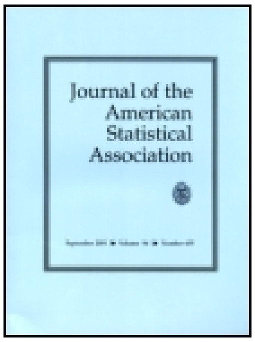

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Quantile Autoregression

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We consider quantile autoregression (QAR) models in which the autoregressive coefficients can be expressed as monotone functions of a single, scalar random variable. The models can capture systematic influences of conditioning variables on the location, scale, and shape of the conditional distribution of the response, and thus constitute a significant extension of classical constant coefficient linear time series models in which the effect of conditioning is confined to a location shift. The models may be interpreted as a special case of the general random-coefficient autoregression model with strongly dependent coefficients. Statistical properties of the proposed model and associated estimators are studied. The limiting distributions of the autoregression quantile process are derived. QAR inference methods are also investigated. Empirical applications of the model to the U.S. unemployment rate, short-term interest rate, and gasoline prices highlight the model's potential.

KEY WORDS: Asymmetric persistence; Autoregression; Comonotonicity; Quantile; Random coefficients.

1. INTRODUCTION

Constant-coefficient linear time series models have played an enormously successful role in statistics, and gradually various forms of random-coefficient time series models have also emerged as viable competitors in particular fields of application. One variant of the latter class of models, although perhaps not immediately recognizable as such, is the linear quantile regression model. This model has received considerable attention in the theoretical literature and can be easily estimated with the quantile regression methods proposed by Koenker and Bassett (1978). Curiously, however, all of the theoretical work dealing with this model that we are aware of focuses exclusively on the iid innovation case that restricts the autoregressive coefficients to be independent of the specified quantiles. In this article we seek to relax this restriction and consider linear quantile autoregression models with autoregressive (slope) parameters that may vary with quantiles $\tau \in [0, 1]$. We hope that these models might expand the modeling options for time series that display asymmetric dynamics or local persistence.

Considerable recent research effort has been devoted to modifications of traditional constant-coefficient dynamic models to incorporate various heterogeneous innovation effects. An important motivation for such modifications is the introduction of asymmetries into model dynamics. It is widely acknowledged that many important economic variables may display asymmetric adjustment paths (e.g., Neftci 1984; Enders and Granger 1998). The observation that firms are more apt to increase than to reduce prices is a key feature of many macroeconomic models. Beaudry and Koop (1993) have argued that positive shocks to the U.S. GDP are more persistent than negative shocks, indicating asymmetric business cycle dynamics over different quantiles of the innovation process. In addition, although it is generally recognized that output fluctuations are persistent, less persistent results are also found at longer horizons (Beaudry and Koop 1993), suggesting some form of "local persistence" (see, inter alia, Delong and Summers 1986; Hamilton 1989; Evans and Wachtel 1993; Bradley and Jansen 1997; Hess and Iwata 1997; Kuan and Huang 2001). A related development is the growing literature on threshold autoregression (TAR) (see, e.g., Balke and Fomby 1997; Tsay 1997; Gonzalez and Gonzalo 1998; Hansen 2000; Caner and Hansen 2001).

We believe that quantile regression methods can provide an alternative way to study asymmetric dynamics and local persistence in time series. We propose a new quantile autoregression (QAR) model in which autoregressive coefficients may take distinct values over different quantiles of the innovation process. We show that some forms of the model can exhibit unit root–like tendencies or even temporarily explosive behavior, but that occasional episodes of mean reversion are sufficient to ensure stationarity. The models lead to interesting new hypotheses and inference apparatus for time series.

The article is organized as follows. Section 2 introduces the model and gives some basic statistical properties of the QAR process. Section 3 develops the limiting distribution of the QAR estimator. Section 4 considers some restrictions imposed on the model by the monotonicity requirement on the conditional quantile functions. Section 5 explores statistical inference, including testing for asymmetric dynamics. Section 6 reports a Monte Carlo experiment on the sampling performance of the proposed inference procedure, and Section 7 gives an empirical application to U.S. unemployment rate time series. The Appendix provides proofs.

2. THE MODEL

There is a substantial theoretical literature, including works by Weiss (1987), Knight (1989), Koul and Saleh (1995), Koul and Mukherjee (1994), Hercé (1996), Hasan and Koenker (1997), and Hallin and Jurečková (1999), dealing with the linear quantile autoregression model. In this model the τth conditional quantile function of the response y_t is expressed as a linear function of lagged values of the response. Here we wish to study estimation and inference in a more general class of QAR models in which all of the autoregressive coefficients are allowed to be τ-dependent and thus are capable of altering the location, scale, and shape of the conditional densities.

2.1 The Model

Let $\{U_t\}$ be a sequence of iid standard uniform random variables, and consider the pth-order autoregressive process,

$$y_t = \theta_0(U_t) + \theta_1(U_t)y_{t-1} + \cdots + \theta_p(U_t)y_{t-p}, \tag{1}$$

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where the θ_j's are unknown functions $[0, 1] \to \mathbb{R}$ that we want to estimate. Provided that the right side of (1) is monotone increasing in U_t, it follows that the τth conditional quantile function of y_t can be written as

$$Q_{y_t}(\tau | y_{t-1}, \dots, y_{t-p}) = \theta_0(\tau) + \theta_1(\tau) y_{t-1} + \cdots + \theta_p(\tau) y_{t-p} \quad (2)$$

or, somewhat more compactly, as

$$Q_{y_t}(\tau | \mathcal{F}_{t-1}) = \mathbf{x}_t^\top \boldsymbol{\theta}(\tau), \quad (3)$$

where $\mathbf{x}_t = (1, y_{t-1}, \dots, y_{t-p})^\top$ and $\mathcal{F}_t$ is the σ-field generated by $\{y_s, s \le t\}$. The transition from (1) to (2) is an immediate consequence of the fact that for any monotone increasing function g and standard uniform random variable, U, we have

$$Q_{g(U)}(\tau) = g(Q_U(\tau)) = g(\tau),$$

where $Q_U(\tau) = \tau$ is the quantile function of U. In the foregoing model, the autoregressive coefficients may be τ-dependent and thus can vary over the quantiles. The conditioning variables not only shift the location of the distribution of y_t, but also may alter the scale and shape of the conditional distribution. We call this model the QAR(p) model.

We argue that QAR models can play a useful role in expanding the modeling territory between classical stationary linear time series models and their unit root alternatives. To illustrate this in the QAR(1) case, consider the model

$$Q_{y_t}(\tau | \mathcal{F}_{t-1}) = \theta_0(\tau) + \theta_1(\tau) y_{t-1}, \quad (4)$$

with $\theta_0(\tau) = \sigma \Phi^{-1}(\tau)$ and $\theta_1(\tau) = \min\{\gamma_0 + \gamma_1 \tau, 1\}$ for $\gamma_0 \in (0, 1)$ and $\gamma_1 > 0$. In this model, if $U_t > (1 - \gamma_0)/\gamma_1$, then the model generates the y_t according to the standard Gaussian unit root model, but for smaller realizations of U_t, we have a mean reversion tendency. Thus the model exhibits a form of asymmetric persistence in the sense that sequences of strongly positive innovations tend to reinforce its unit root-like behavior, whereas occasional negative realizations induce mean reversion and thus undermine the persistence of the process. The classical Gaussian AR(1) model is obtained by setting $\theta_1(\tau)$ to a constant.

The formulation in (4) reveals that the model may be interpreted as somewhat special form of random-coefficient autoregressive (RCAR) model. Such models arise naturally in many time series applications. Discussions of the role of RCAR models have been given by inter alia, Nicholls and Quinn (1982), Tjøstheim (1986), Pourahmadi (1986), Brandt (1986), Karlsen (1990), and Tong (1990). In contrast with most of the literature on RCAR models, in which the coefficients are typically assumed to be stochastically independent of one another, the QAR model has coefficients that are functionally dependent.

Monotonicity of the conditional quantile functions imposes some discipline on the forms taken by the $\boldsymbol{\theta}$ functions. This discipline essentially requires that the function $Q_{y_t}(\tau | y_{t-1}, \dots, y_{t-p})$ be monotone in τ in some relevant region Υ of $(y_{t-1}, \dots, y_{t-p})$ space. The correspondence between the random-coefficient formulation of the QAR model (1) and the conditional quantile function formulation (2) presupposes the monotonicity of the latter in τ. In the region Υ where this monotonicity holds, (1) can be considered a valid mechanism for simulating from the QAR model (2). Of course, model (1) can, even in the absence of this monotonicity, be taken as a valid data-generating mechanism; however, the link to the strictly linear conditional quantile model is no longer valid. At points where the monotonicity is violated, the conditional quantile functions corresponding to the model described by (1) have linear "kinks." Attempting to fit such piecewise linear models with linear specifications can be hazardous. We return to this issue in the discussion of Section 4. In the next section we briefly describe some essential features of the QAR model.

2.2 Properties of the Quantile Autoregression Process

The QAR(p) model (1) can be reformulated in more conventional random-coefficient notation as

$$y_t = \mu_0 + \alpha_{1,t} y_{t-1} + \cdots + \alpha_{p,t} y_{t-p} + u_t, \quad (5)$$

where $\mu_0 = E\theta_0(U_t)$, $u_t = \theta_0(U_t) - \mu_0$, and $\alpha_{j,t} = \theta_j(U_t)$, for $j = 1, \dots, p$. Thus $\{u_t\}$ is an iid sequence of random variables with distribution function $F(\cdot) = \theta_0^{-1}(\cdot + \mu_0)$, and the $\alpha_{j,t}$ coefficients are functions of this u_t innovation random variable. The QAR(p) process (5) can be expressed as an p-dimensional vector autoregression process of order 1,

$$\mathbf{Y}_t = \boldsymbol{\Gamma} + \mathbf{A}_t \mathbf{Y}_{t-1} + \mathbf{V}_t,$$

with

$$\boldsymbol{\Gamma} = \begin{bmatrix} \mu_0 \\ \mathbf{0}_{p-1} \end{bmatrix}, \qquad \mathbf{A}_t = \begin{bmatrix} \mathbf{A}_{p-1,t} & \alpha_{p,t} \\ \mathbf{I}_{p-1} & \mathbf{0}_{p-1} \end{bmatrix},$$

and

$$\mathbf{V}_t = \begin{bmatrix} u_t \\ \mathbf{0}_{p-1} \end{bmatrix},$$

where $\mathbf{A}_{p-1,t} = [\alpha_{1,t}, \dots, \alpha_{p-1,t}]$, $\mathbf{Y}_t = [y_t, \dots, y_{t-p+1}]^\top$, and $\mathbf{0}_{p-1}$ is the $(p-1)$-dimensional vector of 0's. In the Appendix we show that under regularity conditions given in Theorem 1, an $\mathcal{F}_t$-measurable solution for (5) can be found.

To formalize the foregoing discussion and facilitate later asymptotic analysis, we introduce the following conditions:

- A.1 $\{u_t\}$ are iid random variables with mean 0 and variance $\sigma^2 < \infty$. The distribution function of u_t, F, has a continuous density f with $f(u) > 0$ on $\mathcal{U} = \{u : 0 < F(u) < 1\}$.
- A.2 Let $E(\mathbf{A}_t \otimes \mathbf{A}_t) = \boldsymbol{\Omega}_{\mathbf{A}}$; the eigenvalues of $\boldsymbol{\Omega}_{\mathbf{A}}$ have moduli less than unity.
- A.3 Denote the conditional distribution function $\Pr[y_t < \cdot | \mathcal{F}_{t-1}]$ as $F_{t-1}(\cdot)$ and its derivative as $f_{t-1}(\cdot)$; f_{t-1} is uniformly integrable on $\mathcal{U}$.

Theorem 1. Under assumptions A.1 and A.2, the time series y_t given by (5) is covariance stationary and satisfies a central limit theorem,

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{n} (y_t - \mu_y) \Rightarrow \mathrm{N}(0, \omega_y^2),$$

where $\mu_y = \mu_0/(1 - \sum_{j=1}^{p} \mu_j)$, $\omega_y^2 = \lim n^{-1} E[\sum_{t=1}^{n} (y_t - \mu_y)]^2$, and $\mu_j = E(\alpha_{j,t})$, $j = 1, \dots, p$.

To illustrate some important features of the QAR process, we consider the simplest case of a QAR(1) process,

$$y_t = \alpha_t y_{t-1} + u_t, \quad (6)$$

where $\alpha_t = \theta_1(U_t)$ and $u_t = \theta_0(U_t)$, corresponding to (4), the properties of which are summarized in the following corollary.

Corollary 1. If y_t is determined by (6) and $\omega_\alpha^2 = E(\alpha_t)^2 < 1$, then, under assumption A.1, y_t is covariance stationary and satisfies a central limit theorem,

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n} y_t \Rightarrow \mathrm{N}(0, \omega_y^2),$$

where $\omega_y^2 = \sigma^2(1+\mu_\alpha)/((1-\mu_\alpha)(1-\omega_\alpha^2))$ with $\mu_\alpha = E(\alpha_t) < 1$.

In the example given in Section 2.1, $\alpha_t = \theta_1(U_t) = \min\{\gamma_0 + \gamma_1 U_t, 1\} \le 1$, and $\Pr(|\alpha_t| < 1) > 0$, the condition of Corollary 1 holds, and the process y_t is globally stationary but can still display local (and asymmetric) persistency in the presence of certain type of shocks (positive shocks in the example). Corollary 1 also indicates that even with $\alpha_t > 1$ over some range of quantiles, as long as $\omega_\alpha^2 = E(\alpha_t)^2 < 1$, y_t can still be covariance stationary in the long run. Thus, a QAR process may allow for some (transient) forms of explosive behavior while maintaining stationarity in the long run.

Under the assumptions in Corollary 1, by recursively substituting in (6), we can see that

$$y_t = \sum_{j=0}^{\infty} \beta_{t,j} u_{t-j}, \quad \text{where } \beta_{t,0} = 1 \text{ and } \beta_{t,j} = \prod_{i=0}^{j-1} \alpha_{t-i}, \quad \text{for } j \ge 1, \tag{7}$$

is a stationary $\mathcal{F}_t$-measurable solution to (6). In addition, if $\sum_{j=0}^{\infty} \beta_{t,j} v_{t-j}$ converges in L^p, then y_t has a finite pth-order moment. The $\mathcal{F}_t$-measurable solution of (6) gives a doubly stochastic $MA(\infty)$ representation of y_t. In particular, the impulse response of y_t to a shock u_{t-j} is stochastic and is given by $\beta_{t,j}$. On the other hand, although the impulse response of the QAR process is stochastic, it does converge (to 0) in mean square (and thus in probability) as $j \to \infty$, corroborating the stationarity of y_t. If we denote the autocovariance function of y_t by $\gamma_y(h)$, then it is easy to verify that $\gamma_y(h) = \mu_\alpha^{|h|}\sigma_y^2$, where $\sigma_y^2 = \sigma^2/(1-\omega_\alpha^2)$.

Remark 1. Compared with the $QAR(1)$ process y_t, if we consider a conventional AR(1) process with autoregressive coefficient μ_α and denote the corresponding process by $\underline{y}_t$, then the long-run variance of y_t (given by ω_y^2) is (as expected) larger than that of $\underline{y}_t$. The additional variance of the QAR process y_t comes from the variation of α_t. In fact, ω_y^2 can be decomposed into the summation of the long-run variance of $\underline{y}_t$ and an additional term that is determined by the variance of α_t,

$$\omega_y^2 = \underline{\omega}_y^2 + \frac{\sigma^2}{(1-\mu_\alpha)^2(1-\omega_\alpha^2)}\operatorname{var}(\alpha_t),$$

where $\underline{\omega}_y^2 = \sigma^2/(1-\mu_\alpha)^2$ is the long-run variance of $\underline{y}_t$.

We consider estimation and related inference on the QAR model in the next two sections.

3. ESTIMATION

Estimation of the QAR model (3) involves solving the problem

$$\min_{\boldsymbol{\theta}\in\mathbb{R}^{p+1}} \sum_{t=1}^{n} \rho_\tau(y_t - \mathbf{x}_t^\top \boldsymbol{\theta}), \tag{8}$$

where $\rho_\tau(u) = u(\tau - I(u < 0))$ as in work of Koenker and Bassett (1978). Solutions, $\widehat{\boldsymbol{\theta}}(\tau)$, are called autoregression quantiles. Given $\widehat{\boldsymbol{\theta}}(\tau)$, the τth conditional quantile function of y_t, conditional on $\mathbf{x}_t$, can be estimated by

$$\hat{Q}_{y_t}(\tau|\mathbf{x}_t) = \mathbf{x}_t^\top \widehat{\boldsymbol{\theta}}(\tau),$$

and the conditional density of y_t can be estimated by the difference quotients,

$$\hat{f}_{y_t}(\tau|\mathbf{x}_{t-1}) = \frac{(\tau_i - \tau_{i-1})}{\hat{Q}_{y_t}(\tau_i|\mathbf{x}_{t-1}) - \hat{Q}_{y_t}(\tau_{i-1}|\mathbf{x}_{t-1})},$$

for some appropriately chosen sequence of τ's.

If we denote $E(y_t)$ as μ_y and $E(y_t y_{t-j})$ as γ_j, and let $\boldsymbol{\Omega}_0 = E(\mathbf{x}_t\mathbf{x}_t^\top) = \lim n^{-1}\sum_{t=1}^{n} \mathbf{x}_t\mathbf{x}_t^\top$, then

$$\boldsymbol{\Omega}_0 = \begin{bmatrix} 1 & \boldsymbol{\mu}_y^\top \\ \boldsymbol{\mu}_y & \boldsymbol{\Omega}_y \end{bmatrix},$$

where $\boldsymbol{\mu}_y = \mu_y \cdot \mathbf{1}_{p\times 1}$ and

$$\boldsymbol{\Omega}_y = \begin{bmatrix} \gamma_0 & \cdots & \gamma_{p-1} \\ \vdots & \ddots & \vdots \\ \gamma_{p-1} & \cdots & \gamma_0 \end{bmatrix}.$$

In the special case of QAR(1) model (6), $\boldsymbol{\Omega}_0 = E(\mathbf{x}_t\mathbf{x}_t^\top) = \operatorname{diag}[1, \gamma_0]$, $\gamma_0 = E[y_t^2]$. Let $\boldsymbol{\Omega}_1 = \lim n^{-1}\sum_{t=1}^{n} f_{t-1}[F_{t-1}^{-1}(\tau)] \times \mathbf{x}_t\mathbf{x}_t^\top$, and define $\boldsymbol{\Sigma} = \boldsymbol{\Omega}_1^{-1}\boldsymbol{\Omega}_0\boldsymbol{\Omega}_1^{-1}$. The asymptotic distribution of $\widehat{\boldsymbol{\theta}}(\tau)$ is summarized in the following theorem.

Theorem 2. Under assumptions A.1–A.3,

$$\boldsymbol{\Sigma}^{-1/2}\sqrt{n}(\widehat{\boldsymbol{\theta}}(\tau) - \boldsymbol{\theta}(\tau)) \Rightarrow \mathbf{B}_k(\tau),$$

where $\mathbf{B}_k(\tau)$ represents a k-dimensional standard Brownian bridge, $k = p+1$.

By definition, for any fixed τ, $\mathbf{B}_k(\tau)$ is $\mathcal{N}(\mathbf{0}, \tau(1-\tau)\mathbf{I}_k)$. In the important special case with constant coefficients, $\boldsymbol{\Omega}_1 = f[F^{-1}(\tau)]\boldsymbol{\Omega}_0$, where $f(\cdot)$ and $F(\cdot)$ are the density and distribution functions of u_t. We state this result in the following corollary.

Corollary 2. Under assumptions A.1–A.3, if the coefficients α_{jt} are constants, then

$$f[F^{-1}(\tau)]\boldsymbol{\Omega}_0^{1/2}\sqrt{n}(\widehat{\boldsymbol{\theta}}(\tau) - \boldsymbol{\theta}(\tau)) \Rightarrow \mathbf{B}_k(\tau).$$

An alternative form of the model that is widely used in economic applications is the augmented Dickey–Fuller (ADF) regression,

$$y_t = \mu_0 + \delta_{0,t} y_{t-1} + \sum_{j=1}^{p-1} \delta_{j,t}\Delta y_{t-j} + u_t, \tag{9}$$

where, corresponding to (5),

$$\delta_{0,t} = \sum_{s=1}^{p} \alpha_{s,t}, \qquad \delta_{j,t} = -\sum_{s=j+1}^{p} \alpha_{s,t}, \qquad j = 1, \dots, p-1.$$

In the foregoing transformed model, $\delta_{0,t}$ is the critical parameter corresponding to the largest autoregressive root. Letting $\mathbf{z}_t = (1, y_{t-1}, \Delta y_{t-1}, \dots, \Delta y_{t-p+1})^\top$, we may write the quantile regression counterpart of (9) as

$$Q_{y_t}(\tau|\mathcal{F}_{t-1}) = \mathbf{z}_t^\top \boldsymbol{\delta}(\tau), \tag{10}$$

where

$$\boldsymbol{\delta}(\tau) = \big(\alpha_0(\tau), \delta_0(\tau), \delta_1(\tau), \dots, \delta_{p-1}(\tau)\big)^\top.$$

The limiting distributions of the quantile regression estimators $\widehat{\boldsymbol{\delta}}(\tau)$ can be obtained from our previous analysis. If we define

$$\mathbf{J} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & -1 & & -1 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Delta} = \mathbf{J}\boldsymbol{\Sigma}\mathbf{J},$$

then we have, under assumptions A.1–A.3,

$$\boldsymbol{\Delta}^{-1/2}\sqrt{n}(\widehat{\boldsymbol{\delta}}(\tau) - \boldsymbol{\delta}(\tau)) \Rightarrow \mathbf{B}_k(\tau).$$

If we focus our attention on the largest autoregressive root $\delta_{0,t}$ in the ADF-type regression (9) and consider the special case where $\delta_{j,t}$ is constant for $j = 1, \dots, p-1$, then a result similar to Corollary 1 can be obtained.

Corollary 3. Under assumptions A.1–A.3, if $\delta_{j,t}$ is constant for $j = 1, \dots, p-1$, and $\delta_{0,t} \le 1$ and $|\delta_{0,t}| < 1$ with positive probability, then the time series y_t given by (9) is covariance stationary and satisfies a central limit theorem.

4. QUANTILE MONOTONICITY

As in other linear quantile regression applications, linear QAR models should be cautiously interpreted as useful local approximations to more complex nonlinear global models. If we take the linear form of the model too literally, then obviously at some point (or points) there will be "crossings" of the conditional quantile functions, unless these functions are precisely parallel, in which case we are back to the pure location-shift form of the model. This crossing problem appears to be more acute in the autoregressive case than in ordinary regression applications, because the support of the design space (i.e., the set of $\mathbf{x}_t$'s that occur with positive probability) is determined within the model. Nevertheless, we may still regard the linear models specified earlier as valid local approximations over a region of interest.

It should be stressed that the *estimated* conditional quantile functions,

$$\hat{Q}_y(\tau|\mathbf{x}) = \mathbf{x}^\top \widehat{\boldsymbol{\theta}}(\tau),$$

are guaranteed to be monotone at the mean design point, $\mathbf{x} = \bar{\mathbf{x}}$, as shown by Bassett and Koenker (1982) for linear quantile regression models. In our random-coefficient view of the QAR model,

$$y_t = \mathbf{x}_t^\top \boldsymbol{\theta}(U_t),$$

we express the observable random variable y_t as a linear function of conditioning covariates. But rather than assuming that the coordinates of the vector $\boldsymbol{\theta}$ are independent random variables, we adopt a diametrically opposite viewpoint—that they are perfectly functionally dependent, all driven by a single random uniform variable. If the functions $(\theta_0, \dots, \theta_p)$ are all monotonically increasing, then the coordinates of the random vector $\boldsymbol{\alpha}_t$ are said to be comonotonic in the sense of Schmeidler (1986). [Random variables X and Y on a probability space $(\Omega, \mathcal{A}, P)$ are said to be comonotonic if there are monotone functions g and h and a random variable Z on $(\Omega, \mathcal{A}, P)$ such that $X = g(Z)$ and $Y = h(Z)$.] This is often the case, but there are important cases in which this monotonicity fails. What then?

What really matters is that we can find a linear reparameterization of the model that does exhibit comonotonicity over some relevant region of covariate space. Because for any nonsingular matrix $\mathbf{A}$ we can write

$$Q_y(\tau|\mathbf{x}) = \mathbf{x}^\top \mathbf{A}^{-1}\mathbf{A}\boldsymbol{\theta}(\tau),$$

we can choose $p+1$ linearly independent design points $\{\mathbf{x}_s : s = 1, \dots, p+1\}$, where $Q_y(\tau|\mathbf{x}_s)$ is monotone in τ. Then, choosing the matrix $\mathbf{A}$ so that $\mathbf{A}\mathbf{x}_s$ is the sth unit basis vector for $\mathbb{R}^{p+1}$, we have

$$Q_y(\tau|\mathbf{x}_s) = \gamma_s(\tau),$$

where $\boldsymbol{\gamma} = \mathbf{A}\boldsymbol{\theta}$. Now inside the convex hull of our selected points, we have a comonotonic random-coefficient representation of the model. In effect, we have simply reparameterized the design so that the $p+1$ coefficients are the conditional quantile functions of y_t at the selected points. The fact that quantile functions of sums of nonnegative comonotonic random variables are sums of their marginal quantile functions (see, e.g., Denneberg 1994; Bassett, Koenker, and Kordas 2004) allows us to interpolate inside the convex hull. Of course, linear extrapolation is also possible, but we must be cautious about possible violations of the monotonicity requirement in this region.

The interpretation of linear conditional quantile functions as approximations to the local behavior in central range of the covariate space should always be considered provisional; richer data sources can be expected to yield more elaborate nonlinear specifications that would have validity over larger regions. Figure 1 illustrates a realization of the simple QAR(1) model described in Section 2. The black sample path shows 1,000 observations generated from the model (4) with AR(1) coefficients $\theta_1(u) = .85 + .25u$ and $\theta_0(u) = \Phi^{-1}(u)$. The gray sample path

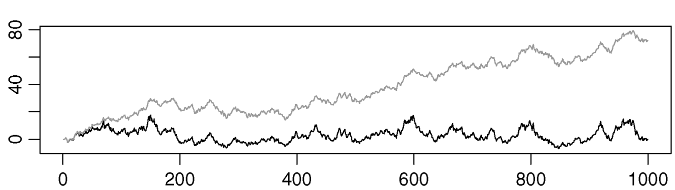

Figure 1. QAR and Unit Root Time Series. This figure contrasts two time series generated by the same sequence of innovations. The gray sample path is a random walk with standard Gaussian innovations; the black sample path illustrates a QAR series generated by the same innovations with random AR(1) coefficient .85 + .25 $\Phi(u_t)$. The latter series, although exhibiting explosive behavior in the upper tail, is stationary, as described in the text.

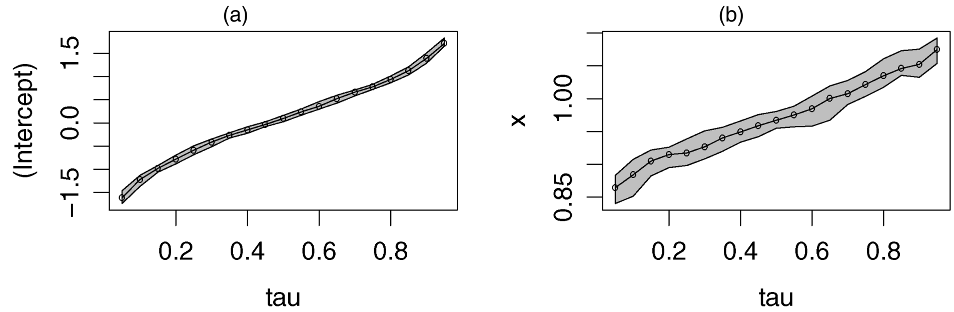

Figure 2. Estimating the QAR Model. This figure illustrates estimates of the QAR(1) model based on the black time series of the previous figure: (a) the intercept estimate at 19 equally spaced quantiles and (b) the AR(1) slope estimate at the same quantiles. The shaded region is a .90 confidence band. Note that the slope estimate quite accurately reproduces the linear form of the QAR(1) coefficient used to generate the data.

depicts the a random walk generated from the same innovation sequence, that is, the same $\theta_0(U_t)$'s but with constant θ_1 equal to 1. It is easy to verify that the QAR(1) form of the model satisfies the stationarity conditions of Section 2.2, and despite the explosive character of its upper-tail behavior, we observe that the series appears quite stationary, at least compared with the random-walk series. Estimating the QAR(1) model at 19 equally spaced quantiles yields the intercept and slope estimates depicted in Figure 2.

Figure 3 depicts estimated linear conditional quantile functions for short-term (3-month) U.S. interest rates using the QAR(1) model superimposed on the AR(1) scatterplot. In this example, the scatterplot clearly shows greater dispersion at higher interest rates, with nearly degenerate behavior at very low rates. The fitted linear quantile regression lines in Figure 3(a) show little evidence of crossing, but at rates below .04 there are some violations of the monotonicity requirement in the fitted quantile functions. Fitting the data using a somewhat more complex nonlinear (in variables) model by introducing a another additive component $\theta_2(\tau)(y_{t-1}-\delta)^2 I(y_{t-1}<\delta)$ with $\delta = 8$ in our example we can eliminate the problem of the crossing of fitted quantile functions. Figure 4 depicts the fitted coefficients of the QAR(1) model and their confidence region, showing that the estimated slope coefficient of the QAR(1) model has a somewhat similar appearance to the simulated example. Even more flexible models may be needed in other settings. A B-spline expansion QAR(1) model for Melbourne daily temperature has been described by Koenker (2000) to illustrate this approach.

The statistical properties of nonlinear QAR models and associated estimators are much more complicated than the linear QAR model that we study in the present article. Despite the possible crossing of quantile curves, we believe that the linear QAR model provides a convenient and useful local approximation to nonlinear QAR models. Such simple QAR models can still deliver important insight about dynamics (e.g., adjustment asymmetries) in economic time series and thus provides a useful tool in empirical diagnostic time series analysis.

5. INFERENCE ON THE QUANTILE AUTOREGRESSION PROCESS

In this section we turn our attention to inference in QAR models. Although other inference problems can be analyzed, here we consider the following inference problems that are of paramount interest in many applications. The first hypothesis is the quantile regression analog of the classical representation of linear restrictions on $\boldsymbol{\theta}$: (1) $H_{01}: \mathbf{R}\boldsymbol{\theta}(\tau) = \mathbf{r}$, with known $\mathbf{R}$ and $\mathbf{r}$, where $\mathbf{R}$ denotes an $(q \times p)$-dimensional matrix and $\mathbf{r}$ is an q-dimensional vector. In addition to the classical inference problem, we are also interested in testing for asymmetric dynamics under the QAR framework. Thus we consider the hypothesis of parameter constancy, which can be formulated in

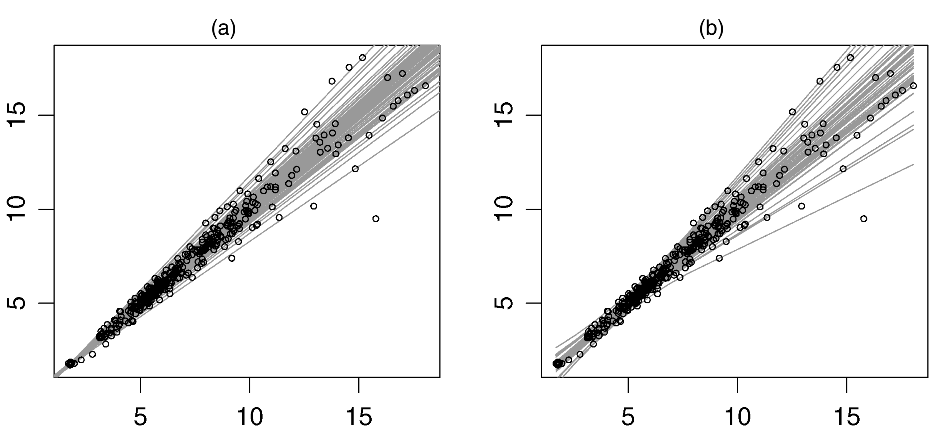

Figure 3. QAR(1) Model of U.S. Short-Term Interest Rate. (a) The AR(1) scatterplot of the U.S. 3-month rate superimposed with 49 equally spaced estimates of linear conditional quantile functions. (b) The model is augmented with a nonlinear (quadratic) component. The introduction of the quadratic component alleviates some nonmonotonicity in the estimated quantiles at low interest rates.

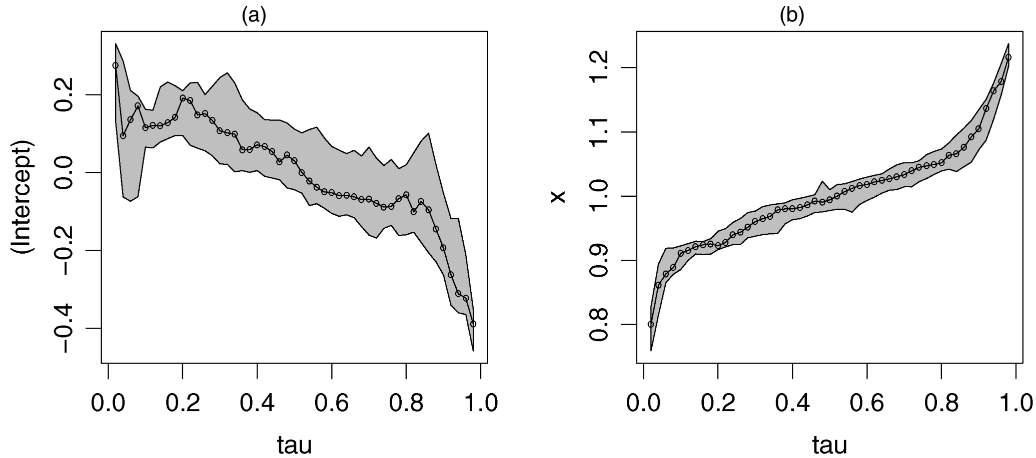

Figure 4. QAR(1) Model of U.S. Short-Term Interest Rate. The QAR(1) estimates of the intercept (a) and slope (b) parameters for 19 equally spaced quantile functions. Note that the slope parameter is, like the prior simulated example, explosive in the upper tail but mean-reverting in the lower tail.

the form of (2) $H_{02}: \mathbf{R}\boldsymbol{\theta}(\tau) = \mathbf{r}$, with unknown but estimable $\mathbf{r}$. We consider both the cases at specific quantiles, τ (e.g., median, lower quartile, upper quartile), and the case over a range of quantiles, $\tau \in \mathcal{T}$.

5.1 The Regression Wald Process and Related Tests

Under the linear hypothesis $H_{01}: \mathbf{R}\boldsymbol{\theta}(\tau) = \mathbf{r}$ and assumptions A.1–A.3, we have

$$\mathbf{V}_n(\tau) = \sqrt{n}[\mathbf{R}\boldsymbol{\Omega}_1^{-1}\boldsymbol{\Omega}_0\boldsymbol{\Omega}_1^{-1}\mathbf{R}^\top]^{-1/2}(\mathbf{R}\widehat{\boldsymbol{\theta}}(\tau) - \mathbf{r}) \Rightarrow \mathbf{B}_q(\tau), \tag{11}$$

where $\mathbf{B}_q(\tau)$ represents a q-dimensional standard Brownian bridge. For any fixed τ, $\mathbf{B}_q(\tau)$ is $\mathcal{N}(\mathbf{0}, \tau(1-\tau)\mathbf{I}_q)$. Thus the regression Wald process can be constructed as

$$W_n(\tau) = n(\mathbf{R}\widehat{\boldsymbol{\theta}}(\tau) - \mathbf{r})^\top[\tau(1-\tau)\mathbf{R}\widehat{\boldsymbol{\Omega}}_1^{-1}\widehat{\boldsymbol{\Omega}}_0\widehat{\boldsymbol{\Omega}}_1^{-1}\mathbf{R}^\top]^{-1} \times (\mathbf{R}\widehat{\boldsymbol{\theta}}(\tau) - \mathbf{r}),$$

where $\widehat{\boldsymbol{\Omega}}_1$ and $\widehat{\boldsymbol{\Omega}}_0$ are consistent estimators of $\boldsymbol{\Omega}_1$ and $\boldsymbol{\Omega}_0$. If we are interested in testing $\mathbf{R}\boldsymbol{\theta}(\tau) = \mathbf{r}$ over $\tau \in \mathcal{T}$, then we may consider, say, the following Kolmogorov–Smirnov (KS)-type sup-Wald test:

$$KSW_n = \sup_{\tau \in \mathcal{T}} W_n(\tau).$$

If we are interested in testing $\mathbf{R}\boldsymbol{\theta}(\tau) = \mathbf{r}$ at a particular quantile $\tau = \tau_0$, then we can conduct a chi-squared test based on the statistic $W_n(\tau_0)$. The limiting distributions are summarized in the following theorem.

Theorem 3. Under assumptions A.1–A.3 and the linear restriction H_{01},

$$W_n(\tau_0) \Rightarrow \chi_q^2 \qquad \text{and} \qquad KSW_n = \sup_{\tau \in \mathcal{T}} W_n(\tau) \Rightarrow \sup_{\tau \in \mathcal{T}} Q_q^2(\tau),$$

where $Q_q(\tau) = \|\mathbf{B}_q(\tau)\|/\sqrt{\tau(1-\tau)}$ is a Bessel process of order q, where $\|\cdot\|$ represents the Euclidean norm. For any fixed τ, $Q_q^2(\tau) \sim \chi_q^2$ is a centered chi-squared random variable with q degrees of freedom.

5.2 Testing for Asymmetric Dynamics

The hypothesis that $\theta_j(\tau)$, $j = 1, \ldots, p$, are constants over τ [i.e., $\theta_j(\tau) = \mu_j$] can be represented as $H_{02}: \mathbf{R}\boldsymbol{\theta}(\tau) = \mathbf{r}$ by taking $\mathbf{R} = [\mathbf{0}_{p\times 1} \vdots \mathbf{I}_p]$ and $\mathbf{r} = [\mu_1, \ldots, \mu_p]^\top$ with unknown parameters $\mu_1, \ldots, \mu_p$. The Wald process and associated limiting theory provide a natural test for the hypothesis $\mathbf{R}\boldsymbol{\theta}(\tau) = \mathbf{r}$ when $\mathbf{r}$ is known. To test the hypothesis with unknown $\mathbf{r}$, an appropriate estimator of $\mathbf{r}$ is needed. In many econometrics applications, a $\sqrt{n}$-consistent estimator of $\mathbf{r}$ is available. If we look at the process

$$\widehat{\mathbf{V}}_n(\tau) = \sqrt{n}[\mathbf{R}\widehat{\boldsymbol{\Omega}}_1^{-1}\widehat{\boldsymbol{\Omega}}_0\widehat{\boldsymbol{\Omega}}_1^{-1}\mathbf{R}^\top]^{-1/2}(\mathbf{R}\widehat{\boldsymbol{\theta}}(\tau) - \widehat{\mathbf{r}}),$$

then, under H_{02}, we have

$$\begin{aligned}\widehat{\mathbf{V}}_n(\tau) &= \sqrt{n}[\mathbf{R}\widehat{\boldsymbol{\Omega}}_1^{-1}\widehat{\boldsymbol{\Omega}}_0\widehat{\boldsymbol{\Omega}}_1^{-1}\mathbf{R}^\top]^{-1/2}(\mathbf{R}\widehat{\boldsymbol{\theta}}(\tau) - \mathbf{r}) \\ &\quad - \sqrt{n}[\mathbf{R}\widehat{\boldsymbol{\Omega}}_1^{-1}\widehat{\boldsymbol{\Omega}}_0\widehat{\boldsymbol{\Omega}}_1^{-1}\mathbf{R}^\top]^{-1/2}(\widehat{\mathbf{r}} - \mathbf{r}) \\ &\Rightarrow \mathbf{B}_q(\tau) - f(F^{-1}(\tau))[\mathbf{R}\boldsymbol{\Omega}_0^{-1}\mathbf{R}^\top]^{-1/2}\mathbf{Z},\end{aligned}$$

where $\mathbf{Z} = \lim \sqrt{n}(\widehat{\mathbf{r}} - \mathbf{r})$. The need to estimate $\mathbf{r}$ introduces a drift component in addition to the simple Brownian bridge process, invalidating the distribution-free character of the original KS test.

To restore the asymptotically distribution-free nature of inference, we use a martingale transformation proposed by Khmaladze (1981) over the process $\widehat{\mathbf{V}}_n(\tau)$. Denote $df(x)/dx$ as $\dot{f}$ and define

$$\dot{\mathbf{g}}(r) = \left(1, (\dot{f}/f)(F^{-1}(r))\right)^\top \qquad \text{and}$$

$$\mathbf{C}(s) = \int_s^1 \dot{\mathbf{g}}(r)\dot{\mathbf{g}}(r)^\top\, dr,$$

we construct a martingale transformation $\mathcal{K}$ on $\widehat{\mathbf{V}}_n(\tau)$, defined as

$$\begin{aligned}\widetilde{\mathbf{V}}_n(\tau) &= \mathcal{K}\widehat{\mathbf{V}}_n(\tau) \\ &= \widehat{\mathbf{V}}_n(\tau) - \int_0^\tau \left[\dot{\mathbf{g}}_n(s)^\top \mathbf{C}_n^{-1}(s) \int_s^1 \dot{\mathbf{g}}_n(r)\, d\widehat{\mathbf{V}}_n(r)\right] ds,\end{aligned} \tag{12}$$

where $\dot{\mathbf{g}}_n(s)$ and $\mathbf{C}_n(s)$ are uniformly consistent estimators of $\dot{\mathbf{g}}(r)$ and $\mathbf{C}(s)$ over $\tau \in \mathcal{T}$, and propose the following KS-type test based on the transformed process:

$$KH_n = \sup_{\tau \in \mathcal{T}} \|\widetilde{\mathbf{V}}_n(\tau)\|. \tag{13}$$

(A Cramer–von Mises-type test based on the transformed process can also be constructed and analyzed in a similar way.) Under the null hypothesis, the transformed process $\widetilde{\mathbf{V}}_n(\tau)$ converges to a standard Brownian motion. (For more discussion of quantile regression inference based on the martingale transformation approach, see Koenker and Xiao 2002 and references therein.) We make the following assumption on the estimators:

A.4. There exist estimators $\dot{\mathbf{g}}_n(\tau)$, $\widehat{\mathbf{\Omega}}_0$ and $\widehat{\mathbf{\Omega}}_1$ satisfying (a) $\sup_{\tau\in} |\dot{\mathbf{g}}_n(\tau) - \dot{\mathbf{g}}(\tau)| = o_p(1)$, and (b) $\|\widehat{\mathbf{\Omega}}_0 - \mathbf{\Omega}_0\| = o_p(1)$, $\|\widehat{\mathbf{\Omega}}_1 - \mathbf{\Omega}_1\| = o_p(1)$, $\sqrt{n}(\widehat{\mathbf{r}} - \mathbf{r}) = O_p(1)$.

Theorem 4. Under the assumptions A.1–A.4 and the hypothesis H_{02},

$$\widetilde{\mathbf{V}}_n(\tau) \Rightarrow \mathbf{W}_q(\tau), \qquad KH_n = \sup_{\tau \in \mathcal{T}} \|\widetilde{\mathbf{V}}_n(\tau)\| \Rightarrow \sup_{\tau \in \mathcal{T}} \|\mathbf{W}_q(\tau)\|,$$

where $\mathbf{W}_q(r)$ is a q-dimensional standard Brownian motion.

The martingale transformation is based on the function $\dot{\mathbf{g}}(s)$, which must be estimated. There are several approaches to estimating the score $\frac{f'}{f}(F^{-1}(s))$. Portnoy and Koenker (1989) studied adaptive estimation and used kernel-smoothing method in estimating the density and score functions; they also discussed uniform consistency of the estimators. Cox (1985) proposed an elegant smoothing spline approach to the estimation of f'/f, and Ng (1994) provided an efficient algorithm for computing this score estimator. Estimation of $\mathbf{\Omega}_0$ is straightforward: $\widehat{\mathbf{\Omega}}_0 = n^{-1}\sum_t \mathbf{x}_t\mathbf{x}_t^\top$. (For estimation of $\widehat{\mathbf{\Omega}}_1$, see, inter alia, Koenker 1994; Powell 1989; Koenker and Machado 1999 for related discussions.)

6. MONTE CARLO

Here we report a Monte Carlo experiment conducted to examine the QAR-based inference procedures. We were particularly interested in time series displaying asymmetric dynamics; thus we considered the QAR model with $p = 1$ and tested the hypothesis that $\alpha_1(\tau)$ is constant over τ.

The data in our experiments were generated from model (6), where u_t are iid random variables. We consider the KS test KH_n given by (13) for different sample sizes ($n = 100$ and 300) and innovation distributions, and choose $\mathcal{T} = [.1, .9]$. Both normal innovations and Student-t innovations are considered. The number of repetitions is 1,000.

Representative results of the empirical size and power of the proposed tests are reported in Tables 1–3. We report the empirical size of this test for three choices of α_t: $\alpha_t = .95$, .9, and .6. The first two choices (.95 and .9) are large and close to unity, so that the corresponding time series display a certain degree of (symmetric) persistence. For models under the alternative, we considered the following four choices of α_t:

$$\alpha_t = \varphi_1(u_t) = \begin{cases} 1, & u_t \ge 0 \\ .8, & u_t < 0, \end{cases}$$

$$\alpha_t = \varphi_2(u_t) = \begin{cases} .95, & u_t \ge 0 \\ .8, & u_t < 0, \end{cases}$$

$$\alpha_t = \varphi_3(u_t) = \min\{.5 + F_u(u_t), 1\},$$

$$\alpha_t = \varphi_4(u_t) = \min\{.75 + F_u(u_t), 1\}. \tag{14}$$

These alternatives deliver processes with different types of asymmetric (or local) persistence. In particular, when $\alpha_t = \varphi_1(u_t)$, $\varphi_3(u_t)$, $\varphi_4(u_t)$, and y_t display unit-root behavior in the presence of positive or large values of innovations but have a mean reversion tendency with negative shocks. The alternative $\alpha_t = \varphi_2(u_t)$ has local to (or weak) unit-root behavior in the presence of positive innovations and behave more stationarily when there are negative shocks.

The construction of tests uses estimators of density and score. We estimate the density (or sparsity) function using the approach of Siddiqui (1960). The density estimation entails choosing a bandwidth. We consider the bandwidth choices suggested by Hall and Sheather (1988) and Bofinger (1975) and rescaled versions of them. A bandwidth rule that Hall and Sheather (1988) suggested based on Edgeworth expansion for studentized quantiles (and using Gaussian plug-in) is

$$h_{HS} = n^{-1/3} z_\alpha^{2/3} \left[\frac{1.5\phi^2(\Phi^{-1}(t))}{2(\Phi^{-1}(t))^2 + 1}\right]^{1/3},$$

where z_α satisfies $\Phi(z_\alpha) = 1 - \alpha/2$ for the construction of $1 - \alpha$ confidence intervals. Another bandwidth selection has been proposed by Bofinger (1975) based on minimizing the mean squared error of the density estimator and is of order $n^{-1/5}$. Plugging in the Gaussian density, we obtain the following bandwidth, which is widely used in practice:

$$h_B = n^{-1/5} \left[\frac{4.5\phi^4(\Phi^{-1}(t))}{(2(\Phi^{-1}(t))^2 + 1)^2}\right]^{1/5}.$$

Monte Carlo results indicate that the Hall–Sheather bandwidth provides a good lower bound, and the Bofinger bandwidth provides a reasonable upper bound for bandwidth in testing parameter constancy. For this reason, we considered bandwidth choices between h_{HS} and h_B. In particular, we considered rescaled versions of h_B and h_{HS} (θh_B and δh_{HS}, where $0 < \theta < 1$ and $\delta > 1$ are scalars) in our Monte Carlo experiment and report representative results. Bandwidth values that are constant over the whole range of quantiles are not recommended. The sampling performance of tests using a constant bandwidth turned out to be poor and inferior to such bandwidth choices as the Hall–Sheather and Bofinger bandwidths which vary over the quantiles. For these reason, we focus on bandwidths h_B, h_{HS}, θh_B, and δh_{HS}. The Monte Carlo results indicate that the test using a rescaled version of the Bofinger bandwidth ($h = .6h_B$) yields good performance in the cases that we study.

We estimated the score function by the method of Portnoy and Koenker (1989) and chose the Silverman (1986) bandwidth in our Monte Carlo. Our simulation results show that the test was more affected by the estimator of the density than that of the score. Intuitively, the estimator of the density plays the role of a scalar and thus has the greatest influence. The Monte Carlo results also indicate that the method of Portnoy and Koenker (1989) coupled with the Silverman bandwidth has reasonably good performance. Table 1 reports the empirical size and power for the case with Gaussian innovations and sample size $n = 100$. Table 2 reports results in the case where the u_t's are Student-t

Table 1. Empirical Size and Power of Tests of Constancy of the Coefficient α With Gaussian Innovations

	Model	$h = 3h_{HS}$	$h = h_{HS}$	$h = h_B$	$h = .6h_B$
Size	$\alpha_t = .95$	.073	.287	.018	.056
	$\alpha_t = .9$	.073	.275	.01	.046
	$\alpha_t = .6$	.07	.287	.012	.052
Power	$\alpha_t = \varphi_1(u_t)$	.474	.795	.271	.391
	$\alpha_t = \varphi_2(u_t)$	.262	.620	.121	.234
	$\alpha_t = \varphi_3(u_t)$	.652	.939	.322	.533
	$\alpha_t = \varphi_4(u_t)$	.159	.548	.046	.114

NOTE: Models for size use the indicated constant coefficient; models for power comparisons are those indicated in (14). The sample size is 100, and the number of replications is 1,000.

innovations (with 3 degrees of freedom) and $n = 100$. Results in Table 2 confirm that using the quantile regression-based approach, power gain can be obtained in the presence of heavy-tailed disturbances. (Such gains obviously depend on choosing quantiles at which there is sufficient conditional density.) Experiments based on larger sample sizes are also conducted. Table 3 reports the size and power for the case with Gaussian innovations and sample size $n = 300$. These results are qualitatively similar to those of Table 1 but also show that as the sample sizes increase, the tests do have improved size and power properties, corroborating the asymptotic theory.

7. EMPIRICAL APPLICATIONS

There have been many claims and observations that some economic time series display asymmetric dynamics. For example, it has been observed that increases in the unemployment rate are sharper than declines. If an economic time series displays asymmetric dynamics systematically, then appropriate models are needed to incorporate such behavior. In this section we apply the QAR model to two economic time series: unemployment rates and retail gasoline prices in the United States. Our empirical analysis indicate that both series display asymmetric dynamics.

7.1 Unemployment Rate

Many studies on unemployment suggest that the response of unemployment to expansionary or contractionary shocks may be asymmetric. An asymmetric response to different types of shocks has important implications for economic policy. In this section we examine unemployment dynamics using the proposed procedures.

The data that we consider are quarterly and annual rates of unemployment in the United States. In particular, we look at (seasonally adjusted) quarterly rates, starting from the first quarter of 1948 and ending at the last quarter of 2003, for a total of 224 observations, as well as the annual rates from 1890 to 1996. Many empirical studies in the unit-root literature have investigated unemployment rate data. Nelson and Plosser (1982) studied the unit-root property of annual U.S. unemployment rates in their seminal work on 14 macroeconomic time series. Evidence based on the unit-root tests suggests that the series is stationary. This series and other types of unemployment rates often have been reexamined in later analysis.

Table 2. Empirical Size and Power of Tests of Constancy of the Coefficient α With t(3) Innovations

	Model	$h = 3h_{HS}$	$h = h_{HS}$	$h = h_B$	$h = .6h_B$
Size	$\alpha_t = .95$	.086	.339	.011	.059
	$\alpha_t = .9$	.072	.301	.015	.043
	$\alpha_t = .6$	.072	.305	.013	.038
Power	$\alpha_t = \varphi_1(u_t)$	.556	.819	.319	.444
	$\alpha_t = \varphi_2(u_t)$	.348	.671	.174	.279
	$\alpha_t = \varphi_3(u_t)$	.713	.933	.346	.55
	$\alpha_t = \varphi_4(u_t)$	.284	.685	.061	.162

NOTE: Configurations are as in Table 1.

Table 3. Empirical Size and Power of Tests of Constancy of the Coefficient α With Gaussian Innovations

	Model	$h = 3h_{HS}$	$h = h_{HS}$	$h = h_B$	$h = .6h_B$
Size	$\alpha_t = .95$	.081	.191	.028	.049
	$\alpha_t = .90$	.098	.189	.030	.056
	$\alpha_t = .60$	.097	.160	.020	.045
Power	$\alpha_t = \varphi_1(u_t)$	.974	.992	.921	.937
	$\alpha_t = \varphi_2(u_t)$	.831	.923	.685	.763
	$\alpha_t = \varphi_3(u_t)$	.998	1.000	.971	.989
	$\alpha_t = \varphi_4(u_t)$	.557	.897	.235	.392

NOTE: Configurations are as in Table 1, except that the sample size is 300.

We first apply regression (10) on the unemployment rates. We use the Bayes information criterion (BIC) of Schwarz (1978) and Rissanen (1978) in selecting the appropriate lag length of the autoregressions. The selected lag length is $p = 3$ for the annual data and $p = 2$ for the quarterly data. The ordinary least squares (OLS) estimation of the largest autoregressive root is .718 for the annual series and .941 for the quarterly rates. We also perform QAR for each decile. Estimates of the largest AR root at each quantile are reported in Table 4. These estimated values vary over different quantiles, displaying asymmetric dynamics.

We then test asymmetric dynamics using the martingale transformation-based KS procedure (13) based on QAR (8). Following a suggestion from the Monte Carlo results, we choose the rescaled Hall and Sheather (1988) bandwidth $3h_{HS}$ and the rescaled Bofinger (1975) bandwidth $.6h_B$ in estimating the density function. The tests are constructed over $\tau \in T = [.05, .95]$, and results are reported in Table 5. The empirical results indicate that asymmetric behavior exist in these series.

7.2 Retail Gasoline Price Dynamics

Our second application investigates the asymmetric price dynamics in the retail gasoline market. We consider weekly data of U.S. regular gasoline retail price from August 20, 1990 to Februry 16, 2004. The sample size is 699. Evidence from OLS-based ADF tests of the null hypothesis of a unit root is mixed. The unit-root null is rejected by the coefficient-based test ADF_α, with a test statistic of -17.14 and critical value of -14.1, but it cannot be rejected by the t-ratio–based test ADF_t, given the test statistic -2.67 and critical value -2.86. Again we use the BIC to select the lag length to obtain $p = 4$ for these tests.

Table 4. Estimates of the Largest AR Root at Each Decile of Unemployment

Frequency	τ	.1	.2	.3	.4	.5	.6	.7	.8	.9
Annual	$\delta_0(\tau)$	.740	.776	.929	.871	.858	.793	.727	.680	.599
Quarterly	$\delta_0(\tau)$	.912	.908	.931	.919	.951	.959	.967	.962	.953

Table 5. Kolmogorov Test of Constant AR Coefficient for Unemployment

Bandwidth	$.6h_B$	$3h_{HS}$	5% CV
Annual	4.89	5.12	4.523
Quarterly	4.46	5.36	3.393

We next consider quantile regression evidence based on the ADF model (9) on persistence of retail gasoline prices. Table 6 reports the estimates of the largest autoregressive roots $\widehat{\delta}_0(\tau)$ at each decile. These results suggest that the gasoline price series has asymmetric dynamics. The estimate takes quite different values over different quantiles. Estimates, $\widehat{\delta}_0(\tau)$, monotonically increase as we move from lower quantiles to higher quantiles. The AR coefficient values at the lower quantiles are relatively small, indicating that the local behavior of the gasoline price would be stationary. However, at higher quantiles, the largest AR root is close to or even slightly above unity, consequently the time series display unit root or locally explosive behavior at upper quantiles.

A formal test of the null hypothesis that gasoline prices have a constant autoregressive coefficient is conducted using the KS procedure (13) based on QAR (2) and martingale transformation (12). Constancy of coefficients is rejected. The calculated KS statistic [using the rescaled Bofinger (1975) bandwidth, $.6h_B$] is 8.347735 (lag length, $p = 4$), considerably larger than the 5% level critical value of 5.56. However, taking into account the possibility of unit-root behavior under the null, we also consider the (coefficient-based) empirical quantile process $U_n(\tau) = n(\widehat{\delta}_0(\tau) - 1)$ and the KS or Cramer–von Mises (CvM)-type tests,

$$QKS_\alpha = \sup_{\tau\in\mathcal{T}} |U_n(\tau)|, \qquad QCM_\alpha = \int_{\tau\in\mathcal{T}} U_n(\tau)^2\, d\tau. \quad (15)$$

Using the results of unit-root quantile regression asymptotics provided by Koenker and Xiao (2004), we have, under the unit-root hypothesis,

$$U_n(\tau) \Rightarrow U(\tau) = \frac{1}{f(F^{-1}(\tau))}\left[\int_0^1 \underline{B}_y^2\right]^{-1} \int_0^1 \underline{B}_y\, dB_\psi^\tau, \quad (16)$$

where $\underline{B}_w(r)$ and $B_\psi^\tau(r)$ are limiting processes of $n^{-1/2} \times \sum_{t=1}^{[nr]} \Delta y_t$ and $n^{-1/2} \sum_{t=1}^{[nr]} \psi_\tau(u_{t\tau})$. We adopt the approach of Koenker and Xiao (2004) and approximate the distributions of the limiting variates by resampling method and construct bootstrap tests for the unit-root hypothesis based on (15).

We consider both the QKS_α and QCM_α tests for the null hypothesis of a constant AR coefficient equal to unity. Both tests firmly reject the null with test statistics of 35.79 and 320.41 and 5% level critical values of 13.22 and 19.72. The critical values were computed based on the resampling procedure described by Koenker and Xiao (2004). These results, together with the point estimates reported in Table 6, indicate that the gasoline price series has asymmetric adjustment dynamics and thus is not well characterized as a constant coefficient unit-root process.

Table 6. Estimated Largest AR Root at Each Decile of Retail Gasoline Price

τ	.1	.2	.3	.4	.5	.6	.7	.8	.9
$\widehat{\delta}_0(\tau)$	.948	.958	.971	.980	.996	1.005	1.016	1.024	1.047

APPENDIX: PROOFS

A.1 Proof of Theorem 2.1

Given a pth-order autoregression process (5), we write $E(\alpha_{j,t}) = \mu_j$ and assume that $1 - \sum \mu_j \neq 0$. Letting $\mu_y = \mu_0/(1 - \sum_{j=1}^p \mu_j)$ and writing

$$\underline{y}_t = y_t - \mu_y,$$

we have

$$\underline{y}_t = \alpha_{1,t}\underline{y}_{t-1} + \cdots + \alpha_{p,t}\underline{y}_{t-p} + v_t, \quad (A.1)$$

where

$$v_t = u_t + \mu \sum_{l=1}^p (\alpha_{l,t} - \mu_l).$$

It is easy to see that $Ev_t = 0$ and $Ev_tv_s = 0$ for any $t \neq s$, because $E\alpha_{l,t} = \mu_l$ and u_t are independent. To derive stationarity conditions for the process $\underline{y}_t$, we first find an $\mathcal{F}_t$-measurable solution for (A.1). We define the $p \times 1$ random vectors

$$\underline{\mathbf{Y}}_t = [\underline{y}_t, \ldots, \underline{y}_{t-p+1}]^\top \qquad \text{and} \qquad \mathbf{V}_t = [v_t, 0, \ldots, 0]^\top$$

and the $p \times p$ random matrix

$$\mathbf{A}_t = \begin{bmatrix} \mathbf{A}_{p-1,t} & \alpha_{p,t} \\ \mathbf{I}_{p-1} & \mathbf{0}_{p-1} \end{bmatrix},$$

where $\mathbf{A}_{p-1,t} = [\alpha_{1,t}, \ldots, \alpha_{p-1,t}]$ and $\mathbf{0}_{p-1}$ is the $(p-1)$-dimensional vector of 0's. Then

$$E(\mathbf{V}_t\mathbf{V}_t^\top) = \begin{bmatrix} \sigma_v^2 & \mathbf{0}_{1\times(p-1)} \\ \mathbf{0}_{(p-1)\times 1} & \mathbf{0}_{(p-1)\times(p-1)} \end{bmatrix} = \mathbf{\Sigma},$$

and the original process can be written as

$$\underline{\mathbf{Y}}_t = \mathbf{A}_t\underline{\mathbf{Y}}_{t-1} + \mathbf{V}_t.$$

By substitution, we have

$$\begin{aligned}\underline{\mathbf{Y}}_t &= \mathbf{V}_t + \mathbf{A}_t\mathbf{V}_{t-1} + \mathbf{A}_t\mathbf{A}_{t-1}\mathbf{V}_{t-2} \\ &\quad + [\mathbf{A}_t\cdots\mathbf{A}_{t-m+1}]\mathbf{V}_{t-m} + [\mathbf{A}_t\cdots\mathbf{A}_{t-m}]\underline{\mathbf{Y}}_{t-m-1} \\ &= \underline{\mathbf{Y}}_{t,m} + \mathbf{R}_{t,m},\end{aligned}$$

where

$$\underline{\mathbf{Y}}_{t,m} = \sum_{j=0}^m \mathbf{B}_j\mathbf{V}_{t-j}, \qquad \mathbf{R}_{t,m} = \mathbf{B}_{m+1}\underline{\mathbf{Y}}_{t-m-1}, \qquad \text{and}$$

$$\mathbf{B}_j = \begin{cases} \prod_{l=0}^{j-1} \mathbf{A}_{t-l}, & j \geq 1 \\ \mathbf{I}, & j = 0. \end{cases}$$

The stationarity of an $\mathcal{F}_t$-measurable solution for y_t involves the convergence of $\{\sum_{j=0}^m \mathbf{B}_j\mathbf{V}_{t-j}\}$ and $\{\mathbf{R}_{t,m}\}$ as m increases, for fixed t. Following a similar analysis as reported by Nicholls and Quinn (1982, chap. 2), we need to verify that $\operatorname{vec} E[\underline{\mathbf{Y}}_{t,m}\underline{\mathbf{Y}}_{t,m}^\top]$ converges as $m \to \infty$. Note that $\mathbf{B}_j$ is independent with $\mathbf{V}_{t-j}$ and $\{u_t,\ t = 0, \pm 1, \pm 2, \ldots\}$ are independent random variables; thus $\{\mathbf{B}_j\mathbf{V}_{t-j}\}_{j=0}^\infty$ is an orthogonal sequence in the sense that $E[\mathbf{B}_j\mathbf{V}_{t-j}\mathbf{B}_k\mathbf{V}_{t-k}] = \mathbf{0}$ for any $j \neq k$. Therefore,

$$\begin{aligned}\operatorname{vec} E[\underline{\mathbf{Y}}_{t,m}\underline{\mathbf{Y}}_{t,m}^\top] &= \operatorname{vec} E\left[\left(\sum_{j=0}^m \mathbf{B}_j\mathbf{V}_{t-j}\right)\left(\sum_{j=0}^m \mathbf{B}_j\mathbf{V}_{t-j}\right)^\top\right] \\ &= \operatorname{vec} E\left[\sum_{j=0}^m \mathbf{B}_j\mathbf{V}_{t-j}\mathbf{V}_{t-j}^\top\mathbf{B}_j^\top\right].\end{aligned}$$

Noting that $\mathrm{vec}(\mathbf{ABC}) = (\mathbf{C}^\top \otimes \mathbf{A})\,\mathrm{vec}(\mathbf{B})$ and $(\prod_{l=0}^{j}\mathbf{A}_l) \otimes (\prod_{k=0}^{j}\mathbf{B}_k) = \prod_{k=0}^{j}(\mathbf{A}_k \otimes \mathbf{B}_k)$, we have

$$\begin{aligned}\mathrm{vec}\,E\left[\sum_{j=0}^{m}\mathbf{B}_j\mathbf{V}_{t-j}\mathbf{V}_{t-j}^\top\mathbf{B}_j^\top\right] &= E\left[\sum_{j=0}^{m}(\mathbf{B}_j\otimes\mathbf{B}_j)\,\mathrm{vec}(\mathbf{V}_{t-j}\mathbf{V}_{t-j}^\top)\right]\\ &= E\left[\sum_{j=0}^{m}\left(\prod_{l=0}^{j-1}\mathbf{A}_{t-l}\right)\otimes\left(\prod_{l=0}^{j-1}\mathbf{A}_{t-l}\right)\mathrm{vec}(\mathbf{V}_{t-j}\mathbf{V}_{t-j}^\top)\right]\\ &= \sum_{j=0}^{m}\prod_{l=0}^{j-1}E(\mathbf{A}_{t-l}\otimes\mathbf{A}_{t-l})\,\mathrm{vec}\,E(\mathbf{V}_{t-j}\mathbf{V}_{t-j}^\top).\end{aligned}$$

If we write

$$\mathbf{A} = E[\mathbf{A}_t] = \begin{bmatrix}\overline{\boldsymbol{\mu}}_{p-1} & \boldsymbol{\alpha}_p\\ \mathbf{I}_{p-1} & \mathbf{0}_{p-1}\end{bmatrix},$$

where $\overline{\boldsymbol{\mu}}_{p-1} = [\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_{p-1}]$, then $\mathbf{A}_t = \mathbf{A} + \boldsymbol{\Xi}_t$, where $E(\boldsymbol{\Xi}_t) = \mathbf{0}$, and

$$\begin{aligned}E(\mathbf{A}_{t-l}\otimes\mathbf{A}_{t-l}) &= E[(\mathbf{A}+\boldsymbol{\Xi}_t)\otimes(\mathbf{A}+\boldsymbol{\Xi}_t)]\\ &= \mathbf{A}\otimes\mathbf{A} + \mathrm{E}(\boldsymbol{\Xi}_t\otimes\boldsymbol{\Xi}_t) = \boldsymbol{\Omega}_{\mathbf{A}},\end{aligned}$$

then

$$\mathrm{vec}\,E\left[\left(\sum_{j=0}^{m}\mathbf{B}_j\mathbf{V}_{t-j}\right)\left(\sum_{j=0}^{m}\mathbf{B}_j\mathbf{V}_{t-j}\right)^\top\right] = \sum_{j=0}^{m}\boldsymbol{\Omega}_{\mathbf{A}}^j\,\mathrm{vec}(\boldsymbol{\Sigma}).$$

The critical condition for the stationarity of the process $\underline{y}_t$ is that $\sum_{j=0}^{m}\boldsymbol{\Omega}_{\mathbf{A}}^j$ converges as $m\to\infty$.

The matrix $\boldsymbol{\Omega}_{\mathbf{A}}$ may be represented in Jordan canonical form as $\boldsymbol{\Omega}_{\mathbf{A}} = \mathbf{P}\boldsymbol{\Lambda}\mathbf{P}^{-1}$, where $\boldsymbol{\Lambda}$ has the eigenvalues of $\boldsymbol{\Omega}_{\mathbf{A}}$ along its main diagonal. If the eigenvalues of $\boldsymbol{\Omega}_{\mathbf{A}}$ have moduli less than unity, then $\boldsymbol{\Lambda}^j$ converges to $\mathbf{0}$ at a geometric rate. Noting that $\boldsymbol{\Omega}_{\mathbf{A}}^j = \mathbf{P}\boldsymbol{\Lambda}^j\mathbf{P}^{-1}$, following a similar analysis as done by Nicholls and Quinn (1982, chap. 2), $\underline{\mathbf{Y}}_t$ (and thus y_t) is stationary and can be represented as

$$\underline{\mathbf{Y}}_t = \sum_{j=0}^{\infty}\mathbf{B}_j\mathbf{V}_{t-j}.$$

The central limit theorem then follows from work of Billingsley (1961) (also see Nicholls and Quinn 1982, thm. A.1.4).

A.2 Proof of Theorem 3.1

If we denote $\widehat{\mathbf{v}} = \sqrt{n}(\widehat{\boldsymbol{\theta}}(\tau) - \boldsymbol{\theta}(\tau))$, then $\rho_\tau(\mathbf{y}_t - \widehat{\boldsymbol{\theta}}(\tau)^\top\mathbf{x}_t) = \rho_\tau(u_{t\tau} - (n^{-1/2}\widehat{\mathbf{v}})^\top\mathbf{x}_t)$, where $u_{t\tau} = y_t - \mathbf{x}_t^\top\boldsymbol{\theta}(\tau)$. Minimization of (8) is equivalent to minimizing

$$Z_n(\mathbf{v}) = \sum_{t=1}^{n}\left[\rho_\tau\left(u_{t\tau} - (n^{-1/2}\mathbf{v})^\top\mathbf{x}_t\right) - \rho_\tau(u_{t\tau})\right]. \tag{A.2}$$

If $\widehat{\mathbf{v}}$ is a minimizer of $Z_n(\mathbf{v})$, then we have $\widehat{\mathbf{v}} = \sqrt{n}(\widehat{\boldsymbol{\theta}}(\tau) - \boldsymbol{\theta}(\tau))$. The objective function $Z_n(\mathbf{v})$ is a convex random function. Knight (1989) (also see Pollard 1991; Knight 1998) showed that if the finite-dimensional distributions of $Z_n(\cdot)$ converge weakly to those of $Z(\cdot)$ and $Z(\cdot)$ has a unique minimum, then the convexity of $Z_n(\cdot)$ implies that $\widehat{\mathbf{v}}$ converges in distribution to the minimizer of $Z(\cdot)$.

We use the following identity: If we denote $\psi_\tau(u) = \tau - I(u<0)$, then for $u \neq 0$,

$$\begin{aligned}\rho_\tau(u-v) - \rho_\tau(u) &= -v\psi_\tau(u) + (u-v)\{I(0>u>v) - I(0<u<v)\}\\ &= -v\psi_\tau(u) + \int_0^v\{I(u\le s) - I(u<0)\}\,ds.\end{aligned} \tag{A.3}$$

Thus the objective function of minimization problem can be written as

$$\begin{aligned}&\sum_{t=1}^{n}\left[\rho_\tau\left(u_{t\tau} - (n^{-1/2}\mathbf{v})^\top\mathbf{x}_t\right) - \rho_\tau(u_{t\tau})\right]\\ &\quad= -\sum_{t=1}^{n}(n^{-1/2}\mathbf{v})^\top\mathbf{x}_t\psi_\tau(u_{t\tau})\\ &\qquad+ \sum_{t=1}^{n}\int_0^{(n^{-1/2}\mathbf{v})^\top\mathbf{x}_t}\{I(u_{t\tau}\le s) - I(u_{t\tau}<0)\}\,ds.\end{aligned}$$

We first consider the limiting behavior of

$$W_n(\mathbf{v}) = \sum_{t=1}^{n}\int_0^{(n^{-1/2}\mathbf{v})^\top\mathbf{x}_t}\{I(u_{t\tau}\le s) - I(u_{t\tau}<0)\}\,ds.$$

For convenience of asymptotic analysis, we denote

$$W_n(\mathbf{v}) = \sum_{t=1}^{n}\xi_t(\mathbf{v}),$$

$$\xi_t(\mathbf{v}) = \int_0^{(n^{-1/2}\mathbf{v})^\top\mathbf{x}_t}\{I(u_{t\tau}\le s) - I(u_{t\tau}<0)\}\,ds.$$

We further define $\overline{\xi}_t(\mathbf{v}) = E\{\xi_t(\mathbf{v})|\mathcal{F}_{t-1}\}$ and $\overline{W}_n(\mathbf{v}) = \sum_{t=1}^{n}\overline{\xi}_t(\mathbf{v})$. Then $\{\xi_t(\mathbf{v}) - \overline{\xi}_t(\mathbf{v})\}$ is a martingale difference sequence.

Note that

$$u_{\tau t} = y_t - \mathbf{x}_t^\top\boldsymbol{\alpha}(\tau) = y_t - F_{t-1}^{-1}(\tau),$$

$$\begin{aligned}\overline{W}_n(\mathbf{v}) &= \sum_{t=1}^{n}E\left\{\int_0^{(n^{-1/2}\mathbf{v})^\top\mathbf{x}_t}[I(u_{t\tau}\le s) - I(u_{t\tau}<0)]\Big|\mathcal{F}_{t-1}\right\}\\ &= \sum_{t=1}^{n}\int_0^{(n^{-1/2}\mathbf{v})^\top\mathbf{x}_t}\left[\int_{F_{t-1}^{-1}(\tau)}^{s+F_{t-1}^{-1}(\tau)}f_{t-1}(r)\,dr\right]ds\\ &= \sum_{t=1}^{n}\int_0^{(n^{-1/2}\mathbf{v})^\top\mathbf{x}_t}\frac{F_{t-1}(s+F_{t-1}^{-1}(\tau)) - F_{t-1}(F_{t-1}^{-1}(\tau))}{s}s\,ds.\end{aligned}$$

Under assumption A.3,

$$\begin{aligned}\overline{W}_n(\mathbf{v}) &= \sum_{t=1}^{n}\int_0^{(n^{-1/2}\mathbf{v})^\top\mathbf{x}_t}f_{t-1}(F_{t-1}^{-1}(\tau))s\,ds + o_p(1)\\ &= \frac{1}{2n}\sum_{t=1}^{n}f_{t-1}(F_{t-1}^{-1}(\tau))\mathbf{v}^\top\mathbf{x}_t\mathbf{x}_t^\top\mathbf{v} + o_p(1).\end{aligned}$$

By our assumptions and stationarity of y_t, we have

$$\overline{W}_n(\mathbf{v}) \Rightarrow \frac{1}{2}\mathbf{v}^\top\boldsymbol{\Omega}_1\mathbf{v}.$$

Using the same argument as that of Herce (1996), the limiting distribution of $\sum_t\xi_t(\mathbf{v})$ is the same as that of $\sum_t\overline{\xi}_t(\mathbf{v})$.

For the behavior of the first term, $n^{-1/2}\sum_{t=1}^{n}\mathbf{x}_t\psi_\tau(u_{t\tau})$, in the objective function, note that $\mathbf{x}_t\in\mathcal{F}_{t-1}$ and $E[\psi_\tau(u_{t\tau})|\mathcal{F}_{t-1}] = 0$, $\mathbf{x}_t\psi_\tau(u_{t\tau})$ is a martingale difference sequence, and thus $n^{-1/2}\times\sum_{t=1}^{n}\mathbf{x}_t\psi_\tau(u_{t\tau})$ satisfies a central limit theorem. Following the arguments of Portnoy (1984) and Gutenbrunner and Jurečková (1992), the QAR process is tight, and thus the limiting variate, viewed as a random function of τ, is a Brownian bridge over $\tau\in\mathcal{T}$,

$$n^{-1/2}\sum_{t=1}^{n}\mathbf{x}_t\psi_\tau(u_{t\tau}) \Rightarrow \boldsymbol{\Omega}_0^{1/2}\mathbf{B}_k(\tau).$$

For each fixed τ, $n^{-1/2}\sum_{t=1}^{n}\mathbf{x}_t\psi_\tau(u_{t\tau})$ converges to a q-dimensional vector-normal variate with covariance matrix $\tau(1-\tau)\boldsymbol{\Omega}_0$. Thus

$$\begin{aligned}Z_n(\mathbf{v}) &= \sum_{t=1}^{n}\big[\rho_\tau\big(u_{t\tau}-(n^{-1/2}\mathbf{v})^\top\mathbf{x}_t\big)-\rho_\tau(u_{t\tau})\big]\\ &= -\sum_{t=1}^{n}(n^{-1/2}\mathbf{v})^\top\mathbf{x}_t\psi_\tau(u_{t\tau})\\ &\quad+\sum_{t=1}^{n}\int_0^{(n^{-1/2}\mathbf{v})^\top\mathbf{x}_t}\{I(u_{t\tau}\le s)-I(u_{t\tau}<0)\}\,ds\\ &\Rightarrow -\mathbf{v}^\top\boldsymbol{\Omega}_0^{1/2}\mathbf{B}_k(\tau)+\frac{1}{2}\mathbf{v}^\top\boldsymbol{\Omega}_1\mathbf{v}=Z(\mathbf{v}).\end{aligned}$$

By the convexity lemma of Pollard (1991) and arguments of Knight (1989), note that $Z_n(\mathbf{v})$ and $Z(\mathbf{v})$ are minimized at $\widehat{\mathbf{v}}=\sqrt{n}(\widehat{\boldsymbol{\alpha}}(\tau)-\boldsymbol{\alpha}(\tau))$ and $\boldsymbol{\Sigma}^{1/2}\mathbf{B}_k(\tau)$. By lemma A of Knight (1989), we have

$$\boldsymbol{\Sigma}^{-1/2}\sqrt{n}(\widehat{\boldsymbol{\alpha}}(\tau)-\boldsymbol{\alpha}(\tau))\Rightarrow\mathbf{B}_k(\tau).$$

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