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Infinite-dimensional algebras and a trigonometric basis for the classical Lie algebras

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This paper explores features of the infinite-dimensional algebras that have been previously introduced. In particular, it is shown that the classical simple Lie algebras (A_N, B_N, C_N, D_N) may be expressed in an "egalitarian" basis with trigonometric structure constants. The transformation to the standard Cartan–Weyl basis, and the particularly transparent $N \to \infty$ limit that this formulation allows is provided.

I. INTRODUCTION

This article is a further contribution to our investigations of a class of infinite-dimensional Lie algebras whose structure constants are simple trigonometric functions.[1] The generators of the basic algebra are indexed by two-vectors $\mathbf{m} = (m_1, m_2)$ which, in the simplest case, are taken to lie on an integral square lattice, although in general the components need not be integral or even real. We also consider cases where they lie on a triangular lattice. The basic algebra is, in a convenient normalization,

$$[K_{\mathbf{m}+\mathbf{b}}, K_{\mathbf{n}+\mathbf{b}}] = (1/k)\sin k(\mathbf{m}\times\mathbf{n})K_{\mathbf{m}+\mathbf{n}+\mathbf{b}} + \mathbf{a}\cdot\mathbf{m}\,\delta_{\mathbf{m}+\mathbf{n},0}, \tag{1.1}$$

where $\mathbf{m}\times\mathbf{n} = m_1 n_2 - m_2 n_1$ and $\mathbf{a},\mathbf{b}$ are arbitrary two-vectors. If $\mathbf{b}$ is a lattice vector then $K_{\mathbf{m}}$ can be redefined to eliminate it. In what follows, we shall assume that this is the case and this has been done.

This algebra admits a superextension, but only if $\mathbf{a} = \mathbf{0}$, i.e., there is no bosonic c-number central extension, by adjoining the relations

$$\{F_{\mathbf{m}}, F_{\mathbf{n}}\} = \cos k(\mathbf{m}\times\mathbf{n})K_{\mathbf{m}+\mathbf{n}},$$
$$[K_{\mathbf{m}}, F_{\mathbf{n}}] = (1/k)\sin k(\mathbf{m}\times\mathbf{n})F_{\mathbf{m}+\mathbf{n}}. \tag{1.2}$$

There are special cases of (1.1) that are centerless with $\mathbf{m}$ on a two-dimensional integral lattice and have $k = 2\pi/N$ for some integer N. As this imposes a modulo-N arithmetic on the structure constants, the generators can be considered to be indexed by a toroidal integral lattice, in the sense that the generators $K_{\mathbf{m}+N\mathbf{a}}$ are identified with $K_{\mathbf{m}}$ for all integral two-vectors $\mathbf{a}$. The generators $\mathscr{K}_{\mathbf{m}}$ resulting from this identification, in a convenient normalization, satisfy the finite algebra

$$[\mathscr{K}_{\mathbf{m}}, \mathscr{K}_{\mathbf{n}}] = \sin(2\pi/N)(\mathbf{m}\times\mathbf{n})\mathscr{K}_{\mathbf{m}+\mathbf{n}}. \tag{1.3}$$

There is a particular realization of the superalgebra, in which $F_{\mathbf{m}}$ and $K_{\mathbf{m}}$ are identified, going back to Weyl[2] and his correspondence rule, given by

$$K_{\mathbf{m}} = (1/2ik)e^{i(2km_1P + m_2X)} = (1/ik)F_{\mathbf{m}}, \tag{1.4}$$

where (X,P) are canonically conjugate quantum variables with $[X,P] = i$. Using the familiar Baker–Campbell–Hausdorf expansion, the product is

$$K_{\mathbf{m}}K_{\mathbf{n}} = (1/2ik)e^{ik(m_1n_2 - m_2n_1)}K_{\mathbf{m}+\mathbf{n}}, \tag{1.5}$$

and therefore

$$K_{\mathbf{m}}K_{\mathbf{n}} = (1/2ik)e^{2ik(m_1n_2 - m_2n_1)}K_{\mathbf{n}}K_{\mathbf{m}}. \tag{1.6}$$

For the finite algebras (1.3) the relationship (1.6), evocative of quantum groups, is familiar from the work of 't Hooft and Belavin.[3] Also see Refs. 4. Naturally, it satisfies (1.1) and (1.2).

In fact, when N is odd, (1.3) is just the algebra of $\mathrm{SU}(N)\times\mathrm{U}(1)$. Although the fact that there exists a basis for $\mathrm{SU}(N)$ in which the structure constants are simple trigonometric functions has been recorded several times, it still elicits surprise, and there are as yet only a few articles[5] developing the theory of semi-simple Lie algebras from this maximal grading "egalitarian" point of view, in which all the generators appear on the same footing.

In this paper, we explore several features of the algebra (1.1) and the finite algebras (1.3), as well as their $N \to \infty$ limit,

$$[L_{\mathbf{m}}, L_{\mathbf{n}}] = (\mathbf{m}\times\mathbf{n})L_{\mathbf{m}+\mathbf{n}} + \mathbf{a}\cdot\mathbf{m}\,\delta_{\mathbf{m}+\mathbf{n},0}. \tag{1.7}$$

This constitutes the algebra of infinitesimal area-preserving diffeomorphisms of the torus, $\mathrm{SDiff}(T^2)$,[6,7] which we have identified with that of $\mathrm{SU}(\infty)$.[1]

(i) We find the Casimir invariants of (1.1), and its related limit algebra (1.7).

(ii) We explain how both algebras may be realized as algebras of differential operators on surface coordinates and show how they act as algebras of derivations.

(iii) We transform the finite algebras to their Cartan–Weyl bases and demonstrate that for N even the algebra (1.3) is that of $\mathrm{U}(N/2)^4$.

(iv) We identify several subalgebras of (1.3) corresponding to $\mathrm{SO}(N)$ and $\mathrm{USp}(N)$ and express them in a similar neat form.

(v) We also consider algebras whose generators are indexed by a triangular lattice.

(vi) Finally, we utilize the surface coordinate formalism to express gauge theories for $\mathrm{SU}(\infty)$, $\mathrm{SO}(\infty)$, and $\mathrm{USp}(\infty)$ and identify the Schild string action present within the Yang–Mills action.

II. CASIMIR INVARIANTS

The construction of Casimir invariants is modeled upon that for the finite algebras (1.3) discussed by Patera and

 0022-2488/90/051088-07$03.00

Zassenhaus.[5] The quadratic Casimir is

$$\sum_{\mathbf{m}} K_{\mathbf{m}} K_{-\mathbf{m}}. \tag{2.1}$$

There are, in general, two Casimir invariants of each degree above the quadratic. They are the real and imaginary parts of

$$\sum_{\mathbf{m},\mathbf{n}} e^{ik\mathbf{m}\times\mathbf{n}} K_{\mathbf{m}} K_{\mathbf{n}} K_{-\mathbf{m}-\mathbf{n}},\ldots,$$

$$\sum_{\mathbf{m}_1,\mathbf{m}_2,\ldots,\mathbf{m}_r} \Big(\prod_{\alpha<\beta} e^{ik(\mathbf{m}_\alpha\times\mathbf{m}_\beta)}\Big) K_{\mathbf{m}_1} K_{\mathbf{m}_2} \cdots K_{\mathbf{m}_r} K_{-\mathbf{m}_1-\mathbf{m}_2-\cdots-\mathbf{m}_r}. \tag{2.2}$$

Taking the imaginary part, a generic coefficient will be of the form

$$\sin(k(\mathbf{m}\times\mathbf{n} + \mathbf{m}\times\mathbf{p} + \mathbf{n}\times\mathbf{p} + \cdots)).$$

By use of the addition formula for sines, this will always be reducible to terms with a typical $\sin k(\mathbf{m}\times\mathbf{n})$ factor. Whenever the remaining factor in such a term is symmetric in $\mathbf{m}$ and $\mathbf{n}$, after use of the commutation relations to make $K_{\mathbf{m}}$ and $K_{\mathbf{n}}$ adjacent, it is easy to see that this contribution to the Casimir may be reduced to one of one degree lower. For example, in the case of the cubic,

$$\sum_{\mathbf{m},\mathbf{n}} \sin(k(\mathbf{m}\times\mathbf{n})) K_{\mathbf{m}} K_{\mathbf{n}} K_{-\mathbf{m}-\mathbf{n}}$$

$$= \sum_{\mathbf{m},\mathbf{n}} \sin^2(k(\mathbf{m}\times\mathbf{n})) K_{\mathbf{m}+\mathbf{n}} K_{-\mathbf{m}-\mathbf{n}}. \tag{2.3}$$

Re-summing over $\mathbf{m}+\mathbf{n}$ and $\mathbf{m}-\mathbf{n}$ we see that the right-hand side diverges, without an infinite renormalization of $K_{\mathbf{m}}$. Such a renormalization, however, would make the cosine-like contributions vanish.

The Casimirs of (1.7) follow by a $k\to 0$ limiting procedure. Again there are apparently two for each degree: one of which can be reduced in degree as above, again with a divergent result.

III. DERIVATIONS

The algebra (1.7) is known to be, in a particular basis optimal for the torus, that of the generic area-preserving (symplectic) reparametrizations of a two-surface. Taking x and p to be local (commuting) coordinates for the surface, and f and g to be differentiable functions of them, a basis-independent realization for the generators of the centerless algebra is[6,7]

$$L_f = \frac{\partial f}{\partial x}\frac{\partial}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial}{\partial x} \Rightarrow \tag{3.1}$$

$$[L_f, L_g] = L_{\{f,g\}}, \quad [L_f, g] = \{f,g\}, \tag{3.2}$$

where

$$\{f,g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} \tag{3.3}$$

is the Poisson bracket of classical phase space. The generator L_f transforms (x,p) to $(x - \partial f/\partial p, p + \partial f/\partial x)$. Infinitesimally, this is a canonical transformation generated by f, which preserves the phase space area element $dx\,dp$. This may be regarded as the flow generated by an arbitrary Hamiltonian f. For a small patch of two-surface, the functions $f(x,p)$ may be expanded in any coordinate basis. If the surface is a torus, the preferred basis is $\exp(inx + imp)$; if it is a sphere, spherical harmonics; if it is a plane, powers; and so on. Nevertheless, any coordinate basis will do for the infinitesimal transformations effected by the algebra in a local patch.

Another realization and application of this algebra appears in the work of Case and Monge,[8] which investigates the algebra of conserved currents of the Kadomtsev–Petviashvili equation. Their algebra is seen to be contained in (1.1) with $k\to 0$, after a change of variables.

We found a basis-independent differential operator realization of K_f, corresponding to (3.1):

$$K_f = \frac{1}{2ik} f\Big(x + ik\frac{\partial}{\partial p}, p - ik\frac{\partial}{\partial x}\Big). \tag{3.4}$$

In the torus basis, this becomes

$$\begin{aligned} K_{(m_1,m_2)} &= \frac{i}{2k}\exp\Big(im_1 x + km_2\frac{\partial}{\partial x} + im_2 p - km_1\frac{\partial}{\partial p}\Big) \\ &= \frac{i}{2k}\exp(im_1 x + im_2 p) \\ &\quad \times \exp\Big(km_2\frac{\partial}{\partial x} - km_1\frac{\partial}{\partial p}\Big), \end{aligned} \tag{3.5}$$

somewhat analogous to the one-variable realization found by Hoppe.[7] Note the triviality in this realization of the Casimir operators, as the indices of each of their terms sum to zero.

Just as the algebra (1.7) may be thought of as the Fourier transform of the Poisson bracket algebra, (3.2), the algebra with general k is the Fourier transform of the "sine bracket" algebra

$$[K_f, K_g] = K_{\sin\{f,g\}}, \tag{3.6}$$

where the analog of the Poisson bracket in this case is the sine, or Moyal, bracket $\sin\{f,g\}$. This is the extension of the Poisson brackets $\{f,g\}$ to statistical distributions on phase space, introduced by Weyl[2] and Moyal.[9] It is generalized convolution that reduces to the Poisson bracket as $\hbar$, replaced by $2k$ in our context, is taken to zero:

$$\begin{aligned} \sin\{f,g\} = \frac{-1}{4\pi^2 k^3}\int & dp'\,dp''\,dx'\,dx''\,f(x',p') \\ & \times g(x'',p'')\sin(1/k)(p(x'-x'') \\ & + x(p''-p') + p'x'' - p''x'). \end{aligned} \tag{3.7}$$

The argument of the sine above is

$$\frac{1}{k}\det\begin{pmatrix} 1 & p & x \\ 1 & p' & x' \\ 1 & p'' & x'' \end{pmatrix} = \frac{1}{k}\int p\cdot dq, \tag{3.8}$$

i.e., $2/k$ times the area of the equilateral phase-space triangle with vertices at (x,p), (x',p'), and (x'',p''). The antisymmetry of f with g is evident in the determinant. The sine brackets satisfy the Jacobi identities,[10] just as their Fourier components (1.1) do, and thus determine a Lie algebra. These brackets help reformulate quantum mechanics in terms of Wigner's phase-space distribution.[11]

The Poisson bracket $\{f,h\}$ acts as a derivation for both the Poisson bracket and the ordinary product fg, i.e.,

$$\delta(fg) = (\delta f)\, g + g(\delta f), \tag{3.9}$$

where $\delta(f) = \{f,h\}$ for a given h. Similarly, the sine bracket acts as a derivation for the sine bracket and also for the cosine bracket as "product."

$$\delta \cos(f,g) = \cos(\delta f, g) + \cos(f, \delta g), \tag{3.10}$$

where $\delta(f) = \sin\{f,h\}$. The cosine bracket $\cos(f,g)$ is the counterpart of (3.7) for the cosine. These relations are a simple consequence of the graded Jacobi identities for the respective algebras. [In the special case where $h = p$, (3.9) is just the Leibniz rule for the differentiation of a product.] The sine bracket relation with $h = e^{ap}$ generates finite central differences.

IV. CARTAN–WEYL BASIS

We have already shown[1] that for N odd the algebra (1.3) describes $\mathrm{U}(N)$, and that for N even it contains a $\mathrm{U}(N/2)$ subalgebra. By finding the combinations of the $\mathscr{K}$'s which form the Cartan–Weyl basis, we shall demonstrate that the full algebra in the N even case is $\mathrm{U}(N/2)^4$.

Patera and Zassenhaus[5] and Pope and Stelle[12] have transformed the algebra between the trigonometrical basis and the standard $\mathrm{GL}(N)$ basis. We, instead, exhibit the connection of the generators $\mathscr{K}_m$ to the Cartan–Weyl basis h_i and e_α, where the h_i are the members of the Cartan subalgebra H, and α is in the root space Σ.

Here we shall carry out the transformation of the finite algebras (1.3) to the Cartan–Weyl basis, first for N odd, showing that it is $\mathrm{SU}(N)\times\mathrm{U}(1)$. The generator $\mathscr{K}_{0,0}$ factors out of the algebra, as it commutes with the other N^2-1 and cannot result as a commutator of any of them. This is the $\mathrm{U}(1)$ part of the algebra.

The Cartan–Weyl basis for $\mathrm{SU}(N)$ has the following commutation relations, in the usual notation:

$$[e_\alpha, e_\beta] = \begin{cases} N_{\alpha\beta}\, e_{\alpha+\beta}, & \text{if } \alpha+\beta \in \Sigma, \\ (e_\alpha, e_{-\alpha}) h_\alpha, & \text{if } \alpha+\beta = 0, \\ 0, & \text{otherwise,} \end{cases} \tag{4.1}$$

$$[h_i, e_\alpha] = \alpha(h) e_\alpha, \tag{4.2}$$

$$[h_i, h_j] = 0. \tag{4.3}$$

In the case of N odd, the combinations of $\mathscr{K}$'s that give this basis are

$$E^p_q = \sum_{j=0}^{N-1} \omega^{\,(2j-q)p} \mathscr{K}_{j,q-j}, \quad \omega^N = 1, \tag{4.4}$$

where $q=0$ and $p = 1,\dots,\frac{1}{2}(N-1)$ for the Cartan subalgebra, and $q = 1,\dots,N-1$ and $p = 0,\dots,N-1$, for the remaining generators. This may be shown by checking the commutation relations as follows:

$$\begin{aligned}
[E^{p_1}_{q_1}, E^{p_2}_{q_2}] &= \sum_{j=0}^{N-1}\sum_{k=0}^{N-1} \omega^{(2j-q_1)p_1 + (2k-q_2)p_2} \times [\mathscr{K}_{j,q_1-j}, \mathscr{K}_{k,q_2-k}] \\
&= \sum_{j=0}^{N-1}\sum_{k=0}^{N-1} \omega^{(2j-q_1)p_1 + (2k-q_2)p_2} \times \sin\frac{2\pi}{N}(jq_2 - kq_1)\mathscr{K}_{j+k,q_1+q_2-j-k} \\
&= \frac{1}{2i}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1} \omega^{(2j-q_1)p_1 + (2k-q_2)p_2} \times (\omega^{jq_2-kq_1} - \omega^{-jq_2+kq_1})\mathscr{K}_{j+k,q_1+q_2-j-k}.
\end{aligned} \tag{4.5}$$

Putting $s = j+k$ and $q = q_1+q_2$ and then resumming, using the invariance modulo N:

$$\begin{aligned}
[E^{p_1}_{q_1}, E^{p_2}_{q_2}] &= \frac{1}{2i}\left(\sum_{s=0}^{N-1}\sum_{k=0}^{s} + \sum_{s=N}^{2N-2}\sum_{k=s-N+1}^{N-1}\right) \times \omega^{(2s-2k-q_1)p_1 + (2k-q_2)p_2} \\
&\quad \times (\omega^{sq_2-kq} - \omega^{-sq_2+kq})\mathscr{K}_{s,q-s} \\
&= \frac{1}{2i}\sum_{s=0}^{N-1}\omega^{(2s-q_1)p_1 - q_2p_2}\sum_{k=0}^{N-1}(\omega^{2k(p_2-p_1)-kq+sq_2} \\
&\quad - \omega^{2k(p_2-p_1)+kg-sq_2})\mathscr{K}_{s,q-s}.
\end{aligned} \tag{4.6}$$

Using $\Sigma_{k=0}^{N-1}\omega^{ak} = N\delta_{a,0}$, and then comparing with our expression for E^p_q, we obtain

$$\begin{aligned}
[E^{p_1}_{q_1}, E^{p_2}_{q_2}] &= \frac{1}{2i}\sum_{s=0}^{N-1}\omega^{(2s-q_1)p_1 - q_2p_2} N(\omega^{sq_2}\delta_{2(p_2-p_1)-q,0} \\
&\quad - \omega^{-sq_2}\delta_{2(p_2-p_1)+q,0})\mathscr{K}_{s,q-s} \\
&= (N/2i)(E^{p_1+q_2/2}_{q_1+q_2}\delta_{2(p_2-p_1)-q,0} \\
&\quad \times - E^{p_1-q_2/2}_{q_1+q_2}\delta_{2(p_2-p_1)+q,0}),
\end{aligned} \tag{4.7}$$

where the $q_2/2$ may be defined as an integer (as N may be added to the power of ω to ensure that this power is even), so the halving of the index is defined by:

$$\frac{q}{2}\begin{cases} q/2, & q \text{ even}, \\ (q+N)/2, & q \text{ odd}. \end{cases} \tag{4.8}$$

And so

$$[E^{p_1}_{q_1}, E^{p_2}_{q_2}] = \begin{cases} \pm (N/2i)E^{p_1 \pm q_2/2}_{q_1+q_2}, & \text{if } 2(p_2-p_1) = \pm(q_1+q_2), \\ (N/2i)(E^{p_1+q_2/2}_0 - E^{p_1-q_2/2}_0), & \text{if } q_1+q_2 = 0 = p_2-p_1, \\ 0, & \text{otherwise,} \end{cases} \tag{4.9}$$

which corresponds to (4.1).

$$[E^{p_1}_0, E^{p_2}_q] = \begin{cases} \pm (N/2i)E^{p_2}_q, & \text{if } 2(p_2-p_1) = \pm q, \\ 0, & \text{otherwise,} \end{cases} \tag{4.10}$$

showing the basis is diagonal (4.2), and

$$[E^{p_1}_0, E^{p_2}_0] = 0, \tag{4.11}$$

as required.

For the case N even, the situation is more complicated. This time four of the $\mathscr{K}$'s disconnect into U(1)'s, $\mathscr{K}_{0,0}$, $\mathscr{K}_{N/2,0}$, $\mathscr{K}_{0,N/2}$, and $\mathscr{K}_{N/2,N/2}$. This leaves N^2-4 generators, which we shall show span four commuting SU$(N/2)$'s. There are slight differences between the cases (a) $N \equiv 0 \bmod 4$ and (b) $N \equiv 2 \bmod 4$, but the principle of construction is the same. As before, the Cartan subalgebra is spanned by the elements whose indices sum to 0 mod $N/2$. Note that there are $2N-4$ such operators, after excluding the four U(1)'s.

For (a), $N \equiv 0 \bmod 4$, the generators in the Cartan–Weyl basis are

$$E_q^{s,p} = \sum_{a=0}^{1}\sum_{j=0}^{N-1} \omega^{pj}(-1)^{s(j+a)+a(j+1)}\mathscr{K}_{j,q-j+aN/2}, \tag{4.12}$$

and for (b), $N \equiv 2 \bmod 4$,

$$E_q^{s,p} = \sum_{a=0}^{1}\sum_{j=0}^{N-1} \omega^{pj}(-1)^{s(j+1)+(a+1)(j+1+p)} \times \mathscr{K}_{j,q-j+aN/2}, \tag{4.13}$$

where the q labels the sum of the indices, $q=0,...,N/2-1$, and s,p take the values $s=0,1, p=0,...,N-1$. Then for case (a) the elements $E_q^{s,p}$ for $s=0,1$; $q+p$ even, odd span the four commuting SU$(N/2)$'s. For case (b), the splitting is into those with $s+q$ even, odd and p even, odd.

That the above combinations are the generators of SU$(N/2)^4$ in the Cartan–Weyl basis may be shown by checking the commutation relations in a similar fashion to the N odd case above. Commuting two of the E's gives an expression which may be resummed so that the coefficients of the $\mathscr{K}$'s can be read off. The resummed expression for (a), with $a=a_1+a_2$, $j=j_1+j_2$, $q=q_1+q_2$, and $p_- = p_1 - p_2$ is

$$[E_{q_1}^{s_1,p_1},E_{q_2}^{s_2,p_2}] = \sum_{a=0}^{1}\sum_{j=0}^{N-1}\sum_{a_1=0}^{1}\sum_{j_1=0}^{N-1} \omega^{j_1p_-+p_2j} \times \sin\frac{2\pi}{N}(qj_1-jq_1) \times(-1)^{(s_1+s_2)(a_1+j_1)+s_2(a+j)+a(j+1)}\mathscr{K}_{j,q-j+(N/2)a}. \tag{4.14}$$

The separation into $s=0,1$ is evident as the only dependence of the coefficient of the $\mathscr{K}$ on a_1 is of the form $(-1)^{(s_1+s_2)a_1}$, so if $s_1 \neq s_2$ then the two terms in that sum exactly cancel. When $s_1=s_2$, the coefficient indexed by a,j becomes

$$\begin{aligned}&(-1)^{s_1(a+j)+a(j+1)}\omega^{p_2j}\frac{1}{i} \times \sum_{j_1=0}^{N-1}\omega^{j_1p_-}(\omega^{j_1q-jq_1}-\omega^{-j_1q+jq_1})\\ &=(-1)^{s_1(a+j)+a(j+1)}\omega^{p_2j}(N/i) \times(\omega^{-jq_1}\delta_{p_-+q,0}-\omega^{jq_1}\delta_{p_--q,0}).\end{aligned}$$

The δ's are both zero if p_1+q_1 and p_2+q_2 have different parity, showing the overall split into four commuting subspaces. This coefficient may be compared that in (4.12), and the commutator rewritten as

$$[E_{q_1}^{s_1,p_1},E_{q_2}^{s_2,p_2}] = (N/i)\delta_{s_1,s_2}(E_{q_1+q_2}^{s_1,p_2-q_1}\delta_{p_1-p_2+q_1+q_2,0} - E_{q_1+q_2}^{s_1,p_2+q_1}\delta_{p_1-p_2-(q_1+q_2),0}). \tag{4.15}$$

Similarly for (b),

$$[E_{q_1}^{s_1,p_1},E_{q_2}^{s_2,p_2}] = \sum_{a=0}^{1}\sum_{j=0}^{N-1}\sum_{a_1=0}^{1}\sum_{j_1=0}^{N-1} \omega^{j_1p_-+p_2j}(-1)^{(s_1+s_2)(j_1+1)+a_1p_-+a(j+p_2+1)+j(s_2+1)+p_-} \times \sin 2\pi/N(qj_1-jq_1)\mathscr{K}_{j,q-j+(N/2)a}. \tag{4.16}$$

This time, the coefficient of a_1 in the exponent is p_-, so the space splits into p even, odd. The coefficient of the $\mathscr{K}$ here is

$$\begin{aligned}&(-1)^{s_1+s_2+(s_2+a+1)j+a(p_1+1)}\omega^{p_2j}\frac{1}{i}\sum_{j_1=0}^{N-1}(-1)^{j_1(s_1+s_2)}\omega^{j_1p_-}(\omega^{j_1q-jq_1}-\omega^{-j_1q+jq_1})\\ &=(-1)^{s_1+s_2+(s_2+a+1)j+a(p_1+1)}\omega^{p_2j}(N/i)(\omega^{-jq_1}\delta_{p_-+q,(s_1+s_2)(N/2)}-\omega^{jq_1}\delta_{p_--q,(s_1+s_2)(N/2)})\\ &=N/i(-1)^{(s_1+q_1+p_1+1)+(j+1)(s_2+q_1)+(a+1)(j+1+p_1)}\begin{pmatrix}\omega^{(p_2-q_1(1+N/2))j}\delta_{p_-+q,(s_1+s_2)(N/2)}\\ -\omega^{(p_2+q_1(1+N/2))j}\delta_{p_--q,(s_1+s_2)(N/2)}\end{pmatrix}.\end{aligned}$$

Both δ functions are zero if s_1+q_1 and s_2+q_2 have different parity. In performing the above manipulations the fact that $\omega^{N/2} = -1$ has been used. Thus, for $p_1 \equiv p_2 \bmod 2$,

$$[E_{q_1}^{s_1,p_1},E_{q_2}^{s_2,p_2}] = (N/i)(-1)^{s_1+q_1+p_1+1}(E_q^{s,p_2-q_1(1+N/2)} \times\delta_{p_-+q,(s_1+s_2)N/2} - E_q^{s,p_2+q_1(1+N/2)}\delta_{p_--q,(s_1+s_2)N/2}), \tag{4.17}$$

where $q \equiv q_1 + q_2 \bmod N/2$, and $s \equiv q + q_1 + q_2 \bmod 2$, so that $s + q \equiv s_1 + q_1 \equiv s_2 + q_2 \bmod 2$.

Saveliev and Vershik[13] have discussed infinite algebras of the type SDiff(T^2) and their generalizations directly in the Cartan–Weyl basis. In our notation, the Cartan–Weyl basis for the infinite algebras L's in (1.7) is

$$E^p_q = \sum_j e^{ipj} L_{1/2(q+j),(1/2)(q-j)}. \tag{4.18}$$

This result may be obtained by checking the commutation relation

$$[E^{p_1}_{q_1}, E^{p_2}_{q_2}] = \tfrac{1}{2}(q_1+q_2)\delta'(p_1-p_2)E^{(1/2)(p_1+p_2)}_{q_1+q_2} - \tfrac{1}{2}(q_1-q_2)\delta(p_1-p_2)\frac{\partial}{\partial p_1}E^{(1/2)(p_1+p_2)}_{q_1+q_2}. \tag{4.19}$$

In Saveliev and Vershik's notation,

$$X_q(f) = \int_{-\infty}^{\infty} f(p) E^p_q \, dp. \tag{4.20}$$

Thus, multiplying equation (4.19) by $f(p_1)g(p_2)$ and performing this integral transform yields precisely their equation for the SU(∞) commutator

$$[X_{q_1}(f), X_{q_2}(g)] = X_{q_1+q_2}\big(q_2 f'(p_1+p_2)g(p_1+p_2) - q_1 f(p_1+p_2)g'(p_1+p_2)\big). \tag{4.21}$$

V. SUBALGEBRAS

In this section, we exhibit all the (nonexceptional) classical Lie algebras in a 'trigonometrical' basis analogous to that of SU(N). Since these can fit as subalgebras in SU(N), we can extract them from it, and hence SO(∞) and USp(∞) out of the limit SU(∞). To simplify the analysis, we introduce matrices J which satisfy (1.3), but with structure constant π/N, instead of $2\pi/N$. These matrices provide a basis for one of the four copies of U$(N)^4$, and hence for SU(N) equally well for N even or odd.

Consider matrices g,h (see Ref. 1),

$$g = \sqrt{\omega}\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \omega & 0 & \cdots & 0 \\ 0 & 0 & \omega^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \omega^{N-1} \end{pmatrix},$$

$$h = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad g^N = h^N = -\mathbb{1}, \tag{5.1}$$

with $\omega = e^{2\pi i/N}$, $\sqrt{\omega} = e^{\pi i/N}$. They obey the identity

$$hg = wgh. \tag{5.2}$$

Then there is a complete set of unitary $N \times N$ matrices

$$J_{(m_1,m_2)} = \omega^{m_1 m_2/2} g^{m_1} h^{m_2}, \tag{5.3}$$

which satisfy

$$\operatorname{Tr} J_{(m_1,m_2)} = 0, \quad \text{except for } m_1 = m_2 = 0 \bmod N, \tag{5.4}$$

and span the algebra of SU(N). Like the Pauli matrices, they close under multiplication to just one such, (a finite group), by virtue of (5.2):

$$J_{\mathbf{m}} J_{\mathbf{n}} = \omega^{\mathbf{n}\times\mathbf{m}/2} J_{\mathbf{m}+\mathbf{n}}. \tag{5.5}$$

They therefore satisfy the algebra

$$[J_{\mathbf{m}}, J_{\mathbf{n}}] = -2i \sin((\pi/N)\mathbf{m}\times\mathbf{n}) J_{\mathbf{m}+\mathbf{n}}. \tag{5.6}$$

It might appear that the fundamental period is $2N$ instead of N. However, note that, by virtue of the symmetry

$$J_{\mathbf{m}+\mathbf{N}(r,s)} = (-1)^{(m_1+1)s + (m_2+1)r} J_{\mathbf{m}}, \tag{5.7}$$

only indices in the fundamental cell $N \times N$ need be considered.

The subalgebras may be written as combinations of J's which close on themselves. Those of most interest are,

$$J_{m_1,m_2} - (-)^a J_{m_1,-m_2} \quad \begin{cases} a = 0, & & \mathrm{SO}(N) \\ a = m_1, & N \text{ even}, & \mathrm{USp}(N) \\ a = m_2, & N \text{ even}, & \mathrm{SO}(N) \\ a = m_1 + m_2, & N = 4M, & \mathrm{USp}(N) \\ a = m_1 + m_2, & N = 4M+2, & \mathrm{SO}(N) \end{cases},$$

$$J_{m_1,m_2} - (-)^a J_{m_2,m_1} \quad \begin{cases} a = 0, & & \mathrm{SO}(N), \\ a = m_1 + m_2, & N \text{ even}, & \mathrm{SO}(N). \end{cases}$$

As an example, consider the second case, with $a = 0$, N odd. We denote

$$J_{[m_1,m_2]} = J_{m_1,m_2} - J_{m_2,m_1}. \tag{5.8}$$

The number of generators of these algebras is $(1/2)N(N-1)$. The commutation relations are

$$[J_{[m_1,m_2]}, J_{[n_1,n_2]}] = -2i\begin{pmatrix} \sin(\pi/N)(m_1 n_2 - m_2 n_1) J_{[m_1+n_1, m_2+n_2]} \\ -\sin(\pi/N)(m_1 n_1 - m_2 n_2) J_{[m_1+n_2, m_2+n_1]} \end{pmatrix}. \tag{5.9}$$

These algebras are presently shown to be SO(N).

It is convenient to label the generators by $q = m_1 + m_2$, the sum of the indices. Those with $q = 0 \bmod N$ all mutually commute, and this is taken as the Cartan subalgebra. Forming the Cartan–Weyl basis amounts to simultaneously diagonalizing the matrices which are the elements of the Cartan subalgebra in the adjoint representation, i.e., the matrices of structure constants on commutation of h with the e_α's,

$$[h, e_\alpha] = \sum_\beta M_\alpha \beta^e \beta. \tag{5.10}$$

These matrices are block diagonal, with a block for each $q = 1,\ldots,N-1$, of size $r = \tfrac{1}{2}(N-1)$. The blocks are all of the form, independent of q, of

$$M_r = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{5.11}$$

Thus the combinations of the generators with a given q proportional to the eigenvectors of M_r are diagonal.

The characteristic polynomial, P_r, of M_r is given (up to sign) by the recurrence relation

$$P_r = \lambda P_{r-1} - P_{r-2}, \tag{5.12}$$

with $P_0 = 1$ and $P_1 = \lambda - 1$. This may be solved by writing $\lambda = 2\cos\phi$, then

$$\begin{aligned} P_r &= \cos r\phi - [(1-\cos\phi/\sin\phi]\sin r\phi \\ &= (1/\sin\phi)(\sin(r+1)\phi - \sin r\phi) \\ &= (2/\sin\phi)\cos(r+\tfrac{1}{2})\phi\sin(\phi/2). \end{aligned} \tag{5.13}$$

This vanishes when $\phi = \phi^k = (2k-1)\pi/N$, $k = 1,..., N$.

We define

$$P_j^k = (\sin\phi^k)^{-1}(\sin(j+1)\phi^k - \sin j\phi^k), \tag{5.14}$$

so that $P_r^k = P_r = 0$. Then the eigenvector of M_r corresponding to the eigenvalue $\lambda^k = 2\cos\phi^k$ is

$$(P_0^k, P_1^k, ..., P_{r-1}^k).$$

Now it is clear that the combinations of generators that diagonalize the basis are

$$S_q^k = \sum_{j=0}^{r-1} P_j^k J_{[1/2(q+2j-1), 1/2(q-2j+1)]}. \tag{5.15}$$

The Cartan elements are $H^\alpha = J_{[\alpha,-\alpha]}$.

Working out the commutation relations in this basis,

$$\begin{aligned} [H^\alpha, S_q^k] &= \left[J_{[\alpha, N-\alpha]}, \sum_{j=0}^{r-1} P_j^k J_{[(1/2)(q+2j-1)(1/2)(q-2j+1)]} \right] \\ &= \sum_{j=0}^{r-1} (\sin\phi^k)^{-1}(\sin(j+1)\phi^k - \sin j\phi^k)[J_{[\alpha,-\alpha]}, J_{[(1/2)(q+2j-1),(1/2)(q-2j+1)]}] \\ &= \frac{-2i\sin(\pi/N)\alpha q}{\sin\phi^k} \sum_{j=0}^{r-1} (\sin(j+1)\phi^k - \sin j\phi^k) \\ &\quad \times (J_{[(1/2)(q+2j-1)+\alpha,(1/2)(q-2j+1)-\alpha]} - J_{[(1/2)(q-2j+1)+\alpha(1/2)(q+2j-1)-\alpha]}). \end{aligned}$$

Now consider the coefficient of $J_{[(1/2)(q+2l-1),(1/2)(q-2l+1)]}$. This is

$$\begin{aligned} &\frac{-2i\sin(\pi/N)\alpha q}{\sin\phi^k}(\sin(l-\alpha)\phi^k - \sin(l-\alpha-1)\phi^k \\ &\quad + \sin(l+\alpha)\phi^k - \sin(l+a-1)\phi^k) \\ &= \frac{-4i\sin(\pi/N)\alpha q\cos\alpha\phi^k}{\sin\phi^k}(\sin l\phi^k - \sin(l-1)\phi^k), \end{aligned}$$

therefore,

$$\begin{aligned} [H^\alpha, S_q^k] &= \frac{-4i\sin(\pi/N)\alpha q\cos\alpha\phi^k}{\sin\phi^k} \\ &\quad \times \sum_{l=0}^{r-1} P_l^k J_{[1/2(q+2l-1),1/2(q-2l+1)]} \\ &= \frac{-4i\sin(\pi/N)\alpha q\cos\alpha\phi^k}{\sin\phi^k} S_q^k, \end{aligned}$$

showing that this is the diagonal basis.

In a recent paper, a similar identification of some subalgebras has been made by Pope and Romans.[14] They introduce a basis for SO(N) and another for USp(N) which are extensions of the 't Hooft[3] basis to make the identification. Furthermore, they identify the infinite limits of these subalgebras with the group of diffeomorphisms acting on two dimensional manifolds with different topology from the torus; in the one case a Klein bottle, in the other a projective plane.

VI. TRIANGULAR LATTICES

As was mentioned in the Introduction, it is possible to realize similar infinite algebras on other lattices. In this section we report results for triangular lattices.

It is convenient to choose a system of barycentric coordinates, and index K by three integers m_1, m_2, m_3, where $m_1 + m_2 + m_3 = 0$. (Barycentric coordinates measure the perpendicular distances of any point from the edges of the fundamental reference triangle, as in the Dalitz plot.)

For this case, the relations are

$$[K_{\mathbf{m}}, K_{\mathbf{n}}] = \sin(k\mathbf{u}\cdot(\mathbf{m}\times\mathbf{n}))K_{\mathbf{m}+\mathbf{n}}, \tag{6.1}$$

where $k = \pi/N$, and $\mathbf{u}$ is the vector (1,1,1). As before, we find finite algebras by identifying generators at lattice points equivalent modulo N in each index. When $N \equiv 0$ mod 3 the generators whose indices are congruent modulo N all disconnect into U(1)'s. This leaves a hexagonal lattice, and the algebras obtained are U$(N/3)^6$. When $N \not\equiv 0$ mod 3, the fundamental lattice vectors of points reduced mod N contains only one disconnected member, (0,0,0), and the remaining $N^2 - 1$ points are associated with generators which close on SU(N). This situation is parallel with that for the square lattice.

VII. LARGE N LIMITS, AND SU(∞) YANG–MILLS

The two-index SU(N) basis considered here has a particularly simple large N limit. As N increases, the fundamental $N \times N$ cell covers the entire index lattice; the operators $\mathscr{K}$ are supplanted by the K's and, in turn, since $k \to 0$, by the operators L of (1.7).

More directly, it is immediately evident by inspection that, as $N \to \infty$, the SU(N) algebra (5.6) goes over to the centerless algebra (1.7) of SDiff(T^2) through the identification:

$$(iN/2\pi)J_{\mathbf{m}} \to L_{\mathbf{m}}. \tag{7.1}$$

An identification of this type was first noted by Hoppe[7] in the context of membrane physics: He connected the infinite N limit of the SU(N) algebra in a special basis to that of SDiff(S^2), i.e., the infinitesimal symplectic diffeomorphisms in the sphere basis. A discussion of the group topology of $SU(N)$, or $SDiff(T^2)$ vs $SDiff(S^2)$, or other two-dimensional manifolds for that matter[14] goes beyond the scope of this type of local analysis. In view of the $SO(N)$ subalgebras described in (5.8) we may also simply identify the $SO(\infty)$ subalgebra with the Poisson bracket subalgebra whose shift potentials f are odd under interchange of x with p, corresponding to Hamiltonians which evolve even functions to even ones, and odd to odd ones. Likewise, $USp(\infty)$ is generated by shift potentials of the form $\exp(im_1x)\sin(m_2p - m_1\pi/2)$, i.e., toroidal phase-space Hamiltonians odd under $p \mapsto -p$, $x \mapsto x + \pi$.

Floratos, Iliopoulos, and Tiktopoulos[15] utilized Hoppe's identification to take the limit of $SU(N)$ gauge theory. Their results are immediately reproduced without ambiguity, again by inspection, on the basis of the orthogonality condition dictated by (5.4) and (5.5);

$$\mathrm{Tr}\, J_{\mathbf{m}} J_{\mathbf{n}} = N\delta_{\mathbf{m}+\mathbf{n},0} \to \mathrm{Tr}\, L_{\mathbf{m}} L_{\mathbf{n}} = -[N^3/(2\pi)^2]\delta_{\mathbf{m}+\mathbf{n},0}. \tag{7.2}$$

As a result, for a gauge field A_μ in an $SU(N)$ matrix normalization with trace 1, the analog of

$$A_\mu \equiv (1/\sqrt{N}) A_\mu^{\mathbf{m}} J_{\mathbf{m}} \to [2\pi/(iN^{3/2})] A_\mu^{\mathbf{m}} L_{\mathbf{m}} = \widetilde{A}_\mu^{\mathbf{m}} L_{\mathbf{m}}, \tag{7.3}$$

where summation over repeated $\mathbf{m}$'s is implied, and the gauge field $\widetilde{A}_\mu^{\mathbf{m}}$ is as defined above. As $N \to \infty$, the indices $\mathbf{m}$ cover the entire integer lattice, and hence we may define

$$a_\mu^{(x,p)} \equiv -\sum_{\mathbf{m}} \widetilde{A}_\mu^{\mathbf{m}} e^{i(m_1x + m_2p)}. \tag{7.4}$$

By Eq. (3.2),

$$[A_\mu, A_\nu] \to [L_{a_\mu}, L_{a_\nu}] = L_{\{a_\mu, a_\nu\}}. \tag{7.5}$$

Hence, by virtue of the linearity of L in its arguments,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \to L_{f_{\mu\nu}}, \tag{7.6}$$

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu + \{a_\mu, a_\nu\}. \tag{7.7}$$

The group trace defining the Yang–Mills Lagrangian density is thus

$$\begin{aligned}\mathrm{Tr}\, F_{\mu\nu}F_{\mu\nu} &\to -N^3/(2\pi)^2 \widetilde{F}_{\mu\nu}^{\mathbf{m}} \widetilde{F}_{\mu\nu}^{-\mathbf{m}} \\ &= -N^3/(16\pi^4) \int dx\, dp \\ &\quad \times \sum_{\substack{m_1,m_2\\ n_1,n_2}} e^{ix(m_1+m_2) + ip(m_2+n_2)} \\ &\quad \times \widetilde{F}_{\mu\nu}^{m_1,m_2} \widetilde{F}_{\mu\nu}^{n_1,n_2}.\end{aligned} \tag{7.8}$$

Thus, in the limit of the gauge theory, the group indices are surface (torus) coordinates, and the fields are rescaled Fourier transforms of the original $SU(N)$ fields; the group composition rule for them is given by the Poisson bracket, and the trace by surface integration. For $SO(\infty)$ and $USp(\infty)$ the a_μ's must have the above-mentioned symmetries.

Now note that an intriguing connection to strings emerges, for the first time *directly at the level of the action*: for gauge fields independent of x^μ (e.g., vacuum configurations), this Lagrangian density reduces to $\{a_\mu, a_\nu\}\{a_\mu, a_\nu\}$, the quadratic Schild–Eguchi action density for strings,[16] where the a_μ now serve as string variables, and the surface serves as the world-sheet. This means that the classical vacuum states of $SU(\infty)$ Yang–Mills are equivalent to the configurations of the classical string. Whether a superstring follows analogously from super-Yang–Mills is an interesting question.

The Lagrangian (7.8) with the sine bracket supplanting the Poisson bracket is also a gauge invariant theory, provided that the gauge transformation also involves the sine instead of the Poisson bracket,

$$\delta a_\mu = \partial_\mu \Lambda - \sin\{\Lambda, a_\mu\}, \tag{7.9}$$

and hence, by virtue of the Jacobi identity,

$$\delta f_{\mu\nu} = -\sin\{\Lambda, f_{\mu\nu}\}. \tag{7.10}$$

It then follows that

$$\delta \int dx\, dp\, f_{\mu\nu} f_{\mu\nu} = -2 \int dx\, dp\, f_{\mu\nu} \sin\{\Lambda, f_{\mu\nu}\} = 0. \tag{7.11}$$

At the moment, however, it is not clear what system is described by the corresponding space-time-independent Lagrangian density $\sin\{a_\mu, a_\nu\}\sin\{a_\mu, a_\nu\}$.

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