Banach-Tarski Paradox

Motivation.

Definition 1.1. Let G be a group acting on a set S. A subset $E \subseteq S$ is said to be **G-paradoxical** if there are pairwise disjoint subsets $A_1, \ldots, A_n, B_1, \ldots, B_m$ of E and elements $x_1, \ldots, x_n, y_1, \ldots, y_m$ of G such that

$$E = \bigcup_{j=1}^{n} x_j \cdot A_j \qquad E = \bigcup_{j=1}^{m} y_j B_j$$

Remark. The union $A_1 \cup \ldots \cup A_n \cup B_1 \cup \ldots \cup B_m$ need not be all of E. Example?

Theorem 1.2. $\mathbb{F}_2$ *(the free group with two generators) is paradoxical.*

Proof. Let a, b denote the generators of $\mathbb{F}_2$. For $x \in \{a, b, a^{-1}, b^{-1}\}$, let $W(x) = \{\text{words starting with } x\}$, and denote ε the empty word. Observe that

$$\mathbb{F}_2 = \{\varepsilon\} \cup W(a) \cup W(b) \cup W(a^{-1}) \cup W(b^{-1})$$

Let $w \in \mathbb{F}_2 \setminus W(a)$. Since w does not start with an a, $a^{-1}w$ is in reduced form and $a^{-1}w \in W(a^{-1})$ so that $w \in aW(a^{-1})$. Thus, $\mathbb{F}_2 = W(a) \cup aW(a^{-1})$, and similarly $\mathbb{F}_2 = W(b) \cup bW(b^{-1})$. □

Definition 1.3. We say a group G acts on S **without non-trivial fixed points** if, given $x \in G$ and $s \in S$ such that $x \cdot s = s$ then $x = e$.

Theorem 1.4. *(AoC) Let G be a paradoxical group acting on S without non-trivial fixed points. Then, S is G-paradoxical.*

Proof. Let $A_1, \ldots, A_n, B_1, \ldots, B_m$, and $x_1, \ldots, x_n, y_1, \ldots, y_m$ be as in Defn 1.1. Let $T \subset S$ be constructed by containing exactly one element from each G-orbit. By construction,

$$\bigcup \{g \cdot T \,:\, g \in G\} = S$$

holds. Let $x, y \in G$ be such that $x \cdot T \cap y \cdot T \neq \emptyset$. Pick $z \in x \cdot T \cap y \cdot T$ so that there are $s, t \in T$ such that $z = x \cdot s = y \cdot t$ and so $y^{-1}x \cdot s = t$, which reveals that s and t are in the same orbit, so that $s = t$ by construction of T. Hence $y^{-1}x \cdot s = s$ and $y^{-1}x = e$, since G acts without non-trivial fixed points, whence $x = y$. Therefore, the sets $\{x \cdot T\}_{x \in G}$ disjointly partition S.
Put, for $1 \leq i \leq n$ and $1 \leq j \leq m$

$$\tilde{A}_i = \bigcup \{x \cdot T \,:\, x \in A_i\} \qquad \tilde{B}_j = \bigcup \{x \cdot T \,:\, x \in B_j\}$$

Then the set $\tilde{A}_1, \ldots, \tilde{A}_n, \tilde{B}_1, \ldots, \tilde{B}_m$ are disjoint subsets of S such that

$$\bigcup_{i=1}^{n} x_i \cdot \tilde{A}_i = \bigcup_{i=1}^{n} \bigcup \{x_i x \cdot T \,:\, x \in A_i\} = \bigcup \{x \cdot T \,:\, x \in G\} = S$$

and similarly $\bigcup_{j=1}^{n} y_j \cdot \tilde{B}_j = S$ □

Remark. If it follows from the two theorems above that if $\mathbb{F}_2$ acts without non-trivial fixed points on a set S, then S is $\mathbb{F}_2$-paradoxical.

Theorem 1.5. *There are rotations A and B about lines through the origin in $\mathbb{R}^3$ such that the subgroup of $SO(3)$ generated by A and B is isomorphic to $\mathbb{F}_2$.*

Proof. (Sketch) Set

$$A = \begin{pmatrix} \frac{1}{3} & -\frac{2\sqrt{2}}{3} & 0 \\ \frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{2\sqrt{2}}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix}$$

Compute A^{-1} and B^{-1}. To show that the subgroup generated by these two matrices is free, we must show that any non-empty reduced word in A, A^{-1}, B, B^{-1} cannot act as the identity. Casework... $\square$

Remark. $SO(N)$ contains a copy of $SO(3)$ for $N \geq 3$, so that $SO(N)$ contains a subgroup isomorphic to $\mathbb{F}_2$ for $N \geq 3$.

Definition 1.6. For $N \geq 2$, write S^{N-1} for the **unit sphere** is $\mathbb{R}^N$.

Theorem 1.7. *(Hausdorff paradox; AoC). There is a countable subset C of S^2 such that $S^2 \setminus C$ is $SO(3)$-paradoxical.*

Proof. Let A and B be rotations about the origin that generate the subgroup G of $SO(3)$ that they generate is isomorphic to $\mathbb{F}_2$. Each rotation $x \in G \setminus \{e\}$ has two fixed points in S^2 (poles). Construct

$$F := \{s \in S^2 \,:\, s \text{ is a fixed point for some } x \in G \setminus \{e\}\}$$

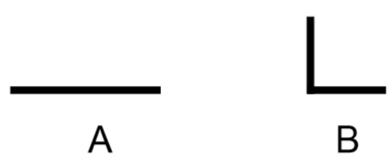
F is countable since G is. Set $C := \bigcup\{x \cdot F \,:\, x \in G\}$ (also countable). Then G acts on $S^2 \setminus C$ without non-trivial fixed points, so by Theorem 1.4, $S^2 \setminus C$ is $SO(3)$-paradoxical. $\square$

Remark. We are one step closer to showing that S^2 is $SO(3)$-paradoxical.

Definition 1.8. Let G be a group acting on a set S, and let A and B be subsets of S. Then, A and B are said to be **G-equidecomposable** if there are disjoint partitions $\{A_1, \ldots, A_n\}$ and $\{B_1, \ldots, B_n\}$ of A and B, and elements $x_1, \ldots, x_n \in G$ such that $x_j \cdot A_j = B_j$.

Notation. If A and B are G-equidecomposable, we shall write $A \sim B$.
Remark. Equidecomposability is an equivalence relation.

Example 1.9. The line segments below are $SO(2)$-equidecomposable.



Example 1.10. Let C be a circle and $x \in C$. Then $C \sim C \setminus \{x\}$. To see this pick θ so that $\theta/2\pi$ is not rational, put R_θ a rotation by θ, and let $H = \{R_\theta^n(x) \,:\, n = 0, 1, 2, \ldots\}$. Observe that $C = (C \setminus H) \cup H$; $C \setminus \{x\} = (C \setminus H) \cup (H \setminus \{x\})$; and $H \setminus \{x\} = R \cdot H$.

Theorem 1.11. *Let $C \subset S^2$ be countable. Then S^2 and $S^2 \setminus C$ are $SO(3)$-equidecomposable.*

Proof. Given C, let ℓ be a line through the origin that does not meet C. Consider the angles $\theta \in [0, 2\pi)$ that satisfy the property:

"There are $x \in C$ and $n \in \mathbb{N}$ such that $\rho \cdot c \in C$ where ρ is a rotation about ℓ by the angle $n\theta$"

Since C is countable, so is the set above, so we may pick $\theta_0 \in [0, 2\pi)$ lacking this property and let ρ be the rotation by θ_0. Then, $\rho^n \cdot C \cap C = \emptyset$ for all $n \in \mathbb{N}$ and so

$$\rho^n \cdot C \cap \rho^m C = \emptyset \quad n \neq m$$

Construct $D = \bigcup_{n=0}^{\infty} \rho^n \cdot C$ and observe that

$$S^2 = D \cup (S^2 \setminus D) \sim \rho \cdot D \cup (S^2 \setminus D) = S^2 \setminus C$$

as desired. □

Definition 1.12. Let G be a group acting on a set S, with $A, B \subseteq S$. We write $A \preceq_G B$ if A and a subset of B are equidecomposable.

We present a Schroder-Bernstein analogue for the relation $\preceq$.

Theorem 1.13. *Let G be a group acting on a set S, and let A and B be subsets of S such that $A \preceq_G B$ and $B \preceq_G A$. Then $A \sim_G B$.*

Proof. (Sketch) Pick bijections $\phi : A \to B_1$ and $\psi : B \to A_1$; set $C_0 = A \setminus A_1$ and $C_{n+1} = \psi(\phi(C_n))$ and $C = \bigcup_{n=0}^{\infty} C_n$. Then

$$A = (A \setminus C) \cup C \sim (B \setminus \phi(C)) \cup \phi(C) = B$$

□

Theorem 1.14. *Let G be a group acting on a set S. Then $E \subset S$ is G-paradoxical if and only if there is a partition $\{A, B\}$ of E such that $A \sim_G E \sim_G B$.*

Proof. ($\Longleftarrow$) Duh.

($\Longrightarrow$) Let $A_1, \ldots, A_n, B_1, \ldots, B_m$, $x_1, \ldots, x_n, y_1, \ldots, y_m$ be as in Definition 1.1. Set

$$A = \bigcup_{j=1}^{n} A_j \qquad B = \bigcup_{j=1}^{m} B_j$$

We claim that $E \preceq_G A$ and $E \preceq_G B$. Set $\tilde{A}_1 = x \cdot A_1$ and inductively define

$$\tilde{A}_j = x_j \cdot A_j \setminus (\tilde{A}_1 \cup \ldots \cup \tilde{A}_{j-1})$$

Then $\{\tilde{A}_1, \ldots, \tilde{A}_n\}$ is a partition of E such that $x_j^{-1} \cdot \tilde{A}_j \subset A_j$ and so $E \sim_G \bigcup_{j=1}^{n} x_j^{-1} \cdot \tilde{A}_j \subset A$, so that $E \preceq_G A$. Likewise, obtain $E \preceq_G B$. Naturally, being subsets of E, A and B satisfy $A \preceq_G E$ and $B \preceq_G E$. Hence, $A \sim_G E$ and $B \sim_G E$. □

Theorem 1.15. *Let G be a group acting on a set S, and let E and E' be subsets of S with $E \sim_G E'$. Then, if E is G-paradoxical, so is E'.*

Proof. Use Theorem 1.14 to get a partition $\{A, B\}$ of E, such that $A \sim_G E \sim_G B$. Then $A \sim_G E \sim_G E' \sim_G B$. □

Theorem 1.16. *S^2 is $SO(3)$-paradoxical.*

Proof. S^2 and $S^2 \setminus C$ are $SO(3)$-equidecomposable (Theorem 1.11). Furthermore, $S^2 \setminus C$ is $SO(3)$-paradoxical (Theorem 1.5). Thus, by Theorem 1.15, we are done. □

Theorem 1.17. *(Weak Banach-Tarski; AoC) Every closed ball in $\mathbb{R}^3$ is paradoxical.*

Proof. It is sufficient to show that $B = \overline{B}(0,1)$ is paradoxical. First, we show that $\overline{B}(0,1) \setminus \{0\}$ is $SO(3)$-paradoxical. Since S^2 is $SO(3)$-paradoxical, get $A_1, \ldots, A_n, B_1, \ldots, B_m \subset S^2$ and $x_1, \ldots, x_n, y_1, \ldots, y_m \in SO(3)$ satisfying Defn 1.1. Set

$$\tilde{A}_j = \{ta : t \in (0,1], a \in A_j\} \qquad j = 1, \ldots n$$

and,

$$\tilde{B}_j = \{tb : t \in (0,1], b \in B_j\} \qquad j = 1, \ldots, m$$

These sets are pairwise disjoint and satisfy

$$\overline{B}(0,1) \setminus \{0\} = \bigcup_{j=1}^{n} x_j \cdot \tilde{A}_j = \bigcup_{j=1}^{m} y_j \tilde{B}_j$$

so that $\overline{B}(0,1) \setminus \{0\}$ is $SO(3)$-paradoxical.
Now, we show that B and $B \setminus \{0\}$ are equidecomposable. Let ℓ be a line through $\left(0, 0, \frac{1}{2}\right)$ parallel to the xy-plane. Let ρ be a rotation about ℓ of infinite order. Set $C = \{\rho^n \cdot 0 : n = 0, 1, 2, \ldots\}$ and observe that $\rho \cdot C = C \setminus \{0\}$. Hence,

$$B = C \cup (B \setminus C) \sim \rho \cdot C \cup (B \setminus C) = B \setminus \{0\}$$

Victory! □