Notes on Estimating Power Spectra

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The power spectrum, or power spectral density, $Sxx(f)$ of a signal $x(t)$ describes the amount of power per unit bandwidth in contained in the signal.

Let us start with something simpler: the standard instantaneous power. We will also specialize to electrical signals, i.e. signals for which $x(t)$ is a voltage and has units of Volts.

The power of a signal is simply

$$[\mathrm{W}] \qquad P = \frac{V^2}{R} = \frac{|x(t)|^2}{R} \qquad \left[\frac{\mathrm{V}^2}{\Omega}\right]$$

where we permit $x(t)$ to be complex to allow it to hold a phase.

Then the average power of some time period T is

$$[\mathrm{W}] \qquad \langle P \rangle = \frac{1}{T}\int_T \mathrm{d}t\, P(t) = \frac{1}{TR}\int_T \mathrm{d}t\ |x(t)|^2 \qquad \left[\frac{1}{\mathrm{s}\cdot\Omega}\cdot \mathrm{s}\cdot \mathrm{V}^2\right] \tag{1}$$

This may be converted into the frequency domain using Parseval's Theorem (App. B):

$$\begin{aligned}
[\mathrm{W}] \qquad \langle P \rangle &= \frac{1}{TR}\int_T \mathrm{d}t\ |x(t)|^2 = \frac{1}{TR}\int \mathrm{d}f\ |X(f)|^2 \qquad \left[\frac{1}{\mathrm{s}\cdot\Omega}\cdot\frac{1}{\mathrm{s}}\cdot \mathrm{V}^2\cdot \mathrm{s}^2\right] \\
&= \int \mathrm{d}f\, \frac{1}{TR}\,|X(f)|^2 \\
[\mathrm{W}] \qquad \langle P \rangle &= \int \mathrm{d}f\, P_{xx}(f) \qquad [\mathrm{Hz}]\left[\frac{\mathrm{W}}{\mathrm{Hz}}\right]
\end{aligned} \tag{2}$$

where we've defined the power spectral density (PSD)

$$P_{xx}(f) \equiv \frac{1}{TR}\,|X(f)|^2 \tag{3}$$

which has units of [W/Hz]. We will also shortly encounter another definition of the power spectral density that does not incorporate the impedance R, since it is a useful intermediate computational step:

$$\begin{aligned}
\left[\frac{\mathrm{V}^2}{\mathrm{Hz}}\right] \qquad S_{xx}(f) &\equiv \frac{1}{T}\,|X(f)|^2 \\
S_{xx}(f) &= P_{xx}(f)\cdot R
\end{aligned} \tag{4}$$

The PSD describes how much power is contained in each unit of bandwidth of a signal $x(t)$. Integrating over a specific bandwidth (i.e. frequency range) gives the amount of power contained in that bandwidth.

In practice, we cannot measure the continuous values of $x(t)$. Instead, we sample the signal at some sampling rate $f_s = 1/T_s$ for some number of samples N over some period of time $T = NT_s = N/f_s$. We need to discretize our expressions. We'll use this map:

$$\begin{aligned}
t &\to n\cdot\Delta t = T_s n \\
f &\to k\cdot\Delta f \qquad \Delta f = f_s/N \\
x(t) &\to x(T_s n) = x[n] \\
X(f) &\to X(k\cdot\Delta f) \\
\mathrm{d}t\ &\to \Delta t = T_s \\
\mathrm{d}f\ &\to \Delta f \\
\int &\to \sum
\end{aligned}$$

where we've assumed that our period of time starts at $t_0 = 0$ for convenience.

Let's start by discretizing the Fourier Transform

$$
\begin{aligned}
X(f) &= \int \mathrm{d}t \, \exp\left(-2\pi i f t\right) x(t) \\
X(k \cdot \Delta f) &= \sum_{n=0}^{N-1} T_s \exp\left(-2\pi i \cdot k\Delta f \cdot T_s n\right) x[n] \\
&= T_s \sum_{n=0}^{N-1} \exp\left(-2\pi i k n / N\right) x[n] \\
[\mathrm{V} \cdot \mathrm{s}] \qquad X(k \cdot \Delta f) &= T_s X[k] \qquad [\mathrm{s} \cdot \mathrm{V}] \qquad (5)
\end{aligned}
$$

so our Fourier Transform is estimated by the sampling period T_s times the Discrete Fourier Transform $X[k]$, and we observe the sampling period is required to align the units.

Substituting in this estimate for $X(f)$ into the PSD (Eq. (3)),

$$
\begin{aligned}
P_{xx}(f) &\equiv \frac{1}{TR} \left|X(f)\right|^2 \\
P_{xx}(k \cdot \Delta f) &= \frac{1}{N T_s R} \left|T_s X[k]\right|^2 \\
&= \frac{T_s}{NR} \left|X[k]\right|^2 \\
\left[\frac{\mathrm{W}}{\mathrm{Hz}}\right] \qquad P_{xx}(k \cdot \Delta f) &= \frac{1}{N f_s R} \left|X[k]\right|^2 \qquad \left[\frac{\mathrm{V}^2}{\mathrm{Hz} \cdot \Omega}\right] \qquad (6)
\end{aligned}
$$

So the power spectral density is related to the DFT of the signal, but we must incorporate the unitless scaling factor $1/N$ and we use the full bandwidth f_s. Intuitively, we think of $\left|X[k]\right|^2 / R$ as something akin to the power in each bin k, so we would want to divide by the binwidth Δf to get back to a spectral quantity, i.e. something like

$$
P_{xx}(k \cdot \Delta f) = \frac{1}{N^2} \frac{\left|X[k]\right|^2}{R \Delta f} \qquad (7)
$$

in which case our unitless correction factor is actually $1/N^2$.

Inserting this into our estimate for average power Eq. (2),

$$
\begin{aligned}
\langle P \rangle &= \int \mathrm{d}f \, P_{xx}(f) \\
&= \sum_k \Delta f \frac{1}{N^2} \frac{\left|X[k]\right|^2}{R \Delta f} \\
&= \sum_k \frac{1}{N^2} \frac{\left|X[k]\right|^2}{R} \qquad \left[\mathrm{bins} \cdot \frac{\mathrm{W}}{\mathrm{bin}}\right] \qquad (8) \\
\langle P \rangle &= \sum_k P_{xx}[k] \qquad (9)
\end{aligned}
$$

where we've defined yet another power spectral density, this time one in units of W/bin:

$$
P_{xx}[k] = \frac{1}{N^2} \frac{\left|X[k]\right|^2}{R} \qquad \left[\frac{\mathrm{W}}{\mathrm{bin}}\right] \qquad (10)
$$

which we can sum over the bins corresponding to our bandwidth of interest to get the amount of power in the band of interest. These quantities all have the same name ("power spectral density"), so the way to distinguish them is via their units. Units are important!

We frequently encounter and are interested in both the per-Hz variety and the per-bin variety, so we need to be able to convert between them. Intuitively, the amount of power in a bin is simply the integral of the PSD (W/Hz) over the bandwidth of one bin (the binwidth Δf [Hz/bin]). If we assume the PSD (W/Hz) is constant, or more precisely we have an estimate of the average value, over the binwidth, then the conversion should be

$$
\left[\frac{\mathrm{W}}{\mathrm{bin}} \cdot \frac{\mathrm{bin}}{\mathrm{Hz}}\right] \qquad \frac{P_{xx}[k]}{\Delta f} = P_{xx}(k\Delta f) \qquad \left[\frac{\mathrm{W}}{\mathrm{Hz}}\right] \qquad (11)
$$

which we can verify by comparing Eq. (10) with Eq. (8).

In practice, we usually use a periodogram method to estimate

$$S_{xx}[k] = \frac{1}{N^2} |X[k]|^2 \qquad \left[\frac{\mathrm{V}^2}{\mathrm{bin}}\right] \tag{12}$$

because we want to tune our estimates for the desired bin width and mitigate windowing errors. The typical units coming out of a periodogram routine are either $\mathrm{V}^2/\mathrm{bin}$ or V^2/Hz, depending on the routine's configuration. To get back to Watts, we simply divide by the impedance R

$$P_{xx}(k \cdot \Delta f) = \frac{S_{xx}(k \cdot \Delta f)}{R} = \frac{S_{xx}[k]}{R\Delta f} = \frac{P_{xx}[k]}{\Delta f} \tag{13}$$

A Units of the Fourier Transform and Discrete Fourier Transform

Let's examine the units of the Fourier Transform. The Fourier Transform is defined as

$$X(f) = \int \mathrm{d}t \, \exp\left(-2\pi i f t\right) x(t)$$

Supposing $x(t)$ is a voltage and has units of Volts,

$$\begin{aligned} X(f) &= \int \mathrm{d}t \, \exp\left(-2\pi i f t\right) x(t) \\ [\mathrm{V} \cdot \mathrm{s}] &= [\mathrm{s}]\,[1]\,[\mathrm{V}] \end{aligned}$$

where the seconds comes from the $\mathrm{d}t$ and the Volts comes from the $x(t)$, while the exponential is unitless.

For the Discrete Fourier Transform, we have

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} \exp\left(-2\pi i k n/N\right) x[n] \\ [\mathrm{V}] &= [1]\,[V] \end{aligned}$$

There is no differential, so we do not pick up a unit from it. Therefore the units of the DFT are Volts on both sides.

B Parseval's Theorem

Parseval's Theorem relates the integrated energy of a signal from a time-domain perspective and from a frequency-domain perspective. Specifically, it states for a signal $x(t)$ and its Fourier Transform $X(f)$:

$$\int \mathrm{d}t \, |x(t)|^2 = \int \mathrm{d}f \, |X(f)|^2 \tag{14}$$

The proof is straightforward:

$$\begin{aligned} \int \mathrm{d}f \, |X(f)|^2 &= \int \mathrm{d}f \, X^*(f) \int \mathrm{d}t \, \exp\left(2\pi i f t\right) x(t) \\ &= \int \mathrm{d}t \, x(t) \int \mathrm{d}f \, \exp\left(2\pi i f t\right) X^*(f) \\ &= \int \mathrm{d}t \, x(t) \left[\int \mathrm{d}f \, \exp\left(-2\pi i f t\right) X(f)\right]^* \\ &= \int \mathrm{d}t \, x(t) x^*(t) \\ &= \int \mathrm{d}t \, |x(t)|^2 \end{aligned}$$

For a discrete signal $x[n]$ and its Discrete Fourier Transform $X[k]$:

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$$

Notably this picks up a $1/N$, which comes from the inverse DFT having a $1/N$. The proof is similar:

$$\begin{aligned}
\sum_{n=0}^{N-1} |x[n]|^2 &= \sum_{n=0}^{N-1} x[n] \left[\frac{1}{N} \sum_{k=0}^{N-1} \exp\left(2\pi ikn/N\right) X[k] \right]^* \\
&= \frac{1}{N} \sum_{k=0}^{N-1} X^*[k] \sum_{n=1}^{N-1} \exp\left(-2\pi ikn/N\right) x[n] \\
&= \frac{1}{N} \sum_{k=0}^{N-1} X^*[k] X[k] \\
&= \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2
\end{aligned}$$

C Limits of Integration

Most of this discussion glosses over the exact limits of integration in order to focus the attention on the transformations. In the end, there are only one reasonable option for limits in each expression, so an exact specification is not strictly necessary. Let us see how this is the case.

Firstly, in Parseval's Theorem (Eq. (14)), the limits of integration must be $-\infty \to \infty$:

$$\int_{-\infty}^{\infty} \mathrm{d}t\ |x(t)|^2 = \int_{-\infty}^{\infty} \mathrm{d}f\ |X(f)|^2$$

which points to the general principle here: unless otherwise specified, consume all of the region of integration (i.e. bandwidth).

Next, we note that the average power Eq. (1) is more precisely

$$\langle P \rangle = \frac{1}{TR} \int_{t_0}^{t_0+T} \mathrm{d}t\ |x(t)|^2 \tag{15}$$

with a region of integration of only width T, but we immediately apply Parseval's theorem (with its infinite region of integration) to convert to the frequency domain. What gives?

The resolution is that in a physical system, we cannot consider an infinite region. We are always limited to measuring some time-limited period T. In such scenarios, we choose a domain expansion that satisfies the needs of our mathematical tools. In this case, we must expand $x(t)$ to have a support of $t : -\infty \to \infty$. The only choice of expansion that does not change the value (or meaning) of $\langle P \rangle$ is to set $x(t) = 0\ \forall\ t \notin (t_0, t_0 + T)$. Of course, $x(t)$ does have non-zero values for t outside of our region of interest, so this is an arbitrary assumption. It is, however, equivalent to multiplying $x(t)$ by a rectangular window function $w(t) = \mathrm{rect}\,(t_0, t_0 + T)$, in which case this identity holds:

$$\int_{t_0}^{t_0+T} \mathrm{d}t\ |x(t)|^2 = \int_{-\infty}^{\infty} \mathrm{d}t\, w(t) x(t) = \int_{-\infty}^{\infty} \mathrm{d}t\, x'(t) \tag{16}$$

And we can rightly apply Parseval's Theorem to $x'(t)$. The multiplication of $w(t)$ has consequences due to the Convolution Theorem, since really our frequency domain will be of $X(f) * W(f)$ and the window function contaminates our representation. We'll have a look at this in more detail when periodogram (i.e. Welch's) methods are discussed. But for getting the concepts and basic definitions down, we can shelve this complication.

D Relationships to Variance

The variance of a signal $x(t)$ is

$$\sigma_x^2 = \mathrm{var}\, x \equiv \langle x^2 \rangle - \langle x \rangle^2$$

For a mean-zero ($\langle x \rangle = 0$) signal, this reduces to

$$\begin{aligned} \sigma_x^2 &= \langle x \rangle^2 \\ \sigma_x^2 &= \frac{1}{T} \int_T \mathrm{d}t \; |x(t)|^2 \end{aligned} \tag{17}$$

Comparing Eq. (17) to Eq. (1), we observe

$$\langle P \rangle = \frac{\sigma_x^2}{R} \tag{18}$$

We can also apply Parseval's Theorem (Eq. (14)) here to get

$$\sigma_x^2 = \frac{1}{T} \int \mathrm{d}f \; |X(f)|^2 \tag{19}$$