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The Pauli matrices in n dimensions and finest gradings of simple Lie algebras of type A_{n-1}

J. Patera and H. Zassenhaus
Centre de Recherches Mathématiques, Université de Montréal, Montréal, Québec, Canada and Department of Mathematics, Ohio State University, Columbus, Ohio 43210

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Properties of the Lie algebra gl(n,$\mathbb{C}$) are described for a basis which is a generalization of the 2×2 Pauli matrices. The 3×3 case is described in detail. The remarkable properties of that basis are the grading of the Lie algebra it offers (each grading subspace is one dimensional) and the matrix group it generates [it is a finite group with the center of SL$(n,\mathbb{C})$ as its commutator group].

I. INTRODUCTION

The purpose of this paper is to exploit recent results in mathematics[1,2] in order to generalize the 2×2 Pauli matrices to the $n\times n$ case. The generalization is unique up to normalization and change of basis. For $n=3$ it is very different from the familiar generalization of the generators of the Lie algebra su(2) to su(3), known as the Gell-Mann matrices.[3]

We start by asking the following question: What is the most important property of the Pauli matrices? A definitive answer to this question cannot be given since "importance" is relative to the purpose one may have in mind, and because the familiar case of 2×2 Pauli matrices is too small in size to really appreciate the analogous properties for larger values of n. However, it is well known that the 2×2 Pauli matrices have other nontrivial properties besides spanning the Lie algebras su(2) and sl(2,$\mathbb{C}$) (real and complex parameters, respectively). We list their properties in Sec. II. The generalization of the Pauli matrices is thus related to what one considers to be the defining important properties of these matrices.

In this paper we adopt the following point of view: The first one of the defining properties of what will henceforth be called the generalization of the Pauli matrices and denoted by $\mathscr{P}_n$ is that they provide a finest grading of the Lie algebra gl$(n,\mathbb{C})$. The role $\mathscr{P}_n$ plays in grading gl$(n,\mathbb{C})$ has two aspects: The adjoint action of $\mathscr{P}_n$ on gl$(n,\mathbb{C})$ provides the grading group, and the generators of the graded gl$(n,\mathbb{C})$ are found among the elements of $\mathscr{P}_n$.

The second defining requirement is that the set of $n\times n$ matrices $\mathscr{P}_n$ generates a subgroup of SL$(n,\mathbb{C})$ with the center of SL$(n,\mathbb{C})$ as its commutator subgroup. It simply means that the group commutator of $\mathscr{P}_n$ must be as large as possible given its role in the grading of gl$(n,\mathbb{C})$. Throughout this paper we try to emphasize those basic properties of $\mathscr{P}_n$ that, in our opinion, should find a reflection in any lasting application of the results in physics.

Until now the role of the gradings in physical applications of Lie algebras and their representations were rarely noticed or emphasized except perhaps for the $\mathbb{Z}_2$ gradings underlying the classification of real forms of simple Lie algebras, the structure of superalgebras, and the Wigner–Inönü contractions of Lie algebras. Also the affinization $\hat{A}$ of finite simple Lie algebra A involves an infinite $\mathbb{Z}$ grading of the algebra $\hat{A}$. Implicitly another type of grading underlies the Cartan or root space decomposition of simple Lie algebras (finite and Kac–Moody ones).

The role of gradings of a Lie algebra in physics cannot be overestimated. In conventional terms it means the existence of preferred bases of the Lie algebra which admit additive quantum numbers. Naturally one wants to know all such bases and all nonequivalent choices available in a given situation. Moreover, such bases "force their way" into physics even if one is not set up to study them. Thus the matrices **A** and **D** below which generate $\mathscr{P}_{2n+1}$ are encountered in physics literature.[4]

In general terms a grading of a Lie algebra L means that L can be written as a direct sum of linear subspaces,

$$L = X_a \oplus X_b \oplus X_c \oplus \cdots, \quad a,b,c,\ldots \in S, \tag{1.1}$$

labeled by a set S of finite sequences of integers or integers to a module $a=\{a_1,a_2,\ldots,a_m\}$, $b=\{b_1,b_2,\ldots,b_m\}$. The set S may be finite or infinite, there may be more than one integer labeling each subspace, etc.; the subspaces are supposed to be not zero, often even of dimension greater than 1. The decomposition (1.1) of L is called a grading provided the nonzero commutation relations of L have the following form:

$$[x_a, y_b] = z_{a+b}, \tag{1.2}$$

for any x_a, y_b of L for which $a,b \in S$, $x_a \in X_a$, $y_b \in X_b$, $[x_a,y_b] \neq 0$ so that $a+b \in S$, $z_{a+b} \in X_{a+b}$. Note that the m-tuple $a+b$ is formed componentwise and it must also be a part of the labeling set S. Practically grading L means to find generators of L and a labeling set S such that (1.2) is satisfied.

In the case of a $\mathbb{Z}_2$ grading the decomposition (1.1) contains exactly two subspaces labeled by integers mod 2. Such gradings most often can be refined to gradings with more than two components, they are coarse gradings. Of interest to us here are the fine gradings, where the sum in (1.1) contains as many subspaces as possible given the requirements of (1.2), among which are the finest gradings in case all subspaces X_i in (1.1) are one-dimensional. The finest gradings of A_n algebras are described here for the first time although we exploit results of Refs. 1 and 2.

Furthermore, it may be possible to grade simultaneously the Lie algebra and its representations, decomposing a representation space V of L into a direct sum of subspaces

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$$V = V_d \oplus V_e \oplus V_f \oplus \cdots, \quad d,e,f \in S, \tag{1.3}$$

with the property

$$x_a | y_d \rangle \in V_{a+d}, \quad a,d,a+d \in S; \ | y_d \rangle \in V_d . \tag{1.4}$$

The relations (1.3), (1.4) contain (1.1), (1.2) as the particular case of the adjoint representation of L.

In quantum mechanics the labels of the set S are the admissible additive quantum numbers, which are eigenvalues of a chosen set of mutually commuting diagonable operators. In the case of a semisimple or reductive Lie algebra L, or of the Kac–Moody algebras, the traditional choice of the "diagonal" operators are the generators (i.e., a basis) of a Cartan subalgebra $\mathfrak{h} = \{h_1,h_2,...,h_r\}$. The remaining generators of the Lie algebra are then taken to be the eigenvectors of $\mathfrak{h}$. This is the traditional scenario which leads to the shift-up and shift-down generators similar to L_+ and L_- generators of the angular momentum theory. If the rank of L is r, then each label has r components. Such a label is called a weight of L; in the case of the Lie algebra these weights are the roots of the algebra, and the decomposition (1.1) of L is a grading called either the root space decomposition or the Cartan decomposition of L. Such a grading is fine but not the finest since $\dim \mathfrak{h} = r > 1$ for all but the 2×2 case. Note how restrictive the grading concept is in comparison with arbitrary decompositions of a Lie algebra into linear subspaces [cf. the matrices (2.2) below], that is, most decompositions do not admit a labeling of the generators with the property (1.2).

Our construction departs from the traditional approach by the observation that the 2×2 Pauli matrices generate a very particular maximal nilpotent subgroup $\mathscr{P}_2$ of SU(2), the quaternion group of order 2^3. This group is non-Abelian and therefore it is not a subgroup of the maximal torus of SL(2,$\mathbb{C}$). However, its adjoint action on the Lie algebra sl(2,$\mathbb{C}$) is Abelian and hence in many standard situations it can be used *instead* of the maximal torus.

Since the features of the general case appear already in the lowest case, $n = 3$, we describe them in detail for the 3×3 example in Secs. III and IV often leaving to the reader the verification of the properties by straightforward computation. In Sec. V the general $(2n+1)\times(2n+1)$ case is dealt with because it is somewhat simpler than the even size generalization presented in the last section. The 4×4 example is also briefly considered there.

The matrices $\mathscr{P}_n$ of any degree n provide a finest grading of A_{n-1}. But not every finest grading of A_{n-1} is conjugate under SL(n,$\mathbb{C}$) to the grading provided by the group $\mathscr{P}_n$. The general theory of Ref. 2 provides the answer that all finest gradings of A_{n-1} (with the exception of some low rank cases) are obtained upon using the Kronecker product groups

$$\mathscr{P}_{m_1} \otimes \mathscr{P}_{m_2} \otimes \cdots \otimes \mathscr{P}_{m_j}, \quad m_1 m_2 \cdots m_j = n .$$

An appropriate name for these matrices would be generalized Dirac matrices since the ordinary Dirac matrices correspond to $\mathscr{P}_2 \otimes \mathscr{P}_2$.

II. PROPERTIES OF THE PAULI MATRICES

The set of matrices

$$\sigma_0 = N' \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = N \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_2 = N \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = N \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{2.1}$$

with any complex nonzero normalization constants N, N' we shall call the Pauli matrices. Sometimes it is convenient to admit also the value $N' = 0$ and thus consider σ_1, σ_2, and σ_3 as the Pauli matrices without the identity matrix σ_0. In physics the most common normalization is $N' = 1$, $N = -i$, which makes all four matrices Hermitian.

A well known 3×3 analog of (1.1) are the Gell-Mann matrices,[3]

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{2.2}$$
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$
$$\lambda_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrices (2.2) generalize (2.1) in that their $\mathbb{R}$- or $\mathbb{C}$-linear combinations span the Lie algebras u(3) and gl(3,$\mathbb{C}$), respectively, just as the Pauli matrices span the Lie algebras u(2) and gl(2,$\mathbb{C}$). However, the Pauli matrices have other remarkable properties not shared by (2.2). They are as follows.

(1) With $N = 1$, $N' = 1$ the Pauli matrices (2.1) (equipped with matrix multiplication) generate the maximal nilpotent subgroup $\mathscr{P}_2$ of SL(2,$\mathbb{C}$), a group of order 2^3. Explicitly the group $\mathscr{P}_2$ consists of the following elements:

$$\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \pm \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{2.3}$$

Note the coincidence of the centers of the groups $\mathscr{P}_2$, SL(2,$\mathbb{C}$), and SU(2). All the matrices (2.3), except the multiples of identity, belong to the same conjugacy class of SL(2,$\mathbb{C}$) elements of order 4 denoted[5] by [11]. It is the unique class of the lowest order regular elements. Note also that the Hermitian normalization of (2.1) would generate a finite group which is quite *different* from $\mathscr{P}_2$.

(2) The adjoint action of the Pauli matrices on themselves is diagonal and does not depend on $N \neq 0$, $N' \neq 0$:

$$\sigma_j \sigma_k \sigma_j^{-1} = \begin{cases} \sigma_k, & \text{if } j = k \text{ or } k = 0 \text{ or } j = 0, \\ -\sigma_k, & \text{if } 0 \neq j \neq k \neq 0 . \end{cases} \tag{2.4}$$

Existence of the group $\mathscr{P}_n$ satisfying (2.4) and the irreducibility of $\mathscr{P}_n$ are the requirements defining the generalization of Pauli matrices in this paper.

Among the interesting consequences of (1) and (2) let us point out the following.

(3) With $N = i/2$ the commutation relations of (2.1) have integer structure constants. The normalization of σ_0 is irrelevant for this property since σ_0 commutes with all the others.

(4) Introducing the following notations for the generators of $\mathrm{sl}(2,\mathbb{C})$:

$$\sigma_1 = (1,0), \quad \sigma_2 = (1,1), \quad \sigma_3 = (0,1),$$

the grading of the algebra is made obvious:

$$[(p,q)(p',q')] = \mathrm{const}(p+p',q+q'), \tag{2.5}$$

where $p,q,p',q',p+p',q+q'$ are integers mod 2.

Let us note the following properties which find some reflection in the generalization.

(5) The Lie algebra su(2) [or $\mathrm{sl}(2,\mathbb{C})$] decomposes into a sum of one-dimensional real (or complex) subspaces generated by $\sigma_1,\sigma_2,\sigma_3$ each of which is a Cartan subalgebra. For $n=2$ this means that $\sigma_1,\sigma_2,\sigma_3$ are diagonable.

(6) The three Cartan subalgebras are pairwise orthogonal,

$$\mathrm{tr}(\sigma_j\sigma_k) = 2N\delta_{jk} \quad (j,k = 1,2,3). \tag{2.6}$$

The properties listed above are not independent of each other. The general theory can be found in Ref. 2.

III. THE GENERALIZATION

We will repeatedly use in Secs. III and IV the constants $\omega = e^{2\pi i/3}$ and $\xi = e^{2\pi i/6}$ and the obvious identities they satisfy.

Consider the following 27 matrices:

$$A_k = \omega^k\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, \quad A_k^- = \omega^{-k}\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},$$

$$B_k = \omega^k\begin{pmatrix}0&\omega&0\\0&0&\omega^2\\1&0&0\end{pmatrix}, \quad B_k^- = \omega^{-k}\begin{pmatrix}0&0&\omega\\1&0&0\\0&\omega^2&0\end{pmatrix},$$

$$C_k = \omega^k\begin{pmatrix}0&\omega^2&0\\0&0&\omega\\1&0&0\end{pmatrix}, \quad C_k^- = \omega^{-k}\begin{pmatrix}0&0&\omega^2\\1&0&0\\0&\omega&0\end{pmatrix},$$

$$D_k = \omega^k\begin{pmatrix}1&0&0\\0&\omega&0\\0&0&\omega^2\end{pmatrix}, \quad D_k^- = \omega^{-k}\begin{pmatrix}1&0&0\\0&\omega^2&0\\0&0&\omega\end{pmatrix},$$

$$I_k = \omega^k\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \quad \text{with } k \text{ an integer mod 3.} \tag{3.1}$$

The set of matrices (3.1) is the 3×3 analog $\mathscr{P}_3$ of the group $\mathscr{P}_2$ of (2.3). Under matrix multiplication they form a subgroup of $\mathrm{SL}(3,\mathbb{C})$ of order 3^3 whose center, $\{I_k, k\equiv 0, \pm 1 \bmod 3\}$, coincides with the center of both $\mathrm{SL}(3,\mathbb{C})$ and SU(3). All but elements of the center belong to the unique $\mathrm{SL}(3,\mathbb{C})$ conjugacy class [111] of lowest order regular elements.[5,6]

Any linearly independent subset of (3.1) is a basis of the Lie algebra $\mathrm{gl}(3,\mathbb{C})$. Our choice of the $\mathrm{sl}(3,\mathbb{C})$ linear generators will be (dropping the subscripts and writing the generators in bold characters)

$$\mathbf{A} = \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix}0&\omega&0\\0&0&\omega^2\\1&0&0\end{pmatrix},$$
$$\mathbf{C} = \begin{pmatrix}0&\omega^2&0\\0&0&\omega\\1&0&0\end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix}1&0&0\\0&\omega&0\\0&0&\omega^2\end{pmatrix},$$
$$\mathbf{A}^- = \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}, \quad \mathbf{B}^- = \begin{pmatrix}0&0&\omega\\1&0&0\\0&\omega^2&0\end{pmatrix}, \tag{3.2}$$
$$\mathbf{C}^- = \begin{pmatrix}0&0&\omega^2\\1&0&0\\0&\omega&0\end{pmatrix}, \quad \mathbf{D}^- = \begin{pmatrix}1&0&0\\0&\omega^2&0\\0&0&\omega\end{pmatrix},$$

The Lie algebra $\mathrm{gl}(3,\mathbb{C})$ is generated by (3.2) and by the identity matrix $\mathbf{I}$. Note that the matrices $\mathbf{B}$ and $\mathbf{B}^-$, $\mathbf{C}$ and $\mathbf{C}^-$ are not inverse to each other, their products are multiples of the identity. Such a choice makes them a particular case of (5.6) below.

It would be possible from now on to consider only Hermitian (or anti-Hermitian) linear combinations of the generators (3.2), but this would reveal little of the general structure and introduces many cumbersome complications (as happens in the angular momentum theory) although it could prove useful in some applications, for instance where the pairwise orthogonality of the generators with respect to the Killing form is required. The Hermitian version of (3.2) is thus

$$\mathbf{A}_+ = \mathbf{A} + \mathbf{A}^- = \begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}, \qquad \mathbf{B}_+ = \mathbf{B} + \omega^2\mathbf{B}^- = \begin{pmatrix}0&\omega&1\\\omega^2&0&\omega^2\\1&\omega&0\end{pmatrix},$$

$$\mathbf{A}_- = i(\mathbf{A} - \mathbf{A}^-) = \begin{pmatrix}0&i&-i\\-i&0&i\\i&-i&0\end{pmatrix}, \quad \mathbf{B}_- = i(\mathbf{B} - \omega^2\mathbf{B}^-) = \begin{pmatrix}0&-i\omega&i\\i\omega^2&0&-i\omega^2\\-i&i\omega&0\end{pmatrix},$$

$$
\begin{aligned}
&\mathbf{D}_+ = \mathbf{D}+\mathbf{D} = \begin{pmatrix} 2 & & \\ & -1 & \\ & & -1 \end{pmatrix}, \qquad \mathbf{C}_+ = \mathbf{C}+\omega\mathbf{C}^- = \begin{pmatrix} 0 & \omega^2 & 1 \\ \omega & 0 & \omega \\ 1 & \omega^2 & 0 \end{pmatrix},\\
&\mathbf{D}_- = i(\mathbf{D}-\mathbf{D}^-) = \sqrt{2}\begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}, \quad \mathbf{C}_- = i(\mathbf{C}-\omega\mathbf{C}^-) = \begin{pmatrix} 0 & -i\omega^2 & i \\ i\omega & 0 & -i\omega \\ -i & i\omega^2 & 0 \end{pmatrix}.
\end{aligned}
\tag{3.2'}
$$

The matrices (3.1), besides spanning the Lie algebra gl(3,$\mathbb{C}$) (under matrix commutation), form at the same time a finite subgroup of SL(3,$\mathbb{C}$) (under matrix multiplication). Thus any matrix (3.1) can be interpreted as a group element or a Lie algebra element. The two interpretations differ by the implied composition law: commutation and linear combinations for the Lie algebra, and matrix multiplication for the group.

The generators (3.2) make obvious a decomposition of the Lie algebra sl(3,$\mathbb{C}$) into a sum of four two-dimensional subspaces

$$\mathrm{sl}(3,\mathbb{C}) = \mathfrak{h}_A + \mathfrak{h}_B + \mathfrak{h}_C + \mathfrak{h}_D\,, \tag{3.3}$$

where the subspaces are spanned by two commuting generators,

$$
\begin{aligned}
&\mathfrak{h}_A = \{\mathbf{A},\mathbf{A}^-\}, \quad \mathfrak{h}_B = \{\mathbf{B},\mathbf{B}^-\},\\
&\mathfrak{h}_C = \{\mathbf{C},\mathbf{C}^-\}, \quad \mathfrak{h}_D = \{\mathbf{D},\mathbf{D}^-\}\,.
\end{aligned}
\tag{3.4}
$$

Hence each of the four subspaces is a Cartan subalgebra of sl(3,$\mathbb{C}$) and, taking suitable linear combinations of generators, also of the su(3). Furthermore, one easily verifies the pairwise orthogonality of the subspaces $\mathfrak{h}_A,\mathfrak{h}_B,\mathfrak{h}_C,\mathfrak{h}_D$ with respect to the Killing form,

$$\mathrm{tr}\,XY = 0, \quad \text{for } X\in\mathfrak{h}_X,\ Y\in\mathfrak{h}_Y,\ X\neq Y\,. \tag{3.5}$$

The commutation relations of the generators (3.2) are summarized in Table I. The nonzero structure constants are cyclotomic integers of the form

$$\xi^k + \xi^{k+1}\,. \tag{3.6}$$

Finally observe that the property (2.4) of the Pauli matrices also generalizes to higher ranks. Namely,

$$X_k Y_{k'} X_k^{-1} = \omega^j Y_{k'} \tag{3.7a}$$

or equivalently

$$X_k Y_{k'} = \omega^j Y_{k'} X_k \tag{3.7b}$$

and also

$$X_k Y_{k'} X_k^{-1} Y_{k'}^{-1} = I_j \tag{3.7c}$$

for any $X_k, Y_{k'}\in\mathscr{P}_3$. The factor ω^j is given in Table I as the power of ξ^{2j} in the structure constant in $[X,Y] = (\xi^{2j}+\xi^{2j+1})Z$.

The finite group $\mathscr{P}_3$ of the matrices (3.1) is obviously non-Abelian. Hence it is not a subgroup of the maximal torus of SL(3,$\mathbb{C}$). Nevertheless its action (3.7a) on the generators of sl(3,$\mathbb{C}$) is Abelian. As a result of that it can be used instead of the maximal torus in many ways.

IV. SOME FURTHER PROPERTIES

A. The cyclotomic quarks and antiquarks

In (3.2) we have a new basis of sl(3,$\mathbb{C}$) with unique properties. Now let us consider the elementary representation theory in terms of the new basis.

The natural (quark) representation of the generators of sl(3,$\mathbb{C}$) coincides with (3.2). Let us choose the basis vectors (quarks) of the three-dimensional representation space as the eigenvectors of the generator $\mathbf{D}$, label them by the power $p \pmod 6$ of ξ in the eigenvalue ξ^p of $\mathbf{D}$, and call them the cyclotomic quarks (most of the relevant numbers related to them in the representation theory are cyclotomic integers). Thus we have the quarks

$$|0\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad |2\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad |4\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix} \tag{4.1}$$

defined by

$$\mathbf{D}|p\rangle = \xi^p|p\rangle, \quad p = \text{even integer mod } 6\,. \tag{4.2}$$

One verifies directly that

$$
\begin{aligned}
&\mathbf{D}|p\rangle = \xi^p|p\rangle, \quad \mathbf{D}^-|p\rangle = \xi^{-p}|p\rangle\,,\\
&\mathbf{A}|p\rangle = |p-2\rangle, \quad \mathbf{A}^-|p\rangle = |p+2\rangle\,,
\end{aligned}
$$

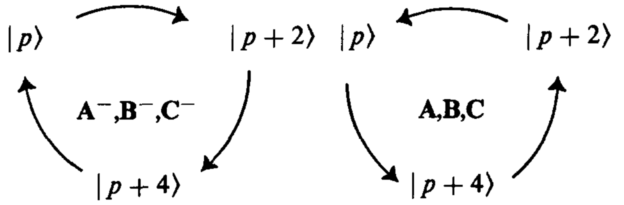
TABLE I. The commutation relations of the sl(3,$\mathbb{C}$) generators (3.2). The 0 blocks on the diagonal indicate the presence of the generators of four Cartan subalgebras in our basis. Only the upper part of the table is shown.

	$\mathbf{A}$	$\mathbf{A}^-$	$\mathbf{B}$	$\mathbf{B}^-$	$\mathbf{C}$	$\mathbf{C}^-$	$\mathbf{D}$	$\mathbf{D}^-$
$\mathbf{A}$	0	0	$(1+\xi)\mathbf{C}^-$	$(1+\xi^5)\mathbf{D}^-$	$(1+\xi^5)\mathbf{B}^-$	$(1+\xi)\mathbf{D}^-$	$(1+\xi)\mathbf{B}^-$	$(1+\xi^5)\mathbf{C}$
$\mathbf{A}^-$	0	0	$(1+\xi^5)\mathbf{D}$	$(1+\xi)\mathbf{C}$	$(1+\xi)\mathbf{D}$	$(1+\xi^5)\mathbf{B}$	$(1+\xi^5)\mathbf{C}^-$	$(1+\xi)\mathbf{B}^-$
$\mathbf{B}$			0	0	$(\xi^4+\xi^5)\mathbf{A}^-$	$(\xi^2+\xi)\mathbf{D}$	$(1+\xi)\mathbf{C}$	$(1+\xi^5)\mathbf{A}$
$\mathbf{B}^-$			0	0	$(\xi^2+\xi)\mathbf{D}$	$(\xi^4+\xi^5)\mathbf{A}$	$(1+\xi^5)\mathbf{A}^-$	$(1+\xi)\mathbf{C}^-$
$\mathbf{C}$					0	0	$(1+\xi)\mathbf{A}$	$(1+\xi^5)\mathbf{B}$
$\mathbf{C}^-$					0	0	$(1+\xi^5)\mathbf{B}^-$	$(1+\xi)\mathbf{A}^-$
$\mathbf{D}$							0	0
$\mathbf{D}^-$							0	0

$$\mathbf{B}|p\rangle = \xi^{\,p}|p-2\rangle, \quad \mathbf{B}^-|p\rangle = \xi^{\,-p}|p+2\rangle,$$
$$\mathbf{C}|p\rangle = \xi^{\,-p}|p-2\rangle, \quad \mathbf{C}^-|p\rangle = \xi^{\,p}|p+2\rangle. \tag{4.3}$$

The relations (4.3) are rewritten in (4.15) below in a compact form using different notation for the generators.

Note the "rotating" action of **A**, **B**, **C** and that of $\mathbf{A}^-$, $\mathbf{B}^-$, $\mathbf{C}^-$ on the quarks and the fact that during commutation the rotations add up. Neither of the generators is a "shift-up" or "shift-down" operator. Symbolically one has



(4.4)

In order to consider other representations, say the antiquark one, it is helpful to distinguish between the abstract generators, which we denote by A, B, C, D, A^-, B^-, C^-, D^-, and their representations. Thus the matrices (3.2) stand for the sl(3,$\mathbb{C}$) generators in the quark representation q. The abstract generators in the antiquark representation $\bar{q}$ are represented by matrices which are the negative transpose of those of (3.2). In particular,

$$q(D) = \mathbf{D}, \quad \bar{q}(D) = -\mathbf{D}^T = -\mathbf{D}. \tag{4.5}$$

Therefore the antiquarks $|p\rangle$ are

$$|1\rangle, \quad |3\rangle, \quad |5\rangle, \tag{4.6}$$

defined by

$$-\mathbf{D}|p\rangle = \xi^{\,p}|p\rangle, \quad p = \text{odd integer mod } 6. \tag{4.7}$$

Thus to every quark $|p\rangle$ there corresponds an antiquark $|p+3\rangle$. Transformation properties of the antiquarks analogous to (4.3) are given in (4.15c).

B. The finest grading of sl(3,$\mathbb{C}$)

Before proceeding further in this direction, it is useful to consider a grading of the Lie algebra sl(3,$\mathbb{C}$) unique to our basis.

The rotating action of the generators on the quarks allows one to decompose gl(3,$\mathbb{C}$) into three subspaces L_d, $d = 0, \pm 1$ mod 3, spanned by the generators

$$L_1 = \{\mathbf{A}^-,\mathbf{B}^-,\mathbf{C}^-\}, \quad L_0 = \{\mathbf{D},\mathbf{D}^-,\mathbf{I}\},$$
$$L_{-1} = \{\mathbf{A},\mathbf{B},\mathbf{C}\}, \tag{4.8}$$

with the grading property

$$[L_r,L_s] \subseteq L_{r+s \bmod 3}. \tag{4.9}$$

The subspaces (4.8) can be defined as eigenspaces of the adjoint action (3.7a) of the generator **D**,

$$L_d = \{X \,|\, \mathbf{D}X\mathbf{D}^{-1} = \omega^d X\}. \tag{4.10}$$

Thus (4.10) allows one to label each gl(3,$\mathbb{C}$) generator by an integer d which can take three values. However, (3.7a) is valid not only for **D** but for any generator (3.2). Therefore we can use any other generator of sl(3,$\mathbb{C}$), or all of them simultaneously, and label the generators by up to eight three-valued integers. In order to label completely all the generators without redundancy of notation, it suffices to use any two of them which do not commute. Choosing in addition to **D**, for instance, the generator **A**, and using its eigenvalues to label the generators, we end up with a new notation for the generators,

$$\mathbf{A} = (0,-1), \quad \mathbf{A}^- = (0,1,), \quad \mathbf{D} = (1,0,),$$
$$\mathbf{B} = (1,-1,), \quad \mathbf{B}^- = (-1,1,), \quad \mathbf{D}^- = (-1,0,),$$
$$\mathbf{C} = (-1,-1,), \quad \mathbf{C}^- = (1,1,), \quad \mathbf{I} = (0,0,), \tag{4.11}$$

where the first label refers to **A** and the second to **D** [cf. (4.10)]. The Abelian property of the adjoint action (3.7a) assures the grading structure of the commutation relations

$$[(k,j),(k',j')] = \text{const}(k+k',j+j') \bmod 3 \tag{4.12}$$

of gl(3,$\mathbb{C}$) with the structure constants given as before in Table I. Since no two generators are labeled by the same symbol in (4.11), the grading (4.12) of gl(3,$\mathbb{C}$) cannot be further refined. We say that it is fine. Note that (4.9) is a coarsening of (4.12) obtained when one ignores the first label. The decomposition (3.3), however, is not a grading. Moreover, since the subspaces $\{(i,j)\}$ generated by each (i,j) are one dimensional the grading is finest. We then have the fine decomposition of the Lie algebra gl(3,$\mathbb{C}$) into a sum of one-dimensional subspaces:

$$\text{gl}(3,\mathbb{C}) = \sum_{i,j=-1}^{1} \{(i,j)\}. \tag{4.13}$$

Finally, note that the grading (4.11) allows us to write the commutation table (Table I) in a compact form. Namely,

$$[(k,j),(k',j')] = (\omega^{kj'} - \omega^{k'j})(k+k',j+j') \quad \bmod 3, \tag{4.14}$$

and that the transformation properties (4.3) of quarks by the generators (r,s) of (4.11) including the matrix elements can be written in a simple form:

$$(r,s)|p\rangle = \xi^{\,rp}|p+2s\rangle. \tag{4.15a}$$

In (3.2) and (4.11) we have identified the abstract generators (r,s) with their matrix (quark) representation $q(r,s)$. Without such convention the relations (4.15a) should have been written as

$$q(r,s)|p\rangle = \xi^{\,rp}|p+2s\rangle \quad (r,s \text{ mod } 3;\ p \text{ even mod } 6). \tag{4.15b}$$

The corresponding relations in the antiquark representation $\bar{q}(r,s)$ of the generators are then

$$\bar{q}(r,s)|p\rangle = -\xi^{\,(p-3)r}|p-2s\rangle \quad (r,s \text{ mod } 3;\ p \text{ odd mod } 6). \tag{4.15c}$$

C. The gl(2,$\mathbb{C}$) and o(3,$\mathbb{C}$) subalgebras of sl(3,$\mathbb{C}$)

There are two maximal subalgebras of gl(3,$\mathbb{C}$) which are often used. Let us now write their generators in our basis of gl(3,$\mathbb{C}$).

First note that the 3×3 matrices E_{ij}, $i,j = 1,2,3$, with 1 at the intersection of the ith row and jth column and 0 elsewhere, can be written as follows:

$$E_{ii} = \frac{1}{3}\sum_{m=1}^{3} \omega^{(1-i)m}\mathbf{D}^m, \quad E_{ik} = E_{ii}\mathbf{A}^{k-i}. \tag{4.16}$$

The subalgebras are now generated for instance by

$$\mathrm{gl}(2,\mathbb{C}):\ E_{32}=E_{33}\mathbf{A}^-,\quad E_{23}=E_{22}\mathbf{A},$$
$$E_{22}-E_{33}=\tfrac{1}{3}(\xi^4+\xi^5)(\mathbf{D}-\mathbf{D}^-)\,,\qquad (4.17)$$
$$2E_{11}-E_{22}-E_{33}=\mathbf{D}+\mathbf{D}^-\,;$$

$$\mathrm{o}(3,\mathbb{C}):\ E_{12}+E_{23}=\tfrac{1}{3}(2+\xi^{-1}\mathbf{D}+\xi\mathbf{D}^-)\mathbf{A}\,,$$
$$E_{21}+E_{32}=-\tfrac{1}{3}(2+\mathbf{D}+\mathbf{D}^{-1})\mathbf{A}^{-1}\,,\qquad (4.18)$$
$$E_{11}-E_{33}=\tfrac{1}{3}((1+\xi^{-1})\mathbf{D}+(1+\xi)\mathbf{D}^{-1})\,.$$

D. The Weyl group and the weight lattice

Among the most important tools of the general representation theory is the Weyl group W and the weight lattices and weight systems of representations. We finish this section by pointing them out in the new basis.

The sl(3,$\mathbb{C}$) weight lattice is usually given as the integer span of the two fundamental weights,

$$Q=\mathbb{Z}\nu_1+\mathbb{Z}\nu_2\,.\qquad (4.19)$$

Here $\mathbb{Z}$ denotes any integer. In our notations the fundamental weights are written as the highest weights of the quarks and antiquarks,

$$\nu_1=1\quad\text{and}\quad \nu_2=\xi\,.\qquad (4.20)$$

Hence the weight lattice Q consists of all the points

$$Q=\mathbb{Z}+\mathbb{Z}\xi=\mathbb{Z}+\mathbb{Z}\omega\,.\qquad (4.21)$$

The Weyl group action in Q is generated by two reflections,

$$\begin{aligned} r_1(a+b\xi)&=r_1(a+b+b\omega)\\ &=-a+(a+b)\xi=b+(a+b)\omega\,,\\ r_2(a+b\xi)&=r_2(a+b+b\omega)\\ &=a+b-b\xi=a-b\omega\,. \end{aligned}\qquad (4.22)$$

In particular all quark sl(3,$\mathbb{C}$) quantum numbers (weights) are found on the same Weyl group orbit,

$$\begin{aligned} &\xi^0=1\leftrightarrow 0\,,\quad \xi^2=r_1\xi^0=-1+\xi\leftrightarrow 2\,,\\ &\xi^4=r_2r_1\xi^0=-\xi\leftrightarrow 4\,. \end{aligned}\qquad (4.23)$$

Similarly one finds the antiquarks on another orbit,

$$\begin{aligned} &\xi\leftrightarrow 1\,,\quad \xi^3=r_1r_2\xi=-1\leftrightarrow 3\,,\\ &\xi^5=r_2\xi=1-\xi\leftrightarrow 5\,. \end{aligned}\qquad (4.24)$$

The standard representation theory can be developed in terms of this basis, irreducible representations are constructed in tensor products of the quark and antiquark ones, etc.

V. THE GENERAL CASE OF gl($2n$+1,$\mathbb{C}$)

The properties of gl(3,$\mathbb{C}$) described in Secs. III and IV are particular cases of those which will be described here. Similar properties of gl($2n$,$\mathbb{C}$) also exist; however, some modification is necessary there. They are described in Sec. VI.

The finite group $\mathscr{P}_{2n+1}$ represented as a group of $(2n+1)\times(2n+1)$ matrices of determinant 1 is generated by the cyclic permutation matrix

$$A=\begin{pmatrix}0&1&0&\cdots&0\\ &&1&0&0\\ \vdots&&&\ddots&\vdots\\ 0&&&&1\\ 1&0&&\cdots&0\end{pmatrix},\quad A^{2n+1}=I\,,\qquad (5.1)$$

and by the diagonal matrix

$$D=\mathrm{diag}\,\{1,\zeta,\zeta^2,\ldots,\zeta^{2n}\},\quad \zeta=e^{2\pi i/2n+1}\,.\qquad (5.2)$$

The group consists of $(2n+1)^3$ matrices given by

$$K_{kad}=\zeta^kA^aD^d,\quad k,a,d\in\mathbb{Z}_{2n+1}\,,\qquad (5.3a)$$

which means that k,a,d assume integral values mod$(2n+1)$. Equivalently we could have chosen

$$K'_{kad}=\zeta^kD^dA^a,\quad k,a,d\in\mathbb{Z}_{2n+1}\,,\qquad (5.3b)$$

instead of (5.3a). The transfer between the two conventions is made as follows. Because

$$ADA^{-1}=\zeta D\Leftrightarrow DAD^{-1}=\zeta^{-1}A\,,\qquad (5.4)$$

one has

$$A^aD^dA^{-a}=\zeta^{ad}D^d\Leftrightarrow D^dA^aD^{-d}=\zeta^{-ad}A^a\qquad (5.5)$$

and therefore

$$K'_{kad}=\zeta^{ad}K_{kad}\,.\qquad (5.6)$$

Rewriting (5.5) in terms of K_{kad}, we establish easily the crucial property of the group $\mathscr{P}_{2n+1}$ which generalizes (2.4) and (3.7). Namely,

$$K_{kad}K_{k'a'd'}(K_{kad})^{-1}=\zeta^{ad'-a'd}K_{k'a'd'}\,,$$
$$a,a',d,d'\in Z_{2n+1}\,.\qquad (5.7)$$

Linear combinations of the matrices (5.3) with complex coefficients span the Lie algebra gl$(2n+1,\mathbb{C})$. A suitable set of generators can be chosen, for example, by putting $k=0$ in (5.3a). To be specific we choose the generators

$$\mathbf{K}_{ad}=K_{0ad},\quad -n\leqslant a,d\leqslant n\,.\qquad (5.8)$$

In particular, the one-dimensional center of gl$(2n+1,\mathbb{C})$ is generated by the identity matrix

$$\mathbf{K}_{00}=K_{000}\,;$$

the matrices A and D are also among the generators

$$A=\mathbf{K}_{10}=K_{010}\,,\quad D=\mathbf{K}_{01}=K_{001}\,.$$

Moreover, the subgroup of SL$(2n+1,\mathbb{C})$ generated by $\mathbf{A},\mathbf{D}$ has as its commutator subgroup the whole center of SL$(2n+1,\mathbb{C})$.

When it is possible to decompose sl$(2n+1,\mathbb{C})$ into the algebraic sum

$$\mathrm{sl}(2n+1,\mathbb{C})=\mathfrak{h}+\sum_{d=-n}^{n}\mathfrak{h}_d\qquad (5.9)$$

of $2n+2$ Cartan subalgebras? It can be done, according to a conjecture in Ref. 1, if and only if $2n+1$ is a prime power. If $2n+1$ is a prime number then we find the following solution for which we conjecture that our solution is the only one that can be refined to a finest grading:

$$\mathfrak{h}_d=\{(\mathbf{K}_{1d})^a,\ 1\leqslant a\leqslant 2n\}\,,$$

while $\mathfrak{h}$ is the Cartan subalgebra of diagonal matrices,

$$\mathfrak{h}=\{K_{0d},1\leqslant d\leqslant 2n\}=\{D,D^2,D^3,\ldots,D^{2n}\}\,.\qquad (5.10)$$

The property (5.7) specialized for the generators, i.e., $k=k'=0$, allows one to label the generators by the eigenvalues of other generators acting as in (5.7). Thus the $(2n+1)^2$ basis elements of $gl(2n+1,\mathbb{C})$ can each be labeled by $(2n+1)^2$ eigenvalues. Avoiding redundancy of notation, it suffices to use eigenvalues of any two generators which generate $\mathscr{P}_{2n+1}$ upon multiplication. Our choice of labeling generators from now on is A and D.

A generator $\mathbf{K}_{ad}$ is labeled by the eigenvalues of the transformations

$$A\,\mathbf{K}_{ad}A^{-1}=\zeta^{d}\mathbf{K}_{ad},\quad D\,\mathbf{K}_{ad}D^{-1}=\zeta^{-a}\mathbf{K}_{ad}\,. \tag{5.11}$$

For simplicity of notation we write

$$\mathbf{K}_{ad}=(d,-a)\,. \tag{5.12}$$

Here $-a$ and d are integers $\bmod(2n+1)$. Note that each generator of $gl(2n+1,\mathbb{C})$ is labeled by a distinct pair $(d,-a)$. The identity $\mathbf{K}_{00}$ is labeled by $(0,0)$.

Consider the commutation relations

$$\begin{aligned}[][\mathbf{K}_{pq},\mathbf{K}_{p'q'}]&=[(q,-p),(q',-p')]\\&=(q,-p)(q',-p')-(q',-p')(q,-p)\,.\end{aligned}$$

Since

$$\begin{aligned}(q,-p)(q',-p')&=A^{p}D^{q}A^{p'}D^{q'}\\&=A^{p+p'}A^{-p'}D^{q}A^{p'}D^{q'}\\&=\zeta^{-p'q}A^{p+p'}D^{q+q'},\end{aligned}$$

all the commutation relations of our generators of $gl(2n+1,\mathbb{C})$ can be written in the explicit form

$$[A^{a}D^{d},A^{a'}D^{d'}]=(\zeta^{-a'd}-\zeta^{-ad'})A^{a+a'}D^{d+d'}, \tag{5.13a}$$

$$[(a,b)(a',b')]=(\zeta^{a'b}-\zeta^{ab'})(a+a',b+b')\,, \tag{5.13b}$$

where the addition of the generator labels a,b,a',b' is understood $\bmod(2n+1)$. The finest grading of $gl(2n+1,\mathbb{C})$ realized by our basis (5.10) is made obvious in (5.11). Note that (5.11) is valid also for $sl(2n+1,\mathbb{C})$ which requires the exclusion of $(0,0)$ from the set of generators of the algebra.

There are $2n$ Casimir operators of $sl(2n+1,\mathbb{C})$. In our basis they are written in an obvious way. Indeed,

$$\begin{aligned}C^{(2)}&=\sum_{\substack{p_1+p_2=0\\q_1+q_2=0}}(p_1,q_1)(p_2,q_2)\,;\\C^{(3)}&=\sum_{\substack{p_1+p_2+p_3=0\\q_1+q_2+q_3=0}}(p_1,q_1)(p_2,q_2)(p_3,q_3)\,;\\&\vdots\\C^{(2n+1)}&=\sum_{\substack{p_1+\cdots+p_{2n+1}=0\\q_1+\cdots+q_{2n+1}=0}}(p_1,q_1)(p_2,q_2)\\&\quad\times\cdots\times(p_{2n+1},q_{2n+1})\,.\end{aligned} \tag{5.14}$$

It is understood that only the generators of $sl(2n+1,\mathbb{C})$ do appear in (5.14), i.e., $(0,0)$ is excluded.

Finally observe that also the relations (4.16) generalize in an obvious way:

$$\begin{aligned}E_{ii}&=\frac{1}{2n+1}\sum_{m=1}^{2n+1}\zeta^{(1-i)m}D^{m},\\E_{ik}&=E_{ii}A^{k-i},\quad 1\leqslant i,k\leqslant 2n+1\,.\end{aligned} \tag{5.15}$$

VI. THE GENERAL CASE OF gl(2n,C)

The development in this case follows the same line as in Sec. V. Differences occur in two ways[2]: the generating matrices A and D have to be modified in order to assure that their determinant is 1, and the orthogonal decomposition of $gl(2n,\mathbb{C})$ into Cartan subalgebras holds only for $n=1$.

The group $\mathscr{P}_{2n}$ of $2n\times 2n$ matrices of determinant 1 is generated by

$$A=\begin{pmatrix}0&1&0&\cdots&0\\&&1&0&0\\\vdots&&&\ddots&\vdots\\0&&&&1\\-1&0&&\cdots&0\end{pmatrix},\quad A^{2n}=-I, \tag{6.1}$$

and by the diagonal matrix

$$\begin{aligned}&D=\operatorname{diag}\{\eta,\eta^3,\ldots,\eta^{4n-1}\},\quad \eta=e^{2\pi i/4n},\\&D^{2n}=-I\,.\end{aligned} \tag{6.2}$$

Similarly as before $\mathscr{P}_{2n}$ consists of $(2n)^3$ matrices given by

$$K_{kad}=\eta^{2k}A^{a}D^{d},\quad k,a,d \text{ integers mod } 2n\,. \tag{6.3}$$

The property (5.7) of the group $\mathscr{P}_{2n}$, which lies at the origin of our interest in it, is written as

$$K_{kad}K_{k'a'd'}(K_{kad})^{-1}=\eta^{2(ad'-a'd)}K_{k'a'd'},\quad j\in\mathbb{Z}_{2n}\,. \tag{6.4}$$

It is verified directly using (6.3) and the relations

$$ADA^{-1}=\eta^{2}D,\quad DAD^{-1}=\eta^{-2}A\,. \tag{6.5}$$

Choosing the labeling elements A,D and using the notations

$$\mathbf{K}_{ad}=K_{0ad},\quad a,d \text{ integers mod } 2n \tag{6.6}$$

for the basis of $gl(2n,\mathbb{C})$, we have

$$\begin{aligned}&A=\mathbf{K}_{10}=(0,-1),\quad D=\mathbf{K}_{01}=(1,0)\,,\\&A^{a}=(\mathbf{K}_{10})^{a}=(0,-a),\quad D^{d}=(K_{01})^{d}=(d,0)\,.\end{aligned} \tag{6.7}$$

The subgroup of $SL(2n,\mathbb{C})$ generated by A,D has as its commutator subgroup again the whole center of $SL(2n,\mathbb{C})$.

A generator $\mathbf{K}_{ad}$ is labeled by the eigenvalues of the transformations

$$A\,\mathbf{K}_{ad}A^{-1}=\eta^{2d}\mathbf{K}_{ad},\quad D\,\mathbf{K}_{ad}D^{-1}=\eta^{-2a}\mathbf{K}_{ad}\,. \tag{6.8}$$

For simplicity of notation we write $\mathbf{K}_{ad}=(d,-a)$ [cf. (5.12)] rather than $\mathbf{K}_{ad}=(\eta^{2d},\eta^{-2a})$.

Then the commutation relations of our basis of $gl(2n,\mathbb{C})$ are given by (5.13a) where $\zeta=e^{2\pi i/2n}$. In the case of (5.13b) one should remember that now, because of the identity $X^{m}=-X^{m+2n}$ for $X=A$ and D, we have

$$[(p,q),(p',q')]=\epsilon(\eta^{2p'q}-\eta^{2pq'})(p+p',q+q')\,. \tag{6.9}$$

Here $\epsilon=-1$ if either $0\leqslant q+q'<2n\leqslant q+q'<4n$ or $0\leqslant q+q'<2n\leqslant q+q'<4n$, and $\epsilon=1$ otherwise. The $2n-1$ Casimir operators of $sl(2n,\mathbb{C})$ have the structure given by (5.12). Also the relations (5.15) hold practically without change taking into account that A is of order $4n$ in the present case,

$$E_{ii} = \frac{1}{2n} \sum_{m=1}^{2n} \zeta^{(1-2i)m} D^{m}, \qquad E_{ik} = E_{ii} A^{k-i}, \quad 1 \leqslant i,k \leqslant 2n. \tag{6.10}$$

Finally let us briefly consider the generalized Pauli matrices of degree 4. The subgroup $\mathscr{P}_4$ of GL(4,C) is of order 4^3. It is faithfully represented by the following 16 matrices each multiplied by ± 1 and $\pm i$:

$$(1,0) = D = \eta \begin{pmatrix} 1 & & & \\ & i & & \\ & & -1 & \\ & & & -i \end{pmatrix}, \quad (2,0) = D^2 = \begin{pmatrix} i & & & \\ & -i & & \\ & & i & \\ & & & -i \end{pmatrix}, \quad (3,0) = D^3 = \eta \begin{pmatrix} i & & & \\ & 1 & & \\ & & -i & \\ & & & -1 \end{pmatrix},$$

$$(0,3) = A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad (0,2) = A^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad (0,1) = A^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$(1,3) = AD = \eta \begin{pmatrix} 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad (2,3) = AD^2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \end{pmatrix},$$

$$(3,3) = AD^3 = \eta \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -1 \\ -i & 0 & 0 & 0 \end{pmatrix},$$

$$(1,2) = A^2D = \eta \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \\ -1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad (2,2) = A^2D^2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix},$$

$$(3,2) = A^2D^3 = \eta \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -1 \\ -i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$(1,1) = A^3D = \eta \begin{pmatrix} 0 & 0 & 0 & i \\ -1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (2,1) = A^3D^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \end{pmatrix},$$

$$(3,1) = A^3D^3 = \eta \begin{pmatrix} 0 & 0 & 0 & -1 \\ -i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & i & 0 \end{pmatrix},$$

$$(0,0) = I = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Here $\eta = \exp(2\pi i/8)$. The 16 matrices above are linearly independent and all but the identity are traceless. Equipped with the commutation relations they generate the Lie algebra gl(4,C). One can also verify that among them one does not find the Dirac matrices given relatively to an uncommon basis. Similarly they do not belong to the symplectic or orthogonal subgroups of GL(4,C).

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