f = 0$ almost everywhere in A.*

Proof. Part 1 is surprisingly difficult, so we delay the proof until after the Monotone Convergence Theorem. We leave the proofs of most of the other parts as an exercise; we give a couple of sample proofs here.

(3). Let $A = B \sqcup C$ for measurable sets B and C. Let $g = f|_B$ and $h = f|_C$. By definition we have

$$\int_A f = \sup \left\{ \int_A r \ : \ r \text{ is simple and dominated by } f \right\}$$

$$\int_B g = \sup \left\{ \int_A s \ : \ s : B \to [0, \infty] \text{ is simple and dominated by } g \right\}$$

$$\int_C h = \sup \left\{ \int_A t : C \to [0, \infty] \ : \ r \text{ is simple and dominated by } h \right\}$$

Let $\epsilon > 0$. Choose simple functions $s : B \to [0, \infty)$ and $t : C \to [0, \infty)$, dominated by g and h, respectively, so that

$$\int_B g - \epsilon < \int_B s \leq \int_B g \qquad \int_C h - \epsilon < \int_C t \leq \int_C h$$

Say $s = \sum_k b_k \chi_{B_k}$ and $t = \sum_l c_l \chi_{C_l}$, so that we let

$$r = \sum_k b_k \chi_{B_k} + \sum_l c_l \chi_{C_l}$$

so that

$$r(x) = \begin{cases} s(x) & x \in B \\ t(x) & x \in C \end{cases}$$

Then,

$$\begin{aligned}\int_A f \geq \int_A r &= \sum_k b_k \lambda(B_k) + \sum_l c_l \lambda(C_k)\\ &= \int_B s + \int_C t\\ &> \left(\int_B g - \epsilon\right) + \left(\int_C h - \epsilon\right)\end{aligned}$$

Since $\epsilon > 0$ was arbitrary,

$$\int_A f \geq \int_B g + \int_C h$$

To show the other inequality, pick $r : A \to [0, \infty)$ simple with $0 \leq r \leq f$ so that

$$\int_A f - \epsilon < \int_A r \leq_A f$$

and argue similarly.

(7) We need to show that if $\int_A f = 0$ then $f = 0$ almost everywhere in A. Let $f : A \to [0, \infty]$ be measurable. Arguing by contrapositive, suppose that we do not have that $f = 0$ a.e. in A. Let

$$E = \{x \in A \,:\, f(x) > 0\}$$

and note that $\lambda(E) > 0$. For each $n \in \mathbb{Z}^+$ let

$$E_n = \left\{x \in A : f(x) > \frac{1}{n}\right\}$$

so that $E_1 \subseteq E_2 \subseteq \ldots$ and $\bigcup_{n=1}^{\infty} E_n = E$. Thus, by continuity from below we have $\lim_{n\to\infty} \lambda(E_n) = \lambda(E) > 0$. Choose some value of $n \in \mathbb{Z}^+$ such that $\lambda(E_n) > 0$. Then, by the comparison property

$$\int_A f = \int_{E_n} f + \int_{A \setminus E_n} f \geq \int_{E_n} \frac{1}{n} = \frac{1}{n}\lambda(E_n) > 0$$

since $\lambda(E_n) > 0$, as desired. ■

Theorem 5.9. ***(Fatou's Lemma)*** *Let $f_n : A \subseteq \mathbb{R} \to [0, \infty]$ be measurable for $n \in \mathbb{Z}^+$. Then*

$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$$

Proof. By the definition of the integral on the left, it suffices to show that for every simple function $s : A \to [0, \infty]$ with $0 \leq s \leq \liminf_{n\to\infty} f_n$ we have $\int_A s \leq \liminf_{n\to\infty} \int_A f_n$. Let $s : A \to [0, \infty)$ be any such simple function. Say $S = \sum_{k=1}^{m} a_k \chi_{A_k}$ where the A_k are disjoint and measurable with $\bigcup_{k=1}^{m} A_k = A$. For all k, for all $x \in A_k$ we have

$$a_k = s(x) \leq \liminf_{n\to\infty} f_n(x)$$

Let $0 < r < 1$ be arbitrary. By the definition of $\liminf_{n\to\infty} f_n(x)$, we can say that $\exists n \in \mathbb{Z}^+$ where for all $l \geq n$, $f_l(x) \geq r a_k$. For $k, n \in \mathbb{Z}^+$, let

$$B_{k,n} = \{x \in A_k \,:\, f_l(x) \geq r a_k \text{ for all } l \geq n\}$$

so that $B_{k,1} \subseteq B_{k,2} \subseteq B_{k,3} \subseteq \ldots$ and $\bigcup_{n=1}^{\infty} B_{k,n} = A_k$. Note that each $B_{k,n}$ is measurable since

$$B_{k,l} = \bigcap_{l \geq n} f_l^{-1}[r a_k, \infty]$$

so that A_k is measurable too and we have $\lim_{n\to\infty} \lambda(B_{k,n}) = \lambda(A_k)$. For all k, n, for $x \in B_{k,n}$ we have $f_l(x) \geq ra_k$ for all $l \geq n$. In particular, taking $l = n$, we have $f_n(x) \geq ra_k$. For fixed n, the sets $B_{k,n}$ are disjoint (since the A_k's are disjoint), so we have

$$f_n(x) \geq \sum_{k=1}^{m} ra_k \chi_{B_{n,k}}$$

so that, by integrating, we have

$$\int_A f \geq \sum_{k=1}^{m} ra_k \lambda(B_{n,k})$$

Taking lim inf of both sides, we get

$$\liminf_{n\to\infty} \int f_n \geq \lim_{n\to\infty} \sum_{k=1}^{m} ra_k \lambda(B_{n,k}) = \sum_{k=1}^{m} ra_k \lambda(A_k) = r \int_A s$$

Since $0 < r < 1$ was arbitrary, we obtain the desired inequality. ∎

Theorem 5.10. *Let $f_n : A \to [0, \infty]$ be measurable functions for each $n \in \mathbb{Z}$. Suppose $\lim_{n\to\infty} f_n(x)$ exists (pointwise) for all $x \in A$ and that $f_n(x) \leq \lim_{n\to\infty}$ for all $x \in A$. Then*

$$\int_A \lim_{n\to\infty} = \lim_{n\to\infty} \int_A f_n$$

Proof. By Fatou's lemma,

$$\int \lim_{n\to\infty} \int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$$

and we have that $f_n(x) \leq \lim_{n\to\infty} f_n(x)$ for all x, so that by the comparison theorem, we have

$$\int f_n(X) \leq \int \lim_{n\to\infty} f_n(x)$$

for all n so that $\limsup_{n\to\infty} f_n(x) \leq \int \lim_{n\to\infty}$. Since we have

$$\limsup_{n\to\infty} \int f_n \leq \int \lim_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$$

it follows that $\lim_{n\to\infty} \int f_n$ exists and $\lim_{n\to\infty} \int f_n = \int \lim_{n\to\infty} f_n$. ∎

Theorem 5.11. ***(Lebesgue's Monotone Convergence Theorem)****. Let $f_n : A \subseteq \mathbb{R} \to [0, \infty]$ be measurable for each $n \in \mathbb{Z}^+$. Furthermore, suppose $f_1 \leq f_2 \leq f_3 \leq \ldots$. Then,*

$$\int \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int f_n$$

Proof. Follows directly from the theorem above. ∎

Theorem 5.12. *Let $f, g : A \subseteq \mathbb{R} \to [0, \infty]$ are measurable, then*

$$\int_A (f + g) = \int_A f + \int_A g$$

Proof. Choose a two sequences of non-decreasing functions approximating f and g and apply the Monotone Convergence Theorem. ∎

Theorem 5.13. *Let $f_n : A \to [0, \infty]$ be measurable. Then*

$$\int_A \sum_{n=1}^{\infty} = \sum_{n=1}^{\infty} \int_A f_n$$

Proof. We may apply the MCT to $g_n = \sum_{k=1}^{n} f_k$. ∎

Theorem 5.14. *Let $A = \bigcup_{n=1}^{\infty} A_n$ where the A_n are disjoint measurable sets in $\mathbb{R}$ and let $f : A \to [0,\infty]$ be measurable. Then*

$$\int_A f = \sum_{n=1}^{\infty} \int_{A_n} f$$

Proof. We apply the MCT to $g_n = \sum_{k=1}^{n} f \cdot \chi_{A_n}$. ∎

Theorem 5.15. *For any measurable function $g : \mathbb{R} \to [0,\infty]$ we can define a measure $\mu : \mathcal{M} \to [0,\infty]$, where $\mathcal{M}(\mathbb{R})$ is the set of all Lebesgue measurable sets, by $\mu(A) = \int_A g$.*

Proof. Follows by theorem above. ∎

Definition 5.16. For a measurable function $f : A \subseteq \mathbb{R} \to [-\infty,\infty]$, we say that f is **Lebesgue integrable** whenever $f^+ = \max(0, f)$ and $f^- = \max(-f, 0)$ have finite integrals. In this case, we define

$$\int_A f = \int_A f^+ - \int f^-$$

Remark. When f is integrable, then the set $f^{-1}(\{\infty\})$ has Lebesgue measure zero.

Theorem 5.17. ***(Properties of the Lebesgue Integral)*** *Let $f, g : A \to [-\infty,\infty]$ be integrable. Then:*

1. *(Linearity) If, for some reason, we were to say that $0 \cdot \infty = 0$ and $\infty - \infty$, then the Lebesgue integral is linear.*
2. *If $f \leq g$ a.e. on A, then $\int_A f \leq \int_A g$.*
3. *(Estimation) $|f|$ is integrable and $\left|\int_A f\right| \leq \int_A |f|$.*
4. *If $A = B \cup C$ where B and C are measurable and disjoint then $\int_A f = \int_B f + \int_C f$*
5. *If $E \subseteq A$ is measurable then $\int_E f = \int_A f \cdot \chi_E$*
6. *If $\lambda(A) = 0$, then $\int_A f = 0$*
7. *If $f = g$ a.e on A, then $\int_A f = \int_A g$*

Proof. Split them up into f^+ and f^- and apply the properties of the Lebesgue integral of the non-negative functions. ∎

Theorem 5.18. ***(Lebesgue Dominated Convergence Theorem)*** *Let $f_n : A \to [-\infty,\infty]$ be measurable for $n \in \mathbb{Z}^+$. Suppose that $\lim_{n\to\infty} f_n(x)$ exists (pointwise) for all $x \in A$. Furthermore, suppose there exists a measurable function $g : A \to [0,\infty]$ such that $|f_n(x)| \leq g(x)$ a.e., and suppose $\int_A g < \infty$. Then,*

$$\int \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int f_n$$

Proof. Let $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in A$. Note that $-g \leq f_n \leq g$ a.e. so that $-g \leq f \leq g$ a.e. and so $0 \leq f^+ \leq g$ and $0 \leq f^- \leq g$ a.e. so f is integrable. We may now apply Fatou's lemma to $g + f_n$, noting that $g + f_n \geq 0$, to get

$$\int g + \int f = \int \lim_{n\to\infty} (g + f_n) = \int \liminf_{n\to\infty} (g + f_n) \leq \liminf \int (g + f_n) = \liminf \left(\int g + \int f_n \right) = \int g + \liminf \int f_n$$

hence

$$\int f \leq \liminf_{n\to\infty} \int f_n$$

We may apply Fatou's lemma to $g - f_n$ to obtain:

$$\int g - \int f = \int \lim(g - f_n) = \int \liminf(g - f_n) \leq \liminf \int (g - f_n) = \int g - \limsup \int f_n$$

Then, $\limsup_{n\to\infty} \int f_n \leq \int f$. Hence

$$\limsup \int f_n \leq \int f \leq \liminf \int f_n$$

which completes the proof. ■

Remark. When $A \subset \mathbb{R}$ is measurable and $f : A \to \mathbb{C}$, we may write $f(x) = u(x) + iv(x)$, where $u, v : A \subseteq \mathbb{R} \to \mathbb{R}$, then we say that f is **Lebesgue integrable** when u and v are both integrable and, in this case, we define the integral of f on A to be

$$\int_A f = \int_A (u + iv) = \int_A u + i \int_A v$$

Part IV

The L_p Spaces

Chapter 6

Some preliminary definitions

Definition 6.1. We expect the reader to be familiar with an **inner product**. We make inner products linear in the first argument, like sensible people do.

Definition 6.2. An **inner product space** is a vector space V endowed with an inner product $\langle \cdot, \cdot \rangle$.

Definition 6.3. A linear map $L : V \to W$ for inner product spaces V, W is called an inner product space homomorphism if $\langle L(u), L(v) \rangle = \langle u, v \rangle$.

Definition 6.4. A **norm** on a vector space W is a function $\|\cdot\| : W \to [0, \infty)$, which is positive definite, has a scaling property, and has a triangle inequality.

Definition 6.5. A **normed linear space** is a vector space with a norm. When V and W are normed linear spaces, a **normed linear space homeomorphism** from V to W is a linear map $L : V \to W$ such that $\|L(u)\| = \|u\|$ for all $u \in V$.

Definition 6.6. A **metric** on a set X is a function $d : X \times X \to [0, \infty)$ such that:

1. $d(x, x) = 0$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

A **metric space** is a a set X with a metric d.

Definition 6.7. A **topology** on a set X is a set T of subsets of X such that:

1. $\emptyset \in T$ and $X \in T$
2. T is closed under finite intersections
3. T is closed under arbitrary unions

A **topological space** is a tuple (X, T), that is a set with a topology.

Theorem 6.8. *Let $\|\cdot\|$ be a norm induced by an inner product. Then, the following are true:*

1. $\|u + v\|^2 = \|u\|^2 + 2Re\langle u, v \rangle + \|v\|^2$
2. ***Pythagorean Theorem*** *If* $\|u\| \, v = 0$ *then* $\|u + v\|^2 = \|u\|^2 + \|v\|^2$
3. ***Parallelogram Law*** $\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2)$
4. ***Polarisation identity.*** *In* $\mathbb{R}$, *we have* $\langle u, v \rangle = \frac{1}{4}\left(\|u + v\|^2 - \|u - v\|^2\right)$. *In* $\mathbb{C}$, *we have*

$$\langle u, v \rangle = \frac{1}{4}\left(\|u + v\|^2 + i\,\|u + iv\|^2 - \|u - v\|^2 - i\,\|u - iv\|^2\right)$$

5. ***Cauchy-Schwarz*** $|\langle u, v \rangle| \leq \|u\| \, \|v\|$, *with equality if and only if u and v are linearly dependent.*

Proof. From MATH 245. ■

Chapter 7

The ℓ_p and L_p spaces

Definition 7.1. Let $R^\omega = \{x = (x_1, x_2, \dots) \ : \ x_k \in \mathbb{R}\}$. For $x \in R^\omega$ and for $p \in [1, \infty)$ we define the ***p*-norm** as

$$\|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}$$

and

$$\|x\|_\infty = \sup_{k \in \mathbb{Z}^+} \{|x_k|\}$$

The "little l-p" spaces are defined as follows.

Definition 7.2. We say:

$$\ell_p = \ell_p(\mathbb{R}) = \{x \in R^\omega \mid \|x\|_p < \infty\}$$

and

$$\ell_\infty = \ell_\infty(\mathbb{R}) = \{x \in \mathbb{R}^\omega \mid \|x\|_\infty < \infty\}$$

Definition 7.3. The **essential supremum** of f is defined by

$$\|f\|_\infty = \inf \left\{a \geq 0 \mid \lambda \left(|f|^{-1}(a, \infty]\right) = 0\right\}$$

Definition 7.4. For a measurable function $f : A \subseteq \mathbb{R} \to [-\infty, \infty]$, we define:

$$\|f\|_p = \left(\int_A |f|^p \right)^{\frac{1}{p}}$$

Definition 7.5. We define the L_p **spaces** as a set of equivalence classes:

$$L_p(A) = L_p(A, \mathbb{R}) = \left\{ \text{measurable } f : A \to [-\infty, \infty] \mid \|f\|_p < \infty \right\} / \sim$$

where we have $f \sim g$ if $f = g$ λ-a.e. on A.

Theorem 7.6. *Let $f : A \subseteq \mathbb{R} \to [-\infty, \infty]$ be measurable. Then the set $E = \{x \in A \mid |f(x)| > \|f\|_\infty\}$ has measure zero.*

Proof. Let E be the set described above. Note that if $y > \|f\|_\infty$, by the approximation property of the infimum, we can choose $a \geq 0$ with $\lambda \left(|f|^{-1}(a, \infty]\right) = 0$ such that $\|f\|_\infty \leq a < y$. Then $|f|^{-1}(y, \infty] \subseteq |f|^{-1}(a, \infty]$, so by monotonicity,

$$\lambda \left(|f|^{-1}(y, \infty]\right) \leq \lambda \left(|f|^{-1}(a, \infty]\right) = 0$$

Now let $E_n = \left\{x \in A \mid |f(x)| > \|f\|_\infty + \frac{1}{n}\right\}$. Then we have $\lambda(E_n) = 0$ for all n, with $E_1 \subseteq E_2 \subseteq E_3 \subseteq \dots$ with $\bigcup_{n=1}^\infty E_n = E$ so that by continuity from below, $\lambda(E) = \lim_{n \to \infty} \lambda(E_n) = 0$. ■

Definition 7.7. For $p, q \in [1, \infty]$ we say that p and q are **conjugate** whenever

$$\frac{1}{p} + \frac{1}{q} = 1$$

Theorem 7.8. ***(Young's Inequality)*** *Let $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then for all $a, b \geq 0$ we have*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Proof. Note that $\frac{1}{p} + \frac{1}{q} = 1$ can be re-written in the following forms:

$$\begin{aligned}
\frac{1}{p} + \frac{1}{q} = 1 &\iff \frac{1}{q} = 1 - \frac{1}{p} - \frac{p-1}{p} \\
&\iff p = q(p-1) \\
&\iff q = p(q-1)
\end{aligned}$$

For $x, y \geq 0$, we have

$$\begin{aligned}
y = x^{p-1} &\iff y^q = x^{q(p-1)} = x^p \\
&\iff x = y^{q-1}
\end{aligned}$$

so the functions $f, g : [0, \infty) \to [0, \infty)$ given by$f(x) = x^{p-1}$ and $g(y) = y^{q-1}$ are inverse functions. It then follows (picture!) that:

$$\begin{aligned}
ab &\leq \int_0^a x^{p-1} dx + \int_0^b y^{q-1} dy \\
&= \frac{a^p}{p} + \frac{b^q}{q}
\end{aligned}$$

as desired. ■

Theorem 7.9. ***(Hölder's inequality)*** *Let $p, q \in [1, \infty]$ be conjugates. Then:*

1. *For all $x, y \in \mathbb{R}^\omega$, we get $\|xy\|_1 \leq \|x\|_p \|y\|_q$,*
2. *For $f, g : A \to \infty[-\infty, \infty]$ measurable, we have $\|fg\|_1 \leq \|f\|_p \|g\|_q$*

Proof. We shall prove part 2. If $\|f\|_p = 0$ or $\|g\|_q = 0$ then the inequality holds trivially. So suppose not.

Case 1: suppose $p, q \in (1, \infty)$. We apply Young's inequality using

$$a = \frac{|f(x)}{\|f\|_p} \qquad b = \frac{|g(x)|}{\|g\|_q}$$

Then,

$$\frac{|f(x)g(x)|}{\|f\|_p \|g\|_q} \leq \frac{|f(x)|^p}{p \|f\|_p^p} + \frac{|g(x)|^q}{q \|g\|_q^q}$$

for all $x \in A$. We may integrate both sides to get,

$$\begin{aligned}
\frac{\|fg\|_1}{\|f\|_p \|g\|_q} &\leq \frac{\|f\|_p^p}{p \|f\|_p^p} + \frac{\|g\|_q^q}{q \|g\|_q^q} \\
&= \frac{1}{p} + \frac{1}{q} \\
&= 1
\end{aligned}$$

hence $\|fg\|_1 \le \|f\|_p \|g\|_q$, as required.

Case 2: Suppose $p = 1$ and $q = \infty$. Let $E = \{x \in A \mid |g(x)| > \|g\|_\infty\}$ and note that $\lambda(E) = 0$. Then,

$$\begin{aligned}
\|fg\|_1 &= \int_A |f(x)g(x)| d\lambda(x) \\
&= \int_{A\setminus E} |f(x)||g(x)| d\lambda(x) \\
&\le \int_{A\setminus E} |f(x)| \|g\|_\infty \, d\lambda(x) \\
&= \|g\|_\infty \int_{A\setminus E} |f(x)| d\lambda(x) \\
&= \|g\|_\infty \int_A |f(x)| d\lambda(x) \\
&= \|f\|_1 \|g\|_\infty
\end{aligned}$$

as required. ∎

Theorem 7.10. *Let $p \in [1, \infty]$. Then:*

1. *For all $x, y \in \mathbb{R}^\omega$, $\|x + y\|_p \le \|x\|_p + \|y\|_q$*
2. *For all measurable functions $f, g : A \subseteq \mathbb{R} \to [-\infty, \infty]$ such that $f + g$ is defined a.e. in A, then we have $\|f + g\|_p \le \|f\|_p + \|g\|_p$.*

Proof. We prove part 2.

Case 1: Suppose $p = 1$. Then

$$\|f + g\|_1 = \int_A |g + g| \le \int_A (|f| + |g|) = \int_A |f| + \int_A |g| = \|f\|_1 + \|g\|_1$$

Case 2: Suppose $1 < p < \infty$. Then,

$$\begin{aligned}
|f(x) + g(x)|^p &= |f(x) + g(x)||f(x) + g(x)|^{p-1} \\
&\le (|f(x)| + |g(x)|)|f(x) + g(x)|^{p-1} \\
&= |f(x)||f(x) + g(x)|^{p-1} + |g(x)||f(x) + g(x)|^{p-1}
\end{aligned}$$

for all $x \in A$. Integrate on A on both sides to get:

$$\begin{aligned}
\|f + g\|_p^p &\le \left\||f||f + g|^{p-1}\right\|_1 + \left\||g||f + g|^{p-1}\right\|_1 \\
&\le \|f\|_p \left\||f + g|^{p-1}\right\|_q + \|g\|_p \left\||f + g|^{p-1}\right\|_q && \text{(by Hölder's inequality)} \\
&= \left(\|f\|_p + \|g\|_p\right) \left\||f + g|^{p-1}\right\|_q
\end{aligned}$$

where we choose q so that $\frac{1}{p} + \frac{1}{q} = 1$. Note that

$$\left\||f + g|^{p-1}\right\|_q = \left(\int_A |f + g|^{q(p-1)}\right)^{1/q} = \left(\int_A |f + g|^p\right)^{\frac{1}{p}\cdot\frac{p}{q}} = \|f + g\|_p^{p/q} = \|f + g\|_p^{p-1}$$

Thus, we get

$$\|f + g\|_p^p \le \left(\|f\|_p + \|g\|_p\right) \|f + g\|_p^{p-1}$$

If $\|f+g\|_p = 0$, then Minkowski's inequality holds. If $\|f+g\|_p \neq 0$, we can divide both sides by $\|f+g\|_p^{p-1}$ to get

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

as required.

Case 3. Suppose that $p = \infty$. We need to show that

$$\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$$

This is left as an exercise to the reader. ■

Theorem 7.11. *When $A \subseteq \mathbb{R}$ is measurable, $L_p(A)$ is a normed linear space using the p-norm.*

Proof. Follows from a suitable application of Minkowski's inequality to the set of equivalence classes in L_p space. ■

Definition 7.12. A metric space is said to be **complete** if all Cauchy sequences converge.

Definition 7.13. A complete inner product space is called a **Hilbert space**.

Definition 7.14. A complete normed linear space is called a **Banach space**.

Theorem 7.15. *Let $A \subseteq \mathbb{R}$ be measurable and let $p \in [1, \infty]$. Then $L_p(A)$ is complete.*

Proof. Case 1: Suppose $p \in [1, \infty)$. Let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $L_p(A)$. That is, for all $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that whenever $k, l \geq m$ then $\|f_k - f_l\|_p < \epsilon$. Since this sequence is Cauchy, extract a subsequence $(f_{n_k})_{k=1}^\infty$ such that $\left\|f_{n_{k+1}} - f_{n_k}\right\|_p \leq \frac{1}{2^k}$. Pick representatives for the subsequence, let

$$g_l(x) = \sum_{k=1}^{l} |f_{n_{k+1}}(x) - f_{n_k}(x)|$$

and let g be defined by a pointwise limit, as follows

$$g(x) = \lim_{l\to\infty} g_l(x) = \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|$$

For each l, by Minkowski's inequality,

$$\|g_l\|_p \leq \sum_{k=1}^{l} \left\|f_{n_{k+1}} - f_{n_k}\right\|_p \leq \sum_{k=1}^{l} \frac{1}{2^k} \leq \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$$

By Fatou's Lemma,

$$\|g\|_p^p = \int |g|^p = \int \lim_{l\to\infty} |g_l|^p = \int_A \liminf |g_l|^p \leq \liminf \int_A |g_l|^p = \liminf \|g_l\|_p^p \leq 1$$

so that $g \in L_p(A)$. Since $\|g\|_p < \infty$ we have $g(x) < \infty$ a.e. on A. Hence

$$\sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)| < \infty$$

for a.e. $x \in A$, so that

$$\sum_{k=1}^{\infty} \left(f_{n_{k+1}}(x) - f_{n_k}(x)\right) < \infty$$

therefore, since the sum is telescoping, $(f_{n_k}(x))_{k=1}^{\infty}$ converges for a.e. $x \in A$. We can now define a function f as the pointwise limit:

$$f(x) = \begin{cases} \lim_{k\to\infty} f_{n_k}(x) & \text{if } (f_{n_k}(x)) \text{ converges} \\ 0 & \text{otherwise} \end{cases}$$

With the given ϵ, choose m so that for all $k, l \geq m$ we get $\|f_k - f_l\|_p < \epsilon$. For any k so that $n_k \geq m$ and then for all $l \geq m$, $\|f_{n_k} - f_l\| < \epsilon$. By Fatou's lemma,

$$\begin{aligned} \|f - f_l\|_p^p &= \int_A |f - f_l|^p \\ &= \int_A \liminf_{l\to\infty} |f_{n_k} - f_l|^p \\ &\leq \liminf_{l\to\infty} \int_A |f_{n_k} - f_l|^p \\ &= \liminf_{l\to\infty} \|f_{n_k} - f_l\|_p^p \\ &< \epsilon^p \end{aligned}$$

Hence, $\|f - f_l\| < \epsilon$ for all $l \geq m$. Note that $f \in L_p(A)$ because since for fixed $l \geq m$, we have that

$$\|f\|_p = \|f - f_l + f_l\|_p \leq \|f - f_l\|_p + \|f_l\| \leq \epsilon + \|f_l\|_p < \epsilon$$

finishing of the completeness proof for finite p.

Case 2: Suppose $p = \infty$. Let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in $L_\infty(A)$. Thus, for all $\epsilon > 0$, there exists an m such that whenever $k, l \geq m$, we get $\|f_k - f_l\| < \epsilon$. Let $B_n = \{x \in A : |f_n(x)| \geq \|f_n\|_\infty\}$ and $C_{kl} = \{x \in A : |f_k(x) - f_l(x)| > \|f_k - f_l\|_\infty\}$ and note that $\lambda(B_n) = \lambda(C_{kl}) = 0$ for all k, l, n. Let

$$E = \bigcup_{n\geq 1} B_n \cup \bigcup_{n\geq 1} C_{kl}$$

and note that $\lambda(E) = 0$. For all $x \in A \setminus E$ we have that $|f_n(x)| \leq \|f_n\|$ and $|f_k(x) - f_l(x)| \leq \|f_k - f_l\|_\infty$.

Let $\epsilon > 0$. Get an $m \in \mathbb{N}$ such that whenever $k, l \geq m$ we have that $\|f_k - f_l\|_\infty$. Then for all k, l, if $k, l \geq m$ we have that $|f_k(x) - f_l(x)| \leq \|f_k - f_l\|_\infty < \epsilon$ for all $x \in A \setminus E$. So $(f_n)_{n\geq 1}$ converges uniformly in $A \setminus E$. Define

$$f(x) = \begin{cases} \lim_{n\to\infty} f_n(x) & x \in A \setminus E \\ 0 & x \notin x \in E \end{cases}$$

For $\epsilon > 0$ and $m \in \mathbb{N}$ as above, we have that, whenever $n \geq m$, we get the estimate $|f(x) - f_n(x)| \leq \|f - f_n\| < \epsilon$. By the triangle inequality,

$$\|f\|_\infty \leq \|f - f_n\|_\infty + \|f_n\| < \infty$$

so that $f \in L_\infty(A)$. This completes the proof that the L_p spaces are complete. Victory! ■

We switch gears and talk about containment relations in the Lebesgue spaces.

Theorem 7.16. *Let $A \subseteq \mathbb{R}$ be measurable with $\lambda(A) < \infty$ and let $1 \leq p < q \leq \infty$. Then $L_q(A) \subset L_p(A)$. Indeed, when $f : A \to [-\infty, \infty]$ is measurable, then $\|f\|_p \leq \lambda(A)^{\frac{1}{p}-\frac{1}{q}} \|f\|_q$.*

Proof. Let $f : A \to [-\infty, \infty]$ be measurable.

Case 1: $p < \infty$ Then,

$$\begin{aligned}
\|f\|_p^p &= \int_A |f|^p \\
&= \|1 \cdot |f|^p\|_1 \\
&= \|1\|_u \, \||f|^p\|_v && \text{(by Holder's inequality for } u, v \text{ conjugate)} \\
&= \left(\int_A 1^u\right)^{1/u} \left(\int_A |f|^{pv}\right)^{1/v} \\
&= \lambda(A)^{1/u} \left(\int_A |f|^{pv}\right)^{1/v}
\end{aligned}$$

We choose $v = \frac{q}{p}$ and u so that $\frac{1}{u} + \frac{1}{v} = 1$, so that $\frac{1}{u} = 1 - \frac{1}{v} = 1 - \frac{p}{q} = \frac{q-p}{q}$ that is $u = \frac{q}{q-p}$. Then we have

$$\begin{aligned}
\|f\|_p^p &\leq \lambda(A)^{1-\frac{p}{q}} \left(\int_A |f|^q\right)^{p/q} \\
&= \lambda(A)^{1-\frac{p}{q}} \|f\|_q^p
\end{aligned}$$

Taking the p-th root, we get $\|f\|_p \leq \lambda(A)^{\frac{1}{p}-\frac{1}{q}} \|f\|_q$.

Case 2: $p = \infty$. Let $E = \{x \in A : |f(x)| > \|f\|_\infty\}$, which has Lebesgue measure zero. Then,

$$\begin{aligned}
\|f\|_p^p &= \int_A |f|^p \\
&= \int_{A \setminus E} |f|^p \\
&\leq \int_{A \setminus E} \|f\|_\infty^p \\
&= \int_A \|f\|_\infty^p \\
&= \lambda(A) \|f\|_\infty^p
\end{aligned}$$

So that $\|f\|_p \leq \lambda(A)^{1/p} \|f\|_\infty$. ∎

Theorem 7.17. *Let $A \subseteq \mathbb{R}$ be measurable. Let $1 \leq p < q < r \leq \infty$. Then $L_p(A) \cap L_r(A) \subseteq L_q(A) \subseteq L_p(A) + L_r(A)$.*

Proof. Let $f \in L_q(A)$. Let $B = \{x \in A : |f(x)| \geq 1\}$ and $C = \{x \in A : |f(x)| < 1\}$ so that B and C are disjoint and $A = B \cup C$. Then let $g = f\chi_B$, $h = f\chi_C$. Then,

$$\begin{aligned}
\|g\|_p^p &= \int_A |g|^p \\
&= \int_A |f\chi_B|^p \\
&= \int_A |f|^p |\chi_B|^p \\
&= \int_A |f|^p \chi_B \\
&= \int_B |f|^p \\
&\leq \int_B |f|^q \\
&\leq \int_A |f|^q \\
&= \|f\|_q^q \\
&< \infty
\end{aligned}$$

and if $r < \infty$ then,

$$\begin{aligned}
\|h\|_r^r &= \int_A |h|^r \\
&= \int_C |f|^r \\
&\leq \int_C |f|^q \\
&\leq \int |f|^q \\
&\leq \|f\|_q^q \\
&< \infty
\end{aligned}$$

If $r = \infty$ then we have $\|h\|_r \leq 1$ since $|h(x)| \leq 1$ for all x.

To show that $L_p(A) \cap L_r(A) \subseteq L_q(A)$ let $f \in L_p(A) \cap L_r(A)$. Suppose first that $r < \infty$. Obtain:

$$\begin{aligned}
\|f\|_q^q &= \int_A |f|^q \\
&= \int_A |f|^{k+l} && \text{(where } k+l=q\text{)} \\
&= \left\| |f|^k |f|^l \right\|_1 \\
&\leq \left\| |f|^k \right\|_u \left\| |f|^l \right\| && \text{(where u,v are conjugates)} \\
&= \left(\int_A |f|^{ku} \right)^{1/u} \left(\int_A |f|^{lv} \right)^{1/v}
\end{aligned}$$

We choose k, l, u, v so that $k + l = q$, u, v conjugate, $ku = p$, $lv = r$, and $l = q - k$ to get

$$u = \frac{r-p}{r-q} \qquad k = \frac{p(r-q)}{r-p} \qquad l = \frac{(q-p)r}{r-p} \qquad v = \frac{r-p}{q-p}$$

This gives,

$$\begin{aligned}
\|f\|_q^q &\leq \left(\int |f|^p\right)^{\frac{r-q}{r-p}} \left(\int |f|^r\right)^{\frac{q-p}{r-p}} \\
&= \|f\|_p^{\frac{p(r-q)}{r-p}} \|f\|_r^{\frac{r(q-p)}{r-p}} \\
&< \infty
\end{aligned}$$

There may be a typo or not, but hopefully there is not.

Suppose $r = \infty$. So $f \in L_p(A)$ and $f \in L_\infty(A)$. Let $E = \{x \in A \ : \ |f(x)| > \|f\|\}$ and note that $\lambda(E) = 0$. Then,

$$\begin{aligned}
\|f\|_q^q &= \int_A |f|^q \\
&= \int_{A \setminus E} |f|^p |f|^{q-p} \\
&\leq |f|^p \, \|f\|_\infty^{q-p} \\
&= \|f\|_p^p \, \|f\|_\infty^{q-p} \\
&< \infty
\end{aligned}$$

■

Definition 7.18. A metric space is **separable** when A has a countable dense subset.

Theorem 7.19. *Let $a, b \in \mathbb{R}$ with $a < b$. Then $L_p([a,b])$ is separable for $1 \leq p < \infty$. However, $L_\infty([a,b])$ is not separable.*

Part V

Banach and Hilbert Spaces

Chapter 8

An insulting review

When $A \subseteq \mathbb{R}$ is measurable, then $L_2(A, \mathbb{R})$ and $L_2(A, \mathbb{C})$ are Hilbert spaces and $L_p(A, \mathbb{R})$ and $L_p(A, \mathbb{C})$ are Banach spaces.

Recall. Let W be an inner product space over $\mathbb{F}$ and $\mathcal{U} \subseteq W$, and let $U = \operatorname{Span}(\mathcal{U})$. If $\mathcal{U}$ is an orthogonal set of non-zero vectors and $x \in U$ with, say,

$$x = \sum_{k=1}^{n} t_k u_k$$

we have

$$t_k = \frac{\langle x, u_k \rangle}{\|u_k\|^2}$$

so, in particular, $\mathcal{U}$ is linearly independent. If $\mathcal{U}$ is orthonormal, then $t_k = \langle x, u_k \rangle$.

Recall. If W is a finite or countable dimensional inner product space, and if $\mathcal{U} = \{u_1, u_2, \ldots\}$ is a basis for W, and if we define $v_1 = u_1$ and

$$v_k = u_k - \sum_{j=1}^{k-1} \frac{\langle u_k, v_j \rangle}{\|v_j\|^2} v_j$$

then $\mathcal{V} = \{v_1, v_2, \ldots\}$ is an orthogonal set of non-zero vectors such that $\{v_1, \ldots, v_k\}$ is an orthonormal basis for its span.

Theorem 8.1. *Every finite or countable dimensional inner product space has an orthonormal basis.*

Theorem 8.2. *If W is a finite or countable inner product space and U is a finite dimensional subspace then every orthonormal basis for U extends to an orthonormal basis for W.*

Theorem 8.3. *Let W be an inner product space. If $\dim W = n$ then W is isomorphic as an inner product space to $\mathbb{F}^n$. If $\dim W = \aleph_0$ then $W \cong \mathbb{F}^\infty$ where*

$$\mathbb{F}^\omega = \{x = (x_1, x_2, \ldots) \ : \ x_k \in \mathbb{F}\}$$

and

$$\mathbb{F}^\infty = \left\{x \in \mathbb{F}^\omega \ : \ \exists n \in \mathbb{Z}^+ \, \forall k \geq n \ x_k = 0\right\} = \operatorname{Span}(\{e_1, e_2, e_3, \ldots\})$$

Recall that when W is an inner product space and $U \subseteq W$ is a subspace, we have that $U^\perp$

$$U^\perp = \{w \in W \ : \ \langle w, u \rangle = 0 \ \forall u \in U\}$$

is a subspace. If $\mathcal{U}$ is a basis for U, then $\mathcal{U}^\perp = U^\perp$. Furthermore, $U \cap U^\perp = \{0\}$. We also have $U \subseteq (U^\perp)^\perp$.

If U is finite dimensional, we have $W = U \oplus U^\perp$ and $U = (U^\perp)^\perp$.

Example 8.4. Let $V = \mathbb{R}^\infty = \mathrm{Span}\left(\{e_1, e_2, e_3, \ldots\}\right)$ and let $U = \mathrm{Span}\left(\{u_2, u_3, u_4, \ldots\}\right)$ where $u_k = e_1 - e_k$. Thus,

$$U = \left\{ x = (x_1, x_2, x_3, \ldots) \in \mathbb{R}^\infty \ : \ \sum_{k=1}^{\infty} x_k = 0 \right\} \subset W$$

Note that $U^\perp = \{0\}$. Also, $W \neq U \oplus U^\perp = U$. Furthermore $(U^\perp)^\perp = W$ so that $U \subset (U^\perp)^\perp$.

Definition 8.5. Let W be an inner product space and let U be a subspace of W with the property that $W = U \oplus U^\perp$. When $x \in W$ is written uniquely as $x = u + v$ with $u \in U$ and $v \in U^\perp$ we say that u is the orthogonal projection of x onto U. We write $u = \mathrm{Proj}_U(x)$.

Recall. For W and U as above, $u = \mathrm{Proj}_U(x)$ is the unique point in U which is nearest to x.

Note. I missed a couple of classes: Friday 8 and Monday 11. I might post notes for these later.

Chapter 9

A few less-insulting results about Banach and Hilbert Spaces

Theorem 9.1. *Let H be a separable Hilbert space. Let $\mathcal{U} = \{u_1, u_2, u_3, \dots\}$ be a countable orthonormal set in H. Let $U = \operatorname{Span}(\mathcal{U})$. Then, the following are equivalent:*

1. *$\mathcal{U}$ is maximal*
2. *$\mathcal{U}$ is dense*
3. *$\forall x \in H$, $x = \sum_{k=1}^{\infty} \langle x, u_k \rangle u_k$*
4. *$\forall x \in H$, $\|x\|^2 = \sum_{k=1}^{\infty} |\langle x, u_k \rangle|^2$*
5. *$\forall x, y \in H$, $\langle x, y \rangle = \sum_{k=1}^{\infty} \langle x, u_k \rangle \overline{\langle y, u_k \rangle}$*

Proof. Might post later. ■

Theorem 9.2. *Let H be a separable Hilbert space:*

1. *If $\dim H = n$ then $H \cong \mathbb{F}^n$*
2. *If $\dim H$ is infinite, then $H \cong \ell_2(\mathbb{F})$ with the 2-norm*

Indeed, an isomorphism is given by $F : H \to \ell_2$, $F(x) = (c_1, c_2, c_3, \dots)$ where $c_k = \langle x, u_k \rangle$.

Definition 9.3. Recall that when X and Y are normed linear spaces and $L : X \to Y$ is a linear transformation, we define the **operator norm** of L to be

$$\|L\| = \sup \{\|Lx\| \ : \ x \in X \ , \ \|x\| \leq 1\}$$

Remark. We can change the less than or equal sign above into equality.

Remark. If X and Y are finite dimensional, then the square root of the largest eigenvalue of $L^T L$ is the operator norm.

Theorem 9.4. *Let X and Y be normed linear spaces and let $L : X \to Y$ be linear. Then the following are equivalent:*

1. *L is continuous at 0*
2. *L is bounded*
3. *L is uniformly continuous*

Proof. (1) $\implies$ (2) Suppose L is continuous at 0. Choose $\delta > 0$ so that $\|x\| \le \delta$ implies that $\|Lx\| \le 1$. Then, for all x, we get that $\|x\| \le 1$ implies that $\|\delta x\| \le \delta$, thus $\delta \|L(x)\| = |L(\delta x)| \le 1$ and $\|Lx\| \le \frac{1}{\delta}$, so that L is bounded with $\|L\| \le \frac{1}{\delta}$.

(2) $\implies$ (3) Suppose L is bounded. Then for all x, y with $x \neq y$,

$$\begin{aligned}\|Lx - Ly\| &= \|L(x-y)\| \\ &= \left\| L\left(\frac{x-y}{\|x-y\|}\right)\right\| \|x-y\| \\ &\le \|L\| \|x-y\|\end{aligned}$$

which is less than ϵ provided that we choose $\delta = \frac{\epsilon}{\|L\|}$.

(3) $\implies$ (1) Trivial. ∎

Theorem 9.5. ***(Uniform boundedness principle)***. *Let X be a Banach space and let Y be a normed linear space. Let S be a set of bounded linear maps $L : X \to Y$. Suppose that for all x there exists a number $m_x > 0$ such that $\|Lx\| \le m_x$ for all $L \in S$. Then S is uniformly bounded.*

Proof. For each $n \in \mathbb{Z}^+$ let

$$A_n = \{x \in X \mid \|Lx\| \le n \quad L \in S\}$$

Note that for each $L \in S$, the set

$$B_{n,L} = \{x \in X \mid \|Lx\| \le n\}$$

is closed. Thus, $A_n = \bigcap_{L\in S} B_{n,L}$ is closed. By the Baire Category Theorem (since X is complete) if all the sets A_n were nowhere dense, then $\bigcup_{n=1}^{\infty} A_n$ would be first category. Choose $n \in \mathbb{Z}^+$ so that A_n is not nowhere dense, so that $A^\circ \neq \emptyset$. Choose $a \in A_n$ and $r > 0$ so that $\overline{B(a,r)} \subseteq A_n$ so that for all $x \in X$

$$\|x - a\| \le r \quad \implies \quad \|Lx\| \le n$$

Then, $\|x\| \le r$ implies that $\|(x+a) - a\| \le r$, implying then that $\|L(x+a)\| \le n$, for all $L \in S$. Thus,

$$\begin{aligned}\|L(x)\| &= \|L(x+a) - L(a)\| \\ &\le \|L(x+a)\| + \|L(a)\| \\ &\le n + n \\ &= 2n\end{aligned}$$

Thus, for all $L \in S$, for all $x \in X$, we have that $\|x\| \le 1$ implies that $\|rx\| \le r$, then implying that $r\|L(x)\| = \|L(rx)\| \le 2n$ and thus $\|Lx\| \le \frac{2n}{r}$. Thus, $\|L\| \le \frac{2n}{r}$ for all L so S is uniformly bounded, as required. ∎

Theorem 9.6. ***(Condensation of singularities)*** *Let X be a Banach space and let Y be a normed linear space. For each $n, m \in \mathbb{Z}^+$, let $L_{n,m} : X \to Y$ be a bounded linear map. Suppose that for each $m \in \mathbb{Z}^+$,*

$$\limsup_{n\to\infty} \|L_{n,m}x\| = \infty$$

then the set

$$E = \left\{ x \in X \;:\; \limsup_{n\to\infty} \|L_{n,m}x\| = \infty \text{ for every } m \in \mathbb{Z}^+ \right\}$$

is a dense $\mathcal{G}_\delta$.

Proof. Fix m and for $l \in \mathbb{Z}^+$ let

$$A_l = A_{l,m} = \left\{x \in X \ : \ \|L_{n,m}x\| \leq l \text{ for all } n \in \mathbb{Z}^+\right\}$$

Note that each A_l is closed. If one of the sets A_l was **not** nowhere dense ($A_l^\circ = \emptyset$), then (as in the previous proof) then we could find $R > 0$ such that $|L_{n,m}| \leq R$ for all $n \in \mathbb{Z}^+$. But if we had $\|L_{n,r}\| \leq R$ for all $n \in \mathbb{Z}^+$, then for all $x \in X$ we would have

$$\|L_{n,m}x\| \leq R\,\|x\|$$

but then

$$\limsup_{n\to\infty} \|L_{n,m}x\| \leq R\,\|x\| < \infty$$

Thus, all of the sets A_l must be nowhere dense. If we let

$$\begin{aligned}
B_m &= \bigcup_{l=1}^{\infty} A_l \\
&= \bigcup_{l=1}^{\infty} A_{l,m} \\
&= \{x \in X \ : \ \exists l \forall n \quad \|L_{n,m}\| \leq l\} \\
&= \left\{x \in X \mid \limsup_{n\to\infty} \|L_{n,m}x\| < \infty\right\}
\end{aligned}$$

and if we let

$$C = \bigcup_{m=1}^{\infty} B_m = \left\{x \in X \ : \ \exists m \quad \limsup_{n\to\infty} \|L_{n,m}x\| < \infty\right\} = X \setminus E$$

since C is a countable union of nowhere dense closed sets, E is a countable intersection of dense open sets. Thus E is a dense $\mathcal{G}_\delta$ by the Baire Category Theorem. ■

Remark. Some people may find the above exciting. I don't know if I am one of those people yet. Maybe when we apply it in the next chapter I will.

Part VI

Fourier Series

Chapter 10

An informal discussion

Fourier series arose first by studying solutions to differential equations. Later on, we cared about questions about convergence and they became interesting from a Pure Mathematics perspective.

Definition 10.1. A real **trigonometric polynomial** is a 2π periodic function $p : \mathbb{R} \to \mathbb{R}$ of the form

$$p(x) = \sum_{n=1}^{m} a_n \cos(nx) + \sum_{n=1}^{m} b_n \sin(nx)$$

and a real **trigonometric series** is a series of the form

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

which is the sequence of partial sums

$$S_m(x) = a_0 + \sum_{n=1}^{m} a_n \cos(nx) + \sum_{n=1}^{m} \sin(nx)$$

We may ask questions about the convergence of trigonometric series: whether it is pointwise, in L^p, or uniform. Assuming, for the moment, that we can integrate term-wise, the coefficients a_n and b_n can be determined by the sum of the series $f(x)$ using the formulas (where δ_{mn} is the Kroenecker delta):

$$\int_{-\pi}^{\pi} 1 dx = 2\pi$$

$$\int_{-\pi}^{\pi} \cos(nx) dx = 0$$

$$\int_{-\pi}^{\pi} \sin(nx) dx = 0$$

$$\int_{-\pi}^{\pi} \cos^2(nx) dx = \pi$$

$$\int_{-\pi}^{\pi} \sin^2(nx) dx = \pi$$

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx) dx = \pi\delta_{mn}$$

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx) dx = \pi\delta_{mn}$$

$$\int_{-\pi}^{\pi} \cos(nx)\sin(mx) dx = 0$$

The formulas above tell us that the set $\{1, \cos(nx), \sin(nx) \ : \ n \in \mathbb{Z}^+\}$ is an orthogonal set of functions, in some function space.

We find that

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)dx$$

Chapter 11

Differential equations

11.1 Forced damped harmonic oscillation

When an object of mass m is attached to a spring of spring constant k and oscillates in a fluid of damping constant c, under the influence of an external force $f(t)$, then the object satisfies the following differential equation:

$$mx'' + cx' + kx = f(t)$$

Let us use Fourier series to solve the DE when $m = 1, c = 2, k = 5$ and f is given by

$$f(t) = \begin{cases} \frac{\pi}{2} + t & -\pi \leq t \leq 0 \\ \frac{\pi}{2} - t & 0 \leq t \leq \pi \end{cases}$$

Using the formulas above, we may compute the Fourier series for $f(t)$ to get

$$f(t) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n^2} \cos(nt)$$

First, we solve the associated homogeneous DE

$$x'' + 2x' + 5x = 0$$

by solving its associated characteristic equation

$$r^2 + 2r + 5 = 0$$

that arises by guessing $x = e^{rt}$; thus, $r = -1 \pm 2i$. This gives us the solutions:

$$\begin{aligned} e^{(-1+2i)t} &= e^{-t} (\cos 2t + i \sin 2t) \\ e^{(-1-2i)t} &= e^{-t} (\cos 2t - i \sin 2t) \end{aligned}$$

We may take linear combinations of the above to get real solutions $e^{-t}\cos(2t)$, $e^{-t}\sin(2t)$, from which the homogenous solution is given by

$$x(t) = Ae^{-t}\cos(2t) + B\sin(2t)$$

for some $A, B \in \mathbb{R}$. Since $f(t)$ is a sum of trigonometric terms, we may use trial and error to solve

$$x'' + 2x' + 5x = \cos nt$$

where we look for solutions of the form $x_n(t) = A_n \cos(nt) + B_n \sin(nt)$, so that

$$\begin{aligned} x_n'(t) &= -nA_n \sin(nt) + nB_n \cos(nt) \\ x_n''(t) &= -n^2 A_n \cos(nt) - n^2 B_n \sin(nt) \end{aligned}$$

If we put this back in the DE, we get:

$$(5-n^2)A_n + 2nB_n = 1$$
$$-2nA_n + (5-n^2)B_n = 0$$

which solves for:

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \begin{pmatrix} \frac{5-n^2}{(5-n^2)^2+4n^2} \\ \frac{2n}{(5-n^2)^2+4n^2} \end{pmatrix}$$

This gives the general solution of the original DE as:

$$x(t) = Ae^{-t}\cos(2t) + Be^{-t}\sin(2t) + \sum_{n \text{ odd}} \frac{4}{\pi n^2}(A_n \cos(nt) + B_n \sin(nt))$$

11.2 One-dimensional wave equation

When an elastic string is stretched to length π and its endpoints are fixed along the x-axis at $x = 0$ and $x = \pi$, then it is displaced vertically until it follows the curve

$$u = u_0(x) = f(x)$$

then it is released and allowed to vibrate; its vertical displacement $u(x,t)$ for $0 \leq x \leq \pi$, and $0 \leq t$ satisfies the one-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (\dagger)$$

for some constant c. Let us solve this DE with boundary conditions

$$u(0,t) = u(\pi,t) \qquad \forall t \geq 0$$

and initial conditions

$$u(x,0) = u_0(x) = f(x)$$

and

$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

We look for a solution of the form

$$u(x,t) = y(x)s(t)$$

The DE in $(\dagger)$ then becomes

$$y(x)s''(t) = c^2 y''(x)s(t)$$

which, provided that $s(t), y(x)$ are non-zero, gives

$$\frac{y''(x)}{y(x)} = \frac{1}{c^2}\frac{s''(t)}{s(t)}$$

The function on the left is constant in t and the function on the right is constant in x, so to be equal for all x and g, they must both be constant. Thus, the DE becomes

$$\frac{y''(x)}{y(x)} = k = \frac{1}{c^2}\frac{s''(t)}{s(t)}$$

This method is known as **separation of variables**.

Chapter 12

Convergence and Fourier Series

Note. I missed the class on March 20th. I might add notes for that class later.

Definition 12.1. Let $T = \mathbb{R}/2\pi\mathbb{Z}$. We define $L_p(T) = L_p(T, \mathbb{R})$ to be the set of **measurable functions** $f : T \to \mathbb{R}$ with finite p-norm modulo equality λ-a.e., where we say that f is measurable if the associated 2π - periodic function is measurable. When $1 \leq p < \infty$, we shall write

$$\|f\|_p^p = \int_T |f|^p = \int_{-\pi}^{\pi} |f|^p = \int_{\theta}^{\theta+2\pi} |f|^p$$

Remark. $L_\infty(T) \subseteq L_p(T) \subseteq L_1(T)$ and $\|f\|_p \leq (2\pi)^{1/p} \|f\|_\infty$.

Definition 12.2. For $f \in L_1(T) = L_1(T, \mathbb{R})$, the **Fourier coefficients** of f are the real numbers given by

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)dx \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)dx \end{aligned}$$

and the **real Fourier series** of f is the trigonometric series whose partial sums are

$$S_m(f)(x) = a_0 + \sum_{n=1}^{m} a_n \cos(nx) + \sum_{n=1}^{m} b_n \sin(nx)$$

Definition 12.3. A **complex-valued trigonometric polynomial** is a function $f : \mathbb{R} \to \mathbb{C}$ of the form

$$f(x) = \sum_{n=-m}^{m} c_n e^{inx}$$

for some $c_n \in \mathbb{C}$. A **complex trigonometric series** is a series of the form

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}$$

which is given by the sequence of partial sums

$$S_m = \sum_{n=-m}^{m} c_n e^{inx}$$

Remark. Hereafter we drop the term "complex", as we shall mostly be dealing with them.

Definition 12.4. We define $L_p(T) = L_p(T, \mathbb{C})$ in the expected fashion. For $f \in L_p(T)$ given by $f = u + iv$ where $u, v : T \to \mathbb{R}$, f is measurable if and only if u and v are measurable and, then, when $1 \le p < \infty$

$$\|f\|_p^p = \int_T |f|^p = \int_{-\pi}^{\pi} |f|^p = \int_{-\pi}^{\pi} |\sqrt{u^2 + v^2}|^p$$

Definition 12.5. When f is the trigonometric polynomial

$$f(x) = \sum_{n=-m}^{m} c_n e^{inx}$$

using the formula

$$\begin{aligned}
\int_{-\pi}^{\pi} e^{ikx} e^{-ilc} dx &= \int_{-\pi}^{\pi} e^{i(k-l)x} dx \\
&= \int_{-\pi}^{\pi} \cos(k-l)x dx + i \int_{-\pi}^{\pi} \sin(k-l)x dx \\
&= \begin{cases} 2\pi & k = l \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

we obtain

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{inx} dx$$

When $f \in L_1(T, \mathbb{C})$, the above integral exists and is finite. When $f \in L_1(T, \mathbb{C})$ we define the **Fourier coefficients** of f to be the complex numbers

$$c_n = c_n(f) = \hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

and the **Fourier series** of f is the trigonometric series

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}$$

Note. When $a_n, b_n \in \mathbb{R}$, then

$$\begin{aligned}
a_0 + \sum_{n=1}^{m} a_n \cos(nx) + \sum_{n=1}^{m} b_n \sin(nx) &= a_0 + \sum_{n=1}^{m} a_n \frac{e^{inx} + e^{-inx}}{2} + \sum_{n=1}^{m} b_n \frac{e^{inx} - e^{-inx}}{2i} \\
&= a_0 + \sum_{n=1}^{m} \frac{a_n - ib_n}{2} e^{inx} + \sum_{n=1}^{m} \frac{a_n + ib_n}{2} e^{-inx} \\
&= \sum_{n=-m}^{m} c_n e^{inx}
\end{aligned}$$

where $c_0 = a_0$, $c_n = \frac{a_n - ib_n}{2}$, $c_{-n} = \bar{c_n} = \frac{a_n + ib_n}{2}$. On the other hand, when $f \in L_1(T, \mathbb{R})$, we have

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx - \frac{i}{2\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx = \frac{1}{2}(a_n - ib_n)$$

Example 12.6. Exercise. Let $a_n \in \mathbb{C}$ and let $S_m = \sum_{n=-m}^{m} a_n e^{inx}$ and let $f \in L_p(T)$ where $1 < p < \infty$. Show that if $S_m \to f$ in $L_p(T)$ then $a_n = c_n(f)$ so that $s_m = s_m(f)$.

Soln. Suppose $S_m \to f$ in $L_p(T)$. Fix n. For $m \geq n$,

$$a_n = c_n(s_m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} s_m(x)e^{-inx}dx$$

so that

$$\begin{aligned}
|c_n(f) - a_n| &= |c_n(f) - c_n(s_m)| \\
&= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x) - S_m(x))e^{-inx}dx\right| \\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x) - S_m(x)|dx \\
&= \frac{1}{2\pi}\|f - S_m\| \\
&\leq \frac{1}{2\pi}(2\pi)^{1-\frac{1}{p}}\|f - S_m\|_p \\
&\to 0
\end{aligned}$$

as desired. ∎

12.1 Applications of the Stone-Weierstrass theorem

Theorem 12.7. *Let X be a compact metric space. If $\mathcal{A} \subseteq C(X, \mathbb{R})$ is a unital algebra that separates points, then $\overline{\mathcal{A}} = C(X, \mathbb{R})$.*

Proof. Exercise. ∎

Theorem 12.8. *Let X be a compact metric space. If $\mathcal{A} \subseteq C(X, \mathbb{C})$ is a unital self-adjoint algebra that separates points, then $\overline{\mathcal{A}} = C(X, \mathbb{C})$.*

Proof. Exercise. ∎

In what follows, we state a few theorems which are easy corollaries of what we already know.

Example 12.9. The polynomials with real coefficients are dense in $C(X, \mathbb{R})$. Likewise, for complex polynomials in $C(X, \mathbb{C})$.

Example 12.10. The set of real-valued trigonometric polynomials is dense in $C(T, \mathbb{R})$, and likewise for complex-valued polynomials in $C(T, \mathbb{C})$.

Theorem 12.11. *For $1 \leq p < \infty$, the set of real trigonometric polynomials is dense in $L_p(T, \mathbb{R})$ and the set of complex trigonometric polynomials is dense in $L_p(T, \mathbb{C})$ (using the p-norm).*

Proof. This follows since $C(T, \mathbb{F})$ is dense in $L_p(T, \mathbb{F})$. ∎

Theorem 12.12. *In $L_2(T, \mathbb{R})$, the set $\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos(nx), \frac{1}{\sqrt{\pi}}\sin(nx) \ : \ n \in \mathbb{Z}^+\}$ is a Hilbert basis and in $L_2(T, \mathbb{C})$, the set $\left\{\frac{1}{\sqrt{2\pi}} \ : \ n \in \mathbb{Z}\right\}$ is an orthonormal basis for $L_2(T, \mathbb{C})$.*

Theorem 12.13. *For $f \in L_2(T) = L_t(T, \mathbb{C})$:*

1. $c_n(f) = \frac{1}{2\pi}\langle f, e^{inx}\rangle = \frac{1}{2\pi}\int_T f(x)e^{-inx}$
2. *Out of all trigonometric polynomials, $p(x) = \sum_{n=-m}^{m} a_n e^{inx}$ the m-th partial sum $S_m(f)$ is the best approximation to f in the p-norm.*

3. *$S_m(f) \to f$ in $L_2(T)$*
4. *$\|f\|_2^2 = 2\pi \sum_{n=-\infty}^{\infty} |c_n(f)|^2$*
5. *$\langle f, g\rangle = 2\pi \sum_{n=-\infty}^{\infty} c_n(f)\overline{c_n(g)}$*
6. *Given $a_n \in \mathbb{C}$ for $n \in \mathbb{Z}$, we have $a_n = c_n(f)$ for some $f \in L_2(T, \mathbb{C})$ if and only if $\sum_{n=-\infty}^{\infty} |a_n|^2 < \infty$.*

12.2 Dirichlet Kernel

Note that for $f \in L_1(T)$,

$$\begin{aligned}
S_m(f) &= \sum_{n=-m}^{m} c_n(f)e^{inx} \\
&= \sum_{n=-m}^{m} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt\right)e^{inx} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\sum_{n=-m}^{m} e^{-int}e^{inx}dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\sum_{n=-m}^{m} e^{in(x-t)} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_m(x-t)dt
\end{aligned}$$

where

$$D_m(u) = \sum_{n=-m}^{m} e^{inu}$$

Definition 12.14. The **m-th Dirichlet kernel** is the function

$$D_m(u) = \sum_{n=-m}^{m} e^{inu}$$

Remark. The convolution of f with g, for $f, g \in L_1(T)$ is

$$f * g = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)g(x-t)dt$$

so we have $S_m(f) = f * D_m$.

Note that the Dirichlet kernel is a geometric sum, so that

$$\begin{aligned}
D_m(u) &= \sum_{n=-m}^{m} e^{inu} \\
&= e^{-imu}\cdot\frac{e^{i(2m+1)u}-1}{e^{iu}-1} \\
&= e^{-imu}\cdot\frac{e^{i(2m+1)u}-1}{e^{iu}-1} \\
&= e^{-imu}\cdot\frac{2ie^{i\frac{(2m+1)u}{2}}\sin\left(\frac{(2m+1)u}{2}\right)}{2ie^{iu/2}\sin\left(\frac{u}{2}\right)} \\
&= \frac{\sin\left(\frac{(2m+1)u}{2}\right)}{\sin\frac{u}{2}}
\end{aligned}$$

where we have used the identity

$$e^{i\theta} - 1 = e^{i\theta/2}\left(e^{i\theta/2} - e^{-i\theta/2}\right) = 2ie^{i\frac{\theta}{2}} \sin\frac{\theta}{2}$$

Note that when $u = 0$ we get $D_m(0) = 2m + 1$.

Theorem 12.15. *A two part lemma.*

1. $\int_{-\pi}^{\pi} D_m(t)dt = 2\pi$
2. $\int_{-\pi}^{\pi} |D_m(t)|dt \geq \frac{8}{\pi}\ln m$

Proof. From the definition,

$$\begin{aligned}
\int_{-\pi}^{\pi} D_m(t)dt &= \int_{-\pi}^{\pi} \sum_{n=-m}^{m} e^{int}dt \\
&= \int_{-\pi}^{\pi} \left(1 + \sum_{n=1}^{m} 2\cos(nt)\right) \\
&= 2\pi
\end{aligned}$$

For the second part, note that

$$\begin{aligned}
\int_{-\pi}^{\pi} |D_m(t)|dt &= \int_{-\pi}^{\pi} \frac{\left|\sin\frac{(2m+1)t}{2}\right|}{\left|\sin\frac{t}{2}\right|} \\
&= 2\int_{0}^{\pi} \frac{\left|\sin\frac{(2m+1)t}{2}\right|}{\sin\frac{t}{2}}dt \\
&\geq 2\int_{0}^{\pi} \frac{\left|\sin\frac{(2m+1)t}{2}\right|}{t/2} \\
&= 2\int_{0}^{(m+1/2)\pi} \frac{|\sin u|}{\frac{u}{2m+1}} \cdot \frac{2}{2m+1}du \qquad\qquad u = \frac{(2m+1)t}{2} \\
&= 4\int_{0}^{(m+1/2)\pi} \frac{|\sin u|}{u}du \\
&\geq 4\int_{0}^{m\pi} \frac{|\sin u|}{u}du \\
&= 4\sum_{n=1}^{m} \int_{(n-1)\pi}^{n\pi} \frac{|\sin u|}{u}du \\
&\geq 4\sum_{n=1}^{m} \int_{(n-1)\pi}^{n\pi} \frac{|\sin u|}{n\pi}du \\
&= \frac{8}{\pi}\sum_{n=1}^{m} \frac{1}{n} \\
&\geq \frac{8}{\pi}\ln(m+1) \\
&\geq \frac{8}{\pi}\ln m
\end{aligned}$$

∎

12.3 Big theorems on convergence

Theorem 12.16. ***(Pointwise divergence of Fourier series)*** *There exists a dense $\mathcal{G}_\delta$ set E in $C(T) = C(T, \mathbb{C})$ (using the ∞-norm) such that for every $f \in E$ the set of points in T at which the Fourier series of $f(x)$ diverges is dense in T.*

Proof. Consider $x = 0$ first. For $m \in \mathbb{Z}^+$ define $F_m : C(T) \to \mathbb{C}$ by $F_m(f) = S_m(f)(0)$. For $f \in C(T)$,

$$\begin{aligned}
|F_m(f)| &= |S_m(f)(0)| \\
&= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_m(-t) dt \right| \\
&\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)| \, |D_m(t)| dt \\
&\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \|f\|_\infty \, |D_m(t)| dt \\
&= \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |D_m(t)| dt \right) \|f\|_\infty
\end{aligned}$$

Thus, $\|F_m\| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_m(t)| dt$. We claim that $\|F_m\| = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_m(t)| dt$. Let

$$s(x) = \begin{cases} 1 & D_m(t) \geq 0 \\ -1 & D_m(t) < 0 \end{cases}$$

Choose continuous functions $g_n C(T)$ with $|g_n(x)| \leq 1$ for all x so that $g_n \to s$ in $L_1(T)$. Then,

$$\begin{aligned}
|F_m(g_n)| &= |S_m(g_n)(0)| \\
&= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} g_n(t) D_m(t) dt \right| \\
&\to \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} s(t) D_m(t) dt \right| && \text{(as } n \to \infty) \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_m(t)| dt
\end{aligned}$$

Therefore, since $\|g_n\|_\infty \leq 1$, we have that

$$\|F_m\| \geq \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_m(t)| dt \geq \frac{8}{\pi} \ln m$$

Since $\|F_m\| \geq \frac{8}{\pi} \ln m$ for all $m \in \mathbb{Z}^+$, the set $S = \{F_m \ : \ m \in \mathbb{Z}^+\}$ is not uniformly bounded. By the contrapositive of the uniform boundedness principle, there exists $f \in C(T)$ such that for all M_f, $\|F_m(f)\| > M$ for all $m \in \mathbb{Z}^+$. Thus, the set $\{|S_m(f)()0|\}_{m \geq 1}$ is unbounded so that $\limsup_{m \to \infty} |S_m(f)(0)| = \infty$. In particular, the Fourier series for f diverges at 0.

We can make life even worse. Denumerate $[0, 2\pi) = \{a_1, a_2, \ldots\}$. For each $n \in \mathbb{Z}^+$, let $f_n = f(x - a_n)$ so that $\limsup_{m \to \infty} |S_m(f_n)(a_n)| = \infty$. By Condensation of Singularities(applies to $L_{nm} : C(f) \to \mathbb{C}$ with $L_{nm}(f) = S_m(f)(a_n)$), the set

$$E = \left\{ f \in C(T) \ : \ \limsup_{m \to \infty} \right\}$$

is a dense $\mathcal{G}_\delta$ set in $C(T)$. Note that for every $f \in E$, we have $\limsup_{m \to \infty} |S_m(f)(a_n)| = \infty$ so the Fourier series of f diverges at every point a_n. ∎

Exercise. As an exercise, determine whether the set of all $x \in T$ such that the Fourier series of f diverges at x for every $f \in E$, is a $\mathcal{G}_\delta$ set.

Definition 12.17. For each $n \in \mathbb{Z}^+$, let $a_n \in \mathbb{C}$. Define $S_m = \sum_{n=1}^{m} a_m$ and $\sigma_l = \frac{1}{l}\sum_{m=1}^{l} s_m$. The latter sequence of sums is called the **Cesaro means**.

Theorem 12.18. *With the notation above, is $(S_m)_{m\geq 1}$ converges, then so does $(\sigma_l)_{l\geq 1}$ and*

$$\lim_{l\to\infty} = \lim_{m\to\infty} S_m = \sum_{n=1}^{\infty} a_n$$

Proof. Suppose $(S_m)_{m\geq 1}$ and say $S_m \to b$. Let $\epsilon > 0$ and choose $k \geq 1$ so that for all $l \geq k$ implies that $|s_l - b| < \epsilon$. Then,

$$\begin{aligned}
\sigma_l &= \frac{S_1 + \ldots + S_k}{l} + \frac{S_{k+1} + \ldots + S_l}{l} \\
&\leq \frac{S_1 + \ldots + S_k}{l} + (b+\epsilon)\frac{l-k}{l} \\
&\to b + \epsilon && \text{(as } l \to \infty)
\end{aligned}$$

and

$$\sigma_l \geq \frac{S_1 + \ldots + S_k}{l} + (b-\epsilon)\frac{l-k}{l} \to b - \epsilon$$

as $l \to \infty$. ∎

Example 12.19. For $a_n = (-1)^n$, for $n \geq 1$, the sum $\sum a_n$ diverges, but $\sigma_l \to -0.5$.

Definition 12.20. For $f \in L_1(T) = L_1(T, \mathbb{C})$, we write the **Cesaro means of f** as

$$\sigma_l(f)(x) = \frac{1}{l+1}\sum_{m=0}^{l} S_m(f)(x)$$

Note. Via the following manipulation:

$$\begin{aligned}
\sigma_l(f)(x) &= \frac{1}{l+1}\sum_{m=0}^{l} S_m(f)(x) \\
&= \frac{1}{l+1}\sum_{m=0}^{l} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) D_m(x-t)dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{1}{l+1}\sum_{m=0}^{l} D_m(x-t)dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) K_l(x-t)dt
\end{aligned}$$

where

$$K_l(u) = \frac{1}{l+1}\sum_{m=0}^{l} D_m(u)$$

Definition 12.21. The above function given by

$$K_l(u) = \frac{1}{l+1}\sum_{m=0}^{l} D_m(u)$$

is called the l-th **Fejer kernel**.

Remarks. Via some easy manipulations we may find that

$$K_l(u) = \frac{\sin^2 \frac{(l+1)u}{2}}{(l+1)\sin^2 \frac{u}{2}}$$

It is easy to see that K_l is non-negative, even, and 2π periodic. Furthermore, since $D_m(0) = 2m+1$, we can note that $K_l(0) = l+1$. Some further manipulations will show that

$$\int_{-\pi}^{\pi} K_l(t)dt = 2\pi$$

and that

$$\sigma_l(f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_l(t)dt$$

Theorem 12.22. ***(Convergence of Cesaro means)*** *Let $f \in L_1(T) = L_1(T, \mathbb{C})$ and consider f as a 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$. Suppose that $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ both exist in $\mathbb{C}$ and write*

$$f(a^+) = \lim_{x\to a^+} f(x) = \lim_{t\to 0^+} \lim_{t\to 0^+} f(a+t) \qquad f(a^-) = \lim_{x\to a^-} f(x) = \lim_{t\to 0^+} f(a-t)$$

Then,

$$\lim_{l\to\infty} \sigma_l(f)(a) = \frac{f(a^+)+f(a^-)}{2}$$

Furthermore, if $f : \mathbb{R} \to \mathbb{C}$ is continuous in $[a,b]$ then $\sigma_l(f)(x) \to f(x)$ uniformly in $[a,b]$

Proof. We have

$$\begin{aligned}
\sigma_l(f)(a) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)K_l(a-t)dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(a+t)F_l(t)dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(a-t)K_l(t)dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(a+t)+f(a-t)}{2} \cdot K_l(t)dt
\end{aligned}$$

Also, since $\int_{-\pi}^{\pi} K_l(t)dt = 2\pi$, we have that,

$$\begin{aligned}
\frac{f(a^+)+f(a^-)}{2} &= \frac{1}{2\pi}\frac{f(a^+)+f(a^-)}{2} \cdot \int_{-\pi}^{\pi} K_l(t)dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(a^+)f(a^-)}{2} \cdot K_l(t)dt
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|\sigma_l(f)(a) - \frac{f(a^+)+f(a^-)}{2}\right| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{f(a+t)-f(a-t)}{2} - \frac{f(a^+)+f(a^-)}{2}\right)K_l(t)dt\right| \\
&= \left|\frac{1}{2\pi}\int_{0}^{\pi}\big((f(a+t)-f(a^+)) + (f(a-t)-f(a^-))\big)K_l(t)dt\right| \\
&\le \frac{1}{2\pi}\int_{0}^{\pi}\big|(f(a+t)-f(a^+)) + (f(a-t)-f(a^-))\big|K_l(t)dt \\
&= I_\delta + J_\delta
\end{aligned}$$

where

$$I_\delta = \frac{1}{2\pi}\int_0^\delta \left|(f(a+t)-f(a^+))+(f(a-t)-f(a^-))\right| K_l(t)dt$$
$$J_\delta = \frac{1}{2\pi}\int_\delta^\pi \left|(f(a+t)-f(a^+))+(f(a-t)-f(a^-))\right| K_l(t)dt$$

for any $\delta \in [0,\pi]$.

Let $\epsilon > 0$. Choose $\delta > 0$ so that for $0 < t < \delta$ we have $|f(a+t)-f(a^+)| < \frac{\epsilon}{4}$ and $|f(a-t)-f(a^-)| < \frac{\epsilon}{4}$ and then, by the MCT,

$$\begin{aligned} I_\delta &\leq \frac{1}{2\pi}\int_0^\delta \left(\frac{\epsilon}{4}+\frac{\epsilon}{4}\right) K_l(t)dt \\ &\leq \frac{1}{2\pi}\cdot\frac{\epsilon}{2}\int_0^\pi K_l(t)dt \\ &= \frac{\epsilon}{2} \end{aligned}$$

Since we have that $K_l(t) \leq \frac{1}{l+1}\frac{\pi^2}{u^2}$, so for $t \geq \delta$, we have $K_l(t) \leq \frac{1}{l+1}\frac{\pi^2}{\delta^2}$. Thus, we can choose $m \in \mathbb{Z}^+$ so that whenever $l \geq m$ we have $\frac{1}{l+1}\frac{\pi^2}{\delta^2} < \frac{\epsilon}{M}$ for $M = \frac{1}{\pi}\|f\|_1 + |f(a^+)+f(a^-)|$. Thus, for $l \geq m$, we have,

$$\begin{aligned} J_\delta &\leq \frac{1}{2\pi}\int_\delta^\pi \left|(f(a+t)+f(a-t)-(f(a^+)+f(a^-)))\right|\cdot\frac{\epsilon}{M}dt \\ &\leq \frac{1}{2\pi}\cdot\frac{\epsilon}{M}\int_0^\pi |(f(a+t)+f(a-t)| + \left|(f(a^+)+f(a^-)))\right| dt \\ &= \frac{1}{2\pi}\cdot\frac{\epsilon}{M}\left(\|f\|_1 + \pi|f(a^+)+f(a^-)|\right) \\ &= \frac{\epsilon}{2} \end{aligned}$$

To prove the uniform convergence for the case where f is continuous at a, argue similarly and use uniform continuity of f on $[a,b]$. ■

Theorem 12.23. *Let $f \in L_1(T) = L_1(T,\mathbb{C})$ and identify f with the corresponding 2π periodic function, Let $a \in \mathbb{R}$ and suppose both one-sided limits exist for f at a. Also, suppose that $\lim_{m\to\infty} S_m(f)(a)$ exists in $\mathbb{C}$. Then*

$$\lim_{m\to\infty} S_m(f)(a) = \frac{f(a^+)+f(a^-)}{2}$$

Proof. If $\lim_{m\to\infty} S_m(f)(a)$ exists, then

$$\lim_{m\to\infty} S_m(f)(a) = \lim_{m\to\infty} \sigma_l(f)(a)$$

■

Remark. This allows us to justify our proof that the sum of the reciprocals of $(2n+1)^2$ sums to $\frac{\pi^2}{8}$.

Note. This ends the material for the final exam.